AMERICAN UNIVERSITY OF BEIRUT

NON-COMMUTATIVE FOURIER TRANSFORM ON $SU(2)$

by
SLEIMAN MOHAMMAD JRADI

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SLEIMAN MOHAMMAD JRADI

Approved by:

Dr. Tlas, Tamer, Assistant Professor — Advisor
Mathematics

Dr. Shayya, Bassam, Professor — Member of Committee
Mathematics

Dr. Egeileh, Michel, Assistant Professor — Member of Committee
Mathematics

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Contents

AN ABSTRACT OF THE THESIS OF

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Major: Mathematics

Title: Non-Commutative Fourier Transform on SU(2)

We study a modification of the usual Fourier transform where the character of the representation of group $\mathbb{R}$ is replaced with $e^{i.\alpha.Tr(Xg)}$ where $X \in su(2), g \in SU(2)$. In particular we analyze how do Schwartz space and the space of distributions behave under it.

Chapter 1

Introduction

1.1 Path Integral Formulation of Quantum Filed Theory

The path integral formulation is based on the partition function which is the expression of the form,

$$Z = \int \mathcal{D}\phi e^{iS(\phi)}$$

where S is an action and $\mathcal{D}\phi$ is a formal, rigorously undefined, measure on the space of fields (i.e functions from the manifold to some space).

1.1.1 Feynman diagrams

Feynman diagrams are diagrammatic notations for specific expressions which appear in the expansion of the partition function. Such diagrams are used in quantum field theory to give us a simple image of interaction between subatomic particles. For example, the following figure [5] describes the interaction between an electron and a positron. The 2 particles collide to produce a photon γ that break to a quark and anti-quark pair after radiating a gluon g.

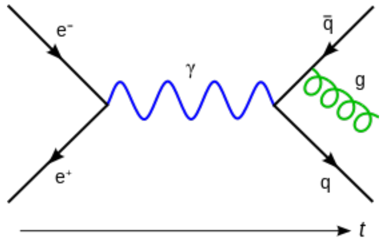

Figure 1.1: Feynman diagram

1.2 Background Field Theory

The BF theory is a topological field theory where the variables of this theory are:

- an $SU(2)$ connection, denoted by$A$.
- an $su(2)$-valued 1-form, denoted B,

where the action is given by

$$S(B, A, M) = \int_M Tr(B \wedge F(A))$$

for a manifold M, where $F(A)$ is a 2-form representing the curvature of A, and the trace is taken in the adjoint representation. The BF theory can be quantized [3] by the path integral formulation:

$$Z(M) = \int \mathcal{D}B\ \mathcal{D}Ae^{iS(B,A,M)}$$

In order to make mathematical sense of the above integral we have to do a discretization by introducing a triangulation of the manifold M, Δ and its topological dual Δ^*, as well as dicretize the fields. We do this by considering the manifold as a collection of tetrahedra glued together. Δ^* isdefined to be the 2-complex where

- every vertex of Δ^* corresponds to a tetrahedron.
- every dual edge corresponds to a face in Δ.
- every dual face corresponds to an edge in Δ.

The discretization of the fields is done in the following way:

- integrate the field B over the edges of our triangulation to get an $su(2)$ element for every edge.
- integrate the field F over the dual faces to get an $su(2)$ element for every dual face of the Δ^*.

The discretized curvature F can be defined through the holonomy around the boundary of the dual face that it is discretized on. That is, if we use the relation $Holonomy = \mathbb{I} + F + ...$, we get $tr(B(\mathbb{I} + F + ...)) = tr(BF)$.[1] Therefore the discretized theory has the following set of variables:

- an $su(2)$ element X_e corresponding to every edge e in the triangulation Δ.
- an $SU(2)$ element g_e^* corresponding to every dual edge e^*.

The product of all the g_{e^*} around a dual face corresponding to the edge e will be denoted by G_e. The discretization above gives the following action:

$$S = \sum_{edges} Tr(X_e G_e)$$

[1]B is self adjoint and so Tr(B)=0

Thus the discretized partition function becomes[2]

$$Z = \prod_e \int dX_e \prod_{e^*} \int dg_e^* \prod_e \frac{(1+\epsilon(G_e))}{2} e^{i(\sum_e Tr(X_e G_e)}$$

The integrals over X_e can be done, and give

$$Z = \prod_{e^*} \int dg_e^* \prod_e \delta(G_e)$$

The above deals with the situation where no particles are present. Let us describe the modification required if there are particles moving in M. Let Γ be a sub-graph of the 1-skeleton Δ where edges of Γ are thought of as world-lines of the particles moving on the edges of the Feynman diagram. We discretize the fields [6] in the following action,

$$S = \int_M Tr(B \wedge F(A)) - \sum_i \int Tr(Bp_i)$$

where $\sum_i \int Tr(Bp_i)$ is taken over over the distinct edges of Γ, to get

$$S = \sum_{edges \notin \Gamma} Tr(X_e G_e) + \sum_{edges \in \Gamma} Tr(X_e G_e {u_e}^{-1} e^{(m_e J_0)} u_e)$$

where ${u_e}^{-1} e^{(m_e J_0)} u_e$ is an $SU(2)$ element corresponding to the $su(2)$ element $m_e u J_0 u^{-1}$ and m_e is the mass of the edge $e \in \Gamma$. Note that there is one u for every edge of Γ.

---

[2]The factor $\frac{(1+\epsilon(G_e))}{2}$ is introduced in order to get a delta function in our integral because otherwise we will get $\delta(G)$ and $\delta(-G)$. See appendix A.

Now the partition function becomes,

$$Z = \prod_e \int dX_e \prod_{e^*} \int dg_e^* \prod_e \frac{(1+\epsilon(G_e))}{2}$$
$$\prod_{e\in\Gamma} \int du_e e^{i(\sum_{e\notin\Gamma} Tr(X_e G_e) + \sum_{e\in\Gamma} Tr(X_e G_e u_e{}^{-1} e^{(m_e J_0)} u_e)}$$

We can integrate over X_e to get

$$Z = \prod_{e^*} \int dg_e^* \prod_{e\in\Gamma} \int du_e \prod_{e\notin\Gamma} \delta(G_e) \prod_{e\in\Gamma} \delta(G_e u_e h_e u_e^{-1})$$

This expression is formally divergent. We consider the following change of variables, $g = \sqrt{1-\kappa^2 \left|\vec{P}\right|^2} + \kappa \vec{P}.\vec{J}$ where P is the non-commutative momenta in the 3-ball $\mathcal{B}_\kappa(\mathbb{R}^3)$ of radius $\dfrac{1}{\kappa} = 4\pi G$, and G is the Newton's constant. Using the gauge fixing procedure [6] is done in lattice gauge theory,[3] the above expression becomes

$$Z = \int \prod_{e\in\Gamma} \frac{\kappa^3 d^3\vec{P}_e}{\pi^2} \prod_{e\in\Gamma} \delta\Big(\left|\vec{P}_e\right|^2 - \frac{\sin^2(m_e\kappa)}{\kappa^2}\Big) \prod_{e\in\Gamma} \delta\Big(\oplus_{e\in\partial v} \vec{P}_e \Big)$$

By refining our triangulation we have assumed that there is one dual edge with non zero group element for each edge of Γ. In other words, the terms $\delta(G_e u_e h_e u_e^{-1})$ and $\delta(G_e)$ combined with Bianchi identity [4] become $\delta(\oplus_{e\in\partial v}\vec{P}_e)$ and $\delta\big(\left|\vec{P}_e\right|^2 - \dfrac{\sin^2(m_e\kappa)}{\kappa^2}\big)$ respectively. This integral then is asymptotic to the amplitude given by the Feynman diagram corresponding to the standard action

[3]The gauge fixing is done by selecting a maximal tree in the dual complex and then using gauge invariance freedom at the vertices of the tree to make all the g's on its edges equal to the identity. see [8] for details.

[4]The Bianchi identity in the discrete corresponds to the fact that the product of G's for all edges which meet at a vertex is equal to the identity.

in scalar field theory as $\kappa \to 0$. So the question now is, does there exist an action S' such that the value of the above integral is equal to $\displaystyle\int \mathcal{D}\phi e^{iS'(\phi)}$ for finite fixed κ?

