AMERICAN UNIVERSITY OF BEIRUT

WELSCHINGER INVARIANTS OF THE PROJECTIVE PLANE

by

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I don't always show it, but I appreciate how much both of you have helped me in my life, granted me love and given me all the things that have gotten me here, no words can ever repay you and no words can describe how much I am grateful for your great support.
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Contents

AN ABSTRACT OF THE THESIS OF

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In this thesis we study the paper Recursive Formulas for Welschinger Invariants of the Projective Plane by Arroyo, Brugallé and López de Medrano. In this paper the authors give a recursive formula to compute Welschinger invariants of generic subsets of the complex projective plane that are invariant under complex conjugation. They do this by reducing the problem to a purely combinatorial one that involves enumerating certain graphs called marked floor diagrams and giving a recursive algorithm for generating these diagrams.

Chapter 1

Introduction

In this thesis we study the paper [?] in which the authors give a recursive formula for computing Welschinger invariants of the projective plane. These invariants give a lower bound for the number of real curves of degree d passing through a generic collection of $3d-1$ points in $\mathbb{CP}^2$ that is invariant under complex conjugation. Welschinger invariants can be considered as real analogues of Gromov-Witten invariants which enumerate complex curves satisfying certain conditions. Before defining these invariants, we will begin by setting some notations.

Convention 1.1. *We will use the following notation:*

- $\mathbb{N} = \{n \in \mathbb{Z} | \ n \geq 0\}$ denotes the set of natural numbers.

- $\mathbb{N}^* = \{n \in \mathbb{Z} | \ n > 0\}$ denotes the set of natural numbers without zero.

- $\mathbb{N}^\infty$ denotes the set of sequences of elements in $\mathbb{N}$ having only finitely many non zero terms.

- e_i denotes the vector in $\mathbb{N}^\infty$ whose i^{th} coordinate is equal to 1 and the remaining coordinates are equal to zero.

- $(\alpha)_i$ denotes the i^{th} coordinate of vector $\alpha \in \mathbb{N}^\infty$.

- If a and b are two integer numbers, $\binom{a}{b}$ denotes the binomial coefficient

$$\binom{a}{b} = \begin{cases} \dfrac{a!}{b!(a-b)!} & \text{if } 0 \leq b \leq a \\ \\ 0 & \text{otherwise.} \end{cases}$$

- If a and $b_1, b_2, \ldots, b_k$ are integer numbers

$$\binom{a}{b_1, b_2, \ldots, b_k}$$

denotes the multinomial coefficient

$$\binom{a}{b_1, b_2, \ldots, b_k} = \prod_{i=1}^{k} \binom{a - \sum_{j=1}^{i-1} b_j}{b_i}$$

- If $\alpha, \alpha_1, \ldots, \alpha_l$ are vectors in $\mathbb{N}^\infty$, then

$$\binom{\alpha}{\alpha_1, \ldots, \alpha_l} = \prod_{i=1}^{\infty} \binom{(\alpha)_i}{(\alpha_1)_i, \ldots, (\alpha_l)_i}$$

In addition;

- $|\alpha| = \sum_{i=1}^{\infty} (\alpha)_i$

- $I\alpha = \sum_{i=1}^{\infty} i(\alpha)_i$

- $I^\alpha = \prod_{i=1}^{\infty} i^{(\alpha)_i}$

and for vectors α and β in $\mathbb{N}^\infty$, we say $\alpha \geq \beta$ if $(\alpha)_i \geq (\beta)_i \ \forall \ i$.

Our interest is to find the number of curves passing through a given set of points with certain conditions.

Let w=$\{p_1, p_2, \ldots, p_{3d-1}\}$ be a generic collection of $3d-1$ points in $\mathbb{CP}^2$. Let $C(w)$ be the set of all irreducible complex rational curves of degree d in $\mathbb{CP}^2$ passing through w. The cardinality of $C(w)$ is independent of w as long as the points of w are in generic position. The cardinality of $C(w)$ is called the genus 0 $Gromov-Wittten$ invariant of degree d.

To define the relative $Gromov-Witten$ invariants we fix a line L in $\mathbb{CP}^2$, an integer degree $d \geq 1$, and two vectors α and β in $\mathbb{N}^\infty$ such that $I\alpha + \mathrm{I}\beta = d$. We let

$w = \{p_1, \ldots, p_{2d-1+|\beta|}\}$ be a collection of $2d-1+|\beta|$ points in $\mathbb{CP}^2$ and let

$w_L = \{p^1_1, \ldots, p^1_{(\alpha)_1}, p^2_1, \ldots, p^2_{(\alpha)_2}, \ldots, p^k_1, \ldots, p^k_{(\alpha)_k}, \ldots\}$ be a collection of $|\alpha|$ points on L.

Let $C(w, \alpha, \beta)$ be the set of all irreducible complex rational curves of degree d in $\mathbb{CP}^2$ passing through the points of w having no singular points on L and intersecting the line L at the points $p^j_i \in w_L$ with multiplicity j for all i and j, and intersecting L at $(\beta)_j$ other points with multiplicity j for all j. The cardinality of $C(w, \alpha, \beta)$ is independent of the choice of points in w and w_L provided that the number of points is fixed and the points are in general position. The Relative Gromov Witten invariant of degree d with respect to α, β is the cardinality of $C(w, \alpha, \beta)$ and is denoted by $N^{\alpha,\beta}(d)$.

We want to consider the real case of the same problem.

Fix a degree $d \geq 1$ let $w = \{p_1, \ldots, p_{3d-1}\}$ be a subset of $\mathbb{CP}^2$ that is invariant under complex conjugation. Let $\mathbb{R}C(w)$ be the set of all irreducible real rational curves of degree d in $\mathbb{CP}^2$ passing through the points of w. The cardinality of $\mathbb{R}C(w)$ is not independent of w and it could vary with the variation of the points in w. Welschinger [**?**] discovered that we can get an invariant if we count the curves in $\mathbb{R}C(w)$ with the appropriate sign. Let $W(C)$ be the number of all nodes on the real algebraic nodal curve C in $\mathbb{CP}^2$ such that if we map the neighborhood of this point to the origin we will get the equation $X^2 + Y^2 = 0$. The Welschinger invariant is the number

$$\sum_{C \in \mathbb{R}C(w)} (-1)^{W(C)}$$

It only depends on the degree d and the number of pairs of complex conjugate points in w.

It is denoted by $W_2(d,r)$ where r is the number of pairs of complex conjugate points in w.

Through Mikhalkin's correspondence theorem [**?**] and Brugallé and Mikhalkin's floor diagrams [**?**], the computation of Gromov-Witten and Welschinger invariants has been reduced to a purely combinatorial problem involving certain graphs called marked floor diagrams.

Chapter 2

Marked Floor Diagrams

In this chapter we will define floor diagrams and state their relation with Welschinger and Gromov Witten invariants of $\mathbb{CP}^2$.

Definition 2.1. *Some graph theoretic notions:*

- *An oriented graph* Γ *is a pair of finite sets* $\overline{Vert}(\Gamma)$*, the set of vertices of* Γ *and* $Edge(\Gamma)$*, the set of edges of* Γ*. An element* e *of* $Edge(\Gamma)$ *is an oriented pair* (v_1, v_2) *of vertices of* Γ*. We say* e *is adjacent to* v_1 *and to* v_2 *and is outgoing from* v_1 *and incoming to* v_2*.*

- *A source is a vertex of* Γ *all of whose adjacent edges are outgoing. We denote by* $Vert^\infty(\Gamma)$ *the set of sources of* Γ*, and we put* $Vert(\Gamma) = \overline{Vert}(\Gamma) \backslash Vert^\infty(\Gamma)$ *and denote by* $Edge^\infty(\Gamma)$ *the set of edges adjacent to a source.*

- *An oriented graph* Γ *is connected if for any two elements* v_1 *and* v_2 *of* $\overline{Vert}(\Gamma)$*, there exists a sequence* $s_1 = v_1, s_2, \ldots, s_{k-1}, s_k = v_2$ *of elements of* $\overline{Vert}(\Gamma)$ *such that* $(s_i, s_{i+1}) \in Edge(\Gamma)$ *or* $(s_{i+1}, s_i) \in Edge(\Gamma) \forall i$*.*

- *A cycle is a sequence* $s_1, s_2, \ldots, s_k$ *of elements of* $\overline{Vert}(\Gamma)$ *such that* $(s_i, s_{i+1}) \in Edge(\Gamma)$ *or* $(s_{i+1}, s_i) \in Edge(\Gamma) \forall i$ *and* $s_1, \ldots, s_{k-1}$ *are distinct,* $s_k = s_1$*.*

- *An oriented tree is a connected oriented graph with no cycles. In an oriented tree* Γ*,* $\#\overline{Vert}(\Gamma) - \#Edge(\Gamma) = 1$*.*

  An oriented tree Γ *is naturally enhanced with a partial ordering. Let* $v, w \in \overline{Vert}(\Gamma)$*, we say* $v \leq w$ *if there is a sequence* $v_1, \ldots, v_k$ *of vertices of* Γ *such that* $v_1 = v, v_k = w$ *and* $(v_i, v_{i+1}) \in Edge(\Gamma)$ *with* $1 \leq i < k$*.*

- *A weighted graph is a graph* Γ *enhanced with a function* w*:*$Edge(\Gamma) \to \mathbb{N}^*$*.* $w(e)$ *is called the weight of the edge* e*.*

- *The divergence at a vertex* v *of* Γ*, denoted by* $div(v)$*, is the sum of the weights of all incoming edges to* v *minus the sum of the weights of all outgoing edges from* v*.*

Definition 2.2. *A floor diagram $\mathcal{D}$ of genus 0 and degree d is a connected weighted oriented tree with the following conditions satisfied:*

- *For any $v \in Vert(\mathcal{D})$ one has div(v) $= 1$.*
- *Each source has a unique adjacent edge.*
- *One has $\sum_{v \in Vert^{\infty}}$ div(v)= -d.*

From this definition one can conclude that the set $Vert(\mathcal{D})$ has exactly d elements. The following lemma and corollary prove this.

Lemma 2.3. *Let $\mathcal{D}$ be a directed graph with vertex set $V = \overline{Vert}(\mathcal{D})$ and edge set $E = Edge(\mathcal{D})$. Then,*

$$\sum_{v \in V} div(v) = 0.$$

Proof:

For $v \in V$, let $I(v)$ be the set of all incoming edges to v and let $O(v)$ be the set of all outgoing edges from v. Note that every edge $e \in E$ belongs to $I(v)$ for exactly one $v \in V$

and belongs to $O(v)$ for exactly one $v \in V$.

$$
\begin{aligned}
\sum_{v\in V} \operatorname{div}(v) &= \sum_{v\in V}\left(\sum_{e\in I(v)} w(e) - \sum_{e\in O(v)} w(e)\right) \\
&= \sum_{v\in V}\sum_{e\in I(v)} w(e) - \sum_{v\in V}\sum_{e\in O(v)} w(e) \\
&= \sum_{e\in E} w(e) - \sum_{e\in E} w(e) \\
&= 0. \quad \blacksquare
\end{aligned}
$$

Corollary 2.4. *If $\mathcal{D}$ is a floor diagram of genus zero and degree d then $\#(Vert(\mathcal{D})) = d$.*

<u>Proof:</u> If $v \in Vert(\mathcal{D})$ then $\operatorname{div}(v) = 1$. By lemma 2.3, we have

$$
\begin{aligned}
0 &= \sum_{v\in\overline{Vert}(\mathcal{D})} \operatorname{div}(v) \\
&= \sum_{v\in Vert(\mathcal{D})} \operatorname{div}(v) + \sum_{v\in Vert^{\infty}(\mathcal{D})} \operatorname{div}(v) \\
&= |Vert(\mathcal{D})| - d. \quad \blacksquare
\end{aligned}
$$

Sketching of floor diagrams:

- Elements of $Vert^{\infty}(\mathcal{D})$ are represented by small horizontal segments.

- Elements of $Vert(\mathcal{D})$ are represented by ellipses.

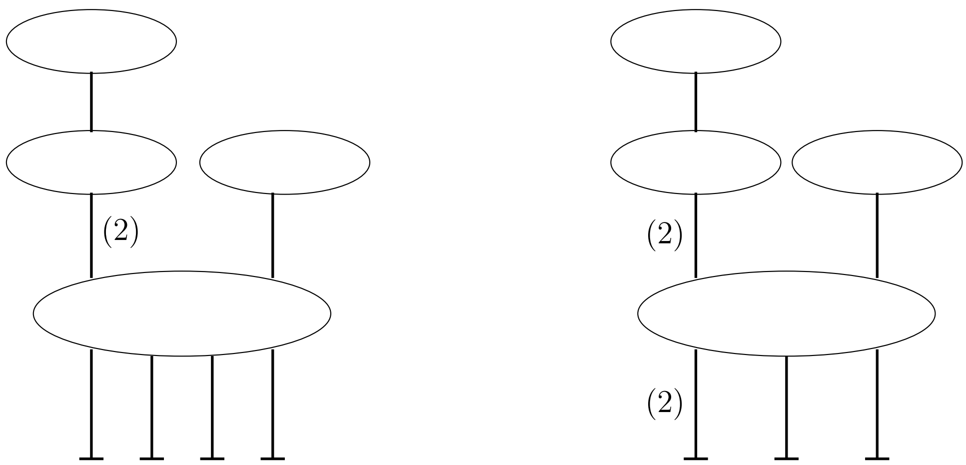
- Elements of $Edge(\mathcal{D})$ are represented by vertical lines.

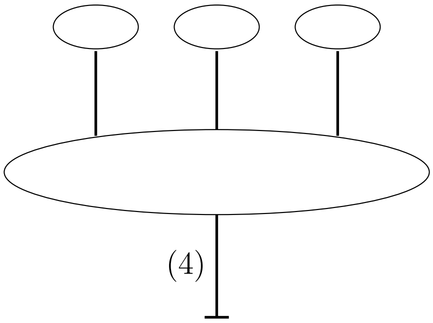
The orientation is implicitly from down to up.

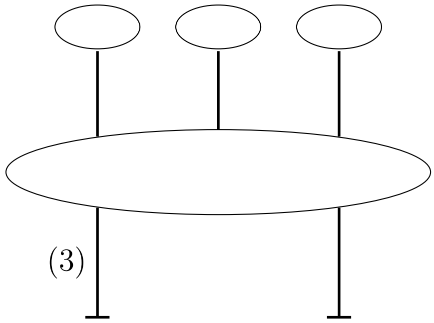
When the weight is not mentioned this means that the edge is of weight equals to one.

Example 2.5.

