AMERICAN UNIVERSITY OF BEIRUT

On the fixed points of the Berezin transform

by
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A thesis
submitted in partial fulfillment of the requirements
for the degree of Master of Science
to the Department of Mathematics
of the Faculty of Arts and Sciences
at the American University of Beirut

Beirut, Lebanon
May 2014

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Acknowledgements

To my advisor Professor Farouk Abi Khuzam, I express sincere gratitude. I thank him for his guidance and assistance throughout my graduate studies, and most of all, for dedicating considerable time to discuss questions and concerns.

To my family and friends, I dedicate my profound gratefulness for their support and encouragement. They have given me the motives to work hard and never give up.

Above all, all praise belongs to God, the Almighty, whose many blessings have made me who I am today.

Contents

AN ABSTRACT OF THE THESIS OF

Fatima Moussa Maziad for Master of Science

Major: Mathematics

Title: On the fixed points of the Berezin transform

Abstract

The Bergman space $\mathbf{A}^2(\Omega)$ consists of functions analytic and square integrable on a region Ω of the complex plane.The Berezin transform T of a function φ in this space is defined as the Berezin transform of the Toeplitz operator having φ as its symbol.A question of interest is to determine fixed points of the Berezin transform.

In this thesis, we present a partial study of work done on this question.We first consider the case where Ω is the open unit disk D and present conditions for which Tu = u and Tu $\geqslant$ u, where u is integrable.We then consider the more difficult case where Ω is an annulus centered at the origin. In the case of a radial function, we present conditions implying either Tu $\geqslant$ u, or Tu $\leqslant$ u.

Chapter 1. Preliminaries

1.1 Introduction

In this chapter we introduce preliminary material that will be needed later on in this thesis. We start by defining Bergman spaces and develop some properties of functions belonging to them. We then consider a very special functional defined on $\mathbf{A}^2(\Omega)$, namely the evaluation functional, and obtain its representation via the Riesz Representation theorem. This gives us the Bergman kernel, which will be used to define the Berezin transform. We develop some properties of the kernel, and end the chapter by examining the connection between the Bergman kernal and the green's function of Ω with a review of some of the properties of the Green's function.

1.2 Bergman spaces

Definition

Let $\Omega \subset \mathbb{C}$ be an open connected set in the complex plane. For $1 \leqslant \mathrm{p} \leqslant \infty$, the Bergman space $\mathbf{A}^{\mathrm{p}}(\Omega)$ consists of all functions f analytic in Ω for which

$$\|f\|_p = \left(\int_\Omega |f(z)|^p dA(z) \right)^{1/p} < \infty,$$

where dA is the Lebesgue measure in $\mathbb{C}$. We shall use the notation A^p when reference to the underlying domain Ω is not necessary. A^p is a normed vector space: $\|f\|_p$ is the norm of the function f.

In the case $p = 2$,the set A^2 can be endowed with an inner product as follows:

If $f, g \in A^2$, define

$$(f, g) = \int_\Omega f(z)\overline{g(z)}dA(z).$$

Then under this inner product, A^2 becomes a complete inner product space, i.e. a Hilbert space. Historically the pioneering work of Stefan Bergman (1895-1977) deals mainly with the case p=2, and consideration of p spaces followed after. If Ω is a bounded domain, any bounded analytic function in Ω belongs to all these spaces, in particular all polynomials, and so, naturally there will be interest in unbounded functions that still belong to A^2.The example $f(z) = \frac{1}{1-z}$, $\Omega = D$ indicates that for an unbounded analytic function, some bound on its growth must be placed in order to guarantee its membership in A^2.Indeed, if $0 < R < 1$, and $D_R = \{z : |z| < R\}$, then for $\frac{1}{1-z}$, a straightforward computation gives

$$\int_{D_R} \frac{1}{|1-z|^2}dA(z) = -2\pi \log{(1-R^2)} \to \infty, R \to 1-,$$

so that $\frac{1}{1-z}$ does not belong to $A^2(D)$.

This last example, shows that functions in a Bergman space cannot grow too rapidly near the boundary, so that some kind of growth condition must be available to guarantee that the function f is in the space.The following theorem provides an upper bound on the growth of functions in $A^p(\Omega)$.

Theorem 1. *[1] Point-evaluation is bounded in each Bergman space* $\mathbf{A}^{\mathrm{p}}(\Omega)$*. More specifically, each function* $f \in \mathbf{A}^{\mathrm{p}}(\Omega)$ *has the property*

$$|f(z)| \leqslant \pi^{-1/p}\delta(z)^{-2/p}\|f\|_p, z \in \Omega,$$

where $\delta(z)$ = *dist(z,dΩ) is the distance from* z *to the boundary.*

Proof. Fix a point z and let $\delta = \delta(z)$. Then the disk

$$D = \{\zeta \in \mathbb{C} : |\zeta - z| < \delta\}$$

lies in Ω.The integrals

$$\int_0^{2\pi} |f(z + re^{i\theta})|^p d\theta$$

are nondecreasing functions of r, so it follows that

$$|f(z)|^p \leqslant \frac{1}{2\pi}\int_0^{2\pi} |f(z + re^{i\theta})|^p d\theta, \quad 0 \leqslant r < \delta.$$

Another integration gives

$$\begin{aligned} \pi\delta^2|f(z)|^p &\leqslant \int_0^{\delta}\int_0^{2\pi} |f(z + re^{i\theta})|^p d\theta r dr \\ &= \int_D |f(\zeta)|^p dA \\ &\leqslant \int_\Omega |f(\zeta)|^p dA = \|f\|_p^p, \end{aligned}$$

which is the stated result. □

A consequence of Theorem 1 is the fact that $A^2(\Omega)$ is a closed subspace of $L^2(\Omega)$.

Corollary 2. *The Bergman space* $A^2(D)$ *is a closed subspace of the Hilbert space* $L^2(D) = \{ f : D \to \mathbb{C}$, f *is measurable and* $\int_D |f(z)|^2 dA(z) < \infty\}$

Proof. Let $f_n \in A^2(D)$, $\|f_n - f\|_2 \to 0$ in $L^2(D)$, then $\exists$ a subsequence $\{f_{n_k}\} \to f$ almost everywhere in D.Let $K \subseteq D$ be compact, then $\exists$ $0 < R < 1$, $K \subseteq \{|z| \leqslant R\}$. But f_n

converges to f in norm then f_n is a Cauchy sequence in norm, so by Theorem 1,

$$\begin{aligned}|f_n(z) - f_m(z)| &\leqslant \frac{1}{\sqrt{\pi}}\frac{1}{1-|z|}\|f_n - f_m\|_2 \\ &\leqslant \frac{1}{\sqrt{\pi}}\frac{1}{1-R}\|f_n - f_m\|_2 \\ sup|(f_n - f_m)(z)| &\leqslant \frac{1}{\sqrt{\pi}}\frac{1}{1-R}\|f_n - f_m\|_2 \to 0\end{aligned}$$

so f_n is a uniform Cauchy sequence on D, so f_n converges uniformly to f on compact subsets of D, and since each f_n is holomorphic, it follows that f is an analytic function and so it belongs to $A^2(D)$. $\square$

In general, for an open connected set Ω , $A^2(\Omega)$ is a closed subspace of $L^2(\Omega)$, and this is shown by taking $K \subseteq \Omega$ to be compact, then $\exists\ \eta > 0$, such that for every $z \in K$, $\delta(z) \leqslant \eta$ and then the result follows by the same reasoning discussed in the case where $\Omega = D$.

It follows that $A^2(\Omega)$ is a Hilbert space as stated in the introduction. The fact that A^2 is a Hilbert space leads naturally to a search for an orthonormal basis, and this will be taken up in the next 2 chapters.
In this thesis we shall consider two major cases where Ω is the open unit disk D, or an open annulus $\mathscr{A}$ centered at the origin. Since both are bounded domains, the Bergman space of the first contains all monomials $\{z^n : n \in \mathbb{N}\}$, while the Bergmann space of the second contains all monomials $\{z^n : n \in \mathbb{Z}\}$. These sets will serve, respectively, as the building blocks for the orthonormal sets in each of those spaces.

1.3 Evaluation functional

We now recall the Riesz representation theorem in the form needed for our purposes.

Theorem 3. *Riesz Representation. If H be a Hilbert space over $\mathbb{C}$ and T is a bounded linear functional on H, then there exists $g \in H$ such that for every $f \in H$ we have*

$$T(f) = < f, g > .$$

Moreover, $\|T\| = \|g\|$, *where* $\|T\|$ *is the operator norm of* T, $\|g\|$ *is the Hilbert norm of* g.

We shall apply the Riesz representation theorem to a very special functional defined on $A^2(\Omega)$, namely the evaluation functional.

Let $z_0 \in D$. To each $f \in A^2(\Omega)$, we associate the complex number $f(z_0)$ through the map k_{z_0} from $A^2(\Omega)$ to $\mathbb{C}$.

$$\begin{array}{rcl} A^2(\Omega) & \xrightarrow{k_{z_0}} & \mathbb{C} \\ f & \xrightarrow{k_{z_0}} & f(z_0) \end{array}$$

It is immediate that $k_{z_0}\ (f+g) = k_{z_0}(f) + k_{z_0}(g)$, and $k_{z_0}\ (cf) = \mathrm{c}k_{z_0}(f)$, so that k_{z_0} is a linear functional on $A^2(\Omega$.

The norms $|k_{z_0}(f)|$ and $\|f\|_2$ are connected through the inequality establish in Theorem 1 :

$$|k_{z_0}(f)| = |f(z_0)| \leqslant \frac{1}{\sqrt{\pi}} \frac{1}{1 - |z_0|} \|f\|_2$$

Therefore, k_{z_0} is a bounded linear functional on $A^2(\Omega)$. Thus, $A^2(\Omega)$ being a Hilbert space and

$$\Phi : A^2(\Omega) \to \mathbb{C}, \quad \Phi(f) = f(z)$$

a bounded linear functional,then the Riesz Representation theorem guarantees the existence of an element in $A^2(\Omega)$, call it k_z such that

$\Phi(f) = f(z) = (f, k_z) = \int_\Omega f(\zeta)\ \overline{k_z(\zeta)}\ dA(\zeta)$.

Properties of the kernel function

1. The kernel function is a reproducing kernel also called the Bergman kernel because it has the following property:

   $f(z) = < f, k(z, .) > = \int_\Omega f(\zeta)\ \overline{k(z,\zeta)}\ dA(\zeta), \quad z \in \Omega,$

   for each function $f \in A^2(\Omega)$.

2. The kernel function is uniquely determined by its reproducing property:

   Suppose there is a function $l(z,\zeta) \in A_2(\Omega)$ that has the reproducing property, then $f(z) = < f, l(z, .) >$ and so $< f, k(z, .) - l(z, .) >$=0 for every $f \in A^2(\Omega)$, but this implies $k(z, .) - l(z, .) = 0$ so that $k(z,\zeta) = l(z,\zeta)$.

3. The kernel function is symmetric:

   Taking $f(\zeta) = \overline{k(w,\zeta)}$ for some $w \in \Omega$, then, $\overline{k(w,z)} = \int_\Omega \overline{k(w,\zeta)}\ k(z,\zeta)\ dA(\zeta) = k(z,w)$.