It can be shown [4] that such an action exists and is given by,

$$S(\phi) = \frac{1}{8\pi\kappa^3} \int \left[\frac{1}{2} \left(\phi \star \left[\frac{\Box - \frac{\sin^2(km)}{\kappa^2}}{\sqrt{1 - \kappa^2 \partial^2}} \right] \phi(x) \right) + \frac{\lambda}{3!} \left(\phi \star \phi \star \phi \right)(x) \right] d^3x$$

where we have defined the Fourier transform from functions defined on $SU(2)$ to functions on $\mathbb{R}^3$ by $f(X) = \displaystyle\int e^{\frac{i}{2\kappa} Tr(Xg)} f(g)\, dg$, where the $\star$ product is the non-commutative product $e^{\frac{i}{2\kappa} Tr(Xg_1)} \star e^{\frac{i}{2\kappa} Tr(Xg_2)} = e^{\frac{i}{2\kappa} Tr(Xg_1 g_2)}$, $\Box$ is the Laplacian on $\mathbb{R}^3$.

Thus it is important to study the Fourier transform on functions defined on $SU(2)$ and the corresponding inverse Fourier transform of functions defined on the Lie algebra $su(2)$ where we utilized the fact that $su(2) \cong \mathbb{R}^3$.

Chapter 2

Schwartz space

2.1 Functions of "Rapid Decrease" on S^1

Let us begin with the case of function on the circle S^1. The set of functions that we will be interested in is the set of smooth functions. We shall denote it by $\mathcal{S}(S^1)$. Note that this is the the collection of smooth (i.e. infinitely-differentiable), complex-valued functions ϕ which clearly satisfy

$$||\phi||_n = \sup_{x \in S^1} |D^n \phi(x)| < \infty \qquad \forall\, n \in \mathbb{N}^1.$$

It would be very convenient if we could replace the norms above with integral-squared ones as then we would be able to employ orthogonality arguments in our analysis. This leads us to define the family of norms

$$||\phi||_{n,2} = \int_{S^1} |D^n \phi|^2.$$

[1] We use here the convention where $\mathbb{N}$ includes the zero.

Now given two collection of seminorms $\{||.||_n\}_{n\in\mathbb{N}}$, $\{||.||_{n,2}\}_{n\in\mathbb{N}}$ we can talk about their equivalence, i.e. whether they define the same topology on the (locally convex) vector space. It can be shown[1] that they do, provided that one can find a collection of constants $\{C_{n,m}\}_{m=0}^{\infty}$, only finitely many of which nonzero, such that for any $\phi \in \mathcal{S}(S^1)$ we have

$$||\phi||_n \leq \sum_{m=0}^{\infty} C_{n,m}||\phi||_{m,2},$$

with a similar inequality going in the other direction.

We now have the following

Lemma 2.1.1. *The two families of norms $\{||.||_n\}_{n\in\mathbb{N}}$ and $\{||.||_{n,2}\}_{n\in\mathbb{N}}$ are equivalent.*

Proof. For one direction we have

$$\begin{aligned}
||\phi||_{n,2} &= ||D^n\phi||_2 \\
&= \Big(\int |D^n\phi(x)|^2\Big)^{1/2} \\
&\leq \Big(\int ||\phi||_n^2\Big)^{1/2} \\
&= 2\pi\, ||\phi||_n
\end{aligned}$$

The other direction is a little more tricky. Let $D^n\phi(x_{0,n}) = min_{x\in S^1} D^n\phi(x)$.

Then

$$
\begin{aligned}
|D^n\phi(x)| &= \left| D^n\phi(x_{0,n}) + \int_{x_{0,n}}^{x} D^{n+1}\phi(x)\,dx \right| \\
&\le |D^n\phi(x_{0,n})| + \left| \int_{x_{0,n}}^{x} D^{n+1}\phi(x)\,dx \right| \\
&\le C_1\,||D^n\phi(x)||_2 + C_2 \left(\int_{x_{0,n}}^{x} \left| D^{n+1}\phi(x) \right|^2 \right)^{1/2} \\
&= C_1\,||\phi(x)||_{n,2} + C_2\,||\phi||_{n+1,2} \\
\implies \quad ||\phi(x)||_n &\le C_1\,||\phi(x)||_{n,2} + C_2\,||\phi||_{n+1,2}
\end{aligned}
$$

Therefore, the two families are equivalent. □

Preposition 2.1.2. *The set* $A_k = \left\{ \dfrac{e^{ikx}}{\sqrt{2\pi}}, k \in \mathbb{N} \right\}$ *is an orthonormal basis for* $L^2(S^1)$*.*

Proof. To prove orthonormality, we will take the inner product between two basis elements. Consider

$$
\begin{aligned}
(e_k, e_m) &= \int_0^{2\pi} \frac{e^{-ikx}.e^{imx}}{2\pi}\,dx \\
&= \int_0^{2\pi} \frac{e^{i(m-k)x}}{2\pi}\,dx \\
&= \begin{cases} 0, & \text{if } m \neq k \\ 1, & \text{if } m = k \end{cases}
\end{aligned}
$$

Therefore, $\{e_k, k \in \mathbb{N}\}$ are orthonormal. To show that $\{e_k\}_{k\in\mathbb{N}}$ is a basis, note that the set of linear combinations of e_k's is an algebra of continuous functions on S^1 that separates points, which vanishes nowhere, so by Stone-Weistrass theorem we get that this set is dense in $C(S^1)$, the set of continuous functions on S^1. So any function $f \in \mathcal{C}(S^1)$ is the uniform limit of some sequence $\{g_k\}_{k\in\mathbb{N}}$ in this algebra. But $\mathcal{C}(S^1)$ is dense in $L^2(S^1)$, this implies that for any $f \in L^2(S^1)$, $f = \lim_{n\to\infty} \sum_{k=0}^{n} a_{k,n}e_k$, but we want to find some coefficients a_k's that are related to f and independent of n. We claim that such a_k's are the Fourier coefficients of f. To see this, let $a_k =< f, e_k >$ and notice that

$$\begin{aligned}
\left\|f - \sum_{k=0}^{n} a_{k,n}e_k\right\|^2 &= \left\|f - \sum_{k=0}^{n} a_{k,n}e_k + \sum_{k=0}^{n} a_k e_k - \sum_{k=0}^{n} a_k e_k\right\|^2 \\
&= \left\|f - \sum_{k=0}^{n} a_k e_k + \sum_{k=0}^{n} (a_k - a_{k,n})e_k\right\|^2 \\
&= \left\|f - \sum_{k=0}^{n} a_k e_k\right\|^2 + \left\|\sum_{k=0}^{n} (a_k - a_{k,n})e_k\right\|^2 \\
&+ 2Re\Big(< f - \sum_{k=0}^{n} a_k e_k, \sum_{l=0}^{n} (a_l - a_{l,n})e_l > \Big)
\end{aligned}$$

But we have,

$$\begin{aligned}
< f - \sum_{k=0}^{n} a_k e_k, \sum_{l=0}^{n} (a_l - a_{l,n})e_l > &=< f, \sum_{l=0}^{n} (a_l - a_{l,n})e_l > \\
&- < \sum_{k=0}^{n} a_k e_k, \sum_{l=0}^{n} (a_l - a_{l,n})e_l > \\
&= \sum_{l=0}^{n} (a_l - a_{l,n})a_l - \sum_{k=0}^{n} a_k(a_k - a_{k,n}) \\
&= 0
\end{aligned}$$

Thus,

$$\left|\left|f - \sum_{k=0}^{n} a_k e_k\right|\right| \leq \left|\left|f - \sum_{k=0}^{n} a_{k,n} e_k\right|\right|$$

Any function $f \in L^2(S^1)$ can be written as, $f = \sum_{k=o}^{\infty} a_k e_k$ where $a_k =< f, e_k >$, and the sum converges in the L^2 sense. Note that this is the Fourier series of a continuously differentiable function on the circle which means that the series is absolutely convergent. Note that the derivatives of f can also be represented by their Fourier series for the same reasoning above. Therefore using integration by parts we get $D^m f = \sum_{k=o}^{\infty} a_k (ik)^m e_k$, and this sum should converge because $f \in \mathcal{S}(S^1)$. Therefore we get that $\sum_{k=o}^{\infty} a_k^2 k^{2m} < \infty$. In particular we have that, $\sup_k |a_k k^m| < \infty$. $\square$

Now we will prove that the map that takes the function f to a sequence of its Fourier coefficients is an ismorphism.