Some floor diagrams of degree 4 and genus 0 are:




(4)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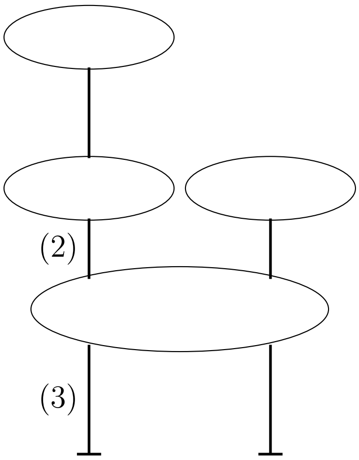
(2)
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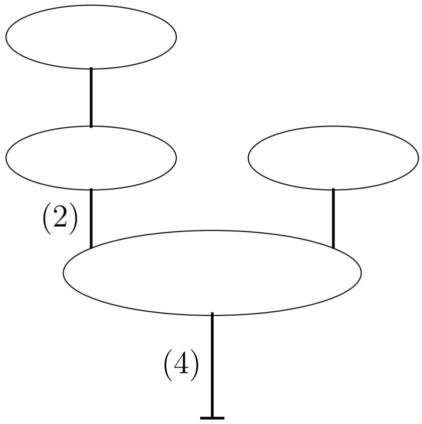
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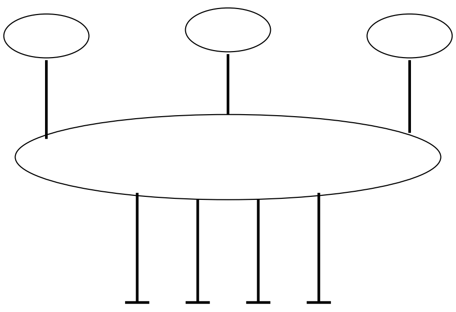

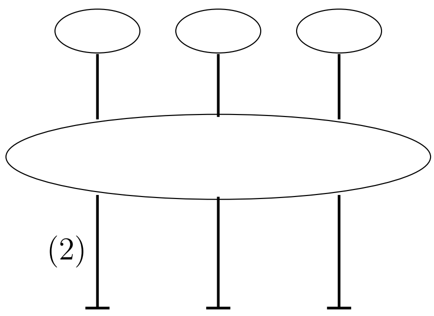
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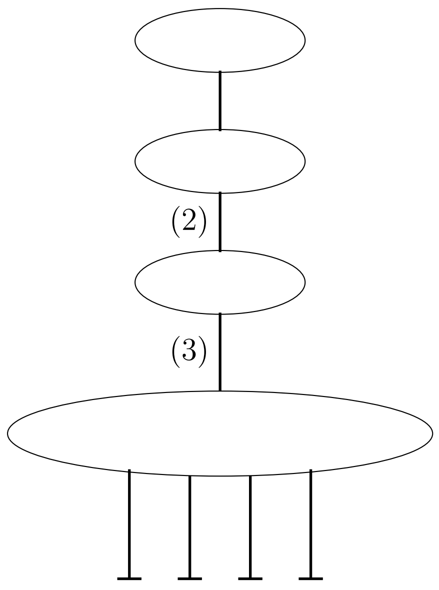
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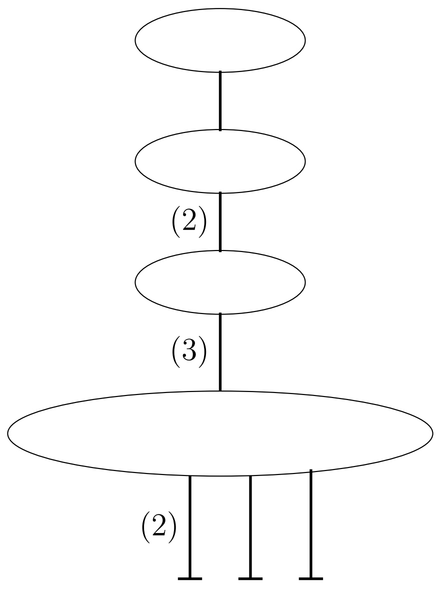
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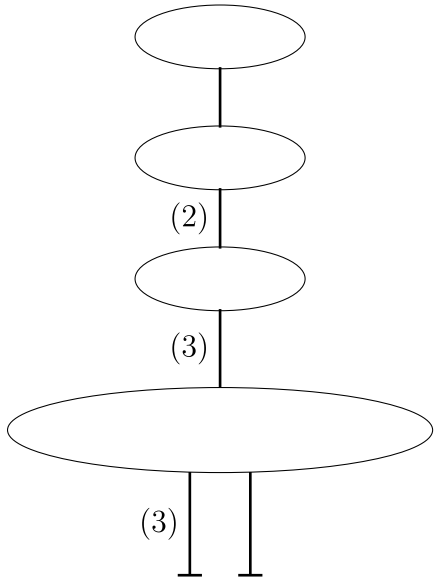
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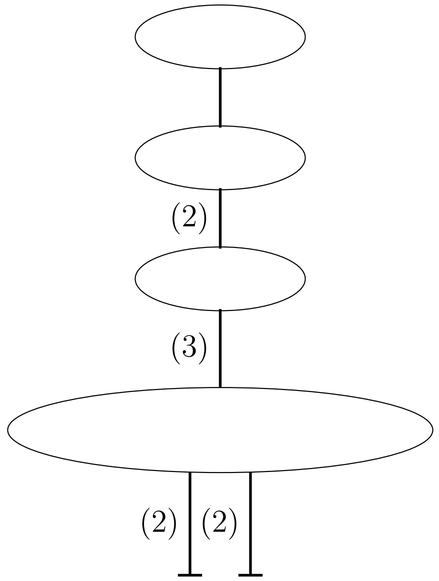

(2)
(3)
(2)
(2)

To find $Gromov-Witten$ and $Welschinger$ invariants using floor diagrams we first need a marking of a floor diagram.

Let $\alpha, \beta, \gamma, \delta$ be four vectors in $\mathbb{N}^{\infty}$, and define $n = 2d - 1 + |\alpha + \beta + 2\gamma + 2\delta|$ and $P = \{1, \dots, n\}$.

Definition 2.6. *Let $\mathfrak{M} : \mathcal{P} \longrightarrow \mathcal{D}$ be a function mapping $\mathcal{P}$ to $\overline{Vert}(\mathcal{D}) \cup Edge(\mathcal{D})$.*

The map $\mathfrak{M}$ is called a marking of $\mathcal{D}$ of type $(\alpha, \beta, \gamma, \delta)$ if the following conditions are satisfied :

1. *the floor diagram $\mathcal{D}$ is of genus 0 and of degree $I\alpha + I\beta + 2I\gamma + 2I\delta$,*

2. *the map $\mathfrak{M}$ is injective, i.e. two distinct elements of $\mathcal{P}$ have distinct images in $\mathcal{D}$,*

3. *if $\mathfrak{M}(i) > \mathfrak{M}(j)$, then $i > j$,*

4. *if $\sum_{j=1}^{k-1}(\alpha)_j + 1 \leq i \leq \sum_{j=1}^{k}(\alpha)_j$ or $|\alpha| + 2\sum_{j=1}^{k-1}(\gamma)_j + 1 \leq i \leq |\alpha| + 2\sum_{j=1}^{k}(\gamma)_j$, then $\mathfrak{M}(i)$ is a source with divergence -k,*

5. *for any $k \geq 1$,there are exactly $(\beta)_k + 2(\delta)_k$ elements of $Edge^{\infty}(\mathcal{D})$ with weight k in the*

image of $\mathfrak{M}$,

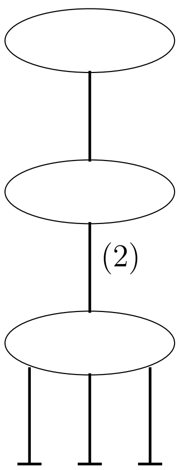
6. *If e is an edge of $\mathcal{D}$ adjacent to a source v, then exactly one of the two elements v or e is in the image of $\mathfrak{M}$.*

A marked floor diagram of type $(\alpha, \beta, \gamma, \delta)$ is a floor diagram coupled with a marking of type $(\alpha, \beta, \gamma, \delta)$.

Example 2.7.

Here is an example to clarify more about a marking $\mathfrak{M}$ of a floor diagram $\mathcal{D}$:

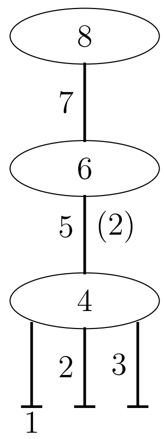
Given the following floor diagram We are going to define a marking of type $(e_1, 2e_1, 0, 0)$ on it.



n= $2d-1+|\alpha+\beta+2\gamma+2\delta| = 8$ and $\mathcal{P}=\{1,\ldots,8\}$

The given diagram is of genus 0 and of degree 3 which is equal to $\mathrm{I}\alpha$ + $\mathrm{I}\beta$ + $2\mathrm{I}\gamma$ + $2\mathrm{I}\delta$.

Condition 4 tells us that, $\mathfrak{M}(1)$ is a source with divergence -1, and this is the only source that is marked. Condition 5 tells us that, since β is equal to $2e_1$ and δ is equal to zero there are two elements of $Edge^{\infty}(\mathcal{D})$ with weight 1 in the image of $\mathfrak{M}$. Condition 6 tells us that these two edges are not adjacent to the marked source. Finally, we use condition 3 to complete the marking.



Two marked floor diagrams $(\mathcal{D},\mathfrak{M})$ and $(\mathcal{D}',\mathfrak{M}')$ are equivalent if there is a homeomorphism of oriented weighted graphs $\Psi : (\mathcal{D},\mathfrak{M}) \longrightarrow (\mathcal{D}',\mathfrak{M}')$ such that $\Psi(\mathfrak{M}(i)) = \mathfrak{M}'(i)$.

Proposition 2.8. *If $\mathfrak{M}$ is a marking of $\mathcal{D}$, then any vertex in $Vert(\mathcal{D})$ and any edge which is not adjacent to a source is in the image of $\mathfrak{M}$.*

<u>**Proof:**</u> Let $\mathfrak{M}$ be a marking of $\mathcal{D}$. We know that $n = 2d - 1 + |\alpha + \beta + 2\gamma + 2\delta|$ is the total number of marked elements in $\mathcal{D}$, $|\alpha + 2\gamma|$ is the number of sources marked, and $|\beta + 2\delta|$ is the number of marked edges adjacent to sources. We want to show that

$\#Vert(\mathcal{D}) + \#(Edge(\mathcal{D}) \backslash Edge^{\infty}(\mathcal{D})) = 2d - 1$.

Since $\mathcal{D}$ is a tree, $\#\overline{Vert}(\Gamma) - \#Edge(\Gamma) = 1$.

Let $\boldsymbol{M}$ be the number of edges that are not adjacent to a source, $\boldsymbol{F}$ the number of edges adjacent to a source, $\boldsymbol{N}$ the number of vertices that are not sources, and $\boldsymbol{S}$ the number of sources.

Then, $\#\overline{Vert}(\Gamma) - \#Edge(\Gamma) = (\boldsymbol{N} + \boldsymbol{S}) - (\boldsymbol{M} + \boldsymbol{F}) = 1$.

By lemma 2.3, we have

$$\sum_{v \in V} \operatorname{div}(v) = \sum_{v \in Vert(\mathcal{D})} \operatorname{div}(v) + \sum_{v \in Vert^{\infty}(\mathcal{D})} \operatorname{div}(v) = 0,$$

which gives $\boldsymbol{N} = d$.

By definition of a floor diagram, each source has a unique edge in $Edge^{\infty}$ connected to it, which means $\boldsymbol{S} = \boldsymbol{F}$ and $\boldsymbol{M} = d - 1$.

So we can conclude that $\boldsymbol{M} + \boldsymbol{N} = 2d - 1$. ■

Now, we are going to define the *complex* multiplicity and *r-real* multiplicity that are assigned to any marked floor diagram.

Definition 2.9. *The complex multiplicity of a marked floor diagram $\mathcal{D}$ of type $(\alpha, \beta, \gamma, \delta)$ is denoted by $\mu^{\mathbb{C}}(\mathcal{D})$ and is defined as*

$$\mu^{\mathbb{C}}(\mathcal{D}) = I^{-2\alpha-\beta-4\gamma-2\delta} \prod_{e \in Edge(D)} w(e)^2.$$

Note that the complex multiplicity does not depend on the marking.

Using the above definition we can now state the Brugallé-Mikhalkin theorem that allows us to find the *Gromov-Witten* invariants :

Theorem 2.10. *[?]*

For any α and β in $\mathbb{N}^{\infty}$, and $d = I\alpha + I\beta$, one has

$$N^{\alpha,\beta}(d) = \sum \mu^{\mathbb{C}}(\mathcal{D}),$$

where the sum is taken over all marked floor diagrams of degree d, genus 0 and type

$(\alpha, \beta, 0, 0)$.

Taking the sum over all marked floor diagrams of degree d, genus 0 and of type $(\alpha, \beta, \gamma, \delta)$, corresponds to evaluating $N^{\alpha+2\gamma,\beta+2\delta}(d)$. To prove this, let $\alpha^{'} = \alpha + 2\gamma$, $\beta^{'} = \beta + 2\delta$. Since $(\beta)_k + 2(\delta)_k = (\beta + 2\delta)_k = (\beta^{'} + 2\delta^{'})_k$, both floor diagrams have the same number of marked edges of weight k in $Edge^{\infty}$. Similarly, since $\alpha + 2\gamma = \alpha^{'}$, both floor diagrams have the same number of marked sources of divergence $-k$ $\forall k \geq 0$. So there is a bijection between the set of all marked floor diagrams of type $(\alpha, \beta, \gamma, \delta)$ and those of type $(\alpha + 2\gamma, \beta + 2\delta, 0, 0)$.

To define the $r - real$ multiplicity of a floor diagram $\mathcal{D}$ we need to define what is an

$r - real$ floor diagram.

Let $\mathcal{D}$ be a floor diagram of type $(\alpha, \beta, \gamma, \delta)$ marked by $\mathfrak{M}$. Let r be a positive integer such that $2d - 1 + |\beta + 2\delta| - 2r \geq 0$. If $i = |\alpha| + 2k - 1$ with $1 \leq k \leq |\gamma|$ or $i = n - 2k + 1$ with $1 \leq k \leq r$, then the set $\{i, i+1\}$ is called an $r - pair$.

Let $\mathfrak{F}(\mathcal{D}, \mathfrak{M}, r)$ be the union of all sets $\{\mathfrak{M}(i), \mathfrak{M}(i+1)\}$ where $\{i, i+1\}$ is an $r - pair$ and $\mathfrak{M}(i)$ and $\mathfrak{M}(i+1)$ are not adjacent.