   Thus, the Kernel function has the symmetry property $k(z,\zeta) = \overline{k(\zeta,z)}$.This shows that $k(z,\zeta)$ is analytic in z and anti-analytic in ζ.

4. $k(z,z)$=$\int_\Omega |k(z,\zeta)|^2\ dA(\zeta)$= $\|k(z)\|_2^2$.

5. Applying the Cauchy Schwarz inequality to $f(z) = (f, k(z, .)) = \int_\Omega f(\zeta)\ k(z,\zeta)\ dA(\zeta)$ we get,

   $|f(z)| \leqslant \sqrt{k(z,z)}\ \|f\|_2$

1.4 Toeplitz Operator

Definition

For $\phi \in L^2(\Omega)$, the Toeplitz operator T_ϕ with symbol ϕ , is the operator defined by $T_\phi(f) = P(\phi f)$, where P is the orthogonal projection of the Hilbert space onto the Bergman space (this is possible by Corollary 2). [3]

$$
\begin{aligned}
P : L^2(\Omega) &\rightarrow A^2(\Omega) \\
T_\phi : A^2(\Omega) &\rightarrow A^2(\Omega) \\
f &\rightarrow T_\phi(f) = P(\phi f)
\end{aligned}
$$

We now give an explicit formula for the orthogonal projection.
If $f \in L^2(\Omega)$, then $f = f_1 + f_2$, where $f_1 \in A^2(\varOmega)$, $f_2 \in \left(A^2(\varOmega)\right)^\perp$, and $< g, f_2 > = 0$ for all $g \in A^2(\varOmega)$. This is true since $A^2(\varOmega)$ is a closed subspace of the Hilbert space $L^2(\Omega)$.
Fix $z \in \Omega$, since $k(z,\zeta) \in A^2(\varOmega)$, then $< k(z,.), f_2 > = 0$, which yields

$$< f, k(z,.) >=< f_1, k(z,.) > + < f_2, k(z,.) >=< f_1, k(z,.) >= f_1(z).$$

Hence,

$$f_1(z) = \int_\Omega f(\zeta)\overline{k(z,\zeta)}dA(\zeta) = P(f(z)).$$

1.5 Berezin Transform

Definition

The Berezin transform associates smooth functions with operators on Hilbert spaces of analytic functions. [2]

For T a bounded operator on the Hilbert space, the Berezin Transform of T, denoted T', is the complex valued function on Ω defined by

$T'(z) = < Tk_z, k_z >$, where k_z is the Bergman kernel function of Ω

The Berezin Transform has been most successful as tool to study operators on the Bergman space, and we will restrict attention from now on to that area [2].

The Berezin Transform of the function ϕ denoted $B\phi$, is defined to be the Berezin transform of the toeplitz operator T_ϕ, more precisely the Berezin transform of a function $\phi \in L^\infty(\Omega)$ is defined by

$$
\begin{aligned}
(B\phi)(z) =< T_\phi k(z,.), k(z,.) > &= \int_\Omega (T_\phi k(z,.))(w)\overline{k(z,w)}dA(w) \\
&= \int_\Omega \Big(\int_\Omega \phi(t)k(z,t)\overline{k(w,t)}dA(t) \Big) \overline{k(z,w)}dA(w) \\
&= \int_\Omega \phi(t)k(z,t)dA(t) \int_\Omega \overline{k(w,t)k(z,w)}dA(w) \\
&= \int_\Omega \phi(t)k(z,t)dA(t) \int_\Omega k(z,w)\overline{k(z,w)}dA(w) \\
&= \int_\Omega \phi(t)k(z,t)\overline{k(z,t)}dA(t) \\
&= \int_\Omega \phi(t)|k(z,t)|^2 dA(t)
\end{aligned}
$$

1.6 Connection with Green's function

We start by recalling the definition of the Green's function. For $\zeta \in \Omega$, The Green's function $G(z,\zeta)$ is the function harmonic in Ω except at ζ, where it has a logarithmic singularity, i.e $G(z,\zeta)$ - log $\frac{1}{|z-\zeta|}$ is harmonic in a neighborhood of ζ. Moreover, it has boundary values $G(z,\zeta) = 0$ for all $z \in \partial\Omega$.

Theorem 4. *[1] Let Ω be a finitely connected bounded domain, and let $G(z,\zeta)$ be the Green's function of Ω. Then the Bergman kernel function is*

$$K(z,\zeta) = -\frac{2}{\pi}\frac{\partial^2 G}{\partial z \partial \overline{\zeta}} \quad , z \neq \zeta$$

Proof. Green's function has the form: $G(z,\zeta) = log \frac{1}{|z-\zeta|} + h(z,\zeta)$, in some neighborhood of ζ, where $h(z,\zeta)$ is a harmonic function of z. Thus,

$$\frac{\partial G}{\partial z}(z,\zeta) = -\frac{1}{2}\frac{1}{z-\zeta} + \frac{\partial h}{\partial z}(z,\zeta)$$

and

$$\frac{\partial^2 G}{\partial z \partial \overline{\zeta}}(z,\zeta) = \frac{\partial^2 h}{\partial z \partial \overline{\zeta}}(z,\zeta) \qquad z \neq \zeta$$

We will make use of the Cauchy-Green theorem

$$\int_{\partial\Omega} F(z) dz = 2i \int_{\Omega} \frac{\partial F}{\partial \overline{z}} dA \qquad F \in C^1(\overline{\Omega})$$

Suppose f is analytic in Ω and continuous in $\overline{\Omega}$ and let

$$\begin{aligned} \Omega_\epsilon &= \Omega - \{|z-\zeta| \leqslant \epsilon\} \\ \Gamma_\epsilon &= \{|z-\zeta| = \epsilon\} \end{aligned}$$

Since G and therefore also $\frac{\partial G}{\partial z}$ vanishes for ζ on $\partial\ \Omega$, and taking the boundary of Ω in the counter-clockwise direction, and Γ_ϵ in the clockwise direction, the Cauchy-Green theorem then gives

$$\int_{\Omega_\epsilon} = \int_{\partial\Omega_\epsilon} = \int_{\partial\Omega+\Gamma_\epsilon}$$

therefore,

$$\frac{1}{2i}\int_{\Gamma_\epsilon} \frac{\partial G}{\partial z}(z,\zeta)f(\zeta)d\zeta = -\int_{\Omega_\epsilon} \frac{\partial^2 G}{\partial z \partial \overline{\zeta}}(z,\zeta)f(\zeta)dA(\zeta)$$

The Cauchy formula then yields,

$$\begin{aligned} \int_{\Gamma_\epsilon} \frac{\partial G}{\partial z}(z,\zeta)f(\zeta)d\zeta &= \int_{\Gamma_\epsilon} \Big(-\frac{1}{2}\frac{1}{z-\zeta} + \frac{\partial h}{\partial z}(z,\zeta)\Big) f(\zeta)d\zeta \\ &= \int_{\Gamma_\epsilon} -\frac{1}{2}\frac{1}{z-\zeta} f(\zeta)d\zeta + \int_{\Gamma_\epsilon} \frac{\partial h}{\partial z}(z,\zeta)f(\zeta)d\zeta \\ &= \pi i f(z), \end{aligned}$$

Since the last integral involving the partial derivative of h is 0, because the 2 functions f and h are both harmonic and so bounded in Ω and therefore when taking the limit as ϵ tends to 0, the integral will go to 0. Therefore, it follows that when taking the limit as ϵ

approaches 0, we get,

$$\begin{aligned}\frac{1}{2i}\pi i f(z) &= -\int_{\Omega_\epsilon} \frac{\partial^2 G}{\partial z \partial \overline{\zeta}}(z,\zeta) f(\zeta) dA(\zeta) \\ f(z) &= -\frac{2}{\pi}\int_{\Omega} \frac{\partial^2 G}{\partial z \partial \overline{\zeta}}(z,\zeta) f(\zeta) dA(\zeta)\end{aligned}$$

for functions f analytic in Ω and continuous in $\overline{\Omega}$. Now, recall that

$$\frac{\partial^2 G}{\partial z \partial \overline{\zeta}}(z,\zeta) = \frac{\partial^2 h}{\partial z \partial \overline{\zeta}}(z,\zeta),$$

where h is a harmonic function.It follows that the partial derivative of the function G of second order, is analytic.

Taking $f(z) = K(z,\eta)$, the above formula gives,

$$-\frac{2}{\pi}\int_{\Omega} \frac{\partial^2 G}{\partial z \partial \overline{\zeta}}(z,\zeta) K(\zeta,\eta) dA(\zeta) = K(z,\eta)$$

but by the reproducing property of the kernel function, the integral is equal to

$$-\frac{2}{\pi}\frac{\partial^2 G}{\partial z \partial \overline{\zeta}}(z,\eta).$$

□

Chapter 2. Fixed points of the Berezin transform (I)

We have seen that $\mathbf{A}^2(\Omega)$ becomes a Hilbert space under the inner product,

$$< f, g >= \int_\Omega f(z)\overline{g(z)}dA, \quad f, g \in \mathbf{A}^2(\Omega)$$

In this Hilbert space setting, we showed in the previous chapter the existence of a kernel function, and we discussed the kernel's main property, namely its reproducing property. In this chapter, we specialize to the case where $\Omega = D$, where D is the open unit disk, and obtain an explicit formula for the corresponding kernel function. We then consider the Berezin transform on D and discuss the relations between harmonic functions and fixed points of the Berezin transform. In addition, we consider possible analogues of the result, when the property of harmonicity is replaced by that of subharmonicity.

2.1 The Kernel function of the unit disk

We start by determining an orthonormal set of functions $\{\phi_n\}$ in $\mathbf{A}^2(D)$.

Observe that the monomials 1,z,z^2,z^3,... form an orthogonal set in $\mathbf{A}^2(D)$. Indeed,

$$\begin{aligned} < z^n, z^m > &= \int_D z^n \overline{z^m} dA(z) = \int_0^{2\pi} \int_0^1 r^{n+m} e^{i(n-m)\theta} r dr d\theta = \frac{2\pi}{n+m+2} \delta_{n,m}. \\ < z^n, z^n > &= \int_0^{2\pi} \int_0^1 r^{2n+1} dr d\theta = \frac{\pi}{n+1}. \end{aligned}$$

Thus, the functions

$$\phi_n(z) = \sqrt{\frac{n+1}{\pi}} z^n, \quad n = 0, 1, 2, 3...$$

are orthonormal in $\mathbf{A}^2(D)$.To show that they form a basis, we show that they span the space. Equivalently, we show that the Parseval's identity holds for every $f \in \mathbf{A}^2(D)$.

$$\sum_{n=0}^{\infty} | < f, \phi_n > |^2 = \|f\|_2^2$$

But this is equivalent to showing the following identity

$$\|f\|_2^2 = \pi \sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1}, \quad f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

which is easily established.