Theorem 2.1.3. *Let s_k be the set of sequences $\{a_k\}_{k\in\mathbb{N}}$ with the property $\sup_{k\in\mathbb{N}} |a_k|\,|k^m| < \infty$, for each m. We topologize s_k with $||\{a_k\}_{k\in\mathbb{N}}||_m = \sqrt{\sum_{k=0}^{\infty} k^{2m} |a_k|^2}$. Then the map $\phi : \mathcal{S}(S^1) \to s_k$ given by $\phi(f) = \{a_k\}$ is an isomorphism.*

Proof. To see that the map ϕ is continuous, Note that

$$\begin{aligned}
||\phi(f)||_m &= ||\{a_k\}||_m \\
&= \sqrt{\sum_{k=0}^{\infty} k^{2m} |a_k|^2} \\
&= ||D^m f||_2
\end{aligned}$$

Let $f_1, f_2 \in \mathcal{S}(S^1)$ and $c \in \mathbb{R}$ such that $f_1 = \sum_{k=0}^{\infty} a_k e_k$ and $f_2 = \sum_{k=0}^{\infty} b_k e_k$.We start by showing that ϕ is linear,

$$\begin{aligned}
\phi(cf_1 + f_2) &= \phi(c\sum_{k=0}^{\infty} a_k e_k + \sum_{k=0}^{\infty} b_k e_k) \\
&= \phi(\sum_{n=0}^{\infty} (ca_k + b_k e_k) \\
&= \{ca_k + b_k\} \\
&= c\{a_k\} + \{b_k\} \\
&= c\phi(f_1) + \phi(f_2)
\end{aligned}$$

To see that ϕ is one-to-one, note that

$$\begin{aligned}
&\phi(f_1) = \phi(f_2) \\
\implies & a_k = b_k \\
\implies & a_k = b_k \qquad \forall k \in \mathbb{N} \\
\implies & f_1 = \sum_{k=0}^{\infty} a_k e_k \\
& \quad = \sum_{k=0}^{\infty} b_k e_k \\
& \quad = f_2
\end{aligned}$$

Hence, ϕ is one-to-one.To prove that ϕ is onto, consider a sequence $\{a_k\} \in s_k$. We claim that the function $g = \sum_{n=0}^{\infty} a_k e_k$ is in $\mathcal{S}(S^1)$. Note that this sum converges in the L^2 sense and therefore we can exchange integration and summation when integrating such sums. For this, consider

$$\begin{aligned}
||g||_{n,2} &= \left(\int (D^n g)^2 \right)^{\frac{1}{2}} \\
&= \left(\int (\sum a_k D^n e_k)^2 \right)^{\frac{1}{2}} \\
&= \left(\int (\sum a_k (ik)^n e_k)^2 \right)^{\frac{1}{2}} \\
&= \left(\sum (\int (a_k^2 (ik)^{2n} e_k^2) \right)^{\frac{1}{2}} \\
&= \left(\sum a_k^2 (ik)^{2n} \right)^{\frac{1}{2}}
\end{aligned}$$

But since $\sup_{k \in \mathbb{N}} |a_k| \, |k|^m < \infty$, then the series $\left(\sum a_k^2 (ik)^{2n} \right)^{\frac{1}{2}}$ converges.Therefore g is in the Schwatrz space over S^1.

$\square$

As a consequence of this theorem, every function in the $\mathcal{S}(S^1)$ space can be represented as a rapidly decreasing sequence of real numbers. We will show that a similar result applies for distributions on S^1. A "Schwartz" distribution on S^1 is a linear continuous functional that takes smooth functions on S^1 to $\mathbb{R}$. We will denote the set of "Schwartz" distributions by $\mathcal{S}'(S^1)$.

Theorem 2.1.4. *Consider the distribution $T \in \mathcal{S}'(S^1)$.Let $b_k = T(e_k) \forall k \in \mathbb{N}$. Then for some $m \in \mathbb{N}, |b_k| \leq Ck^m$. Conversely if $|b_k| \leq Ck^m$ for some m and for all k there is a unique $T \in \mathcal{S}'$ with $T(e_k) = b_k$.*

Proof. Since $T \in \mathcal{S}'(S^1)$, we have $|T(e_k)| \leq \sum_{j=0}^{L} C_j \, ||e_k||_{m_j}$. But $||e_k||_{m_j} = k^{m_j} \, ||e_k|| = k^{m_j}$. Let $m = \max_j(m_j)$, then $|T(e_k)| = |b_k| \leq Ck^m$ for some constant C.

Now suppose $|b_k| \leq Ck^m$for all k. For any sequence $\{a_k\}$ satisfying $\sup_{k\in\mathbb{N}} |a_k| \, |k|^m < \infty$, define $B(\{a_k\}) = \sum_{k=0}^{\infty} b_k a_k$. Then

$$
\begin{aligned}
|B(\{a_k\})| &\leq \sum_{k=0}^{\infty} |b_k| \, |a_k| \\
&\leq C \sum_{k=0}^{\infty} k^m \, |a_k| \\
&\leq C \left(\sum_{k=0}^{\infty} k^{2m+2} \, |a_k|^2 \right)^{\frac{1}{2}} \left(\sum_{k=0}^{\infty} \frac{1}{k^2} \right)^{\frac{1}{2}} \\
&\leq C \frac{\pi^2}{6} \, ||a_k||_{m+1}
\end{aligned}
$$

Thus B defines a linear continuous functional on sequences $\{a_k\}$ that satisfies the above property. We have proved that any Schwartz function on S^1 can be represented with such sequences. Therefore, there is $T \in \mathcal{S}'$ with $T(\sum_{k=0}^{\infty} a_k e_k) = \sum_{k=0}^{\infty} a_k b_k$ where $T(e_k) = b_k$. To see that T is unique, let T_1 be another distribution such that $T_1(e_k) = b_k$. Note that T and T_1 are equal when acting on finite sums like $\sum_{k=0}^{N} c_k e_k$, but any function $f \in \mathcal{S}(S^1)$ can be written as $f = \sum_{k=0}^{\infty} a_k e_k$ where this sum converges uniformly to f. Then $T = T_1$, and thus T is unique. $\square$

2.2 Schwartz functions on $\mathbb{R}^3$

Now we will talk about functions of rapid decrease on $\mathbb{R}^3$. The set of such functions is denoted by $\mathcal{S}(\mathbb{R}^3)$.

Definition 2.2.1. *A complex-valued smooth function f is said to be a Schwartz function on $\mathbb{R}^3$ if it satisfies*

$$||\phi||_{m,n} = sup_{x\in\mathbb{R}^3}|x^m D^n \phi(x)| < \infty \qquad \forall\, m, n \in \mathbb{N}$$

Hence, the Schwartz functions are those functions that decrease along with their derivatives faster than the inverse of any polynomial. It can be shown [1] that the similar results as above apply for the the Schwartz space $\mathcal{S}(\mathbb{R}^3)$ and its dual space, i.e both the Schwartz space and its dual can be represented by rapidly decreasing sequences. Our concern now is to prove that $\mathcal{S}(\mathbb{R}^3)$ maps to itself under the Fourier transform.

Definition 2.2.2. *Let $f \in \mathcal{S}(\mathbb{R}^3)$. The Fourier transform of f is the function $\mathcal{F}(f)$ and is given by*

$$\mathcal{F}(f) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} e^{-i\xi.x} f(x)\, dx$$

where $\xi.x = \xi_1 x_1 + \xi_2 x_2 + \xi_3 x_3$.

Note that

$$
\begin{aligned}
|\mathcal{F}(f)| &= \frac{1}{(2\pi)^{\frac{3}{2}}} \left| \int_{\mathbb{R}^3} e^{-i\xi.x} f(x)\, dx \right| \\
&\leq \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} \left| e^{-i\xi.x} f(x) \right| \, dx \\
&= \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} |f(x)| \, dx \\
&< \infty
\end{aligned}
$$

Thus the above integral makes sense since every Schwartz function f is in $L^1(\mathbb{R}^3)$. This will allow us to define the inverse Fourier transform of a function $f \in \mathcal{S}(\mathbb{R}^3)$ by

$$
\mathcal{F}^{-1}(f) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} e^{i\xi.x} f(x)\, dx
$$

Note that the above definition makes sense because,

$$
\begin{aligned}
\left|\mathcal{F}^{-1}(f)\right| &\leq \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} |f(x)| \frac{1+x^2}{1+x^2} \, dx \\
&\leq \sup_{x\in\mathbb{R}^3} (1+x^2)\, |f(x)| \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} \frac{1}{1+x^2} \, dx \\
&\leq \infty
\end{aligned}
$$

Our goal now is to prove that $\mathcal{F}$ and $\mathcal{F}^{-1}$ are the inverses of each other.

