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QUINTESSENTIAL INFLATION IN MIMETIC DARK MATTER

by
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AN ABSTRACT OF THE THESIS OF

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Quintessential inflation models have been introduced drastically in the literature to account for the existence of the cosmological constant and to combine it with inflation. It was first introduced by Einstein in an *ad hoc* way to account for a static behavior of the universe. Later on it was used to also try to link inflation with the later on stages of the Universe. A new Quintessential inflation model is the topic of this thesis. The model will be a specific situation of the more general model called "Mimetic Dark Matter" introduced by Chamseddine and Mukhanov in 2013. It was shown in [15]that introducing a potential is essential to study the dynamics of the universe. In this thesis, the potential used will be defined on two different intervals, one before the end of inflation, and the other after it, with parameters that will be fixed in a way to produce 60 e-folds inflation and an energy density that represents a cosmological constant. The results agree with inflation during the first phase, both for the scale factor and the energy density. Moreover, an energy density will decreases like $1/t^2$ after inflation and then converges to a constant; the constant which represents energy density of Quintessence (or Dark energy). In addition, quantum cosmological perturbations will be discussed in a way similar to that presented in [15], but the detailed calculations are not carried out in here, they will be left for future work. Finally, a comparison of the model with that given by Peebles and Vilenkin in [11] is carried out, to finish with a conclusion and potential future work to enhance the model.

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Chapter 1

Introduction

When Albert Einstein first formulated his theory of gravity, the General Theory of Relativity, he added a term of the form $\Lambda g_{\mu\nu}$ to his equations,where Λ is a negative number known as the cosmological constant [1], and $g_{\mu\nu}$ is the metric of space-time, and thus the equations of motion became:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = T_{\mu\nu}. \tag{1.1}$$

(all of the terms will be explained in due term). Einstein introduced this term to explain why the universe is static, which was the general assumption at that time. On the other hand, in 1929, Astronomer Edwin Hubble published a paper in the Proceedings of the National Academy of Science, entitled :"*A relation between distance and radial velocity among extra-galactic nebulae* [2]. In this paper, Hubble discovered that the universe is expanding! From that moment on, people used this constant to explain the effects of the expansion of the universe. However, there was still something missing with this picture. This constant was introduced in an *ad hoc* way. So, how valid is such a constant, and what's the physical meaning behind it? Moreover, the Cosmic Microwave Background Radiation(CMB) was discovered in 1964, and this was done separately by astrophysicists A. G. Doroshkevich and Igor Novikov,and by astronomers Arno Penzias and Robert Wilson [3]. This discovery supported the idea of an expanding universe, and the steady-state model was abandoned. However, the Homogeneity of the CMB caused yet another problem, which became known as the "Horizon Problem" [4](see chapter 3 for details). In order to solve the problem, Alan Guth, and later on Andrei Linde, presented a solution to the problem: "The Inflation Phase". In this model, what happens is an extreme rapid accelerating expansion of the universe during a very short period of time at its very early age. To describe inflation, a scalar field ϕ called the "inflaton" [5] was used, which will result in the following Lagrangian:

$$L = R + \frac{1}{2}\dot{\varphi}^2 + V(\varphi) \tag{1.2}$$

where the dot represents derivative with respect to time, and $V(\varphi)$ is the potential that dictates the dynamics of the scalar inflaton field. Guth used the FRW(Friedman-Robertson-Walker) metric, $ds^2 = dt^2 - a(t)\delta_{ij}dx^i dx^j$, and from the equations of motion, a very important consequence appeared. This model showed that the scale factor $a(t)$, which shows how the universe expands, had a positive second derivative, which means an accelerating Universe! Thus, one of the implications of inflation is an accelerating universe.

The discovery of the CMB brought yet another complication to the picture. The small anisotropy spectrum in the CMB indicated a spatially flat universe, while it was also assumed at the time that the universe is closed (i.e. it has positive spatial curvature) [6]. This result implies that there is some shortfall in the energy density of the universe. Such a behavior can be explained by a cosmological constant, but with a negative pressure effect. However, this still raises the question of where did that constant come from. In order to answer such a question, P. J. E. Peebles and Bharat Ratra wrote a paper in 1988, entitled:" *Cosmological consequences of a rolling homogeneous scalar field*". Following a similar path as Guth, Peebles and Rahatra introduced a dynamical scalar field to the Lagrangian of general relativity, that could account for the shortfall in the energy density. This scalar field is called

Quintessence [7]. The result was, in addition to fixing the energy density, an accelerated universe.

The fact that both the inflaton and the quintessence fields resulted in an expanding universe, triggered the possibility of both of them having some relation, if not being one and the same entity. And from there on, numerous models have been formulated about "*Quintessential Inflation*". Using one scalar field, the inflationary phase of the universe, the accelerated expansion and the energy density crisis where solved in one hit. Moreover, in 1998, a new discovery that came to support these models. Two independent projects, the Supernova Cosmology Project and the High-Z Supernova Search Team simultaneously obtained results suggesting an accelerated expansion of the Universe. This discovery encouraged people to work more on such models and develop them [8]-[11].