Note that as r increases, so does the number of $r - pairs$.

Let $\rho_{\mathcal{D},\mathfrak{M},r} : \mathcal{D} \longrightarrow \mathcal{D}$ be the bijection defined as:

$\rho_{\mathcal{D},\mathfrak{M},r}(\mathfrak{M}(i)) =$

$$\begin{cases} \mathfrak{M}(i) & \text{if } \mathfrak{M}(i) \ \in \mathcal{D} \backslash \mathfrak{F}(\mathcal{D}, \mathfrak{M}, r) \\ \mathfrak{M}(j) & \text{if } \{i, j\} \text{ is an } r\text{-pair and } \{\mathfrak{M}(i), \mathfrak{M}(j)\} \subset \mathfrak{F}(\mathcal{D}, \mathfrak{M}, r) \end{cases}$$

The map $\rho_{\mathcal{D},\mathfrak{M},r}$ is an involution, that is, the inverse of ρ is itself.

Definition 2.11. *We say a marked floor diagram $\mathcal{D}$ of type $(\alpha,\beta,\gamma,\delta)$ is $r-real$ if :*

- $(\mathcal{D},\mathfrak{M})$ *and* $(\mathcal{D},\rho_{\mathcal{D},\mathfrak{M},r}\circ\mathfrak{M})$ *are equivalent, and*

- *Exactly* $2(\delta)_k$ *edges of weight* k *are in* $Edge^{\infty}(\mathcal{D})\bigcap\mathfrak{F}(\mathcal{D},\mathfrak{M},r)$ *for any* $k\geq 1$.

Now we have all the tools to give the formula of $r-real$ multiplicity which is denoted by $\mu_r^{\mathbb{R}}(\mathcal{D},\mathfrak{M})$.

Definition 2.12. *Let* $(\mathcal{D},\mathfrak{M})$ *be an* $r-real$ *marked floor diagram of type* $(\alpha,\beta,\gamma,\delta)$. *If all edges of* $\mathcal{D}$ *having even weight are in* $\mathfrak{F}(\mathcal{D},\mathfrak{M},r)$, *then*

$$\mu_r^{\mathbb{R}}(\mathcal{D},\mathfrak{M})=(-1)^{\frac{\#(Vert(\mathcal{D})\bigcap\mathfrak{F}(\mathcal{D},\mathfrak{M},r))}{2}}\mathbf{I}^{-\delta}\prod_{e\in Edge(\mathcal{D})\cap\mathfrak{M}(\{n-2r+1,\ldots,n\})}w(e).$$

If there exists an edge of even weight that does not belong to $\mathfrak{F}(\mathcal{D},\mathfrak{M},r)$, *then*

$$\mu_r^{\mathbb{R}}(\mathcal{D},\mathfrak{M})=0.$$

If $(\mathcal{D},\mathfrak{M})$ *is not* $r-real$ *then* $\mu_r^{\mathbb{R}}(\mathcal{D},\mathfrak{M})=0$.

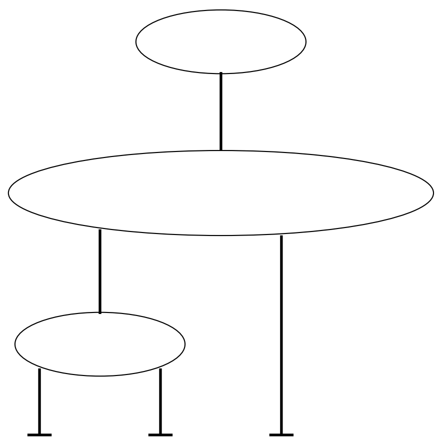
Note that when $r = 0$ and $\gamma = 0$ the 0-*real* multiplicity of a $0 - real$ marked floor diagram does not depend on the marking. In this case, $\mathfrak{F}(\mathcal{D}, \mathfrak{M}, 0)$ is empty and $Edge(\mathcal{D})$ $\subset \mathfrak{M}(\{1, \ldots, n\})$. So, if $\mathcal{D}$ has an edge of even weight, then $\mu_0^{\mathbb{R}}(\mathcal{D}, \mathfrak{M}) = 0$. Otherwise, $\mu_0^{\mathbb{R}}(\mathcal{D}, \mathfrak{M}) = I^{-\delta}$.

When $r = 0$ and $\gamma \neq 0$, the union of all $0 - pairs$ is the set $S = \{|\alpha| + 1, \ldots, |\alpha| + 2|\gamma|\}$.

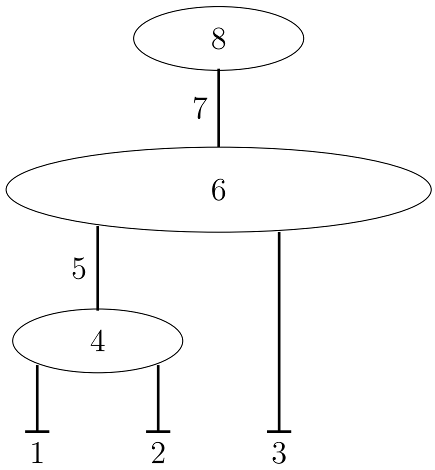
By definition of the marking, each element of S is mapped to a source so the multiplicity depends on the marking.

Example 2.13.

Let $\mathcal{D}$ be the floor diagram of type $(e_1, 0, e_1, 0)$ given by:

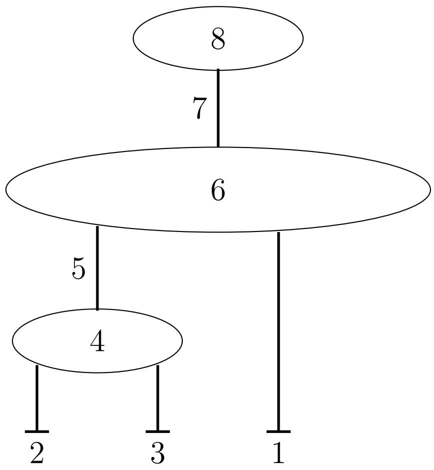


Let $\mathfrak{M}$ and $\mathfrak{M}'$ be two markings on $\mathcal{D}$ given by:



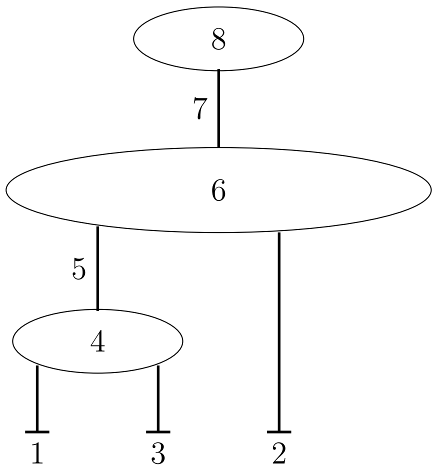
$(\mathcal{D}, \mathfrak{M})$

and



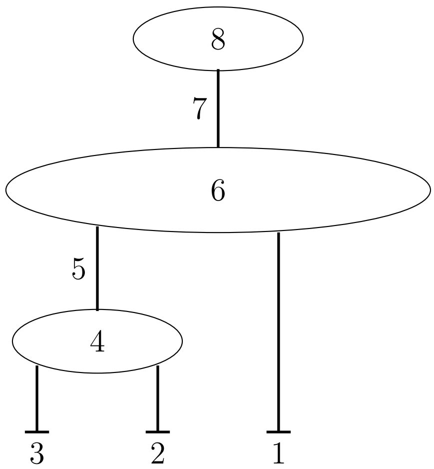
$(\mathcal{D}, \mathfrak{M}')$

For $r = 0$, $\{2, 3\}$ is the only $0 - pair$ so $\rho(\mathcal{D}, \mathfrak{M})$ and $\rho(\mathcal{D}, \mathfrak{M}')$ are:



$(\mathcal{D}, \rho_{\mathcal{D},\mathfrak{M},0} \circ \mathfrak{M})$

and



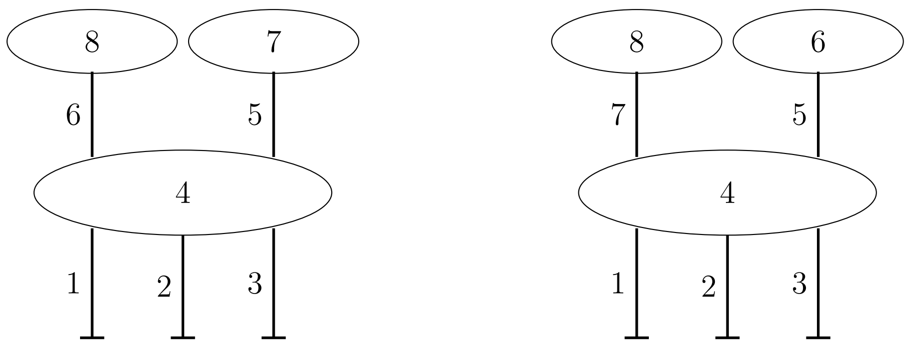
$(\mathcal{D}, \rho_{\mathcal{D},\mathfrak{M}',0} \circ \mathfrak{M}')$

$(\mathcal{D}, \mathfrak{M})$ and $(\mathcal{D}, \rho_{\mathcal{D},\mathfrak{M},0} \circ \mathfrak{M})$ are not equivalent so $(\mathcal{D}, \mathfrak{M})$ is not $0 - real$ and its $0 - real$ multiplicity is zero. On the other hand, $(\mathcal{D}, \mathfrak{M}')$ and $(\mathcal{D}, \rho_{\mathcal{D},\mathfrak{M}',0} \circ \mathfrak{M}')$ are equivalent so $(\mathcal{D}, \mathfrak{M}')$ is $0 - real$ and its $0 - real$ multiplicity is 1.

For $r \geq 1$, the $r - real$ multiplicity of an $r - real$ marked floor diagram depends not only on the underlying floor diagram but also on the marking.

Example 2.14.

Let $(\mathcal{D}, \mathfrak{M})$ and $(\mathcal{D}, \mathfrak{M}^{'})$ be given by:



The underlying floor diagram $\mathcal{D}$ is of degree 3, genus 0 and type $(0, 3e_1, 0, 0)$. So, for $r = 1$, there is only one $1 - pair$,$\{7, 8\}$. For the marking $\mathfrak{M}$, $\mathfrak{F}(\mathcal{D}, \mathfrak{M}, 1) = \{\mathfrak{M}(7), \mathfrak{M}(8)\}$ and $\rho(\mathcal{D}, \mathfrak{M})$ is not equivalent to $(\mathcal{D}, \mathfrak{M})$, so $(\mathcal{D}, \mathfrak{M})$ is not 1-*real*. For $\mathfrak{M}^{'}$, $\mathfrak{M}^{'}(7)$ and $\mathfrak{M}^{'}(8)$ are adjacent, so $\mathfrak{F}(\mathcal{D}, \mathfrak{M}^{'}, 1) = \phi$ and $(\mathcal{D}, \mathfrak{M}^{'})$ is $1 - real$ with $1 - real$ multiplicity equal to 1.

Theorem 2.15. *[?] For any $d \geq 1$ and any $r \geq 0$ such that $3d - 1 - 2r \geq 0$, one has*

$$W_2(d,r) = \sum \mu_r^{\mathbb{R}}(\mathcal{D}, \mathfrak{M})$$

where the sum is taken over all $r-$ *real marked floor diagrams of degree d, genus 0 and of type* $(0,(d-2i)e_1,0,ie_1)$ *with* $0 \leq i \leq \frac{d}{2}$.

Chapter 3

Recursive Formulas

Definition 3.1. *Given two integers $l \geq 0$ and $d \geq 0$ and two vectors $\alpha, \beta \in \mathbb{N}^\infty$, let*

$\mathbf{S}(d,l,\alpha,\beta)$ *be the set of vectors* $(d_1,\ldots,d_l,k_1,\ldots,k_l,\alpha_1,\ldots,\alpha_l,\beta_1,\ldots,\beta_l) \in (\mathbb{N}^*)^{2l} \times (\mathbb{N}^\infty)^{2l}$

satisfying:

- $\forall i, (d_i, k_i, \alpha_i, \beta_i) \leq (d_{i+1}, k_{i+1}, \alpha_{i+1}, \beta_{i+1})$ *for the lexicographic order,*
- $\sum d_i = d - 1$,
- $\sum \alpha_i \leq \alpha$,
- $\forall i, \beta_i \geq e_{k_i}$,
- $\sum(\beta_i - e_{k_i}) = \beta$,

- $\forall i, I\alpha_i + I\beta_i = d_i$.

For $s \in \mathbf{S}(d,l,\alpha,\beta)$, let $\simeq_s$ be the equivalence relation on the set $\{1, \ldots, l\}$ defined by

$$i \simeq_s j \Leftrightarrow (d_i, k_i, \alpha_i, \beta_i) = (d_j, k_j, \alpha_j, \beta_j).$$

For each of the equivalence classes of $\simeq_s$, evaluate the factorial of its cardinality, and denote by $\sigma(s)$ the product of these factorials.

Theorem 3.2. *[?] Caporaso-Harris*

The numbers $N^{\alpha,\beta}(d)$ are given by the initial value $N^{e_1,0}(1)$=1 and the relation

$$N^{\alpha,\beta}(d) = \sum_{k/\beta \geq e_k} kN^{\alpha+e_k,\beta-e_k}(d) +$$
$$\sum_{l\geq 0, s\in S(d,l,\alpha,\beta)} \left[\frac{1}{\sigma(s)} \binom{2d-2+|\beta|}{2d_1-1+|\beta_1|, \ldots, 2d_l-1+|\beta_l|} \binom{\alpha}{\alpha_1, \ldots, \alpha_l} \prod_{i=1}^{l} (\beta_i)_{k_i} k_i N^{\alpha_i,\beta_i}(d_i) \right].$$

Now, we are going to state a recursive formula that allows us to compute the numbers $W_2(d,r)$. This recursive formula does not explicitly involve the numbers $W_2(d,r)$, but some related numbers defined as follows.

Given four vectors α, β, γ and δ in $\mathbb{N}^\infty$, let $d = I\alpha + I\beta + 2I\gamma + 2I\delta$ and let r be a non-negative integer such that $2d - 1 + |\beta + 2\delta| - 2r \geq 0$. Let

$$C^{\alpha,\beta,\gamma,\delta}(d, r) = \sum \mu_r^{\mathbb{R}}(\mathcal{D}, \mathfrak{M})$$

where the sum is taken over all marked floor diagrams of type $(\alpha, \beta, \gamma, \delta)$.

Recall that if $(\mathcal{D}, \mathfrak{M})$ is not $r - real$ then $\mu_r^{\mathbb{R}}(\mathcal{D}, \mathfrak{M})$=0.

