AMERICAN UNIVERSITY OF BEIRUT

A Graph Theoretical Description of the Universal Grobner Bases of Toric Ideals of Graphs

by
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Contents

AN ABSTRACT OF THE THESIS OF

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Title:
A Graph Theoretical Description of the Universal Grobner Bases of Toric Ideals of Graphs

Abstract
We are going to study the paper: On the Universal Grobner Bases of Toric Ideals of Graphs by Christos Tatakis and Apostolos Thoma which gives a graph theoretical characterization of the elements of the universal Grobner basis of the toric ideal of a graph as well as a bound on their degrees.

Chapter 1

Introduction

We are going to study the paper: On the Universal Grobner Bases of Toric Ideals of Graphs [2]. In chapter 2, we introduce toric ideals associated to finite subsets of $\mathbb{N}^n$ and state a relation between their sets of circuits, Grobner bases and Graver bases. In chapter 3, we define graphs, state their basic properties, and characterize toric ideals associated to graphs. In chapter 4, we give the form of binomials that belong to the universal Grobner basis of the toric ideal of a graph. In chapter 5, we determine the largest degree of any binomial in the Graver basis and in the universal Grobner basis for $n \geq 4$. In chapter 6, we give counter examples to the true circuit conjecture and examples of primitive walks that do not belong to the universal Grobner basis.

Chapter 2

Toric Ideals

A monomial in a collection of variables $x_1, \ldots, x_m$ is a product $x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_m^{\alpha_m}$ where the α_i are non-negative integers. Alternatively, we can write a monomial as x^{α} where $x = x_1 \ldots x_m$ and $\alpha = (\alpha_1, \ldots, \alpha_m)$ is the vector of exponents in the monomial. The total degree of a monomial x^{α} is the sum of the exponents $\alpha_1 + \cdots + \alpha_m$ and is denoted by $|\alpha|$.

Example 2.1.

$x_1^2 x_3 x_4^3$ is a monomial in the variables x_1, x_2, x_3, x_4 with $\alpha = (2, 0, 1, 3)$ and $|\alpha| = 6$.

Let K be any field. A polynomial in the variables $x_1, \ldots, x_m$ is a finite linear combination of monomials with coefficients in K. The polynomial ring $K[x_1, \ldots, x_m]$ is the collection of all polynomials in $x_1, \ldots, x_m$ with coefficients in K.

Definition 2.2. *[1] A monomial order on $K[x_1, \ldots, x_m]$ is any relation $<$ on the set of* *monomials x^α in $K[x_1, \ldots, x_m]$ (or equivalently on the exponent vector $\alpha \in \mathbb{Z}^m_{\geq 0}$) satisfying:*

1. *$>$ is a total ordering relation which implies that the terms appearing in any polynomial can be uniquely listed in increasing or decreasing order under $>$.*

2. *$>$ is compatible with multiplication in $K[x_1, \ldots, x_m]$, in the sense that if $x^\alpha > x^\beta$ and x^γ is any monomial, then $x^\alpha x^\gamma = x^{\alpha+\gamma} > x^{\beta+\gamma} = x^\beta x^\gamma$.*

3. *$>$ is a well-ordering, that is every nonempty collection of monomials has a smallest element under $>$.*

In the polynomial ring $K[x_1, \ldots, x_m]$, we set up an ordering on the variables x_i:

$$x_1 > x_2 > \cdots > x_m.$$

Definition 2.3. *Lexicographic Order: Let x^α and x^β be monomials in $K[x_1, \ldots, x_m]$. We say $x^\alpha >_{lex} x^\beta$ if the leftmost nonzero entry in the difference $\alpha - \beta \in \mathbb{Z}^m$ is positive.*

Example 2.4.

In $K[x, y, z]$, with $x > y > z$, we have $x^6y^3z^2 >_{lex} x^4y^7z^{11}$.

Consider the polynomial ring $K[x_1, \ldots, x_m]$. Fix a monomial order $<$ on $K[x_1, \ldots, x_m]$.

The leading term of the polynomial $q(x_1, \ldots, x_m)$ in $K[x_1, \ldots, x_m]$, denoted by $LT(q)$, is the monomial term of greatest order with respect to $<$ in $q(x_1, \ldots, x_m)$. Let I be an ideal in $K[x_1, \ldots, x_m]$. The ideal generated by the set of leading terms of the polynomials in I is denoted by $LT(I)$.

Definition 2.5. *A Grobner basis for an ideal I in $K[x_1, \ldots, x_m]$ is a generating set $\{q_1, \ldots, q_n\}$ of I such that the leading terms of $q_1, \ldots, q_n$ generate $LT(I)$, that is, if $Q = \{q_1, \ldots, q_n\}$is a Grobner basis of an ideal I, then the ideal $LT(I)$ is generated by the set $\{LT(q_1), \ldots, LT(q_n)\}$.*

A polynomial q_i is monic if the coefficient of $LT(q_i)$ is one. The set Q of generators is reduced if, for $i = 1, \ldots, n$, q_i is monic and $LT(q_i)$ does not divide any monomial in (q_j) for all $j \neq i$.

Definition 2.6. *The universal Grobner basis of an ideal I is the union of all reduced Grobner bases of I with respect to all term orders.*

Theorem 2.7. *The universal Grobner basis of I is a finite subset of I and it is a Grobner basis for I with respect to all term orders [5].*

Theorem 2.8. *Every ideal in $K[x_1, \ldots, x_m]$ has a universal Grobner basis [4, 8].*

Fix a monomial order on $K[x_1, \ldots, x_m]$. For $i = 1, \ldots, m$, let $a_i = (b_1, \ldots, b_n)$ be vectors in $\mathbb{N}^n$ and let $A = \{a_i \quad | \quad 1 \leq i \leq m\} \quad \subseteq \quad \mathbb{N}^n$ and $\mathbb{N}A = \{l_1a_1 + \cdots + l_m a_m \quad | \quad l_i \in \mathbb{N}\}$. To each variable and monomial in $K[x_1, \ldots, x_m]$ we assign a vector in $\mathbb{N}^n$ called the $A - degree$ as follows. The $A - degree$ of each variable x_i is $deg_A(x_i) = a_i$ for $i = 1, \ldots, m$. For $u = (u_1, \ldots, u_m) \in \mathbb{N}^m$, the $A - degree$ of the monomial $x^u = x_1^{u_1} \ldots x_m^{u_m}$ is defined as $deg_A(x^u) = u_1a_1 + \cdots + u_m a_m$.

Definition 2.9. *The toric ideal I_A associated to A in the polynomial ring $K[x_1, \ldots, x_m]$ is the prime ideal generated by all binomials $x^u - x^v$ such that $deg_A(x^u) = deg_A(x^v)$. For binomials of this form, we set $deg_A(x^u - x^v) := deg_A(x^u)$.*

Definition 2.10. *The binomial* $x^u - x^v$ *is irreducible if whenever we factorize it into* $p \times q$*,* p *or* q *is a unit in* K*.*

Definition 2.11. *If the binomial* $x^u - x^v$ *in* I_A *is irreducible and there exists no other binomial* $x^w - x^z$ *in* I_A *such that* x^w *divides* x^u *and* x^z *divides* x^v*, then* $x^u - x^v$ *is called primitive. The Graver basis of* I_A *consists of the set of primitive binomials in* I_A *and is denoted by* Gr_A*.*

Definition 2.12. *Recall that a polynomial in the variables* $x_1, \ldots, x_m$ *is a finite linear combination of monomials with coefficients in* K*. The support of a monomial* $x_1^{\alpha_1} \ldots x_m^{\alpha_m}$ *is* $supp(x^\alpha) = \{x_i : \alpha_i \neq 0\}$*. The support of a polynomial is the union of the supports of its monomials.*

Definition 2.13. *The binomial* $x^u - x^v$ *has minimal support in* I_A *if its support does not properly contain the support of any other binomial in* I_A*. If the binomial* $x^u - x^v$ *is irreducible and has minimal support in* I_A *then* $x^u - x^v$ *is said to be a circuit. The set of circuits is denoted by* C_A*.*

The universal Grobner basis of I_A is denoted by U_A.

Theorem 2.14. *The connection between the set of circuits, the universal Grobner basis, and the Graver basis of* I_A *is given by* $C_A \subset U_A \subset Gr_A$ *[5].*

Chapter 3

Graphs and Their Toric Ideals

Definition 3.1. *A graph G is an ordered pair $(V(G), E(G))$ of sets. The set $V(G)$ is nonempty and its elements are called the vertices of G. The elements of $E(G)$ are called the edges of G. Each edge e in $E(G)$ joins two vertices in $V(G)$. If the edge e joins the vertices u and v, then $e = uv$ where u and v are the end vertices of e and u and v are said to be adjacent. The edge e is incident with each one of its end vertices. The graph G is finite if both $V(G)$ and $E(G)$ are finite.*

Definition 3.2. *A loop is an edge whose end vertices are the same.*

Definition 3.3. *If two or more edges of a graph G have the same end vertices then these edges are multiple edges of G.*

Definition 3.4. *A simple graph is a graph that has no loops and no multiple edges.*

Definition 3.5. *A complete graph K_n is a simple graph with n vertices such that every two vertices are adjacent.*

For the rest of this chapter, G is a finite simple graph with vertex set $V(G) = \{v_1, \dots, v_n\}$ and edge set $E(G) = \{e_1, \dots, e_m\}$.