Moreover, during the same period of time, work has been done on explaining yet another cosmological phenomena presented by nature, now known as "Dark Matter". These are hypothetical particles that have been introduced in order to explain strange gravitational effects that have been detected in nearby galaxies through gravitational lensing [12]. The first detection of Dark matter was done by Dutch Astronomer Jan Oort, and later on many detections were made. Several models have been introduced for these hypothetical particles, the most successful one (or more accurately the most used) is the WIMPs(weakly interacting massive particles), since these particles don't seem to interact through any force other than the gravitational force [13]. In addition, other candidates for these particles are supersymmetric particles. Yet non of these particles have been detected so far. However, in 2013, Chamseddine and Mukhanov presented a new model that could explain such a behavior, which they called "*Mimetic Dark Matter*"(MDM) [14]. In this model, Chamseddine and Mukhanov separated the physical metric into two pieces:

$$g_{\mu\nu} = \tilde{g}_{\mu\nu}(\tilde{g}^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi) \equiv P\tilde{g}_{\mu\nu} \tag{1.3}$$

where $\tilde{g}_{\mu\nu}$ is an auxiliary metric and ϕ is a scalar field. One can see that this metric remains invariant under a conformal transformation: $\tilde{g}_{\mu\nu} \to \Omega(x)^2\tilde{g}_{\mu\nu}$ with $\Omega(x)$ being an arbitrary function of space-time. This means that the conformal degree of freedom of the metric, which is represented by the first derivative of the scalar field, has been isolated. By writing the action:

$$S = -\frac{1}{2}\int d^4x\sqrt{-g(\tilde{g}_{\mu\nu})}[R(g_{\mu\nu}(\tilde{g}_{\mu\nu},\phi)) + L_m] \tag{1.4}$$

where R is the Ricci scalar (the same R that has been introduced since the beginning of this chapter), and L_m is the Lagrangian of the matter content, one can determine the equations of motion from the least action principle. These equations will split into two parts: the first one is a traceless equation, which results from the variation of the action with respect to the auxiliary metric, and the other part is a differential equation of the trace that results from varying the action with respect to the scalar field. The striking result is that these equations will give rise to a dynamical conformal degree of freedom, and so, even in the absence of the L_m part, it can produce effects similar to dark matter. From here, we say that this conformal degree of freedom *mimics* dark matter. Moreover, Chamseddine, Mukhanov and Vikman extended the model by introducing a potential for the scalar field to the Lagrangian, and thereby mimicking other gravitational properties of the normal matter [15]. From this, the purpose of this thesis is to produce a quintessential inflation in the frame work of MDM.

The organization of the thesis will proceed as follows: In chapter two, a brief introduction to General Relativity and Cosmology is presented. Then, in chapter three, a discussion of Inflation and Quintessence will be introduced, along with a brief presentation of cosmological perturbations, though they are not an essential part of this thesis. In chapter four, Mimetic dark matter model and its extension are described. The quintessential inflation model in MDM is explained in chapter five, followed by a comparison with the quintessential inflation model presented by P.J.E.Peebles and A. Vilenkin, as an example. Finally, a conclusion that summarizes the results, with possible future works is given in chapter 6.

Chapter 2

A (very) Brief Introduction to General Relativity and Cosmology

2.1 Geometry and Dynamics

One of the most remarkable properties of our universe is its homogeneity and isotropy at large scales. The observable universe is of the order of 3000Mpc($1Mpc \simeq 3.26 \times 10^6 lightyears \simeq 3.08 \times 10^{24} cm$). The homogeneity and isotropy of the universe appears on scales of the order of $100Mpc$, while on smaller scales, inhomogeneities begin to appear, such as galaxies and clusters [16]. This is known as *the Cosmological Principle.* Another important implication of the Cosmological Principle is that we are not at the center of the Universe, any other point is as much of a center as we are. We begin by investigating the different types of curvatures that a space can have.

2.1.1 Types of Curvature

The metric of spacetime is like the clock and the meter stick at each point in our space. It defines an invariant quantity called *the line element*:

$$ds^2 = \sum_{\mu\nu=0}^{3} g_{\mu\nu} dx^\mu dx^\nu \equiv g_{\mu\nu} dx^\mu dx^\nu \tag{2.1}$$

where the last equality implies that the Einstein summation convention is used throughout the thesis. The fact that our space is homogeneous and isotropic allows us to represent the universe as a time-ordered sequence of 3D space like surfaces, that are by themselves also homogeneous and isotropic(it's like cutting a bread loaf). There are 3 types of such maximally symmetric spaces: flat spaces, positively curved spaces and negatively curved spaces. We shall examine the line elements and the characteristic of these spaces here.

