AMERICAN UNIVERSITY OF BEIRUT

A Cosmological Solution to Mimetic Dark Matter

by

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AN ABSTRACT OF THE DISSERTATION OF

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An introductory analysis to general relativity, cosmology, and mimetic dark matter is provided. In chapter 3 where cosmology is discussed, the need of an inflationary theory is motivated. Inflation is showed to possibly solve the flatness problem and the horizon problem of the cosmic microwave background (CMB) radiation. Based on an exponential potential, a solution is obtained representing a scale factor of a matter-dominated universe satisfying the condition of inflation for some initial conditions. Furthermore, a more general potential is suggested that incorporates inflation too. Then, with this general potential, a solution to the 0-i perturbed Einstein's differential equation that includes mimetic dark matter is provided in chapter 5. Finally, quantum perturbations are mentioned briefly. The constants involved in this model are tuned to be in agreement with the amplitude fluctuation of the CMB. Therefore, dark matter, inflation, and CMB can exist in one model without introducing any extra degrees of freedom.

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Chapter 1

Introduction

The nature of dark matter is not well understood. Some models were proposed based on modifications of Einsteins General Theory of Relativity, others are based on proposing the existence of new particles, while others are based on much more fancy complicated mathematical theories and ideas that are not supported by observations. A recent modification of General Relativity was proposed in [1] suggests that the metric has a conformal invariance when it is expressed in terms of a field which is not dynamical, that's to say it satisfies Hamilton-Jacobi equation. The equations of motion that result are similar to Einstein's equations of motion with an extra mode term that mimics cold dark matter even in the absence of normal matter. Hence, it is called Mimetic Dark Matter (MDM). This work was followed by a paper [2] that shows the compatibility of MDM with cosmology. It demonstrates how MDM can generate inflation, bouncing universes, and quintessence.

Inflation is an attractive theory because it explains the curvature flatness of the observable universe and the homogeneity of the CMB to a certain degree, and it dilutes monopoles (if Grand Unification Theories (GUT) exist) so that they are not observed today. Moreover, some of the predictions of inflation were confirmed experimentally by checking indirect consequences of inflation on primordial density perturbations in the CMB by the BOOMERanG experiment (Balloon Observations Of Millimetric Extragalactic Radiation ANd Geophysics) [3], the Cosmic Background Explorer [4], then by the Wilkinson Microwave Anisotropy Probe [5], and recently by the Planck satellite [6][7]. The Ekpyrotic universe model [8] was proposed in order to solve the singularity problem and the problems that inflation solves. It is a contracting phase preceding the big bang. A duality between the two models has been established in [9]. According to Planck satellite, the data favors modeling inflation by using one single field [6]. However, calculating the spectral index for scalar perturbations in the ekpyrotic model with a single field [10] [11] yields a blue-shifted spectrum ($n_s = 3$) in disagreement with Planck's data which is red-shifted spectrum. However, by using multiple fields, the ekpyrotic model's cosmological perturbations become in agreement with observations. Moreover, some argue that cosmological perturbations are ill-defined or aren't generated by ekpyrotic and cyclic models [12] [13]. For cyclic universes, the second-law of thermodynamics dictates that the subsequent cycles must have a larger entropy than the previous ones; hence, it hinders a truly cyclic universe because one would reach a point in the past where the cycle is infinitesimally small [14]. Hence, inflation is still more coherent with observations when it comes to modeling data by a single field in order to obtain the spectral index with a red tilt. In addition, Planck's data favors a plateau-like potential with an upper limit; therefore, challenging the initial conditions that give rise to inflation in the first place making it less likely to occur [15]. A suggested extension to the idea of inflation is provided in [16] to resolve this problem.

Searching for dark matter is still more convincing than pursuing other theories such as MOND (MOdified Newtonian Dynamics) because you can't build a coherent and consistent cosmological model from MOND (it doesn't give the CMB angular power spectrum for instance). MOND violates the equivalence principle, conservation of momentum (Newton's third law), Lorentz invariance, sometimes the cosmological principle, superposition of gravitational fields, and the existence of escape velocity [17]. Moreover, it doesn't work in galaxy clusters [18]. MOND is a kind of an effective theory that misses an explanation similar to Kepler's law that requires Newtonian mechanics to explain it [19].

The aim of this thesis is to link MDM to an experiment in order to verify that the model works without introducing any extra degrees of freedom. This might be achieved by trying to find a potential where both 70 e-folds inflation and the amplitude fluctuations of the CMB could be obtained. In order to get the toolkit for this work, chapter 2 provides an introduction to general theory of relativity which is the basic playground to understand gravity and things at the large scale. Then, an introduction to cosmology and inflation is presented in chapter 3. Moreover, MDM is explained along with its applications to cosmology in chapter 4. And finally, the goal of this thesis is presented in chapter 5 where both inflation and CMB can be obtained without introducing any extra degrees of freedom.

Chapter 2

General Relativity

General Relativity (GR) has some assumptions about the properties of spacetime such as it's a C^∞ n-dimensional and pseudo-Riemannian manifold. The differentiability condition insures the existence of differential equations, while the type of manifold insures that the metric is compatible with Minkowski space in special relativity. In this section, a quick introduction to the basics of GR are introduced. Note that in all of this work the speed of light and reduced Planck constant are set according to natural units, i.e. $c = \hbar = 1$. For more elaborate information about the basics check [20] and [21].

2.1 Metric

The line element in GR for a pseudo-Riemannian metric $g_{\mu\nu}$

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu; \qquad \mu, \nu = 0, 1, ..., (n-1) \tag{2.1}$$

The line element contains information about the system that we are dealing with. For instance, if we have an isotropic and homogeneous universe, then we will find spherical symmetry. Moreover, if it doesn't depend on time, then the system has conservation of energy.

The determinant is defined as

$$g \equiv |detg_{\mu\nu}| \tag{2.2}$$

2.2 Equation of motion

Consider the action

$$S = \int ds = \int \sqrt{g_{\mu\nu}dx^{\mu}dx^{\nu}} = \int \sqrt{g_{\mu\nu}(x(\lambda))\frac{dx^{\mu}}{d\lambda}\frac{dx^{\nu}}{d\lambda}}d\lambda \tag{2.3}$$

where λ is an affine parameter. By varying the action with respect to λ and then set the affine parameter λ to proper time τ, we would get the geodesic equation of GR [20]

$$\frac{d^2x^{\mu}}{d\tau^2} + \Gamma^{\mu}_{\rho\sigma}\frac{dx^{\rho}}{d\tau}\frac{dx^{\sigma}}{d\tau} = 0 \tag{2.4}$$

and

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2}g^{\sigma\rho}(\partial_{\mu}g_{\nu\rho} + \partial_{\nu}g_{\rho\mu} - \partial_{\rho}g_{\mu\nu}) \tag{2.5}$$

where Γ is called Christoffel symbol. Note that τ is related to λ by $\lambda = a\tau + b$ (since they are affine parameters) where a and b are constants.

2.3 Covariant Derivative

By going to curved spacetime, we have to update our partial derivatives and volume elements in integration in order to connect different space vectors that are defined at each point on the manifold or because partial derivatives are not invariant under general coordinates transformation because of the derivative of the basis vectors. Hence, partial derivatives are upgraded to covariant derivative for a general tensor as

$$\nabla_\sigma T^{\mu_1\mu_2\ldots\mu_k}_{\nu_1\nu_2\ldots\nu l} = \partial_\sigma T^{\mu_1\mu_2\ldots\mu_k}_{\nu_1\nu_2\ldots\nu l} + \Gamma^{\mu_1}_{\sigma\lambda} T^{\lambda\mu_2\ldots\mu_k}_{\nu_1\nu_2\ldots\nu l} + \ldots - \Gamma^{\lambda}_{\sigma\nu_1} T^{\mu_1\mu_2\ldots\mu_k}_{\lambda\nu_2\ldots\nu l} - \ldots \tag{2.6}$$

And the volume element upgrades to $\sqrt{|g|}d^n x$ in n dimensions. Furthermore, our metric must also be upgraded from $\eta_{\mu\nu}$ to $g_{\mu\nu}$

2.4 Riemann and Ricci tensors

Riemann tensor can be constructed out of Christoffel symbols as

$$R^{\rho}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma} \tag{2.7}$$

And the Ricci tensor can be obtained by contracting the upper index with the middle lower index as

$$R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu} \tag{2.8}$$

2.5 Einstein-Hilbert Action

Einstein's equation can be derived by varying this action with respect to the metric

$$S = -\frac{1}{2}\int \sqrt{-g}\big[R(g_{\mu\nu}) + \mathcal{L}_m\big] d^4x \tag{2.9}$$

where R is called the Ricci scalar, $\mathcal{L}_m(g_{\mu\nu})$ is the Lagrangian of matter, and $8\pi G = 1$. The resulting equation is

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = T_{\mu\nu} \tag{2.10}$$

where $R_{\mu\nu}$ is Ricci tensor and $T_{\mu\nu}$ is the energy-momentum tensor. This equation can be interpreted as matter or energy curves spacetime and also matter and energy follow the curvature along their geodesic lines.

The energy-momentum tensor can be approximated as a perfect fluid; hence it can be expressed as

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + p g_{\mu\nu} \tag{2.11}$$

2.6 Conformal Transformation

If Ω is a smooth and strictly positive function, then the metric $\tilde{g}_{\alpha\beta} = \Omega^2 g_{\alpha\beta}$ is generated by a *conformal transformation* from $g_{\alpha\beta}$. The Christoffel symbols, Ricci tensor, Ricci scalar, and Weyl tensor that arise from $\tilde{g}_{\alpha\beta}$ and $g_{\alpha\beta}$ are related by a set of equations [21] and [22],

$$\Gamma^\rho_{\mu\nu} \to \bar{\Gamma}^\rho_{\mu\nu} = \Gamma^\rho_{\mu\nu} + \Omega^{-1}(\delta^\rho_\mu \nabla_\nu \Omega + \delta^\rho_\nu \nabla_\mu \Omega - g_{\mu\nu} g^{\rho\alpha} \nabla_\alpha \Omega) \tag{2.12}$$

$$\begin{aligned} R^\nu_\mu \to \bar{R}^\nu_\mu = {} & \Omega^{-2} R^\nu_\mu - (n-2)\Omega^{-1} \nabla_\mu \nabla_\rho (\Omega^{-1}) g^{\rho\nu} \\ & + (n-2)^{-1} \Omega^{-n} \nabla_\rho \nabla_\sigma (\Omega^{n-2}) g^{\rho\sigma} \delta^\nu_\mu \end{aligned} \tag{2.13}$$

$$\begin{aligned} R \to \bar{R} = {} & \Omega^{-2} R + 2(n-1)\Omega^{-3} \nabla_\mu \nabla_\nu \Omega g^{\mu\nu} \\ & + (n-1)(n-4)\Omega^{-4} \nabla_\mu \Omega \nabla_\nu \Omega g^{\mu\nu} \end{aligned} \tag{2.14}$$

$$C^\sigma_{\mu\nu\rho} \to \bar{C}^\sigma_{\mu\nu\rho} = C^\sigma_{\mu\nu\rho} \tag{2.15}$$

Chapter 3

Cosmology

Our modern view of cosmology rests heavily on two ideas. The first idea is that the universe on large scales appears homogeneous and isotropic. The second idea is that the universe is expanding. Once these assumptions are set, a metric can be written called the Friedmann-Lemaitre-Robertson-Walker (FLRW) metric, and equations of motion can be derived from Einstein's equation (2.10).