Definition 3.6. *A walk in G of length q is a sequence $w = (v_{i_1}v_{i_2}, v_{i_2}v_{i_3}, \dots, v_{i_q}v_{i_{q+1}})$ of q edges of G joined end to end. The walk w connects the vertices v_{i_1} and $v_{i_{q+1}}$. If the length of w is even (respectively odd), then w is even (respectively odd). If $v_{i_{q+1}} = v_{i_1}$, then w is a closed walk.*

The inverse of the walk $w = (v_{i_1}v_{i_2}, \dots, v_{i_q}v_{i_{q+1}})$ is $(v_{i_{q+1}}v_{i_q}, \dots, v_{i_2}v_{i_1})$ and is denoted by $-w$. The walk $-w$ connects $v_{i_{q+1}}$ to v_{i_1}.

Remark: Although G is simple and thus has no multiple edges, the same edge e can appear more than once in a walk w. In such case, e is called multiple edge of the walk w.

Definition 3.7. *A path is a walk $w = (v_{i_1}v_{i_2}, \dots, v_{i_q}v_{i_{q+1}})$ in G such that $v_{i_j} \neq v_{i_k}$ for*

$j \neq k$.

Definition 3.8. *A cycle is a closed walk $w = (v_{i_1}v_{i_2}, \ldots, v_{i_q}v_{i_1})$ such that $v_{i_k} \neq v_{i_j}$ for all $1 \leq k < j \leq q$.*

A graph G is connected if every two vertices in G are connected by a walk in G.

Definition 3.9. *A subgraph S of a graph G is said to be maximal with respect to a given property P if S has property P and no other subgraph of G containing S has this property.*

Definition 3.10. *A connected component of a graph G is a maximal connected subgraph of G.*

Let G be a simple graph with $V(G) = \{v_1, \ldots, v_n\}$ and $E(G) = \{e_1, \ldots, e_m\}$. Let $K[e_1, \ldots, e_m]$ be the polynomial ring in the m variables $e_1, \ldots, e_m$ over an arbitrary field K. We identify each vertex v_i with the corresponding standard coordinate vector ϵ_i in $\mathbb{Z}^n$. For each edge $e = v_iv_j \in E(G)$, let $a_e = \epsilon_i + \epsilon_j$ and let $A_G = \{a_e | e \in E(G)\}$. The $A-degree$ of an edge $e = v_iv_j \in E(G)$ is $deg_A(e) = \epsilon_i + \epsilon_j$ and that of a monomial e^α is $\sum_{j=1}^{m} \alpha_j deg_A(e_j)$.

Let I_G be the toric ideal in $K[e_1, \ldots, e_m]$ generated by $e^u - e^v$ such that

$deg_A(e^u) = deg_A(e^v)$.

Let $w = (e_{i_1}, \ldots, e_{i_{2q}})$ be an even closed walk of the graph G. Let $E^+(w) = \prod_{k=1}^{q} e_{i_{2k-1}}$ and $E^-(w) = \prod_{k=1}^{q} e_{i_{2k}}$. Consider the binomial $B_w = \prod_{k=1}^{q} e_{i_{2k-1}} - \prod_{k=1}^{q} e_{i_{2k}}$.

Notice that $E^+(w) = e_{i_1}e_{i_3}\ldots e_{i_{2q-1}}$ and $E^-(w) = e_{i_2}e_{i_4}\ldots e_{i_{2q}}$. So we get the following:

$$\begin{aligned} deg_A E^+(w) &= 1 \times a_{e_{i_1}} + 1 \times a_{e_{i_3}} + 1 \times a_{e_{i_{2q-1}}} \\ &= \epsilon_{i_1} + \epsilon_{i_2} + \epsilon_{i_3} + \epsilon_{i_4} + \cdots + \epsilon_{i_{2q-1}} + \epsilon_{i_{2q}} \\ deg_A E^-(w) &= 1 \times a_{e_{i_2}} + 1 \times a_{e_{i_4}} + 1 \times a_{e_{i_{2q}}} \\ &= \epsilon_{i_2} + \epsilon_{i_3} + \epsilon_{i_4} + \epsilon_{i_5} + \cdots + \epsilon_{i_{2q}} + \epsilon_{i_{2q+1}} \end{aligned}$$

Since w is closed, $v_{i_1} = v_{i_{2q+1}}$ and hence $\epsilon_{i_1} = \epsilon_{i_{2q+1}}$. We deduce $deg_A E^+(w) = deg_A E^-(w)$ and so $B_w \in I_G$. In fact, I_G is generated by binomials of this form.[7]

The walk w can be considered to be a subgraph of G with vertices the vertices of w and edges the edges of w.

Consider the walk $w' = (e_{j_1}, \ldots, e_{j_t})$. We say w' is a subwalk of w and divides w if the edges of w' are also edges of w and if w' is of smaller length than the length of w.

Definition 3.11. *Let $w = (e_{i_1}, \ldots, e_{i_{2q}})$ be an even closed walk. Let $w^+ = \{e_{i_j} | j$ is odd $\}$ and $w^- = \{e_{i_j} | j$ is even$\}$. The edges of w^+ are said to be the odd edges of w and those of w^- are said to be the even edges of w. The walk w is primitive if $w^+ \cap w^- = \phi$ and there does not exist any even closed subwalk w' of smaller length such that $E^+(w')$ divides $E^+(w)$ and $E^-(w')$ divides $E^-(w)$.*

The binomial B_w corresponding to the walk w is primitive if and only if w is primitive.

Example 3.12.

Let $w = e_1e_2e_6e_7e_8e_9e_2e_3e_4e_5$ in figure 3.1. Then $E^+(w) = e_1e_6e_8e_2e_4$ and $E^-(w) = e_2e_7e_9e_3e_5$. Notice that $e_2 \in w^+ \cap w^-$ so $w^+ \cap w^- \neq \phi$. Let $w' = e_1e_2e_2e_3e_4e_5$. The edges of w' are also edges of w and w' is of smaller length than w, so w' is a subwalk of w. Also, $E^+(w') = e_1e_2e_4$ and $E^-(w') = e_2e_3e_5$. Hence $E^+(w')$ divides $E^+(w)$ and $E^-(w')$ divides $E^-(w)$. Therefore, w is not a primitive walk.

Definition 3.13. *If removing an edge (respectively vertex) of a graph yields a subgraph having more connected components than the original graph, we call this edge (respectively*

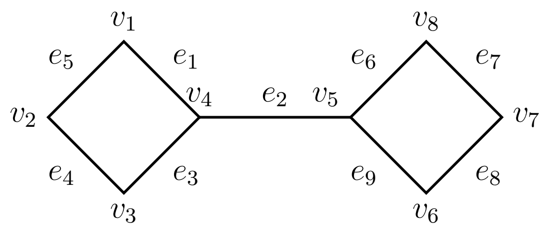

Figure 3.1:

vertex) a cut edge (respectively cut vertex).

Example 3.14.

If we remove e_2 from the graph in figure 3.1, we will have two disjoint connected subgraphs as shown in figure 3.2. Therefore e_2 is a cut edge.

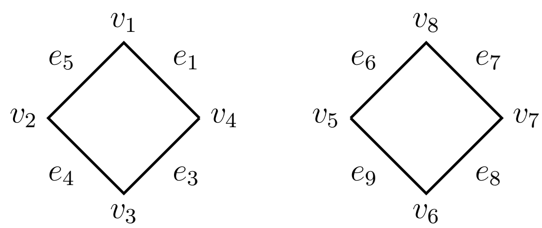

Figure 3.2:

If we remove v_4 in figure 3.1 we will get two disjoint connected subgraphs as shown in figure 3.3. So v_4 is a cut vertex.

Definition 3.15. *A biconnected graph is a connected graph having no cut vertex. A*

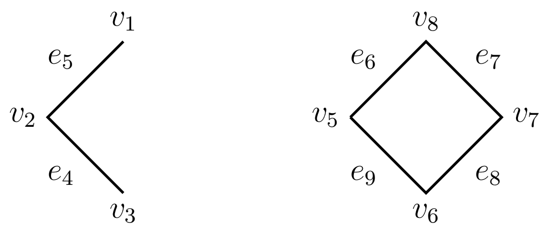

Figure 3.3:

maximal biconnected subgraph of a graph G is called a block of G.

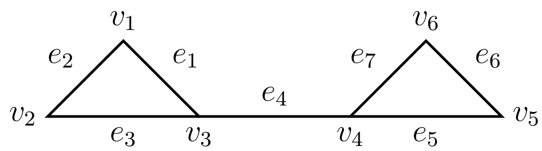

Figure 3.4:

Example 3.16.

The blocks of the graph in figure 3.4 are:

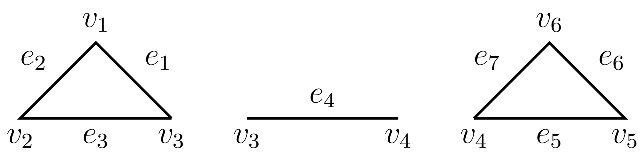

Figure 3.5:

Definition 3.17. *Let w be an even closed walk in G and B a block of w. If B contains two edges incident to a vertex v and both edges belong to w^+ or to w^- then v is called a sink of*

B.

Suppose w is a primitive walk and e is a cut edge of w. Since the walk w is primitive, then e is either in w^+ or w^- but not in both. Also, if the cut edge e appears only once, then there must exist another edge in the closed walk w joining the two connected components resulting from removing e from w. In this case, e is no longer a cut edge, so e appears at least twice in w. This means that both end vertices of the cut edge e in the primitive walk are sinks.

Example 3.18.

Consider the primitive walk $w = e_1e_2e_3e_4e_5e_6e_7e_4$ and the cut edge e_4 in the graph of figure 3.4. Notice that $w^+ = \{e_1, e_3, e_5, e_7\}$ and $w^- = \{e_2, e_4, e_6\}$. The end vertices of e_4 are v_3 and v_4. The vertex v_3 is common between the two odd edges e_1 and e_3 of B_1 and the vertex v_4 is common between the two odd edges e_5 and e_7 of B_3. Therefore both v_3 and v_4 are sinks.