a) *flat spaces:* the line element of three-dimensional Euclidean space E^3 is

$$ds^2 = \delta_{ij} dx^i dx^j \tag{2.2}$$

where δ_{ij} is the Kronecker delta. This line element is invariant under translation($x^i \rightarrow x^i + a^i, a^i \equiv constant$) or under rotation($x^i \rightarrow R^i_k x^k$ with $R^i_k R^k_l = \delta^k_l$ are rotation matrices)

b)*positively curved space:* To represent a 3D positively curved space, one can study the line element of a 3D sphere (S^3) embedded in a 4D Euclidean space (E^4):

$$ds^2 = dx^2 + dy^2 + dz^2 + dw^2$$

$$x^2 + y^2 + z^2 + w^2 = a^2. \tag{2.3}$$

Now, take the infinitesimal difference of the second equation in (2.3), to get:

$$xdx + ydy + zdz + wdw = 0 \Rightarrow wdw = -\frac{\vec{x}.\vec{dx}}{\sqrt{a^2 - \vec{x}^2}}$$

where $\vec{x} = (x, y, z)$ and $\vec{dx} = (dx, dy, dz)$. Plugging this in the line element of (2.3), we then have:

$$ds^2 = \vec{dx}^2 + \frac{(\vec{x}.\vec{dx})^2}{a^2 - \vec{x}^2} \tag{2.4}$$

The fact that the line element of the sphere is invariant under 4D rotation implies homogeneity and isotropy of the surface of the 3-sphere

c) *negatively curved space:*this is represented as a hyperboloid H^3, embedded in a 4D Lorentzian space $SO(1,3)$,

$$x^2 + y^2 + z^2 - w^2 = a^2$$

$$ds^2 = dx^2 + dy^2 + dz^2 - dw^2 \tag{2.5}$$

. Applying the same procedure as the one we did for the positively curved space, we get

$$ds^2 = \vec{dx}^2 - \frac{(\vec{x}.\vec{dx})^2}{a^2 + \vec{x}^2} \tag{2.6}$$

The homogeneity and isotropy of the surface of the hyperboloid is a result of the invariance of the line element under a Lorentz transformation(or a pseudo-rotation). One can do the following transformation to the coordinates $\vec{x} \to a\vec{x}$ and $w \to aw$, and thus we get the line element for both the sphere and the hyperboloid:

$$ds^2 = a^2(\vec{dx}^2 \pm dw^2)$$

$$\vec{x}^2 \pm w^2 = \pm 1 \tag{2.7}$$

which means the $\vec{x}$ and w are now dimensionless. One can unify the three types of curvatures in one line element:

$$ds^2 = a^2[\vec{dx}^2 + k\frac{(\vec{x}.\vec{dx})^2}{1 - kx^2}] \equiv a^2\gamma_{ij}dx^i dx^j \tag{2.8}$$

with

$$\gamma_{ij} = \delta_{ij} + k\frac{x_i x_j}{1 - kx^k x_k} \tag{2.9}$$

and

$$k = \begin{cases} 1 & \text{positively curved} \\ 0 & \text{flat} \\ -1 & \text{negatively curved} \end{cases} \tag{2.10}$$

. Of course it is more helpful and easier to use spherical coordinates, since then the symmetry of the space will be manifested much clearly:

$$\vec{dx}^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

$$\vec{x}.\vec{dx} = rdr \tag{2.11}$$

using this in the line element (2.6), we get:

$$ds^2 = a^2[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2] \tag{2.12}$$

where $d\Omega$ is the solid angle given by $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. Another way of writing this line element, which will prove to be more convenient when dealing with Cosmology, is done by redefining the radial element using the transformation:$d\chi = \frac{dr}{\sqrt{1-kr^2}}$, and we get as a result:

$$ds^2 = a^2[d\chi^2 + S(k)^2 d\Omega^2], \tag{2.13}$$

where

$$S(k) = \begin{cases} sinh(\chi) & k = -1 \\ \chi & k = 0 \\ \sin(\chi) & k = +1 \end{cases} \tag{2.14}$$

. We now go to the most used metric(if not the only one) in Cosmology, the Friedman-Robertson-Lemaitre-Walker(FRW) metric.

2.1.2 FRW Metric

The FRW metric is defined by adding to the line element a term of the form $a^2\gamma_{ij}dx^idx^j$ with "a" varying with time. This will result in the line element:

$$ds^2 = dt^2 - a(t)^2\gamma_{ij}dx^idx^j. \tag{2.15}$$

It is essential to note that the metric component g_{0i} is set to 0 to insure isotropy of the metric. Moreover, we could have included a term g_{00} in the metric, but it is not necessary since this term can always be absorbed into t by redefining it as $dt' = \sqrt{g_{00}}dt$. Moreover, the ten independent components of the space-time metric have been reduced to simply two: the *scale factor* $a(t)$ and the curvature parameter k. The coordinates x^i are called the *comoving coordinates*, which are related to the physical coordinates by $x^i_{physical} = a(t)x^i$. An important implication of this distinction appears in the physical velocity of a system:

$$v^i_{physical} = \frac{dx^i_{physical}}{dt} = a(t)\frac{dx^i}{dt} + \frac{da}{dt}x^i \equiv v^i_{pec} + Hx^i. \tag{2.16}$$

