ON SYSTEMATIZATION OF SOLUTIONS BY NUCLEI OF STRAIN IN THREE-DIMENSIONAL ELASTICITY

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PREFACE

The linear theory of elasticity deals with stresses and deformations in solid bodies. The basic problem in the theory of elasticity is to solve the general equations of equilibrium combined with boundary conditions which fix the values either of the stress or of the displacement. There are several types of solutions of the linear theory of elasticity, but an interesting group of solutions is Nuclei of Strain. The fundamental solution for a single force applied at a point in an isotropic solid of indefinite extent was given by Kelvin. From his solution, other nuclei can be obtained by processes of superposition, differentiation, and integration.

To reduce the amount of guesswork as much as possible through systematizing the solutions of three-dimensional problems of elasticity by nuclei of strain is the concern of the present thesis. This need for such a systematization was felt by the author while attacking the problem of a concentrated force at an interior point in quarter-space with fixed boundary.

In the introductory chapter, we present the basic equations of equilibrium in terms of Galerkin vectors, from which the components of displacements and stress can be computed. For convenience, the basic equations were stated in terms of displacements, rather than stresses. In chapter II, we present the fundamental nuclei of strain, their representation in terms of Galerkin vectors, and equations for computing displacements of nuclei of strain. Formulas for combination of nuclei of strain to produce zero components of displacement in x- and y- directions and a non-zero one term component in z-direction at the boundary are derived in chapter III, after giving the formula for obtaining the n^{th} partial derivative of R^{2p} with respect to z. The chapter is concluded with a proof for the formulae.

In Chapter IV, similar formulae for combinations of nuclei producing at the boundary zero components of displacement in y- and z directions and a non-zero one term component in x-direction are derived. The derivation of formulae for combinations of nuclei producing at the boundary zero component of displacement in x- and z-directions and a non-zero one term component in y-direction is also included. At the end of chapter IV, applications of the formulæ in solving some problems by nuclei of strain are given as a concluding section.

It is worthy of mention that the derived formulæ were written in a way easy for programming. The author had in mind the application of Computer's methods to systematize and generalize facts about nuclei of strain, some of which were obtained by Mark Lesley in his Masters Thesis (available at the American University of Beirut, Lebanon).

The author hopes that the results reported here will help in the complete and final systematization of solutions of three-dimensional elasticity problems by Nuclei of Strain.

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SYMBOLS USED IN THE THESIS

The symbols are collected here for reference. They are listed according to the order of their appearance in the text.

K	Resultant body force per unit volume
T	Stress dyadic
S_x, S_y, S_z	The three vectors of force
σ_{xx}, σ_{yy}, σ_{zz}	Normal components of stress
σ_{xz}, σ_{xy}, σ_{yz}	Shearing components of stress
$\underline{i}$, $\underline{j}$, $\underline{k}$	Unit vectors along x-, y-, z-axis respectively
G	Modulus of rigidity
∇	Gradient
δ_{zx}, $\delta_x^{(n)}$	Partial derivative with respect to subscript variable, with a superscript for order of differentiation.
ν	Poisson's Ratio
$\underline{u}$	Displacement vector
u_x, u_y, u_z	Components of Displacement
Δ	Laplace's Operator
$\underline{F}$	Galerkin vector
F_x, F_y, F_z	Galerkin Functions
ϕ	Strain Potential
R	Distance from (x,y,z) to the origin
X, Y, Z, C	Single forces in x-, y-, z-directions, Center of Compression

$\sigma_x X$, $\sigma_x^{(n)} Z$ etc.	Nuclei derived from single forces as indicated by operations.
R_1, R_2	Distance from (x,y,z) to (0,0,c), distance from (x,y,z) to (0,0,-c).
X_2, Z_2 etc.	Nuclei located at (0,0,-c)
X_1, Z_1 etc.	" " " (0,0,c).
R.	$(x^2 + y^2 + c^2)^{\frac{1}{2}}$.
$\left[\frac{n}{2}\right]$, $\left[\frac{n}{2} - j + 1\right]$	Greatest integer in $\frac{n}{2}$, $(\frac{n}{2} - j + 1)$ respectively.
N	Combinations of nuclei producing a non-zero one term z component of displacement, zero x- and y- displacement at z = 0. See p. 17.
$\xrightarrow{z = 0}$	Produces at z = 0
t	Numerical coefficients of terms in the partial derivatives of $\frac{1}{R}$ with respect to z. See p. 18.
$n^{2q}_{4k+4q+1}$, $n^{2q+1}_{4k+4q+3}$	Numerical coefficients of values of $2Gu_z$ corresponding to the combinations $N^{2q}_{4k+4q+1}$, $N^{2q+1}_{4k+4q+3}$ respectively.
m	Numerical coefficient of terms in the partial derivatives of $\sigma_x \frac{1}{R}$ with respect to z. See p. 18.
r	Numerical coefficient of terms in the partial derivatives of $\frac{x^2}{R^5}$ with respect to z. See p. 19.
d	Numerical coefficient of terms in the partial derivatives of $\frac{x^3}{R^7}$ with respect to z. See p. 19.
$c \to z$, $z \to c$	c is replaced by z, z is replaced by c.

$\alpha_{2k,2j/2q+1+2j}$ Numerical coefficients of terms in the $(2k)^{th}$ partial derivative of $t^{2q}_{2q+1+2j} \times \frac{x^{2j}}{R^{2q+1+2j}}$ $j = 0,1,2,\ldots q$, with respect to z. See p. 31.

γ Combinations of nuclei producing a non-zero one term x-component of displacement, zero components of displacement in y- and z directions at $z = 0$. See p. 38.

Chapter I

INTRODUCTION

1. HISTORICAL BACKGROUND

In 1821, the first attempt to deduce general equations of equilibrium was made by Navier. Starting with the picture of molecular interaction, Navier deduced three differential equations for displacements in the interior of an isotropic elastic solid . These equations contained only one elastic constant. It was Cauchy who, in 1822, derived the basic differential equations used today for displacements in an isotropic material. [6]*

The fundamental particular solution of the problem of a single force acting at a point in the interior of an infinite solid was given by Lord Kelvin in 1848. But, it was E. Betti who first applied the method of singularities to the theory of elasticity. Betti started his work with a certain reciprocal theorem from which he deduced a formula for determining the average strain produced by given forces applied in a body [1] . Starting with Kelvin's solution, an infinite number of solutions, known as nuclei of strain, can be obtained by superposition, differentiation and integration. [2] . Later on, Boussinesq and Cerruti derived solutions for the first, second and certain types of mixed boundary-value problems of the semi-infinite region bounded by a plane. They used potential theory

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* Numbers inside square brackets refer to correspondingly numbered entries in the bibliography.

in finding solutions for special problems[4]. Boussinesq, in 1878, solved the problem of a normal force acting on the plane surface of a large solid. While Cerruti solved the corresponding problem of a tangential force in 1882.

Love in his Treatise [1] obtained a certain function, called the strain function, which has been found useful in simplifying the derivations in the problems cited above. However, in 1930, Galerkin set up three strain functions which were interpreted as the three components of a vector, called the Galerkin vector. Mindlin, in 1936, used the Galerkin vector to solve the problem of a single force applied at an interior point in a semi-infinite elastic solid. The force acted at some finite distance from the surface in any direction. (See [6]). Recently, Mindlin and Cheng in [2] found out the stress functions for forty nuclei of strain, which were derived from the solution of the single force in the interior of the semi-infinite solid.

In 1953, Mindlin used the Papkovitch function approach to solve his problem of a force at a point in the interior of a semi-infinite solid. Similarly, Rongved solved in 1955 the same problem with the condition that the boundary is fixed. In 1956, W. Hijab showed that the problems with mixed boundary conditions can also be solved using Papkovitch functions.

In 1959-1960, an analysis was made by Chattarji and Dutt for the stresses due to a nucleus in the form of a center of dilatation, and another in the form of a center of rotation in an elastic infinite solid with a rigid spherical inclusion. Use was made of the stress function approach to an axisymmetric problem of elasticity. Other special problems dealing with nuclei of strain have been solved during the past few years.

2. EQUILIBRIUM EQUATIONS OF ELASTICITY. (see, [6])

The complete system of equations of equilibrium of a homogeneous isotropic elastic solid is made up of the following equations:

a) Differential equations of Equilibrium

If $\underline{K} = \underline{i}k_x + \underline{j}k_y + \underline{k}k_z$ is the resultant body force per unit of volume, then

$$\text{div.}\ \underline{T} + \underline{K} = 0 \qquad (1)$$

where $\underline{T} = s_x\underline{i} + s_y\underline{j} + s_z\underline{k}$ called the stress dyadic

in which $s_x = \underline{i}\,\sigma_{xx} + \underline{j}\,\sigma_{xy} + \underline{k}\,\sigma_{xz}$ * ---------(x,y,z; i,j,k)

where σ_{xx}, σ_{yy}, σ_{zz} are normal stresses and σ_{xy}, σ_{xz}, σ_{yz} are shearing stresses,

and $\underline{i}$, $\underline{j}$, $\underline{k}$ are the unit vectors along x-y- and z-axes respectively.

b) Generalized Hooke's Law

One statement of Hooke's Law is the following:

$$s_x = G(\nabla u_x + \delta_x \underline{u}) + \underline{i}\,\frac{2\nu G}{1-2\nu}\ \text{div}\ \underline{u} \ \text{-------}(x,y,z;\ i,j,k) \qquad (2)$$

where G is the modulus of rigidity, ν is Poisson's ratio, ∇ is the Gradient, and $\underline{u}$ is the displacement vector

$$\underline{u} = \underline{i}u_x + \underline{j}u_y + \underline{k}u_z.$$

* The notation -------(x,y,z; i,j,k) after an equation will mean that the equation remains true after cyclic interchange of x,y,z and i,j,k simultaneously. Then the equation written stands for three.

Here, we have used the symbol ∂ to stand for partial differentiation with a subscript to denote the variable of differentiation. This will be used throughout the thesis with a superscript denoting the order of differentiation. When the superscript is omitted, the order is meant to be the first.

To the above equations combined with boundary conditions, one must adjoin the equations of compatibility given in [4].

Using the above system of equations, one can derive the so-called basic equation of equilibrium

$$G\left(\Delta + \frac{1}{1-2\nu} \nabla \operatorname{div}\right) \underline{u} + \underline{K} = 0$$

where Δ is Laplace's operator.

When the body forces are negligible, the basic equation of equilibrium takes the form:

$$G\left(\Delta + \frac{1}{1-2\nu} \nabla \operatorname{div}\right) \underline{u} = 0 \qquad (3).$$

Note: we will assume zero body forces throughout this thesis.

c) Galerkin vector

To solve a problem is to find a vector satisfying (3). One form of solution, useful in this thesis, is the Galerkin vector, with Galerkin functions as components. The displacement vector is derived from the Galerkin vector by the formula:

$$2G\underline{u} = \left[2(1-\nu)\Delta - \nabla \operatorname{div}\right] \underline{F} \qquad (4)$$

where $\underline{F} = \underline{i}F_x + \underline{j}F_y + \underline{k}F_z$ is the Galerkin vector, and F_x, F_y, F_z are the Galerkin functions. In order that (4) satisfies the basic equation (3),

$$\Delta\Delta\underline{F} = 0.$$

A vector function $\underline{F}$ satisfying $\Delta\Delta \underline{F} = 0$ is a biharmonic vector function. The components of displacement are defined from (4) by the equation:

$$2Gu_x = 2(1-\nu)\Delta F_x - \sigma_x \operatorname{div} \underline{F} \text{ --------- } (x,y,z) \qquad (5).$$

The components of stress can be obtained from (5) by the aid of Hooke's Law. In the present thesis, we will be concerned with components of displacement. Therefore we will omit any discussion or formulae for stresses, which can be derived at any time by Hooke's Law.

If the Galerkin vector is not only a biharmonic vector, function, but also harmonic

i.e. $\Delta \underline{F} = 0$, then the displacement vector becomes:

$$2G\underline{u} = - \nabla \operatorname{div}. \underline{F} \qquad (6).$$

One can write that the strain potential $\phi = - \operatorname{div}. \underline{F}$, so that the displacements are defined by

$$2G\underline{u} = \nabla \phi \qquad (6').$$

Chapter II

NUCLEI OF STRAIN

1. SOME FUNDAMENTAL NUCLEI OF STRAIN [1] THEIR GALERKIN VECTORS

a) Single force

A single force is acting at the origin of an elastic solid with an arbitrary point (x,y,z). It can be shown that the displacements and stresses are singular at the origin. Therefore, this point is deleted from the solid by enclosing it in a sphere S of small radius; the solutions in the remaining region correspond to the deformation present in a solid with a cavity S subjected to the action of forces with a resultant equal to the single force, such that the displacements and stresses vanish at infinite distance from the origin.

The Galerkin vectors for the single force are

$\underline{F} = \underline{i}R$	Single force in x-direction
$\underline{F} = \underline{j}R$	" " " y-direction
$\underline{F} = \underline{k}R$	" " " z -direction

where R is the distance from (x,y,z) to the origin [7].

b) Double Force in x-direction

Two equal and opposite forces are applied in the direction of x-axis at (0,0,0) and (h,0,0). Passing to a limit by supposing that h is diminished indefinitely, we describe the singularity as a double force in x-direction. The displacements and stresses are then the partial derivatives with respect to x of the corresponding displacements and stresses for the single force.

The Galerkin vector producing these displacements is found by a similar differentiation of the vector for the single force.

$$\underline{F} = \underline{i}\ \frac{x}{R} \qquad \text{double force in x-direction}$$

Similarly, the Galerkin vectors for other directions can be found out.

c) <u>Double force in x-direction with moment about y-axis</u>

A force acts at the origin in the direction of x-axis, and an equal but opposite force is superposed at (0,h,0). Passing to a limit as before, we get a double force in x-direction with moment about z-axis. The forces applied to the body in the neighbourhood of the origin are equivalent to a couple about the axis of z. The components of displacement are the partial derivatives with respect to y of the corresponding components for the single force.

The Galerkin vector is $\underline{F} = \underline{i}\ \frac{y}{R}$ (double force in x-direction with moment about z-axis).

c') <u>Center of Rotation</u>

We may combine two double forces with moment, the moments being about the same axis and of the same sign, and the directions of the forces being orthogonal. The singularity is a center of rotation about the axis of z.

d) <u>Center of Compression</u>

We may superpose three double forces without moment, having their directions parallel to the axes of coordinates, at the origin. The singularity is described as a "center of Compression". The point of singularity is enclosed in a cavity within the body.

The Galerkin vector is obtained as in the previous cases:

$$\underline{F} = \underline{i}\frac{x}{R} + \underline{j}\frac{y}{R} + \underline{k}\frac{z}{R} \qquad \text{Center of Compression.}$$

Starting with a single force or a center of compression, one may differentiate or integrate an arbitrary number of times with the condition that the displacements should vanish at infinite distance from the origin. Doing this, one may obtain an indefinite number of nuclei. It is upon this fact and the validity of the principle of superposition that most of the results obtained in this thesis depend.

2. <u>NOTATION FOR NUCLEI OF STRAIN</u>

We will use the capital letters X, Y, Z to denote the single forces in x- y- and z directions respectively. We will also use the capital letter C to denote the center of compression. As these nuclei are considered basic, all names for other nuclei are derived from them.

The name for any nucleus derived from the four fundamental ones is found by prefixing to the name of the fundamental nucleus from which it is derived, the operators carried in the derivation of that nucleus.