Lemma 2.2.3. *The map $\mathcal{F}$ is a linear continuous function from $\mathcal{S}(\mathbb{R}^3)$ to $\mathcal{S}(\mathbb{R}^3)$.*

Proof. The fact that $\mathcal{F}$ is linear follows from the linearity of the integral. to prove that $\mathcal{F}(f) \in \mathcal{S}(\mathbb{R}^3)$, note that

$$\begin{aligned}
\xi^m D_\xi^n \mathcal{F}(f) &= \xi^m D^n \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} e^{-i\xi.x} f(x)\, dx \\
&= \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} \xi^m D_\xi^n e^{-i\xi.x} f(x)\, dx \\
&= \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} \xi^m (-ix)^n e^{-i\xi.x} f(x)\, dx \\
&= \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} \frac{(-ix)^n}{(-i)^m} (-i\xi)^m e^{-i\xi.x} f(x)\, dx \\
&= \frac{(-i)^m}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} (D_x^m e^{-i\xi.x})(-ix)^n f(x)\, dx
\end{aligned}$$

Using integration by parts we get,

$$\xi^m D_\xi^n \mathcal{F}(f) = \frac{(-i)^m}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} e^{-i\xi.x} D_x^m((-ix)^n f(x))\, dx$$

Therefore, we have

$$\begin{aligned}
||\mathcal{F}(f)||_{m,n} &= \sup_x \left| \frac{(-i)^m}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} e^{-i\xi.x} D_x^m((-ix)^n f(x))\, dx \right| \\
&\leq \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} |D_x^m((-ix)^n f(x))|\, dx \\
&< \infty
\end{aligned}$$

Note that we have exchanged differentiation and integration because we have,

$$
\begin{aligned}
\frac{\partial}{\partial \xi_j} \int_{\mathbb{R}^3} e^{-i\xi.x} f(x)\,dx &= \lim_{h\to 0} \frac{\int_{\mathbb{R}^3} e^{-i(\xi+h).x} f(x)\,dx - \int_{\mathbb{R}^3} e^{-i\xi.x} f(x)\,dx}{h} \\
&= \lim_{h\to 0} \int_{\mathbb{R}^3} f(x) \frac{(e^{-i(\xi+h).x} - e^{-i\xi.x})\,dx}{h} \\
&\leq \lim_{h\to 0} \int_{\mathbb{R}^3} |xf(x)|\;dx
\end{aligned}
$$

In the last step we used the mean value theorem, and we have that $f(x)\dfrac{(e^{-i(\xi+h).x} - e^{-i\xi.x})}{h}$ is dominated by $|xf(x)|$ which is an integrable function. Therefore we can bring the limit inside the integral by the dominated convergence theorem, and thus we can interchange differentiation and integration.

Hence, $\mathcal{F}$ takes $\mathcal{S}(\mathbb{R}^3)$ to $\mathcal{S}(\mathbb{R}^3)$. Moreover, we have that

$$
\begin{aligned}
||\mathcal{F}(f)||_{m,n} &\leq \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} \frac{(1+x^2)^k}{(1+x^2)^k} \left|D_x^m((-ix)^n f(x))\right|\;dx \\
&\leq \frac{1}{(2\pi)^{\frac{3}{2}}} \sup_x\,(1+x)^k \left|D_x^m((-ix)^n f(x))\right| \int_{\mathbb{R}^3} \frac{1}{(1+x^2)^k}\;dx
\end{aligned}
$$

Choosing k large enough so that $\displaystyle\int_{\mathbb{R}^3} \frac{1}{(1+x^2)^k}\;dx < \infty$, we get that the function $\mathcal{F}$ is continuous since

$$
||\mathcal{F}(f)||_{m,n} \leq \sum_{j=1}^{L} C_j\,||f||_{m_j,n_j}
$$

where C_j, m_j, and n_j are some constants.

$\square$

Theorem 2.2.4. *The function $\mathcal{F}(f)$ is a bijection from $\mathcal{S}(\mathbb{R}^3)$ to $\mathcal{S}(\mathbb{R}^3)$.*

Proof. Any Schwartz function on $\mathbb{R}^3$ can be represented [1] as,

$$f = \sum_{k=0}^{\infty} a_k \phi_k$$

where a_k's are the Hermite coefficients, ϕ_k's are Hermite functions, and the sum converges in the L^2 sense. The Hermite functions are given by

$$\phi_k = (-1)^n e^{\frac{1}{2}x^2} \left(\frac{d}{dx}\right)^n e^{-x^2}$$

where $\phi_k = \left(x - \dfrac{d}{dx}\right)\phi_{k-1}$. Now we will show that the Fourier transform of a Hermite function is another Hermite function. We claim that $\mathcal{F}(\phi_k) = (i)^k \phi_k$. For this, notice that

$$\begin{aligned}
\mathcal{F}(\phi_0) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi x} e^{-\frac{1}{2}x^2}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{(x-i\xi x)^2}{2} - \frac{\xi^2}{2}}\, dx \\
&= e^{-\frac{\xi^2}{2}}
\end{aligned}$$

Assume that $\mathcal{F}(\phi_k) = (i)^k \phi_k$, and consider

$$\begin{aligned}
\mathcal{F}(\phi_{k+1}) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi x} \left(x - \frac{d}{dx}\right) \phi_k\, dx \\
&= \frac{1}{\sqrt{2\pi}} \left(\int_{\mathbb{R}} e^{-i\xi x} x \phi_k\, dx - \int_{\mathbb{R}} e^{-i\xi x} \phi_k'\, dx \right) \\
&= \frac{1}{\sqrt{2\pi}} \left(\int_{\mathbb{R}} -(-i) \frac{d}{d\xi} e^{-i\xi x} \phi_k\, dx - \left[e^{-i\xi x} \phi_k \right]_{-\infty}^{\infty} - \int_{\mathbb{R}} i\xi e^{-i\xi x} \phi_k\, dx \right) \\
&= -i(\xi - \frac{d}{d\xi}) \mathcal{F}(\phi_k) \\
&= -i^{k+1} \phi_{k+1}
\end{aligned}$$

Here we have computed the integral over $\mathbb{R}$, but the same result follows over $\mathbb{R}^3$ because the integral can be split into 3 integrals, each over $\mathbb{R}$. To show smoothness, Consider

$$\begin{aligned}
\frac{\partial}{\partial \xi_j}\mathcal{F}(f) &= \frac{1}{(2\pi)^{\frac{3}{2}}}\frac{\partial}{\partial \xi_j}\int_{\mathbb{R}^3} e^{-i\xi.x} f(x)\, dx \\
&= \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathbb{R}^3} \frac{\partial}{\partial \xi_j} e^{-i\xi.x} f(x)\, dx \\
&= \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathbb{R}^3} (-ix_j) e^{-i\xi.x} f(x)\, dx \\
&= \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathbb{R}^3} e^{-i\xi.x}((-ix_j) f(x))\, dx \\
&= \mathcal{F}(-ix_j f)
\end{aligned}$$

But $x_j f$ is a Schwartz function, hence $\mathcal{F}(f)$ is a bijection. $\square$

Similar results follow for the map $\mathcal{F}^{-1}$. Therefore, $\mathcal{S}(\mathbb{R}^3)$ maps onto under the Fourier transform.

Chapter 3

The Lie Group $SU(2)$

3.1 $SU(2)$ and $su(2)$

Definition 3.1.1. *$SU(2)$ is the group of all 2x2 complex unitary matrices of determinant equal to one.*

$$SU(2) = \{A \in GL(2,\mathbb{C}) | det(A) = 1, AA^* = A^*A = \mathbb{I}\}$$

where A^* is the complex conjugate of the transpose matrix of A. i.e

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^* = \begin{bmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{bmatrix}$$

Definition 3.1.2. *The Lie algebra of $SU(2)$ is the vector space tangent to $SU(2)$ at the identity. It is defined [2] by*

$$su(2) = \{X \in GL(2,\mathbb{C}) | e^{tX} \in SU(2)\}$$

So we have

$$e^{tX}.(e^{tX})^* = e^{t(X+X^*)} = \mathbb{I} \qquad \forall t \in \mathbb{R}$$

$$\implies X = -X^*$$

Also we have that

$$det(e^{tX}) = 1$$

$$\implies e^{tr(X)} = 1$$

$$\implies tr(X) = 0$$

Therefore $su(2)$ is the set of all traceless 2×2 skew-Hermitian complex matrices:

$$su(2) = \{X \in GL(2,\mathbb{C}) | X = -X^*, tr(X) = 0\}$$

The following matrices

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

are called the Pauli matrices where $\{i\sigma_j, j = 1,2,3\}$ form a basis [2] for the Lie algebra $su(2)$. The Lie algebra structure is given via the commutator

$$[X,Y] = XY - YX$$

Therefore, any vector in $su(2)$ can be written as

$X = i\vec{a}.\vec{\sigma} = ia_1\sigma_1 + ia_2\sigma_2 + ia_3\sigma_3$ where $\vec{a} = (a_1, a_2, a_3) \in \mathbb{R}^3$.