Thus, there are two contributions to the velocity of an object: the *peculiar velocity* $v^i_{pec} = \dfrac{dx^i}{dt}$, and the *Hubble flow* Hx^i, where H is the *Hubble parameter*

$$H = \frac{\dot{a}}{a}. \tag{2.17}$$

The peculiar velocity is the one measured by an observer moving with the *Hubble flow*; a comoving observer. Now that we have the form of the FRW metric, we can simply insert (2.12) into (2.15), to get:

$$ds^2 = dt^2 - a(t)^2[\frac{dr^2}{1-kr^2} + r^2d\Omega^2]. \tag{2.18}$$

Finally, to set the FRW metric in the most appropriate way, we use (2.13) and (2.14) in (2.18), and the metric becomes:

$$ds^2 = dt^2 - a(t)^2[d\chi^2 + S(k)^2 d\Omega^2]. \tag{2.19}$$

Now, let

$$dt = a(t)d\tau \tag{2.20}$$

where τ is called the conformal time. Then, FRW metric becomes:

$$ds^2 = a(t)^2[d\tau^2 - (d\chi^2 + S(k)^2 d\Omega^2)] \tag{2.21}$$

which looks like a Minkowski metric, multiplied by a conformal factor that is varying with time. Now we will go to a very important physical parameter, that shows to be very helpful, the *redshift parameter*.

2.1.3 Redshift

Light that we receive from distant objects has helped us in discovering many of their properties [16]. The *cosmological redshift* is the Doppler shift associated with the relative motion of galaxies due to the expansion of the Universe. One can use two equivalent interpretations to describe this light: particle-like or wave-like interpretation.

Using the particle picture, the wavelength of the photon is inversely proportional to its momentum, $\lambda = \dfrac{h}{p}$ and p, the momentum of the photon, is inversely proportional to the scale factor(the derivation of this information is going to be presented in section 2).Thus, the wavelength of the photon is proportional to the scale factor. So the wavelength emitted at time t_1 will be related to that emitted at time t_0 by:

$$\lambda(t_0) = \frac{a(t_0)}{a(t_1)}\lambda(t_1) \tag{2.22}$$

and since $a(t_0) > a(t_1)$, then $\lambda(t_0) > \lambda(t_1)$, and thus there is a redshift in the wavelength of the light that we receive.

On the other hand, if we use the wave picture, one can say that if a signal of conformal duration $\Delta\tau$ is emitted at time τ_1 from a source at a comoving distance d, we receive this signal at time $\tau_0 = \tau_1 + d$ (the speed of light $c = 1$)since, according to (2.21), $\Delta\tau = \Delta\chi$ (the line element of light is 0). In the comoving frame, the conformal duration measured by us is the same as the source's, however the physical duration is not. From (2.19), we know that $\Delta t_1 = a(t_1)\Delta\tau$ and $\Delta t_0 = a(t_0)\Delta\tau$, thus, since $\lambda = \Delta t$, we get:

$$\frac{\lambda_0}{\lambda_1} = \frac{a(\tau_0)}{a(\tau_1)} \tag{2.23}$$

The *redshift parameter* is defined as the fractional shift in wavelength of a photon emitted by a distant galaxy at time t_1 and observed on Earth today [16]:

$$z = \frac{\lambda_0 - \lambda_1}{\lambda_1} \tag{2.24}$$

and so, if we set $a(t_0) = 1$, we get:

$$1 + z = \frac{1}{a(t_1)} \tag{2.25}$$

One important consequence of this is that, if we are looking at nearby sources, we can expand the scale factor as a power series in time $a(t) = a(t_0)[1+(t-t_0)H_0]+...]$, and thus, the redshift becomes:

$$z \simeq H_0 d \tag{2.26}$$

where d is nothing but the physical distance from the observer to the source. This is another statement of the *Hubble Law*, which states that the receding velocity of a distant object is proportional to the distance from that object, with the constant of proportionality being the *Hubble constant* at the moment of measurement. Now, we go on to investigate the kinematics and Dynamics of the space-time, which uses Einstein's General Theory of Relativity.

2.1.4 Kinematics and Dynamics

Kinematics

According *Einstein's Principle of Equivalence*, particles that are moving under gravitational forces *only*, are considered freely falling [18]. This implies that they follow *geodesics* in space-time, and this is applied to any space-time, in particular the FRW. The geodesic equation is defined as:

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0 \tag{2.27}$$

where x^μ is the position 4-vector, ds is the line element and [18]

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2}g^{\alpha\lambda}(g_{\lambda\beta},_\gamma + g_{\lambda\gamma},_\beta - g_{\beta\gamma},_\lambda) \tag{2.28}$$

where "," represents partial derivatives. This quantity is called *the Christoffel Symbol*, it is a measure of how the basis vectors transform under a coordinate transformation, since basis vectors might depend on the coordinate in some coordinate systems(a good example is spherical basis vectors). The geodesic equation can be derived either by varying the action of a massive relativistic particle ($S = -m\int ds$) or by using the parallel transport equation of the tangent vector to a curve parametrized by s. Defining the *covariant derivative* of a 4-vector A^{μ}as

$$\nabla_{\alpha}A^{\mu} = \partial_{\alpha}A^{\mu} + \Gamma^{\mu}_{\alpha\beta}A^{\beta}$$

one can write the geodesic equation as:

$$P^{\mu}\nabla_{\mu}P^{\nu} = 0 \tag{2.29}$$

where P^{μ} is the 4-momentum of the particle.