Definition 3.19. *The incidence matrix of G is the $n \times m$ matrix $M = (m)_{ij}$ such that*

$m_{ij} = 1$ if v_i and e_j are incident and $m_{ij} = 0$ otherwise.

A row of M all of whose entries are zero represents an isolated vertex. Since each edge in G has two end vertices, then the sum of entries in each column is two.

Definition 3.20. *The valence of a vertex v of a graph G is the number of edges incident with v.*

By the definition of the incidence matrix M, the valence of a vertex v_i is the number of non zero entries in the i^{th} row of M.

Definition 3.21. *Let $\alpha = (\alpha_1, \ldots, \alpha_m)$ be a vector in $\mathbb{Z}^m$. The support of α is $supp(\alpha) = \{i \in \{1, \ldots, m\} \quad | \quad \alpha_i \neq 0\}$. Let S be a subset of $\mathbb{Z}^m$. The vector α is elementary in S if $supp(\alpha)$ does not properly contain $supp(\beta)$ for any nonzero vector β in S.*

The vector α can be written as $\alpha = \alpha_+ - \alpha_-$ where α_+ and α_- are two non negative vectors in $\mathbb{Z}^m$ with disjoint supports.

Let M be the incidence matrix of G and N the kernel of M in $\mathbb{Z}^m$, that is,

$N = \{\alpha \in \mathbb{Z}^m \quad | \quad M\alpha = 0\}$.

Proposition 3.22. *The vector $\alpha = \alpha_+ - \alpha_-$ belongs to N which is the kernel of M if and only if $deg_A(e^{\alpha_+}) = deg_A(e^{\alpha_-})$.*

Proof: Recall that if $e_j = v_l v_k$ then $deg_A(e_j) = \epsilon_l + \epsilon_k$ and that in M, $M_{ij} = 1$ if $i = l$ or $i = k$, and $M_{ij} = 0$ otherwise. So $deg_A(e_j) = M_{1j}\epsilon_1 + \cdots + M_{nj}\epsilon_n = \sum_{i=1}^{n} M_{ij}\epsilon_i$. For $\gamma \in \mathbb{N}^m$, $deg_A(e^\gamma) = \sum_{j=1}^{m} \gamma_j \sum_{i=1}^{n} M_{ij}\epsilon_i = \sum_{i=1}^{n} \epsilon_i \sum_{j=1}^{m} M_{ij}\gamma_j = M\gamma$. Therefore,

$deg_A(e^{\alpha_+}) = deg_A(e^{\alpha_-})$

$\Leftrightarrow M\alpha_+ = M\alpha_-$

$\Leftrightarrow \alpha$ belongs to N. ■

Recall that if $\beta = (\beta_1, \ldots, \beta_m)$ is a vector in $\mathbb{N}^m$, then $e^\beta = e_1^{\beta_1} \ldots e_m^{\beta_m}$ and

$supp(e^\beta) = \{e_i : \beta_i \neq 0\}$.

Definition 3.23. *For $\alpha \in N$, let G_α be the subgraph of G with vertex set*

$V_\alpha = \{v \in V(G) \quad | \quad v \in f^{\alpha_+}\}$ *and edge set*

$E_\alpha = \{e_i \in E(G) \quad | \quad e_i \in supp(e^{\alpha_+}) \bigcup supp(e^{\alpha_-})\}.$

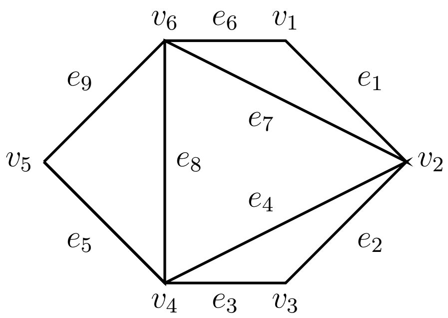

Figure 3.6:

Example 3.24.

The incidence matrix of the graph G in figure 3.6 is

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

The kernel of M is $N = \{r(1,-1,1,-1,0,-1,1,0,0) + s(1,0,0,-1,0,-1,0,1,0) +$

$t(1,-1,1,0,-1,-1,0,0,1) \quad | \quad r,s,t \in \mathbb{Z}\}$.

Consider $\alpha = (1,-1,1,-1,0,-1,1,0,0)$ which is a vector in N. Notice that

$\alpha_+ = (1,0,1,0,0,0,1,0,0)$ and $\alpha_- = (0,1,0,1,0,1,0,0,0)$. Then $e^{\alpha_+} = e_1e_3e_7 = v_1v_2^2v_3v_4v_6$

and $e^{\alpha_-} = e_2e_4e_6 = v_2^2v_3v_4v_1v_6$. The subgraph G_α has vertex set $V(G_\alpha) = \{v_1, v_2, v_3, v_4, v_6\}$

and edge set $E(G_\alpha) = \{e_1, e_2, e_3, e_4, e_6, e_7\}$.

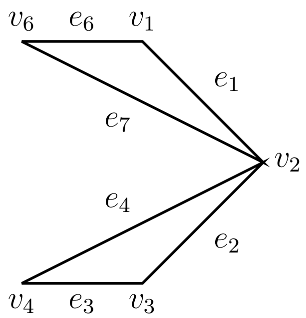

Figure 3.7: G_α

Remark: Let $\alpha \in N$. If G_α is connected, then α defines an even closed walk w in G_α such that each edge e_j in G_α is traversed α_j times in w. If G_α has k connected components $G_\alpha^1, \ldots, G_\alpha^k$, then α can be decomposed into $\alpha = c_1, \ldots, c_k$ where $c_1, \ldots, c_k$ are vectors with pairwise disjoint supports corresponding to $G_\alpha^1, \ldots, G_\alpha^k$. Each c_i defines an even closed walk w_i in G_α^i.

Proposition 3.25. *[7] Consider a graph G and its incidence matrix M. Let N be the kernel of M in $\mathbb{Z}^q$. If the vector α is an elementary vector in N, then G_α is:*

1. *an even cycle, or*

2. *two odd cycles intersecting in exactly one vertex, or*

3. *two vertex disjoint odd cycles joined by a path.*

Proof:

If α is an elementary vector of N then G_α must be connected by the previous remark.

Suppose that α is an elementary vector of N such that G_α is not an even cycle. We want to show that G_α turns out to be two odd cycles intersecting in exactly one vertex or two vertex disjoint odd cycles joined by a path. We do this by eliminating all other possibilities.

Case 1: Suppose G_α is an odd cycle $(x_0x_1, x_1x_2, \ldots, x_{2q}x_0)$. Let $deg_A(x_i) = y_i$. Then $deg_A(e^{\alpha_+}) = y_0 + y_1 + y_2 + y_3 + \cdots + y_{2q} + y_0 \neq y_1 + y_2 + y_3 + \cdots + y_{2q} = deg_A(e^{\alpha_-})$ so α does not belong to N. Therefore G_α can not be an odd cycle.

In cases $2, 3$ we consider the case where G_α is a path $(x_0x_1, x_1x_2, \ldots, x_{n-1}x_n)$.

Case 2: Suppose α is the path $(x_0x_1, x_1x_2, \ldots, x_{n-1}x_n)$. Then $deg_A(x_0)$ will appear in $deg_A(e^{\alpha+})$ but will not appear in $deg_A(e^{\alpha-})$. So α does not belong to N. So α can not be a path.

Case 3: α is a closed walk on G_α. Note that α must be even. If α starts at x_0 then $x_{n-1}x_n$ must be traversed at least twice consecutively since α is closed and α must pass through x_n and x_nx_{n-1} is the only edge adjacent to x_n. Removing this double occurrence of the edge x_nx_{n-1} from α gives an even closed subwalk so α is not elementary in N.

So the remaining case is that G_α properly contains a cycle. Here we also have two cases to consider.

Case 4: Suppose that G_α contains an even cycle $w = (x_0x_1, \ldots, x_{2n+1}x_0)$. Let $y_i = deg_A(x_i)$. So

$$deg_A(e^{\beta+}) = y_0 + \cdots + y_{2n+1}$$

and

$$deg_A(e^{\beta-}) = y_1 + \cdots + y_{2n+1} + y_0$$

Since e^{β_+} and e^{β_-} are equal, $\beta = \beta_+ - \beta_-$ belongs to N by propostion (3.22). The cycle w is a subgraph of G_α, so $supp(\beta)$ is properly contained in $supp(\alpha)$. Therefore α is not elementary in N, which is a contradiction. Therefore, G_α does not contain an even cycle.

Case 5: G_α consists of an odd cycle w together with some other vertices and edges. Let H_α be the subgraph of G_α whose edges are the edges of G_α not in w and whose vertices are those in G_α not in w together with the end vertices of the edges of H_α. Since $G_\alpha = w \bigcup H_\alpha$ is connected, there is at least one vertex in $w \bigcap H_\alpha$. Let $w = (x_0x_1, \ldots, x_nx_0)$ with $x_n \in H_\alpha$. Let $x_nx_{n+1}, \ldots, x_{l-1}x_l$ be a maximal path in H_α. If x_l has valence one in G_α then it contributes either to $deg_A(e^{\alpha_+})$ or to $deg_A(e^{\alpha_-})$ but not to both. This implies that $deg_A(e^{\alpha_+})$ and $deg_A(e^{\alpha_-})$ are not equal so α does not belong to N. Hence G_α contains an edge x_lx_k for some $k < l-1$.