For example,

$\sigma_z Z$ denotes a double force in z-direction ;

$\sigma_x^{(2)} X$ denotes a "triple" force in x-direction;

$\sigma_z X$ denotes a double force in x-direction with moment about y-axis;

$\sigma_z^{(n)} Z$ denotes a "multiple" (n times) force in z-direction.

3. DISPLACEMENTS OF NUCLEI DERIVED FROM THE SINGLE FORCES, AND C.

a) Nuclei derived from X

The Galerkin vector for the single force X is

$$\underline{F} = \underline{i}R .$$

The vector of any nucleus derived from X is found from the above equation by carrying on it the same operators that were performed on the displacements of X in the derivation. Therefore, the vector will be of the form:

$$\underline{F} = \underline{i}\, F_x .$$

Then the components u_x, u_y, and u_z of displacement are computed using equation (5). They are:

$$\begin{aligned} 2Gu_x &= 2(1-\nu)\Delta F_x - \delta_x^{(2)} F_x \\ 2Gu_y &= \qquad\qquad - \delta_{yx} F_x \\ 2Gu_z &= \qquad\qquad - \delta_{zx} F_x . \end{aligned} \qquad (7)$$

b) Nuclei derived from Y.

The Galerkin vector for the single force Y is

$$\underline{F} = \underline{j}R .$$

Therefore, the vector of any nucleus derived from Y is of the form:

$$\underline{F} = \underline{j}F_y .$$

The components u_x, u_y and u_z of displacement are computed from (5).

$$2Gu_x = \qquad - \sigma_{xy}F_y$$

$$2Gu_y = 2(1-\nu)\Delta F_y - \sigma_y^{(2)} F_y \qquad (8)$$

$$2Gu_z = \qquad - \sigma_{zy}F_y$$

c) Nuclei derived from Z.

The displacements of Z are found from the Galerkin vector for Z which is

$$\underline{F} = \underline{k}R .$$

The Galerkin vector for any nucleus derived from Z is of the form:

$$\underline{F} = \underline{k}F_z .$$

The displacements are computed by the equations:

$$2Gu_x = \qquad - \sigma_{xz}F_z$$

$$2Gu_y = \qquad - \sigma_{yz}F_z \qquad (9)$$

$$2Gu_z = 2(1-\nu)\Delta F_z - \sigma_z^{(2)} F_z .$$

d) Nuclei derived from C.

We have previously mentioned that the Galerkin vector for C may be taken to be:

$$\underline{F} = \underline{i}\frac{x}{R} + \underline{j}\frac{y}{R} + \underline{k}\frac{z}{R} ;$$

but $\nabla R = \underline{i}\frac{x}{R} + \underline{j}\frac{y}{R} + \underline{k}\frac{z}{R}$.

Therefore, the Galerkin vector for C becomes :

$$\underline{F} = \nabla R .$$

The Galerkin vector for any nucleus derived from C is found as described in part (a). This vector will be the gradient of a function found by carrying the same operators on R. Let F denote this function. Then the vector of any nucleus derived from C is of the form

$$\underline{F} = \nabla F .$$

Substituting in equation (4), the displacement vector in terms of F is

$$2G\underline{u} = \left[2(1-\nu)\Delta - \nabla \operatorname{div} \right] \nabla F.$$

Since div $\nabla = \Delta$, the displacement vector becomes:

$$2G\underline{u} = (1-2\nu) \nabla \Delta F .$$

To introduce the potential function, we will begin with a Galerkin vector for C :

$$\underline{F} = - \frac{1}{2(1-2\nu)} \nabla R .$$

So that the displacement vector becomes:

$$2G\underline{u} = - \nabla(\tfrac{1}{2} \Delta F) \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_(10)} .$$

$\frac{1}{2} \Delta F$ is called the potential function.

Therefore the components of displacement are

$$2Gu_x = - \partial_x(\tfrac{1}{2} \Delta F) \text{ -----------(x,y,z)} \quad \text{\_\_\_\_\_\_\_\_\_\_(10')} .$$

Chapter III

FORMULAE FOR VANISHING IN-PLANE DISPLACEMENTS

1. DERIVATION OF A FORMULA FOR $\delta_z^{(n)} R^{2p}$

In order that we can find general formulae for combinations of nuclei of strain producing two zero components of displacement and one non-zero term for the third component of displacement, a formula for the n^{th} partial derivative of R^{2p}, where p belongs to the rational field, with respect to one variable (x,y, or z) is required. Since

$$R^{2p} = (x^2 + y^2 + z^2)^p,$$

then

$$\delta_z^{(n)} R^{2p} = \frac{d^n}{dz^n} (b^2 + z^2)^p, \text{ where } b^2 = x^2 + y^2.$$

Starting with the formula given by Ryskik and Gradstein [3]:

$$\frac{d^n}{dz^n} (1 + az^2)^p = \frac{p(p-1)(p-2)\ldots(p-n+1)(2a\ z)^n}{(1 + az^2)^{n-p}} \left[1 + \frac{n(n-1)}{1!(p-n+1)} \cdot \frac{1 + az^2}{4az^2} + \frac{n(n-1)(n-2(n-3)}{2!(p-n+1)(p-n+2)} \cdot \left(\frac{1 + az^2}{4az^2}\right)^2 + \ldots \right] \qquad (11)$$

we derive a formula for $\frac{d^n}{dz^n} (b^2 + z^2)^p$.

$$(b^2 + z^2) = b^2(1 + \frac{z^2}{b^2})$$

$$\Rightarrow (b^2 + z^2)^p = b^{2p}(1 + \frac{z^2}{b^2})^p$$

$$\text{Let } b^2 = \frac{1}{a} \Rightarrow b^{2p} = (\frac{1}{a})^p = a^{-p}$$

$$\therefore (b^2 + z^2)^p = a^{-p}(1 + az^2)^p$$

$$\frac{d^n}{dz^n}\,(b^2+z^2)^p = a^{-p}\,\frac{d^n}{dz^n}\,(1+az^2)^p$$

Using (11), we get:

$$\frac{d^n}{dz^n}(b^2+z^2)^p = a^{-p}\,\frac{p(p-1)(p-2)\ldots(p-n+1)(2az)^n}{(1+az^2)^{n-p}}\left[1+\frac{n(n-1)}{1!(p-n+1)}\cdot\frac{1+az^2}{4az^2}\right.$$

$$\left.+\frac{n(n-1)(n-2)(n-3)}{2!(p-n+1)(p-n+2)}\cdot\left(\frac{1+az^2}{4az^2}\right)^2+\cdots\right].$$

Using the fact that $(1+az^2) = a(b^2+z^2)$, and simplifying, we get:

$$\frac{d^n}{dz^n}\,(b^2+z^2)^p = \frac{p(p-1)(p-2)\ldots(p-n+1)(2z)^n}{(b^2+z^2)^{n-p}}\left[1+\frac{n(n-1)}{1!(p-n+1)}\cdot\frac{b^2+z^2}{4z^2}\right.$$

$$\left.+\frac{n(n-1)(n-2)(n-3)}{2!(p-n+1)(p-n+2)}\cdot\left(\frac{b^2+z^2}{4z^2}\right)^2+\cdots\right].$$

Therefore

$$\sigma_z^{(n)}\,R^{2p} = \frac{p(p-1)(p-2)\ldots(p-n+1)(2z)^n}{R^{2(n-p)}}\left[1+\frac{n(n-1)}{1!(p-n+1)}\cdot\frac{R^2}{4z^2}\right.$$

$$\left.+\frac{n(n-1)(n-2)(n-3)}{2!(p-n+1)(p-n+2)}\left(\frac{R^2}{4z^2}\right)^2+\cdots\right] \qquad \text{———— (11')}.$$

Expanding and writing the first and general term of (11'), we get:

$$\sigma_z^{(n)}\,R^{2p} = \frac{p(p-1)(p-2)\ldots(p-n+1)\,2^n z^n}{R^{2(n-p)}}$$

$$+\frac{p(p-1)(p-2)\ldots(p-n+2)(n)(n-1)\,2^{n-2}z^{n-2}}{R^{2(n-p-1)}} \qquad \text{———— (12)}$$

$$+\frac{p(p-1)(p-2)\ldots(p-n+3)(n)(n-1)(n-2)(n-3)\,2^{n-5}z^{n-4}}{R^{2(n-p-2)}}+\cdots$$

$$+\frac{p(p-1)(p-2)\ldots(p-n+\left[\frac{n}{2}\right]-j+2)\,2^n.z^n}{R^{2(n-p)}}\cdot\frac{n(n-1)(n.2)\ldots(2j-1)}{\left[\frac{n}{2}-j+1\right]!}\left(\frac{R^2}{4z^2}\right)^{\left[\frac{n}{2}-j+1\right]}$$

$$+\cdots+\frac{p(p-1)(p-2)\ldots(p-n+\left[\frac{n}{2}\right]+1)\,2^n.z^n}{R^{2(n-p)}}\cdot\frac{n(n-1)(n-2)\ldots1}{\left[\frac{n}{2}\right]!}\left(\frac{R^2}{4z^2}\right)^{\left[\frac{n}{2}\right]}.$$

where $\left[\frac{n}{2}\right]$ stands for the greatest integer in $\frac{n}{2}$,

$\left[\frac{n}{2}-j+1\right]$ stands for the greatest integer in $(\frac{n}{2}-j+1)$,

and $j = 1,2,3,\ldots,\frac{n}{2}$.

It is more convenient to split (12) into 2 formulae. One formula gives the derivatives for an even order of differentiation; and the other gives the derivatives for an odd order.

Setting $n = 2k$ in (12), we get after simplication, whenever possible,

$$\delta_z^{(2k)} R^{2p} = \{p(p-1)(p-2)\ldots(p-2k+1)\} \frac{2^{2k} z^{2k}}{R^{2(2k-p)}}$$
$$+\{p(p-1)(p-2)\ldots(p-2k+2)\}\ k(2k-1)2^{2k-1} \times \frac{z^{2k-2}}{R^{2(2k-p-1)}}$$
$$+\{p(p-1)(p-2)\ldots(p-2k+3)\}\ k(2k-1)(2k-2)(2k-3)2^{2k-4} \times \frac{z^{2k-4}}{R^{2(2k-p-2)}} + \ldots \qquad (12a)$$
$$+\frac{\{p(p-1)(p-2)\ldots(p-k-j+2)\}}{(k-j+1)!}\ k(2k-1)(2k-2)\ldots(2j-1)2^{2j-1} \times \frac{z^{2j-2}}{R^{2(k-p+j-1)}}$$
$$+\ldots+\frac{\{p(p-1)(p-2)\ldots(p-k+1)\}}{(k-1)!}\frac{2(2k-1)(2k-2)\ldots1}{R^{2(k-p)}}$$

where

$$j = 1,2,3,\ \ldots,\ k.$$

Setting $n = 2k+1$ in (12), we get after simplication

$$\delta_z^{(2k+1)} R^{2p} = \{p(p-1)(p-2)\ldots(p-2k)\}\ 2^{2k+1} \times \frac{z^{2k+1}}{R^{2(2k+1-p)}}$$
$$+\{p(p-1)(p-2)\ldots(p-2k+1)\}\ 2^{2k}(2k+1)(k) \times \frac{z^{2k-1}}{R^{2(2k-p)}}$$
$$+\{p(p-1)(p-2)\ldots(p-2k+2)\}\ (2k+1)(k)(2k-1)(2k-2)2^{2k-3} \times \frac{z^{2k-3}}{R^{2(2k-p-1)}} + \ldots \qquad (12b)$$
$$+\left\{\frac{p(p-1)(p-2)\ldots(p-k-j+1)}{(k-j+1)!}\right\}(2k+1)2k(2k-1)\ldots(2j)2^{2j-1} \times \frac{z^{2j-1}}{R^{2(k-p+j)}} + \ldots$$
$$+\frac{p(p-1)(p-2)\ldots(p-k)2^2(2k+1)(2k-1)(2k-2)\ldots}{(K-1)!}\frac{2z}{R^{2(k+1-p)}}$$

where $j = 1, 2, 3, \ldots, k.$

<u>Particular cases of $\delta_z^{(n)} R^{2p}$</u>

For different rational values of p, we get the n^{th} partial derivatives for different powers of R. Setting $p = -\frac{1}{2}$, a formula for $\delta_z^{(n)} \frac{1}{R}$ is obtained.

$$\delta_z^{(2k)} R^{-1} = \frac{-\frac{1}{2}(-\frac{1}{2}-1)(-\frac{1}{2}-2)\ldots(-\frac{1}{2}-2k+1)\,2^{2k}\,z^{2k}}{R^{2(2k\;\frac{1}{2})}}$$

$$+\frac{-\frac{1}{2}(-\frac{1}{2}-1)(-\frac{1}{2}-2)\ldots(-\frac{1}{2}-2k+2)\,2^{2k-1}\,z^{2k-2}\,(k)(2k-1)}{R^{2(2k+\frac{1}{2}-1)}}$$

$$+\frac{-\frac{1}{2}(-\frac{1}{2}-1)(-\frac{1}{2}-2)\ldots(-\frac{1}{2}-2k+3)(k)(2k-1)(2k-2)(2k-3)\,2^{2k-4}\,z^{2k-4}}{R^{2(2k+\frac{1}{2}-2)}}+\ldots$$

$$+\frac{-\frac{1}{2}(-\frac{1}{2}-1)(-\frac{1}{2}-2)\ldots(-\frac{1}{2}-k-j+2)(k)(2k-1)(2k-2)\ldots(2j-1)\,2^{2j-1}\,z^{2j-2}}{(k-j+1)!\;R^{2(k+\frac{1}{2}+j-1)}}+\ldots$$

$$+\frac{-\frac{1}{2}(-\frac{1}{2}-1)(-\frac{1}{2}-2)\ldots(-\frac{1}{2}-k+1)\,2(2k-1)(2k-2)\ldots1}{(k-1)!\;R^{2(k+\frac{1}{2})}}$$

$$=1.3.5.7.\ldots.(4k-1)\,\frac{z^{2k}}{R^{4k\;1}} - \{1.3.5.7\ldots(4k-3)\}\,k(2k-1)\,\frac{z^{2k-2}}{R^{4k-1}}$$

$$+1.3.5.7.(4k-5) \times \frac{k(2k-1)(k-1)(2k-3)\,z^{2k-4}}{2R^{4k-3}}$$

$$+\ldots+(-1)^{k+j-1}\{1.3.5.7.\ldots.(2k+2j-3)\}(k)(2k-1)(2k-2)\ldots(2j-1)\,\frac{2^{j-k}\,z^{2j-2}}{(k-j+1)!\,R^{2k+2}}$$

$$\ldots\;(-1)^k\,1.3.5.7\ldots\frac{(2k-1)\,2^{1-k}(2k-1)(2k-2)\ldots.1z^0}{(k-1)!\qquad R^{2k+1}} \qquad \text{(12c)}$$

Similarly, formulas for $\delta_z^{(n)} R$, $\delta_z^{(n)} R^{-3}$ and $\delta_z^{(n)} R^{-5}$ can be obtained when $p = \frac{1}{2}$, $-\frac{3}{2}$ and $-\frac{5}{2}$ respectively.

<u>Useful consequences</u>

The first term of $\delta_z^{(2k)} R^{2p}$ is some multiple of $\frac{z^0}{R^{2(k-p)}}$.

The second " " " " " " " $\frac{z^2}{R^{2(k-p+1)}}$.