For the FRW metric, one can calculate the Christoffel Symbols,to get:

$$\Gamma^{0}_{ij} = a\dot{a}\gamma_{ij}; \qquad \Gamma^{i}_{0j} = \frac{\dot{a}}{a}\delta^{i}_{j}; \qquad \Gamma^{i}_{jk} = \frac{1}{2}\gamma^{il}(\gamma_{lj},_{k} + \gamma_{lk},_{j} - \gamma_{jk},_{l}) \tag{2.30}$$

and the remaining terms are 0. And so, if we consider the 0^{th} component of the geodesic equation, with the help of these Christoffel symbols, we get:

$$E\frac{dE}{dt} = -\frac{\dot{a}}{a}p^{2} \tag{2.31}$$

where p is the three-momentum of the particle. From the energy-momentum relation, and if we consider photon (massless) particles, we get $E^{2} = p^{2}$ and so $EdE = pdp$. From here, we get:

$$\frac{\dot{p}}{p} = -\frac{\dot{a}}{a} \Rightarrow p \propto \frac{1}{a} \tag{2.32}$$

and this is from where the redshift is derived. Now we proceed to the dynamics.

Dynamics

The dynamics of the universe are determined by *Einstein's equations of motion* (1.1). The l.h.s determines how space-time is curved, while the r.h.s gives information about the matter content of the universe. The *Cosmological Principle* will guide us in determining the form of the energy-momentum tensor $T^{\mu\nu}$. Isotropy of the universe implies that the off-diagonal elements of $T^{\mu\nu}$ should vanish, since otherwise we will be in a situation where there is more matter distribution in one direction than in another. From that last notion, we infer that the spatial components of the stress-energy tensor should have the same value. Moreover, homogeneity implies that these quantities should be independent of space, and thus they should vary only with time. Therefore, we get the energy-momentum tensor of a perfect fluid to be a perfect representative of the matter content of the universe:

$$T^{\mu\nu} = \begin{bmatrix} \rho(t) & 0 & 0 & 0 \\ 0 & -P(t) & 0 & 0 \\ 0 & 0 & -P(t) & 0 \\ 0 & 0 & 0 & -P(t) \end{bmatrix}. \tag{2.33}$$

This can be written in a more compact way as:

$$T^{\mu\nu} = (\rho(t) + P(t))U^{\mu}U^{\nu} - P(t)\delta^{\mu\nu} \tag{2.34}$$

where $U^{\mu} = \dfrac{dx^{\mu}}{ds}$ is the relative 4-velocity of the fluid with respect to an observer, while ρ and P are the energy density and pressure of the fluid in its rest frame of, respectively. Moreover, energy and momentum conservation are manifested through the relation:

$$\nabla_{\mu}T^{\mu\nu} = T^{\mu\nu},_{\mu} + \Gamma^{\mu}_{\mu\lambda}T^{\lambda\nu} + \Gamma^{\nu}_{\mu\lambda}T^{\mu\lambda}. \equiv 0 \tag{2.35}$$

Now, if we are in an FRW universe, to get the evolution of the energy density, one has to use eq.(2.35) with $\nu = 0$, with the help of the Christoffel Symbols and components of $T^{\mu\nu}$, we get:

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) = 0 \tag{2.36}$$

Now we will consider the energy density profile of the three different types of entities in the universe: matter, radiation and the so called dark energy.

-Matter:

In cosmology, matter means entities that have $|P| \ll \rho$, and thus one can approximate $P \simeq 0$, and then, inserting this in (2.36), we get:

$$\rho \propto a^{-3} \tag{2.37}$$

-Radiation:

Radiation denotes species that have a relation between pressure and energy density of the form: $P = \frac{1}{3}\rho$ (this result actually follows from thermodynamical calculations [19] of a gas of photons). This means that the momentum of the particle is much bigger than its mass, and therefore the energy density becomes:

$$\rho \propto a^{-4} \tag{2.38}$$

this class of species includes, to a first approximation, neutrinos.