Suppose $0 \le k < n$, that is, we return to a vertex in the cycle w other than x_n. Since w is an odd cycle, exactly one of the subwalks $(x_nx_0, x_0x_1, \ldots, x_{k-1}x_k)$ and $(x_kx_{k+1}, \ldots, x_{n-1}x_n)$ has the same parity as the path $(x_nx_{n+1}, \ldots, x_lx_k)$ and forms an even cycle w' with this path. If $e^{\beta_+} - e^{\beta_-}$ is the binomial corresponding to w', then $deg_A(e^{\beta_+}) = deg_A(e^{\beta_-})$. Therefore $\beta = \beta_+ - \beta_-$ belongs to N and $supp(\beta)$ is contained in $supp(\alpha)$. Then α is not an elementary vector in N which is a contradiction.

Therefore $n \leq k < l$. Consider the cycle $w'' = (x_k x_{k+1}, \ldots, x_l x_k)$. If w'' is an even cycle, then again we have an even cycle which is a subgraph of G_α. So α is not elementary and this is a contradiction. So w'' must be an odd cycle and we get in G_α two odd cycles, w and w'' that intersect in exactly one vertex if $k = n$ or are vertex disjoint and joined by a path if $k > n$.

The graph G_α can not contain any additional edges for otherwise it will properly contain an even closed walk. ■

Corollary 3.26. *Consider a finite connected graph G. If the binomial $B \in I_G$ is a circuit then $B = B_w$ where w is:*

1. *an even cycle, or*

2. *two odd cycles intersecting in exactly one vertex, or*

3. two vertex disjoint odd cycles joined by a path.

Proof: Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in \mathbb{Z}^m$ be an elementary vector in N. Let $\alpha_+ = (\overline{\alpha_1}, \overline{\alpha_2}, \ldots, \overline{\alpha_q})$ and $\alpha_- = (\overline{\overline{\alpha_1}}, \overline{\overline{\alpha_2}}, \ldots, \overline{\overline{\alpha_q}})$ where $\overline{\alpha_i} = max(0, \alpha_i)$ and $\overline{\overline{\alpha_i}} = -min(0, \alpha_i)$. Let $B_w = e_1^{\overline{\alpha_1}} e_2^{\overline{\alpha_2}} \ldots e_q^{\overline{\alpha_q}} - e_1^{\overline{\overline{\alpha_1}}} e_2^{\overline{\overline{\alpha_2}}} \ldots e_q^{\overline{\overline{\alpha_q}}}$ be the binomial corresponding to α where $w^+ = e^{\alpha_+}$ and $w^- = e^{\alpha_-}$. It is clear that B_w has minimal support in I_G if and only if α is elementary in N. By the previous proposition, we deduce that if B_w is a circuit then w will have one of the three forms 1 or 2 or 3. ■

Theorem 3.27. *[3] Let G be a graph and let w be an even closed walk of G. The walk w is primitive if and only if:*

1. Every block of w is a cycle or cut edge,

2. Every multiple edge of w is a double edge and a cut edge of w,

3. Every cut vertex of w belongs to exactly two blocks and is a sink of both.

Proof: Let w be a primitive walk in G. Let B be a block of w that is not a cut edge. Suppose B is not a cycle. Let $w = (e_{i_1}, \ldots, e_{i_{2s}})$ and let $w_B = (e_{i_{j_1}}, \ldots, e_{i_{j_q}})$ be the subwalk corresponding to the block B with $j_1 < j_2 < \cdots < j_q$. The edges of w_B are the edges of w that belong to B. If any two blocks in G intersect in more than one vertex, then when we remove one of these vertices, G will still be connected. Therefore, any two blocks of G intersect in at most one point which is a cut vertex of G. So w_B is a closed subwalk of w. If w_B is a closed walk but not a cycle, then there exists at least one vertex that appears twice in w_B. Due to the biconnectivity of block B, there must be at least two vertices that appear twice in w_B. Take these vertices to be u and v. The only way to write w_B is in the following order

$$w_B = (v, w_1, u, w_2, v, w_3, u, w_4, v)$$

where $w_1, \ldots, w_4$ are subwalks of w, or else u and v are cut vertices of B. Any closed subwalk w' of a primitive walk w must be of odd length or else $E^+(w')$ divides $E^+(w)$ and $E^-(w')$ divides $E^-(w)$ and then w is not primitive anymore. So the closed subwalks (v, w_1, u, w_2, v) and (v, w_3, u, w_4, v) are of odd lengths. Then the lengths of the two walks in each of the pairs (w_1, w_2) and (w_3, w_4) have opposite parities. Therefore, the length of w_3

has the same parity as the length of w_1 or the length of w_2. The first edge of w_1 is the first edge of w_B therefore it is in w^+. Since (v, w_1, u, w_2) is of odd length, the last edge of w_2 is in w^+. Suppose for example that w_2 and w_3 both have odd lengths, then $\gamma_1 = (v, w_3, u, w_2, v)$ is an even closed subwalk of w such that $E^+(\gamma_1)$ divides $E^+(w)$ and $E^-(\gamma_1)$ divides $E^-(w)$ and so w is not primitive anymore. This contradicts the fact that w is primitive, thus w_B can not be a closed walk that is not a cycle. So every block of w is a cut edge or a cycle. This proves 1.

Now we are going to show that every multiple edge of the walk w is a double edge and a cut edge of w. Let $e = uv$ be a multiple edge of w. Since w is primitive, then e belongs either to w^+ or to w^-. There are only two distinct ways in which e may appear, namely, $(\dots, u, e, v, \dots)$ and $(\dots, v, e, u, \dots)$. We have two cases to study.

The first one is thet e appears twice in the same way. Without loss of generality, we may assume the sequence (u, e, v) occurs twice in w, and we can write w as $(u, e, v, w_1, u, e, v, \dots)$. The first time e appears is as the first edge of w, so e belongs to

w^+, and since w is primitive, e belongs to w^+ in every time it appears. Therefore the first and last edges of w_1 belong to w^- and so w^- has odd length. So $\gamma = (u, e, v, w_1, u)$ is an even closed subwalk of w such that $E^+(\gamma)$ divides $E^+(w)$ and $E^-(\gamma)$ divides $E^-(w)$. So w is not primitive which is a contradiction. Then this case is impossible.

Case 2: The edge e appears exactly twice in two opposite ways $(\dots, u, e, v, \dots)$ and $(\dots, v, e, u, \dots)$. So e is a double edge and we can write w as $(u, e, v, w_1, v, e, u, w_2, u)$. Again, notice that e belongs to w^+ every time it appears, so the first and last edges of w_1 and w_2 (which are of odd lengths) belong to w^-. If e is not a cut edge of w, then w_1 and w_2 must intersect in at least one vertex. Suppose y is a common vertex of w_1 and w_2, then we can decompose w_1 into two distinct subwalks w_1' and w_1'' both having end vertices v and y. Similarly, decompose w_2 into two distinct subwalks w_2' and w_2'' both having end vertices u and y. So $w = (u, e, v, w_1', y, w_1'', v, e, u, w_2', y, w_2'', u)$. Since w_1 and w_2 have odd lengths, one of the walks in each of the pairs (w_1', w_1'') and (w_2', w_2'') will have even length and the other walk will have odd length. By the same discussion as before we will have that one of the two subwalks $(u, e, v, w_1', y, w_2'', u)$ or $(u, e, v, w_1', y, -w_2', u)$ is an even closed subwalk γ_1 of w such that $E^+(\gamma_1)$ divides $E^+(w)$ and $E^-(\gamma_1)$ divides $E^-(w)$. So w is not primitive which is a contradiction. Therefore e is a double edge of the walk w and a cut edge of w. This proves 2.

Now let v be a cut vertex of w. Since the removal of a cut vertex increases the number of connected components of a graph, v is a common vertex of at least two blocks in w and the walk w can be written as $w = (v, e_1, \ldots, e_s, v, e_{s+1}, \ldots, e_t, v, \ldots)$ such that e_1 and e_s are in the same block and $\{e_i \quad | \quad 1 \leq i \leq s\} \cap \{e_i \quad | \quad s+1 \leq i \leq t\} = \phi$. The edge e_1 is the first edge in the walk w, so e_1 belongs to w^+. If e_s belongs to w^-, then the walk $w_B = (v, e_1, \ldots, e_s, v)$ is an even closed walk such that $E^+(w_B)$ divides $E^+(w)$ and $E^-(w_B)$ divides $E^-(w)$. This is a contradiction to the primitiveness of w, so e_s belongs to w^+. Since both e_1 and e_s belong to w^+, the subwalk γ is of odd length and v is a common vertex of two odd edges and so v is a sink. By the same discussion, we notice that both e_{s+1} and e_t belong to w^- and that $(v, e_{s+1}, \ldots, e_t, v)$ is an odd walk. Therefore the walk

$w' = (v, e_1, \ldots, e_s, v, e_{s+1}, \ldots, e_t, v)$ is an even closed walk such that $E^+(w')$ divides $E^+(w)$ and $E^-(w')$ divides $E^-(w)$. Since w is primitive, $w = w'$. Therefore, the cut vertex v belongs to exactly two blocks of w and it is a sink of both. This proves 3.