Therefore,the set $\{\phi_n\}$ is an orthonormal basis in $\mathbf{A^2}$. The Kernel function $K(z, \zeta)$ being an analytic function in ζ is thus expressible as a series in this basis. Using the reproducing property of the kernel, we find that

Theorem 4. *[1] The Kernel function of the unit disk has the following representation*

$$K(z,\zeta) = \sum_{n=0}^{\infty} \overline{\phi}_n(z)\phi_n(\zeta),$$

Proof. Since $\{\phi_n\}$ is an orthonormal basis, then each function $f \in \mathbf{A}^2(D)$ has a unique expansion in the form,

$$f(\zeta) = \sum_{n=1}^{\infty} c_n\phi_n(\zeta)$$

Fix $z \in D$, take $f(\zeta) = K(z,\zeta)$ then we have $K(z,\zeta) = \sum_{n=0}^{\infty} c_n(z)\ \phi_n(\zeta)$. Using the reproducing property of the kernel,

$$\phi_m(\zeta)\overline{K(z,\zeta)} = \sum_{n=0}^{\infty} \overline{c_n(z)\phi_n(\zeta)}\phi_m(\zeta)$$

We integrate with respect to ζ,

$$\begin{aligned} \phi_m(z) &= \overline{c_m(z)}\|\phi_m\|^2 \\ \overline{c_m(z)} &= \phi_m(z) \end{aligned}$$

$\square$

Since $\phi_n(z)$= $\sqrt{\frac{n+1}{\pi}}\ z^n$, we get

$$\begin{aligned} K(z,\zeta) &= \sum_{n=0}^{\infty} \overline{\phi_n(z)}\phi_n(\zeta) \\ &= \sum_{n=0}^{\infty} \sqrt{\frac{n+1}{\pi}}\overline{z^n} \quad \sqrt{\frac{n+1}{\pi}}\zeta^n \\ &= \frac{1}{\pi}\sum_{n=0}^{\infty}(n+1)(\overline{z}\zeta)^n \\ &= \frac{1}{\pi}\frac{1}{(1-\overline{z}\zeta)^2} \end{aligned}$$

We next normalize the kernel function obtained and from now on the kernel function $K(z, \zeta)$ will be denoted by

$$K(z, \zeta) = \frac{1}{\sqrt{\pi}} \frac{1 - |z|^2}{(1 - \overline{z}\zeta)^2}$$

2.2 Berezin transform on the unit disk

Definition

For each $z \in D$, we have the biholomorphic involution $\phi_z : D \to D$ given by, $\phi_z(\zeta) = \frac{z-\zeta}{1-\overline{z}\zeta}$ [3]. These involutions provide an alternative form of transform of a function. So we can define the Berezin transform Tu of any $u \in L^1(\mathrm{dA})$, by

$$Tu(z) = \frac{1}{\pi} \int_D u \circ \phi_z dA = \frac{1}{\pi} \int_D u \circ \phi_z(\zeta) dA(\zeta).$$

We show that the above definition is equivalent to the one presented in the previous chapter, by performing a simple change of variable. [3]

Proposition 5.

$$\begin{aligned} Tu(z) &= \int_D u(\zeta) |K(z, \zeta)|^2 dA(\zeta) \\ &= \frac{(1 - |z|^2)^2}{\pi} \int_D \frac{u(\zeta)}{|1 - \overline{\zeta} z|^4} dA(\zeta) \quad [3] \end{aligned}$$

Proof. Recall that the function ϕ_z is a biholomorphic convolution. Let

$$\begin{aligned} y &= \phi_z(\zeta) \\ \zeta &= \phi_z(y) \end{aligned}$$

Then,

$$dA(y) = |\phi_z'(\zeta)|^2 dA(\zeta) = \frac{(1-|z|^2)^2}{|1-\overline{z}\zeta|^4} dA(\zeta).$$

Thus,

$$Tu(z) = \int_D u(\zeta) \frac{(1-|z|^2)^2}{|1-\overline{z}\zeta|^4} dA(\zeta) = \int_D u \circ \phi_z(y) dA(y)$$

$\square$

2.3 Harmonicity and fixed points

Theorem 6. *If u is harmonic in D, and $u \in L^1(D)$ then $Tu(z) = u(z)$, $\forall\ z \in D$.*

Proof. Let $\zeta = re^{i\theta}$, then

$$\begin{aligned} Tu(z) &= \frac{1}{\pi} \int_D u \circ \phi_z dA \\ &= \frac{1}{\pi} \int_0^1 \int_0^{2\pi} u \circ \phi_z(re^{i\theta}) r dr d\theta. \end{aligned}$$

ϕ_z is a biholomorphic convolution in ζ, and u is harmonic,thus $u \circ \phi_z$ is harmonic. The mean value property for harmonic functions gives us that

$$\frac{1}{2\pi} \int_0^{2\pi} u \circ \phi_z(z_0 + re^{i\theta}) d\theta = u \circ \phi_z(z_0),$$

and so we get

$$Tu(z) = \int_0^1 u(\phi_z(0)) 2r dr = u(\phi_z(0)) = u(z).$$

$\square$

The converse of Theorem 6, runs very deep and has been the subject of a number of investigations, culminating finally in a result of [4]. Because of its importance we devote the next section to a detailed review of it.

2.3.1 From fixed point to harmonicity

Theorem 7. *[4] If $u \in L^1(D)$, with $Tu = u$ then u must be harmonic*

Proof. for $f \in L^1(D)$, $z \in D$, $k = 0, 1, 2...$, define the Toeplitz operators by

$$(T_k f)(z) = (k+1) \int_D \frac{(1-|z|^2)^{k+2}(1-|\zeta|^2)^k}{|1-z\overline{\zeta}|^{2(k+2)}} f(\zeta) dA(\zeta).$$

Note that $T_0 f(z)$ represents the original Toeplitz operator $Tf(z)$ that we are working with in this paper.

We define further, $M = \{ \ f \in L^1(D) \ / \ T_0 f = f \ \}$.

We note that M is a closed subspace of $L^1(D)$. Next, we establish a relation between the Laplacian of f belonging to M and the Toeplitz operators and we obtain the following relation,

$$\Delta_M T_k f = 8(T_k f - T_{k+1} f) \qquad (*)$$

and so, for all $f \in M$ we get,

$$\Delta_M f = \Delta_M T_0 f = 8(T_0 f - T_1 f) = 8(f - T_1 f).$$

This shows that the map $\Delta_M : M \to L^1(D)$ is bounded since $f \in L^1(D)$ and T_1 is a bounded operator.

Using the fact that the Toeplitz operators commute for $k = 0$ and $k = 1$ i.e $T_0 T_1 = T_1 T_0$ we get,

$$T_0\Delta_M f = 8(T_0 f - T_0 T_1 f) = 8(f - T_1 T_0 f) = 8(f - T_1 f) = \Delta_M f$$

Thus Δ_M carries M into M, so it is a bounded linear operator. Rewriting $(*)$ in the following way,

$$T_k f = \Big(1 - \frac{\Delta_M}{4k(1+k)}\Big)T_{k-1} f \qquad (**)$$

we get by induction that for $f \in M$

$$T_k f = G_{k,1}\Delta_M T_0 f$$

where

$$G_{k,1}(\lambda) = \prod_{j=1}^{k} 1 - \frac{\lambda}{4j(1+j)}$$

As $k \to \infty$, the functions $G_{k,1}$ converge uniformly to $G_1(\lambda) = \prod_{j=1}^{\infty}$ (1- $\frac{\lambda}{4j(1+j)}$) on compact subsets of $\mathbb{C}$. It can then be shown that $\lim_{k\to\infty} \|f - T_k f\| = 0$ and so we conclude that as $k \to \infty$, $(**)$ becomes

$$f = G_1(\Delta_M) f$$

and so $G_1(\Delta_M)$ is the identity operator on M.

In the next step of the proof, let $\lambda \in \mathbb{C}$ be such that $\lambda = -4\beta(1-\beta)$, we define then the following sets,

$$\begin{aligned}
\Sigma_{1,1} &= \{\beta \in \mathbb{C} : -1 < Re\beta < 1\} \\
\Omega_{1,1} &= \{\beta \in \mathbb{C} : Re\beta < 8\} \\
X_\lambda &= \{f \in L^1(D) : \Delta_M f = \lambda f\} \\
E_1 &= \{\lambda \in \Omega_{1,1} : G_1(\lambda) = 1\}
\end{aligned}$$

We note that the reason behind choosing the set $\Sigma_{1,1}$ is that $X_\lambda \cap L^1(D) \neq 0$ if and only if $\beta \in \Sigma_{1,1}$, and the set $\Omega_{1,1}$ is the image of $\Sigma_{1,1}$ under the map that takes $\beta \to \lambda$.
We claim that the set E_1 is the set of all eigenvalues of Δ_M, i.e if $\Delta_M f = \lambda f$ then $\lambda \in E_1$.
Next, we let Q be the monic polynomial that has a simple zero at every point of the set E_1 and no other zeros in $\mathbb{C}$, since $G_1'(\lambda) \neq 0$ in $\Omega_{1,1}$ then there is an entire function H satisfying

$$H(\lambda)Q(\lambda) = G_1(\lambda) - 1$$

and $H(\lambda) \neq 0$ at every $\lambda \in E_1$.
We define the spectrum of an operator A, denoted by $\sigma(A)$ to be the set of all eigenvalues of A, so in our case $E_1 = \sigma(\Delta_M)$. From this, we see that $H(\sigma(\Delta_M))$ does not contain 0. We apply the Spectral mapping theorem to conclude that $0 \notin \sigma(H(\Delta_M))$, and thus, it follows that $H(\Delta_M)$ is 1-1.
Using the fact that $G_1(\Delta_M)$ is the identity operator on M we get,

$$H(\Delta_M)Q(\Delta_M) = G_1(\Delta M) - I = 0$$

This means that the range of $Q(\Delta_M)$ is in the null-space of $H(\Delta_M)$ which as we just showed equals to 0. We conclude that $Q(\Delta_M) = 0$ on M. Finally we state a lemma and a proposition and make use of them to finish the proof.

Lemma 8. *[4] Suppose that*

1. *X is a vector space over $\mathbb{C}$,*

2. *$T : X \to X$ is linear,*

3. *$Q(\lambda) = \prod_{i=1}^{N} (\lambda - \alpha_i)$, $\alpha_i \in \mathbb{C}$, $\alpha_i \neq \alpha_j$, if $i \neq j$,*

4. *$Q(T) = 0$,*

5. *Y_i is the null-space of $(T - \alpha_i I)$.*

Then $X = X_1 \bigoplus ... \bigoplus Y_N$.