-Dark Energy:

Today, as was mentioned in the introduction, the universe seems to be dominated by a creature that contributes negative pressure to the equations of motion. This creature is *Dark Energy*, and so with $P = -\rho$, the energy density of such a creature behaves as:

$$\rho \propto a^{0} \tag{2.39}$$

To finish up this chapter, we have to look at a very important set of equations in Cosmology, the *Friedmann Equations*

Friedmann Equations:

The *Einstein tensor* is given by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$$

(ignoring the cosmological constant for the moment) where $R_{\mu\nu}$ is a symmetric 2^{nd} rank tensor called the *Ricci* tenor and R is the Ricci scalar (the trace of the Ricci tensor).The Ricci tensor is given by [18] :

$$R_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu},_{\lambda} - \Gamma^{\lambda}_{\lambda\mu},_{\nu} + \Gamma^{\lambda}_{\lambda\rho}\Gamma^{\rho}_{\mu\nu} - \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\rho}. \tag{2.40}$$

The Ricci tensor results from contracting the first and the third indicies of the Riemann curvature tensor $R^{\alpha}_{\beta\alpha\gamma}$. The Riemann curvature tensor indicates whether a space-time is inherently curved, or if the curvature is coming from the coordinate system chosen. From the FRW metric, we get the following results for the Ricci tensor and scalar:

$$R_{00} = -3\frac{\ddot{a}}{a}; \qquad R_{0i} = 0; \qquad R_{ij} = -[\frac{\ddot{a}}{a} + 2(\frac{\dot{a}}{a})^2 + 2\frac{k}{a^2}]g_{ij} \tag{2.41}$$

The fact that R_{i0} is 0 and that R_{ij} is proportional to g_{ij} is consistent with isotropy and symmetry of the FRW metric. From these results, we get the Ricci tensor to be:

$$R = -6[\frac{\ddot{a}}{a} + (\frac{\dot{a}}{a})^2 + \frac{k}{a^2}] \tag{2.42}$$

Finally, inserting all of these into Einstein's equations, with the stress-energy tensor of a perfect fluid that was described, we get the Friedmann equations:

$$(\frac{\dot{a}}{a})^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \tag{2.43}$$

	w	$\rho(a)$	$a(t)$	$a(\tau)$
RD	$\frac{1}{3}$	a^{-4}	$t^{1/2}$	τ
MD	0	a^{-3}	$t^{2/3}$	τ^2
ΛD	-1	a^0	e^{Ht}	$-\tau^{-1}$

Figure 2.1: The different behaviors of the energy density for Radiation dominated(RD), matter dominated(MD) and dark energy dominated(ΛD) universes . $w$is defined such as $p = w\rho$, this is known as the *equation of state*

$$\ddot{a} = \frac{-4\pi G}{3}(\rho + 3P)a \tag{2.44}$$

If we consider a flat universe (k=0), the energy density today would be (from (2.41)):

$$\rho_{crit,0} = \frac{3H_0^2}{8\pi G} \tag{2.45}$$

Eq. (2.45)is known as the *critical energy density.*

In this chapter, we have seen the different types of curvatures that a space might have. We then investigated the Cosmological principle and its effect on the equations that will govern the evolution of the universe. And we ended up with describing Friedmann equations, which are essential for cosmological studies. Now that we have all the tools in hand, we can dwell into the mysteries of *Inflation.*

Chapter 3

Inflation and Quintessence

3.1 Inflation

3.1.1 The Horizon Problem

The causal size of a region of spacetime is determined by how far light can travel in that region. To see this, we go back to the FRW metric.Due to spherical symmetry, we can set $\theta = \phi = constant$, and using conformal time $dt = a(t)d\tau$, we obtain:

$$ds^2 = a(\tau)^2[d\tau^2 - d\chi^2]. \tag{3.1}$$

For a photon, the line element is $ds^2 = 0$, which means that $\Delta\tau = \pm\Delta\chi$, where the $+$ sign represents an outgoing photon, and the $-$ an incoming photon. Now, there is a crucial concept that is used when discussing causality; The *Particle Horizon*.

-*Particle Horizon:* The maximum comoving distance that light can travel between conformal times τ_1 and $\tau_2 > \tau_1$ is $\Delta\tau = \tau_2 - \tau_1$. So, if the "Big Bang" singularity started at $t_i = 0$, then the maximum conformal distance that light can travel, such that it is received by an observer at time t, is:

$$\chi_{ph} = \tau - \tau_i = \int_{t_i}^{t} \frac{dt}{a(t)}. \tag{3.2}$$

The boundary of this region of space is called the *particle horizon.* Another way to see it is that it's the intersection of the past light cone of a particle with the surface $\tau = \tau_i$, since there, only particles whose worldline intersects with the past light cone can have influence on the particle under consideration.(There is another horizon, compliment to the particle horizon, called the *event horizon* but it is irrelevant to the purpose of this thesis). Moreover, we can write equation (3.2) in a way that relates it to the *comoving Hubble radius*, $(aH)^{-1}$:

$$\chi_{ph} = \int_{\ln(a_i)}^{\ln(a)} (aH)^{-1} d\ln(a) \tag{3.3}$$

.For a flat universe dominated by a fluid that has $p = w\rho$ (such relations are called *equations of state*), then from (2.36), we get:

$$\rho = \rho_0 a^{3(1+w)}$$

and hence, inserting this into Friedman's equation (2.43), multiplying it by a, we get:

$$(aH)^{-1} = H_0^{-1} a^{(1+3w)/2}. \tag{3.4}$$

All familiar matter sources satisfy what is called the *Strong energy Condition*(SEC) $(1 + 3w) > 0$, which means that for these sources, the Hubble radius is always increasing. In this case, the integral (3.3) becomes:

$$\chi_{ph} = \frac{2H_0^{-1}}{1+3w}[a^{(1+3w)/2} - a_i^{(1+3w)/2}] \rightarrow \quad \frac{2H_0^{-1}}{1+3w}[a^{(1+3w)/2}] \tag{3.5}$$