Now, we will show that if B_w satisfies (1) and (2) and (3) then B_w is primitive. Suppose that w is an even closed walk that satisfies the three conditions but is not primitive. The walk w is not primitive which implies the existence of a primitive subwalk w' that has a smaller length than that of w and such that $E^+(w')$ divides $E^+(w)$ and $E^-(w')$ divides $E^-(w)$. By the first part of proof and since w' is primitive, it satisfies (1) and (2) and (3). We want to show the graphs w and w' have the same blocks. Let $B_{w'}$ be a block of w'. Suppose $B_{w'}$ is not totally contained in a single block of w, then $B_{w'}$ is not biconnected anymore and so not a block. Therefore there exists a block B_w of w that contains $B_{w'}$. In fact, $B_w = B_{w'}$:

By 1, $B_{w'}$ is either a cycle or a cut edge of w'. Assume $B_{w'} = \{e\}$ is a cut edge of w'. If e is not a multiple edge of w', then when we remove e, the number of components of w' does not increase and the graph will still be connected. So e must be a double edge of w'. Since w' is a subwalk of w such that $E^+(w')$ divides $E^+(w)$ and $E^-(w')$ divides $E^-(w)$, then e is also a multiple edge of w. Now by (2), e is a cut edge of w and hence a block in w. If $B_{w'}$ is a cycle in w' then it is a cycle in B_w and so $B_{w'} = B_w$. Therefore, if B is a block in w' then B is a block in w.

Next we show that every block of w is a block of w'. Suppose there exists a block B in w that is not a block in w'. If every block of w that is not a block of w' does not have any common vertex with w', then w is not connected anymore and this is impossible. So there exists a block B of w which is not a block of w' and has at least one common vertex v with w'. Let B' be the block of w' containing v. Since B' is also a block of w, v is a cut vertex of w and by property 3 it is a sink of B and B'. Since $E^+(w')$ divides $E^+(w)$ and $E^-(w')$ divides $E^-(w)$, v is also a sink of w'. Since w' is primitive, v must belong to another block B'' of w' (otherwise, either w' is odd or contains multiple edges that are not cut edges). But we just proved that a block of w' is a block of w. Therefore B, B', and B'' are distinct blocks of w having common vertex the sink v which is a contradiction to property 3 so the graphs w and w' are the same and $E^+(w') = E^+(w)$ and $E^-(w') = E^-(w)$. This means that the walks w and w' have the same length which contradicts our assumption. So the walk w is primitive. ■

Chapter 4

Universal Grobner Bases of Graphs

Throughout this chapter, G is a finite simple graph with vertex set $V(G) = \{v_1, \ldots, v_n\}$ and edge set $E(G) = \{e_1, \ldots, e_m\}$.

Definition 4.1. *Consider a primitive walk w. A cyclic block B of w is called pure if the edges of B are either all in w^+ or all in w^-.*

Proposition 4.2. *If w is an even primitive walk in G and contains a pure cyclic block, then B_w is not in the universal Grobner basis of I_G.*

Proof: Suppose that w contains a pure cyclic block B, see for example figure 4.1. Write w as $(w_1, \gamma_1, \ldots, w_s, \gamma_s)$ where $\gamma_1, \ldots, \gamma_s$ are edges of B. Since B is pure we may assume that its edges are in w^-. This, together with the fact that w is even, implies that w_i are subwalks of w of odd length.

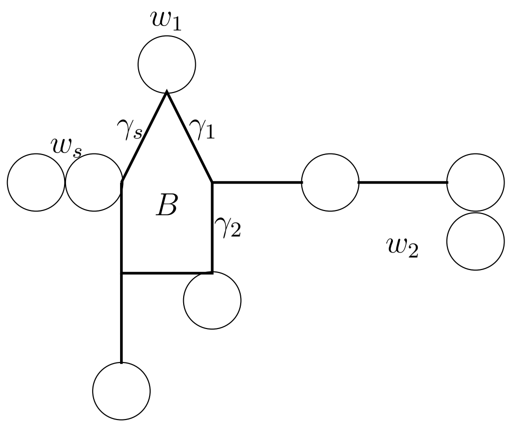

Figure 4.1:

For a walk w_i, we have $E^+(w_i) = \prod_{e_{i_{2k-1}} \in w_i} e_{i_{2k-1}}$ and $E^-(w_i) = \prod_{e_{i_{2k}} \in w_i} e_{i_{2k}}$.

Then for the walk w we have

$$B_w = E^+(w_1)E^+(w_2)\dots E^+(w_s) - \gamma_1\gamma_2\dots\gamma_s E^-(w_1)E^-(w_2)\dots E^-(w_s)$$

Consider the walk $W_i = (w_i, \gamma_i, w_{i+1}, \gamma_i)$. The subwalks w_i are of odd length, so W_i has even length. For W_i, consider the corresponding binomial

$F_i = E^+(w_i)E^+(w_{i+1}) - \gamma_i^2 E^-(w_i)E^-(w_{i+1})$ for $i = 1, 2, \dots, s-1$. Also consider the binomial $F_s = E^+(w_s)E^+(w_1) - \gamma_s^2 E^-(w_s)E^-(w_1)$. Notice that $F_1, \dots, F_s$ belong to the toric ideal I_G. Assume that B_w belongs to a reduced *Grobner* basis for I_G with respect to a term order $<$.

We have two cases:

- Case one: $E^+(w_1)E^+(w_2)\dots E^+(w_s) > \gamma_1\gamma_2\dots\gamma_s E^-(w_1)E^-(w_2)\dots E^-(w_s)$, then $E^+(w) = E^+(w_1)E^+(w_2)\dots E^+(w_s)$ is the leading term in the binomial B_w. Suppose that $\gamma_i^2 E^-(w_i)E^-(w_{i+1}) < E^+(w_i)E^+(w_{i+1})$. Then $E^+(W_i) = E^+(w_i)E^+(w_{i+1})$ is the leading term in the binomial F_i. Notice that $E^+(W_i)$ divides $E^+(w)$, that is, $LT(F_i)$ divides $LT(B_w)$ where F_i belongs to I_G and B_w belongs to a reduced *Grobner* basis of I_G. This is a contradiction since we can not have the leading term of any element in the reduced *Grobner* basis for I_G divisible by the leading term of any element in I_G. Then $E^+(w_i)E^+(w_{i+1}) < \gamma_i^2 E^-(w_i)E^-(w_{i+1})$ for all i. If we substitute the values of i in the latter inequality we get

$$E^+(w_1)E^+(w_2) < \gamma_1^2 E^-(w_1)E^-(w_2)$$

$$E^+(w_2)E^+(w_3) < \gamma_2^2 E^-(w_2)E^-(w_3)$$

$$\vdots$$

$$E^+(w_{s-1})E^+(w_s) < \gamma_{s-1}^2 E^-(w_{s-1})E^-(w_s)$$

$$E^+(w_s)E^+(w_1) < \gamma_s^2 E^-(w_s)E^-(w_1)$$

If we multiply all these inequalities we get

$$(E^+(w_1)E^+(w_2)\dots E^+(w_s))^2 < (\gamma_1\gamma_2\dots\gamma_s E^-(w_1)E^-(w_2)\dots E^-(w_s))^2$$

which is a contradiction to case one.

- Case two: $E^+(w_1)E^+(w_2)\dots E^+(w_s) < \gamma_1\gamma_2\dots\gamma_s E^-(w_1)E^-(w_2)\dots E^-(w_s)$, then

$E^-(w) = \gamma_1\gamma_2\dots\gamma_s E^-(w_1)E^-(w_2)\dots E^-(w_s)$ is the leading term in the binomial B_w.

The number s is either even or odd. First consider the case where $s = 2k$. Let

$H = \gamma_1\gamma_3\dots\gamma_{2k-1} - \gamma_2\gamma_4\dots\gamma_{2k}$. So H belongs to I_G. The two monomials in H divide

$\gamma_1\gamma_2\dots\gamma_s E^-(w_1)E^-(w_2)\dots E^-(w_s)$. Therefore, $LT(H)$ divides $LT(B_w)$, where H

belongs to I_G and B_w belongs to the reduced *Grobner* basis for I_G, which is a contradiction. Now consider the case where $s = 2k+1$. For $i = 1, \ldots, s$ let $H_i = E^+(w_i)\gamma_{i+1}\gamma_{i+3}\ldots\gamma_{i+2k-1} - E^-(w_i)\gamma_i\gamma_{i+2}\ldots\gamma_{i+2k}$ such that $\gamma_j = \gamma_l$ if $j = l \bmod (2k+1)$. The binomial H_i belongs to I_G. Suppose that $E^+(w_i)\gamma_{i+1}\gamma_{i+3}\ldots\gamma_{i+2k-1} < E^-(w_i)\gamma_i\gamma_{i+2}\ldots\gamma_{i+2k}$, then $E^-(w_i)\gamma_i\gamma_{i+2}\ldots\gamma_{i+2k}$ is the leading term of H_i. Notice that $E^-(w_i)\gamma_i\gamma_{i+2}\ldots\gamma_{i+2k}$ divides $\gamma_1\gamma_2\ldots\gamma_s E^-(w_1)E^-(w_2)\ldots E^-(w_s)$. Therefore $LT(H_i)$ divides $LT(B_w)$, where H_i belongs to I_G and B_w belongs to the reduced *Grobner* basis of I_G, which is a contradiction. Therefore,

$$E^+(w_i)\gamma_{i+1}\gamma_{i+3}\ldots\gamma_{i+2k-1} > E^-(w_i)\gamma_i\gamma_{i+2}\ldots\gamma_{i+2k}.$$

If we substitute the different values of i in the latter inequality we get

$$E^+(w_1)\gamma_2\gamma_4\ldots\gamma_{2k} > E^-(w_1)\gamma_1\gamma_3\ldots\gamma_{1+2k}$$

$$E^+(w_2)\gamma_3\gamma_5\dots\gamma_{1+2k} > E^-(w_2)\gamma_2\gamma_4\dots\gamma_{2+2k}$$

$$E^+(w_3)\gamma_4\gamma_6\dots\gamma_{2+2k} > E^-(w_3)\gamma_3\gamma_5\dots\gamma_{3+2k}$$

$$\vdots$$

$$E^+(w_{2k})\gamma_{2k+1} > E^-(w_{2k})\gamma_{2k}$$

$$E^+(w_{2k+1}) > E^-(w_{2k+1})\gamma_{2k+1}$$

If we multiply these inequalities and get rid of the common factors, we get

$$E^+(w_1)E^+(w_2)\dots E^+(w_s) > \gamma_1\gamma_2\dots\gamma_s E^-(w_1)E^-(w_2)\dots E^-(w_s)$$

This is contradiction to case two.