Proposition 9. *[5] The function $G_1 - 1$ does not vanish on $\sum = \{z \in \mathbb{C} : 0 \leqslant Rez \leqslant 1\}$*

Proof. $G_1(\lambda) = \frac{\sin(\pi\beta)}{\pi\beta(1-\beta)}$, where λ = -4 β (1-β).
Since $G_1(0) = G_1(1) = 1$, it suffices to show that $G_1 - 1$ does not vanish in $\sum - \{0, 1\}$.
Using the product expansion of sine, we have

$$\frac{\sin(\pi\beta)}{\pi\beta(1-\beta)} = \prod_{k=2}^{\infty} \Big(1 - \frac{\beta(1-\beta)}{k(1-k)}\Big) = \prod_{k=1}^{\infty} \Big(1 + \frac{\beta(1-\beta)}{k(1+k)}\Big).$$

The function $\beta \to \beta(1-\beta)$ maps $\sum$ onto the parabolic region

$$P = \{z \in \mathbb{C} : Re(z) \geqslant (Im(z))^2\}$$

Let $z \in \sum$ and denote $s = z(1-z) \in P$. Suppose $s = x + iy$ and $y = Im(s) > 0$. Then

$$\begin{aligned} 0 < \arctan \frac{y/k(k+1)}{1 + x/k(k+1)} &\leqslant \arctan \frac{y}{k(1+k) + y^2} \\ &\leqslant \arctan \frac{y}{k^2 + y^2} \\ &\leqslant \frac{y}{k^2 + y^2} \end{aligned}$$

so $1 + z(1-z)/k(k+1) = R_k e^{i\theta_k}$, for some $R_k > 0$ and $0 < \theta_k \leqslant y/(k^2 + y^2)$. It follows that $sin(\pi z)/\pi z(1-z) = Re^{i\theta}$ for some $R \geqslant 0$ and

$$\begin{aligned} 0 < \theta \leqslant \sum_{k=1}^{\infty} \frac{y}{k^2 + y^2} &\leqslant \frac{y}{1+y^2} + \int_1^{\infty} \frac{y}{k^2 + y^2} dk \\ &= \frac{y}{1+y^2} + \frac{\pi}{2} - \arctan \frac{1}{y} \\ &< \pi \end{aligned}$$

Particularly, $Re^{i\theta} \neq 1$. Similarly, $\sin(\pi z)/(\pi z)(1-z) \neq 1$ if $s = z(1-z) \in P$ and $Im(s) < 0$. Finally, if $s \in P \cap \mathbb{R} = [0, +\infty)$, we get

$$\prod_{k=1}^{\infty} \Big(1 + \frac{s}{k(k+1)}\Big) \geqslant \prod_{k=1}^{\infty} \big(1 + \frac{0}{k(k+1)}\Big) = 1.$$

with equality occurring only when $s = 0$, i.e when $z = 0$, or $s = 1$. $\square$

We apply lemma 8 to the space M and the operator Δ_M. Then every $f \in M$ is a sum

$$f = \sum_{\lambda \in E_1} f_\lambda$$

in which $f_i \in M \bigcap X_i$.

proposition 10 shows that $E_1 = 0$, hence $Q(\lambda) = \lambda$ and we reach the conclusion that $\Delta_M = 0$ for every $f \in M$ i.e, $T_0 f = f$ implies that f is harmonic. $\square$

It is important to point out that with more stringent conditions on the function u, e.g $u \in C^2(D)$, a much simpler proof of theorem 6 may be obtained. This proof relies on a formula that relates the Laplacian of a function $u \in C^2(D)$ to its Berezin transform on a disk.

Proposition 10. *[3] Suppose $u \in C^2(D)$ and $0 < r < 1$, Then*

$$u(0) = \frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})d\theta + \frac{2}{\pi}\int_{|\zeta|\leqslant r} \Delta u(\zeta)\log\frac{|\zeta|}{r}dA(\zeta)$$

Proof. Starting with Green's theorem,

$$\int_C Mdx + Ndy = \int\int_{D_r} \Big(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\Big)dA,$$

where D_r is the circle with radius r and circumference C, along with the following identity,

$$\int\int_{D_r} (\Delta u)dA = \int_C \frac{\partial u}{\partial n}ds,$$

where n is the outernormal, and so

$$\frac{\partial u}{\partial n} = (\nabla\vec{u}).\vec{n} = (u_x, u_y).\vec{n} = (u_x, u_y).(\frac{x}{r}, \frac{y}{r}) = \frac{1}{r}(xu_x + yu_y).$$

With these 2 equations, taking $N{=}u_x$, and $M{=}{-}u_y$ we get,

$$\int\int_{D_r} (u_{xx} + u_{yy})dA = \int_C u_x dy + u_y dx = \int_C (xu_x + yu_y)\frac{ds}{r}.$$

In polar coordinates,

$$u(x,y) = u(rcos\theta, rsin\theta) = U(r,\theta)$$

$$r\frac{\partial U}{\partial r} = r\frac{\partial u}{\partial x}cos\theta + r\frac{\partial u}{\partial y}sin\theta = xu_x + yu_y = r\frac{\partial u}{\partial n}.$$

$$\frac{\partial u}{\partial n} = \frac{\partial U}{\partial r}.$$

Note that the Laplacian we used is $\Delta u(x,y) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 4\,\frac{\partial^2 u}{\partial z \partial \bar{z}} = 4\Delta u(z)$, therefore,

$$\int\int_{D_r} 4(\Delta u)dA = \int_{C_r} \frac{\partial u}{\partial n} ds = \int_0^{2\pi} \frac{\partial U}{\partial r} r d\theta = r\frac{\partial}{\partial r}\int_0^{2\pi} U(r,\theta)d\theta.$$

Let ρ be the radius of D_ρ such that $\rho < r$, then the same formula obtained above can be written in terms of ρ, so

$$\frac{1}{\rho}\int\int_{D_\rho} 4(\Delta u)dA = \frac{\partial}{\partial \rho}\int_0^{2\pi} U(\rho,\theta)d\theta.$$

Integrate with respect to ρ on $[0, r]$,

$$\int_0^r \frac{1}{\rho}\Big(\int\int_{D_\rho} 4(\Delta u)dA\Big)d\rho = \int_0^{2\pi} U(\rho,\theta)d\theta\Big|_{\rho=0}^{\rho=r} = \int_0^{2\pi} U(r,\theta)d\theta - 2\pi U(0)$$

and so this yields:

$$\frac{1}{2\pi}\int_0^{2\pi} U(r,\theta)d\theta - U(0) = \frac{2}{\pi}\int_0^r \frac{1}{\rho}\Big(\int\int_{D_\rho} (\Delta u)dA\Big)d\rho.$$

We now work the right hand side of the last equality obtained,

$$\begin{aligned}
\frac{2}{\pi}\int_0^r \frac{1}{\rho}\Big(\int\int_{D_\rho}(\Delta u)dA\Big)d\rho &= \frac{2}{\pi}\int_0^{2\pi} d\theta\Big(\frac{1}{\rho}\int_0^{2\pi}\int_0^{\rho}(\Delta u)(se^{i\theta})sds\Big)d\rho \\
&= \frac{2}{\pi}\int_0^{2\pi} d\theta\Big(\log|\rho|\int_0^{\rho}(\Delta u)(se^{i\theta})sds\Big)\Big|_{\rho=0}^{\rho=r} \\
&- \frac{2}{\pi}\int_0^{2\pi} d\theta\Big(\int_0^{r}\log|\rho|(\Delta u)(\rho e^{i\theta})\Big)\rho d\rho \\
&= \frac{2}{\pi}\int_0^{2\pi} d\theta(\log r)\int_0^{r}(\Delta u)(se^{i\theta})sds \\
&- \frac{2}{\pi}\int_0^{2\pi} d\theta\lim_{\rho\to 0}(\log|\rho|)\int_0^{\rho}(\Delta u)(se^{i\theta})sds \\
&- \frac{2}{\pi}\int_0^{2\pi} d\theta\int_0^{r}(\log|\rho|)(\Delta u)(\rho e^{i\theta}\rho d\rho
\end{aligned}$$

We next compute the limit above as follows, by continuity of (Δu)

$$\begin{aligned}
\Big|\log|\rho|\int_0^{\rho}(\Delta u)(se^{i\theta})sds\Big| &\leqslant |log|\rho||\int_0^{\rho}|(\Delta u)(se^{i\theta})||sds| \\
&\leqslant |log|\rho||\int_0^{\rho}Msds = M|\log|\rho||\frac{\rho^2}{2}\to 0 \quad as \quad \rho\to 0
\end{aligned}$$

So the limit is 0, the equality above becomes then,

$$\begin{aligned}
\frac{2}{\pi}\int_0^r \frac{1}{\rho}\Big(\int\int_{D_\rho}(\Delta u)dA\Big)d\rho &= \frac{2}{\pi}\int_0^{2\pi} d\theta \int_0^r (\log r)(\Delta u)(\rho e^{i\theta})\rho d\rho \\
&- \frac{2}{\pi}\int_0^{2\pi} d\theta \int_0^r (\log|\rho|)(\Delta u)(\rho e^{i\theta})\rho d\rho \\
&= \frac{2}{\pi}\int_0^{2\pi} d\theta \int_0^r (\log r - \log|\rho|)(\Delta u)(\rho e^{i\theta})\rho d\rho \\
&= \frac{2}{\pi}\int_0^{2\pi} d\theta \int_0^r (\Delta u)(\rho e^{i\theta}) \log\frac{r}{|\rho|}\rho d\rho \\
&= \frac{2}{\pi}\int_{|\rho|\leqslant r} \Delta u(\rho)\log\frac{r}{|\rho|}dA(\rho)
\end{aligned}$$

Finally, we get

$$u(0) = \frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})d\theta + \frac{2}{\pi}\int_{|\rho|\leqslant r}\Delta u(\rho)\log\frac{r}{|\rho|}dA(\rho).$$

□

Next, we multiply both sides of the equation derived by $2r$ and integrate on r from 0 to 1. We obtain

$$(Tu)(0) - u(0) = \int_{|\rho|<1}\Delta u(\rho)K(\rho)dA(\rho) \qquad (**)$$

where

$$K(\rho) = \frac{4}{\pi}\int_{|\rho|}^{1} r\log\frac{r}{|\rho|}dr \;=\; \frac{4}{\pi}\frac{r^2}{2}\log\frac{r}{|\rho|}\Big|_{|\rho|}^{1} - \frac{4}{\pi}\int_{|\rho|}^{1}\frac{r}{2}dr$$
$$= \frac{2}{\pi}\log\frac{1}{|\rho|} - \frac{r^2}{\pi}\Big|_{|\rho|}^{1}$$
$$= \frac{1}{\pi}\Big[\log\frac{1}{|\rho|^2} - (1-|\rho|^2)\Big].$$

Next, we check what conditions are required on u so that the above equality is satisfied.

We look at the Kernel K obtained, this is mainly done in [3].

Let $f(x) = \log\frac{1}{x} - (1-x)$, then application of Taylor's formula with remainder shows that

$$\begin{aligned} f(x) &= f(1) + f'(1)(x-1) + f''(t)\frac{(x-1)^2}{2!}, \qquad x<t<1 \\ &= \frac{1}{t^2}\frac{(x-1)^2}{2!} = \frac{1}{2t^2}(x-1)^2, \qquad 0<x<t<1 \end{aligned}$$

Then $f(x) \geqslant 0$ for $0 < x < 1$.