The third term of $\delta_z{}^{(2k)}R^{2p}$ is some multiple of $\frac{z^4}{R^{2(k-p+2)}}$.

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The $(j+1)^{th}$ term " " " " " $\frac{z^{2j-2}}{R^{2(k-p+j-1)}}$.

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The last term " " " " " $\frac{z^{2k}}{R^{2(2k-p)}}$.

The common ratio is $\frac{z^2}{R^2}$. Therefore, the number N of terms in the $(2k)^{th}$ partial derivative of R^{2p} with respect to z is $N = k + 1$. Similarly, the number of terms in the $(2k+1)^{th}$ partial derivative of R^{2p} with respect to z is $= N = k + 1$.

2. <u>PRELIMINARY DISCUSSION: STATEMENT OF PROBLEM, GROUPS OF FORMULAE</u>

A concentrated force in z-direction is applied at (0,0,c) in a half-space occupying the region $z \geqslant 0$, and bounded by the plane $z = 0$. (x,y,z) is considered as an arbitrary point of the body,

Nuclei will be situated at (0,0,c) and at (0,0,-c) which is symmetrical to (0,0,c) with respect to the boundary $z = 0$.

We will use,

R_2 to denote the distance from (x,y,z) to (0,0,-c),

R_1 to denote the distance from (x,y,z) to (0,0,c), and

$R_{\bullet} = (x^2 + y^2 + c^2)^{\frac{1}{2}}$.

Any nucleus subscripted with 2 is located at (0,0,-c); whereas any nucleus subscripted with 1 is located at (0,0,c).

To remove the displacements at $z = 0$ of a force at (0,0,c), nuclei are superposed at (0,0,-c). The displacements of Z_2, X_2, $\delta_z Z_2$, C_2 etc. are obtained from those of Z, X, $\delta_z Z$, C etc., respectively, by replacing $z + c$ and R_2 instead of z and R in their corresponding functions.

The formulae for combinations of nuclei fall into three groups. One group of formulae, called Group A, produces at $z = 0$, a non-zero one term z-displacement and vanishes in-plane displacements. The second group of formulae, called Group B, produces at $z = 0$ a non-zero one-term x-component of displacement and zero components of displacement in y- and z directions. The last group, called Group C, produces zero displacements in x- and z-directions, and a non-zero one-term y-component of displacement. We treat in this chapter group A of formulae for combinations of nuclei of strain. Different cases will be considered according to the different values of $2Gu_z$. $2Gu_x = 2Gu_y = 0$ in all these cases; therefore, they will often be omitted.

3. GROUP A: FORMULAE FOR COMBINATIONS OF NUCLEI OF STRAIN CORRESPONDING TO DIFFERENT TYPES OF ONE-TERM $2Gu_z$ AND $2Gu_x = 2Gu_y = 0$ AT THE BOUNDARY $z = 0$.

Notation

We use N to denote combinations of nuclei producing a non-zero one-term z-component of displacement at $z = 0$ $(2Gu_x = 2Gu_y = 0)$. Different types of combinations are denoted by subscripts and superscripts added to N. We let

(i) $N^{2q}_{4k+4q+1}$ denote the combinations of nuclei

$$\xrightarrow[z=0]{} 2Gu_z = 3.5.7\ldots(4q-1)(4q+1)\ldots(4q+4k-1)\frac{x^{2q}c^{4k+2q-1}}{R_\bullet^{4k+4q+1}}$$

(ii) $N^{2q}_{4k+4q+3}$ denote the combinations of nuclei

$$\xrightarrow[z=0]{} 2Gu_z = -3.5.7\ldots(4q-1)(4q+1)\ldots(4q+4k+1)\frac{x^{2q}c^{4k+2q+1}}{R_\bullet^{4k+4q+3}}$$

respectively, by replacing $z + c$ and R_2 instead of z and R in their corresponding functions.

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(iii) $N^{2q+1}_{4k+4q+3}$ denote the combinations of nuclei

$$\xrightarrow[z=0]{} 2Gu_z = -\frac{3.5.7\ldots(4q-1)(4q+1)\ldots(4q+4k+1)x^{2q+1}c^{4k+2q-1}}{R_{\bullet}^{4k+4q+3}}$$

(iv) $N^{2q+1}_{4k+4q+5}$ denote the combinations of nuclei

$$\xrightarrow[z=0]{} 2Gu_z = \frac{3.5.7\ldots(4q-1)(4q+1)\ldots(4q+4k+3)x^{2q+1}c^{4k+2q+1}}{R_{\bullet}^{4k+4q+5}}$$

where the superscript stands for the power of x in the numerator of $2Gu_z$ and the subscript denotes the power of R in the denominator of the z-displacement.

We also use $n^{2q}_{4k+4q+1}$, $n^{2q}_{4k+4q+3}$, $n^{2q+1}_{4k+4q+3}$ and $n^{2q+1}_{4k+4q+5}$ to stand for the numerical coefiicients of the values of $2Gu_z$ corresponding to the combinations $N^{2q}_{4k+4q+1}$, $N^{2q}_{4k+4q+3}$, $N^{2q+1}_{4k+4q+3}$ and $N^{2q+1}_{4k+4q+5}$ respectively.

We use t to denote numerical coefficients of the terms in the partial derivatives of $\frac{1}{R}$ with respect to z. A superscript is used to denote the order of defferentiation with respect to z. To differentiate between the terms of the derivatives, a certain subscript is added to t. The subscript stands for the power of R in a certain term of the indicated derivative. For instance, t^{2k}_{4k-1} denotes the numerical coefficient of the term in $\frac{1}{R^{4k-1}}$ in the $(2k)^{th}$ partial derivative of $\frac{1}{R}$ with respect to z.

We let m denote numerical coefficients of the terms in the partial derivatives of $-\frac{x}{R^3} = \delta_x \frac{1}{R}$ with respect to z. Superscripts and subscripts are used as with t. For example,

m^{2k+1}_{4k+5} denotes the numerical coefficient of the term in $\frac{x}{R^{4k+5}}$ in the $(2k+1)^{th}$ partial derivative of $-\frac{x}{R^3}$ with respect to z.

We use r to denote the numerical coefficients of the terms in the partial derivatives of $\frac{x^2}{R^5}$ with respect to z. Subscripts and superscripts are used as before. For instance,

r^{2k}_{4k+5} denotes the numerical coefficient of the term in $\frac{x^2}{R^{4k+5}}$ in the $(2k)^{th}$ partial derivative of $\frac{x^2}{R^5}$ with respect to z.

Finally we use d to stand for numerical coefficients of terms in $\frac{x^3}{R^{4k+5}}$ in the partial derivatives of $-\frac{x^3}{R^7}$ with respect to z. Subscripts and superscripts are used as before. For example, d^{2k+1}_{4k+5} denotes the numerical coefficient of the term in $\frac{x^3}{R^{4k+5}}$ in the $(2k+1)^{th}$ partial derivative of $-\frac{x^3}{R^7}$ with respect to z.

Case 1: $2Gu_z$ is a multiple of $\frac{1}{R^{2n+1}}$.

We begin with the combination $Z_2 - cC_2$. Using equations (9) and (10'), we find that $2Gu_x = 2Gu_y = 0$ and $2Gu_z = (3-4\nu)\frac{1}{R}$ at $z = 0$. To produce higher powers of R, partial derivatives of the starting combination (after c is replaced by z) with respect to z will be considered.

$\delta_z \left[Z_2 - cC_2\right]_{c \to z}$ produces $2Gu_z = (3-4\nu)\delta_z \frac{1}{R} = (3-4\nu)\frac{-z}{R^3}$, $2Gu_x = 2Gu_y = 0$

At $z = 0$, $\delta_z\left[Z_2 - cC_2\right]_{c \to z} = \delta_z Z_2 - C_2 - c\,\delta_z C_2$ produces $2Gu_z = (3-4\nu)\left\{-\frac{c}{R^3}\right\}$, $2Gu_x = 2Gu_y = 0$. When the order n of differentiation with respect to z is 2k, we have at $z = 0$,

$$\delta_z^{(2k+1)}\left[Z_2 - cC_2\right]_{c \to z} = \delta_z^{(2k+1)} Z_2 - (2k+1)\,\delta_z^{(2k)} C_2 - c\,\delta_z^{(2k+1)} C_2$$

produces $2Gu_z = (3-4\nu)\,\delta_z^{(2k+1)} \left.\frac{1}{R}\right]_{z \to c}$

Note: $(3-4\nu)$ is a factor in all values of $2Gu_z$. Henceforth, it will be omitted and on in values of $2Gu_z$.

As a sample for further reference, a list of z-components of displacement corresponding to combinations derived from Z_2-cC_2 by differentiation with respect to z is given in table I. These displacements are computed by aid of formula (12c).

In order that the z-component of displacement consists of <u>one</u> term only, linear combinations of the combinations of nuclei listed in table I are required to annul unnecessary terms in the values of $2Gu_z$. The pattern in which the terms are arranged gives immediately the method of annulment of these terms. In table II, several linear combinations of nuclei, producing at $z = 0$, $2Gu_z =$ some multiple of $\frac{1}{R_\bullet^{2n+1}}$, are given to show the method of constructing the general formula.

<u>The formulae for case 1 are:</u>

$$(i)\quad c^{2k-1}\left[\delta_z^{(2k)} Z_2 - 2k\,\delta_z^{(2k-1)} C_2 - c\,\delta_z^{(2k)} C_2\right] - \frac{t^{2k}_{4k-1}}{n^o_{4k-1}}\left[N^o_{4k-1}\right]$$

$$- \frac{t^{2k}_{4k-3}}{n^o_{4k-3}}\left[N^o_{4k-3}\right] - \cdots - \frac{t^{2k}_{2k+1}}{n^o_{2k+1}}\left[N^o_{2k+1}\right]$$

$$\xrightarrow[z=0]{} 2Gu_z = 3.5.7\ldots(4k-1)\frac{c^{4k-1}}{R_\bullet^{4k+1}} \qquad \text{(13i)}$$

where $k = 1,2,3,\ldots$

Using the summation sign, we get:

$$(i)\quad c^{2k-1}\left[\delta_z^{(2k)} Z_2 - 2k\,\delta_z^{(2k-1)} C_2 - c\,\delta_z^{(2k)} C_2\right]$$

$$- \sum_{i=0}^{k-1} \frac{t^{2k}_{4k-(2i+1)}}{n^o_{4k-(2i+1)}}\left[N^o_{4k-(2i+1)}\right]$$

$$\xrightarrow[z=0]{} 2Gu_z = 3.5.7\ldots(4k-1)\frac{c^{4k-1}}{R_\bullet^{4k+1}} \qquad \text{(13i)}$$

$$2Gu_x = 2Gu_y = 0\,.$$

TABLE I

DISPLACEMENTS CORRESPONDING TO THE FIRST EIGHT COMBINATIONS OF NUCLEI DERIVED FROM $Z_2 - cC_2$

BY DIFFERENTIATION WITH RESPECT TO Z

$$Z_2 - cC_2 \xrightarrow[z=0]{} 2Gu_z = \frac{1}{R}$$

$$\delta_z Z_2 - C_2 - c\,\delta_z C_2 \xrightarrow[z=0]{} 2Gu_z = -\frac{c}{R_{\circ}^3}$$

$$\delta_z^2 Z_2 - 2\,\delta_z C_2 - c\,\delta_z^2 C_2 \xrightarrow[z=0]{} 2Gu_z = -\frac{1}{R_{\circ}^3} + \frac{3c^2}{R_{\circ}^5}$$

$$\delta_z^3 Z_2 - 3\,\delta_z^2 C_2 - c\,\delta_z^3 C_2 \xrightarrow[z=0]{} 2Gu_z = \frac{9c}{R_{\circ}^5} - 15\frac{c^3}{R_{\circ}^7}$$

$$\delta_z^4 Z_2 - 4\,\delta_z^3 C_2 - c\,\delta_z^4 C_2 \xrightarrow[z=0]{} 2Gu_z = \frac{9}{R_{\circ}^5} - 90\frac{c^2}{R_{\circ}^7} + 3\cdot5\cdot7\,\frac{c^4}{R_{\circ}^9}$$

$$\delta_z^5 Z_2 - 5\,\delta_z^4 C_2 - c\,\delta_z^5 C_2 \xrightarrow[z=0]{} 2Gu_z = -3^2\cdot5^2\,\frac{c}{R_{\circ}^7} + 2\cdot3\cdot5^2\cdot7\,\frac{c^3}{R_{\circ}^9} - 3\cdot5\cdot7\cdot9\,\frac{c^5}{R_{\circ}^{11}}$$

$$\delta_z^6 Z_2 - 6\,\delta_z^5 C_2 - c\,\delta_z^6 C_2 \xrightarrow[z=0]{} 2Gu_z = -3^2\cdot5^2\,\frac{1}{R_{\circ}^7} + 3^3\cdot5^2\cdot7\frac{c^2}{R_{\circ}^9} - 3^4\cdot5^2\cdot7\frac{c^4}{R_{\circ}^{11}} + 3^3\cdot5\cdot7\cdot11\frac{c^6}{R_{\circ}^{13}}$$

$$\delta_z^7 Z_2 - 7\,\delta_z^6 C_2 - c\,\delta_z^7 C_2 \xrightarrow[z=0]{} 2Gu_z = 3^2\cdot5^2\cdot7^2\,\frac{c}{R_{\circ}^9} - 3^2\cdot5^2\cdot7^2\cdot9\,\frac{c^3}{R_{\circ}^{11}}$$

$$+3^2\cdot5\cdot7^2\cdot9\cdot11\frac{c^5}{R_{\circ}^{13}} - 3\cdot5\cdot7\cdot9\cdot11\cdot13\frac{c^7}{R_{\circ}^{15}}$$

$$\delta_z^8 Z_2 - 8\,\delta_z^7 C_2 - c\,\delta_z^8 C_2 \xrightarrow[z=0]{} 2Gu_z = 3^2\cdot5^2\cdot7^2\,\frac{1}{R_{\circ}^9} - 2^2\cdot3^2\cdot5^2\cdot7^2\cdot9\,\frac{c^2}{R_{\circ}^{11}}$$

$$+2\cdot3^2\cdot5^2\cdot7^2\cdot9\cdot11\frac{c^4}{R_{\circ}^{13}} - 2^2\cdot3\cdot5\cdot7^2\cdot9\cdot11\cdot13\frac{c^6}{R_{\circ}^{15}}$$

$$+3\cdot5\cdot7\cdot9\cdot11\cdot13\cdot15\,\frac{c^8}{R^{17}}$$

A factor of $(3-4\nu)$ is omitted in values for $2Gu_z$.

$2Gu_x = 2Gu_y = 0$ forall combinations.