So B_w does not belong to any reduced *Grobner* basis of I_G and it does not belong to the universal *Grobner* basis of I_G. ■

For $n \leq 8$, $U_{K_n} = Gr_{K_n}$ [2]. For $n = 9$, U_{K_n} is not equal to Gr_{K_n} since K_9 contains a primitive walk with a pure cyclic block. The following example shows this primitive walk.

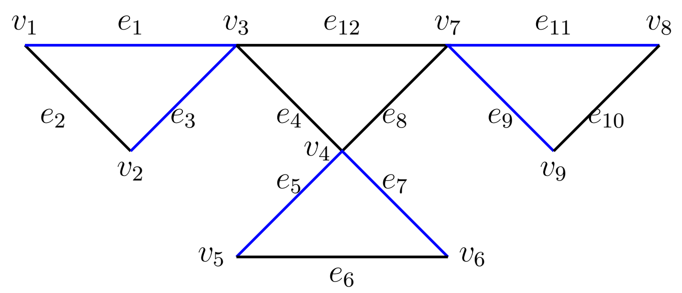

Figure 4.2:

Example 4.3.

Consider the walk $w = (e_1e_2e_3e_4e_5e_6e_7e_8e_9e_{10}e_{11}e_{12})$ in the graph of figure 4.2 . Then $w^+ = e_1e_3e_5e_7e_9e_{11}$ and $w_- = e_2e_4e_6e_8e_{10}e_{12}$.

The binomial B_w corresponding to the walk w in G_w is primitive since:

- The blocks in G_w are given in figure 4.3. Each block is a cycle.
- There does not exist any multiple edge in w.
- Every cut vertex belongs to exactly two blocks and is a sink of both. The cut vertices are v_3, v_4, and v_7. The sink v_3 belongs to blocks A and B. The vertex v_3 is the

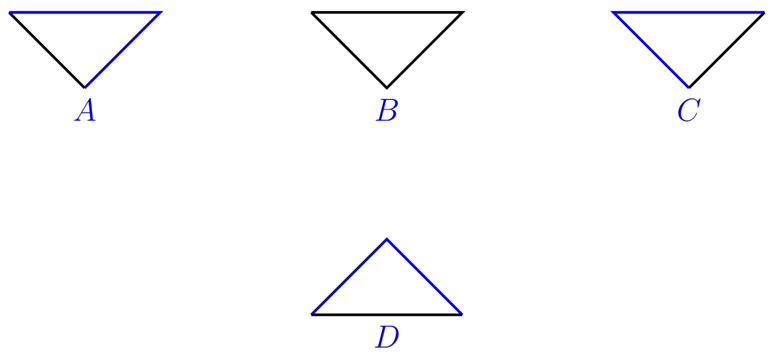

Figure 4.3:

intersection of the odd edges e_1 and e_3 in block A so v_3 is a sink of A. Similarly, v_3 is the intersection of the two even edges e_4 and e_{12} in block B so v_3 is a sink of B. Also, v_4 belongs to the two blocks B and D and is a sink of both and v_7 belongs to the blocks B and C and is a sink of both.

The primitive walk w contains a pure cyclic block B such that all the edges in B are odd. Therefore, B_w does not belong to the universal *Grobner* basis.

Definition 4.4. *Let* $T = \{y_1, \ldots, y_q\}$ *and* $S = \{x_1, \ldots, x_p\}$ *be two sets of variables and let* $<_T$ *and* $<_S$ *be two monomial orders defined on* $K[y_1, \ldots, y_q]$ *and* $K[x_1, \ldots, x_p]$*. The elimination order corresponding to* $((S, <_S), (T, <_T))$ *is the monomial order* $<$ *on* $K[x_1, \ldots, x_p, y_1, \ldots, y_q]$ *defined as follows:*

Consider the two monomials $s = x^{\alpha}y^{\beta}$ and $s' = x^{\alpha'}y^{\beta'}$.

- *If $\beta = 0$ and $\beta' \neq 0$ then $s' > s$.*

- *If β and β' are both nonzero and $y^{\beta} = y^{\beta'}$ or if $\beta = \beta' = 0$, then $s > s'$ if and only if $x^{\alpha} >_s x^{\alpha'}$.*

Let w be a primitive walk in G. Let $S = E(G) \bigcap w$ and $T = E(G) \backslash S$. Let $<_T$ be any monomial order on T. Define a monomial order $<_s$ on S as follows. Enumerate all the cyclic blocks of w with any enumeration $B_1, \ldots, B_{l_0}$. The edges in $w^+ \cap B_i$ are the odd edges of w that are in block B_i and the edges in $w^- \cap B_i$ are the even edges of w that are in block B_i. The number of edges in $w^+ \cap B_i$ is denoted by t_i^+, and the number of edges in $w^- \cap B_i$ is denoted by t_i^-. If the walk w has p edges, then define $W = (w_{ij})$ to be the $(l_0 \times p)$ matrix such that

$$w_{ij} = \begin{cases} 0 & \text{if } e_j \notin B_i \\ t_i^- & \text{if } e_j \in w^+ \cap B_i \\ t_i^+ & \text{if } e_j \in w^- \cap B_i \end{cases}$$

Suppose there is a column in W with more than one nonzero entry. This means that the edge corresponding to this column belongs to more than one block which is impossible since each edge belongs to exactly one block. Then each column of W has at most one nonzero entry. Let $[u]$ be the vector u represented as a column vector. We consider that $e^u <_w e^v$ if and only if the first nonzero coordinate of $W[u-v]$ is negative. If the first nonzero coordinate of $W[u-v]$ is positive, then consider $W[v-u]$ to compare e^u and e^v. Then the first nonzero coordinate of $W[v-u]$ is negative, and hence $e^v <_w e^u$. In the case where $W[u-v]=0$, order e^u and e^v by any term order. Let $<_w$ be the elimination order corresponding to $((S,<_S),(T,<_T))$.

Lemma 4.5. *Let w be a mixed primitive walk and let z be a primitive walk. If $E^+(z)$ divides $E^+(w)$ then $E^-(z)$ does not divide any of $E^+(w)$ and $E^-(w)$.*

Proof: Let z be primitive walk and w be a primitive mixed walk such that $E^+(z)$ divides $E^+(w)$. If $E^-(z)$ divides $E^-(w)$, then, for the primitive walk z, we have $E^+(z)$ divides $E^+(w)$ and $E^-(z)$ divides $E^-(w)$, and so w is not primitive. Therefore $E^-(z)$ does not divide $E^-(w)$. Now if $E^-(z)$ divides $E^+(w)$, then there exists in w a pure cyclic block all of whose edges are odd and so w is not mixed. Therefore $E^-(z)$ does not divide $E^+(w)$.

Theorem 4.6. *Let w be a primitive walk in G. The binomial B_w belongs to the universal Grobner basis of I_G if and only if w is mixed.*

Proof: If w is not mixed, then w has a pure cyclic block. Therefore, by proposition 4.2, B_w does not belong to the universal *Grobner* basis of I_G. So if B_w belongs to the universal *Grobner* basis of I_G, then w is mixed.

Now suppose that w is a mixed primitive walk. We need to show that B_w belongs to the universal *Grobner* basis of I_G. Since the universal *Grobner* basis is the union of all reduced *Grobner* bases with respect to all term orders, then it is enough to show that B_w belongs to the reduced *Grobner* basis of I_G with respect to the term order $<_w$.

By lemma 4.5, to show that B_w belongs to the reduced *Grobner* basis of I_G with respect to the term order $<_w$, it is enough to show that if B_z is a primitive binomial such that $E^+(z)$ divides $E^+(w)$ and z is not equal to w, then $E^-(z) >_w E^+(z)$. Let B_z be a primitive binomial such that $E^+(z)$ divides $E^+(w)$.

Suppose that z is not a subset of w. Since $E^+(z)$ divides $E^+(w)$, then z^+ is a subset of w^+. Therefore there exists an edge of z^- that is not an edge of w. By the elimination order we get that $E^-(z) >_w E^+(z)$.

Now suppose that z is a subset of w. We have two cases. The first case is that there exists at least one i such that $B_i \bigcap z$ is not empty and $B_i \bigcap z^+$ is a proper subset of $B_i \bigcap w^+$. Notice that since $E^+(z)$ divides $E^+(w)$ then $B_i \bigcap z^+ \subseteq B_i \bigcap w^+$. The second case is the negation of the first one and it is that for every i, either $B_i \bigcap z$ is empty or $B_i \bigcap z^+$ is equal to $B_i \bigcap w^+$.

- Case I: We first consider the second case. The graph w is the union of its blocks. We will first show that there are integers i and j such that $B_i \bigcap z = \phi$ and $B_j \bigcap z^+$ is equal to $B_j \bigcap w^+$.

  If $B_i \bigcap z$ is empty for all B_i then z is empty which is impossible.

  Suppose now that for all i we have that $B_i \bigcap z^+$ is equal to $B_i \bigcap w^+$, that is, z is a primitive walk such that z is a subset of w and $E^+(z)$ divides $E^+(w)$ and $B_i \bigcap z^+$ is equal to $B_i \bigcap w^+$. B_i is a block of a primitive walk w so B_i is a cycle or a cut edge.