3.1 FLRW Metric

FLRW metric is

$$ds^2 = -dt^2 + a^2(t)\Big[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\Big] \tag{3.1}$$

where $a(t)$ is the scale factor function which informs us how big is the space-like slice at time t, and k is the curvature constant. If $k = 1$, then the observable universe would be closed with a positive curvature; if $k = -1$, then the observable universe would be open with a negative curvature; and if $k = 0$, then the observable universe would be flat with no curvature. By observation [3][4][5][7], the observable universe is considered flat.

The metric (3.1) with $k = 0$ is

$$ds^2 = -dt^2 + a^2(t)\Big[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\Big] \tag{3.2}$$

3.2 Friedmann equations

By plugging the components of the FLRW metric (3.1) in (2.5) and (2.8), and note that Christoffel symbols are symmetric in the lower indices we obtain

$$\begin{aligned}
\Gamma^0_{11} &= \frac{a\dot{a}}{1-kr^2}, \quad \Gamma^0_{22} = a\dot{a}r^2, \quad \Gamma^0_{33} = a\dot{a}r^2\sin^2\theta \\
\Gamma^1_{10} &= \Gamma^2_{20} = \Gamma^3_{30} = \frac{\dot{a}}{a} \\
\Gamma^1_{22} &= -r(1-kr^2), \quad \Gamma^1_{33} = -r(1-kr^2\sin^2\theta) \\
\Gamma^2_{12} &= \Gamma^3_{13} = \frac{1}{r} \\
\Gamma^2_{33} &= -\sin\theta\cos\theta, \quad \Gamma^3_{23} = \frac{\cos\theta}{\sin\theta}
\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}
R_{00} &= -3\frac{\ddot{a}}{a} \\
R_{11} &= \frac{a\ddot{a} + 2\dot{a}^2 + 2k}{1-kr^2} \\
R_{22} &= r^2(a\ddot{a} + 2\dot{a}^2 + 2k) \\
R_{33} &= r^2(a\ddot{a} + 2\dot{a}^2 + 2k)\sin^2\theta
\end{aligned} \tag{3.4}$$

Einstein's equation (2.10) can be re-written (by taking the trace of both sides) as

$$R_{\mu\nu} = 8\pi G(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T) \tag{3.5}$$

By plugging the metric (3.1) components, (2.11), and (3.4) in (3.5), two equations are obtained for $\mu\nu = 00$ and $\mu\nu = ij$ respectively

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) \tag{3.6}$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \tag{3.7}$$

These two equation are called the *Friedmann equations*. The first equation describes acceleration of the scale factor, while the second describes the velocity of the scale factor. The rate of expansion can be realized by the *Hubble parameter*

$$H = \frac{\dot{a}}{a} \tag{3.8}$$

The value of the Hubble parameter now is denoted by H_0. There is uncertainty in the value of H_0; hence, it's expressed as [23]

$$H_0 = 100\, h\, km\, s^{-1}\, Mpc^{-1} = \frac{h}{3000}\, Mpc^{-1} \tag{3.9}$$

where h falls between $0.5 \leq h \leq 0.8$ and $1pc \approx 3.1 \times 10^{16}$ meters. Moreover, if the universe is flat $k = 0$, then a critical density is defined as

$$\rho_c(t) = \frac{3H^2}{8\pi G} \tag{3.10}$$

and

$$\Omega(t) \equiv \frac{\rho}{\rho_c} \tag{3.11}$$

where Ω is a density parameter that can represent the total density. The present critical density now is [23]

$$\rho_c(t_0) = 1.88\, h^2 \times 10^{-29} g\, cm^{-3} \tag{3.12}$$

In order to solve the system of Friedmann equations (3.6) (3.7), we must have knowledge of how the density ρ evolves with time, and an equation of state that relates density and pressure.
A particle horizon is defined as the greatest comoving distance for an observer at a given time to receive signals that travel at the speed of light

$$\chi_{ph}(\tau) = \tau - \tau_i = \int_{t_i}^{t} \frac{dt}{a(t)} \tag{3.13}$$

At any epoch, events separated by more than twice the particle horizon cannot have a common cause; they are said to be out of causal contact. Hence, particle horizons set a limit distance to which past events can be observed.

3.3 The Fluid Equation

The fluid equation can be derived from the first law of thermodynamics applied to a volume V with a unit comoving radius [24]

$$dE + pdV = TdS \tag{3.14}$$

where E is energy, (3.14) becomes

$$E = \frac{4\pi}{3} a^3 \rho \tag{3.15}$$

The change in energy and volume with respect to time

$$\frac{dE}{dt} = 4\pi a^2 \rho \frac{da}{dt} + \frac{4\pi}{3} a^3 \frac{d\rho}{dt} \tag{3.16}$$

$$\frac{dV}{dt} = 4\pi a^2 \frac{da}{dt} \tag{3.17}$$

Assuming an adiabatic process $dS = 0$, and substituting (3.16) and (3.17) in (3.14) yields

$$\dot{\rho} + 3H\big(\rho + p\big) = 0 \tag{3.18}$$

Equation (3.18) is called the fluid equation or continuity equation.

3.4 Types of Universes

We will consider four relevant possibilities for the equation of state that relates density and pressure. In the first three cases $k = 0$, which represents a good approximation.

- Matter:

For matter we treat it as a dust, that's to say, not moving with high speeds; hence, $p \approx 0$. The fluid equation (3.18) becomes

$$\dot{\rho} + 3\frac{\dot{a}}{a}\rho = 0 \Rightarrow \quad \frac{1}{a^3}\frac{d}{dt}(\rho a^3) = 0 \Rightarrow \quad \frac{d}{dt}(\rho a^3) = 0 \Rightarrow \quad \rho = \frac{\rho_0}{a^3} \tag{3.19}$$

where ρ_0 is a proportionality constant. This result can be obtained from intuition by realizing that the density falls off proportionally with the volume. Now to obtain $a(t)$, we substitute (3.19) in Friedmann equation (3.7)

$$\dot{a}^2 = \frac{8\pi G\rho_0}{3}\frac{1}{a} \tag{3.20}$$

Assuming a power-law solution to this differential equation yields

$$a(t) = \left(\frac{t}{t_0}\right)^{\frac{2}{3}} \tag{3.21}$$

and the density

$$\rho(t) = \frac{\rho_0}{a^3} = \frac{\rho_0 t_0^2}{t^2} \tag{3.22}$$

The universe in this scenario is called a matter-dominated universe where it expands forever, however $H(t)$ is decreasing as

$$H(t) \equiv \frac{\dot{a}}{a} = \frac{2}{3t} \tag{3.23}$$

- Radiation:

The relationship between pressure and density for radiation [25]

$$p = \frac{\rho}{3} \tag{3.24}$$

hence, the fluid equation (3.18) becomes

$$\dot{\rho} + 4\frac{\dot{a}}{a}\rho = 0 \Rightarrow \quad \frac{1}{a^3}\frac{d}{dt}(\rho a^4) = 0 \Rightarrow \quad \frac{d}{dt}(\rho a^4) = 0 \Rightarrow \quad \rho = \frac{\rho_0}{a^4} \tag{3.25}$$

Performing the same analysis we obtain

$$a(t) = \left(\frac{t}{t_0}\right)^{\frac{1}{2}} \tag{3.26}$$

and

$$\rho(t) = \frac{\rho_0}{a^4} = \frac{\rho_0 t_0^2}{t^2} \tag{3.27}$$

Note that a radiation-dominated universe expands slower than a matter-dominated universe because of its pressure. Moreover, the density for each scenario decreases as t^2.

- Vacuum or Constant Density:

The constant density behavior is similar to the cosmological constant Λ (Vacuum) behavior up to a constant. It corresponds to $p = -\rho$. The Friedmann equation (3.7) becomes

$$\frac{\dot{a}}{a} = \sqrt{\frac{8\pi G}{3}\rho_0} = H_0 \tag{3.28}$$

The solution to this differential equation is straightforward

$$a(t) = a_0 e^{H_0 t} \tag{3.29}$$

- $k \neq 0$

If $k \neq 0$ and the first term in (3.7) is negligible after a sufficient time, then the universe becomes curvature-dominated. Hence, if $k = -1$ (open universe), then Friedmann equation (3.7) becomes

$$\left(\frac{\dot{a}}{a}\right) = \frac{1}{a^2} \tag{3.30}$$

The solution to (3.30) is

$$a(t) = a_0 t \tag{3.31}$$

This implies that the universe is going to expand just because of the curvature. It's called free expansion. However, if $k = 1$ (closed universe), then in the beginning of time the first term in (3.7) dominates; and hence, the universe is expanding because H^2 is positive; however, after a sufficient amount of time, the second term in (3.7) dominates; and hence the universe starts collapsing because H^2 is negative now.

Let's assume that we are in a matter-dominated universe. Then, the evolution of the scale factor for the three different values of k with respect to time is shown in Figure 3.1 [26].

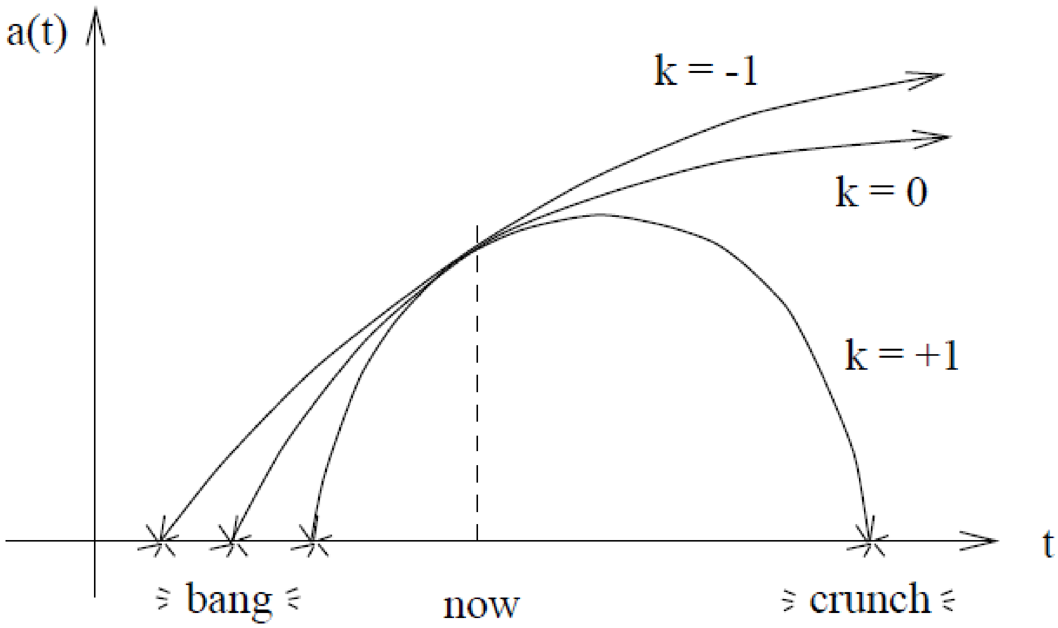

Figure 3.1: Three different scenarios for the end of the universe

To summarize, Table 3.1 lists the evolution and properties of different universes chronologically dominant as the universe expands (as $a(t)$ increases).