Since $t < 1$ and for $0 < x < 1$ we get

$$f(x) \geqslant \frac{1}{2}(1-x)^2$$

And for $\frac{1}{2} < x < 1$ we have $t > \frac{1}{4}$ and so we get

$$f(x) \leqslant 2(1-x)^2$$

Thus,$(**)$ holds for $u \in C^2(D)$ and for

$$\int_{|\rho|} |u(\rho)|dA(\rho) < \infty$$

and,

$$\int_{|\rho|<1} |\Delta u(\rho)|(1-|\rho|^2)^2 dA(\rho) < \infty$$

Apply now $(**)$ with $u \circ \phi_z$ instead of u, and get

$$\begin{aligned} Tu(z) - u(z) &= \int_{|\rho|<1} \Delta(u \circ \phi_z)(\rho) K(\rho) dA(\rho) \\ &= \int_{|\rho|<1} (\Delta u)(\phi_z(\rho))|\phi_z'(\rho)|^2 \end{aligned}$$

and so we restate the results of Proposition 10 in the form of a theorem,

Theorem 11. *[3] For $u \in C^2(D)$,*

$$\int_{|\rho|} |u(\rho)| dA(\rho) < \infty$$

and,

$$\int_{|\rho|<1} |\Delta u(\rho)|(1-|\rho|^2)^2 dA(\rho) < \infty$$

Then

$$(Tu)(0) - u(0) = \int_{|\rho|<1} \Delta u(\rho) K(z,\rho) dA(\rho)$$

where

$$K(z,\rho) = \frac{1}{\pi}\Big[\log \frac{1}{|\phi_z(\rho)|^2} - (1 - |\phi_z(\rho)|^2)\Big]$$

Note that $K(z,\rho)$ is obtained by setting $w = \phi_z(\rho)$ then $dw = |\phi_z'(\rho)|^2 \, dA(\rho)$ and so

$$\int_{|\rho|<1} \Delta(u \circ \phi_z)(\rho) K(\rho) dA(\rho)$$

becomes

$$\int_{|\rho|<1} \Delta u(\rho) K(z,\rho) dA(\rho)$$

Proposition 12. *[3] The kernel $K(z,\zeta)$ obtained in the previous theorem, satisfies*

$$K(z,\zeta) \geqslant \frac{1}{2\pi}\Big[\frac{(1-|z|^2)(1-|\zeta|^2)^2}{|1-\overline{z}\zeta|^2}\Big] \quad (1)$$

and,

$$K(z,\zeta) \leqslant \frac{2}{\pi}\Big[\frac{(1-|z|^2)(1-|\zeta|^2)}{|1-\overline{z}\zeta|^2}\Big]^2, \qquad if \quad \frac{(1-|z|^2)(1-|\zeta|^2)}{|1-\overline{z}\zeta|^2} < \frac{1}{2} \quad (2)$$

Before proving the 2 inequalities, we prove the following equation and we make use of it to prove proposition 12.

$$1-|\phi_z(\zeta)|^2 = \frac{(1-|z|^2)(1-|\zeta|^2)}{|1-\overline{z}\zeta|^2} \quad (3)$$

Starting with the left hand side, a simple calculation yields

$$\begin{aligned} 1-|\phi_z(\zeta)|^2 &= 1-\frac{(z-\zeta)(\overline{z}-\overline{\zeta})}{|1-\overline{z}\zeta|^2} \\ &= 1-\frac{|z|^2-z\overline{\zeta}-\zeta\overline{z}+|\zeta|^2}{|1-\overline{z}\zeta|^2} \\ &= \frac{1-|\zeta|^2-|z|^2+|z|^2|\zeta|^2}{|1-\overline{z}\zeta|^2} \\ &= \frac{(1-|z|^2)(1-|\zeta|^2)}{|1-\overline{z}\zeta|^2} \end{aligned}$$

Proof. we start by proving the first inequality (1) .We have seen that for a function f defined by $f(x) = \log\frac{1}{x}$ - $(1-x)$, the following holds,

$$f(x) = \log\frac{1}{x} - (1-x) \geqslant \frac{1}{2}(1-x)^2$$

So applying this inequality to the function $K(z,\zeta)$ we prove (1) as follows,

$$\begin{aligned}
K(z,\zeta) &= \frac{1}{\pi}\Big[\log\frac{1}{|\phi_z(\zeta)|^2} - (1-|\phi_z(\zeta)|^2)\Big]\\
&\geqslant \frac{1}{2\pi}(1-|\phi_z(\zeta)|^2)^2\\
&\geqslant \frac{1}{2\pi}\Big[\frac{(1-|z|^2)(1-|\zeta|^2)}{|1-\overline{z}\zeta|^2}\Big]^2,
\end{aligned}$$

where the last last step follows by (3).

To prove (2) we also refer back to the function $f(x) = \log\frac{1}{x}$ - $(1-x)$ discussed previously, and we have seen that the following holds,

$$f(x) = \log\frac{1}{x} - (1-x) \leqslant 2(1-x)^2, \qquad \frac{1}{2} < x < 1$$

So applying this inequality to the function $K(z,\zeta)$, with x replaced by $|\phi_z(\zeta)|^2$ we get (2) ,

$$\begin{aligned}
K(z,\zeta) &= \frac{1}{\pi}\Big[\log\frac{1}{|\phi_z(\zeta)|^2} - (1-|\phi_z(\zeta)|^2)\Big]\\
&\leqslant \frac{2}{\pi}(1-|\phi_z(\zeta)|^2)^2\\
&\leqslant \frac{2}{\pi}\Big[\frac{(1-|z|^2)(1-|\zeta|^2)}{|1-\overline{z}\zeta|^2}\Big]^2
\end{aligned}$$

where the last step also follows by (3).(2) holds for $\frac{1}{2} < |\phi_z(\zeta)|^2 < 1$, that is

$$1-|\phi_z(\zeta)|^2 \;<\; \frac{1}{2}$$

thus

$$\frac{(1-|z|^2)(1-|\zeta|^2)}{|1-\overline{z}\zeta|^2} \;<\; \frac{1}{2}$$

as specified in the theorem. $\square$

Having found that a function u is a fixed point of the Berezin transform if it is harmonic, it is natural to consider possible analogues of this when the function is subharmonic. It turns out that subharmonicity of u implies the inequality $Tu \geqslant u$. However the converse of this statement turns out not to be true globally.

2.4 Subharmonicity and fixed points

Proposition 13. *If u is subharmonic, $u \in L^1(D)$, then $Tu \geqslant u$ in D.*

Proof. It is known that if $u \in C^2\ (\Omega)$ then u is subharmonic if and only if $\Delta u \geqslant 0$. Since

$$\Delta(u \circ \phi_z) = \big((\Delta u) \circ \phi_z\big)|\phi_z'|^2| \geqslant 0,$$

it follows that, if u is subharmonic in D then $u \circ \phi_z$ is subharmonic in D.
Using the mean value inequality for subharmonic functions we obtain

$$Tu(z) = \frac{1}{\pi}\int_0^1\int_0^{2\pi} u \circ \phi_z(re^{i\theta})rdrd\theta \geqslant \int_0^1 2r(u \circ \phi_z)(0)dr = u(z)$$

$\square$

We next show that the converse of the proposition
13 does not hold, by providing a function u that is not subharmonic but that satisfies $Tu \geqslant u$.

Example [3]

Define a continuous function G by,

$$G(a) = \int_D |\phi_a| \frac{dA}{\pi}.$$

Note that $G(0) = \frac{2}{3}$, so there exists $\delta > 0$ such that $G(a) > \frac{1}{2}$ if $|a| < \delta$.

Let u be any strictly convex function that is continuous and integrable on [0,1) such that $u(0) = u(\alpha) = 0$ for 0< $\alpha < \frac{1}{2}$, then $u(r) < 0$ for 0< r $< \alpha$, and u has a minimum at a unique point β.

We further assume that $\beta < \delta$, and regard u as a radial function on D. We next prove that any such u satisfies $Tu \geqslant u$:

First suppose that $|a| \leqslant \beta$, then $u(a) = u(|a|) < 0$ because $|a| \leqslant \beta \leqslant \alpha$ and $u(r) < 0$ for 0 < r $< \alpha$.

On the other hand, $G(a) = \int_D |\phi_a| \frac{dA}{\pi} \geqslant \frac{1}{2} > \alpha$.

By Jensen's inequality,

$$u(\int_D |\phi_a| \frac{dA}{\pi}) \leqslant \int_D u \circ \phi_a \frac{dA}{\pi}.$$

Hence $Tu(a) \geqslant u(a)$ in this case.

The second case is for $|a| > \beta$, we have

$$\begin{aligned} a &= \int_D \phi_a \frac{dA}{\pi} \\ |a| &\leqslant \int_D |\phi_a| \frac{dA}{\pi} \end{aligned}$$

The equality above is shown by use of the mean value property on the disk,

$$f(z_0) = \frac{1}{\pi r^2} \int_{D(z_0,r)} f(z) dA(z).$$

Therefore $u(a) \leqslant u(\int |\phi_a| \frac{dA}{\pi})$ because u is strictly increasing on $(\beta,1)$, then by Jensen's

$$u(a) \leqslant u(\int_D u \circ \phi_a \frac{dA}{\pi}).$$

And so also, $Tu(a) \geqslant u(a)$ in this case.

Clearly, u is not subharmonic because $u(0) = 0$ and

$$\frac{1}{2\pi} \int_0^{2\pi} u(re^{i\theta}) d\theta = u(r) < 0, \qquad 0 < r < \alpha.$$

While it is not true that the condition $Tu \geqslant u$ implies subharmonicity of u, it does imply some subharmonicity near the boundary. We consider this next, but in order to do so, we will need some estimates on integrals. The following lemma is stated in [3] without proof. Because of its importance, we supply a proof below.

Lemma 14. *[3] There exists $c_0 > 0$ such that*

$$\int_{|\zeta|<1} \frac{(1 - |\zeta|^2)^2}{|1 - \overline{z}\zeta|^4} dA(\zeta) \geqslant c_0 \log \frac{1}{1 - |z|}$$

Proof. 1. We show that

$$\int_0^{2\pi} \frac{d\theta}{|1 - ae^{i\theta}|^4} = 2\pi \frac{(1 + |a|^2)}{(1 - |a|^2)^3}.$$

Let $z = e^{i\theta}$ then $d\theta = \frac{dz}{iz}$

$$|1 - ae^{i\theta}|^4 = (1 - az)^2 (1 - \overline{az})^2 = (1 - za)^2 (1 - \frac{\overline{a}}{z})^2$$

It follows that

$$\begin{aligned}\int_0^{2\pi}\frac{d\theta}{|1-ae^{i\theta}|^4} &= \frac{1}{i}\int_{|z|=1}\frac{\frac{1}{z}dz}{(1-az)^2(1-\frac{\overline{a}}{z})^2}\\ &= \frac{1}{i}\int_{|z|=1}\frac{zdz}{(1-az)^2(z-\overline{a})^2}\end{aligned}$$

$\overline{a}\in\{|z|<1\}$ where as $\frac{1}{a}\notin\{|z|<1\}$, so by Cauchy formula,

$$\begin{aligned}\frac{1}{i}\int_{|z|=1}\frac{zdz}{(1-az)^2(z-\overline{a})^2} &= \frac{1}{i}\int_{|z|=1}\frac{f(z)}{(z-\overline{a})^2} \qquad \left(f(z)=\frac{z}{(1-\overline{a}z)^2}\right)\\ &= \frac{1}{i}2\pi i f'(\overline{a})\\ &= 2\pi\frac{(1-|a|^4)}{(1-|a|^2)^4}\\ &= 2\pi\frac{(1+|a|^2)}{(1-|a|^2)^3}\end{aligned}$$