For any $d \geq 1$ and $0 \leq r \leq \frac{3d-1}{2}$, Theorem 2.15 states that

$$W_2(d, r) = \sum_{i=0}^{\frac{d}{2}} C^{0,(d-2i)e_1,0,ie_1}(d, r)$$

A vector $\alpha \in \mathbb{N}^\infty$ is said to be *odd* if $(\alpha)_{2i} = 0$ for all $i \geq 1$.

Proposition 3.3. *If α or β is not odd, then* $C^{\alpha,\beta,\gamma,\delta}(d, r) = 0$.

Proof:

Let $(\mathcal{D}, \mathfrak{M})$ be a marked floor diagram of type $(\alpha, \beta, \gamma, \delta)$. If α is not odd then $(\alpha)_{2i} \neq 0$ for some $i \geq 1$. As a result we will obtain a marked source of divergence $-2i$. This means that the edge adjacent to this source is of even weight, and is not in $\mathfrak{F}(\mathcal{D}, \mathfrak{M}, r)$ since it is unmarked. Therefore $\mu_r^{\mathbb{R}}(\mathcal{D}, \mathfrak{M}) = 0$ and consequently $C^{\alpha,\beta,\gamma,\delta}(d, r) = 0$. If $(\beta)_{2i} \neq 0$ for some $i \geq 1$, then $\mathcal{D}$ has $(\beta)_{2i} + 2(\delta)_{2i}$ edges in $Edge^{\infty}(\mathcal{D})$ of weight $2i$, but only $2(\delta)_{2i}$ of them belong to $\mathfrak{F}(\mathcal{D}, \mathfrak{M}, r)$. Therefore $(\mathcal{D}, \mathfrak{M})$ is not $r - real$ and its multiplicity is zero.

We need some definitions to state the next theorem.

Definition 3.4. *Given three integer numbers $l \geq 0$, $m \geq 0$ and $r \geq 0$ and four vectors α, β, γ and δ in $\mathbb{N}^{\infty}$, let $\mathbf{S}_w(l, m, r, \alpha, \beta, \gamma, \delta)$ be the set of vectors*

$(d_1, \ldots, d_l, k_1, \ldots, k_l, \gamma_1, \ldots, \gamma_l, \delta_1, \ldots, \delta_l, d_1', \ldots, d_m', k_1', \ldots, k_m', r_1', \ldots, r_m', \alpha_1', \ldots, \alpha_m',$

$\beta_1', \ldots, \beta_m', \gamma_1', \ldots, \gamma_m', \delta_1', \ldots, \delta_m')$ *in* $(\mathbb{N}^*)^{2l} \times (\mathbb{N}^{\infty})^{2l} \times (\mathbb{N}^*)^{2m} \times \mathbb{N}^m \times (\mathbb{N}^{\infty})^{4m}$ *satisfying :*

- $\forall i, (d_i', k_i', r_i', \alpha_i', \beta_i', \gamma_i', \delta_i') \leq (d_{i+1}', k_{i+1}', r_{i+1}', \alpha_{i+1}', \beta_{i+1}', \gamma_{i+1}', \delta_{i+1}')$ *for the lexicographic order,*

- $\sum \alpha_i' \leq \alpha$,

- $\forall i, k_i'$ *is odd,*

- $\forall i, \beta_i^{'} \geq e_{k_i^{'}}$,

- $\sum(\beta_i^{'} - e_{k_i^{'}}) = \beta$,

- $\sum \gamma_i^{'} \leq \gamma$,

- $\sum \delta_i^{'} \leq \delta$,

- $\forall i, I\alpha_i^{'} + I\beta_i^{'} + 2I\gamma_i^{'} + 2I\delta_i^{'} = d_i^{'}$,

- $(d_1, \ldots, d_l, k_1, \ldots, k_l, \gamma_1, \ldots, \gamma_l, \delta_1, \ldots, \delta_l) \in \mathbf{S}(\frac{d - \sum d_i^{'} + 1}{2}, l, \gamma - \sum \gamma_i^{'}, \delta - \sum \delta_i^{'})$,

- $\sum(2d_i - 1 + |\delta_i|) + \sum r_i^{'} = r$,

- $\forall i, 2d_i^{'} - 1 + |\beta_i^{'} + 2\delta_i^{'}| - r_i^{'} \geq 0$.

By definition any element s of $\boldsymbol{S}_w(l, m, r, \alpha, \beta, \gamma, \delta)$ defines an element $s^{'}$ of $\boldsymbol{S}(\frac{d - \sum d_i^{'} + 1}{2}, l, \gamma - \sum \gamma_i^{'}, \delta - \sum \delta_i^{'})$. We denote by $\sigma(s)$ the integer $\sigma(s^{'})$ given in definition 3.1. For $s \in \boldsymbol{S}_w(l, m, r, \alpha, \beta, \gamma, \delta)$, let $\mathcal{E}(s)$ be the set of all j in $\{1, \ldots, m\}$ such that $\beta_j^{'} \geq e_{k_j^{'}}$ and $2d_j^{'} - 1 + |\beta_j^{'} + 2\delta_j^{'}| - r_j^{'} = 1$. Given an element $j \in \{1, \ldots, m\}$, we denote by $\simeq_s^j$ the restriction to $\{1, \ldots, m\} \backslash \{j\}$ of the equivalence relation $\simeq_s$. For each of the equivalent classes of $\simeq_s^j$, evaluate the factorial of its cardinality, and denote by $\sigma^{'}(s,j)$ the product of these factorials.

Definition 3.5. *Given two integer numbers $l \geq 0$ and $r \geq 0$ and four vectors α, β, γ and δ in $\mathbb{N}^\infty$, let $\tilde{\mathbf{S}_w}(l, r, \alpha, \beta, \gamma, \delta)$ be the set of vectors*

$(d_1, \ldots, d_l, k_1, \ldots, k_l, \gamma_1, \ldots, \gamma_l, \delta_1, \ldots, \delta_l, d_1^{'}, k_1^{'}, r_1^{'}, \gamma_1^{'}, \delta_1^{'})$ in

$(\mathbb{N}^*)^{2l} \times (\mathbb{N}^\infty)^{2l} \times (\mathbb{N}^*)^2 \times \mathbb{N} \times (\mathbb{N}^\infty)^2$ satisfying:

- $\gamma_1^{'} \leq \gamma$,
- $0 \leq \delta_1^{'} - e_{k_1^{'}} \leq \delta$,
- $I\alpha + I\beta + 2I\delta_1^{'} + 2I\delta_1^{'} = d_1^{'}$,
- $(d_1, \ldots, d_l, k_1, \ldots, k_l, \gamma_1, \ldots, \gamma_l, \delta_1, \ldots, \delta_l) \in \mathbf{S}(\frac{d-d_1^{'}}{2}, l, \gamma - \gamma_1^{'}, \delta - \delta_1^{'} + e_{k_1^{'}})$,
- $\sum(2d_i - 1 + |\delta_i|) + r_1^{'} = r$,
- $2d_1^{'} - 1 + |\beta + 2\delta_1^{'}| - r_1^{'} \geq 0$.

By definition, any element s of $\widetilde{\boldsymbol{S}}_w(l,r,\alpha,\beta,\gamma,\delta)$ defines an element $s^{'}$ of

$\boldsymbol{S}(\frac{d-d_1^{'}}{2}, l, \gamma-\gamma_1^{'}, \delta-\delta_1^{'}+e_{k_1^{'}})$. We denote by $\sigma(s)$ the integer $\sigma(s^{'})$ given in definition 3.1.

Given an element s of $\boldsymbol{S}_w(l,m,r,\alpha,\beta,\gamma,\delta)$ or $\widetilde{\boldsymbol{S}}_w(l,r,\alpha,\beta,\gamma,\delta)$, we put

$\Theta(s) =$

$$\binom{r}{2d_1-1+|\delta_1|,\ldots,2d_l-1+|\delta_l|,r_1^{'},\ldots,r_m^{'}}\binom{\alpha}{\alpha_1^{'},\ldots,\alpha_m^{'}}\binom{\gamma}{\gamma_1,\ldots,\gamma_l,\gamma_1^{'},\ldots,\gamma_m^{'}}\prod_{i=1}^{l}(-1)^{d_i}(\delta_i)_{k_i}k_i 2^{2d_i-2+|\delta_i+\gamma_i|}N^{\gamma_i,\delta_i}(d_i)$$

where if s is in $\widetilde{\boldsymbol{S}}_w(l,r,\alpha,\beta,\gamma,\delta)$, then there are no $\alpha_i^{'}$ elements and we set the value of the corresponding multinomial coefficient equal to 1.

Theorem 3.6. *The numbers $C^{\alpha,\beta,\gamma,\delta}(d,r)$, with α and β odd, are given by the initial values $C^{e_1,0,0,0}(0,1) = C^{0,e_1,0,0}(1,1) = 1$ and the relations:*

- *Case I: if $2d-1+|\beta+2\delta|-2r > 0$, then*

$$C^{\alpha,\beta,\gamma,\delta}(d,r) = \sum_{k odd|\beta\geq e_k} C^{\alpha+e_k,\beta-e_k,\gamma,\delta}(d,r) + \sum_{l,m\geq 0 s\in\mathbf{S}_w(l,m,r,\alpha,\beta,\gamma,\delta)}$$
$$\left[\frac{\Theta(s)}{\sigma(s)\sigma^{'}(s)}\prod_{i=1}^{m}(\beta_i^{'})_{k_i^{'}}C^{\alpha_i^{'},\beta_i^{'},\gamma_i^{'},\delta_i^{'}}(d_i^{'},r_i^{'})\binom{2d-2+|\beta+2\delta|-2r}{2d_1^{'}-1+|\beta_1^{'}+2\delta_1^{'}|-2r_1^{'},\ldots,2d_m^{'}-1+|\beta_m^{'}+2\delta_m^{'}|-2r_m^{'}}\right];$$

- *Case II : if $2d-1+|\beta+2\delta|-2r = 0$, then*

$$C^{\alpha,\beta,\gamma,\delta}(d,r) = \sum_{k|\delta\geq e_k} kC^{\alpha,\beta,\gamma+e_k,\delta-e_k}(d,r-1) +$$

$$\sum_{l,m\geq 0,\mathbf{K}odd|\beta\geq e_{\mathbf{K}},s\in\mathbf{S}_w(l,m,r-1,\alpha,\beta-e_{\mathbf{K}},\gamma,\delta)} \frac{\mathbf{K}\Theta(s)}{\sigma(s)\sigma'(s)}\prod_{i=1}^{m}(\beta_i')_{k_i'}C^{\alpha_i',\beta_i',\gamma_i',\delta_i'}(d_i',r_i') +$$

$$\sum_{l,m\geq 0,s\in\mathbf{S}_w(l,m,r-1,\alpha,\beta,\gamma,\delta),j\in\mathcal{E}(s)} \frac{\Theta(s)k_j'C^{\alpha_j'+e_{k_j'},\beta_j'-e_{k_j'},\gamma_j',\delta_j'}(d_j',r_j')}{\sigma(s)\widetilde{\sigma}'(s,j)}\prod_{i=1,i\neq j}^{m}(\beta_i')_{k_i'}C^{\alpha_i',\beta_i',\gamma_i',\delta_i'}(d_i',r_i') -$$

$$\sum_{l\geq 0,s\in\tilde{\mathbf{S}}_w(l,r-1,\alpha,\beta,\gamma,\delta)} \frac{\Theta(s)}{\sigma(s)}2^{|\gamma-\gamma_1'-\sum_{i=1}^{l}|+1}(\delta_1')_{k_1'}k_1'C^{\alpha,\beta,\delta_1',\delta_1'}(d_1',r_1')$$

<u>Proof of theorem 3.6:</u>

Let $r\geq 0$ and $d>0$ and let α,β,γ and δ be four vectors in $\mathbb{N}^\infty$ with α and β odd. We will consider two cases.

Case I: when $2d-1+|\beta+2\delta|-2r>0$.

Recall that

$$C^{\alpha,\beta,\gamma,\delta}(d,r)=\sum_{(\mathcal{D},\mathfrak{M})}\mu_r^{\mathbb{R}}(\mathcal{D},\mathfrak{M})$$

where the sum is taken over all marked floor diagrams of type $(\alpha,\beta,\gamma,\delta)$. Note that $\mathfrak{M}(|\alpha+2\gamma|+1)$ is not a source, because the maximum number assigned to a source is equal to $|\alpha+2\gamma|$. So $|\alpha+2\gamma|+1$ is assigned either to an edge (adjacent to a source or not) or to a vertex which is not a source. The option of having $\mathfrak{M}(|\alpha+2\gamma|+1)$ as an edge which is not adjacent to a source is discarded because if this was the case then condition 3 in definition 2.6 would fail. Condition 3 in definition 2.6 also implies that if

$\mathfrak{M}(|\alpha + 2\gamma| + 1) = v \in Vert(\mathcal{D})$, then v must be adjacent to at least one unmarked edge in

$Edge^{\infty}(\mathcal{D})$ and cannot be adjacent to any marked edge in $Edge^{\infty}(\mathcal{D})$. So one can write:

$$C^{\alpha,\beta,\gamma,\delta}(d,r) = \sum_{(\mathcal{D},\mathfrak{M})\in \boldsymbol{A}} \mu_r^{\mathbb{R}}(\mathcal{D},\mathfrak{M}) + \sum_{(\mathcal{D},\mathfrak{M})\in \boldsymbol{B}} \mu_r^{\mathbb{R}}(\mathcal{D},\mathfrak{M})$$

where

$\boldsymbol{A}$ is the set of all $r - real$ marked floor diagrams of degree d and type $(\alpha, \beta, \gamma, \delta)$ such that

$\mathfrak{M}(|\alpha + 2\gamma| + 1) \in Edge^{\infty}(\mathcal{D})$, and

$\boldsymbol{B}$ is the set of all $r - real$ marked floor diagrams of degree d and type $(\alpha, \beta, \gamma, \delta)$ such that

$\mathfrak{M}(|\alpha + 2\gamma| + 1) \in Vert(\mathcal{D})$.

Case I-A: Marked floor diagrams in $\boldsymbol{A}$.

Let $K = \{k : \beta \geq e_k\}$ and let $\boldsymbol{A}_k$ be the set of all $r - real$ marked floor diagrams of degree

d and type $(\alpha + e_k, \beta - e_k, \gamma, \delta)$. Let $\Phi : \boldsymbol{A} \to \bigcup_{k\in K} \boldsymbol{A}_k$ be the bijection defined as follows:

If $(\mathcal{D}, \mathfrak{M}) \in \boldsymbol{A}$ and k is the weight of the edge $\mathfrak{M}(|\alpha + 2\gamma| + 1)$, let

$$\Phi(\mathcal{D},\mathfrak{M}) = (\mathcal{D},\mathfrak{M}')$$

where $\mathfrak{M}'$ is the marking of $\mathcal{D}$ of type $(\alpha+e_k, \beta-e_k, \gamma, \delta)$ defined as follows.