Universe Type	Evolution	$a(t)$
Radiation-dominated	$\propto \frac{1}{a^4}$	$\propto t^{\frac{1}{2}}$
Matter-dominated	$\propto \frac{1}{a^3}$	$\propto t^{\frac{2}{3}}$
Curvature-dominated	$\frac{1}{a^2}$	$\propto t^*$
Vacuum-dominated	$\propto$ constant	$\propto e^{H_0 t}$

* For negative curvature $k = -1$

Table 3.1: Properties of different types of universes

Furthermore, if we define an equation of state to be

$$w = \frac{p}{\rho} \tag{3.32}$$

then we substitute this equation of state in the fluid equation (or continuity equation) (3.18) and solve the differential equation we would get

$$\rho \propto a^{-3(1+w)} \tag{3.33}$$

Equation (3.33) generalizes all the cases that we have mentioned above by specifying the equation of state.

$*$ Non-relativistic matter $w = 0 \rightarrow \rho \propto a^{-3}$

$*$ Relativistic matter (e.g. radiation) $w = \frac{1}{3} \rightarrow \rho \propto a^{-4}$

$*$ Vacuum energy $w = -1 \rightarrow \rho \propto \textit{constant}$

It's informing to look at the thermal history of the universe that's given in Figure 3.2 [27]

Event	time t	redshift z	temperature T
EW phase transition	20 ps	10^{15}	100 GeV
QCD phase transition	20 μs	10^{12}	150 MeV
Neutrino decoupling	1 s	6×10^9	1 MeV
Electron-positron annihilation	6 s	2×10^9	500 keV
Big Bang nucleosynthesis	3 min	4×10^8	100 keV
Matter-radiation equality	60 kyr	3400	0.75 eV
Recombination	260–380 kyr	1100–1400	0.26–0.33 eV
CMB decoupling	380 kyr	1000–1200	0.23–0.28 eV
Reionization	100–400 Myr	11–30	2.6–7.0 meV
Dark energy-matter equality	9 Gyr	0.4	0.33 meV
Present	13.7 Gyr	0	0.24 meV

Figure 3.2: Major events in the thermal history of the universe

3.5 Problems with the Traditional Big Bang Theory

The traditional big bang theory solved some of the problems in cosmology such as the existence of the CMB and the expansion of the universe. However, there have been some persisting problems that cannot be explained by the traditional big bang theory. These problems are the flatness and horizon problems. There were some other problems like the prediction that at high temperatures in the beginning of the universe an abundance of magnetic monopoles and heavy particles according to GUT (if GUT exist) were created however today they aren't observed. The first two problems are problems that are imposed by nature; a.k.a external problems. However, the third problem is generated from problems of theories beyond the Standard Model of particle physics; a.k.a internal problems [28].

- Flatness Problem:

We can write Friedmann equation (3.7) as

$$\Omega_{tot}(t) - 1 = \frac{k}{a^2 H^2} \tag{3.34}$$

The evolution of the denominator on the RHS for matter-dominated and radiation dominated universe are

$$\begin{aligned} a^2 H^2 &\propto t^{-\frac{2}{3}} \qquad \text{matter-dominated unvierse} \\ a^2 H^2 &\propto t^{-1} \qquad \text{radiation-dominated unvierse} \end{aligned} \tag{3.35}$$

and hence

$$\begin{aligned} \Omega_{tot}(t) - 1 &\propto t^{\frac{2}{3}} \qquad \text{matter-dominated unvierse} \\ \Omega_{tot}(t) - 1 &\propto t \qquad \text{radiation-dominated unvierse} \end{aligned} \tag{3.36}$$

We can notice that the difference $\Omega_{tot} - 1$ increases with time for both scenarios. However, we know that the universe is flat from experiment. Moreover, we can deduce that the flat curvature case is an unstable configuration because if Ω_{tot} deviates little bit from 1 it will start evolving with time.

Another way of viewing the problem is to examine how much accurate Ω_{tot} must be in the beginning of time to produce the flatness that we observe today. In other words, it's to what extent the initial condition on Ω_{tot} must be tuned without an explanation for the tuning. Let's assume a universe that has only radiation to simplify the calculation and illustrate the point clearly. Let's look at the ratio

$$\frac{\Omega_{tot}(t_p) - 1}{\Omega_{tot}(t_0) - 1} = \left(\frac{\dot{a}(t_0)}{\dot{a}(t_p)}\right)^2 = \left(\frac{t_p}{t_0}\right) \tag{3.37}$$

where $t_0 \simeq 4 \times 10^{17}$ sec and t_p is the time of the process in question. Hence, the ratio (3.37) for [29] [30]

$*$ At decoupling ($t \simeq 10^{13}$ sec) requires $\Omega_{tot}(t_{dec}) - 1 \leq 10^{-5}$

$*$ At nucleosynthesis ($t \simeq 1$ sec) requires $\Omega_{tot}(t_{nuc}) - 1 \leq 10^{-18}$

As we go back in time, the accuracy of Ω to be initial exactly 1 increases without an explanation.

- Horizon Problem:

The temperatures across two opposite points in the CMB are almost the same. The straightforward and intuitive explanation for this phenomenon is that these two points reached thermal equilibrium because they have been in contact before. However, if we compare the time at which these two points were emitted (decoupling or recombination time) and the present time, and calculate the distance between them via (3.13)

$$\int_{t_*}^{t_{dec}} \frac{dt}{a(t)} \ll \int_{t_{dec}}^{t_{present}} \frac{dt}{a(t)} \tag{3.38}$$

then, it can be inferred that the two points couldn't have the time to reach thermal equilibrium. Equation (3.38) means that the distance light could travel before the microwave background was released is much smaller than the present horizon distance. Hence, the question now is: How can two regions that haven't been in contact before be at thermal equilibrium?

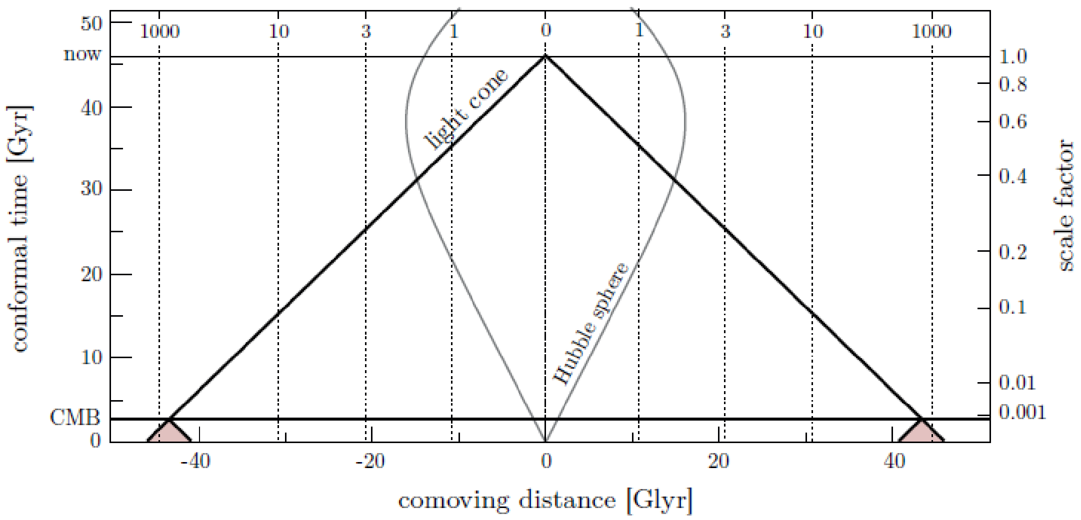

Figure 3.3: The past light cones of two points at recombination don't overlap

In Figure 3.3 [27], the problem is illustrated how two particle horizons in a causal diagram don't overlap yet they are in thermal equilibrium. According to the hot big bang theory, the x-axis in Figure 3.3 represents a singularity for the beginning of the universe. The Hubble sphere is the distance beyond which the recession velocity is greater the the speed of light, and it's not a horizon [31].

3.6 Inflation

Inflation was designed to solve problems in the traditional cosmology such as the flatness and horizon problem [32]. Let's define inflation, then see how it solves the previously mentioned problems.

Inflation can be defined in many ways that are equivalent. It's a period of time when the scale factor was accelerating

$$\ddot{a} > 0 \quad \leftrightarrow \quad \frac{d}{dt}(\dot{a}) > 0 \quad \leftrightarrow \quad \frac{d}{dt}(aH) > 0 \tag{3.39}$$

If we apply the condition of inflation (3.39) to (3.6)

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) \tag{3.40}$$

we notice that $\rho + 3p < 0$ in order for inflation to occur. Density is always positive; hence, pressure must be negative $p < -\frac{\rho}{3}$. Moreover, inflation must hold for a period of time, after that it must have a graceful exit to convert its energy to conventional matter. An accelerating stage for the scale factor can be achieved if we have a cosmological constant term $\frac{\Lambda}{3}$ in (3.7), and after solving it for late times, the scale factor obtained

$$a(t) = e^{Ht} = e^{\sqrt{\frac{\Lambda}{3}}t} \tag{3.41}$$

However, driving inflation from a cosmological constant has some problems such as it doesn't have a graceful exit.

3.6.1 Flatness Problem Revisited with Inflation

If we plug (3.41) in (3.34), then we would get

$$\Omega_{tot}(t) - 1 \propto e^{-2Ht} \tag{3.42}$$

Hence, this exponential evolution ensures to make Ω_{tot} to be very close to 1; and hence, a minuscule deviation is only possible at very late times as shown in Figure 3.4 [23].