Case 1: Suppose that B_i is a cycle. If $B_i \bigcap z^-$ is not equal to $B_i \bigcap w^-$, then B_i is not a block of z since it is not biconnected in z anymore. So every edge in $B_i \bigcap z^+$ is a cut edge of z and then a double edge of z. Hence for every edge e that belongs to $B_i \bigcap z^+$, we have that e^2 divides $E^+(z)$. On the other hand, B_i is a cyclic block of w, then each edge in B_i is an edge of w. Therefore e^2 does not divide $E^+(w)$ which is a contradiction since $E^+(z)$ divides $E^+(w)$. So if B_i is a cycle and $B_i \bigcap z^+$ is equal to $B_i \bigcap w^+$, then $B_i \bigcap z^-$ is equal to $B_i \bigcap w^-$. Hence $B_i \bigcap z$, $B_i \bigcap w$, and B_i are equal.

Case 2: B_i be a cut edge e_0 of w. Without loss of generality, let e_0 be in w^+, then $B_i \bigcap w^+$ is equal to e_0 and $B_i \bigcap w^-$ is empty. Then $B_i \bigcap z^+$ is equal to e_0 and e_0 belongs to z^+. Since z is primitive, then e_0 does not belong to z^- and so $B_i \bigcap z$ is empty. Therefore, if B_i is a cut edge and $B_i \bigcap z^+$ is equal to $B_i \bigcap w^+$, then $B_i \bigcap z^-$ is equal to $B_i \bigcap w^-$. Hence $B_i \bigcap z$ is equal to B_i.

Therefore for every block B_i such that $B_i \bigcap z^+$ is equal to $B_i \bigcap w^+$, we get that $B_i \bigcap z$ is equal to B_i. But if $B_i \bigcap z$ is equal to B_i for all i, then z and w are equal which is impossible.

Then there are integers i and j such that $B_i \bigcap z = \phi$ and $B_j \bigcap z^+$ is equal to $B_j \bigcap w^+$. Let A be the subgraph of w consisting of all blocks B_i such that $B_i \bigcap z = \phi$ and C the subgraph consisting of all blocks B_i such that $B_i \bigcap z$ is equal to B_i as the second subgraph C of w. Then w is the union of A and C. Since the graph w represents a walk, then it is connected. So there exists a block B_i in A and another one B_j in C that are adjacent and have a common vertex v. The vertex v is a common vertex of two blocks, then it is a sink of both, that is, v is the intersection of two even edges or two odd edges in each of B_i and B_j. If $z = (e_{i_1}, e_{i_2}, \ldots, e_{i_{2q}})$ then $E^+(z) = e_{i_1} e_{i_3} \ldots e_{i_{2q-1}}$ and $E^-(z) = e_{i_2} e_{i_4} \ldots e_{i_{2q}}$. Then

$deg_A E^+(z) = \epsilon_{i_1} + \epsilon_{i_2} + \epsilon_{i_3} + \epsilon_{i_4} + \cdots + \epsilon_{i_{2q-1}} + \epsilon_{i_{2q}}$ and

$deg_A E^-(z) = \epsilon_{i_2} + \epsilon_{i_3} + \epsilon_{i_4} + \epsilon_{i_5} + \cdots + \epsilon_{i_{2q}} + \epsilon_{i_{2q+1}}$ where $\{v_{i_j}, v_{i_{j+1}}\}$ are the end vertices of the edge e_{i_j} in z. So if v_i is the intersection of two odd edges in B_j then $2\epsilon_i$ appears in $deg_A E^+(z)$ and does not appear in $deg_A E^-(z)$. While if v_i is the intersection of two even edges in B_j then $2\epsilon_i$ appears in $deg_A E^-(z)$ and does not appear in $deg_A E^+(z)$. Then $deg_A E^+(z)$ and $deg_A E^-(z)$ are not equal. So B_z does not belong to I_G which is a contradiction.

- Case II: Now we consider the case where there exists at least one i such that $B_i \bigcap z$ is not empty and $B_i \bigcap z^+$ is a proper subset of $B_i \bigcap w^+$. Let i be the smallest integer such that $B_i \bigcap z$ is not empty and $B_i \bigcap z^+$ is a proper subset of $B_i \bigcap w^+$, that is if $j = 1, \ldots, i-1$, then either $B_j \bigcap z$ is empty or $B_j \bigcap z^+$ is equal to $B_j \bigcap w^+$. Let w_j be the j^{th} row of W.

  The entries in w_j are determined according to whether each edge belongs to B_j or not. The entries in $[z^+]$ are either one, corresponding to odd edges in z, or zero, corresponding to other edges of w. The entries in $[z^-]$ are either one, corresponding to even edges in z, or zero corresponding to other edges of w. Consider $w_j[z^+]$.

If, $j = 1, \ldots, i-1$, $B_j \bigcap z$ is empty, then all entries corresponding to edges of z^+ in w_j are zero. The nonzero entries in w_j correspond to edges that are not in z^+. But the entries in w_j that correspond to edges that are not in z^+ are multiplied by entries in $[z^+]$ that correspond to edges that are not in z^+ and these are equal to zero. So $w_j[z^+]$ is zero. Similarly, $w_j[z^-]$ is zero. Then the first $i-1$ coordinates of $W[z^+ - z^-]$ are zero.

If $j = 1, \ldots, i-1$ and $B_j \bigcap z^+$ is equal to $B_j \bigcap w^+$, then by previous argument, $B_j \bigcap z^-$ is equal to $B_j \bigcap w^-$. By definition of W, the entries in w_j that correspond to edges in $B_j \bigcap z^+$ are equal to t_j^-. When we multiply $[z^+]$ by w_j, the entries in w_j that correspond to odd edges of z are multiplied by entries in $[z^+]$ that correspond to odd edges of z and these are the only entries in $[z^+]$ equal to one. Since the number of edges in the intersection of B_j and z^+ is the same number of edges in $B_j \bigcap w^+$ which is t_j^+, then $w_j[z^+]$ is equal to $t_j^- t_j^+$. Similarly, $w_j[z^-]$ is equal to $t_j^- t_j^+$.

If $j = i, \ldots, l_0$ and $B_j \bigcap z^+$ is a proper subset of $B_j \bigcap w^+$, then we have two cases.

First consider the case when $B_j \bigcap z$ is not equal to B_j. Let e be an edge in $B_j \bigcap z$ and suppose e belongs to z^+. Since $B_i \bigcap z^+$ is a proper subset of $B_i \bigcap w^+$, then there is at least one edge in B_i that is not an edge of z. Therefore every edge in $B_i \bigcap z$ is a cut edge and then a double edge. So e^2 divides $E^+(z)$ and then e^2 divides $E^+(w)$ which is impossible since B_i is a cyclic block and all its edges are single ones. Then e can not be in z^+. Therefore, if e is an edge in $B_i \bigcap z$, then e is in z^-. As in the argument before, $w_j[z^+]$ is zero and $w_j[z^-]$ is greater than zero. According to the elimination order, $E^-(z) >_w E^+(z)$.

Now suppose that $B_j \bigcap z$ is equal to B_j. Since $B_j \bigcap z^+$ is a proper subset of $B_j \bigcap w^+$, then the number of edges in $B_j \bigcap z^+$ is less than t_j^+. So as in the previous argument, we will get $w_j[z^+]$ is less than $t_j^- t_j^+$. Since $B_j \bigcap z^-$ is B_j after $B_j \bigcap z^+$ is removed, then the number of edges in $B_j \bigcap z^-$ is larger than t_j^-. Therefore, $w_j[z^-]$ is greater than $t_j^- t_j^+$. Then $w_j[z^+] <_w w_j[z^-]$. Again by elimination order $E^-(z) >_w E^+(z)$.

Therefore, B_w belongs to the reduced *Grobner* basis with respect to the elimination order

$<_w$. So B_w belongs to the universal *Grobner* basis of I_A. ■

Corollary 3.26 gives us a characterization of the form of circuits in C_A and theorem 3.27 gives us a characterization of the form of primitive walks in Gr_A. Moreover, by theorem 4.6, we obtain that a graph G with the property that the universal *Grobner* basis and the *Graver* basis of I_G are equal is a primitive walk that is free from any pure block. So we are able to construct graphs such that the universal *Grobner* basis is equal to the *Graver* basis by considering primitive walks that do not contain any pure blocks or if there are any then make subdivisions in some edges of the pure block so that it is not pure anymore.

Therefore, we are able to construct graphs such that their corresponding toric ideals have specific properties concerning the elements in C_G, U_G, and in Gr_G and the relation between them.

Chapter 5

Degree Bounds

The degree of a monomial $x_1^{\alpha_1} \dots x_n^{\alpha_n}$ is $\alpha_1 + \cdots + \alpha_n$. Note that this is the sum of the components of the vector $deg_A(x^\alpha)$ where $A = \{e_1, \dots, e_n\}$. The degree of a binomial $x^\alpha + x^\beta$ is $max\{deg(x^\alpha), deg(x^\beta)\}$.

Let K_n be the complete graph on n vertices.