Theorem 3.1.3. *$SU(2)$ is homeomorphic to the 3-sphere S^3.*

Proof. Let $X = a\vec{n}.\vec{\sigma} \in su(2)$, and $J_k = i\sigma_k$ where $\vec{n}$ is the normal vector of $\vec{a} = (a_1, a_2, a_3)$ and a its magnitude.First note that $(\vec{n}.\vec{\sigma})^n = \begin{cases} I & \text{if } n \text{ is even} \\ \vec{n}.\vec{\sigma} & \text{if } n \text{ is odd} \end{cases}$

Now consider

$$\begin{aligned}
e^{a\vec{n}.\vec{J}} &= \sum_n \frac{(a\vec{n}.\vec{J})^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{(a\vec{n}.\vec{J})^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(a\vec{n}.\vec{J})^{2n+1}}{(2n+1)!} \\
&= \mathbb{I} \sum_{n=0}^{\infty} \frac{(-1)^n (a)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(-1)^n (a)^{2n+1}}{(2n+1)!} \\
&= \mathbb{I} \cos a + \vec{n}.\vec{J} \sin a \\
&= \mathbb{I} \cos a + i\vec{n}.\vec{\sigma} \sin a
\end{aligned}$$

Thus

$$x_0 = \cos a \qquad x_1 = n_1 \sin a \qquad x_2 = n_2 \sin a \qquad x_3 = n_3 \sin a$$

and

$$x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1$$

Therefore, we have that $SU(2)$ is homeomorphic to the 3-sphere S^3. $\square$

Here we used the fact that the exponential function is surjective [2] so that every element in $SU(2)$ can be written as e^{tX} where $X \in su(2)$ and $t \in \mathbb{R}$. Here is a direct proof for the theorem above not using this fact.

Proof. Any 2×2 complex matrix can be written as
$A = c_0\mathbb{I} + ic_1\sigma_1 + ic_2\sigma_2 + ic_3\sigma_3$, where $c_0, c_1, c_2,$ and $c_3 \in \mathbb{C}$. This can be seen easily since

$$\begin{aligned} c_0\mathbb{I} + ic_1\sigma_1 + ic_2\sigma_2 + ic_3\sigma_3 &= c_0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + ic_1 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + ic_2 \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \\ &+ ic_3 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} c_0 + ic_3 & ic_1 + c_2 \\ ic_1 - c_2 & c_0 - ic_3 \end{bmatrix} \end{aligned}$$

So if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is any 2×2 complex matrix, then we have

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c_0 + ic_3 & ic_1 + c_2 \\ ic_1 - c_2 & c_0 - c_3 \end{bmatrix}$$
$$\implies \begin{cases} a = c_0 + ic_3 \\ b = ic_1 + c_2 \\ c = ic_1 - c_2 \\ d = c_0 - ic_3 \end{cases} \implies \begin{cases} c_0 = \dfrac{a+d}{2} \\ c_1 = \dfrac{b+c}{2i} \\ c_2 = \dfrac{b-c}{2} \\ c_3 = \dfrac{a-d}{2i} \end{cases}$$

Now, let $A \in SU(2)$, then we have $A = c_0\mathbb{I} + ic_1\sigma_1 + ic_2\sigma_2 + ic_3\sigma_3$, $det(A) = 1$,

and $AA^* = A^*A = \mathbb{I}$. Note that if
$A = c_0\mathbb{I} + ic_1\sigma_1 + ic_2\sigma_2 + ic_3\sigma_3 = \begin{bmatrix} c_0 + ic_3 & ic_1 + c_2 \\ ic_1 - c_2 & c_0 - ic_3 \end{bmatrix}$, then
$A^* = \begin{bmatrix} \bar{c_0} - i\bar{c_3} & -i\bar{c_1} - \bar{c_2} \\ -i\bar{c_1} + \bar{c_2} & \bar{c_0} + i\bar{c_3} \end{bmatrix} = \bar{c_0}\mathbb{I} - i\bar{c_1}\sigma_1 - i\bar{c_2}\sigma_2 - i\bar{c_3}\sigma_3$. Write $A = c_0\mathbb{I} + i\vec{c}.\vec{\sigma}$,
and $A^* = \bar{c_0}\mathbb{I} - i\bar{\vec{a}}.\sigma$, where $\vec{c} = (c_1, c_2, c_3)$ and $\bar{\vec{c}} = (\bar{c_1}, \bar{c_2}, \bar{c_3})$. Note first that

$$\begin{aligned}
(\vec{a}.\vec{\sigma})(\vec{b}.\vec{\sigma}) &= (a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3)(b_1\sigma_1 + b_2\sigma_2 + b_3\sigma_3) \\
&= a_1b_1\mathbb{I} + ia_1b_2\sigma_3 - ia_1b_3\sigma_2 - ia_2b_1\sigma_3 + a_2b_2\mathbb{I} + ia_2b_3\sigma_1 \\
&+ ia_3b_1\sigma_2 - ia_3b_2\sigma_1 + a_3b_3\mathbb{I} \\
&= (\vec{a}.\vec{b})\mathbb{I} + i((a_2b_3 - a_3b_2)\sigma_1 - (a_1b_3 - a_3b_1)\sigma_2 + (a_1b_2 - a_2b_1)\sigma_3) \\
&= (\vec{a}.\vec{b})\mathbb{I} + i(a \times b).\vec{\sigma}
\end{aligned}$$

Then

$$AA^* = (c_0\mathbb{I} + i\vec{c}.\vec{\sigma})(\bar{c_0}\mathbb{I} - i\bar{\vec{a}}.\sigma) = (||c_0||^2 + ||c||^2)\mathbb{I} + i(\bar{c_0}\vec{c} - c_0\bar{\vec{c}} + (\vec{c} \times \bar{\vec{c}})).\sigma = \mathbb{I}$$

Therefore, $||c_0||^2 + \vec{c}.\bar{\vec{c}} = 1$, and $\bar{c_0}\vec{c} - c_0\bar{\vec{c}} + (\vec{c} \times \bar{\vec{c}}) = 0$. But we know that $\vec{c} \times \bar{\vec{c}} + \bar{c_0}\vec{c} - c_0\bar{\vec{c}} = 0$ if and only if $\vec{c} \times \bar{\vec{c}} = 0$ since $\vec{c} \times \bar{\vec{c}}$ is orthogonal to both $\vec{c}$ and $\vec{e}$. Thus, $\vec{c}$ and $\bar{\vec{c}}$ are parallel and we have $\vec{c} = e^{i\theta}\bar{\vec{c}}$ for some $\theta \in \mathbb{R}$.Then we will have $e^{-i\frac{\theta}{2}}\vec{c} = \vec{x}$ is real. Also we have

$$\bar{c_0}\vec{c} - c_0\bar{\vec{c}} = 0 \implies \bar{c_0}e^{i\theta}\bar{\vec{c}} - c_0\bar{\vec{c}} = 0 \implies c_0 = e^{i\theta}\bar{c_0} \implies e^{-i\frac{\theta}{2}}c_0 = x_0 \text{ is real .}$$

and

$$||c_0||^2 + ||\vec{c}||^2 = 1 \implies x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1.$$

Hence, $A = c_0\mathbb{I} + i\vec{c}.\vec{\sigma} = e^{i\frac{\theta}{2}}(x_0 + i\vec{x}.\sigma)$. But $det(A) = 1 \implies det$

$$e^{i\frac{\theta}{2}} \begin{bmatrix} x_0 + ix_3 & ix_1 + x_2 \\ ix_1 - x_2 & x_0 - ix_3 \end{bmatrix} = e^{i\theta}(x_0^2 + x_1^2 + x_2^2 + x_3^2) = e^{i\theta} = 1 \implies \theta = 2k\pi, k \in \mathbb{Z}.$$

Therefore,

$$SU(2) = \{x_0\mathbb{I} + i\vec{x}.\vec{\sigma} |\ (x_0, x_1.x_2.x_3) \in \mathbb{R}^4, x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1\}$$

$\square$

3.2 Haar Measure on $SU(2)$

Haar measure is used to define integrals of functions defined on locally compact groups. A function f defined on a group G is said to be compactly supported if it vanishes outside a compact set. Denote by $\mathfrak{F}_0(G)$ the space of all continuous compactly supported functions on the group G. For every $f \in \mathfrak{F}_0(G)$ we assign the integral $\int_G f(g)\,dg$. We say that this integration is left invariant if the following holds

$$\int_G f(g)\,dg = \int_G f(a^{-1}g)\,dg$$

for all $f \in \mathfrak{F}_0(G)$ and all $a \in G$. Every compact group has a unique invariant measure and the integral $\int_G 1\,dg$ is finite. $SU(2)$ is a compact Lie group and therefore [2] it has a unique invariant measure. So our job is to find this integral so we can integrate functions defined on the Lie group $SU(2)$.