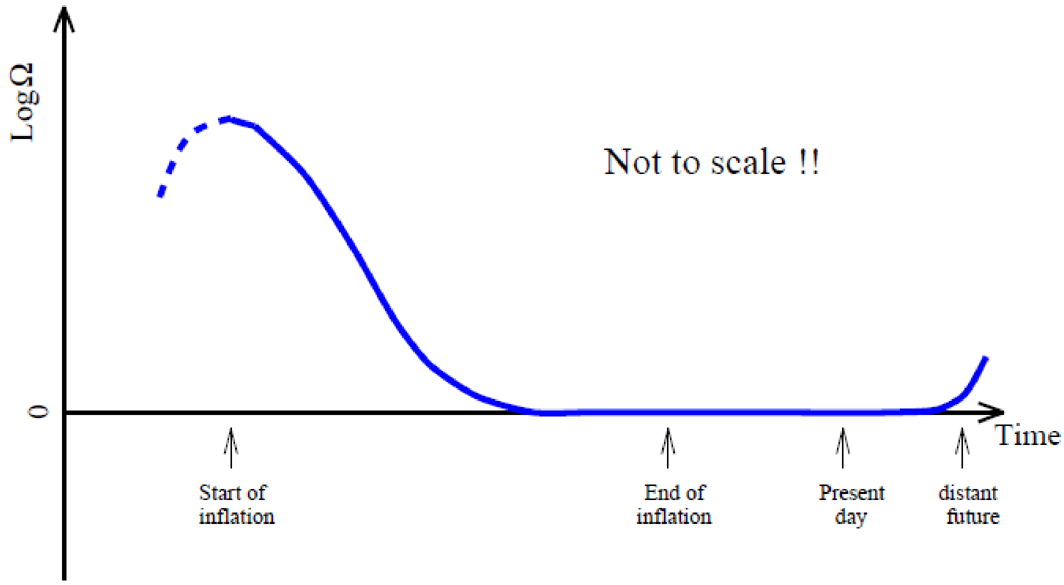

Figure 3.4: Inflation damps the amplitude of Ω_{tot} to 1

Therefore, we can deduce that at early times the universe was stretched to become flat to an extent that the increase in (3.36) doesn't affect the flatness of the universe much. You can think of a manifold that has a complicated structure in the beginning. After stretching it extremely enough, this manifold would look flat to an observer on its surface. It's similar to an observer who sees the surface of Earth to be flat because it's stretched enough that the observer can't perceive the global geometry of Earth.

3.6.2 Horizon Problem Revisited with Inflation

Inflation pushes the singularity of the Big Bang to a negative conformal time value; thus, giving time for two opposite regions to be in thermal contact. Figure

3.5 [27] illustrates how two regions in the CMB are in causal contact because inflation extends the slants of the particle horizon cones to the past.

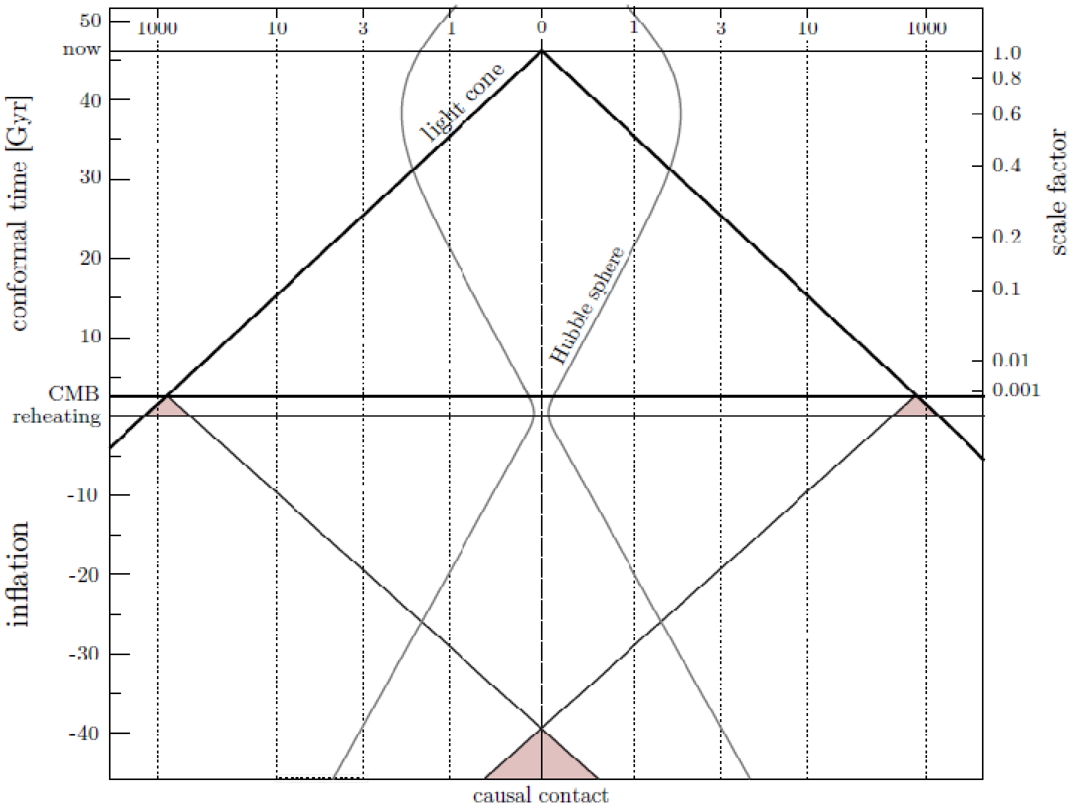

Figure 3.5: Inflation pushes the singularity to a negative conformal time value

Inflation insures that the period of time between the beginning of inflation and decoupling is larger than the period between decoupling and the present moment

$$\int_{t_*}^{t_{dec}} \frac{dt}{a(t)} \gg \int_{t_{dec}}^{t_{present}} \frac{dt}{a(t)} \tag{3.43}$$

For inflation, $\tau = 0$ represents not a singularity, but a transition phase between an inflationary period (that solves the flatness and horizon problems) and the traditional Big Bang theory. Hence, in Figure 3.5, the Hubble sphere isn't equal to zero at the reheating period because it's not a singularity. Moreover, the boundary of the Hubble sphere during inflation can be obtained by considering the third definition of inflation in equation (3.39) that describes the shrinking of the Hubble sphere in comoving coordinates.

In order to measure how much inflation is required to solve the flatness and horizon problem, cosmologists refer to the number of e-foldings as

$$N = \int_{t_i}^{t_f} H dt \tag{3.44}$$

where t_i and t_f are the times when inflation began and ended respectively.

3.7 Inflation Scalar Field

We are going to deal with a flat universe $k = 0$ from now on as dictated by inflation. The action for a scalar field with a potential

$$S = -\int \sqrt{-g}\Big[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + V(\phi)\Big]d^4x \tag{3.45}$$

The energy-momentum tensor is by definition

$$T_{\mu\nu} \equiv \frac{-2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} \tag{3.46}$$

We vary the action (3.45) with respect to the metric and get

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\Big(\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - V(\phi)\Big) \tag{3.47}$$

Then, equating (2.11) and (3.47), we deduce

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{3.48}$$

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi) \tag{3.49}$$

Substituting (3.48) in (3.7)

$$H^2 = \frac{8\pi G}{3}\Big(\frac{1}{2}\dot{\phi}^2 + V(\phi)\Big) \tag{3.50}$$

Because we have a homogeneous and isotropic universe which are symmetries of our FLRW metric, for a perfect fluid the diagonal terms are equal to zero ($-\dot{\phi}\partial_i\phi = T^i_0 = 0$). Hence, the inflation field should not depend on spatial coordinates

$$\phi = \phi(t) \tag{3.51}$$

Varying the action with respect to the scalar field ϕ

$$\delta_\phi S = -\int \sqrt{-g}\Big[\frac{1}{2}g^{\mu\nu}\delta(\partial_\mu\phi)\partial_\nu\phi + V_{,\phi}\delta\phi\Big]d^4x \tag{3.52}$$

where $\frac{\partial V}{\partial \phi} = V_{,\phi}$. Integrating by parts and $\delta_\phi S = 0$ yields [33]

$$\frac{\delta_\phi S}{\delta\phi} = 0 = \int \Big[\partial_\mu\Big(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi\Big) - \sqrt{-g}V_{,\phi}\Big]d^4x \tag{3.53}$$

Note that the D'Alembert operator gives

$$\Box\phi \equiv \frac{1}{\sqrt{-g}}\partial_\mu\Big(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi\Big) = V_{,\phi} \tag{3.54}$$

$\sqrt{-g} = a^3 r^2 \sin\theta$ for metric (3.2); hence,

$$\begin{aligned} \Box\phi &= \frac{1}{a^3 r^2 \sin\theta}\partial_\mu\left(a^3 r^2 \sin\theta g^{\mu\nu}\partial_\nu\phi\right) \\ &= -\ddot{\phi} - 3H\dot{\phi} + \frac{1}{a^2}\nabla^2\phi = V_{,\phi} \end{aligned} \tag{3.55}$$

However, according to (3.51) there should be no spatial derivatives for the scalar field. Therefore, we arrive to a very important equation

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0 \tag{3.56}$$

Equations (3.50) and (3.56) make a system of equations that can inform us about the evolution of the scalar field ϕ and the scale factor $a(t)$ for a given potential.

3.8 Slow-roll Approximation

Equation (3.56) can be viewed as the equation for the harmonic oscillator with a friction term that damps its initial velocities and acceleration. We can re-write (3.6) as

$$\frac{\ddot{a}}{a} = H^2 + \dot{H} = H^2(1-\epsilon) \tag{3.57}$$

where

$$\epsilon = -\frac{\dot{H}}{H^2} = \frac{3}{2}(w_\phi + 1) = \frac{M_{Pl}^2}{2}\frac{\dot{\phi}^2}{H^2} \tag{3.58}$$

and where $8\pi G = \frac{1}{M_{Pl}^2}$ and M_{Pl}^2 is the reduced Planck mass. The de Sitter limit is when

$$\frac{\ddot{a}}{a} \gg 0 \quad \Longrightarrow \quad w_\phi \to 0 \quad \Longrightarrow \quad \epsilon \to 0 \quad \Longrightarrow \quad \dot{\phi}^2 \ll V(\phi) \tag{3.59}$$

Moreover, the potential is greater than the kinetic energy; thus, $H \propto \sqrt{V}$. This entails that to make inflation sustain for a long period of time

$$|\ddot{\phi}| \ll |3H\dot{\phi}|, |V_{,\phi}| \tag{3.60}$$

Hence, equations (3.56) and (3.50) are simplified to

$$H^2 = \frac{V}{3M_{Pl}^2} \tag{3.61}$$

$$3H\dot{\phi} = -V_{,\phi} \tag{3.62}$$

Substituting these equations in (3.59)

$$\frac{M_{Pl}^2}{2}\frac{\dot{\phi}^2}{H^2} = \epsilon \ll 1 \quad \Longrightarrow \quad \frac{M_{Pl}^2}{2}\left(\frac{V_{,\phi}}{V}\right)^2 = \epsilon_V \ll 1 \tag{3.63}$$

Let's define a parameter to capture condition (3.60)

$$\eta = -\frac{\ddot{\phi}}{H\dot{\phi}} \ll 1 \tag{3.64}$$

and

$$\eta_V = M_{Pl}^2 \frac{V_{,\phi\phi}}{V} \ll 1 \tag{3.65}$$

Hence, the slow-roll conditions are

$$\epsilon_V \ll 1 \quad , \quad \eta_V \ll 1 \tag{3.66}$$

It's important to distinguish between the *Hubble Slow-Roll Parameters* (HSRP) ϵ and η, and the *Potential Slow-Roll Parameters* (PSRP) ϵ_V and η_V. The first type has clearer geometrical interpretation and some analytic properties; while the second type is suitable to study inflation for a given potential [34]. They can be related as [35]

$$\epsilon \approx \epsilon_V \quad , \quad \eta \approx \eta_V - \epsilon_V \tag{3.67}$$

Note that the first parameter in (3.66) refers to the slope of the potential; thus, it makes sure that it's very flat (de Sitter limit (3.59)) in order to roll slowly for a long period of time. It also informs us when inflation should start. The second parameter refers to the curvature of the potential and informs us how much inflation is required.