2. We show that the following holds for $|z|<1$

$$\int_{|\zeta|<1}\frac{(1-|\zeta|^2)^2}{|1-\overline{z}\zeta|^4}d\zeta = 2\pi\int_0^1\frac{r(1-r^2)^2(1+|rz|^2)}{(1-|rz|^2)^3}dr$$

So,

$$\begin{aligned}\int_{|\zeta|<1}\frac{(1-|\zeta|^2)^2}{|1-\overline{z}\zeta|^4}d\zeta &= \int_0^1\int_0^{2\pi}\frac{(1-r^2)^2}{|1-\overline{z}re^{i\theta}|^4}rdrd\theta\\ &= \int_0^1 r(1-r^2)^2\int_0^{2\pi}\frac{d\theta}{|1-\overline{z}re^{i\theta}|^4}\\ &= 2\pi\int_0^1\frac{r(1-r^2)^2(1+|rz|^2)}{(1-|rz|^2)^3}dr\end{aligned}$$

3. We try to determine the right hand side of the equation obtained in part 2,

$$\begin{aligned}
\frac{2}{(1-x)^3} &= \sum_{n=0}^{\infty}(n+2)(n+1)x^n \\
\frac{2(1+x)}{(1-x)^3} &= \sum_{n=0}^{\infty}\Big[(n+2)(n+1)x^n + (n+2)(n+1)x^{n+1}\Big] \\
&= 2+\sum_{n=0}^{\infty}\Big[(n+3)(n+2)x^{n+1} + (n+2)(n+1)x^{n+1}\Big] \\
&= 2+\sum_{n=0}^{\infty}\Big[2x^{n+1}(n+2)^2\Big]
\end{aligned}$$

Replacing x in the above by $r^2|z|^2$,

$$\frac{1+r^2|z|^2}{(1-r^2|z|^2)^3} = 1+\sum_{n=0}^{\infty} r^{2n+2}(n+2)^2|z|^{2n+2}$$

multiplying the above by $r\ (1-r^2)^2$,

$$\frac{r(1-r^2)^2(1+r^2|z|^2)}{(1-r^2|z|^2)^3} = (r-2r^3+r^5)+\sum_{n=0}^{\infty}(n+2)^2(r^{2n+3}-2r^{2n+5}+r^{2n+2})$$

and finally integrating the last equation from 0 to 1 we get,

$$\begin{aligned}
2\pi\int_0^1 \frac{r(1-r^2)^2(1+r^2|z|^2)}{(1-r^2|z|^2)^3}dr &= 2\pi\Big[\frac{1}{6}+\frac{1}{2}\sum_{n=0}^{\infty}(n+2)^2(\frac{1}{n+2}-\frac{2}{n+3}+\frac{1}{n+4})|z|^{2n+2}\Big] \\
&= 2\pi\Big[\frac{1}{6}+\frac{1}{2}\sum_{n=0}^{\infty}(n+2)^2|z|^{2n+2}\Big(\frac{2}{(n+2)(n+3)(n+4)}\Big)\Big] \\
&= 2\pi\Big[\frac{1}{6}+\sum_{n=0}^{\infty}\frac{(n+2)|z|^{2n+2}}{(n+3)(n+4)}\Big]
\end{aligned}$$

Now it is easily observed that

$$\frac{n+1}{(n+2)(n+3)} \geqslant \frac{1}{6n}$$

Therefore it follows by parts 1, 2 , and 3

$$\begin{aligned}
\int_{|\zeta|<1} \frac{(1-|\zeta|^2)^2}{|1-\bar{z}\zeta|^4} d\zeta &\overset{by1\&2}{=} 2\pi \int_0^1 \frac{r(1-r^2)^2(1+r^2|z|^2)}{(1-r^2|z|^2)^3} dr \\
&\overset{by3}{=} 2\pi \Big[\frac{1}{6} + \sum_{n=0}^{\infty} \frac{(n+2)|z|^{2n+2}}{(n+3)(n+4)}\Big] \\
&\geqslant 2\pi (\frac{1}{6} + \frac{1}{6} \sum_{n=0}^{\infty} \Big[\frac{|z|^{2n}}{n}\Big] \\
&= \frac{\pi}{3}(1 + 2\log\frac{1}{1-|z|}) \\
&\geqslant \frac{2\pi}{3} \log \frac{1}{1-|z|}
\end{aligned}$$

Thus, the lemma is proved for $c_0 = \frac{2\pi}{3}$

□

Previously, we provided a counterexample showing that for $Tu \geqslant u$, u need not be subharmonic. The next theorem shows that the condition $Tu \geqslant u$ does imply some sort of an almost subharmonicity near the boundary, i.e we show that *limsup* $\Delta u \geqslant 0$ given that $Tu \geqslant u$.

Theorem 15. *[3] Suppose that $u \in C^2(D)$,*

$$\begin{aligned}
&\textstyle\int_{|\zeta|<1} \quad |u(\zeta)| dA(\zeta) < \infty, \\
&\textstyle\int_{|\zeta|<1} \quad |\Delta u(\zeta)|(1-|\zeta|^2)^2 dA(\zeta) < \infty,
\end{aligned}$$

and that $\limsup_{z\to\zeta_0} \Delta u(z) < 0$ *for some $\zeta_0 \in \partial D$. Then there exists $\delta > 0$ such that $Tu(z) < u(z)$ for all $z \in D$ such that $|z-\zeta_0| < \delta$.*

Proof. we assume that $\zeta_0 = 1$.There exists $a > 0$ and $\epsilon > 0$ such that if $z \in D$ and $|z-1| <$

ϵ then $\Delta u(z) \leqslant -a$.

Let $D(1,\epsilon) = \{z \in D{:}|z-1| < \epsilon\}$ and $D(1,\epsilon)' = \{D/\{D(1,\epsilon)\}\ \}$, then we have

$$\begin{aligned}\int_D \Delta u(\zeta)K(z,\zeta)dA(\zeta) &= \int_{D(1,\epsilon)} \Delta u(\zeta)K(z,\zeta)dA(\zeta)\\ &+ \int_{D(1,\epsilon)'} \Delta u(\zeta)K(z,\zeta)dA(\zeta).\end{aligned}$$

We start with the second integral,so for $|z-1| < \frac{\epsilon}{2}$ and $\zeta \in D(1,\epsilon)'$,

$$\frac{(1-|z|^2)(1-|\zeta|^2)}{|1-\overline{\zeta}z|^2} \leqslant C(1-|z|^2) < \frac{1}{2}$$

if $(1-|z|^2)$ is sufficiently small, and so by the inequality (2) in proposition 12, we have that $K(z,\zeta) \leqslant C(1-|z|^2)^2(1-|\zeta|^2$, which yields

$$\Big|\int_{D(1,\epsilon)'} \Delta u(\zeta)K(z,\zeta)dA(\zeta)\Big| \leqslant C(1-|z|^2)^2 \int_{D(1,\epsilon)'} |\Delta u(\zeta)|(1-|\zeta|^2)^2 dA(\zeta).$$

Note that this is $O((1-|z|^2))$.

Next, we deal with the first integral,

$$\begin{aligned}\int_{D(1,\epsilon)} \Delta u(\zeta)K(z,\zeta)dA(\zeta) &\leqslant -a\int_{D(1,\epsilon)} K(z,\zeta)dA(\zeta)\\ &= -a\int_D K(z,\zeta)dA(\zeta) + a\int_{D(1,\epsilon)'} K(z,\zeta)dA(\zeta)\\ &\leqslant \frac{-a}{2\pi}\int_D \frac{(1-|z|^2)^2(1-|\zeta|^2)^2}{|1-\overline{\zeta}z|^4}dA(\zeta)\\ &+ \frac{2a}{\pi}\int_{D(1,\epsilon)'} \frac{(1-|z|^2)^2(1-|\zeta|^2)^2}{|1-\overline{\zeta}z|^4}dA(\zeta)\\ &\leqslant -C_0 a(1-|z|^2)^2\log\frac{1}{1-|z|} + O((1-|z|^2)^2)\end{aligned}$$

Therefore,

$$Tu(z) - u(z) \leqslant -C_0 a(1 - |z|^2)^2 \log \frac{1}{1 - |z|} + O((1 - |z|^2)^2),$$

which becomes negative as z approaches 1. $\square$

Chapter 3. Fixed points of the Berezin transform (II)

In this chapter, we consider analogues of results in Chapter 2, where the disk is replaced by an annulus $\mathscr{A}$. We start by obtaining the Bergman Kernel function for the annulus and we define the Berezin transform of the annulus in terms of the kernel function, then we consider the main problem in this chapter which is to determine the fixed points of the Berezin transform in the annulus. This problem remains open at present, but we are able to give some necessary conditions for a function u to be a fixed point of the berezin transform.

We recall that the reproducing kernel K_z is the unique function in $L_a^2(\mathscr{A})$ such that for every $f \in \mathbf{A}^2$,

$$< f, K_z >= f(z) = \int_{\Omega} f(\zeta)\overline{K_z(\zeta)}dA(\zeta).$$

3.1 The kernel function of the annulus

We start by determining an orthogonal set of functions $\{\phi_n\}$ in $\mathbf{A}^2(\mathscr{A})$.

Observe that the monomials $\{z^n : n \in \mathbb{Z}\}$ form an orthogonal set in $\mathbf{A}^2(\mathscr{A})$.

Indeed,

$$
\begin{aligned}
< z^n, z^m > \quad &= \quad \int_{\mathscr{A}} z^n \overline{z^m} dA(z) = \int_0^{2\pi} \int_R^1 r^{n+m} e^{i(n-m)\theta} r dr d\theta = \frac{2\pi(1 - R^{n+m+2})}{n+m+2} \delta_{n,m}. \\
< z^n, z^n > \quad &= \quad \int_0^{2\pi} \int_R^1 r^{2n+1} dr d\theta = \frac{\pi(1 - R^{2n+2})}{n+1} \qquad\qquad n \neq -1 \\
< \frac{1}{z}, \frac{1}{z} > \quad &= \quad \int_R^1 \int_0^{2\pi} \frac{1}{r} dr d\theta = 2\pi \int_R^1 \frac{1}{r} dr = 2\pi \log \frac{1}{R} \qquad n = -1
\end{aligned}
$$

Thus the functions

$$
\phi_n(z) = \begin{cases} z^n \sqrt{\frac{n+1}{\pi(1-R^{2n+2})}} & : n \neq -1 \\ \sqrt{\frac{1}{2\pi \log \frac{1}{R}}} \frac{1}{z} & : n = -1 \end{cases}
$$

are orthonormal in $\mathbf{A}^2(\mathscr{A})$. To show that they form a basis, we show that they span the space, by showing that Parseval's identity holds for every $f \in \mathbf{A}^2(\mathscr{A})$.And this is equivalent to showing the following identity holds

$$
\|f\|^2 = \sum_{n=-\infty}^{+\infty} \frac{\pi(1 - R^{2n+2})}{n+1} |a_n|^2, \qquad f(z) = \sum_{n=-\infty}^{+\infty} a_n z^n,
$$

which is easily established. Therefore the set $\{\phi_n\}$ is an orthonormal basis in $\mathbf{A}^2(\mathscr{A})$. Next we make use of the reproducing property of the kernel function to derive a representation for the kernel function of the annulus, and finally obtain its formula explicitly.