<u>TABLE II</u>

A LIST OF COMBINATIONS OF NUCLEI PRODUCING AT THE BOUNDARY $z=0$, ONE NON-VANISHING z-DISPLACEMENT OF THE FORM OF $\frac{1}{R_\bullet^{4k+1}}$ OR $\frac{1}{R_\bullet^{4k+3}}$

$$Z_2 - cC_2 \xrightarrow{z=0} 2Gu_z = \frac{1}{R_\bullet}$$

$$\partial_z Z_2 - C_2 - c\,\partial_z C_2 \xrightarrow{z=0} 2Gu_z = -\frac{c}{R_\bullet^3}$$

$$c\left[\partial_z^2 Z_2 - 2\partial_z C_2 - c\,\partial_z^2 C_2\right] - \left[\partial_z Z_2 - C_2 - c\,\partial_z C_2\right] \xrightarrow{z=0} 2Gu_z = \frac{3c^3}{R_\bullet^5}$$

$$c^2\left[\partial_z^3 Z_2 - 3\partial_z^2 C_2 - c\,\partial_z^3 C_2\right] - 3\left[c\{\partial_z^2 Z_2 - 2\partial_z C_2 - c\,\partial_z^2 C_2\} - \{\partial_z Z_2 - C_2 - c\,\partial_z C_2\}\right]$$

$$\xrightarrow{z=0} 2Gu_z = -15\frac{c^5}{R_\bullet^7}$$

$$c^3\left[\partial_z^4 Z_2 - 4\partial_z^3 C_2 - c\,\partial_z^4 C_2\right] - 6\left[c^2\{\partial_z^3 Z_2 - 3\partial_z^2 C_2 - c\,\partial_z^3 C_2\} - 3(c\{\partial_z^2 Z_2 - 2\partial_z C_2\right.$$

$$\left. - \partial_z Z_2 - C_2 - \partial_z C_2\})\right] - 3\left[c\{\partial_z^2 Z_2 - 2\partial_z C_2 - c\,\partial_z^2 C_2\}\right.$$

$$\left. - \{\partial_z Z_2 - C_2 - c\,\partial_z C_2\}\right] \xrightarrow{z=0} 2Gu_z = 3.5.7\,\frac{c^7}{R_\bullet^9}$$

$$c^4\left[\partial_z^5 Z_2 - 5\partial_z^4 C_2 - c\,\partial_z^5 C_2\right] - 10\left[c^3\{\partial_z^4 Z_2 - 4\partial_z^3 C_2 - c\,\partial_z^4 C_2\} - 6\left(c^2\{\partial_z^3 Z_2 - 3\partial_z^2 C_2 - c\,\partial_z^3 C_2\}\right.\right.$$

$$- 3(c\{\partial_z^2 Z_2 - 2\partial_z C_2 - c\,\partial_z^2 C_2\} - \{\partial_z Z_2 - C_2 - \partial_z C_2\})\Big) - 3\Big(c\{\partial_z^2 Z_2 - 2\partial_z C_2 - c\,\partial_z^2 C_2\}$$

$$\left.- \{\partial_z Z_2 - C_2 - c\,\partial_z C_2\}\Big)\right] - 15\left[c^2\{\partial_z^3 Z_2 - 3\partial_z^2 C_2 - c\,\partial_z^3 C_2\} - 3\Big(c\{\partial_z^2 Z_2 - 2\partial_z C_2 - c\,\partial_z^2 C_2\}\right.$$

$$\left.- \{\partial_z Z_2 - C_2 - c\,\partial_z C_2\}\Big)\right] \xrightarrow{z=0} 2Gu_z = -3.5.7.9\,\frac{c^9}{R_\bullet^{11}}$$

A factor of $(3-4\nu)$ is omitted in the values for $2Gu_z$.

$$\text{(ii)}\quad c^{2k}\left[\delta_z^{(2k+1)} Z_2 - (2k+1)\delta_z^{(2k)} C_2 - c\,\delta_z^{(2k+1)} C_2\right] - \frac{t^{2k+1}_{4k+1}}{n^o_{4k+1}}\left[N^o_{4k+1}\right]$$

$$- \frac{t^{2k+1}_{4k-1}}{n^o_{4k-1}}\left[N^o_{4k-1}\right] - \frac{t^{2k+1}_{4k-3}}{n^o_{4k-3}}\left[N^o_{4k-3}\right] - \dots - \frac{t^{2k+1}_{2k+3}}{n^o_{2k+3}}\left[N^o_{2k+3}\right]$$

$$\xrightarrow[z=0]{} 2Gu_z = -\,1.3.5.7\ldots(4k+1)\,\frac{c^{4k+1}}{R_o^{4k+3}} \qquad \text{(13ii)}$$

In compact form, the formula is:

$$c^{2k}\left[\delta_z^{(2k+1)} Z_2 - (2k+1)\delta_z^{(2k)} C_2 - c\,\delta_z^{(2k+1)} C_2\right] - \sum_{i=0}^{k-1} \frac{t^{2k+1}_{4k+1-2i}}{n^o_{4k+1-2i}}\left[N^o_{4k+1-2i}\right]$$

$$\xrightarrow[z=0]{} 2Gu_z = -\,1.3.5.7\ldots(4k+1)\,\frac{c^{4k+1}}{R_o^{4k+3}},$$

$$2Gu_x = 2Gu_y = 0 .$$

Case 2: $2Gu_z$ is a multiple of $\frac{x}{R_o^{2n+3}}$.

In this case, x appears in its first power. Therefore, we will begin with the combination of nuclei $\delta_x\left[Z_2 - cC_2\right]$. This produces at $z = 0$, $2Gu_z = -\frac{x}{R_o^3}$. By differentiating this displacement n times with respect to z, one gets a term of the form of $\frac{x}{R_o^{2n+3}}$ at $z = 0$, $\delta_z^{(n)} \frac{x}{R_o^3} = -x\,\delta_z^{(n)} R^{-3}$. The terms of $\delta_z^{(n)} R^{-3}$ are obtained by formulae (12a) and (12b) after setting $p = -\frac{3}{2}$. A sample of those displacements is given in Table III. The formulae for case 2 are obtained by similar methods to case 1.

TABLE III

DISPLACEMENTS CORRESPONDING TO COMBINATIONS OF NUCLEI DERIVED FROM $\sigma_x \left[Z_2 - cC_2\right]$ BY DIFFERENTIATION WITH RESPECT TO Z

$$\sigma_x Z_2 - c\,\sigma_x C_2 \xrightarrow{z=0} 2Gu_z = -\frac{x}{R^3}$$

$$\sigma_{zx} Z_2 - \sigma_x C_2 - c\,\sigma_{zx} C_2 \xrightarrow{z=0} 2Gu_z = \frac{3xc}{R^5}$$

$$\sigma_z^2 \sigma_x Z_2 - 2\sigma_z \sigma_x C_2 - c\,\sigma_z^2 \sigma_x C_2 \xrightarrow{z=0} 2Gu_z = \frac{3x}{R^5} - 3.5\,\frac{xc^2}{R^7}$$

$$\sigma_z^3 \sigma_x Z_2 - 3\sigma_z^2 \sigma_x C_2 - c\,\sigma_z^2 \sigma_x C_2 \xrightarrow{z=0} 2Gu_z = -\frac{45xc}{R^7} + 3.5.7\,\frac{xc^3}{R^9}$$

$$\sigma_z^4 \sigma_x Z_2 - 4\sigma_z^3 \sigma_x C_2 - c\,\sigma_z^4 \sigma_x C_2 \xrightarrow{z=0} 2Gu_z = -\frac{45x}{R^7} + 2.3^2.5.7\,\frac{xc^2}{R^9} - 3.5.7.9\,\frac{xc^4}{R^{11}}$$

$$\sigma_z^5 \sigma_x Z_2 - 5\sigma_z^4 \sigma_x C_2 - c\,\sigma_z^5 \sigma_x C_2 \xrightarrow{z=0} 2Gu_z = 3^2.5^2.7\,\frac{xc}{R^9} - 2.3.5^2.7.9\,\frac{xc^3}{R^{11}}$$

$$+3.5.7.9.11\,\frac{xc^5}{R^{13}}$$

$$\sigma_z^6 \sigma_x Z_2 - 6\sigma_z^5 \sigma_x C_2 - c\,\sigma_z^6 \sigma_x C_2 \xrightarrow{z=0} 2Gu_z = 3^2.5^2.7\,\frac{x}{R^9} - 3^3.5^2.7.9\,\frac{xc^2}{R^{11}}$$

$$+3^2.5^2.7.9.11\,\frac{xc^4}{R^{13}} - 3.5.7.9.11.13\,\frac{xc^6}{R^{15}}$$

A factor of $(3-4\nu)$ is omitted in all values for $2Gu_z$.

$2Gu_x = 2Gu_y = 0$ for all these combinations.

We have:

$$\partial_x Z_2 - c\,\partial_x C_2 \quad \xrightarrow[z=0]{} 2Gu_z = -\frac{x}{R^3}.$$

$$\partial_{zx} Z_2 - \partial_x C_2 - c\,\partial_{zx} C_2 \quad \xrightarrow[z=0]{} 2Gu_z = \frac{3xc}{R^5}.$$

$$c\left[\partial_z^2\,\partial_x Z_2 - 2\,\partial_{zx} C_2 - c\,\partial_z^2\,\partial_x C_2\right] - \left[\partial_{zx} Z_2 - \partial_x C_2 - c\,\partial_{zx} C_2\right] \xrightarrow[z=0]{} 2Gu_z = -15\,\frac{xc^3}{R^7}.$$

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.when $k \neq 0$.

(i) $$c^{2k-1}\left[\partial_z^{(2k)}\,\partial_x Z_2 - 2k\,\partial_z^{(2k-1)}\,\partial_x C_2 - c\,\partial_z^{(2k)}\,\partial_x C_2\right] - \frac{m^{2k}_{4k+1}}{n^{1}_{4k+1}}\left[N^{1}_{4k+1}\right]$$

$$-\frac{m^{2k}_{4k-1}}{n^{1}_{4k-1}}\left[N^{1}_{4k-1}\right] - \ldots - \frac{m^{2k}_{2k+3}}{n^{1}_{2k+3}}\left[N^{1}_{2k+3}\right]$$

$$\xrightarrow[z=0]{} 2Gu_z = -1.3.5\ldots(4k+1)\,\frac{xc^{4k-1}}{R^{4k+3}}. \qquad \text{(14i)}$$

Using the summation sign, the formula becomes:

$$c^{2k-1}\left[\partial_z^{(2k)}\,\partial_x Z_2 - 2k\,\partial_z^{(2k-1)}\,\partial_x C_2 - c\,\partial_z^{(2k)}\,\partial_x C_2\right] - \sum_{i=0}^{k-1}\left\{\frac{m^{2k}_{4k-(2i-1)}}{n^{1}_{4k-(2i-1)}}\left[N^{1}_{4k-(2i-1)}\right]\right\}$$

$$\xrightarrow[z=0]{} 2Gu_z = -1.3.5\ldots(4k+1)\,\frac{xc^{4k-1}}{R^{4k+3}},$$

$$2Gu_x = 2Gu_y = 0.$$

(ii) $$c^{2k}\left[\partial_z^{(2k+1)}\,{}_x Z_2 - (2k+1)\,\partial_z^{(2k)}\,\partial_x C_2 - c\,\partial_z^{(2k+1)}\,\partial_x C_2\right]$$

$$-\sum_{i=0}^{k-1}\left\{\frac{m^{2k+1}_{4k-(2i-3)}}{n^{1}_{4k-(2i-3)}}\left[N^{1}_{4k-(2i-3)}\right]\right\}$$

$$\xrightarrow[z=0]{} 2Gu_z = 2.3.5.7\ldots(4k+3)\,\frac{xc^{4k+1}}{R^{4k+5}},$$

$$2Gu_x = 2Gu_y = 0. \qquad \text{(14ii)}$$

TABLE IV

DISPLACEMENTS CORRESPONDING TO COMBINATIONS OF NUCLEI DERIVED FROM $\sigma_x^2\left[Z_2 - cC_2\right]$ BY DIFFERENTIATION WITH RESPECT TO Z

$$\sigma_x^2 Z_2 - c\,\sigma_x^2 C_2 \xrightarrow[z=0]{} 2Gu_z = -\frac{1}{R^3} + \frac{3x^2}{R^5}$$

$$\sigma_z\sigma_x^2 Z_2 - \sigma_x^2 C_2 - c\,\sigma_z\sigma_x^2 C_2 \xrightarrow[z=0]{} 2Gu_z = \frac{3c}{R^5} - 15\frac{x^2c}{R^7}$$

$$\sigma_z^2\sigma_x^2 Z_2 - 2\sigma_z\sigma_x^2 C_2 - c\,\sigma_z^2\sigma_x^2 C_2 \xrightarrow[z=0]{} 2Gu_z = \frac{3}{R^5} - 15\frac{c^2}{R^7} - 15\frac{x^2}{R^7} + 15\cdot 7\frac{x^2c^2}{R^9}$$

$$\sigma_z^3\sigma_x^2 Z_2 - 3\sigma_z^2\sigma_x^2 C_2 - c\,\sigma_z^3\sigma_x^2 C_2 \xrightarrow[z=0]{} 2Gu_z = -45\frac{c}{R^7} + 15\cdot 7\frac{c^3}{R^9} + 3\cdot 15\cdot 7\frac{x^2c}{R^9} - 15\cdot 7\cdot 9\frac{x^2c^3}{R^{11}}$$

$$\sigma_z^4\sigma_x^2 Z_2 - 4\sigma_2^3\sigma_x^2 C_2 - c\,\sigma_z^4\sigma_x^2 C_2 \xrightarrow[z=0]{} 2Gu_z = -\frac{45}{R^7} + 2\cdot 7\cdot 45\frac{c^2}{R^9} - 15\cdot 7\cdot 9\frac{c^4}{R^{11}}$$

$$+3\cdot 15\cdot 7\frac{x^2}{R^9} - 2\cdot 3\cdot 15\cdot 7\cdot 9\frac{x^2c^2}{R^{11}}$$

$$+15\cdot 7\cdot 9\cdot 11\frac{x^2c^4}{R^{13}}$$

$$\sigma_z^5\sigma_x^2 Z_2 - 5\sigma_z^4\sigma_x^2 C_2 - c\,\sigma_z^5\sigma_x^2 C_2 \xrightarrow[z=0]{} 2Gu_z = 3^2\cdot 5^2\cdot 7\cdot\frac{c}{R^9} - 2\cdot 3\cdot 5^2\cdot 7\cdot 9\frac{c^3}{R^{11}}$$

$$+3\cdot 5\cdot 7\cdot 9\cdot 11\frac{c^5}{R^{13}} - 3^2\cdot 5^2\cdot 7\cdot 9\cdot\frac{x^2c}{R^{11}}$$

$$+2\cdot 3\cdot 5^2\cdot 7\cdot 9\cdot 11\frac{x^2c^3}{R^{13}} - 3\cdot 5\cdot 7\cdot 9\cdot 11\cdot 13\frac{x^2c^5}{R^{15}}$$

A multiplying constant $(3-4\nu)$ is omitted throughout the values of $2Gu_z$. $2Gu_x = 2Gu_y = 0$ for all these combinations.

Case 3: $2Gu_z$ is a multiple of $\dfrac{x^2}{R_{\bullet}^{2n+5}}$.

In this case, terms with x^2 in the numerator are required. So we begin with $\sigma_x^2\left[Z_2 - cC_2\right]$. This combination of nuclei gives at $z = 0$, $2Gu_z = -\dfrac{1}{R_{\bullet}^3} + \dfrac{3x^2}{R_{\bullet}^5}$. Partial derivatives with respect to z of this displacement are then considered. For the n^{th} partial derivative with respect to z, we have

$$\sigma_z^{(n)}\left[-\frac{1}{R_{\bullet}^3} + \frac{3x^2}{R_{\bullet}^5}\right] = \sigma_z^{(n)} - \frac{1}{R^3} + \sigma_z^{(n)}\frac{3x^2}{R^5} = -\sigma_z^{(n)}R^{-3} + 3x^2\sigma_z^{(n)}R^{-5}.$$

The terms of displacement are computed using formula (12). A sample of these derivatives is given in table IV.