Chapter 4

Mimetic Dark Matter

In a recent work [1], Einstein's theory of gravity was reformulated by introducing a physical metric that's built from an auxiliary metric and first-order partial derivatives of a scalar field. By working out the equations for the new metric; a new term emerges in Einstein's equation that represents a new longitudinal mode that can mimic cold dark matter even in the absence of normal matter and without introducing any extra degrees of freedom. This section has detailed calculations of equations that are in [1][2].

4.1 Equations of motion

Let's define a physical metric $g_{\mu\nu}$ in terms of an auxiliary metric $\tilde{g}_{\mu\nu}$ and a scalar field ϕ

$$g_{\mu\nu} = (\tilde{g}^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi)\tilde{g}_{\mu\nu} \equiv P\tilde{g}_{\mu\nu} \tag{4.1}$$

Note that the physical metric $g_{\mu\nu}$ is invariant under conformal transformation $\tilde{g}_{\mu\nu} \to \Omega^2\tilde{g}_{\mu\nu}$.

Einstein-Hilbert action (2.9) is now defined as

$$S = -\frac{1}{2}\int \sqrt{-g(\tilde{g}_{\mu\nu},\phi)}\big[R(g_{\mu\nu}(\tilde{g}_{\mu\nu},\phi)) + \mathcal{L}_m\big]d^4x \tag{4.2}$$

Let's write the variation of the metric in terms on the auxiliary metric and the scalar field

$$\begin{aligned}
\delta g_{\alpha\beta} &= P\delta\tilde{g}_{\alpha\beta} + \tilde{g}_{\alpha\beta}\delta P \\
&= P\delta\tilde{g}_{\alpha\beta} + \tilde{g}_{\alpha\beta}\Big(\delta\tilde{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + 2\tilde{g}^{\mu\nu}\partial_\mu\delta\phi\partial_\nu\phi\Big) \\
&= P\delta\tilde{g}_{\alpha\beta} + \tilde{g}_{\alpha\beta}\Big(-\tilde{g}^{\kappa\mu}\tilde{g}^{\lambda\nu}\delta\tilde{g}_{\mu\nu}\partial_\kappa\phi\partial_\lambda\phi + 2\tilde{g}^{\kappa\lambda}\partial_\kappa\delta\phi\partial_\lambda\phi\Big) \\
&= P\delta\tilde{g}_{\mu\nu}\delta^\mu_\alpha\delta^\nu_\beta - \delta\tilde{g}_{\mu\nu}\tilde{g}_{\alpha\beta}(\tilde{g}^{\kappa\mu}\tilde{g}^{\lambda\nu}\partial_\kappa\phi\partial_\lambda\phi) + 2\tilde{g}_{\alpha\beta}\tilde{g}^{\kappa\lambda}\partial_\kappa\delta\phi\partial_\lambda\phi \\
&= P\delta\tilde{g}_{\mu\nu}\Big(\delta^\mu_\alpha\delta^\nu_\beta - g_{\alpha\beta}g^{\kappa\mu}g^{\lambda\nu}\partial_\kappa\phi\partial_\lambda\phi\Big) + 2g_{\alpha\beta}g^{\kappa\lambda}\partial_\kappa\delta\phi\partial_\lambda\phi
\end{aligned} \tag{4.3}$$

Then we vary (4.2) with respect to the physical metric as in (4.3)

$$\begin{aligned}
\delta S &= -\frac{1}{2}\int d^4x\sqrt{-g}(G^{\alpha\beta} - T^{\alpha\beta}) \\
&\times \Big(\underbrace{P\delta\tilde{g}_{\mu\nu}\Big(\delta^\mu_\alpha\delta^\nu_\beta - g_{\alpha\beta}g^{\kappa\mu}g^{\lambda\nu}\partial_\kappa\phi\partial_\lambda\phi\Big)}_{①} + \underbrace{2g_{\alpha\beta}g^{\kappa\lambda}\partial_\kappa\delta\phi\partial_\lambda\phi}_{②}\Big)
\end{aligned} \tag{4.4}$$

Setting the variation of the metric to zero, the first term reduces to

$$(G^{\alpha\beta} - T^{\alpha\beta})① \Longrightarrow P\delta\tilde{g}_{\mu\nu}\Big[(G^{\mu\nu} - T^{\mu\nu}) - (G - T)g^{\mu\alpha}g^{\nu\beta}\partial_\alpha\phi\partial_\beta\phi\Big] = 0 \tag{4.5}$$

while the second term after integrating by parts (and noting that $\delta\phi$ at infinity vanishes) reduces to

$$(G^{\alpha\beta} - T^{\alpha\beta})② \Longrightarrow \frac{1}{\sqrt{-g}}\partial_\kappa\Big(\sqrt{-g}g^{\kappa\lambda}\partial_\lambda\phi(G - T)\Big) = 0 \tag{4.6}$$

Since $g^{\mu\nu} = \frac{1}{P}\tilde{g}^{\mu\nu}$, the scalar field satisfies the condition

$$g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = 1 \tag{4.7}$$

Notice that equation (4.5) can be re-expressed as

$$G^{\mu\nu} = T^{\mu\nu} + \tilde{T}^{\mu\nu} \tag{4.8}$$

where

$$\tilde{T}^{\mu\nu} = (G - T)g^{\mu\alpha}g^{\nu\beta}\partial_\alpha\phi\partial_\beta\phi \tag{4.9}$$

Comparing (4.9) with (2.11) leads to

$$\rho \equiv G - T, \quad U^\mu \equiv g^{\mu\alpha}\partial_\alpha\phi, \quad p \equiv 0 \tag{4.10}$$

Hence, $\tilde{T}^{\mu\nu}$ mimics the behavior of dust because the expressions in (4.10) are the properties of non-relativistic matter. For instance, if we have

$$ds^2 = dt^2 - \gamma_{ij}dx^i dx^j \tag{4.11}$$

where $\gamma_{ij} = a^2\delta_{ij}$ is a three dimensional metric and a is the scale factor. Note that this metric is similar to metric (3.2). If we take $\phi = t$ (synchronous gauge), which means that the hypersurfaces of constant ϕ are the same as the hypersurfaces of constant t, then equation (4.6) becomes

$$\partial_0\Big(\sqrt{det\gamma}(G - T)\Big) = 0 \tag{4.12}$$

Solving this

$$G - T = \frac{C(x^i)}{a^3} \tag{4.13}$$

Although in the absence of matter, equation (4.13) yields a density $(G - T)$ of how matter evolves. Hence, we have the behavior of dark matter without introducing any extra degrees of freedom and without introducing any new kind of matter. Moreover, if inflation is present for 60 e-folds, then the energy density of MDM would dilute. In the coming sections this problem is addressed again and solutions are suggested.

4.2 Applications to Cosmology

Consider the action [2] [36] [37]

$$S = \int d^4x\sqrt{-g}\Big[-\frac{1}{2}R(g_{\mu\nu}) + \lambda(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 1) - V(\phi) + \mathcal{L}_m(g_{\mu\nu,...})\Big] \tag{4.14}$$

where the cosmological constant can be absorbed in the potential $V(\phi)$ because it shifts $V(\phi)$ by a constant. Varying the action (4.14) with respect to λ (Lagrange multiplier) yields (4.7) which is the normalization condition $U^\mu U_\mu = -1$, where the scalar field ϕ represents the velocity potential. However, varying with respect to $g^{\mu\nu}$ yields

$$G_{\mu\nu} - 2\lambda\partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}V(\phi) = T_{\mu\nu} \tag{4.15}$$

Taking the trace of this equation gives the Lagrange multiplier

$$\lambda = \frac{1}{2}(G - T - 4V) \tag{4.16}$$

hence equation (4.15) becomes

$$G_{\mu\nu} = (G - T - 4V)\partial_\mu\phi\partial_\nu\phi + g_{\mu\nu}V(\phi) + T_{\mu\nu} \tag{4.17}$$

Note that equations (4.7) and (4.17) are equivalent to Einstein's equations with an extra longitudinal mode. Note that the degrees of freedom in MDM are the same as General Relativity according to the Hamiltonian formulation of Mimetic Gravity [38]; hence, there are no extra degrees of freedom.
By taking the covariant derivative of (4.17), and using the energy-momentum conservation $\nabla^\nu T_{\mu\nu} = 0$ and the Bianchi identity $\nabla^\nu G_{\mu\nu} = 0$ we get

$$\nabla^\nu\Big[(G - T - 4V)\partial_\mu\phi\partial_\nu\phi + g_{\mu\nu}V(\phi)\Big] = 0 \tag{4.18}$$

Applying the covariant derivative to (4.7)

$$\nabla^\rho(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi) = 2g^{\mu\nu}(\nabla^\rho\partial_\mu\phi)\partial_\nu\phi = 2g^{\mu\nu}(\nabla_\mu\partial^\rho\phi)\partial_\nu\phi = 0 \tag{4.19}$$

with $\partial_\mu\phi \neq 0$ for at least one index yields

$$\nabla^\nu\Big((G - T - 4V)\partial_\nu\phi\Big) = -\frac{\partial V}{\partial\phi} \tag{4.20}$$

The first and second terms in (4.17) can be identify as the new $\tilde{T}_{\mu\nu}$ with

$$\tilde{p} = -V \tag{4.21}$$

$$\tilde{\rho} = G - T - 3V \tag{4.22}$$

4.2.1 MDM Differential Equation

Consider metric (4.11) and a general solution to condition (4.7) with $T_{\mu\nu} = 0$

$$\phi = t \tag{4.23}$$

where the integration constant is removed for generality. Applying equations (4.21), (4.22), and (4.23) to (4.20)

$$\frac{1}{\sqrt{-g}}\partial_\nu\Big(\sqrt{-g}(\tilde{\rho} - V)\partial^\nu\phi\Big) = \frac{1}{a^3}\frac{d}{dt}\Big(a^3(\tilde{\rho} - V)\Big) = -\dot{V} \tag{4.24}$$