Theorem 5.1. *The largest degree of any binomial in the Graver basis and in the universal Grobner basis for I_{K_n} is $d_n = n - 2$ for $n \geq 4$.*

Proof: Consider the graph K_n. A primitive walk is the union of its blocks that are either cycles or cut edges. If a primitive walk is not a cycle, then it has at least two cyclic blocks. Let w be a primitive walk consisting of l_0 cyclic blocks $B_1, \dots, B_{l_0}$and l_1 cut edges. Let t_i be the number of vertices in B_i and observe that this is also the number of edges in B_i. Let

$l = l_0 + l_1$ be the number of blocks in w. The number of cut vertices in w is $l - 1$. Each cut vertex belongs to exactly two blocks. So the total number of vertices of w is $t_1 + \cdots + t_{l_o} + 2l_1 - (l - 1)$. Since w is a primitive walk of the gragh K_n then $t_1 + \cdots + t_{l_0} + 2l_1 - (l - 1) \leq n$. Since $B_w = E^+(w) - E^-(w)$ where w is an even closed walk, then $deg(B_w) = deg(E^+(w)) = deg(E^-(w))$ where each of $E^+(w)$ and $E^-(w)$ contains exactly half of the edges of w. Each edge of a cyclic block is a single edge of w, while a cut edge is a double edge of w. Then the total number of edges of w is

$$t_1 + \cdots + t_{l_0} + 2l_1$$

So we get

$$2deg(B_w) = t_1 + \cdots + t_{l_0} + 2l_1$$

By the above inequality we have that

$$2deg(B_w) = t_1 + \cdots + t_{l_0} + 2l_1 \leq n + l - 1$$

Then the largest degree of B_w is attained if and only if $2deg(B_w) = n + l_{max} - 1$ where l_{max} is the largest possible number of blocks in w, in particular, B_w must pass through all vertices of K_n. Notice that

$t_1 + \cdots + t_{l_0} + 2l_1 \leq n + l - 1$

$\Leftrightarrow t_1 + t_{l_0} + 2l_1 + (2l_0 - 2l_0) - l \leq n - 1$

$\Leftrightarrow (t_1 - 2) + \cdots + (t_{l_0} - 2) + l \leq n - 1$

If the walk w is a cycle then $l_0 = 1$ and $deg(B_w) \leq n/2$.

Now if the walk w is not a cycle then $l_0 \geq 2$. Also every cyclic block has at least three vertices, then $t_i \geq 3$. So we get

$$(t_1 - 2) + \cdots + (t_{l_0} - 2) \geq 2$$

So we get $l \leq n-3$. Therefore the largest degree of any binomial in the *Graver* basis is $n-2$. Since the universal *Grobner* basis is contained in the *Graver* basis, it follows that the largest degree of any binomial in the universal *Grobner* basis is $n-2$. ■

Corollary 5.2. *Let G be a graph with $n \geq 4$ vertices. The degree d of any binomial in the Graver basis and in the universal Grobner basis for I_G is at most $n-2$.*

Proof: By theorem 4.1, the largest degree of any binomial in the *Graver* basis and in the universal *Grobner* basis of toric ideals of the complete graph K_n is $n-2$ for $n \geq 4$. Since any graph with n vertices is a subgraph of K_n, the corollary follows directly. ■

Remark: Since the maximum degree d_n for I_{K_n} is attained by a circuit with $n-5$ cut edges and two cyclic blocks with three vertices each, then the largest degree of any binomial in the *Graver* basis and in the universal *Grobner* basis for I_G is $n-2$ if and only if G contains a circuit with $n-5$ cut edges and two cyclic blocks of three vertices each provided that $n > 4$.

Chapter 6

True Circuit Conjecture

In July 1995, B. Sturmfels made the conjecture that circuits have maximal degree among the elements of the *Graver* basis [8]. After that, S. Hosten and R. Thomas gave a counter example. Then B. Sturmfels changed the conjecture into the true circuit conjecture which we will state after the following definition.

Definition 6.1. *Let A be a finite subset of $\mathbb{N}^n$. Let C be a circuit in C_A and consider the subset $supp(C)$ of A. The lattice $\mathbb{Z}(supp(C))$ has finite index in the lattice $R(supp(C)) \bigcap \mathbb{Z}A$. This index is called the index of the circuit C and denoted by $index(C)$. The true degree of the circuit C is the product $degree(C).index(C)$.*

The true circuit conjecture of B. Sturmfels states that the maximal true degree of any circuit in C_A is greater than or equal to the degree of any element in the *Graver* basis of the toric ideal I_G.

The following are counter examples to the true circuit conjecture [6].

Example 6.2.

Let G be a graph consisting of a cycle of length p and p pairwise disjoint cycles of odd length q. Each of which has a unique vertex in common with the cycle whose length is p. Consider the walk w which passes once through every edge of the graph G. The length of the walk w is the sum of the lengths of cycles in w. So the length of w is $qp + p = p(q + 1)$ which is even since q is odd. Notice that w is an even closed walk, and that every block of w is a cycle, and that there are no multiple edges in w, and that every cut vertex of w belongs to exactly two blocks and is a sink of both. Therefore B_w belongs to the *Graver* basis of I_G. The degree of B_w is $\frac{p(q+1)}{2}$.

Now, we are going to consider the circuits in G. Any walk c consisting of two odd cycles, each having length q, joined by a path of length $p - 1$ is a circuit and is of maximal length. The degree of B_c is $\frac{2q+2(p-1)}{2} = p + q - 1$ and it is the maximum degree of any circuit in G.

Since p and q are lengths of cycles, each of p and q is greater than 2. Therefore

$$\frac{p(q+1)}{2} - (p+q-1) > \frac{pq+p-2p-2q+2}{2} > \frac{4+pq-2p-2q}{2} = \frac{(p-2)(q-2)}{2} > 0$$

So we get that $\frac{p(q+1)}{2} > q+p-1$. Thus there exists an element B_w in the *Graver* basis of I_G whose degree is larger than the maximum degree of all circuits in I_G. Notice that choosing q and p to be large makes the difference of degrees to be large also.

In order to see the contradiction with the true circuit conjecture, we need to consider the true degree of our chosen circuit c. By computation, it turns out that the degree of B_c is equal to its true degree. So G contains a primitive walk whose degree is larger than the maximum true degree of all circuits in G. Therefore, by giving values to q and p, we will have an infinite number of counter examples to the true circuit conjecture.

Since w contains a pure block which is the cycle of length p, then by theorem 3.4, B_w does not belong to the universal *Grobner* basis of I_G. So we are going to consider a slightly different graph G' in order to have an element in the universal *Grobner* basis whose degree is larger than the true degree of all circuits in G'.

Let G' be the graph consisting of a cycle of length p and $p-2$ odd cycles of length q such that each one is attached to a vertex of the initial cycle in the center. Let w' be the walk that passes once through every edge of G'. Then w' is a primitive walk that does not have any pure block. So w' is mixed and it belongs to the universal *Grobner* basis of $I_{G'}$. The degree of w' is $deg(w') = \frac{p+(p-2)q}{2}$.

The walk c' consisting of the two odd cycles joined by a path of length $p-3$ is the circuit of maximal length. The circuit $B_{c'}$ has the largest degree of all circuits in G' which is $deg(B_{c'}) = \frac{2q+2(p-3)}{2} = q+p-3$.

As in the previous argument, since p and q are each greater than two, then $\frac{p+(p-2)q}{2} > q+p-3$. By computation, we get that the true degree of $B_{c'}$ is equal to its degree. So there is an element $B_{w'}$ in the universal *Grobner* basis whose degree is larger then the maximum true degree of all circuits in G'. Therefore G' gives a family of infinitely many counter examples to the true circuit conjecture. Moreover, if p and q are large, then the difference between the degree of $B_{w'}$ and that of any of the circuits becomes large also.

Example 6.3.

Figure 6.1 shows the graph G for $p = 5$ and $q = 3$. Let

$w = (e_1e_2e_3e_4e_5e_6e_7e_8e_9e_{10}e_{11}e_{12}e_{13}e_{14}e_{15}e_{16}e_{17}e_{18}e_{19}e_{20})$. The walk w is closed and its length is 20 which is even. The blocks in w are all the cycles. There is no multiple edge in w. Every cut vertex belongs to exactly two blocks and is a sink of both blocks. So B_w belongs to the *Graver* basis of I_G. The degree of B_w is 10. In G, the circuit

$w_1 = (e_2e_3e_4e_5e_9e_{13}e_{17}e_{18}e_{19}e_{20}e_{17}e_{13}e_9e_5)$ has maximal length. The degree of B_{w_1} is 7 and it is the largest degree of any circuit in G. Then $deg(B_w) > deg(B_{w_1})$, which implies that the degree of a primitive walk in I_G is greater than the degree of any of the circuits in G. The primitive walk w contains a pure cyclic block, the cycle in the center of the graph. So B_w does not belong to the universal *Grobner* basis of I_G.

Example 6.4.

Let G' be the subgraph of the graph G in figure 6.1 defined by the walk

$w_2 = (e_1e_2e_3e_4e_5e_6e_7e_8e_9e_{10}e_{11}e_{12}e_{13}e_{17})$. The walk w_2 is primitive. The edge $e_{17} \in w_2^-$ while e_1, e_5, e_9, e_{13} are edges in w_2^+. So the primitive walk w_2 does not have any pure cyclic block and it is mixed. Therfore B_{w_2} belongs to the universal *Grobner* basis of $I_{G'}$. The degree of B_{w_2} is 7. The largest degree of any circuit in G' is 6. Therefore, there is an element in the universal *Grobner* basis of I_G such that its degree is larger than the degree of any circuit in I_G.

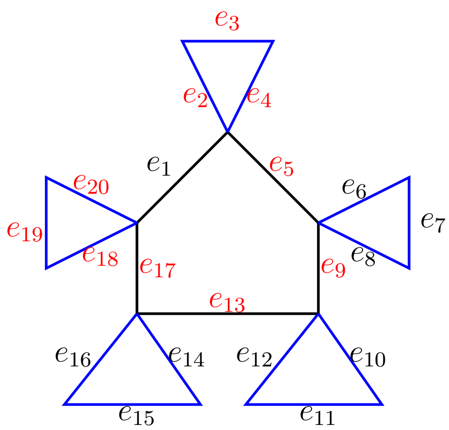

Figure 6.1: G

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