In order that the displacement consists of one term only, all terms before the last in the derivatives should be annulled. Terms of the form of $\dfrac{1}{R_{\bullet}^{2n+1}}$ are annulled using combinations of the type 1.

For example:

$$c\left[\sigma_x^2 Z_2 - c\,\sigma_x^2 C_2\right] - \left[\sigma_z Z_2 - C_2 - c\,\sigma_z C_2\right] \xrightarrow[z=0]{} 2Gu_z = \frac{3x^2c}{R_{\bullet}^5}$$

$$c^2\left[\sigma_z\,\sigma_x^2 Z_2 - \sigma_x^2 C_2 - c\,\sigma_z\,\sigma_x C_2\right] - c\left\{\sigma_z^2 Z_2 - 2\sigma_z C_2 - c\,\sigma_z^2 C_2\right\} - \left\{\sigma_z Z_2 - C_2 - c\,\sigma_z C_2\right\}$$

$$\xrightarrow[z=0]{} 2Gu_z = -15\frac{x^2c^3}{R_{\bullet}^7}.$$

<u>The formulae are:</u>

$$\text{(i)}\quad c^{2k+1}\left[\sigma_z^{(2k)}\,\sigma_x^2 Z_2 - 2k\,\sigma_z^{(2k-1)}\,\sigma_x^2 C_2 - c\,\sigma_z^{(2k)}\,\sigma_x^2 C_2\right] - \frac{3r_{4k+3}^{2k}}{n_{4k+3}^2}\left[N_{4k+3}^2\right]$$

$$-\frac{3r_{4k+1}^{2k}}{n_{4k+1}^2}\left[N_{4k+1}^2\right] - \dots - \frac{3r_{2k+5}^{2k}}{n_{2k+5}^2}\left[N_{2k+5}^2\right] - \frac{m_{4k+3}^{2k}}{n_{4k+3}^o}\left[N_{4k+3}^o\right] \qquad \text{(15i)}$$

$$-\frac{m_{4k+1}^{2k}}{n_{4k+1}^o}\left[N_{4k+1}^o\right] - \dots - \frac{m_{2k+3}^{2k}}{n_{2k+3}^o}\left[N_{2k+3}^o\right] \xrightarrow[z=0]{} 2Gu_z = 3.5.7\dots(4k+3)\frac{x^2c^{4k+1}}{R_{\bullet}^{4k+5}}.$$

Using the summation sign, the formula becomes:

$$c^{2k+1}\left[\sigma_z^{(2k)}\,\sigma_x^2 Z_2 - 2k\,\sigma_z^{(2k-1)}\,\sigma_x^2 C_2 - c\,\sigma_z^{(2k)}\,\sigma_x^2 C_2\right]$$

$$-\sum_{i=0}^{k-1}\frac{3r^{2k}_{4k-1(2i-3)}}{n^{2}_{4k-(2i-3)}}\left[N^{2}_{4k-(2i-3)}\right]-\sum_{i=0}^{k}\frac{m^{2k}_{4k-(2i-3)}}{n^{o}_{4k-(2i-3)}}\left[N^{o}_{4k-(2i-3)}\right]$$

$$\xrightarrow[z=0]{} 2Gu_z = 3.5.7...(4k+3)\,\frac{x^2c^{4k+1}}{\underset{\bullet}{R}^{4k+5}},$$

$$2Gu_x = 2Gu_y = 0.$$

$$\text{(ii)}\quad c^{2k+2}\left[\sigma_z^{(2k+1)}\,\sigma_x^2 Z_2 - (2k+1)\,\sigma_z^{(2k)}\,\sigma_x^2 C_2 - c\,\sigma_z^{(2k+1)}\,\sigma_x^2 C_2\right]$$

$$-\sum_{i=0}^{k-1}\frac{3r^{2k+1}_{4k-(2i-5)}}{n^{2}_{4k-(2i-5)}}\left[N^{2}_{4k-(2i-5)}\right]-\sum_{i=0}^{k}\frac{m^{2k+1}_{4k-(2i-5)}}{n^{o}_{4k-(2i-5)}}\left[N^{o}_{4k-(2i-5)}\right]$$

$$\xrightarrow[z=0]{} 2Gu_z = -\,3.5.7...(4k+5)\,\frac{x^2c^{4k+3}}{\underset{\bullet}{R}^{4k+7}},$$

$$2Gu_x = 2Gu_y = 0. \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_(15ii)}$$

<u>Case 4: $2Gu_z$ is a multiple of $\frac{x^3}{\underset{\bullet}{R}^{2n+7}}$.</u>

In case 4, x^3 appears in the numerator. This case is treated as the previous case. The combination that we start with is $\sigma_x^3\left[Z_2 - cC_2\right]$. This combination of nuclei produces at $z=0$, $2Gu_z = \frac{9x}{\underset{\bullet}{R}^5} - \frac{15x^3}{\underset{\bullet}{R}^7}$.

So $\sigma_z^{(n)}\left[\frac{9x}{\underset{\bullet}{R}^5} - \frac{15x^3}{\underset{\bullet}{R}^7}\right] = 9x\,\sigma_z^{(n)}R^{-5} - 15x^3\,\sigma_z^{(n)}R^{-7}$.

The terms of the displacements corresponding to combinations of nuclei, which are derivatives with respect to z of the starting one are computed by aid of formulae (12a) or (12b).

The formulae for case 4 are:

(i) $$c^{2k+1}\left[\delta_z^{(2k)}\,\delta_x^3 Z_2 - 2k\,\delta_z^{(2k-1)}\,\delta_x^3 C_2 - c\,\delta_z^{(2k)}\,\delta_x^3 C_2\right] - \frac{15d^{2k}_{4k+5}}{n^3_{4k+5}}\left[N^3_{4k+5}\right]$$

$$- \frac{15d^{2k}_{4k+3}}{n^3_{4k+3}}\left[N^3_{4k+3}\right] - \cdots - \frac{15d^{2k}_{2k+7}}{n^3_{2k+7}}\left[N^3_{2k+7}\right] - \frac{9r^{2k}_{4k+5}}{n^1_{4k+5}}\left[N^1_{4k+5}\right]$$

$$- \frac{9r^{2k}_{4k+3}}{n^1_{4k+3}}\left[N^1_{4k+3}\right] - \cdots - \frac{9r^{2k}_{2k+5}}{n^1_{2k+5}}\left[N^1_{2k+5}\right] \qquad \text{(16i)}$$

$$\xrightarrow[z=0]{} 2Gu_z = -\,3.5.7\ldots(4k+5)\,\frac{x^3c^{4k+1}}{R^{4k+7}_{\bullet}}$$

or $$c^{2k+1}\left[\delta_z^{(2k)}\,\delta_x^3 Z_2 - 2k\,\delta_z^{(2k-1)}\,\delta_x^3 C_2 - c\,\delta_z^{(2k)}\,\delta_x^3 C_2\right]$$

$$- \sum_{i=0}^{k-1} \frac{15d^{2k}_{4k-(2i-5)}}{n^3_{4k-(2i-5)}}\left[N^3_{4k-(2i-5)}\right] - \sum_{i=0}^{k} \frac{9r^{2k}_{4k-(2i-5)}}{n^1_{4k-(2i-5)}}\left[N^1_{4k-(2i-5)}\right]$$

$$\xrightarrow[z=0]{} 2Gu_z = -\,3.5.7\ldots(4k+5)\,\frac{x^3c^{4k+1}}{R^{4k+7}_{\bullet}},$$

$$2Gu_x = 2Gu_y = 0\,.$$

(ii) $$c^{2k+2}\left[\delta_z^{(2k+1)}\,\delta_x^3 Z_2 - (2k+1)\delta_z^{(2k)}\,\delta_x^3 C_2 - c\,\delta_z^{(2k+1)}\,\delta_x C_2\right]$$

$$- \sum_{i=0}^{k-1} \frac{15d^{2k+1}_{4k-(2i-7)}}{n^3_{4k-(2i-7)}}\left[N^3_{4k-(2i-7)}\right] - \sum_{i=0}^{k} \frac{9r^{2k+1}_{4k-(2i-7)}}{n^1_{4k-(2i-7)}}\left[N^1_{4k-(2i-7)}\right]$$

$$\xrightarrow[z=0]{} 2Gu_z = 3.5.7\ldots(4k+7)\,\frac{x^3c^{4k+3}}{R^{4k+9}_{\bullet}}\,.$$

$$\qquad \text{(16ii)}$$

A formula for

(i) $2Gu_z$ = a multiple of $\frac{x^{2q}}{\dot{R}^{4q+1}}$

(ii) $2Gu_z$ = a multiple of $\frac{x^{2q+1}}{\dot{R}^{4q+3}}$

Before discussing the general case, we deduce a formula for the starting combinations when $2Gu_z$ = a multiple of $\frac{x^{2q}}{\dot{R}^{4q+1}}$ or $\frac{x^{2q+1}}{\dot{R}^{4q+3}}$. By replacing c by x in the terms of $2Gu_z$, and changing the variable of differentiation to x in table I, we get a new table for combinations of nuclei such that their corresponding $2Gu_z$ are of the form of $\frac{x^{2q}}{\dot{R}^{2n+4q+1}}$ or $\frac{x^{2q+1}}{\dot{R}^{2n+4q+3}}$. Therefore, the formula is constructed similarly to previous cases.

The formulae with the desired conditions are therefore:

$$\text{(i)}\quad c^{2q-1}\left[\sigma_x^{(2q)} Z_2 - c\,\sigma_x^{(2q)} C_2\right] - \frac{t^{2q}_{4q-1}}{n^{2q-2}_{4q-1}}\left[N^{2q-2}_{4q-1}\right] - \frac{t^{2q}_{4q-3}}{n^{2q-4}_{4q-3}}\left[N^{2q-4}_{4q-3}\right]$$

$$- \ldots - \frac{t^{2q}_{2q+1}}{n^{o}_{2q+1}}\left[N^{o}_{2q+1}\right]$$

$$\xrightarrow[z=0]{} 2Gu_z = 3.5.7\ldots(4q-1)\,\frac{x^{2q}c^{2q-1}}{\dot{R}^{4q+1}} \qquad \text{(17i)}$$

$$\text{or}\quad c^{2q-1}\left[\sigma_x^{(2q)} Z_2 - c\,\sigma_x^{(2q)} C_2\right] - \sum_{i=0}^{q-1} \frac{t^{2q}_{4q-(2i+1)}}{n^{2q-2(i+1)}_{4q-(2i+1)}}\left[N^{2q-2(i+1)}_{4q-(2i+1)}\right]$$

$$\xrightarrow[z=0]{} 2Gu_z = 3.5.7\ldots(4q-1)\,\frac{x^{2q}c^{2q-1}}{\dot{R}^{4q+1}},$$

$$2Gu_y = 2Gu_x = 0;$$

$$\text{(ii)}\quad c^{2q-1}\left[\sigma_x^{(2q+1)} Z_2 - c\,\sigma_x^{(2q+1)} C_2\right] - \frac{t_{4q+1}^{2q+1}}{n_{4q+1}^{2q-1}}\left[N_{4q+1}^{2q-1}\right]$$

$$- \frac{t_{4q-1}^{2q+1}}{n_{4q-1}^{2q-3}}\left[N_{4q-1}^{2q-3}\right] - \cdots - \frac{t_{2q+3}^{2q+1}}{n_{2q+3}^{1}}\left[N_{2q+3}^{1}\right]$$

$$\xrightarrow[z=0]{} 2Gu_z = -\,3.5.7\ldots(4q+1)\,\frac{x^{2q+1}c^{2q-1}}{\overset{\bullet}{R}{}^{4q+3}} \qquad \text{(17ii)}$$

$$\text{or}\quad c^{2q-1}\left[\sigma_x^{(2q+1)} Z_2 - c\,\sigma_x^{(2q+1)} C_2\right] - \sum_{i=0}^{q-1} \frac{t_{4q-(2i-1)}^{2q+1}}{n_{4q-(2i-1)}^{2q-(2i+1)}}\left[N_{4q-(2i-1)}^{2q-(2i+1)}\right]$$

$$\xrightarrow[z=0]{} 2Gu_z = -\,3.5.7\ldots(4q+1)\,\frac{x^{2q+1}c^{2q-1}}{\overset{\bullet}{R}{}^{4q+3}},$$

$$2Gu_x = 2Gu_y = 0\,.$$

<u>The General Case:</u>

(i) $\quad 2Gu_z$ = a multiple of $\dfrac{x^{2q}}{\overset{\bullet}{R}{}^{4k+4q+1}}$

(ii) $\quad 2Gu_z$ = a multiple of $\dfrac{x^{2q}}{\overset{\bullet}{R}{}^{4k+4q+3}}$

(iii) $\quad 2Gu_z$ = a multiple of $\dfrac{x^{2q+1}}{\overset{\bullet}{R}{}^{4k+4q+3}}$

(iv) $\quad 2Gu_z$ = a multiple of $\dfrac{x^{2q+1}}{\overset{\bullet}{R}{}^{4k+4q+5}}$

<u>Notation</u>

We use $\alpha^{2k,2j/2q+1+2j}$ to denote numerical coefficients of the terms in the $(2k)^{th}$ partial derivative of $t_{2q+1+2j}^{2q} \times \dfrac{x^{2j}}{\overset{\bullet}{R}{}^{2q+1+2j}}$ $(j = 0,1,\ldots,2q)$ with respect to z. The first superscript stands for the order of differentiation with respect to z, and the second superscript shows the function to be differentiated (the numerator stands for the power of x, and the denominator stands for the power of R). To differentiate between the different terms of the indicated derivative, a subscript is added to α to denote a certain term. The subscript denotes the power of R of that term in the above described derivative.

For instance,

$\alpha_{4k+4q+1}^{2k,2q/4q+1}$ denotes the coefficient of the term in $\frac{1}{\dot{R}^{4k+4q+1}}$ in the $(2k)^{th}$ partial derivative of $t_{4q+1}^{2q} \frac{x^{2q}}{\dot{R}^{4q+1}}$ with respect to z.

$\alpha^{2k+1,2j+1/2q+3+2j}$ $(j = 0,1,2,\ldots,q)$ is similarly defined.