Solving this equation and integrating by parts lead to

$$\tilde{\rho} = \frac{3}{a^3}\int a^2 V da \tag{4.25}$$

The Friedmann equation can be obtained from the $\mu\nu = 0$ component in (4.17) in the absence of matter ($T_{\mu\nu} = 0$)

$$H^2 = \frac{\tilde{\rho}}{3} = \frac{1}{a^3}\int a^2 V da \tag{4.26}$$

The $\mu\nu = ij$ components yield

$$2\dot{H} + 3H^2 = V(t) \tag{4.27}$$

This equation can be obtained by multiplying (4.26) by a^3 and differentiating with respect to time. It's better to convert the integral in (4.26) to an ordinary differential equation. This is accomplished by introducing a new variable

$$y = a^{\frac{3}{2}} \tag{4.28}$$

then

$$H = \frac{2}{3}\frac{\dot{y}}{y} \quad , \quad \dot{H} = \frac{2}{3}\left(\frac{\ddot{y}}{y} - \left(\frac{\dot{y}}{y}\right)^2\right) \tag{4.29}$$

hence

$$\ddot{y} - \frac{3}{4}V(t)y = 0 \tag{4.30}$$

By solving this equation, we can know how $a(t)$ evolves with time given any potential. Therefore, using this equation we can find cosmological solutions for different potentials easily.

4.2.2 MDM Cosmological Perturbations

Let's consider the metric perturbations of MDM. We can notice that the off-diagonal elements in $\tilde{T}_{\mu\nu}$ are zero in first order. The perturbed metric in the Newtonian gauge is [39]

$$ds^2 = (1 + 2\Phi)dt^2 - (1 - 2\Phi)a^2(t)\delta_{ij}dx^i dx^j \tag{4.31}$$

and

$$\phi = t + \delta\phi \tag{4.32}$$

is the first-order perturbation in the scalar field. A term $\frac{1}{2}\gamma(\Box\phi)^2$ (where γ is a constant, and $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$) is added to action (4.14) for reasons to be mentioned later. The action becomes [2]

$$S = \int d^4x\sqrt{-g}\Big[-\frac{1}{2}R(g_{\mu\nu}) + \lambda(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 1) - V(\phi) + \frac{1}{2}\gamma(\Box\phi)^2\Big] \quad (4.33)$$

Let's vary the action with respect to the metric term by term

$$\delta S_1 = -\frac{1}{2}\int d^4x\sqrt{-g}G_{\mu\nu}\delta g^{\mu\nu} \quad (4.34)$$

$$\begin{aligned} \delta S_2 &= \int d^4x\Big[\delta(\sqrt{-g})\lambda(\underbrace{g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi}_{=1}-1) + \delta g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\lambda\sqrt{-g}\Big] \\ &= \int d^4x\sqrt{-g}\partial_\mu\phi\partial_\nu\phi\lambda\delta g^{\mu\nu} \end{aligned} \quad (4.35)$$

$$\begin{aligned} \delta S_3 &= -\int d^4x\delta(\sqrt{-g})V(\phi) \\ &= \int d^4x\sqrt{-g}\frac{1}{2}g_{\mu\nu}V\delta g^{\mu\nu} \end{aligned} \quad (4.36)$$

Note that we can use this useful identity

$$\Box\phi = \frac{1}{\sqrt{-g}}\partial_\mu\Big((\sqrt{-g}g^{\mu\nu}\partial_\nu)\phi\Big) = \chi \quad (4.37)$$

Hence

$$\begin{aligned}
\delta S_4 &= \underbrace{\frac{1}{2}\gamma\int d^4x\delta(\sqrt{-g})\chi^2}_{\delta S_{4A}} + \underbrace{\frac{1}{2}\gamma\int d^4x\sqrt{-g}\delta(\chi^2)}_{\delta S_{4B}} && (4.38)\\
\delta S_{4A} &= \frac{1}{2}\gamma\int d^4x\frac{-\sqrt{-g}}{2}g_{\mu\nu}\chi^2\delta g^{\mu\nu}\\
&= -\frac{1}{4}\gamma\int d^4x\sqrt{-g}g_{\mu\nu}\chi^2\delta g^{\mu\nu} && (4.39)\\
\delta S_{4B} &= \gamma\int d^4x\sqrt{-g}\chi\delta(\chi)]\\
&= \gamma\int d^4x\sqrt{-g}\chi\left[\delta\Big(\frac{1}{\sqrt{-g}}\Big)\sqrt{-g}\chi + \frac{1}{\sqrt{-g}}\delta(\sqrt{-g}\chi)\right.\\
&= \underbrace{\frac{1}{2}\gamma\int d^4x\sqrt{-g}g_{\mu\nu}\chi^2\delta g^{\mu\nu}}_{\delta S_{4B1}} + \underbrace{\gamma\int d^4x\chi\delta\Big(\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu)\phi\Big)}_{\delta S_{4B2}} && (4.40)\\
\delta S_{4B2} &= \gamma\int d^4x\chi\left[\partial_\mu\Big((\delta(\sqrt{-g})g^{\mu\nu}\partial_\nu)\phi\Big) + \partial_\mu\Big((\sqrt{-g}\delta g^{\mu\nu}\partial_\nu)\phi\Big)\right]\\
&= \gamma\int d^4x\chi\left[\partial_\mu\left(\Big(\frac{-1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}g^{\mu\nu}\partial_\nu\Big)\phi\right) + \partial_\mu\Big((\sqrt{-g}\delta g^{\mu\nu}\partial_\nu)\phi\Big)\right]\\
&= -\gamma\int d^4x\chi\partial_\mu\Big((\sqrt{-g}\delta g^{\mu\nu}\partial_\nu)\phi\Big) && (4.41)
\end{aligned}$$

Integrating by parts yields

$$\delta S_{4B2} = -\int d^4x\sqrt{-g}\frac{1}{2}\gamma(\chi_{,\nu}\phi_{,\mu} + \chi_{,\mu}\phi_{,\nu})\delta g^{\mu\nu} \tag{4.42}$$

Adding up all the variations

$$\begin{aligned}
\delta S &= \delta S_1 + \delta S_2 + \delta S_3 + \delta S_4\\
&= \delta S_1 + \delta S_2 + \delta S_3 + \delta S_{4A} + \delta S_{4B1} + \delta S_{4B2} && (4.43)
\end{aligned}$$

and then set $\frac{\delta S}{\delta g^{\mu\nu}} = 0$ we obtain

$$G^{\mu}_{\nu} = \tilde{T}^{\mu}_{\nu} \tag{4.44}$$

where

$$\tilde{T}^{\mu}_{\nu} = \Big(V + \gamma\Big(\phi_{,\alpha}\chi^{,\alpha} + \frac{1}{2}\chi^{2}\Big)\Big)\delta^{\mu}_{\nu} + 2\lambda\phi_{,\nu}\phi^{,\mu} - \gamma(\phi_{,\nu}\chi^{,\mu} + \chi_{,\nu}\phi^{,\mu}) \tag{4.45}$$

Equations (4.7) and (4.44) determine the scalar field, metric, and Lagrange multiplier λ. They are equivalent to Einstein's equation without any extra degree of freedom. Let's consider (4.23) with an integration constant as a general solution for (4.7) for metric (4.11), we obtain

$$\chi = \Box\phi = \ddot{\phi} + 3H\dot{\phi} = 3H \tag{4.46}$$

Let's consider (4.32) in (4.7) in the Newtonian gauge metric (4.31). Note that for metric (4.31) we have

$$g_{00} = (1+2\Phi) \quad , \quad g_{ij} = -(1-2\Phi)a^{2}\delta_{ij} \tag{4.47}$$

$$g^{00} = (1-2\Phi) \quad , \quad g^{ij} = -(1+2\Phi)a^{-2}\delta^{ij} \tag{4.48}$$

equation (4.7) becomes

$$(g_0^{\mu\nu} + g_1^{\mu\nu} + ...)(\delta^0_{\mu} + \partial_{\mu}\delta\phi)(\delta^0_{\nu} + \partial_{\nu}\delta\phi) = 1 \tag{4.49}$$

hence, the zeroth order

$$g_0^{\mu\nu}\delta^0_{\mu}\delta^0_{\nu} = g_0^{00}\delta^0_0\delta^0_0 = 1 \tag{4.50}$$

and the first order

$$\begin{aligned}
& g_0^{\mu\nu}\Big(\delta_\mu^0\partial_\nu\delta\phi+\delta_\nu^0\partial_{\mu\delta\phi}\Big)+g_1^{\mu\nu}\delta_\nu^0\delta_\mu^0=0 \\
\Rightarrow\;& 2g_0^{00}\partial_0\delta\phi+g_1^{00}=0 \\
\Rightarrow\;& g_1^{00}=-2\partial_0\delta\phi g_0^{00}=-2a^{-2}\Phi \\
\Rightarrow\;& \partial_0\delta\phi=\Phi\Rightarrow\partial^0\delta\phi=-\Phi
\end{aligned} \tag{4.51}$$

The expressions for $\tilde{T}^\mu_\nu$ for different components

$$\tilde{T}^0_0 = V+3\gamma\Big(\frac{3H^2}{2}-\dot{H}+2\lambda\Big) \tag{4.52}$$

$$\tilde{T}^0_i = 0 \tag{4.53}$$

$$\tilde{T}^i_j = \Big(V+\frac{3}{2}(2\dot{H}+3H^2)\Big)\delta^i_j \tag{4.54}$$

Perturbing $\tilde{T}^\mu_\nu$ we obtain

$$\begin{aligned}
\delta\tilde{T}^\mu_\nu &= \Big[\delta V+\gamma\Big(\delta(\phi_{,\alpha})\chi^{,\alpha}+\phi_{,\alpha}\delta(\chi^{,\alpha})+\chi\delta(\chi)\Big)\Big]\delta^\mu_\nu \\
&+ 2\lambda\Big(\delta(\phi_{,\nu})\phi^{,\mu}+\phi_{,\nu}\delta(\phi^{,\mu})\Big) \\
&- \gamma\Big(\delta(\phi_{,\nu})\chi^{,\mu}+\phi_{,\nu}\delta(\chi^{,\mu})+\delta(\chi_{,\nu})\phi^{,\mu}+\chi_{,\nu}\delta(\phi^{,\mu})\Big) \\
&= \Big[\delta V+\gamma\Big(\delta(\phi_{,0})\chi^{,0}+\delta(\phi_{,i})\chi^{,i}+\underbrace{\phi_{,i}\delta(\chi^{,i})}_{=0}+\phi_{,0}\delta(\chi^{,0})+\chi\delta(\chi)\Big)\Big]\delta^\mu_\nu \\
&+ 2\lambda\Big(\delta(\phi_{,\nu})\phi^{,\mu}+\phi_{,\nu}\delta(\phi^{,\mu})\Big) \\
&- \gamma\Big(\delta(\phi_{,\nu})\chi^{,\mu}+\phi_{,\nu}\delta(\chi^{,\mu})+\delta(\chi_{,\nu})\phi^{,\mu}+\chi_{,\nu}\delta(\phi^{,\mu})\Big)
\end{aligned} \tag{4.55}$$