- $\mathfrak{M}'(i)$=$\mathfrak{M}(i)$ if $i \le \sum_{j=1}^{k}(\alpha)_j$ or $i \ge |\alpha+2\gamma|+2$;
- $\mathfrak{M}'\left(\sum_{j=1}^{k}(\alpha)_j+1\right)$ is the source adjacent to $\mathfrak{M}\left(|\alpha+2\gamma|+1\right)$;
- $\mathfrak{M}'(i) = \mathfrak{M}(i-1)$ if $\sum_{j=1}^{k}(\alpha)_j+2 \le i \le |\alpha+2\gamma|+1$;

Note that $n-2r+1 > |\alpha+2\gamma|+1$ since $2d-1+|\beta+2\delta|-2r > 0$. So $\mathfrak{M}(i) = \mathfrak{M}'(i)\forall i \ge n-2r+1$. Moreover, the elements in $\mathfrak{F}(\mathcal{D},\mathfrak{M},r)$ are exactly the same elements of $\mathfrak{F}(\mathcal{D},\mathfrak{M}',r)$. And clearly, the weight of the edges and the vector δ never change. So,

$$\mu_r^{\mathbb{R}}(\mathcal{D},\mathfrak{M}) = (-1)^{\frac{\#(Vert(\mathcal{D})\bigcap\mathfrak{F}(\mathcal{D},\mathfrak{M},r))}{2}}\boldsymbol{I}^{-\delta}\prod_{e\in Edge(\mathcal{D})\cap\mathfrak{M}(\{n-2r+1,\ldots,n\})} w(e) = \mu_r^{\mathbb{R}}(\mathcal{D},\mathfrak{M}').$$

Therefore,

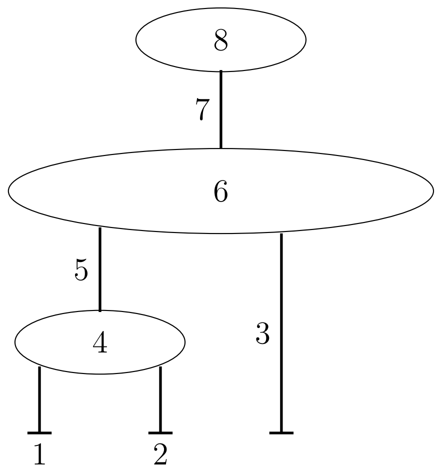
$$\sum_{(\mathcal{D},\mathfrak{M})\in\boldsymbol{A}}\mu_r^{\mathbb{R}}(\mathcal{D},\mathfrak{M}) = \sum_{k|\beta\ge e_k}\sum_{(\mathcal{D},\mathfrak{M}')\in\boldsymbol{A}_k}\mu_r^{\mathbb{R}}(\mathcal{D},\mathfrak{M}')$$

Note that since β is odd, k must be odd, hence one has

$$\sum_{(\mathcal{D},\mathfrak{M})\in \boldsymbol{A}} \mu_r^{\mathbb{R}}(\mathcal{D},\mathfrak{M}) = \sum_{kodd|\beta\geq e_k} C^{\alpha+e_k,\beta-e_k,\gamma,\delta}(d,r).$$

Example 3.7.

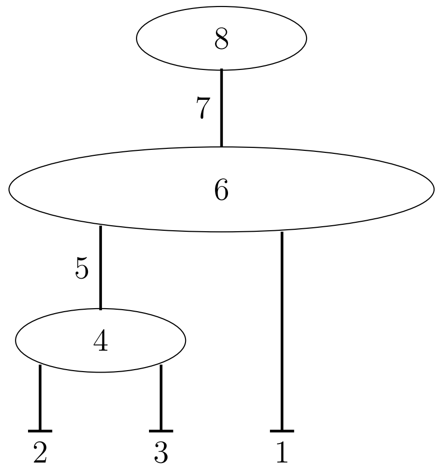
Let $(\mathcal{D},\mathfrak{M})$ be the following $2-real$ marked floor diagram of degree 3, genus 0 and type $(0,e_1,e_1,0)$.



First, $2d-1+|\beta+2\delta|-2r=2>0$, so we are in case I. Second, $\mathfrak{M}(|\alpha+2\gamma|+1)=\mathfrak{M}(3)$ is an edge adjacent to a source, so we are in set $\boldsymbol{A}$.

The 2-pairs are $\{1,2\}$, $\{5,6\}$ and $\{7,8\}$. Since $\mathfrak{M}(5)$ is adjacent to $\mathfrak{M}(6)$, and $\mathfrak{M}(7)$ is adjacent to $\mathfrak{M}(8)$, but $\mathfrak{M}(1)$ is not adjacent to $\mathfrak{M}(2)$, $\mathfrak{F}(\mathcal{D},\mathfrak{M},2)=\{\mathfrak{M}(1),\mathfrak{M}(2)\}$. All edges in $\mathfrak{M}(\{5,6,7,8\})$ have weight 1. Applying the formula of $r-real$ multiplicity, we get $\mu_2^{\mathbb{R}}(\mathcal{D},\mathfrak{M})$=1.

Applying the new marking $\mathfrak{M}'$ on $\mathcal{D}$, we will obtain:



$\mathfrak{M}'(1)$ is the source adjacent to $\mathfrak{M}(3)$. $\mathfrak{M}'(2)=\mathfrak{M}(1)$, $\mathfrak{M}'(3)=\mathfrak{M}(2)$ and $\mathfrak{M}'(i)=\mathfrak{M}(i)$ for $i>3$. To find $\mu_2^{\mathbb{R}}(\mathcal{D},\mathfrak{M}')$: Note that $(\mathcal{D},\mathfrak{M}')$ is of type $(e_1,0,e_1,0)$, $\mathfrak{F}(\mathcal{D},\mathfrak{M}',2)=\{\mathfrak{M}(2),\mathfrak{M}(3)\}$. All edges in $\mathfrak{M}(\{5,6,7,8\})$ have weight 1. So $\mu_2^{\mathbb{R}}(\mathcal{D},\mathfrak{M}')$=1.

Case I-B: Marked floor diagrams in $\boldsymbol{B}$.

Let l and m be two non-negative integer numbers such that the set $\boldsymbol{S}_w(l,m,r,\alpha,\beta,\gamma,\delta)$ is not empty. For each s in $\boldsymbol{S}_w(l,m,r,\alpha,\beta,\gamma,\delta)$, let $\mathfrak{B}(s)$ be the set of all $2l+m$-tuples,

$\big((\mathcal{D}_1, \mathfrak{M}_1), \dots, (\mathcal{D}_{2l}, \mathfrak{M}_{2l}), (\mathcal{D}_1^{'}, \mathfrak{M}_1^{'}), \dots, (\mathcal{D}_m^{'}, \mathfrak{M}_m^{'})\big)$ where $(\mathcal{D}_{2i-1}, \mathfrak{M}_{2i-1})$ and $(\mathcal{D}_{2i}, \mathfrak{M}_{2i})$ are two equivalent marked floor diagrams of degree d_i and type $(\gamma_i, \delta_i, 0, 0)$, and $(\mathcal{D}_i^{'}, \mathfrak{M}_i^{'})$ is an $r_i^{'} - real$ marked floor diagram of degree $d_i^{'}$ and type $(\alpha_i^{'}, \beta_i^{'}, \gamma_i^{'}, \delta_i^{'})$. Let

$\Phi_i : (\mathcal{D}_{2i-1}, \mathfrak{M}_{2i-1}) \longrightarrow (\mathcal{D}_{2i}, \mathfrak{M}_{2i})$ be a homeomorphism establishing the equivalence of the two diagrams.

To construct several elements of $\boldsymbol{B}$, take an element of $\mathfrak{B}(s)$ and proceed in the following way:

1. For all i in $\{1, \dots, l\}$ choose an element a_i of $Edge^{\infty}(\mathcal{D}_{2i-1})$ which is in the image of $\mathfrak{M}_{2i-1}$ and of weight k_i. Since $\delta_i \geq e_{k_i}$, it is always possible to choose such an a_i.

   The total number of ways of selecting $(a_1, \dots, a_l)$ is $\prod_{i=1}^{l} (\delta_i)_{k_i}$.

2. For all i in $\{1, \dots, m\}$ choose an element $a_i^{'}$ of $Edge^{\infty}(\mathcal{D}_i^{'})$ which is in the image of $\mathfrak{M}_i^{'}$ but not in $\mathfrak{F}(\mathcal{D}_i^{'}, \mathfrak{M}_i^{'}, r_i^{'})$, and of weight $k_i^{'}$. Since $\beta_i^{'} \geq e_{k_i^{'}}$, it is always possible to choose such an $a_i^{'}$.

   The total number of ways of selecting $(a_1^{'}, \dots, a_m^{'})$ is $\prod_{i=1}^{m} (\beta_i^{'})_{k_i}$.

3. Construct a new oriented tree $\widetilde{\mathcal{D}}$ out of

   $(\mathcal{D}_1, \mathfrak{M}_1), \ldots, (\mathcal{D}_{2l}, \mathfrak{M}_{2l}), (\mathcal{D}_1^{'}, \mathfrak{M}_1^{'}), \ldots, (\mathcal{D}_m^{'}, \mathfrak{M}_m^{'})$ by identifying all the sources adjacent to the edges $a_i, \phi(a_i)$, and $a_j^{'}$. Denote this vertex by v.

4. By adding sources and edges adjacent to the vertex v, complete $\widetilde{\mathcal{D}}$ into a unique floor diagram $\mathcal{D}$ of degree d, genus 0, with $(\alpha)_j + (\beta)_j + 2(\gamma)_j + 2(\delta)_j$ edges in $Edge^{\infty}(\mathcal{D})$ of weight j for all $j \geq 1$. Denote by $v_1, \ldots, v_t$ the sources added.

5. Define $\alpha_{m+1}^{'}$=$\alpha - \sum_{i=1}^{m} \alpha_i^{'}$ and $\gamma_{m+1}^{'} = \gamma - \sum_{i=1}^{l} \gamma_i - \sum_{i=1}^{m} \gamma_i^{'}$.

6. For all $j \geq 1$, choose a partition $(I_i^j)_{1 \leq i \leq m+1}$ of the set $\{1, \ldots, (\alpha)_j\}$ such that $\#I_i^j = (\alpha_i^{'})_j$ for all i.

   The number of possible choices is $\prod\limits_{j=1}^{\infty} \binom{(\alpha)_j}{(\alpha_1^{'})_j, \ldots, (\alpha_m^{'})_j} = \binom{\alpha}{\alpha_1^{'}, \ldots, \alpha_m^{'}}$.

7. For all $j \geq 1$, choose a partition $(\hat{I}_i^j)_{1 \leq i \leq l} \cup (\widetilde{I}_i^j)_{1 \leq i \leq m+1}$ of the set $\{1, \ldots, (\gamma)_j\}$ such that $\#\hat{I}_i^j = (\gamma_i)_j$ and $\#\widetilde{I}_i^j = (\gamma_i^{'})_j$ for all i.

   The number of possible choices is $\binom{\gamma}{\gamma_1, \ldots, \gamma_l, \gamma_1^{'}, \ldots, \gamma_m^{'}}$.

8. Choose a partition $(J_i)_{1\leq i\leq m}$ of the set $\{1,\ldots,2d-2+|\beta+2\delta|-2r\}$ such that

   $\#J_i = 2d_i^{'}-1+|\beta_i^{'}+2\delta_i^{'}|-2r_i^{'}$ for all i.

   The number of possible choices is $\binom{2d-2+|\beta+2\delta|-2r}{2d_1^{'}-1+|\beta_1^{'}+2\delta_1^{'}|-2r_1^{'},\ldots,2d_m^{'}-1+|\beta_m^{'}+2\delta_m^{'}|-2r_m^{'}}$.

9. Choose a partition $(\hat{J}_i)_{1\leq i\leq l}\cup(\widetilde{J}_i)_{1\leq i\leq m}$ of the set $\{1,\ldots,r\}$ such that

   $\#\hat{J}_i = 2d_i-1+|\delta_i|$ and $\#\widetilde{J}_i = r_i^{'}$ for all i.

   The number of possible choices is $\binom{r}{2d_1-1+|\delta_1|,\ldots,2d_l-1+|\delta_l|,r_1^{'},\ldots,r_m^{'}}$.

10. For all i in $\{1,\ldots,l\}$, choose a vector $\mathcal{E}_i$ in $\{0,1\}^{2d_i-1+|\gamma_i+\delta_i|}$.

    The number of possible choices is $\prod_{i=1}^{l} 2^{d_i-1+|\gamma_i+\delta_i|} = 2^l\prod_{i=1}^{l} 2^{2d_i-2+|\gamma_i+\delta_i|}$.

11. Choose a marking $\mathfrak{M}$ of $\mathcal{D}$ of type $(\alpha,\beta,\gamma,\delta)$ such that

    a- $\mathfrak{M}(|\alpha|+2|\gamma|+1)=v$,

    b- for all j ≥ 1 and all k in I_{m+1}^j, $\mathfrak{M}\left(\sum_{i=1}^{j-1}(\alpha)_t+k\right)$ is a source v_q (see step (4)) of $\mathcal{D}$ of divergence $-j$. Note that different choices of v_q produce equivalent marked floor diagrams, c- for all $j \geq 1$ and all k in $\widetilde{I}^j_{m+1}$, $\mathfrak{M}(|\alpha| + 2\sum_{t=1}^{j-1}(\gamma)_t + 2k - 1)$ and $\mathfrak{M}(|\alpha| + 2\sum_{t=1}^{j-1}(\gamma)_t + 2k)$ are two sources v_q and $v_{q'}$ of $\mathcal{D}$ of divergence $-j$. Note that different choices of v_q and $v_{q'}$ produce equivalent floor diagrams,

d- for all $j \geq 1$ and all i in $\{1, \ldots, m\}$, if k is the h-th element (for the natural ordering of I^j_i) of I^j_i, then

$$\mathfrak{M}\left(\sum_{t=1}^{j-1}(\alpha)_t + k\right) = \mathfrak{M}'_j\left(\sum_{t=1}^{j-1}(\alpha'_j)_t + h\right)$$

Since $I^j_i \subset \{1, \ldots, (\alpha)_j\}$ and the cardinality of I^j_i is $(\alpha'_i)_j$, $1 \leq h \leq (\alpha'_i)_j$ and $1 \leq k \leq (\alpha)_j$. So we get the following inequalities:

$1 \leq \sum_{t=1}^{j-1}(\alpha)_t + k \leq |\alpha|$, these correspond to sources in $(\mathcal{D}, \mathfrak{M})$.