Lemma 3.2.1. *The Haar measure on $SU(2)$ is $dg = \frac{1}{2\pi}\sin^2\theta \sin\psi d\theta \wedge d\psi \wedge d\phi$.*

This expression is an invariant 3-form, the measure is obtained in the usual way from this volume form.

Proof. We have seen that every element $g \in SU(2)$ can written as,

$$g = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}$$

where $||a||^2 + ||b||^2 = 1$. We claim that $g^{-1}dg$ is left invariant where dg is 2×2 matrix of differential forms of $SU(2)$, i.e

$$dg = \begin{bmatrix} da & db \\ -d\bar{b} & d\bar{a} \end{bmatrix}$$

Here we say that the matrix $g^{-1}dg$ is left invariant if each of its entries is left invariant. Consider the function $f(g) = g^{-1}dg$, and let h be a constant element of $SU(2)$. Note that

$$\begin{aligned} f(h^{-1}g) &= (h^{-1}g)^{-1}d(h^{-1}g) \\ &= g^{-1}hh^{-1}dg \\ &= g^{-1}dg \\ &= f(g) \end{aligned}$$

So,

$$g^{-1}dg = \begin{bmatrix} \bar{a} & -b \\ \bar{b} & a \end{bmatrix} \begin{bmatrix} da & db \\ -d\bar{b} & d\bar{a} \end{bmatrix}$$
$$= \begin{bmatrix} \bar{a}da + bd\bar{b} & \bar{a}db - bd\bar{a} \\ \bar{b}da - ad\bar{b} & \bar{b}db + ad\bar{a} \end{bmatrix}$$

Now let us find the wedge product of the second entry, third entry, and the fourth one

$$\begin{aligned}(\bar{a}db - bd\bar{a}) \wedge (\bar{b}db + ad\bar{a}) \wedge (\bar{b}da - ad\bar{b}) &= (a\bar{a} + b\bar{b})db \wedge d\bar{a} \wedge (\bar{b}da - ad\bar{b}) \\ &= db \wedge d\bar{a} \wedge (\bar{b}da - ad\bar{b})\end{aligned}$$

Note that $a\bar{a} + \bar{b}b = 1$. So, if we differentiate this equation we get

$$\begin{aligned} & ad\bar{a} + \bar{a}da + bd\bar{b} + \bar{b}db = 0 \\ \implies & d\bar{b} = -\frac{1}{b}(ad\bar{a} + \bar{a}da + \bar{b}db)\end{aligned}$$

Putting $d\bar{b}$ back in the wedge product above, we get that

$$\begin{aligned}(ad\bar{a} + bd\bar{b}) \wedge (-adb + bda) \wedge (\bar{a}d\bar{b} - \bar{b}d\bar{a}) &= db \wedge d\bar{a} \wedge (\bar{b}da - ad\bar{b}) \\ &= db \wedge d\bar{a} \wedge (\bar{b}da + \frac{a}{b}(ad\bar{a} + \bar{a}da + \bar{b}db) \\ &= \frac{b\bar{b} + a\bar{a}}{b} db \wedge d\bar{a} \wedge da\end{aligned}$$

is the left invariant form that we are trying to find. Let us compute this wedge

product in spherical coordinates. We have proved above that
$g = \begin{bmatrix} x_0 + ix_3 & ix_1 + x_2 \\ ix_1 - x_2 & x_0 - ix_3 \end{bmatrix}$, where

$$x_0 = \cos\theta$$

$$x_1 = \sin\theta\cos\psi$$

$$x_2 = \sin\theta\sin\psi\cos\phi$$

$$x_3 = \sin\theta\sin\psi\sin\phi$$

where $0 \le \theta \le \pi, 0 \le \psi \le \pi$, and $0 \le \phi \le 2\pi$. So, we get

$$a = \cos\theta + i\sin\theta\sin\psi\sin\phi \qquad b = \sin\theta\sin\psi\cos\phi + i\sin\theta\cos\psi$$

Then,

$$\begin{aligned} db\wedge d\bar{a}\wedge da &= db\wedge d(\cos\theta - i\sin\theta\sin\psi\sin\phi)\wedge d(\cos\theta + i\sin\theta\sin\psi\sin\phi) \\ &= db\wedge(-2id(\cos\theta)\wedge d(\sin\theta\sin\psi\sin\phi)) \\ &= db\wedge(-2i((-\sin^2\theta\cos\psi\sin\phi)d\theta\wedge d\psi - (\sin^2\theta\sin\psi\cos\phi)d\theta\wedge d\phi)) \\ &= 2i\sin^2\theta d(\sin\theta\sin\psi\cos\phi + i\sin\theta\cos\psi) \\ &\wedge(\cos\psi\sin\phi d\theta\wedge d\psi + \sin\psi\cos\phi d\theta\wedge d\phi) \\ &= 2i\sin^2\theta((\sin\theta\sin\psi\cos\psi\sin^2\phi)d\phi\wedge d\theta\wedge d\psi \\ &+ (\sin\theta\cos\psi\sin\psi\cos^2\phi)d\psi\wedge d\theta\wedge d\phi + (-i\sin\theta\sin^2\psi\cos\phi)d\psi d\theta d\phi) \\ &= 2i\sin^2\theta\sin\psi(\sin\theta\cos\phi(\sin^2\phi + \cos^2\phi) - i\sin\theta\sin\psi\cos\phi)d\phi\wedge d\theta\wedge d\psi \\ &= 2i\sin^2\theta\sin\psi(ib)d\phi\wedge d\theta\wedge d\psi \end{aligned}$$

Hence,

$$\frac{1}{b} db \wedge d\bar{a} \wedge da = 2\sin^2\theta \sin\psi d\phi \wedge d\theta \wedge d\psi$$
$$= 2\sin^2\theta \sin\psi d\theta \wedge d\psi \wedge d\phi$$

We multiply the measure by $\dfrac{1}{4\pi^2}$ so we can have $\displaystyle\int_G 1\, dg = 1$. Therefore the Haar measure on $SU(2)$ is given by $dg = \dfrac{1}{2\pi}\sin^2\theta \sin\psi d\theta \wedge d\psi \wedge d\phi$. □

Note here that if the function f depends only on θ, then we have

$$\begin{aligned}\int_G f(g)\, dg &= \int f(\theta)\frac{1}{2\pi}\sin^2\theta \sin\psi d\theta d\psi d\phi \\ &= 2(2\pi)\frac{1}{2\pi^2}\int f(\theta)\sin^2\theta d\theta \\ &= \frac{2}{\pi}\int f(\theta)\sin^2\theta d\theta\end{aligned}$$

We can find the Haar measure on $SU(2)$ in a simpler way. Since every element in $SU(2)$ can be written as

$$g = \begin{bmatrix} x_0 + ix_3 & ix_1 + x_2 \\ ix_1 - x_2 & x_0 - ix_3 \end{bmatrix}$$

Then we will have

$$\begin{aligned}\int_G f(g)\, dg &= \frac{1}{2\pi^2}\int f(x_0, x_1, x_2, x_3)\frac{\partial(x_0, x_1, x_2, x_3)}{\partial(r, \theta.\psi, \phi)}\, dx \\ &= \frac{1}{2\pi^2}\int f(x_0, x_1, x_2, x_3)\sin^2\theta \sin\psi\, d\theta d\psi d\phi\end{aligned}$$

Here we also multiplied the integral on the right by $\dfrac{1}{2\pi^2}$ in order to normalize the measure.