Hence

$$\begin{aligned}\delta\tilde{T}^0_i &= 2\lambda\delta(\phi_{,i}) - \gamma\delta(\phi_{,i})\chi^{,0} - \gamma\delta(\chi_{,i}) \\ &= 2\lambda\delta(\phi_{,i}) - 3\gamma\delta(\phi_{,i})\dot{H} - \gamma\delta(\chi_{,i}) \end{aligned} \quad (4.56)$$

Moreover, let's calculate χ and $\delta\chi$ by using metric (4.31)

$$\begin{aligned}\chi &= \frac{1}{\sqrt{-g}}\partial_\mu\Big(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi\Big) \\ &= \frac{1}{a^3(1-2\Phi)\sqrt{1-4\Phi^2}}\partial_\mu\Big(a^3(1-2\Phi)\sqrt{1-4\Phi^2}g^{\mu\nu}\partial_\nu\phi\Big) \\ &= 3H(1-2\Phi)\dot{\phi} - 4\dot{\Phi}\dot{\phi} - 4(1+2\Phi)\Phi\dot{\Phi}\dot{\phi} + (1-2\Phi)\ddot{\phi} \\ &+ \frac{2\Phi_{,i}(1+2\Phi)a^{-2}\delta^{ij}\phi_{,j}}{1-2\Phi} + \frac{4\Phi\Phi_{,i}\delta^{ij}\phi_{,j}}{1-2\Phi} \\ &- 2\Phi_{,i}a^{-2}\delta^{ij}\phi_{,j} - (1+2\Phi)a^{-2}\delta^{ij}\phi_{,ij} \end{aligned} \quad (4.57)$$

and

$$\begin{aligned}\delta\chi &= 3H\delta\dot{\phi} - 6H\Phi - 4\dot{\Phi} + \delta\ddot{\phi} - \frac{\Delta}{a^2}\delta\phi \\ &= -3\delta\ddot{\phi} - 3H\delta\dot{\phi} - \frac{\Delta}{a^2}\delta\phi \end{aligned} \quad (4.58)$$

Einstein's $0-0$ equation using (4.17) and (4.52)

$$\begin{aligned}G^0_0 &= 3H^2 \\ H^2 &= \frac{V}{3} + \gamma(\frac{3}{2}H^2 - \dot{H}) + \frac{2}{3}\lambda \end{aligned} \quad (4.59)$$

and the $i-j$ equation using (4.17) and (4.54)

$$\begin{aligned} G^i_j &= 2\frac{\ddot{a}}{a}+\left(\frac{\dot{a}}{a}\right)^2 = 2\dot{H}+3H^2 \\ 2\dot{H}+3H^2 &= V+\frac{3}{2}(2\dot{H}+3H^2) \end{aligned} \tag{4.60}$$

and hence we obtain

$$2\dot{H}+3H^2 = \frac{2}{2-3\gamma}V \quad \text{or} \quad H^2 = \frac{2V}{3(2-3\gamma)}-\frac{2}{3}\dot{H} \tag{4.61}$$

Moreover, by substituting (4.61) in (4.59) we obtain

$$\lambda = \dot{H}(3\gamma-1) \tag{4.62}$$

On the one hand, the diagonal Einstein's equations remain the same up to a constant. On the other hand, the $0-i$ Einstein's perturbed equation using (4.51), (4.53), (4.56),(4.58), and (4.62) [40] [41]

$$\begin{aligned} 2(\dot{\Phi}+H\Phi)_{,i} &= 2\lambda\delta\phi_{,i}-3\gamma\dot{H}\delta\phi_{,i}-\gamma\delta\chi_{,i} \\ 2\dot{\Phi}+2H\Phi &= 2\lambda\delta\phi-3\gamma\dot{H}\delta\phi-\gamma\delta\chi \\ 2\delta\ddot{\phi}+2H\delta\dot{\phi} &= 3\gamma(\dot{H}\delta\phi+\delta\ddot{\phi}+H\delta\dot{\phi})-2\dot{H}\delta\phi+\gamma\frac{\Delta}{a^2}\delta\phi \end{aligned} \tag{4.63}$$

hence

$$\delta\ddot{\phi}+H\delta\dot{\phi}+\dot{H}\phi-\frac{c_s^2}{a^2}\Delta\delta\phi = 0 \tag{4.64}$$

where

$$c_s^2 = \frac{\gamma}{2-3\gamma} \tag{4.65}$$

Convert (4.64) to Fourier space, and then for short wavelength perturbation H and $\dot{H}$ are neglected because $\lambda_{ph} = \frac{a}{k} \ll c_s H^{-1}$,

$$\delta\phi_k \propto e^{\pm i c_s k t} \tag{4.66}$$

However, for long wavelength perturbation, , the term $\frac{c_s^2}{a^2}\Delta\delta\phi$ is neglected because $\lambda_{ph} = \frac{a}{k} \gg c_s H^{-1}$ yielding,

$$\delta\phi = A\frac{1}{a}\int a dt \tag{4.67}$$

and

$$\Phi = \delta\dot{\phi} = A\frac{d}{dt}\Big(\frac{1}{a}\int a dt\Big) = A\Big(1 - \frac{H}{a}\int a dt\Big) \tag{4.68}$$

Note that the addition of $\frac{1}{2}\gamma(\Box\phi)^2$ in the Lagrangian makes it possible to distinguish between short and long wavelengths perturbation solutions.

In order to obtain quantum fluctuations, the action must be expanded to second order and the effects of gravity are neglected because these quantum perturbations are small in scale; hence, they can be considered in flat spacetime. Even if the effects of gravity are included in the analysis, the same result would be obtained. Action (4.33) to second order and integrating by parts yield

$$S = -\frac{1}{2}\int d^4x\Big(\frac{\gamma}{c_s^2}\delta\phi'\Delta\delta\phi' + ...\Big) \tag{4.69}$$

where the $\frac{1}{c_s^2}$ factor is included in order for mimetic dark matter not to vanish for higher derivatives. To choose the canonically normalized quantum fluctuation variable [42] [39] the $\delta\phi'\Delta\delta\phi'$ term can be considered as X'. Hence, $X = \delta\phi\Delta\delta\phi'$ converted to the Fourier space $k^2\delta\phi\delta\phi'$. Therefore, the variable is

$$v_k \sim \frac{\sqrt{\gamma}}{c_s}\, k\, \delta\phi_k \tag{4.70}$$

with vacuum fluctuation

$$\delta v_k \sim \frac{1}{\sqrt{\omega_k}} \sim \frac{1}{\sqrt{c_s k}} \tag{4.71}$$

and hence,

$$\delta\phi_k \sim \sqrt{\frac{c_s}{\gamma}}\, k^{-\frac{3}{2}} \tag{4.72}$$

During inflation,

$$\frac{1}{a}\int a\, dt \simeq H^{-1} \tag{4.73}$$

Matching long wavelength perturbations (4.67) with quantum perturbations (4.72), we obtain

$$A_k \sim \sqrt{\frac{c_s}{\gamma}}\, \frac{H_{c_s k\sim Ha}}{k^{3/2}} \tag{4.74}$$

Hence, the gravitational potential in comoving scales $\lambda \sim 1/k$

$$\Phi_\lambda \sim A_k\, k^{3/2} \sim \sqrt{\frac{c_s}{\gamma}}\, H_{c_s k\sim Ha} \tag{4.75}$$

where $H_{c_s k\sim Ha}$ is evaluated during inflation because quantum perturbations crossed the horizon during the inflationary stage and then during matter-dominated universe the horizon catches up with the quantum perturbations that were frozen out.

Chapter 5

A Cosmological Solution to MDM

5.1 Scale factor

Plugging in the exponential potential [23],

$$V = V_o e^{-\kappa t} \tag{5.1}$$

in equation (4.30) and define V_o to be a constant α

$$\ddot{y} - \frac{3}{4}\alpha e^{-\kappa t} y = 0 \tag{5.2}$$

Apply the transformation $s = \frac{\sqrt{-3\alpha}}{\kappa} e^{\frac{-\kappa t}{2}}$ to get

$$
\begin{aligned}
\frac{dy}{dt} &= \frac{dy}{ds}\frac{ds}{dt} = \frac{dy}{ds}\Big(- \frac{\sqrt{-3\alpha}}{2}\, e^{-\frac{\kappa t}{2}} \Big) \\
\frac{d^2y}{dt^2} &= \frac{d}{dt}\big(\frac{dy}{dt}\big) = \frac{dy}{ds}\Big(\frac{\kappa}{4}\sqrt{-3\alpha}\, e^{\frac{-\kappa t}{2}}\Big) + \underbrace{\frac{d}{dt}\big(\frac{dy}{ds}\big)}_{=\frac{ds}{dt}\frac{d^2y}{ds^2}}\Big(- \frac{\sqrt{-3\alpha}}{2}\, e^{-\frac{\kappa t}{2}} \Big) \\
&= \frac{dy}{ds}\Big(\frac{\kappa}{4}\sqrt{-3\alpha}\, e^{\frac{-\kappa t}{2}}\Big) + \frac{d^2y}{ds^2}\Big(- \frac{\sqrt{-3\alpha}}{2}\, e^{-\frac{\kappa t}{2}} \Big)\Big(- \frac{\sqrt{-3\alpha}}{2}\, e^{-\frac{\kappa t}{2}} \Big) \\
&= \frac{dy}{ds}\Big(\frac{\kappa}{4}\sqrt{-3\alpha}\, e^{\frac{-\kappa t}{2}}\Big) + \frac{d^2y}{ds^2}\Big(- \frac{3\alpha}{4}\, e^{-\kappa t} \Big) \qquad (5.3)
\end{aligned}
$$

Equation (5.2) becomes

$$
\begin{aligned}
-\frac{3\alpha}{4}\, e^{\kappa t}\frac{d^2y}{ds^2} + \frac{\kappa}{4}\sqrt{-3\alpha}\, e^{-\frac{\kappa t}{2}}\,\frac{dy}{ds} - \frac{3}{4}\alpha\, e^{\kappa t} y &= 0 \\
\frac{\kappa^2 s^2}{4}\frac{d^2y}{ds^2} + \frac{\kappa^2 s}{4}\frac{dy}{ds} + \frac{\kappa^2 s^2}{4} y &= 0 \qquad (5.4)
\end{aligned}
$$