$1 \leq \sum_{t=1}^{j-1}(\alpha'_j)_t + h \leq |\alpha'_i|$, these correspond to sources in $(\mathcal{D}'_i, \mathfrak{M}'_i)$.

e- for all $j \geq 1$ and all i in $\{1, \ldots, l\}$, if k is the h-th element of $\widehat{I}^j_i$, then

$$\mathfrak{M}\left(|\alpha| + 2\sum_{t=1}^{j-1}(\gamma)_t + 2k - 1 + (\mathcal{E}_i)_{\sum_{t=1}^{j-1}(\gamma_i)_t + h}\right) = \mathfrak{M}_{2i-1}\left(\sum_{t=1}^{j-1}(\gamma_i)_t + h\right)$$

and

$$\mathfrak{M}\left(|\alpha| + 2\sum_{t=1}^{j-1}(\gamma)_t + 2k - (\mathcal{E}_i)_{\sum_{t=1}^{j-1}(\gamma_i)_t + h}\right) = \phi_i \circ \mathfrak{M}_{2i-1}\left(\sum_{t=1}^{j-1}(\gamma_i)_t + h\right).$$

Since $\hat{I}_i^j \subset \{1, \ldots, (\gamma)_j\}$ and the cardinality is $\#\hat{I}_i^j = (\gamma_i)_j$, $1 \leq h \leq (\gamma_i)_j$ and

$1 \leq k \leq (\gamma)_j$, we get the following inequalities:

$|\alpha| + 1 \leq |\alpha| + 2\sum_{t=1}^{j-1}(\gamma)_t + 2k - 1 + (\mathcal{E}_i)_{\sum_{t=1}^{j-1}(\gamma_i)_{t+h}} \leq |\alpha| + 2|\gamma|$, these correspond to

marked sources in $(\mathcal{D}, \mathfrak{M})$.

$1 \leq \sum_{t=1}^{j-1}(\gamma_i)_t + h \leq |\gamma_i|$, these correspond to all the marked sources in $(\mathcal{D}_{2i-1}, \mathfrak{M}_{2i-1})$.

$1 + |\alpha| \leq |\alpha| + 2\sum_{t=1}^{j-1}(\gamma)_t + 2k - (\mathcal{E})_{\sum_{t=1}^{j-1}(\gamma_i)_{t+h}} \leq |\alpha| + 2|\gamma|$, these correspond to marked

sources in $(\mathcal{D}, \mathfrak{M})$.

$1 \leq \sum_{t=1}^{j-1}(\gamma_i)_t + h \leq |\gamma_i|$, these correspond to all the marked sources in $(\mathcal{D}_{2i}, \mathfrak{M}_{2i})$.

We can conclude that there are exactly $(\gamma_i)_j$ marked sources of divergence j in

$(\mathcal{D},\mathfrak{M})$ corresponding to the marked sources in $(\mathcal{D}_{2i-1},\mathfrak{M}_{2i-1})$. The edges adjacent to these marked sources contribute to $\mu^{\mathbb{C}}(\mathcal{D}_{2i-1})$ but not to $\mu^{\mathbb{R}}(\mathcal{D}_{2i-1},\mathfrak{M}_{2i-1})$. The product of the weights of these edges is I^{γ_i}. The same is true for $(\mathcal{D}_{2i},\mathfrak{M}_{2i})$. Moreover, if $\mathfrak{M}_{2i-1}(c)$ is a marked source in $(\mathcal{D}_{2i-1},\mathfrak{M}_{2i-1})$ then $\mathfrak{M}_{2i-1}(c)$ and $\phi\circ\mathfrak{M}_{2i-1}(c)$ correspond to images of an $r-pair$ in $(\mathcal{D},\mathfrak{M})$ and belong to $\mathfrak{F}(\mathcal{D},\mathfrak{M},r)$.

f- for all $j\geq 1$ and all i in $\{1,\dots,m\}$, if k is the h-th element of $\widetilde{I}_i^j$, then

$$\mathfrak{M}\left(|\alpha|+2\sum_{t=1}^{j-1}(\gamma)_t+2k-1\right)=\mathfrak{M}_i^{'}\left(|\alpha_i^{'}|+2\sum_{t=1}^{j-1}(\gamma_i^{'})_t+2h-1\right)$$

and

$$\mathfrak{M}\left(|\alpha|+2\sum_{t=1}^{j-1}(\gamma)_t+2k\right)=\mathfrak{M}_i^{'}\left(|\alpha_i^{'}|+2\sum_{t=1}^{j-1}(\gamma_i^{'})_t+2h\right).$$

Since $\widetilde{I}_i^j\subset\{1,\dots,(\gamma)_j\}$ and the cardinality of $\widetilde{I}_i^j$ is $\#\widetilde{I}_i^j=(\gamma_i^{'})_j$, $1\leq h\leq(\gamma_i^{'})_j$ and $1\leq k\leq(\gamma)_j$.So we get the following inequalities:

$$|\alpha| + 1 \leq |\alpha| + 2\sum_{t=1}^{j-1}(\gamma)_t + 2k - 1 \leq |\alpha| + 2|\gamma| - 1$$

$$|\alpha| + 2 \leq |\alpha| + 2\sum_{t=1}^{j-1}(\gamma)_t + 2k \leq |\alpha| + 2|\gamma|$$

All these correspond to marked sources of $(\mathcal{D}, \mathfrak{M})$ that belong to $\mathfrak{F}(\mathcal{D}, \mathfrak{M}, r)$.

$$|\alpha_i^{'}| + 1 \leq |\alpha_i^{'}| + 2\sum_{t=1}^{j-1}(\gamma_i^{'})_t + 2h - 1 \leq |\alpha_i^{'}| + 2|\gamma_i^{'}| - 1$$

$$|\alpha_i^{'}| + 2 \leq |\alpha_i^{'}| + 2\sum_{t=1}^{j-1}(\gamma_i^{'})_t + 2h \leq |\alpha_i^{'}| + 2|\gamma_i^{'}|$$

These correspond to all the marked sources of $(\mathcal{D}_i^{'}, \mathfrak{M}_i^{'})$ that belong to $\mathfrak{F}(\mathcal{D}_i^{'}, \mathfrak{M}_i^{'}, r_i^{'})$.

g- for all i in $\{1, \ldots, m\}$, if k is the h-th element of J_i, then

$$\mathfrak{M}\left(|\alpha| + 2|\gamma| + k + 1\right) = \mathfrak{M}_i^{'}\left(|\alpha_i^{'}| + 2|\gamma_i^{'}| + h\right).$$

Since $J_i \subset \{1, \ldots, 2d - 2 + |\beta + 2\delta| - 2r\}$ and the cardinality

$\#J_i = 2d_i^{'} - 1 + |\beta_i^{'} + 2\delta_i^{'}| - 2r_i^{'}$; $1 \leq h \leq 2d_i^{'} - 1 + |\beta_i^{'} + 2\delta_i^{'}| - 2r_i^{'}$ and

$1 \leq k \leq 2d - 2 + |\beta + 2\delta| - 2r$, we get the following inequalities:

$$|\alpha| + 2|\gamma| + 2 \leq |\alpha| + 2|\gamma| + k + 1 \leq |\alpha| + 2|\gamma| + 2d - 2 + |\beta + 2\delta| - 2r + 1 = n - 2r$$

$$|\alpha_i^{'}| + 2|\gamma_i^{'}| + 1 \leq |\alpha_i^{'}| + 2|\gamma_i^{'}| + h \leq |\alpha_i^{'}| + 2|\gamma_i^{'}| + 2d_i^{'} - 1 + |\beta_i^{'} + 2\delta_i^{'}| - 2r_i^{'} = n_i^{'} - 2r_i^{'}.$$

h- for all i in $\{1,\ldots,l\}$, if k is the h-th element of $\widehat{J}_i$, then (recall that $n = 2d - 1 + |\alpha + \beta + 2\gamma + 2\delta|$)

$$\mathfrak{M}(n - 2k + 1 + (\mathcal{E}_i)_{|\gamma_i|+h}) = \mathfrak{M}_{2i-1}(2d_i + |\gamma_i| + |\delta_i| - h)$$

and

$$\mathfrak{M}(n - 2k + 2 - (\mathcal{E}_i)_{|\gamma_i|+h}) = \phi_i \circ \mathfrak{M}_{2i-1}(2d_i + |\gamma_i| + |\delta_i| - h)$$

Since $\hat{J}_i \subset \{1,\ldots,r\}$ and has cardinality $\#\hat{J}_i = 2d_i - 1 + |\delta_i|$,$1 \le h \le 2d_i - 1 + |\delta_i|$, we get the following inequality:

$|\gamma_i| + 1 \le 2d_i + |\gamma_i| + |\delta_i| - h \le 2d_i + |\gamma_i| + |\delta_i| - 1 = n_i$.

This shows that $\mathfrak{M}_{2i-1}(2d_i + |\gamma_i| + |\delta_i| - h)$ is not a source.

Also, since $\hat{J}_i \subset \{1,\ldots,r\}$ and $k \in \hat{J}_i$, we get the following inequalities:

$n - 2r + 1 \le n - 2k + 1 + (\mathcal{E}_i)_{|\gamma_i|+h} \le n$

$n - 2r + 2 \le n - 2k + 2 + (\mathcal{E}_i)_{|\gamma_i|+h} \le n + 1$.

This shows that all marked elements of $(\mathcal{D}_{2i}, \mathfrak{M}_{2i})$ and $(\mathcal{D}_{2i-1}, \mathfrak{M}_{2i-1})$ that are not sources correspond to elements in $\mathcal{D}$ marked by a number greater than $n - 2r$.

Furthermore, for these elements, $\mathfrak{M}_{2i-1}(j)$ and $\phi_i \circ \mathfrak{M}_{2i-1}(j)$ is an $r - pair$ in $(\mathcal{D}, \mathfrak{M})$.

i- for all i in $\{1, \ldots, m\}$, if k is the h-the element of $\widetilde{J}_i$, then

$$\mathfrak{M}(n - 2k + 1) = \mathfrak{M}_i^{'}(n_i^{'} - 2h + 1)$$

$$and$$

$$\mathfrak{M}(n - 2k + 2) = \mathfrak{M}_i^{'}(n_i^{'} - 2h + 2)$$

Since $\widetilde{J}_i \subset \{1, \ldots, r\}$ and the cardinality, $\#\widetilde{J}_i = r_i^{'}$, $1 \leq h \leq r_i^{'}$ and $1 \leq k \leq r$, we get the following inequalities:

$n_i^{'} - 2r_i^{'} + 1 \leq n_i^{'} - 2h + 1 \leq n_i^{'} - 1,$

$n_i^{'} - 2r_i^{'} + 2 \leq n_i^{'} - 2h + 2 \leq n_i^{'},$

$n - 2r + 1 \leq n - 2k + 1 \leq n - 1$, and

$n-2r+2 \leq n-2k+2 \leq n.$

By $(d),(f),(g)$ and (i) we can say that the edges in $\mathcal{D}_i^{'}$ corresponding to edges in $\mathcal{D}$

marked by a number in the set $\{1,\ldots,n-2r\}$ are precisely the elements of

$\mathfrak{M}_i^{'}(1,\ldots,n_i^{'}-2r_i^{'}).$

This gives us several elements of $\boldsymbol{B}$.

Proposition 3.8. *For each element* $(\mathcal{D},\mathfrak{M})$ *constructed above:*

$$\mu_r^{\mathbb{R}}(\mathcal{D},\mathfrak{M}) = \prod_{i=1}^{l}(-1)^{d_i}k_i\mu^{\mathbb{C}}(\mathcal{D}_{2i},\mathfrak{M}_{2i})\prod_{i=1}^{m}\mu_{r_i^{'}}^{\mathbb{R}}(\mathcal{D}_i^{'},\mathfrak{M}_i^{'}).$$

<u>Proof:</u> Recall that

$$\mu_r^{\mathbb{R}}(\mathcal{D},\mathfrak{M}) = (-1)^{\frac{\#(Vert(\mathcal{D})\bigcap\mathfrak{F}(\mathcal{D},\mathfrak{M},r))}{2}}\boldsymbol{I}^{-\delta}\prod_{e\in Edge(\mathcal{D})\cap\mathfrak{M}(\{n-2r+1,\ldots,n\})} w(e).$$

The factor $\boldsymbol{I}^{-\delta}$ can be written as:

$$
\begin{aligned}
\boldsymbol{I}^{-\delta} = \;& \boldsymbol{I}^{-\sum \delta_i'} . \boldsymbol{I}^{-(\delta - \sum \delta_i')} \\
= \;& \boldsymbol{I}^{-\sum_{i=1}^{m} \delta_i'} . \boldsymbol{I}^{\sum_{i=1}^{l} (e_{k_i} - \delta_i)} \\
= \;& \boldsymbol{I}^{-\sum_{i=1}^{m} \delta_i'} \prod_{i=1}^{l} \boldsymbol{I}^{e_{k_i}} . \boldsymbol{I}^{-\delta_i} \\
= \;& \boldsymbol{I}^{-\sum_{i=1}^{m} \delta_i'} \prod_{i=1}^{l} k_i \boldsymbol{I}^{-\delta_i}.
\end{aligned}
$$

On the other hand, $\#(Vert(\mathcal{D}) \cap \mathfrak{F}(\mathcal{D}, \mathfrak{M}, r)) = \sum_{i=1}^{l} \#(Vert(\mathcal{D}_{2i}) \cap \mathfrak{F}(\mathcal{D}, \mathfrak{M}, r)) + \sum_{i=1}^{l} \#(Vert(\mathcal{D}_{2i-1}) \cap \mathfrak{F}(\mathcal{D}, \mathfrak{M}, r)) + \sum_{i=1}^{m} \#(Vert(\mathcal{D}_i') \cap \mathfrak{F}(\mathcal{D}, \mathfrak{M}, r))$.

From step (h) of the algorithm, we have that $Vert(\mathcal{D}_{2i}) \cap \mathfrak{F}(\mathcal{D}, \mathfrak{M}, r) = Vert(\mathcal{D}_{2i})$. By Corollary 2.4 , $\#Vert(\mathcal{D}_{2i}) = d_i$. Similarly, $\#(Vert(\mathcal{D}_{2i-1}) \cap \mathfrak{F}(\mathcal{D}, \mathfrak{M}, r) = d_i$.