<u>The Formulae with the Desired Conditions are:</u>

$$\text{(i)} \quad c^{2k+1}\left[\sigma_z^{(2k)}\sigma_x^{(2q)}Z_2 - 2k\sigma_z^{(2k-1)}\sigma_x^{(2q)}C_2 - c\sigma_z^{(2k)}\sigma_x^{(2q)}C_2\right]$$

$$= \frac{\alpha_{4k+4q-1}^{2k,2q/4q+1}}{n_{4k+4q-1}^{2q}}\left[N_{4k+4q-1}^{2q}\right] - \frac{\alpha_{4k+4q-3}^{2k,2q/4q+1}}{n_{4k+4q-3}^{2q}}\left[N_{4k+4q-3}^{2q}\right] - \ldots$$

$$- \frac{\alpha_{2k+4q+1}^{2k,2q/4q+1}}{n_{2k+4q+1}^{2q}}\left[N_{2k+4q+1}^{2q}\right] - \sum_{j=0}^{j=q-1}\left\{\frac{\alpha_{4k+2q+2j+1}^{2k,2j/2q+1+2j}}{n_{4k+2q+2j+1}^{2j}}\left[N_{4k+2q+2j+1}^{2j}\right]\right.$$

$$\left. + \frac{\alpha_{4k+2q+2j-1}^{2k,2j/2q+1+2j}}{n_{4k+2q+2j-1}^{2j}}\left[N_{4k+2q+2j-1}^{2q}\right] + \ldots + \frac{\alpha_{2k+2q+2j+1}^{2k,2j/2q+1+2j}}{n_{2k+2q+2j+1}^{2j}}\left[N_{2k+2q+2j+1}^{2j}\right]\right\}$$

$$\xrightarrow[z=0]{} 2Gu_z = 3.5.7\ldots(4q-1)(4q+1)(4q+3)\ldots(4q+4k-1)\ \frac{x^{2q}c^{4k+2q-1}}{\dot{R}^{4k+4q+1}}.$$

Using again the summation sign on k, the formula in its compact form is:

(i) $$c^{2k+2q-1}\left[\sigma_z^{(2k)}\sigma_x^{(2q)}Z_2-2k\sigma_z^{(2k-1)}\sigma_x^{(2q)}C_2-c\sigma_z^{(2k)}\sigma_x^{(2q)}C_2\right]$$

$$-\sum_{i=0}^{k-1}\frac{\alpha^{2k,2q/4q+1}_{4k+4q-1-2i}}{n^{2q}_{4k+4q-1-2i}}\left[N^{2q}_{4k+4q-1-2i}\right] \qquad \text{(18i)}$$

$$-\sum_{j=0}^{q-1}\sum_{i=0}^{k}\frac{\alpha^{2k,2j/2q+1+2j}_{4k+2q+2j+1-2i}}{n^{2j}_{4k+2q+2j+1-2i}}\left[N^{2j}_{4k+2q+2j+1-2i}\right]$$

$$\xrightarrow[z=0]{}2Gu_z = 3.5.7\ldots(4q-1)(4q+1)(4q+3)\ldots(4q+4k+1)\,\frac{x^{2q}c^{4k+2q-1}}{R_\bullet^{4k+4q+1}},$$

$$2Gu_x = 2Gu_y = 0.$$

(ii) $$c^{2k+2q}\left[\sigma_z^{(2k+1)}\sigma_x^{(2q)}Z_2-(2k+1)\sigma_z^{(2k)}\sigma_x^{(2q)}C_2-c\sigma_z^{(2k+1)}\sigma_x^{(2q)}C_2\right]$$

$$-\sum_{i=0}^{k-1}\frac{\alpha^{2k+1,2q/4q+1}_{4k+4q+1-2i}}{n^{2q}_{4k+4q+1-2i}}\left[N^{2q}_{4k+4q+1-2i}\right] \qquad \text{(18ii)}$$

$$-\sum_{j=0}^{q-1}\sum_{i=0}^{k}\frac{\alpha^{2k+1,2j/2q+1+2j}_{4k+2q+2j+3-2i}}{n^{2j}_{4k+2q+2j+3-2i}}\left[N^{2j}_{4k+2q+2j+3-2i}\right]$$

$$\xrightarrow[z=0]{}2Gu_z = -\,3.5.7\ldots(4q-1)(4q+1)(4q+3)\ldots(4q+4k+1)\,\frac{x^{2q}c^{4k+2q+1}}{R_\bullet^{4k+4q+3}}.$$

(iii) $$c^{2k+2q-1}\left[\sigma_z^{(2k)}\sigma_x^{(2q+1)}Z_2-2k\sigma_z^{(2k-1)}\sigma_x^{(2q+1)}C_2-c\sigma_z^{(2k)}\sigma_x^{(2q+1)}C_2\right]$$

$$-\sum_{i=0}^{k-1}\frac{\alpha^{2k,2q+1/4q+3}_{4k+4q+1-2i}}{n^{2q+1}_{4k+4q+1-2i}}\left[N^{2q+1}_{4k+4q+1-2i}\right]-\sum_{j=0}^{q-1}\sum_{i=0}^{k}\frac{\alpha^{2k,2j+1/2q+3+2j}_{4k+2q+2j+3-2i}}{n^{2j+1}_{4k+2q+2j+3-2i}}\left[N^{2j+1}_{4k+2q+2j+3-2i}\right]$$

$$\xrightarrow[z=0]{}2Gu_z = -\,3.5.7\ldots(4q-1)(4q+1)(4q+3)\ldots(4q+4k+1)\,\frac{x^{2q+1}c^{4k+2q-1}}{R_\bullet^{4k+4q+3}}.$$

(18iii)

$$(iv)\quad c^{2k+2q}\left[\delta_z^{(2k+1)}\delta_x^{(2q+1)}Z_2-(2k+1)\delta_z^{2k}\delta_x^{(2q-1)}C_2-c\,\delta_z^{(2k+1)}\delta_x^{(2q+1)}C_2\right]$$

$$-\sum_{i=0}^{k-1}\frac{\alpha^{2k+1,2q+1/4q+3}_{4k+4q+3-2i}}{n^{2q}_{4k+4q+3-2i}}\left[N^{2q+1}_{4k+4q+3-2i}\right]$$

$$-\sum_{j=0}^{q-1}\sum_{i=0}^{k}\frac{\alpha^{2k+1,2j+1/2q+3+2j}_{4k+2q+2j+5-2i}}{n^{2j+1}_{4k+2q+2j+5-2i}}\left[N^{2j+1}_{4k+2q+2j+5-2i}\right] \text{ ------------ (18iv}$$

$$\xrightarrow[z=0]{} 2Gu_z = 3.5.7\ldots(4q-1)(4q+1)(4q+3)\ldots(4q+4k+3)\,\frac{x^{2q+1}c^{4k+2q+1}}{R_{\bullet}^{4k+4q+5}}.$$

Other cases:

The formulae for combinations of nuclei producing at $z = 0$, $2Gu_z$ = a multiple of $\frac{y^h}{R_{\bullet}^{2n+2h+1}}$ at $z = 0$, are found from those of group A by replacing x by y whenever x appears in the formulae and the discussion given before.

It has shown before that the starting combination that corresponds to $2Gu_z$, being a multiple of $\frac{x^{2q}}{R_{\bullet}^{4q+1}}$, is $\delta_x^{2q}\left[Z_2-cC_2\right]$; i.e. the variable in the numerator suggests the variable of differentiation of $Z_2 - cC_2$, and the power of the variable in the numerator gives the order of differentiation with respect to that variable. This fact is used, henceforth, to find the starting combinations corresponding to different values for $2Gu_z$, $2Gu_x$ of $2Gu_y$. For instance, to obtain xy in the numerator of the value of $2Gu_z$, we start with the combination $\delta_{xy}\left[Z_2 - cC_2\right]$. Similarly, the combination $\delta_x^{(h)}\delta_y^{(m)}\left[Z_2 - cC_2\right] \xrightarrow[z=0]{} 2Gu_z$ = a multiple of $\frac{x^h y^m}{R_{\bullet}^{2h+2m+1}}$. As before, partial derivatives of this displacement with respect to z are then obtained so that terms with higher powers of R are found out. The formulae for this case can, therefore, be obtained similarly to previous cases in group A.

4. PROOFS FOR THE DERIVED FORMULAE

We will supply a proof for formulae (13i) and (13ii) by mathematical induction. Proofs for all other formulae are similar, therefore one proof is sufficient. In order that the second Principle of Finite Induction can be used, formulae (13i) and (13ii) are combined together. Therefore, what is required to be proved is the following:

$$c^{n-1}\left[\delta_z^{(n)} Z_2 - n\delta_z^{(n-1)} C_2 - c\delta_z^{(n)} C_2\right] - \sum_{i=0}^{\left[\frac{n}{2}\right]} \frac{t^n_{2n+1-2(\left[\frac{n}{2}\right]-j+1)}}{n^o_{2n+1-2(\left[\frac{n}{2}\right]-j+1)}} \left[N^o_{2n+1-2(\left[\frac{n}{2}\right]-j+1)}\right]$$

$$\xrightarrow[z=0]{} 2Gu_z = (-1)^n\, 1.3.5\ldots(2n-1)\, \frac{c^{2n-1}}{R^{2n+1}}$$

$$2Gu_x = 2Gu_y = 0$$

where $\left[\frac{n}{2} - j + 1\right]$ and $\left[\frac{n}{2}\right]$ are the greatest integer in $(\frac{n}{2} - j + 1)$ and $\frac{n}{2}$ respectively.

Call the above statement P(n). Assume the truth of all P(n) where n = 1,2,3,...,m-1. It is required to prove P(m):-

$$c^{m-1}\left[\delta_z^{(m)} Z_2 - m\delta_z^{(m-1)} C_2 - c\delta_z^{(m)} C_2\right] - \sum_{j=0}^{\left[\frac{m}{2}\right]} \frac{t^m_{2m+1-2(\left[\frac{m}{2}-j+1\right])}}{n^o_{2m+1-2(\left[\frac{m}{2}-j+1\right])}} \left[N^o_{2m+1-2(\left[\frac{m}{2}-j+1\right])}\right]$$

$$\xrightarrow[z=0]{} 2Gu_z = (-1)^m\, 1.3.5\ldots(2m-1)\, \frac{c^{2m-1}}{R^{2m+1}}$$

$$2Gu_x = 2Gu_y = 0$$

where $\left[\frac{m}{2} - j + 1\right]$ and $\left[\frac{m}{2}\right]$ are the greatest integer in $(\frac{m}{2} - j + 1)$ and $\frac{m}{2}$ respectively.

Proof:

$$\sigma_z^{(m)} Z_2 - m\sigma_z^{(m-1)} C_2 - c\,\sigma_z^{(m)} C_2 \quad \xrightarrow[z=0]{} 2Gu_z = \sigma_z^{(m)} \frac{1}{R}\Big]_{z\to c},\quad 2Gu_x = 2Gu_y = 0;$$

but

$$\sigma_z^{(m)} \frac{1}{R}\Big]_{z\to c} = \sum_{j=1}^{\left[\frac{m}{2}\right]+1} t^m_{(2m+1)-2\left(\left[\frac{m}{2}-j+1\right]\right)} \times \frac{c^{m-2\left(\left[\frac{m}{2}-j+1\right]\right)}}{R_{\bullet}^{(2m+1)-2\left(\left[\frac{m}{2}-j+1\right]\right)}}$$

Therefore,

$$c^{m-1}\left[\sigma_z^{(m)} Z_2 - m\sigma_z^{(m-1)} C_2 - c\,\sigma_z^{(m)} C_2\right]$$

$$\xrightarrow[z=0]{} 2Gu_z = \sum_{j=0}^{\left[\frac{m}{2}\right]+1} t^m_{(2m+1)-2\left(\left[\frac{m}{2}-j+1\right]\right)} \times \frac{c^{(2m-1)-2\left(\left[\frac{m}{2}-j+1\right]\right)}}{R_{\bullet}^{(2m+1)-2\left(\left[\frac{m}{2}-j+1\right]\right)}} \qquad \text{———(19)}.$$

Since P(1), P(2),...P(m-1) are assumed to be true, therefore,

$$\frac{N^o_{(2m+1)-2\left(\left[\frac{m}{2}-j+1\right]\right)}}{n^o_{(2m+1)-2\left(\left[\frac{m}{2}-j+1\right]\right)}}$$ is the combination of nuclei of strain

$$\xrightarrow[z=0]{} 2Gu_z = \frac{c^{2m-1-2\left(\left[\frac{m}{2}-j+1\right]\right)}}{R_{\bullet}^{2m+1-2\left(\left[\frac{m}{2}-j+1\right]\right)}},\quad 2Gu_x = 2Gu_y = 0. \qquad \text{———(19')}$$

Using (19), and (19'), we get that:

$$c^{m-1}\left[\sigma_z^{(m)} Z_2 - m\sigma_z^{(m-1)} C_2 - c\,\sigma_z^{m} C_2\right] - \sum_{j=0}^{\left[\frac{m}{2}\right]} \frac{t^m_{2m+1-2\left(\left[\frac{m}{2}-j+1\right]\right)}}{n^o_{2m+1-2\left(\left[\frac{m}{2}-j+1\right]\right)}} \left[N^o_{2m+1-2\left(\left[\frac{m}{2}-j+1\right]\right)}\right]$$

$$\xrightarrow[z=0]{} 2Gu_z = \sum_{j=0}^{\left[\frac{m}{2}\right]+1} t^m_{(2m+1)-2\left(\left[\frac{m}{2}-j+1\right]\right)} \times \frac{c^{(2m-1)-2\left(\left[\frac{m}{2}-j+1\right]\right)}}{R_{\bullet}^{(2m+1)-2\left(\left[\frac{m}{2}-j+1\right]\right)}}$$

$$- \sum_{j=0}^{\left[\frac{m}{2}\right]} t^m_{(2m+1)-2\left(\left[\frac{m}{2}-j+1\right]\right)} \frac{c^{(2m-1)-2\left(\left[\frac{m}{2}-j+1\right]\right)}}{R_{\bullet}^{2m+1-2\left(\left[\frac{m}{2}-j+1\right]\right)}} = t^m_{2m+1} \times \frac{c^{2m-1}}{R_{\bullet}^{2m+1}}$$

and $2Gu_x = 2Gu_y = 0$.

$t^m_{2m+1} \times \frac{c^{2m-1}}{R_{\bullet}^{2m+1}}$ is the last term in $c^{m-1}\,\sigma_z^{(m)} \frac{1}{R}\Big]_{z\to c}$.

Therefore:

$$t^{m}_{2m+1} = (-1)^{m}\, 1.3.5\ldots(2m-1), \text{ computed from the formula for } \delta_z^{(m)} \frac{1}{R},$$

and $$2Gu_z = (-1)^m\, 1.3.5\ldots(2m-1)\, \frac{c^{2m-1}}{R^{2m+1}} \text{ and } 2Gu_x = 2Gu_y = 0.$$

This proves P(m).

Hence P(n) is true for all n, where n is a positive integer.

Chapter IV

FORMULAE FOR ONE NON-VANISHING IN-PLANE DISPLACEMENTS

1. GROUP B: FORMULAE FOR COMBINATIONS OF NUCLEI OF STRAIN CORRESPONDING TO DIFFERENT TYPES OF ONE TERM X-COMPONENT OF DISPLACEMENT AND ZERO COMPONENTS OF DISPLACEMENT IN Y _ AND Z DIRECTIONS AT THE BOUNDARY Z=0

Different cases will be discussed under the heading Group B according to the different values of $2Gu_x$. However, in all these cases, $2Gu_y = 2Gu_z = 0$. Therefore, whenever these components are omitted, they are equal to zero.

Notation

We use γ to denote combinations of nuclei producing a non-zero one-term x-component of displacement at $z = 0$. Subscripts and superscripts are used with γ to denote different types of combinations. We let

(i) $\gamma^{2q}_{4k+4q+3}$ denote the combinations of nuclei

$$\xrightarrow[z=0]{} 2Gu_x = -1.3.5.7\ldots(4q-1)(4q+1)\ldots(4q+4k+1)\frac{x^{2q}c^{4k+2q+1}}{R^{4k+4q+3}}$$

(ii) $\gamma^{2q}_{4k+4q+5}$ denote the combinations of nuclei

$$\xrightarrow[z=0]{} 2Gu_x = 3.5.7\ldots(4q-1)(4q+1)\ldots(4q+4k+3)\ \frac{x^{2q}c^{4k+2q+3}}{R^{4k+4q+5}}$$

(iii) $\gamma^{2q+1}_{4k+4q+5}$ denote the combinations of nuclei

$$\xrightarrow[z=0]{} 2Gu_x = 3.5.7\ldots(4q-1)(4q+1)\ldots(4q+4k+3)\ \frac{x^{2q+1}c^{4k+2q+1}}{R^{4k+4q+5}}$$

(iv) $\gamma^{2q+1}_{4k+4q+7}$ denote the combinations of nuclei

$$\xrightarrow[z=0]{} 2Gu_x = -3.5.7\ldots(4q-1)(4q+1)\ldots(4q+4k+5)\ \frac{x^{2q+1}c^{4k+2q+3}}{R^{4k+4q+7}}$$

Here the superscript stands for the power of x in the numerator of $2Gu_x$ and the subscript denotes the power of R in the denominator of the x-displacement.