Finally, multiply both sides by $\frac{4}{\kappa^2}$, differential equation (5.2) transforms to [43]

$$
s^2\frac{d^2y}{ds^2} + s\frac{dy}{ds} + s^2 y = 0 \qquad (5.5)
$$

The solution to this differential equation is well-known by Bessel's functions,

$$
y(t) = C_1 J_0\left(\frac{\sqrt{-3\alpha}}{\kappa} e^{\frac{-\kappa t}{2}}\right) + C_2 Y_0\left(\frac{\sqrt{-3\alpha}}{\kappa} e^{\frac{-\kappa t}{2}}\right) \qquad (5.6)
$$

where C_1 and C_2 are constants. The form of $a(t)$ is,

$$
a(t) = \left[C_1 J_0\left(\frac{\sqrt{-3\alpha}}{\kappa} e^{\frac{-\kappa t}{2}}\right) + C_2 Y_0\left(\frac{\sqrt{-3\alpha}}{\kappa} e^{\frac{-\kappa t}{2}}\right)\right]^{\frac{2}{3}} \qquad (5.7)
$$

It can be deduced that for $t \longrightarrow \infty$, $a(t) \propto t^{\frac{2}{3}}$ which is similar to the scaling factor of a matter-dominated universe. Note that we can always add a cosmological constant in the Lagrangian in order to get a universe that's dominated by a cosmological constant as it's observed nowadays. On the other hand, for $t \longrightarrow 0$,

$$
\begin{aligned}
y(t) &= C_1 e^{\sqrt{\frac{3\alpha}{4}}t} + C_2 e^{-\sqrt{\frac{3\alpha}{4}}t} \qquad (5.8) \\
a(t) &= \left[C_1 e^{\sqrt{\frac{3\alpha}{4}}t} + C_2 e^{-\sqrt{\frac{3\alpha}{4}}t}\right]^{\frac{2}{3}} \qquad (5.9)
\end{aligned}
$$

For α positive, $a(t)$ grows exponentially as in an inflationary universe. However, for α negative, $a(t)$ leads to an oscillatory universe in the beginning of time. The energy density of mimetic matter can be obtained as

$$
\tilde{\rho} = 3\Big(\frac{\dot{a}}{a}\Big)^2 = -\alpha e^{-\kappa t}\Big[\frac{C_1 J_{-1}(\frac{\sqrt{-3\alpha}}{\kappa}e^{\frac{-\kappa t}{2}}) + C_2 Y_{-1}(\frac{\sqrt{-3\alpha}}{\kappa}e^{\frac{-\kappa t}{2}})}{C_1 J_0(\frac{\sqrt{-3\alpha}}{\kappa}e^{\frac{-\kappa t}{2}}) + C_2 Y_0(\frac{\sqrt{-3\alpha}}{\kappa}e^{\frac{-\kappa t}{2}})}\Big]^2 \qquad (5.10)
$$

and the pressure,

$$
\tilde{p} = -V(t) = -\alpha e^{-\kappa t} \qquad (5.11)
$$

and the equation of state is

$$
w = \frac{\tilde{p}}{\tilde{\rho}} = \Big[\frac{C_1 J_0(\frac{\sqrt{-3\alpha}}{\kappa}e^{\frac{-\kappa t}{2}}) + C_2 Y_0(\frac{\sqrt{-3\alpha}}{\kappa}e^{\frac{-\kappa t}{2}})}{C_1 J_{-1}(\frac{\sqrt{-3\alpha}}{\kappa}e^{\frac{-\kappa t}{2}}) + C_2 Y_{-1}(\frac{\sqrt{-3\alpha}}{\kappa}e^{\frac{-\kappa t}{2}})}\Big]^2 \qquad (5.12)
$$

Moreover, it can be deduced from (5.2) that the density, pressure, and equation of state evolve like dust in a matter-dominated universe for $t \longrightarrow \infty$

$$
\begin{aligned}
\tilde{\rho} &= \frac{4}{3t^2} \qquad (5.13) \\
\tilde{p} &= -V(t) = 0 \qquad (5.14) \\
w &= 0 \qquad (5.15)
\end{aligned}
$$

For $t \longrightarrow 0$ and C_1 is not much greater than C_2,

$$\tilde{\rho} = \alpha \tag{5.16}$$

$$\tilde{p} = -\alpha \tag{5.17}$$

$$w = -1 \tag{5.18}$$

The equation of state for $t \longrightarrow 0$ is at the Phantom Divide Line similar to the equation of state of a cosmological constant that drives inflation but without a graceful exit. In order to trigger inflation in the beginning of time, $\ddot{a}(t) > 0$ must be satisfied. The acceleration equation is,

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\big(\varepsilon + 3p\big) \tag{5.19}$$

Hence, $\rho + 3p < 0$ must be true. Density is always positive; therefore, we must have negative pressure satisfying

$$p < -\frac{\rho}{3} \tag{5.20}$$

This is valid for t very small and positive α. A 70 e-folds inflation can be generated in this picture for any α because it satisfies the inequality. Let's consider another potential [2]

$$V(t) = \frac{\alpha t^{2n}}{e^{t\kappa} + 1} \qquad \text{for } n > -1 \tag{5.21}$$

given that $e^{t\kappa} >> t^{2n}$ always for positive time and suitable n. As $t \longrightarrow \infty$ and $t \longrightarrow 0$ it evolves as $a(t) \propto t^{\frac{2}{3}}$, and as $t \longrightarrow -\infty$ it generates inflation satisfying the 70 e-folds condition with

$$a(t) \propto e^{-\sqrt{\frac{\alpha}{3(n+1)^2}}\, t^2} \tag{5.22}$$

with

$$H = \frac{\dot{a}}{a} = -\sqrt{\frac{\alpha}{3}}\; t^n \tag{5.23}$$

Note that at $t \longrightarrow \infty$ both potentials (5.1) and (5.21) behaves the same because at $t \longrightarrow \infty$ (5.21) can be approximated as (5.1). In order to give an estimate for α, calculate (3.44) for 70 e-folds for (5.22), and noting that $t_i^2 > t_f^2$ for this model because inflation starts from $-\infty$; and hence

$$\alpha \simeq \Big(\frac{540(n+1)}{t_i^{n+1}}\Big)^2 \tag{5.24}$$

5.2 Perturbative Solution of the Scalar Field in the Newtonian Gauge

Considering a plane wave perturbation $\propto e^{ikx}$, equation (4.64) becomes,

$$\delta\ddot{\phi}_k + \frac{\dot{a}}{a}\delta\dot{\phi}_k + \Big(\frac{c_s^2 k^2}{a^2} + \frac{\ddot{a}}{a} - \big(\frac{\dot{a}}{a}\big)^2\Big)\delta\phi_k = 0 \tag{5.25}$$

By taking the limit of $t \longrightarrow \infty$ in (5.7) is similar when taking the limit of the argument of Bessel's function to zero because of the decaying exponential function inside the argument of Bessel's functions. So for small x [44]

$$J_0(s) \;\rightarrow\; 1 \tag{5.26}$$

$$Y_0(s) \;\rightarrow\; \frac{2}{\pi}\Big[\ln\big(\frac{s}{2}\big) + 0.5772...\Big] \tag{5.27}$$

Hence, the scaling factor (5.7) becomes as $t \longrightarrow \infty$,

$$a(t) = \Big[C_1 + C_2 \frac{2}{\pi}\Big(\ln\big(\frac{\sqrt{-3\alpha}}{2\kappa} e^{\frac{-\kappa t}{2}}\big) + 0.5772\Big)\Big]^{\frac{2}{3}} \tag{5.28}$$

This equation can be expressed again as,

$$a(t) = [C_1 + C_2 \frac{2}{\pi}(\frac{-\kappa t}{2} + \beta)]^{\frac{2}{3}} = [C_1' + C_2' t]^{\frac{2}{3}} \tag{5.29}$$

where $\beta = 0.5772 + \ln(\frac{\sqrt{-3\alpha}}{2\kappa})$ is just a constant, and $C_1' = C_1 + C_2 \frac{2}{\pi}\beta$ and $C_2' = -C_2 \frac{\kappa}{\pi}$. By substituting (5.29) in (4.64), and solving the differential equation, we can get an idea about the evolution of $\delta\phi$ at a very large time-scale and for different wavelengths. For short wavelength perturbation H and $\dot{H}$ are neglected because $\lambda_{ph} = \frac{a}{k} \ll c_s H^{-1}$,

$$\delta\phi \propto e^{\pm i c_s k t} \tag{5.30}$$

However, for long wavelength perturbation, the term $\frac{c_s^2}{a^2}\Delta\delta\phi$ is neglected because $\lambda_{ph} = \frac{a}{k} \gg c_s H^{-1}$; and hence, the solution to equation (4.64) is

$$\delta\phi = D_1 \pi + D_2 \beta - D_2 t \kappa \tag{5.31}$$

Equation (5.31) can also be obtained by a second method; if we plug equation (5.29) in (4.67), and choose $A \propto \kappa$ we would get equation (5.31) again. Note that the perturbation amplitude grows as a function of time only.

5.3 Quantum perturbations

In order to obtain the gravitational potential for comoving scales in this model from quantum perturbations, substitute (5.22), (5.23), and (5.24) in (4.75) with absolute value

$$\Phi_\lambda \sim \sqrt{\frac{c_s}{\gamma}} \times \sqrt{\frac{1}{3}} \, \frac{540(n+1)}{t_i^{n+1}} \, t^n|_{t:c_s k \sim Ha} \tag{5.32}$$

Note that γ is just a constant in the action (4.33). Hence, by choosing n, γ, and t_i appropriately, one can fit the value of the gravitational potential to be equal to the measured value $\propto 10^{-5}$ in CMB experiments [4], [5], and [6].

Chapter 6

Conclusion

In this thesis, an introduction to general relativity, cosmology, and mimetic dark matter was provided. Then, an exponential potential was plugged in the differential equation of MDM that relates any potential to any scaling factor in cosmology. At the limit of time goes to infinity, the density, pressure, and equation of state behave like dust in a matter-dominated universe, and in the limit of time goes to zero, a condition on the density can trigger inflation satisfying the 70 e-folds condition but for some initial conditions. Another general potential is given that satisfies the 70 e-folds condition. Furthermore, a first-order fluctuation is obtained. This can be accomplished by taking the limit of $a(t)$ at infinity and plugging it in the $0 - i$ perturbed Einstein's equation of a scalar field in the Newtonian gauge. Finally, it is worth noting that after performing quantum perturbations, the obtained amplitude fluctuation from MDM can be tuned to be of the same order as the CMB. Hence, MDM can provide a model for dark matter, inflation with 70 e-folds at early times, and CMB amplitude fluctuation at later times without introducing any new degrees of freedom.

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