From step (f), $\#(Vert(\mathcal{D}_i') \cap \mathfrak{F}(\mathcal{D}, \mathfrak{M}, r)) = \#(Vert(\mathcal{D}_i') \cap \mathfrak{F}(\mathcal{D}_i', \mathfrak{M}_i', r_i'))$. Therefore,

$$
(-1)^{\frac{\#(Vert(\mathcal{D}) \bigcap \mathfrak{F}(\mathcal{D}, \mathfrak{M}, r))}{2}} = (-1)^{\sum_{i=1}^{m} \frac{\#(Vert(\mathcal{D}_i') \cap \mathfrak{F}(\mathcal{D}_i', \mathfrak{M}_i', r_i'))}{2}} \prod_{i=1}^{l} (-1)^{d_i}.
$$

Now, we want to treat $\prod_{e \in Edge(\mathcal{D}) \cap \mathfrak{M}(\{n-2r+1, \ldots, n\})} w(e)$.

$\prod_{e \in Edge(\mathcal{D}) \cap \mathfrak{M}(\{n-2r+1, \ldots, n\})} w(e) =$

$$\prod_{i=1}^{l} \prod_{e\in Edge(\mathcal{D}_{2i})\cap\mathfrak{M}(\{n-2r+1,\ldots,n\})} w(e)^2 \prod_{i=1}^{m} \prod_{e\in Edge(\mathcal{D}'_i)\cap\mathfrak{M}(\{n-2r+1,\ldots,n\})} w(e).$$

From (h), the first part of the product is equal to $\prod_{i=1}^{l} \prod_{e\in Edge(\mathcal{D}_{2i})\cap\mathfrak{M}(\{1,\ldots,n\})} w(e)^2$ which is

equal to $\prod_{i=1}^{l} \left[\prod_{e\in Edge(\mathcal{D}_{2i})\cap\mathfrak{M}(\{1,\ldots,n\})} w(e)^2 \prod_{e\in Edge(\mathcal{D}_{2i})\setminus\mathfrak{M}(\{1,\ldots,n\})} w(e)^2 \boldsymbol{I}^{-2\gamma_i} \right]$ by (e).

From $(d), (f), (g)$ and (i) the second factor is equal to $\prod_{i=1}^{m} \prod_{e\in Edge(\mathcal{D}'_i)\cap\mathfrak{M}'_i(\{n'_i-2r'_i+1,\ldots,n'_i\})} w(e)$.

So $\mu_r^{\mathbb{R}}(\mathcal{D},\mathfrak{M}) =$

$$\prod_{i=1}^{l}(-1)^{d_i} \prod_{i=1}^{l} \left[\prod_{e\in Edge(\mathcal{D}_{2i})\cap\mathfrak{M}(\{1,\ldots,n\})} w(e)^2 \prod_{e\in Edge(\mathcal{D}_{2i})\setminus\mathfrak{M}(\{1,\ldots,n\})} w(e)^2 \boldsymbol{I}^{-2\gamma_i} \right] \prod_{i=1}^{l} k_i \boldsymbol{I}^{-\delta_i}$$

$$(-1)^{\sum_{i=1}^{m} \frac{\#(Vert(\mathcal{D}'_i)\cap\mathfrak{F}(\mathcal{D}'_i,\mathfrak{M}'_i,r'_i))}{2}} \boldsymbol{I}^{-\sum_{i=1}^{m}\delta'_i} \prod_{i=1}^{m} \prod_{e\in Edge(\mathcal{D}'_i)\cap\mathfrak{M}(\{n-2r+1,\ldots,n\})} w(e)$$

$$= \prod_{i=1}^{l}(-1)^{d_i} k_i \mu^{\mathbb{C}}(\mathcal{D}_{2i},\mathfrak{M}_{2i}) \prod_{i=1}^{m} \mu^{\mathbb{R}}_{r'_i}(\mathcal{D}'_i,\mathfrak{M}'_i). \quad \blacksquare$$

The total number of, not necessarily distinct, marked floor diagrams generated by the

algorithm is equal to

$$2^l \prod_{i=1}^{l}(\delta_i)_{k_i} 2^{2d_i-2+|\gamma_i+\delta_i|} \prod_{i=1}^{m}(\beta'_i)_{k'_i} \binom{\alpha}{\alpha'_1,\ldots,\alpha'_m} \binom{\gamma}{\gamma_1,\ldots,\gamma_l,\gamma'_1,\ldots,\gamma'_m} \binom{2d-2+|\beta+2\delta|-2r}{2d'_1-1+|\beta'_1+2\delta'_1|-2r'_1,\ldots,2d'_m-1+|\beta'_m+2\delta'_m|-2r'_m}$$

$$\binom{r}{2d_1-1+|\delta_1|,\ldots,2d-1+|\delta_l|,r'_1,\ldots,r'_m}.$$

Each marked floor diagram is generated $2^l\sigma(s)\sigma'(s)$ times.

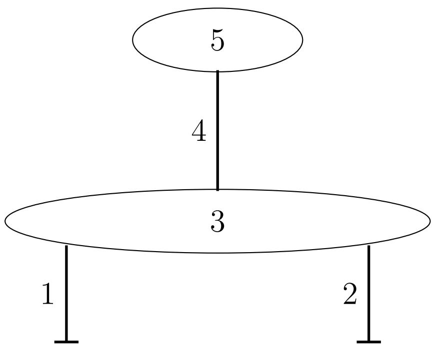
The factor 2^l comes from the equivalence between the floor diagrams $\mathcal{D}_{2i-1}$ and $\mathcal{D}_{2i}$.

$\sigma(s)$ comes from the equivalence relation $(d_i, k_i, \alpha_i, \beta_i) = (d_j, k_j, \alpha_j, \beta_j) \Leftrightarrow i \simeq_s j$ given in definition 3.1. Similarly for $\sigma^{'}(s)$.

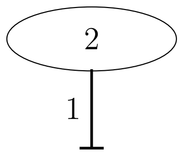
Dividing by $2^l \sigma(s)\sigma^{'}(s)$ then summing up over all $s \in \boldsymbol{S}_w(l, m, r, \alpha, \beta, \gamma, \delta)$ we get what is required.

Example 3.9.

Let $s = (1, 1, 0, e_1, 2, 1, 0, 0, 2e_1, 0, 0) \in \boldsymbol{S}_w(1, 1, 2, 4e_1, e_1, 0, 0)$ and consider the element $\Delta = ((\mathcal{D}_1, \mathfrak{M}_1), (\mathcal{D}_2, \mathfrak{M}_2), (\mathcal{D}_1^{'}, \mathfrak{M}_1^{'}))$ of $\mathfrak{B}(s)$ given by:

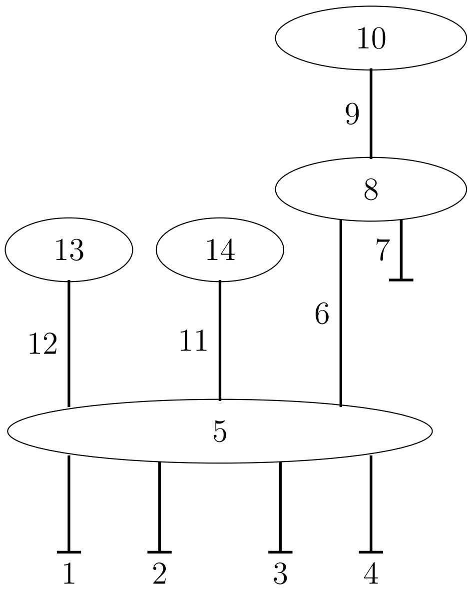


$(\mathcal{D}_1^{'}, \mathfrak{M}_1^{'})$



$(\mathcal{D}_1, \mathfrak{M}_1); (\mathcal{D}_2, \mathfrak{M}_2)$

The non-equivalent elements of $\boldsymbol{B}$ obtained from Δ are: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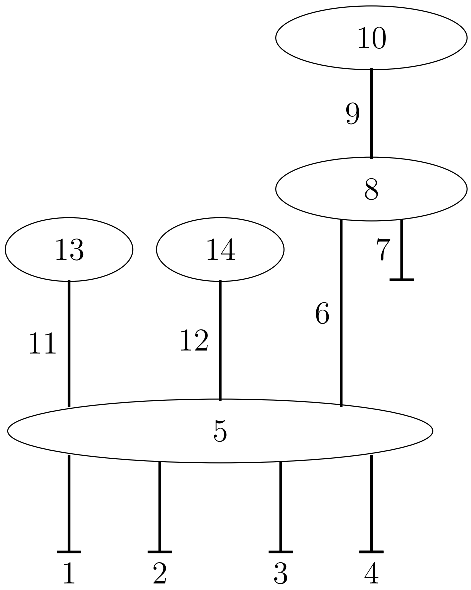
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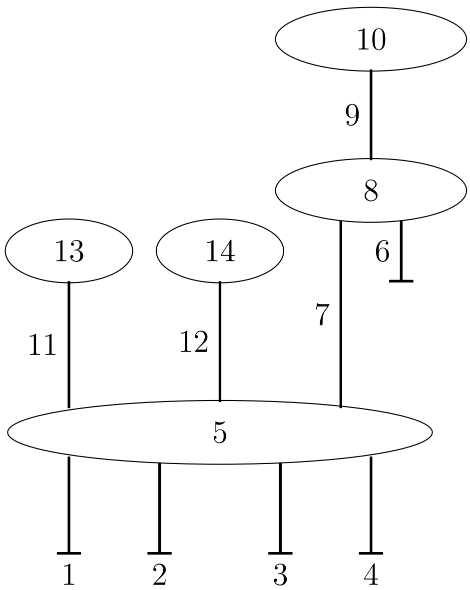
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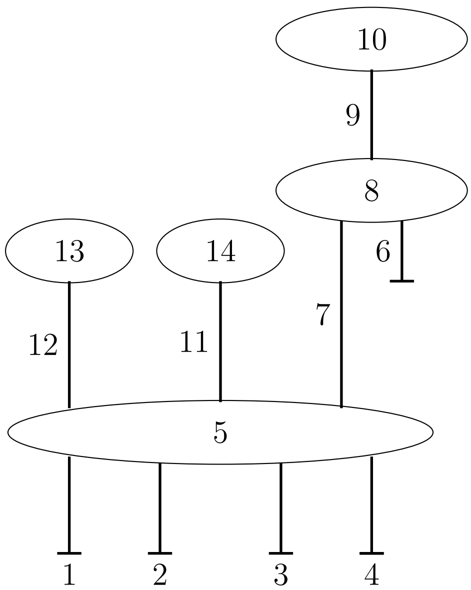
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Case II: when $2d-1+|\beta+2\delta|-2r=0$.

In this case, if $(\mathcal{D},\mathfrak{M})$ is an $r-real$ marked floor diagram of degree d and type $(\alpha,\beta,\gamma,\delta)$, then $\{|\alpha+2\gamma|+1,|\alpha+2\gamma|+2\}$is an $r-pair$ of $(\mathcal{D},\mathfrak{M})$. The four terms in the right hand side of the formula in Theorem 3.6, equation (2), come from consideration of four sub cases, Let $\boldsymbol{A}^{'}$, $\boldsymbol{B}^{'}$, $\boldsymbol{C}^{'}$ and $\boldsymbol{D}^{'}$ be the sets of all $r-real$ marked floor diagrams of degree d and type $(\alpha,\beta,\gamma,\delta)$ satisfying respectively

- both $\mathfrak{M}(|\alpha+2\gamma|+1)$ and $\mathfrak{M}(|\alpha+2\gamma|+2)$ are in $Edge^{\infty}(\Gamma)$,
- $\mathfrak{M}(|\alpha+2\gamma|+1)$ is in $Edge^{\infty}(\Gamma)$ and $\mathfrak{M}(|\alpha+2\gamma|+2)$ is in $Vert(\Gamma)$,
- $\mathfrak{M}(|\alpha+2\gamma|+1)$ is in $Vert(\Gamma)$ and $\mathfrak{M}(\alpha+2\gamma|+2)$ is in $Edge(\Gamma)$,
- both $\mathfrak{M}(|\alpha+2\gamma|+1)$ and $\mathfrak{M}(|\alpha+2\gamma|+2)$ are in $Vert(\Gamma)$.

Case II-A': Marked floor diagrams in $\boldsymbol{A}^{'}$.

There exists a bijection $\Phi^{'}$ from the set $\boldsymbol{A}^{'}$ to the union of all $(r-1)-real$ marked floor diagrams of degree d and type $(\alpha,\beta,\gamma+e_k,\delta-e_k)$ for k such that $\delta\geq e_k$. If $(\mathcal{D},\mathfrak{M})$ is a marked floor diagram in $\boldsymbol{A}'$ and k is the weight of the edge $\mathfrak{M}(|\alpha+2\gamma|+1)$, we define $\Phi'(\mathcal{D},\mathfrak{M})$=$(\mathcal{D},\mathfrak{M}')$ where

- $\mathfrak{M}'(i)=\mathfrak{M}(i)$ if $i \leq |\alpha| + 2\sum_{j=1}^{k}\gamma_j$ or $i \geq |\alpha+2\gamma|+3$,
- $\mathfrak{M}'(|\alpha| + 2\sum_{j=1}^{k}\gamma_j + 1)$ is the source adjacent to $\mathfrak{M}(|\alpha+2\gamma|+1)$,
- $\mathfrak{M}'(|\alpha + 2\sum_{j=1}^{k}\gamma_j + 2)$ is the source adjacent to $\mathfrak{M}(|\alpha+2\gamma|+2)$,
- $\mathfrak{M}'(i)$=$\mathfrak{M}(i-2)$ if $|\alpha| + 2\sum_{j=1}^{k}\gamma_j + 3 \leq i \leq |\alpha+2\gamma|+2$.