Case 1: $2Gu_x$ is a multiple of $\frac{1}{R^{2n+3}}$.

We start with the combination of nuclei $(3-4\nu)\,\delta_z X_2 - 4c(1-\nu)\,\delta_x C_2 + \delta_x Z_2$. Using equations (9) and (10'), we find that $2Gu_x = 4(1-\nu)(3-4\nu)\frac{-c}{R^3}$ at $z=0$ that is, $2Gu_x = 4(1-\nu)(3-4\nu)\,\delta_z \frac{1}{R}\Big]_{z\to c}$.
Partial derivatives of the starting combination (after c is replaced by z) should be obtained in order that their corresponding x-displacement would contain higher powers of R.

(i) $$\delta_z^{(2k)}\Big[(3-4\nu)\,\delta_z X_2 - 4c(1-\nu)\,\delta_x C_2 + \delta_x Z_2\Big]_{c\to z} = (3-4\nu)\,\delta_z^{(2k+1)} X_2$$
$$-8k(1-\nu)\,\delta_z^{(2k-1)}\delta_x C_2 - 4c(1-\nu)\,\delta_z^{(2k)}\delta_x C_2 + \delta_z^{(2k)}\delta_x Z_2$$
$$\xrightarrow[z=0]{} 2Gu_x = 4(1-\nu)(3-4\nu)\,\delta_z^{(2k+1)}\frac{1}{R}\Big]_{z\to c};$$

(ii) $$\delta_z^{(2k+1)}\Big[(3-4\nu)\,\delta_z X_2 - 4c(1-\nu)\,\delta_x C_2 + \delta_x Z_2\Big]_{c\to z} = (3-4\nu)\,\delta_z^{(2k+2)} X_2$$
$$-4(2k+1)(1-\nu)\,\delta_z^{(2k)}\delta_x C_2 - 4c(1-\nu)\,\delta_z^{(2k+1)}\delta_x C_2 + \delta_z^{(2k+1)}\delta_x Z_2$$
$$\xrightarrow[z=0]{} 2Gu_x = 4(1-\nu)(3-4\nu)\,\delta_z^{(2k+2)}\frac{1}{R}\Big]_{z\to c}.$$

Note: $4(1-\nu)(3-4\nu)$ is a factor in the x-displacements $2Gu_x$. From here and hereafter this factor will be omitted.

A sample of the first eight partial derivatives of $\frac{1}{R}$ with respect to z (z is then replaced by c) is given in table I as the values of $2Gu_z$ at $z = 0$ corresponding to the combinations of nuclei which are the first eight partial derivatives of $\Big[Z_2 - cC_2\Big]_{c\to z}$ with respect to z. But they are also the values of $2Gu_x$ at $z = 0$ corresponding to the combinations of nuclei

$$\sigma_z^{(n)}\left[(3-4\nu)\sigma_z X_2 - 4c(1-\nu)\sigma_x C_2 + \sigma_x Z_2\right]_{c\to z} \quad \text{where } n = 0,1,2,\dots,7.$$

To obtain a non-vanishing term as a value for the x-component of displacement, linear combinations of the nuclei given above are found similar to those in table II.

The formulae for case 1($q = 0$ in notation (i) and (ii)) are:

(i) $$c^{2k}\left[(3-4\nu)\sigma_z^{(2k+1)}X_2 - 8k(1-\nu)\sigma_z^{(2k-1)}\sigma_x C_2 - 4c(1-\nu)\sigma_z^{(2k)}\sigma_x C_2 + \sigma_z^{(2k)}\sigma_x Z_2\right] - \sum_{i=0}^{k-1}\frac{t^{2k+1}_{4k-(2i-1)}}{n^{o}_{4k-(2i-1)}}\left[\gamma^{o}_{4k-(2i-1)}\right]$$

$$\xrightarrow[z=0]{} 2Gu_x = -1.3.5\dots(4k+1)\frac{c^{4k+1}}{R_{\bullet}^{4k+3}},$$

$$2Gu_y = 2Gu_z = 0; \qquad \text{(20i)}$$

(ii) $$c^{2k+1}\left[(3-4\nu)\sigma_z^{(2k+2)}X_2 - 4(2k+1)(1-\nu)\sigma_z^{(2k)}\sigma_x C_2 - 4c(1-\nu)\sigma_z^{(2k+1)}\sigma_x C_2 + \sigma_z^{(2k+1)}\sigma_x Z_2\right] - \sum_{i=0}^{k}\frac{t^{2k+2}_{4k-(2i-3)}}{n^{o}_{4k-(2i-3)}}\left[\gamma^{o}_{4k-(2i-3)}\right]$$

$$\xrightarrow[z=0]{} 2Gu_x = 3.5.7\dots(4k+3)\frac{c^{4k+3}}{R_{\bullet}^{4k+5}},$$

$$2Gu_y = 2Gu_z = 0. \qquad \text{(20ii)}$$

Case 2: $2Gu_x$ = a multiple of $\frac{x}{R^{2n+5}}$.

In this case, x appears in its first power. Therefore, the starting combination is $\delta_x\left[(3-4\nu)\delta_z X_2 - 4c(1-\nu)\delta_x C_2 + \delta_x Z_2\right]$. The corresponding displacements at z = 0 are $2Gu_x = \frac{3xc}{R^5}$, and $2Gu_y = 2Gu_z = 0$. As before, partial derivatives of the starting x-displacement with respect to z should be obtained.

$$\delta_z^{(n)}\delta_x\left[(3-4\nu)\delta_z X_2 - 4c(1-\nu)\delta_x C_2 + \delta_x Z_2\right]_{c\to z} \xrightarrow[z=0]{} 2Gu_x = \delta_z^{(n)}\delta_x \frac{-c}{R^3}\Big]_{z\to c}$$

$$= \delta_z^{(n+1)}\delta_x \frac{1}{R}\Big]_{z\to c} = \delta_z^{(n+1)}\frac{-x}{R^3}\Big]_{z\to c} = -x\,\delta_z^{(n+1)} R^{-3}\Big]_{z\to c}.$$

These partial derivatives can be computed by aid of formula (12). A sample of these derivatives is given in table III; but the combinations of nuclei are different. The first three linear combinations of nuclei producing one-term x-component of displacement of the form of $\frac{x}{R^{2n+5}}$ are given below:

$$(3-4\nu)\delta_{zx}X_2 - 4c(1-\nu)\delta_x^2 C_2 + \delta_x^2 Z_2 \xrightarrow[z=0]{} 2Gu_x = \frac{3xc}{R^5}$$

$$c\left[(3-4\nu)\delta_z^2\delta_x X_2 - 4(1-\nu)\delta_x^2 C_2 - 4c(1-\nu)\delta_z\delta_x^2 C_2 + \delta_z\delta_x^2 Z_2\right] -$$

$$-\left[(3-4\nu)\delta_{zx}X_2 - 4c(1-\nu)\delta_x^2 C_2 + \delta_x^2 Z_2\right] \xrightarrow[z=0]{} 2Gu_x = \frac{-15xc^3}{R^7}$$

$$c^2\left[(3-4\nu)\delta_z^3\delta_x X_2 - 2.4(1-\nu)\delta_z\delta_x^2 C_2 - 4c(1-\nu)\delta_z^2\delta_x^2 C_2 + \delta_z^2\delta_x^2 Z_2\right]$$

$$-3\left[c\left\{(3-4\nu)\delta_z^2\delta_x X_2 - 4(1-\nu)\delta_x^2 C_2 - 4c(1-\nu)\delta_z\delta_x^2 C_2 + \delta_z\delta_x^2 Z_2\right\}\right.$$

$$\left.-\left\{(3-4\nu)\delta_{zx}X_2 - 4c(1-\nu)\delta_x^2 C_2 + \delta_x^2 Z_2\right\}\right] \xrightarrow[z=0]{} 2Gu_x = 3.5.7\frac{xc^5}{R^9}.$$

The following linear combination is written using the notation γ for combinations of nuclei:

$$c^3\left[(3-4\nu)\sigma_z^4\sigma_x X_2-3.4(1-\nu)\sigma_z^2\sigma_x^2 C_2-4c(1-\nu)\sigma_z^3\sigma_x^2 C_2+\sigma_z^3\sigma_x^2 Z_2\right]$$

$$-6\left[\gamma\begin{matrix}1\\9\end{matrix}\right]-3\left[\gamma\begin{matrix}1\\7\end{matrix}\right]\xrightarrow[z=0]{}2Gu_x=-3.5.7.9\,\frac{xc^7}{R_{\bullet}^{11}}.$$

<u>The formula for case 2 is split into the following two formulae:</u>

(i) $$c^{2k}\left[(3-4\nu)\sigma_z^{(2k+1)}\sigma_x X_2-8k(1-\nu)\sigma_z^{(2k-1)}\sigma_x^2 C_2\right.$$

$$\left.-4c(1-\nu)\sigma_z^{(2k)}\sigma_x^2 C_2+\sigma_z^{(2k)}\sigma_x^2 Z_2-\sum_{i=0}^{k-1}\frac{m^{2k+1}_{4k-(2i-3)}}{n^1_{4k-(2i-3)}}\left[\gamma^1_{4k-(2i-3)}\right]\right]$$

$$\xrightarrow[z=0]{}2Gu_x=3.5.7\ldots(4k+3)\,\frac{c^{4k+1}x}{R_{\bullet}^{4k+3}},$$

$$2Gu_y=2Gu_z=0; \qquad \text{(21i)}$$

(ii) $$c^{2k+1}\left[(3-4\nu)\sigma_z^{(2k+2)}\sigma_x X_2-4(2k+1)(1-\nu)\sigma_z^{(2k)}\sigma_x^2 C_2\right.$$

$$\left.-4c(1-\nu)\sigma_z^{(2k+1)}\sigma_x^2 Z_2+\sigma_z^{(2k+1)}\sigma_x^2 Z_2-\sum_{i=0}^{k}\frac{m^{2k+2}_{4k-(2i-5)}}{n^1_{4k-(2i-5)}}\left[\gamma^o_{4k-(2i-5)}\right]\right]$$

$$\xrightarrow[z=0]{}2Gu_x=-3.5.7\ldots(4k+5)\,\frac{c^{4k+3}x}{R_{\bullet}^{4k+7}},$$

$$2Gu_y=2Gu_z=0. \qquad \text{(21ii)}$$

<u>Case 3: $2Gu_x$ is a multiple of $\frac{x^2}{R_{\bullet}^{2n+7}}$.</u>

In this case, x^2 appears in the numerator. The starting combination of nuclei is

$$\sigma_x^2\left[(3-4\nu)\sigma_z X_2-4c(1-\nu)\sigma_x C_2+\sigma_x Z_2\right]=(3-4\nu)\sigma_x^2\sigma_z X_2-4c(1-\nu)\sigma_x^3 C_2+\sigma_x^3 Z_2.$$

The corresponding x-component of displacement at $z = 0$ is

$$2Gu_x = \sigma_x^2 \left.-\frac{c}{R^3}\right]_{z\to c} = \delta_z \sigma_x^2 \left.\frac{1}{R}\right]_{z\to c} = \delta_z\left[-\frac{1}{R^3}+\frac{3x^2}{R^5}\right]_{z\to c} =$$

$$-\left[\delta_z R^{-3} + 3x^2 \delta_z R^{-5}\right]_{z\to c}.$$

Partial derivatives of this x-component of displacement are taken. The result is:

$$\delta_z^{(n)}\left[(3-4\nu)\,\sigma_x^2 \delta_z X_2 - 4c(1-\nu)\,\delta_x^3 C_2 + \sigma_x^3 Z_2\right]$$

$$\xrightarrow[z=0]{} 2Gu_x = \left[-\delta_z^{(n+1)} R^{-3} + 3x^2 \delta_z^{(n+1)} R^{-5}\right]_{z\to c}.$$

A few of these partial derivatives are given in table III as the values of $2Gu_z$ at $z = 0$ corresponding to partial derivatives of $\left[Z_2 - cC_2\right]_{c\to z}$ with respect to z.

A sample of combinations of nuclei $\xrightarrow[z=0]{} 2Gu_x$ = a multiple of $\frac{x^2}{R^{2n+7}}$ is given in table V.

<u>The Formulae for Case 3 are:</u>

(i)
$$c^{2k+2}\left[(3-4\nu)\,\delta_z^{(2k+1)} \delta_x^2 X_2 - 8k(1-\nu)\,\delta_z^{(2k-1)} \delta_x^3 C_2 - 4c(1-\nu)\,\delta_z^{(2k)} \delta_x^3 C_2 + \delta_z^{(2k)} \delta_x^3 Z_2\right] - \sum_{i=0}^{k-1} \frac{3r^{2k+1}_{4k-(2i-5)}}{n^2_{4k-(2i-5)}}\left[\gamma^2_{4k-(2i-5)}\right] - \sum_{i=0}^{k} \frac{m^{2k+1}_{4k-(2i-5)}}{n^o_{4k-(2i-5)}}\left[\gamma^o_{4k-(2i-5)}\right]$$

$$\xrightarrow[z=0]{} 2Gu_x = -3.5.7\ldots(4k+5)\,\frac{x^2 c^{4k+3}}{R^{4k+7}},$$

$$2Gu_y = 2Gu_z = 0; \qquad \text{(22i)}$$

Table V

A LSIT OF COMBINATIONS OF NUCLEI PRODUCING AT THE BOUNDARY $Z = 0$, ZERO DISPLACEMENTS IN Y- AND Z DIRECTIONS AND A ONE NON-VANISHING X-DISPLACEMENT OF THE FORM OF $\frac{x^2}{R_0^{2n+7}}$

$$c^2\left[(3-4\nu)\sigma_z^2\sigma_x^2 X_2 - 4c(1-\nu)\sigma_x^3 C_2 + \sigma_x^3 Z_2\right] - \gamma_5^0 \xrightarrow{z=0} 2Gu_x = -15\frac{x^2c^3}{R_0^7}$$

$$c^3\left[(3-4\nu)\sigma_z^2\sigma_x^2 X_2 - 4(1-\nu)\sigma_x^3 C_2 - 4c(1-\nu)\sigma_z\sigma_x^3 C_2 + \sigma_z\sigma_x^3 Z_2\right]$$

$$-\gamma_7^2 - \gamma_7^0 - \gamma_5^0 \xrightarrow{z=0} 2Gu_x = 3.5.7\,\frac{x^2c^5}{R_0^9}$$

$$c^4\left[(3-4\nu)\sigma_z^3\sigma_x^2 X_2 - 2.4(1-\nu)\sigma_z\sigma_x^3 C_2 - 4c(1-\nu)\sigma_z^2\sigma_x^3 C_2 + \sigma_z^2\sigma_x^3 Z_2\right]$$

$$-3\gamma_9^2 - \gamma_9^0 - 3\gamma_7^0 \xrightarrow{z=0} 2Gu_x = -3.5.7.9\,\frac{x^2c^7}{R_0^{11}}$$

$$c^5\left[(3-4\nu)\sigma_z^4\sigma_x^2 X_2 - 3.4(1-\nu)\sigma_z^2\sigma_x^3 C_2 - 4c(1-\nu)\sigma_z^3\sigma_x^3 C_2 + \sigma_z^3\sigma_x^3 Z_2\right]$$

$$-6\gamma_{11}^2 - 3\gamma_9^2 - \gamma_{11}^0 - 6\gamma_9^0 - 3\gamma_7^0 \xrightarrow{z=0} 2Gu_x = 3.5.7.9.11\,\frac{x^2c^9}{R_0^{13}}$$