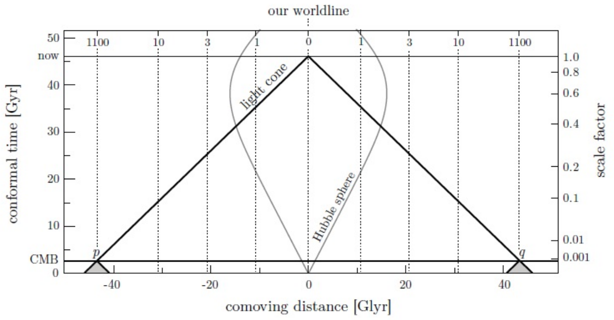

Figure 3.1: An illustration of the Horizon problem. We are situated at the space-like slice called now. The intersection of our past light cone with the CMB defines the regions where we can receive light from the CMB. However, the past light cones of points p and q don't overlap before hitting the singularity, which means they were never in causal contact, yet they have the same properties! This is the horizon problem

since as $a_i \to 0$ and when SEC is satisfied, the lower limit of the integral tends to 0. All of this seemed at the beginning nice and charming, yet the discovery of the CMB brought a problem. This is most easily explained in the diagram above. The isotropy in the CMB is really puzzling (anisotropies are of the order of one part in ten thousands, yet these will show to be useful later on). Since there is a finite time between the Big Bang singularity and the time of formation of the CMB, this implies that any two points along the space-like curve at t_f that are separated by more than 1 Gly (10^9 light-years)had never had their past light cones intersecting. This implies that these points were never in causal contact! How could it be that such points were never in causal contact, yet they knew that they had to be at the same temperature? This is the *Horizon Problem*.

As stated earlier, the fact that the *Hubble radius is always increasing* means that the SEC must be satisfied, and therefore this should also be happening between the Big Bang singularity and the time of formation of the CMB. Thus, a possible solution to the problem is *a decreasing Hubble sphere during the period* $t_i - t_f$, where t_f is

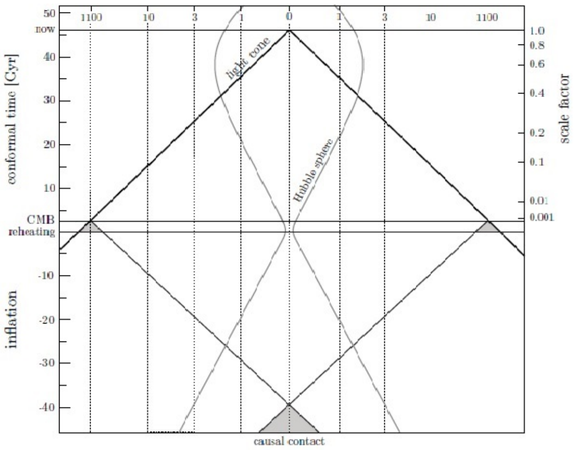

Figure 3.2: The mechanism of Inflation. An decrease in the Hubble radius to the extent that it becomes smaller than the particle Horizon will insure that there is causal contact between two widely separated points on the CMB surface, which will explain the homogeneity of the CMB, and therefore solve the horizon problem

the time of formation of the CMB.

$$\frac{d}{dt}(aH)^{-1} < 0. \tag{3.6}$$

Here rises a violation of the SEC ($1+3w < 0$). This means that the *particle Horizon* is dominated by the lower limit in the integral (3.3), meaning that there is much more time between the big bang singularity and the formation of the CMB; time that is sufficient to insure causal connection between the different parts of the CMB. This mechanism is called *Inflation*.

To know how much (roughly) inflation should take place in order to solve the problem, we start with the fact that:

$$(a_0H_0)^{-1} < (a_IH_I)^{-1}. \tag{3.7}$$

Now, at the time of formation of the CMB, the universe was radiation dominated, so we can neglect the contribution from other species, and so we get:

$$\frac{a_0 H_0}{a_E H_E} \sim \frac{a_0}{a_E}(\frac{a_E}{a_0})^2 \sim \frac{T_0}{T_E} \sim 10^{-28} \tag{3.8}$$

where the second equality follows from the fact that for a radiation dominated universe, $H \propto a^{-2}$, and the third equality follows from $a \propto T$ for radiation. T_0 is the temperature of the universe at the present time ($\sim 10^{-3}$ eV) and T_E is the temperature at the end of inflation($\sim 10^{15} GeV$)[16] Therefore, the following equation

$$(a_I H_I)^{-1} > (a_0 H_0)^{-1} \sim 10^{28}(a_E H_E)^{-1} \tag{3.9}$$

means that there should be a 10^{28} increase in the size of the *Hubble sphere* so that inflation solves the problem. This is the statement that *inflation needs almost 60 e-folds* to solve the *Horizon Problem* (assuming that H is a constant, so that $H_I = H_E$, and then taking the logarithm of (3.9)).