Theorem 15. *The kernel function of the annulus has the following representation*

$$
K(z, \zeta) = \sum_{n=-\infty}^{+\infty} \overline{\phi_n(\zeta)} \phi_n(z)
$$

We omit the proof of this theorem for it is identical to the case of the unit disk which is established and proved in the previous chapter.

Therefore, the Kernel function of the annulus is given by

$$\begin{aligned}
K(z,\zeta) &= \overline{\phi_{-1}(z)}\phi_1(\zeta) + \sum \overline{\phi_n(z)}\phi_n(\zeta) \\
&= -\frac{1}{2\pi \log R}(\overline{z}\zeta)^{-1} + \sum \sqrt{\frac{n+1}{\pi(1-R^{2n+2})}}\overline{z^n} \quad \sqrt{\frac{n+1}{\pi(1-R^{2n+2})}}\zeta^n \\
&= -\frac{1}{2\pi \log R}(\overline{z}\zeta)^{-1} + \frac{1}{\pi}\sum \frac{(n+1)}{1-R^{2n+2}}(\overline{z}\zeta)^n
\end{aligned}$$

where the sum is taken over all $n \neq -1$.

From now on, K_w will denote the normalized kernel function of the annulus, for $w \in \mathscr{A}$.

3.2 The Annulus mean value property

Lemma 16. *[6] Suppose that $u(z) = f(z) + \overline{g(z)} + \log|z|$ is a harmonic functions in $L^2(\mathscr{A})$, with f and g analytic on $\mathscr{A}$. Then f and g belong to $L^2_a(\mathscr{A})$.*

Proof. without loss of generality, we assume that u is a real valued harmonic function.

Hence $u = ReF + c\log|z|$ for some analytic function F.

If we express F as a Laurent series, $F(z) = \sum_{n=-\infty}^{\infty} 2\, a_n z^n$, then u has the form

$$u(z) = \frac{1}{2}\Big(F(z) + \overline{F(z)}\Big) + c\log|z|$$

Let $z = re^{i\theta}$, replace $F(z)$ by its Laurent series, evaluate $\overline{F(z)}$, and replace n by $-n$, we get

$$
\begin{aligned}
u(re^{i\theta}) &= \frac{1}{2}\Big(\sum_{n=-\infty}^{\infty} 2a_n r^n e^{in\theta} + \sum_{n=-\infty}^{\infty} 2\overline{a}_{-n} r^{-n} e^{in\theta}\Big) + c\log r \\
u(re^{i\theta}) &= \sum_{n=-\infty}^{\infty} (a_n r^n + \overline{a}_n r^{-n}) e^{in\theta} + c\log r.
\end{aligned}
$$

Since

$$
\begin{aligned}
\int_0^{2\pi} |u(re^{i\theta})|^2 \frac{d\theta}{2\pi} &= (|c|logr + 2Re(a_0))^2 \\
&+ \sum_{|n|\geqslant 1}\sum_{|m|\geqslant 1}\int_0^{2\pi} (a_n r^n + \overline{a}_{-n} r^{-n})(\overline{a}_m r^m + a_{-m} r^{-m}) e^{i(n-m)\theta}\frac{d\theta}{2\pi}
\end{aligned}
$$

it follows that,

$$
\infty > \int_{\mathscr{A}} |u(z)|^2 dA(z) = K + \int_R^1 \int_0^{2\pi} \Big(\sum_{|n|\geqslant 1} |a_n|^2 r^{2n} + |a_{-n}|^2 r^{-2n} + a_n a_{-n} + \overline{a}_{-n}\overline{a}_n\Big) r d\theta dr \qquad (*)
$$

where k is a positive constant.We need to show that $F \in L_a^2(\mathscr{A})$, so we make use of the last inequality,

$$
\begin{aligned}
\int_R^1 \sum_{|n|\geqslant 1} (a_n a_{-n}) &= \int_R^1 \int_0^{2\pi} \Big(\sum_{|n|\geqslant 1} a_n r^n e^{in\theta}\Big)\Big(\sum_{|m|\geqslant 1} a_{-m} r^{-m} e^{-im\theta}\Big)\frac{d\theta}{2\pi} \\
&= \frac{1}{4}\int_R^1 \int_0^{2\pi} \Big(F(re^{i\theta}) - F(0)\Big)^2 \frac{d\theta}{2\pi},
\end{aligned}
$$

so $\int_R^1 \sum_{|n|\geqslant 1}(a_n a_{-n})$ converges, since F is analytic on $\mathscr{A}$. Therefore, from $(*)$ it follows that

$$
\int_R^1 \sum_{|n|\geqslant 1} |a_n|^2 r^{2n+1} dr < \infty,
$$

and so as claimed, $F(z) = \sum_{n=-\infty}^{\infty} 2\ a_n\ z^n \in L_a^2(\mathscr{A})$ $\square$

3.3 Berezin transform on the annulus

For an integrable function u, we introduce its Berezin transform Tu on the annulus by

$$Tu(z) = \int_{\mathscr{A}} u(\zeta)|K_z(\zeta)|^2 dA(\zeta)$$

We recall from the first chapter that the Toeplitz operator with symbol ϕ, denoted T_ϕ can be defined by $T_\phi\ f = P\ (\phi f)$, where P is the orthogonal projection from $L^2(\Omega)$ onto $\mathbf{A}^2(\Omega)$, for all ϕ belonging to $L^\infty(\Omega)$.

Proposition 17. *[6] If $\phi = f_1 + \overline{f_2}$ and $\psi = g_1 + \overline{g_2}$ are bounded harmonic functions on $\mathscr{A}$ satisfying $T_\phi T_\psi = T_\psi T_\phi$ then the function $u = f_1\overline{g_2}$ - $g_1\overline{f_2}$ must satisfy $Tu = u$ in $\mathscr{A}$.*

Proof. $T_\phi T_\psi = T_\psi T_\phi$ implies that

$$\begin{aligned} < T_\psi K_w, T_{\overline{\phi}} K_w > &= < T_\phi T_\psi K_w, K_w > \\ &= < T_\psi T_\phi K_w, K_w > \\ &= < T_\phi K_w, T_{\overline{\psi}} K_w > \end{aligned}$$

for all $z \in \mathscr{A}$.

Now, we express $T_\psi K_w$ and $T_{\overline{\phi}} K_w$ in terms of the functions g_1 , g_2 and the kernel.

$$\begin{aligned}
T_\psi K_w &= P(\psi K_w)(z) \\
&= P(g_1 + \overline{g_2})K_w(z) \\
&= g_1(z)K_w + P(\overline{g_2}K_w)(z) \\
&= g_1(z)K_w + < P(\overline{g_2}K_w), K_z > \\
&= g_1(z)K_w + < K_w, g_2 K_z > \\
&= g_1(z)K_w + \overline{< g_2 K_z, K_w >} \\
&= g_1(z)K_w + \overline{g_2(w)}K_w(z).
\end{aligned}$$

Similarly, we find that $T_{\overline{\phi}} K_w = f_2(z)K_w(z) + \overline{f_1(w)}\ K_w(z)$, so that

$$\begin{aligned}
< T_\psi K_w, T_{\overline{\phi}} K_w > &= \int_{\mathscr{A}} T_\psi K_w T_\phi \overline{K_w} dA(z) \\
&= \int_{\mathscr{A}} [g_1(z)K_w + \overline{g_2(w)}K_w(z)] \quad [f_2(z)K_w(z) + \overline{f_1(w)}K_w(z)] dA(z) \\
&= \int_{\mathscr{A}} |K_w(z)|^2 \Big(g_1(z)\overline{f_2(z)} + g_1(z)f_1(w) + \overline{g_2(w)f_2(z)} + \overline{g_2(w)}f_1(w) \Big) dA(z)
\end{aligned}$$

But,

$$\begin{aligned}
\int_{\mathscr{A}} g_1(z)f_1(w)|K_w(z)|^2 dA(z) &= f_1(w) < g_1 K_w, K_w > \\
&= f_1(w)g_1(w)K_w(w) \\
&= f_1(w)g_1(w)\|K_w\|^2 \\
&= f_1(w)g_1(w)
\end{aligned}$$

and so finally,

$$\begin{aligned} < T_\psi K_w, T_{\overline{\phi}} K_w > &= \int_{\mathscr{A}} g_1(z)\overline{f_2(z)}|K_w(z)|^2 dA(z) + f_1(w)g_1(w) \\ &+ f_1(w)\overline{g_2(w)} + \overline{f_2(w)g_2(w)} \qquad (*) \end{aligned}$$

by interchanging ψ and ϕ and using commutativity, we get

$$< T_\psi K_w, T_{\overline{\phi}} K_w > = < T_\phi K_w, T_{\overline{\psi}} K_w >$$

The last equality along with $(*)$ gives,

$$\int_{\mathscr{A}} (f_1\overline{g_2} - g_1\overline{f_2})(z)|K_w(z)|^2 dA(z) = (f_1\overline{g_2} - g_1\overline{f_2})(w)$$

for all $w \in \mathscr{A}$.

Therefore the function u satisfies $Tu = u$ and so it is a fixed point of the Berezin transform. □

The functions u satisfying $Tu = u$ are said to have the annulus mean value property. We have seen in the previous chapter that $Tu = u$ in D implies that u is harmonic on D for all functions integrable on D. One may naturally wonder if the same holds true for functions integrable on $\mathscr{A}$. To start with, we look for harmonic functions that have the annulus mean value property.

Proposition 18. *[6] If u is a harmonic function in $L^2(\mathscr{A})$, then $Tu = u$ in $\mathscr{A}$ if and only if $u = f + \overline{g}$, where f and g are analytic functions on $\mathscr{A}$.*

Proof. If $u(z) = f(z) + \overline{g(z)} \in L^2(\mathscr{A})$, then by Lemma 16, it is easily seen that $Tf = f$ and $T\overline{g} = \overline{g}$.

Suppose now that $Tu = u \in \mathscr{A}$ and suppose that $u = f + \overline{g} + \log|z|$. In [6] the author points out a communication to him by Ahern, to the effect that $\log|z|$ does not have the annulus mean value property and therefore the function u cannot have the logarithmic term $\log|z|$ if it satisfies $Tu = u$. $\square$

Now we strengthen the assumption on u by assuming it is continuous on the closure of $\mathscr{A}$ and we try to see whether with this condition, harmonicity can be obtained from the annulus mean value property of u.

Proposition 19. *[6] If $u \in C(\overline{\mathscr{A}})$. Then $Tu = u$ and*

$$\int_0^{2\pi} u(Re^{i\theta})\frac{d\theta}{2\pi} = \int_0^{2\pi} u(e^{i\theta})\frac{d\theta}{2\pi}$$

if and only if $u = f + \overline{g}$, where f and g are analytic on $\mathscr{A}$.