Chapter 4

Non-Commutative Fourier Transform

We have seen that $X \in su(2)$ can be written as $X = i\vec{a}.\vec{\sigma}$ and $g \in SU(2)$ can be written as $g = \cos\theta\mathbb{I} + i\vec{n}.\vec{\sigma}\sin\theta$. We will calculate the trace of Xg which is the character of our transform.

$$
\begin{aligned}
Xg &= (i\vec{a}.\vec{\sigma})(\cos\theta\mathbb{I} + i\vec{n}.\vec{\sigma}\sin\theta) \\
&= \cos\theta(i\vec{a}.\vec{\sigma})\mathbb{I} + i(i\vec{a}.\vec{\sigma})(\vec{n}.\vec{\sigma}) \\
&= \cos\theta(i\vec{a}.\vec{\sigma})\mathbb{I} + i[(i\vec{a}.\vec{n})\mathbb{I} + i(\vec{a}\times\vec{n}).\vec{\sigma}]\sin\theta
\end{aligned}
$$

Since σ_i's are traceless, then we have

$$tr(Xg) = tr(-(\vec{a}.\vec{n})\sin\theta\mathbb{I}) = -(\vec{a}.\vec{n})\sin\theta.$$

Definition 4.0.2. *The Fourier transform of functions from* $\mathbb{R}^3$ *to* $SU(2)$ *is given via*

$$F(g) = \int e^{i\alpha(tr(Xg))} f(X)\, dX$$

where $X = \vec{x}.\vec{\sigma}$, $\vec{x} = (x_1, x_2, x_3)$, *and* $\alpha \in \mathbb{N}$.

For example, consider the function $f(x) = e^{-|x|^2}$. The Fourier transform of this function would be

$$\begin{aligned}
F(\vec{n}, \theta) &= \int e^{i\alpha(tr(Xg)} e^{-|X|^2}\, dX \\
&= \int e^{-i\alpha(\vec{x}.\vec{n})\sin\theta} e^{-|X|^2}\, dX \\
&= \int e^{-i\alpha(x_1 n_1)\sin\theta} e^{-|x_1|^2}\, dx_1 . \int e^{-i\alpha(x_2 n_2)\sin\theta} e^{-|x_2|^2}\, dx_2 . \int e^{-i\alpha(x_3 n_3)\sin\theta} e^{-|x_3|^2}\, dx_3 \\
&= \int e^{-(x_1 + \frac{\alpha n_1 \sin\theta}{2})^2} e^{-\alpha^2 n_1^2 \sin^2\theta}\, dx_1 \int e^{-(x_2 + \frac{\alpha n_2 \sin\theta}{2})^2} e^{-\alpha^2 n_2^2 \sin^2\theta}\, dx_2 \\
&\int e^{-(x_3 + \frac{\alpha n_3 \sin\theta}{2})^2} e^{-\alpha^2 n_3^2 \sin^2\theta}\, dx_3 \\
&= \sqrt{\pi} e^{-\alpha^2 n_1^2 \sin^2\theta} e^{-\alpha^2 n_2^2 \sin^2\theta} e^{-\alpha^2 n_3^2 \sin^2\theta} \\
&= \sqrt{\pi} e^{-\alpha^2 \sin^2\theta}
\end{aligned}$$

Consider the map

$$\begin{aligned}
F_\alpha : S(\mathbb{R}^3) &\to S(SU(2)) \\
f(x) &\to \int_{\mathbb{R}^3} e^{i\alpha(tr(Xg)} f(x)\, dx
\end{aligned}$$

where $S(\mathbb{R}^3)$ and $S(SU(2))$ are the sets of smooth functions on $\mathbb{R}^3$ and $SU(2)$ respectively. This map takes the function $f(x)$ defined on $\mathbb{R}^3$ to the function

$F_\alpha(f)$ on $SU(2)$.

Preposition 4.0.3. *F_α is a non invertible map for a fixed α.*

Proof. We will show this by proving that F_α is not injective. Let $\xi = \vec{n}\sin\theta$, then we have $Tr(Xg) = -(\vec{x}.\vec{n})\sin\theta = -\vec{x}.\vec{\xi}$. Consider the function $e^{i\alpha\vec{x}.\vec{\xi_0}}$ where $\xi_0 \notin S^3$, and take its image under F_α

$$
\begin{aligned}
F_\alpha(e^{i\alpha\vec{x}.\vec{\xi_0}}) &= \int_{\mathbb{R}^3} e^{i\alpha(tr(Xg)} e^{i\alpha\vec{x}.\vec{\xi_0}}\, dx \\
&= \int_{\mathbb{R}^3} e^{-i\alpha\vec{x}.\vec{\xi}} e^{i\alpha\vec{x}.\vec{\xi_0}}\, dx \\
&= \int_{\mathbb{R}^3} e^{-i\alpha\vec{x}.(\vec{\xi}-\vec{\xi_0})}\, dx \\
&= \delta(\xi - \xi_0) \\
&= 0 \text{ on } S^3 \text{ since } \xi_0 \notin S^3 \ .
\end{aligned}
$$

Therefore, F_α is not injective since $e^{i\alpha\vec{x}.\vec{\xi_0}} \neq 0$ on $\mathbb{R}^3$. □

Theorem 4.0.4. *The map $\psi : f \to \{F_\alpha(f)\}_{\alpha\in\mathbb{N}}$ where f is Schwartz, is invertible.*

Proof. Consider the set $U_\alpha = \{F_\alpha(f), f \text{ is schwartz } \}$ and the map $\pi_{\alpha\beta} : U_\beta \to U_\alpha$ for $\alpha < \beta$ with the following properties:

- $\pi_{\alpha\alpha} : U_\alpha \to U_\alpha$ is the identity map.
- $\pi_{\alpha\beta} : U_\beta \to U_\alpha$ is the projection of the map F_β on the smaller sphere.
- $\pi_{\alpha\beta} \circ \pi_{\beta\gamma} = \pi_{\alpha\gamma}$ for all $\alpha < \beta < \gamma$.

We claim that the Schwartz space over $\mathbb{R}^3$ is the inverse limit of the system $((U_\alpha)_{\alpha\in\mathbb{N}}, \pi_{ij})$.Note that

$\varprojlim U_\alpha = \{F(f) \in \prod_{\alpha \in \mathbb{N}} U_\alpha \mid \pi_{\alpha\beta}(F_\beta(f)) = F_\alpha(f) \;\forall\; \beta \geq \alpha$ in $\mathbb{N}$. Now consider the map

$$\phi : S(\mathbb{R}^3) \to \varprojlim U_\alpha$$
$$f \to (F_{\alpha 1}(f), F_{\alpha 2}(f), ...)$$

The map ϕ is linear and a bijection.

$$\begin{aligned}\phi(f_1 + f_2) &= (F_{\alpha 1}(f_1) + F_{\alpha 1}(f_2), F_{\alpha 2}(f) + F_{\alpha 2}(f_2), ...)\\ &= (F_{\alpha 1}(f_1), F_{\alpha 2}(f_1), ...) + (F_{\alpha 1}(f_2), F_{\alpha 2}(f_2), ...)\\ &= \phi(f_1) + \phi(f_2)\end{aligned}$$

ϕ is surjective by construction so it remains to show that ϕ is injective. Let $f \in S(\mathbb{R}^3)$ such that $\phi(f) = 0$. This implies that $\int_{\mathbb{R}^3} e^{-i\alpha\vec{x}.\vec{\xi}} f(x)\, dx = 0 \;\forall\; \alpha \in \mathbb{N}$. So the usual Fourier transform of f will vanish, and thus $f = 0$. Therefore we have $S(\mathbb{R}^3)$ is isomorphic to $\varprojlim U_\alpha$. So gluing all the U_α's together will give us the Schwartz space over $\mathbb{R}^3$.
In other words, if we vary α over $\mathbb{N}$, then $\alpha \sin\theta\vec{n}$ will vary over $\mathbb{R}^3$. Therefore, we can treat the map F_α, where α is arbitrary, as the regular Fourier transform. We know that the Schwartz space is invariant under the regular Fourier transform and $\mathcal{F}(S(\mathbb{R}^3)) = S(\mathbb{R}^3)$ is an isomorphism. □

Preposition 4.0.5. *f is Schwartz $\implies$ $F_\alpha(f)$ is smooth and decays faster than any polynomial as $\alpha \to \infty$.*