A factor of $4(1-\nu)(3-4\nu)$ is omitted throughout all values of $2Gu_x$. $2Gu_y = 2Gu_z = 0$ for all combinations.

$$\text{(ii)}\quad c^{2k+3}\Big[(3-4\nu)\,\delta_z^{(2k+2)}\,\delta_x^{2}X_2-4(2k+1)(1-\nu)\,\delta_z^{(2k)}\,\delta_x^{3}C_2$$

$$-4c(1-\nu)\,\delta_z^{(2k+1)}\,\delta_x^{3}C_2+\delta_z^{(2k+1)}\,\delta_x^{3}Z_2\Big]-\sum_{i=0}^{k}\frac{3r^{2k+2}_{4k-(2i-7)}}{n^{2}_{4k-(2i-7)}}\Big[\gamma^{2}_{4k-(2i-7)}\Big]$$

$$-\sum_{i=0}^{k+1}\frac{m^{2k+2}_{4k-(2i-7)}}{n^{o}_{4k-(2i-7)}}\Big[\gamma^{o}_{4k-(2i-7)}\Big]\xrightarrow[z=0]{}2Gu_x=3.5.7\ldots(4k+7)\,\frac{x^2c^{4k+5}}{R_{\bullet}^{4k+9}},$$

$$2Gu_y=2Gu_z=0.\qquad\text{\_\_\_\_\_\_\_\_\_(22ii)}.$$

A formula for (i) $2Gu_x$ = a multiple of $\frac{x^{2q}}{R_{\bullet}^{4q+3}}$

(ii) $2Gu_x$= a multiple of $\frac{x^{2q+1}}{R_{\bullet}^{4q+5}}$

It has been shown before that

$$(3-4\nu)\,\delta_z X_2-4c(1-\nu)\,\delta_x C_2+\delta_x Z_2\xrightarrow[z=0]{}2Gu_x=\delta_z\frac{1}{R}\Big]_{z\to c}=-\frac{c}{R_{\bullet}^{3}}.$$

Partial derivatives of $\delta_z\frac{1}{R}\Big]_{z\to c}$ with respect to x give for $2Gu_x$ values of the form of $\frac{x^{2q}}{R_{\bullet}^{4q+3}}$ or $\frac{x^{2q+1}}{R_{\bullet}^{4q+5}}$ at z = 0. A sample of these derivatives is given in table III after interchanging x and c in the values of $2Gu_z$. Then they give values for $2Gu_x$ corresponding to partial derivatives of $(3-4\nu)\delta_z X_2-4c(1-\nu)\delta_x C_2+\delta_x Z_2$ with respect to x.

<u>The formula with the desired conditions is given in two seperate formulae:</u>

$$\text{(i)}\quad c^{2q}\Big[(3-4\nu)\,\delta_x^{(2q)}\,\delta_z X_2-4c(1-\nu)\,\delta_x^{(2q+1)}C_2+\delta_x^{(2q+1)}Z_2\Big]$$

$$-\sum_{i=0}^{q-1}\frac{m^{2q}_{4q+1-2i}}{n^{2q-2i-2}_{4q+1-2i}}\Big[\gamma^{2q-2i-2}_{4q+1-2i}\Big]\xrightarrow[z=0]{}2Gu_x=-3.5.7\ldots(4q+1)\,\frac{c^{2q+1}x^{2q}}{R_{\bullet}^{4q+3}},$$

$$2Gu_y=2Gu_z=0\,;\qquad\text{\_\_\_\_\_\_\_\_\_(23i)}$$

(ii) $$c^{2q}\left[(3-4\nu)\sigma_x^{(2q+1)}\sigma_z X_2-4c(1-\nu)\sigma_x^{(2q+2)}C_2+\sigma_x^{(2q+2)}Z_2\right]$$

$$-\sum_{i=0}^{q-1}\frac{m^{2q+1}_{4q+3-2i}}{n^{2q-1-2i}_{4q+3-2i}}\left[\gamma^{2q-1-2i}_{4q+3-2i}\right]\xrightarrow[z=0]{}2Gu_x=3.5.7\ldots(4q+3)\;\frac{c^{2q+1}x^{2q+1}}{R_{\bullet}^{4q+3}}\;.$$

(23ii)

The general case: (i) $2Gu_x$ is a multiple of $\dfrac{x^{2q}}{R_{\bullet}^{2n+4q+3}}$

(ii) $2Gu_x$ is a multiple of $\dfrac{x^{2q+1}}{R_{\bullet}^{2n+4q+5}}$

The formulae with the desired conditions are:

(ia) $$c^{2k+2q}\Big[(3-4\nu)\sigma_z^{(2k+1)}\sigma_x^{(2q)}X_2-8k(1-\nu)\sigma_z^{(2k-1)}\sigma_x^{(2q+1)}C_2$$

$$-4c(1-\nu)\sigma_z^{(2k)}\sigma_x^{(2q+1)}C_2+\sigma_z^{(2k)}\sigma_x^{(2q+1)}Z_2\Big]$$

$$-\sum_{i=0}^{k-1}\frac{\alpha^{2k+1,2q/4q+1}_{4k+4q+1-2i}}{n^{2q}_{4k+4q+1-2i}}\left[\gamma^{2q}_{4k+4q+1-2i}\right]$$

$$-\sum_{j=0}^{q-1}\sum_{i=0}^{k}\frac{\alpha^{2k+1,2j/2q+1+2j}_{4k+4q+1-2j-2i}}{n^{2j}_{4k+4q+1-2j-2i}}\left[\gamma^{2j}_{4k+4q+1-2j-2i}\right]$$

$$\xrightarrow[z=0]{}2Gu_x=-3.5.7\ldots(4q-1)(4q+1)\ldots(4k+4q+1)\frac{x^{2q}c^{4k+2q+1}}{R_{\bullet}^{4k+4q+3}},$$

$$2Gu_y=2Gu_z=0;$$

(24ia)

(ib) $$c^{2k\ 2q\ 1}\Big[(3-4\nu)\sigma_z^{(2k+2)}\sigma_x^{(2q)}X_2-4(2k+1)(1-\nu)\sigma_z^{(2k)}\sigma_x^{(2q+1)}C_2$$

$$-4c(1-\nu)\sigma_z^{(2k+1)}\sigma_x^{(2q+1)}C_2+\sigma_z^{(2k+1)}\sigma_x^{(2q+1)}Z_2\Big]$$

$$-\sum_{i=0}^{k}\frac{\alpha^{2k+2,2q/4q+1}_{4k+4q+3-2i}}{n^{2q}_{4k+4q+3-2i}}\left[\gamma^{2q}_{4k+4q+3-2i}\right]-\sum_{j=0}^{q-1}\sum_{i=0}^{k}\frac{\alpha^{2k+2,2j/2q+1+2j}_{4k+4q+3-2j-2i}}{n^{2j}_{4k+4q+3-2j-2i}}\left[\gamma^{2j}_{4k+4q+3-2j-2i}\right]$$

$$\xrightarrow[z=0]{}2Gu_x=3.5\ldots(4q-1)(4q+1)\ldots(4k+4q+3)\;\frac{x^{2q}\;c^{4q+2q+3}}{R_{\bullet}^{4k+4q+5}},$$

$$2Gu_y=2Gu_z=0\,.$$

(24ib)

(ii) The formula for combinations of nuclei corresponding to $2Gu_x$ being a multiple of $\frac{x^{2q+1}}{R^{4k+4q+5}}$ could be deduced from formulae (ia) by replacing $2q$ by $2q+1$, except for the power of c. $2Gu_x$ would then be the negative of that given in case (ia). Similarly, the formula for combinations of nuclei corresponding to $2Gu_x$ being a multiple of $\frac{x^{2q+1}}{R^{4k+4q+7}}$ could be deduced from formula (ib) by replacing $2q$ by $2q+1$, except for the power of c. $2Gu_x$ would then be the negative of that in (ib)

Other Cases

The formulae for combinations of nuclei $\xrightarrow[z=0]{}$ $2Gu_x = $ a multiple of $\frac{y}{R^{2n+2h+3}}$ are found from those of group B by replacing x by y whenever x appears in the formulae. The discussion treating mixed variables in the numerator of the value of the x-displacemeht holds here too.

2. GROUP C: DERIVATION OF FORMULAE FOR COMBINATIONS OF NUCLEI CORRESPONDING TO DIFFERENT TYPES OF ONE-TERM Y-DISPLACEMENT AND ZERO-DISPLACEMENTS IN X - AND Z DIRECTIONS AT THE BOUNDARY Z = 0.

The formulae for combinations of nuclei of strain corresponding to $2Gu_y$ = one term of different forms and $2Gu_x = 2Gu_z = 0$ at $z = 0$ can be deduced easily from formulae for group B. Replacing x (capital, subscript and variable) by y (capital, subscript and variable) whenever x appear in chapter IV under the heading group B, the formulae follow immediately.

3. DIRECT APPLICATION OF THE FORMULAE IN THE SOLUTION OF FIRST BOUNDARY HALF-SPACE PROBLEMS IN NUCLEI OF STRAIN

We find out the right combinations of nuclei of strain that solve problems of different types of nuclei applied at (0,0,c) to produce zero displacements at the boundary $z = 0$.

1. (a) Double force in z-direction

We start with the combination $\sigma_z Z_1 - \sigma_z Z_2$. This produces at $z = 0$, by aid of equation (9):

$$2Gu_x = 2Gu_y = 0,$$

$$2Gu_z = 2(1-4\nu)\frac{c}{R_\bullet^3} + \frac{6c^3}{R_\bullet^5} .$$

To annull the terms in $2Gu_z$, formulae (13i) and (13ii) are used. They give that:

$$N_3^o \xrightarrow[z=0]{} 2Gu_z = (3-4\nu)\frac{-c}{R_\bullet^3}, \quad 2Gu_x = 2Gu_y = 0;$$

$$N_5^o \xrightarrow[z=0]{} 2Gu_z = (3-4\nu)\frac{3c^3}{R_\bullet^5} \quad \text{"} \quad \text{"} .$$

The combination of nuclei with the desired conditions is:

$$\sigma_z Z_1 - \sigma_z Z_2 + \frac{2(1-4\nu)}{(3-4\nu)}\left[N_3^o\right] - \frac{2}{(3-4\nu)}\left[N_5^o\right]$$

where

$$N_3^o = \sigma_z Z_2 - C_2 - \sigma_z C_2$$

and

$$N_5^o = c\left[\sigma_z^2 Z_2 - 2\sigma_z C_2 - c\sigma_z^2 C_2\right] - \left[\sigma_z Z_2 - C_2 - \sigma_z C_2\right]$$

Some of the nuclei have been written twice. Therefore, the final combination is:

$$\sigma_z Z_1 + \frac{1-4\nu}{3-4\nu}\left\{\sigma_z Z_2\right\} - \frac{4(1-2\nu)}{(3-4\nu)} C_2 + \frac{8\nu c}{3-4\nu}\sigma_z C_2 - \frac{2c}{3-4\nu}\left[\sigma_z^2 Z_2 - c\sigma_z^2 C_2\right].$$

1. (b) Double Force in x-direction

Starting with $\sigma_x X_1 - \sigma_x X_2$, equation (7) gives at $z = 0$:

$$2Gu_x = 2Gu_y = 0;$$

$$2Gu_z = \frac{-2c}{R_\circ^3} + \frac{6cx^2}{R_\circ^5}.$$

To annul the final term, formula (15ii) gives when $k = 0$ that:

$$c\left[\sigma_x^2 Z_2 - c\,\sigma_x^2 C_2\right] - N_3^\circ \xrightarrow[z=0]{} 2Gu_z = \frac{3x^2c}{R_\circ^5}(3-4\nu),\ 2Gu_x = 2Gu_y = 0.$$

To annull the initial term, formula (13ii) gives when $k = 0$ that:

$$\sigma_z Z_2 - C_2 - c\,\sigma_z C_2 \xrightarrow[z=0]{} 2Gu_z = -\frac{c}{R_\circ^3}(3-4\nu), \qquad 2Gu_x = 2Gu_y = 0.$$

Therefore, the combination of nuclei with the desired conditions is:

$$\sigma_x X_1 - \sigma_x X_2 - \frac{2}{3-4\nu}\left[N_5^2\right] - \frac{2}{3-4\nu}\left[N_3^\circ\right] =$$

$$\sigma_x X_1 - \sigma_x X_2 - \frac{2c}{3-4\nu}\left[\sigma_x^2 Z_2 - c\,\sigma_x^2 C_2\right].$$

1. (c) Double Force in y-direction

The solution is the same as (b), after replacing x subscripts and capitals by y subscripts and capitals respectively.

2. (a) Double force in z-direction with moment about y-axis.

We start with $\sigma_x Z_1 + \sigma_x Z_2$. By equation (9), this combination gives at $z = 0$:

$$2Gu_x = 2Gu_y = 0;$$

$$2Gu_z = -(3-4\nu)\frac{2x}{R_\circ^3} - \frac{6xc^2}{R_\circ^5}.$$

To annul the initial term,

$$\sigma_x Z_2 - c\sigma_x C_2 \xrightarrow[z=0]{} 2Gu_z = -\frac{x}{R^3},\quad 2Gu_x = 2Gu_y = 0.$$

To annul the final term, formula (14ii) gives when $k = 0$

$$\sigma_{zx} Z_2 - \sigma_x C_2 - c\,\sigma_{zx} C_2 \xrightarrow[z=0]{} 2Gu_z = \frac{3xc}{R^5},\quad 2Gu_x = 2Gu_y = 0.$$

The combination with desired conditions is

$$\sigma_x Z_1 + \sigma_x Z_2 - 2(3-4\nu)\left[\sigma_x Z_2 - c\,\sigma_x C_2\right] + 2c\left[\sigma_{zx} Z_2 - \sigma_x C_2 - c\,\sigma_{zx} C_2\right].$$

After simplification, the final required combination is :

$$\sigma_x Z_1 - (5-8\nu)\,\sigma_x Z_2 - 4c(1-2\nu)\,\sigma_x C_2 + 2c(\sigma_{zx} Z_2 - c\,\sigma_{zx} C_2).$$

2.(b) <u>Double force in x-direction with moment about z-axis</u>

$\sigma_y X_1 - \sigma_y X_2$ produces at $z = 0$:

$$2Gu_x = 0$$

$$2Gu_y = 0$$

$$2Gu_z = \frac{6xyc}{R^5}.$$

The combination $\sigma_{xy}\left[Z_2 - cC_2\right] \xrightarrow[z=0]{} 2Gu_z = \frac{3xy}{R^5}(3-4\nu)$ $2Gu_x = 2Gu_y = 0.$

The combination of nuclei with desired conditions is:

$$\sigma_y X_1 - \sigma_y X_2 - \frac{2c}{3-4\nu}\left[\sigma_{xy} Z_2 - c\,\sigma_{xy} C_2\right].$$

Similarly, many problems of the same type can be solved very easily without depending on guesswork.

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