Note that $n-2r = 2d-1+|\alpha+\beta+2\gamma+2\delta|-2r = |\alpha+\beta|$. So $\{|\alpha+2\gamma|+1, |\alpha+2\gamma|+2\}$ is an $r-pair$. Since $(\mathcal{D},\mathfrak{M})$ is $r-real$, $\mathfrak{M}(|\alpha+2\gamma|+1)$ and $\mathfrak{M}(|\alpha+2\gamma|+2)$ have the same weight. Also note that $\mathfrak{F}(\mathcal{D},\mathfrak{M}',r-1)=\mathfrak{F}(\mathcal{D},\mathfrak{M},r)$ so $(\mathcal{D},\mathfrak{M}')$ is $(r-1)$-*real* and $v \in Vert(\mathcal{D}) \cap \mathfrak{F}(\mathcal{D},\mathfrak{M},r) \Leftrightarrow v \in Vert(\mathcal{D}) \cap \mathfrak{F}(\mathcal{D},\mathfrak{M}',r-1)$.

$$\prod_{e\in\mathfrak{M}(\{n-2r+1,\ldots,n\})} w(e) = k^2 \prod_{e\in\mathfrak{M}(\{n-2r+3,\ldots,n\})} w(e) = k^2 \prod_{e\in\mathfrak{M}'(\{n-2(r-1),\ldots,n\})} w(e)$$

$I^{-\delta} = I^{-(\delta-e_k)-e_k} = \frac{1}{k}I^{-(\delta-e_k)}$

Therefore $\mu_r^{\mathbb{R}}(\mathcal{D},\mathfrak{M}) = k\mu_{r-1}^{\mathbb{R}}(\mathcal{D},\mathfrak{M}')$. Note that since $(\mathcal{D},\mathfrak{M})$ is $r-real$, the weight of the edges $\mathfrak{M}(|\alpha+2\gamma|+1)$ and $\mathfrak{M}(|\alpha+2\gamma|+2)$ is the same. One has $\mu_r^{\mathbb{R}}(\mathcal{D},\mathfrak{M}) = k\mu_r^{\mathbb{R}}(\mathcal{D},\mathfrak{M}')$, hence the first sum of the right-hand side of the formula in Theorem 3.6, equation (2), is

given by $\sum_{(\mathcal{D},\mathfrak{M})\in \boldsymbol{A}'} \mu_r^{\mathbb{R}}(\mathcal{D},\mathfrak{M})$.

Case II-B': Marked floor diagrams in $\boldsymbol{B}^{'}$.

Choose $K \geq 1$ such that $\beta \geq e_K$. Let l and m be two non-negative integers such that the set $\boldsymbol{S}_w(l,m,d,k,r-1,\alpha,\beta-e_K,\gamma,\delta)$ is not empty, and let $\mathfrak{B}(s)$ be as in Case I-B for any s in $\boldsymbol{S}_w(l,m,d,k,r-1,\alpha,\beta-e_K,\gamma,\delta)$. Starting from an element of $\mathfrak{B}(s)$, construct several elements of $\boldsymbol{B'}$ as in the step (1)-(11) of Case I-B, except for the following modifications:

(8$\boldsymbol{B}^{'}$) define $J_i = \phi$ for all i in $\{1,\ldots,m\}$,

(9$\boldsymbol{B}^{'}$) Choose a partition $(\hat{J}_i)_{1\leq i\leq m} \cup (\widetilde{J}_i)_{1\leq i\leq m}$ of the set $\{1,\ldots,r-1\}$ such that $\#\hat{J}_i = 2d_i - 1 + |\delta_i|$ and $\#\widetilde{J}_i = r_i^{'}$ for all i.

(11$\boldsymbol{B}^{'}$)(a) $\mathfrak{M}(|\alpha+2|\gamma|+2) = v$, and $\mathfrak{M}(|\alpha|+2|\gamma|+1)$ is an edge in $Edge^{\infty}(\mathcal{D})$ of weight K adjacent to v.

By construction, one has

$$\mu_r^{\mathbb{R}}(\mathcal{D},\mathfrak{M}) = K\prod_{i=1}^{l}(-1)^{d_i}k_i\mu^{\mathbb{C}}(\mathcal{D}_{2i},\mathfrak{M}_{2i})\prod_{i=1}^{m}\mu_{r_i^{'}}^{\mathbb{R}}(\mathcal{D}_i^{'},\mathfrak{M}_i^{'}).$$

Now, the second sum in the right-hand side of the formula in 3.6, equation (2), follows from all possible choices in our construction.

Case II-C': Marked floor diagrams in $\boldsymbol{C}'$.

Let l and m be two non-negative integers such that the set $\boldsymbol{S}_w(l, m, d, k, r-1, \alpha, \beta, \gamma, \delta)$ is not empty, and define the set $\mathfrak{B}(s)$ as in Case I-B for any s in $\boldsymbol{S}_w(l, m, d, k, r-1, \alpha, \beta, \gamma, \delta)$. Then, starting from an element of $\mathfrak{B}(s)$ we construct several elements of $\boldsymbol{C}'$ as in the step (1)-(11) of Case I-B, except for the following modifications

($0\boldsymbol{C}'$) Choose j in $\{1, \ldots, m\}$ such that $\beta_j \geq k'_j$, the weight of $\mathfrak{M}'_j(|\alpha'_j| + 2|\gamma'_j| + 1)$ is k'_j, and $2d'_j - 1 + |\beta'_j + 2\gamma'_j| - 2r'_j$=1.

($2\boldsymbol{C}'$) For all $1 \leq i \leq m$, $i \neq j$, choose an element a'_i of $Edges^{\infty}(\mathcal{D}'_i)$ which is in the image of $\mathfrak{M}'_i$ but not in $\mathfrak{F}(\mathcal{D}'_i, \mathfrak{M}', r'_i)$, and of weight k'_i.

($8\boldsymbol{C}'$) Define $J_i = \phi$ for all i in $\{1, \ldots, m\}$.

($9\boldsymbol{C}'$) Choose a partition $(\hat{J}_i)_{1 \leq i \leq m} \cup (\widetilde{J}_i)_{1 \leq i \leq m}$ of the set $\{1, \ldots, r-1\}$ such that $\#\hat{J}_i = 2d_i - 1 + |\delta_i|$ and $\#\widetilde{J}_i = r'_i$ for all i.

$(11\boldsymbol{C}^{'})$ (a) $\mathfrak{M}(|\alpha+2|\gamma|+1)=v$, and $\mathfrak{M}(|\alpha|+2|\gamma|+2)=\mathfrak{M}_j^{\prime}(\alpha_j^{\prime}|+2|\gamma_j^{\prime}|+1)$.

By construction, one has

$$\mu_r^{\mathbb{R}}(\mathcal{D},\mathfrak{M})=k_j^{'}\prod_{i=1}^{l}(-1)^{d_i}k_i\mu^{\mathbb{C}}(\mathcal{D}_{2i},\mathfrak{M}_{2i})\prod_{i=1}^{m}\mu_{r_i^{'}}^{\mathbb{R}}(\mathcal{D}_i^{'},\mathfrak{M}_i^{'}).$$

Now, the third sum in the the right-hand side of the formula in 3.6, equation (2), follows from all possible choices in our construction.

Case II-D': Marked floor diagrams in $\boldsymbol{D}^{'}$.

In this case, by "cutting" the vertices $\mathfrak{M}(|\alpha+2|\gamma|+1)$ and $\mathfrak{M}(|\alpha|+2|\gamma|+2)$ from $(\mathcal{D})$, one obtains several marked floor diagrams of genus 0 and of lower degrees. Since $(\mathcal{D})$ is a tree and is $r-real$, exactly one of these marked floor diagrams is adjacent to both cut vertices, and all the other are naturally coupled in pairs by the map $\rho_{\mathcal{D},\mathfrak{M},r}$. Moreover, any edge in $Edge^{\infty}(\mathcal{D})$ adjacent to the vertex $\mathfrak{M}(|\alpha|+2|\gamma|+2)$ since the diagram is $r-real$. In particular, both edges have the same weight.

Let l be a non-negative integer such that the set $\widetilde{\boldsymbol{S}}_w(l,r-1,\alpha,\beta,\gamma,\delta)$ is not empty. For s

in this set, define the set $\mathfrak{B}(s)$ as in Case I-B with $m = 1$.

Starting from an element of $\mathfrak{B}(s)$, we construct several elements of $\mathcal{D}'$ in the following way

1. For all $0 \leq i \leq l$ choose an element a_i of $Edge^\infty(\mathcal{D}_{2i-1})$ which is in the image of $\mathfrak{M}_{2i-1}$ and of weight k_i.

2. Choose an edge a_1' of $\mathcal{D}_1'$ in $Edge^\infty(\mathcal{D}_1') \cap \mathfrak{F}(\mathcal{D}_1', \mathfrak{M}_1', r_1')$ of weight k_1'.

3. Construct a new oriented tree $\widetilde{\mathcal{D}}$ out of $(\mathcal{D}_1, \mathfrak{M}_1), \ldots, (\mathcal{D}_{2l}, \mathfrak{M}_{2l}), (\mathcal{D}_1', \mathfrak{M}_1'), \ldots, (\mathcal{D}_m', \mathfrak{M}_m')$ by identifying all the sources adjacent to the edges a_i and a_1', and all the sources adjacent to the edges $\phi_i(a_i)$ and $\rho_{\mathcal{D}_1', \mathfrak{M}_1', r_1'}(a_1')$. Denote by v and v' the 2 vertices added.

4. Construct a degree d and genus 0 floor diagram $\mathcal{D}$ out of $\widetilde{D}$ by adding sources $v_1, \ldots, v_t, v_1', \ldots, v_t'$ and edges $(v_1, v), \ldots, (v_t, v), (v_1', v'), \ldots, (v_t', v')$, such that $\mathcal{D}$ has $(\alpha)_j + (\beta)_j + 2(\gamma)_j + 2(\delta)_j$ edges in $Edge^\infty(\mathcal{D})$ of weight j for all $j \geq 1$, and such that there are as many edges (v_i, v) of weight j as edges (v_i', v') of weight j for all $j \geq 1$.

5. Define $\gamma_2^{'} = \gamma - \sum_{i=1}^{l} \gamma_i - \gamma_1^{'}$.

6. For all $j \geq 1$, choose a partition $(\hat{I}_i^j)_{1 \leq i \leq l} \cup (\widetilde{I}_1^j) \cup (\widetilde{J}_2^j)$ of the set $\{1, \dots, (\gamma)_j\}$ such that $\#\hat{I}_i^j = (\gamma_i)_j$ and $\#\widetilde{I}_i^j = (\gamma_i^{'})_j$.

7. Choose a partition $(\hat{J}_i)_{1 \leq i \leq l} \cup (\widetilde{J}_1)$ of the set $\{1, \dots, r-1\}$ such that $\#\hat{J}_i = 2d_i - 1 + |\delta_i|$ and $\#\widetilde{J}_1 = r_1^{'}$.

8. Choose a vector $\mathcal{E}$ in $\{0,1\}^{|\gamma_2^{'}|}$.

9. For all i in $\{1, \dots, l\}$, choose a vector $\mathcal{E}_i$ in $\{0,1\}^{2d_i - 1 + |\gamma_i + \delta_i|}$.

10. Choose a marking $\mathfrak{M}$ of $\mathcal{D}$ of type $(\alpha, \beta, \gamma, \delta)$ such that

    (a) $\mathfrak{M}(|\alpha + 2|\gamma| + 1) = v$, and $\mathfrak{M}(|\alpha| + 2|\gamma| + 2) = v^{'}$,

    (b) For all $j \geq 1$ and if k is the $h-th$ element of $\hat{I}_2^j$, then

    $\mathfrak{M}\left(|\alpha| + 2\sum_{t=1}^{j-1}(\gamma)_t + 2k + (\mathcal{E})_{\sum_{t=1}^{j-1}(\gamma_i^{'})_t + h}\right)$ and

    $\mathfrak{M}\left(|\alpha| + 2\sum_{t=1}^{j-1}(\gamma)_t + 2k - (\mathcal{E})_{\sum_{t=1}^{j-1}(\gamma_i^{'})_t + h}\right)$ are respectively a source v_q and $v_q^{'}$ of divergence $-j$,

(c) If $1 \leq i \leq |\alpha|, \mathfrak{M}(i) = \mathfrak{M}_1'(j)$,

(d) for all $j \geq 1$ and all i in $\{1, \ldots, l\}$, if k is the $h-th$ element of $\hat{I}_i^j$, then

$$\mathfrak{M}\left(|\alpha| + 2\sum_{t=1}^{j-1}(\gamma)_t + 2k - 1 + (\mathcal{E}_i)_{\sum_{t=1}^{j-1}(\gamma_i)_t + h}\right) = \mathfrak{M}_{2i-1}\left(\sum_{t=1}^{j-1}(\gamma_i)_t + h\right)$$

$$\mathfrak{M}\left(|\alpha| + 2\sum_{t=1}^{j-1}(\gamma)_t + 2k + (\mathcal{E}_i)_{\sum_{t=1}^{j-1}(\gamma_i)_t + h}\right) = \phi_i \circ \mathfrak{M}_{2i-1}\left(\sum_{t=1}^{j-1}(\gamma_i)_t + h\right)$$

(e) for all $j \geq 1$, if k is the $h-th$ element of $\widetilde{I}_i^j$, then

$$\mathfrak{M}\left(|\alpha| + 2\sum_{t=1}^{j-1}(\gamma)_t + 2k - 1\right) = \mathfrak{M}_1'\left(|\alpha_1'| + 2\sum_{t=1}^{j-1}(\gamma_1')_t + 2h - 1\right)$$

$$\mathfrak{M}\left(|\alpha| + 2\sum_{t=1}^{j-1}(\gamma)_t + 2k\right) = \mathfrak{M}_1'\left(|\alpha_1'| + 2\sum_{t=1}^{j-1}(\gamma_1')_t + 2h\right)$$

(f) for all i in $\{1, \ldots, l\}$, if k is the h-th element of $\widehat{J}_i$, then

$$\mathfrak{M}(n - 2k + 1 + (\mathcal{E}_i)_{|\gamma_i| + h}) = \mathfrak{M}_{2i-1}(2d_i - |\gamma_i| + |\delta_i| - h)$$

$$\mathfrak{M}(n-2k+2+(\mathcal{E}_i)_{|\gamma_i|+h}) = \phi_i \circ \mathfrak{M}_{2i-1}(2d_i - |\gamma_i| + |\delta_i| - h)$$

(g) if k is the h-the element of $\widetilde{J}_1$, then

$$\mathfrak{M}(n-2k+1) = \mathfrak{M}_1^{'}(n_1^{'} - 2h + 1)$$

$$\mathfrak{M}(n-2k+2) = \mathfrak{M}_1^{'}(n_1^{'} - 2h + 2)$$

In this way, we construct all marked floor diagrams in $\mathcal{D}^{'}$. Moreover, any element of $\mathcal{D}^{'}$ is obtained exactly $\sigma(s)\sigma(s)^{'}$ times for some s in $\widetilde{\boldsymbol{S}}_w(l, r-1, \alpha, \beta\gamma, \delta)$.

By construction, one has

$$\mu_r^{\mathbb{R}}(\mathcal{D}, \mathfrak{M}) = -k_1^{'} \mu_{r_1^{'}}^{\mathbb{R}}(\mathcal{D}_1^{'}, \mathfrak{M}_1^{'}) \prod_{i=1}^{l} (-1)^{d_i} k_i \mu^{\mathbb{C}}(\mathcal{D}_{2i}, \mathfrak{M}_{2i})$$

Now, the fourth sum in the right-hand side of the formula in 3.6, equation (2), follows from all possible choices in our construction.

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