Proof. Suppose that the averages of u on both circles are the same. Then the poisson extension of $u|_{\partial\mathscr{A}}$ is a harmonic function v in $\mathscr{A}$. We denote by v again its continuous extension to the boundary. Applying proposition 18 to the function v, we get that $Tv = v$ in $\mathscr{A}$, and so $T(u - v) = (u - v)$.
We now show that if $w = u - v = 0$ on $\partial\mathscr{A}$, and w satisfies $Tw = w$ then $w = 0$ on $\mathscr{A}$ i.e u $= v$ on $\mathscr{A}$.
We recall that the Kernel is a normalized function and so,

$$\int_{\mathscr{A}} |k_z(\zeta)|^2 dA(\zeta) = 1$$

$Tw = w$ implies the following

$$Tw(z) = \int_{\mathscr{A}} w(\zeta)|k_z(\zeta)|^2 dA(\zeta) = w(z) = \int_{\mathscr{A}} w(z)|k_z(\zeta)|^2 dA(\zeta)$$

and so we get

$$\int_{\mathscr{A}} [w(\zeta) - w(z)]|k_z(\zeta)|^2 dA(\zeta) = 0 \qquad (*)$$

for all $z \in \mathscr{A}$. We show that this last equality implies that $w(z) = 0$ for all $z \in \overline{\mathscr{A}}$.
Since w is continuous on $\overline{\mathscr{A}}$, it attains its maximum/minimum value at a point $z_0 \in \overline{\mathscr{A}}$. If $z_0 \in \partial\overline{\mathscr{A}}$, then the maximum value is 0, and the minimum value is also 0 so the function in this case must be identically 0 inside $\mathscr{A}$ as well as on the boundary.
If $z_0 \notin \partial\overline{\mathscr{A}}$, so $z_0 \in \mathscr{A}$ then we can apply $(*)$

$$\int_{\mathscr{A}} [w(\zeta) - w(z_0)]|k_{z_0}(\zeta)|^2 dA(\zeta) = 0$$

So, $[w(\zeta) - w(z_0)]|k_{z_0}(\zeta)|^2$ is a continuous function of ζ which is $\leqslant 0$ for all ζ and its integral equals to 0, therefore it must be identically 0, so $w = 0$ and therefore $u = v$ on $\mathscr{A}$. It follows that u is a harmonic function since v is harmonic, and so $u = f + \overline{g}$ for some functions f and g analytic on $\mathscr{A}$.
Conversely, if $u = f + \overline{g} \in C(\overline{\mathscr{A}})$, then f and g belong to the Hardy space $H^2(\mathscr{A})$ by Lemma 16. Cauchy's theorem guarantees that the averages of u on both boundaries are the same. $\square$

For $L_n(z) = z^n$ for $n \in \mathbb{Z}$, we define the following function

$$(u * L_n)(z) = \int_0^{2\pi} u(ze^{-i\phi})L_n(e^{i\phi})\frac{d\phi}{2\pi}$$

Lemma 20. *[6] If $u \in C(\overline{\mathscr{A}})$,$L_n(z) = z^n$, $n \neq 0$, and $Tu = u$ then*

1. *$(u * L_n) \in C(\overline{\mathscr{A}})$*

2. *$T(u * L_n) = u * L_n$*

3. *$Average_{\partial D_R}$ $(u * l_n) = 0 = Average_{\partial D}$ $(u * L_n)$*

4. $u * L_n$ *is a harmonic function.*

5. Δu *is a radial function on* $\mathscr{A}$*.*

Proof. 1. Since $u \in C(\overline{\mathscr{A}})$ and $L_n(z)$ is a continuous function, it follows that $(u * L_n) \in C(\overline{\mathscr{A}})$.

2. From the Laurent series expansion of the kernel function obtained before,

$$K(z,\zeta) = \frac{1}{\pi}\sum \frac{(n+1)}{1-R^{2n+2}}(\overline{z}\zeta)^n - \frac{1}{2\pi\log R}(\overline{z}\zeta)^{-1}$$

we see that

$$\begin{aligned} K_{ze^{i\phi}}(we^{-i\phi}) &= \frac{1}{\pi}\sum \frac{n+1}{1-R^{2n+2}}(we^{-i\phi}\overline{z}e^{i\phi}) - \frac{1}{2\pi\log R}((we^{-i\phi}\overline{z}e^{i\phi}) \\ &= K_z(w) \end{aligned}$$

where the sum is taken over all $n \neq -1$.Therefore,

$$\begin{aligned} T(u * L_n)(z) &= \int_0^{2\pi}\Big[\int_{\mathscr{A}} u(we^{i\phi})|K_z(w)|^2 dA(w)\Big]L_n(e^{i\phi})\frac{d\phi}{2\pi} \\ &= \int_0^{2\pi} u(ze^{-i\phi})L_n(e^{i\phi})\frac{d\phi}{2\pi} \\ &= u * L_n(z) \end{aligned}$$

3. We write explicitly the average of the function $u * L_n(z)$ on the boundary of the unit

disk as well as on the disk of radius R, using Fubini's theorem.

$$
\begin{aligned}
Average_{\partial D_R} &= \int_0^{2\pi} (u * l_n)(Re^{i\theta})\frac{d\theta}{2\pi} \\
&= \int_0^{2\pi} \Big(\int_0^{2\pi} u(Re^{i(\theta-\phi)}) L_n(e^{i\phi})\frac{d\phi}{2\pi}\Big)\frac{d\theta}{2\pi} \\
&= \int_0^{2\pi} \Big(\int_0^{2\pi} u(Re^{i(\theta-\phi)}) \frac{d\theta}{2\pi}\Big) L_n(e^{i\phi})\frac{d\phi}{2\pi} \\
&= \int_0^{2\pi} \Big(\int_{-\phi}^{2\pi-\phi} u(Re^{it}) \frac{dt}{2\pi}\Big) L_n(e^{i\phi})\frac{d\phi}{2\pi} \qquad (t = \theta - \phi) \\
&= \Big(\int_0^{2\pi} u(Re^{it}) \frac{dt}{2\pi}\Big)\Big(\int_0^{2\pi} L_n(e^{i\phi})\frac{d\phi}{2\pi}\Big) \\
&= 0
\end{aligned}
$$

The last equality follows since,

$$
\int_0^{2\pi} L_n(e^{i\phi})\frac{d\phi}{2\pi} = \int_0^{2\pi} R^n e^{in\theta}\frac{d\phi}{2\pi} = 0
$$

Similarly, we get the same result when $R = 1$, i.e on the boundary of the unit disk. Therefore the averages on both circles are equal to 0.

4. Since the function $(u * l_n)$ has the annulus mean value property ($T(u * L_n) = u * L_n$ by part 2) and has equal average on both circles (by part 3) then Proposition 19 implies that $(u * l_n)$ is a harmonic function.

5. $(u * L_n)$ being harmonic by part 4, we get

$$
\Delta(u * L_n)(z) = \int_0^{2\pi} \Delta[u(ze^{-i\phi})]L_n(e^{i\phi})\frac{d\phi}{2\pi} = \int_0^{2\pi} (\Delta u)(ze^{-i\phi})e^{in\phi}\frac{d\phi}{2\pi} = 0
$$

Let $z = re^{i\theta}$, and make a change of variable to get,

$$\int_0^{2\pi} (\Delta u)(re^{i\phi})e^{in\phi}\frac{d\phi}{2\pi} = 0$$

for all n different from 0. Thus Δu is a radial function on the annulus.

$\square$

We next show that under these conditions, the function u is the sum of a radial and a harmonic function on $\mathscr{A}$, but first we state a lemma that will be used later on.

Lemma 21. *[6] If $u \in C(\overline{\mathscr{A}})$, then $\Delta(\mathscr{R}(u)) = \mathscr{R}(\Delta u)$, where*

$$\mathscr{R}(u)(w) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta}w)d\theta$$

is the radicalization of u.

Corollary 22. *[6] If $u \in C(\overline{\mathscr{A}})$ and Δu is a radial function, then u can be expressed as the sum of a radial function plus a harmonic function.*

Proof. lemma 21 gives,

$$\Delta(\mathscr{R}(u)) = \mathscr{R}(\Delta u) = \Delta u,$$

where the last equality follows since Δu is itself a radial function. And so,
$\Delta(u - \mathscr{R}(u)) = 0$, which means that $u - \mathscr{R}(u)$ is harmonic. Therefore, u can be written as a radial function plus a harmonic function in the following way,

$$u = \mathscr{R}(u) + (u - \mathscr{R}(u)),$$

$\square$

Using the fact that $\mathscr{R}(u(z)) = (u * L_0)(z)$, it follows that if If $u \in C(\overline{\mathscr{A}})$ satisfies $Tu = u$ in $\mathscr{A}$, then $u = \mathscr{R}(u) + f + \overline{g}$, where f and g are analytic functions on $\mathscr{A}$.[6]

3.4 Convex radial functions and fixed points

In attempting to determine the fixed points of the Berezin transform, it is natural to consider radial functions. But even in this case, difficulties occur. In this section, we present sufficient conditions on a radial function u that guarantees that $Tu \geqslant u$ or that $Tu \leqslant u$.

Proposition 23. *Suppose $f\colon [R,1] \to \mathbb{R}$ is a convex increasing function. If $u\colon \mathscr{A} \to \mathbb{R}$ is defined by $u(z) = f(|z|)$, then $Tu \geqslant u$. If instead, f is a concave decreasing function, then $Tu \leqslant u$.*

Proof. Fix $z \in \mathscr{A}$ and write $d\mu(\zeta) = |k_z(\zeta)|^2 dA(\zeta)$. Then, $\int_{\mathscr{A}} d\mu(\zeta) = 1$. Since f is convex, an application of Jensen's inequality followed by use of the increasing nature of f gives

$$\begin{aligned}
(Tu)(z) &= \int_{\mathscr{A}} u(\zeta)|k_z(\zeta)|^2 dA(\zeta) \\
&= \int_{\mathscr{A}} f(|\zeta|)d\mu(\zeta) \\
&\geqslant f\Big(\int_{\mathscr{A}} |\zeta| d\mu(\zeta)\Big) \\
&\geqslant f\Big(|\int_{\mathscr{A}} \zeta |k_z(\zeta)|^2 dA(\zeta)|\Big) \\
&= f(|z|) = u(z).
\end{aligned}$$

A similar proof can be supplied when f is concave decreasing. □

Remark: One reason why the function $\log r$ presents some difficulties in deciding that it is not a fixed point of the Berezin transform on the annulus, is that it is concave but increasing.

Bibliography

[1] Peter Duren and Alexander Schuster, *Bergman Spaces*, American Mathematical Society, Vol. 100, (1935).

[2] Sheldon Axler, *Berezin transform*, Encyclopedia of Mathematics, Supplement Volume III, Kluwer (2001), 67-68.

[3] Patrick Ahern and Željko Čučković, *A mean value inequality with applications to Bergman space operators*, Pacific Journal of Mathematics, Vol. 173, No. 2, (1996), 295-305.

[4] Patrick Ahern, Manuel Flores, and Walter Rudin, *An invariant Volume-mean-value property*, Journal of Functional Analysis, Vol. 111, No.2, (1993), 380-397.

[5] Miroslav Engliš, *Functions invariant under the Berezin transform*, Journal of Function Analysis, 121 (1994), 233-254.

[6] Željko Čučković, *Commuting Toeplitz Operators on the Bergman Space of an Annulus*, Michigan Math Journal, 43 (1996), 355-365.