Proof. First note that

$$\begin{aligned}
\frac{\partial}{\partial\theta}F_\alpha(f) &= \frac{\partial}{\partial\theta}\int_{\mathbb{R}^3} e^{i\alpha(tr(Xg)}f(x)\,dx \\
&= \frac{\partial}{\partial\theta}\int_{\mathbb{R}^3} e^{-i\alpha(\vec{x}.\vec{n})\sin\theta}f(x)\,dx \\
&= \int_{\mathbb{R}^3}\frac{\partial}{\partial\theta}e^{-i\alpha(\vec{x}.\vec{n})\sin\theta}f(x)\,dx \\
&= -i\alpha\int_{\mathbb{R}^3}\vec{x}.\vec{n}\cos\theta e^{-i\alpha(\vec{x}.\vec{n})\sin\theta}f(x)\,dx \\
&= -i\alpha\cos\theta F_\alpha(\vec{x}.\vec{n}f)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial}{\partial n_1}F_\alpha(f) &= \frac{\partial}{\partial n_1}\int_{\mathbb{R}^3} e^{i\alpha(tr(Xg)}f(x)\,dx \\
&= \frac{\partial}{\partial n_1}\int_{\mathbb{R}^3} e^{-i\alpha(\vec{x}.\vec{n})\sin\theta}f(x)\,dx \\
&= \int_{\mathbb{R}^3}\frac{\partial}{\partial n_1}e^{-i\alpha(\vec{x}.\vec{n})\sin\theta}f(x)\,dx \\
&= -i\alpha\sin\theta\int_{\mathbb{R}^3} e^{-i\alpha(\vec{x}.\vec{n})\sin\theta}x_1 f(x)\,dx \\
&= -i\alpha\sin\theta F_\alpha(x_1 f)
\end{aligned}$$

The chart above does not work for all values of θ, so we have to use some other chart to guarantee that F_α is differentialble for all values of θ and n. We write here $\vec{n}\sin\theta$ in Cartesian coordinates $\Longrightarrow$ $\vec{\xi} = (\xi_1, \xi_2, \xi_3) = \vec{n}\sin\theta$. Then we have

$$
\begin{aligned}
\frac{\partial}{\partial \xi_j} F_\alpha(f) &= \frac{\partial}{\partial \xi_j} \int_{\mathbb{R}^3} e^{i\alpha(tr(Xg)} f(x)\, dx \\
&= \frac{\partial}{\partial \xi_j} \int_{\mathbb{R}^3} e^{-i\alpha(\vec{x}.\vec{n}) \sin\theta} f(x)\, dx \\
&= \int_{\mathbb{R}^3} \frac{\partial}{\partial \xi_j} e^{-i\alpha(\vec{x}.\vec{\xi})} f(x)\, dx \\
&= -i\alpha \int_{\mathbb{R}^3} x_j e^{-i\alpha(\vec{x}.\vec{\xi})} f(x)\, dx \\
&= -i\alpha F_\alpha(x_j f)
\end{aligned}
$$

Here we can interchange the partial derivative and the integral because $\left|e^{-i\alpha(\vec{x}.\vec{n}) \sin\theta} f(x)\right| \leq |f(x)|$ and $f(x)$ is integrable over $\mathbb{R}^3$ since f is Schwartz. Therefore by Lebesgue dominated convergence theorem, we can interchange differentiation and integration. Also note that if f is Schwartz, then $x_j f$ is also Schwartz. So it follow that the function F_α is smooth. Sending α to infinity is the same as sending ξ to infinity and the resulting function will be Schwartz function. Thus it will decrease faster than the inverse of any polynomial. □

We have seen that the Schwartz space $\mathcal{S}(\mathbb{R}^3)$ is invariant under the usual Fourier transform. That is, on $\mathcal{S}(\mathbb{R}^3)$, we can deal with the transform of a function and its inverse transform in the same way. On the other hand, we have seen that we have to know the Fourier transform of a function $f \in \mathcal{S}(\mathbb{R}^3)$ for all $\alpha \in \mathbb{N}$, in order to define the inverse Fourier transform from "Schwartz" functions on $SU(2)$ to $\mathcal{S}(\mathbb{R}^3)$. Therefore, knowledge of the function on $SU(2)$ is not enough to recover the original function that was transformed because there would other functions that will give the same result under the Fourier transform. The same result was attained via different route in [7].

Appendix A

Identity for the Delta function on SU(2)

Below we will show the following Delta function identity,

$$\int e^{iTr(Xg)} \frac{(1+\epsilon(g))}{8\pi} \, d^3x = \delta(g)$$

where $X \in su(2)$, $g \in SU(2)$, and $\epsilon(g) = sign(\cos(\theta))$.

Note that :

$$\int e^{iTr(Xg)} \, d^3x = \int e^{i\vec{x}.\vec{n}.\sin\theta} \, d^3x = (2\pi)^3 \delta(\vec{n} \sin\theta)$$

Notice that

$$\begin{aligned}\int f(x,y,z)\frac{\delta(|\vec{x}|)}{4\pi\left|\vec{x}\right|^2}\,dxdydz &= \int f(r,\theta,\phi)\frac{\delta(r)}{4\pi r^2}\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}\,drd\theta d\phi \\ &= \frac{1}{4\pi}\int f(0,\theta,\phi)\sin\phi\,d\theta d\phi\end{aligned}$$

But we know that $f(0,\theta,\phi)$ in spherical coordinates is equal to $f(0,0,0)$ in cartesian coordinates.Therefore, integrating over θ and ϕ

$$\begin{aligned}\frac{1}{4\pi}\int f(0,\theta,\phi)\sin\phi\,d\theta d\phi &= \frac{1}{4\pi}f(0,0,0)\int \sin\phi\,d\theta d\phi \\ &= f(0,0,0)\end{aligned}$$

We also know that $\int f(x)\delta(\vec{x})\,dxdydz = f(0,0,0)$. Therefore we get the identity:$\delta(\vec{X}) = \frac{\delta(\left|\vec{X}\right|)}{4\pi X^2}$.

Then

$$(2\pi)^3\delta(\vec{n}\sin\theta) = \frac{2\pi^2}{(\sin^2(\theta))}\sum_{n\in\mathcal{Z}}\delta(\theta-n\pi)\frac{1}{|\cos(\theta)|}$$

But $\theta\in[0,\pi]$, thus

$$(2\pi)^3\delta(\vec{n}\sin\theta) = \frac{2\pi^2}{(\sin^2(\theta))}(\delta(\theta)+\delta(\theta-\pi))$$

Now we claim that $\dfrac{2\pi^2}{(\sin^2(\theta))}\delta(\theta) = \delta(g)$. We know that the normalized Haar measure on $SU(2)$ is given by

$$dg = \frac{1}{2\pi^2}\sin^2\theta \sin\psi \, d\theta d\psi d\phi$$

where $0 \leq \theta \leq \pi, 0 \leq \psi \leq \pi$, and $0 \leq \phi \leq 2\pi$. Notice that

$$\begin{aligned}\int f(g)\frac{2\pi^2}{(\sin^2(\theta))}\delta(\theta)\, dg &= \int f(n,\theta)\frac{2\pi^2}{(\sin^2(\theta))}\delta(\theta)\frac{1}{2\pi^2}\sin^2\theta\sin\psi\, d\theta d\psi d\phi \\ &= \int f(n,0)\sin\psi\, d\psi d\phi\end{aligned}$$

But $g = \cos\theta\mathbb{I} + i\vec{n}.\vec{\sigma}\sin\theta$, so for $\theta = 0$ we get $g = \mathbb{I}$. Thus

$$\begin{aligned}\int f(n,0)\sin\psi\, d\psi d\phi &= f(\mathbb{I})\int \sin\psi\, d\psi d\phi \\ &= 4\pi f(\mathbb{I}) \\ &= 4\pi\int f(g)\delta(g)\, dg\end{aligned}$$

Therefore we get $4\pi\delta(g) = \dfrac{2\pi^2}{(\sin^2(\theta))}\delta(\theta)$. Similarly we have $4\pi\delta(-g) = \dfrac{2\pi^2}{(\sin^2(\theta))}\delta(\theta - \pi)$. Similarly we get,

$$\int e^{iTr(Xg)}\epsilon(g)\, d^3x = 4\pi(\delta(g) - \delta(-g))$$

Thus

$$\int e^{iTr(Xg)}\frac{(1+\epsilon(g))}{8\pi}\, d^3x = \delta(g)$$

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