There are a few consequences of this mechanism, we will site only the two that are relevant for us

First Condition: notice that

$$\frac{d}{dt}(aH)^{-1} = -\frac{\ddot{a}}{a^2} < 0 \Rightarrow \ddot{a} > 0. \tag{3.10}$$

The above equation means that the universe is expanding.

Second condition: let

$$\epsilon = -\frac{\dot{H}}{H^2} = \frac{3}{2}(1 + \frac{P}{\rho}) \tag{3.11}$$

where the 2^{nd} equality follows from the second Friedman equation. Inserting the continuity equation, $\dot{\rho} = -3H(\rho + P)$, into ϵ, we get

$$\frac{d \ln \rho}{d \ln a} = 2\epsilon << 1 \tag{3.12}$$

hence, the energy density is almost constant during inflation. Now, we proceed to describe the dynamics of inflation, using the Lagrangian formalism.

3.1.2 Inflation Formalism

As mentioned in the last section, in order for inflation to solve the Horizon problem, $\epsilon = -\frac{\dot{H}}{H^2} = -\frac{d\ln H}{dN} < 1$, where $dN = d\ln a = Hdt$ measures the number of e-folds N of inflationary expansion. This means that ϵ must remain small for a sufficiently long time. This condition is achieved by having:

$$\eta \equiv \frac{d\ln\epsilon}{dN} = \frac{\dot{\epsilon}}{H\epsilon} < 1 \tag{3.13}$$

To study the dynamics of inflation, the Lagrangian of a scalar field represents the L_m introduced in (1.4), and the Lagrangian becomes nothing but (1.2). By varying this Lagrangian with respect to the metric, we get the following energy-momentum tensor [17]

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}(\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - V(\phi)) \tag{3.14}$$

which implies an energy density and pressure:

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi); \qquad P = \frac{1}{2}\dot{\phi}^2 - V(\phi). \tag{3.15}$$

Inserting the energy density in the Friedmann equation for a flat universe, $H^2 = \frac{1}{3M_{pl}^2}\rho$ ($M_{pl} = \sqrt{\frac{\hbar c}{8\pi G}} = 2.4 \times 10^{18} GeV$ is the Planck mass), and the result:

$$H^2 = \frac{1}{3M_{pl}^2}[\frac{1}{2}\dot{\phi}^2 + V(\phi)]. \tag{3.16}$$

Taking the time derivative of (3.16), and combining it with the second Friedman equation, $\dot{H} = \frac{-1}{2M_{pl}^2}(\rho_\phi + P_\phi) = -\frac{1}{2}\frac{\dot{\phi}^2}{2M_{pl}^2}$, we get the Klein-Gordon equation:

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0 \tag{3.17}$$

where the prime denotes derivative with respect to ϕ. A very important notion one has to mention before wrapping up this section, is the idea of *slow role inflation.*

Slow role Inflation

One way to insure sure that the inflation parameter ϵ is less than one is by having a small contribution from the kinetic energy of the scalar field to the energy density. This means that the acceleration of the scalar field must be small, and this is studied through the parameter

$$\delta \equiv \frac{\ddot{\phi}}{H\dot{\phi}}. \tag{3.18}$$

Thus, in the *slow role approximation*, $\frac{1}{2}\dot{\phi}^2 << V(\phi)$, ww get the new Friedmann and Klein-Gordon equations, respectively:

$$H^2 \approx \frac{V}{3M_{pl}^2}; \qquad 3H\dot{\phi} \approx V' \tag{3.19}$$

These approximations simplify the calculations extensively, and still give a good picture of what's going on during inflation. Now that inflation has been described extensively, we move on to Quintessence. One can see that the formalism used will be almost the same, for both of them are described by a scalar field. Nonetheless, the properties of these fields will be different at the asymptotes, and it will be shown that they meet somewhere.

3.2 Quintessence

A very important dimensionless parameter used in observational astronomy is the *density parameter*

$$\Omega = \frac{\rho}{\rho_{crit}} \tag{3.20}$$

with ρ_{crit} being defined in (2.45). Observations have shown that the density parameter of baryonic (i.e normal) matter, cold dark matter and radiation are $\Omega_b = 0.05; \quad \Omega_c = 0.27; \quad \Omega_r = 9.4 \times 10^{-5}$, respectively. However, the sum of these values were not enough to close the universe(i.e to have a total density parameter equals 1) [20]. This means that there must be some other energy component missing that we are not taking into consideration. Moreover, as mentioned before, there are small anisotropies in the temperature spectrum of the CMB. These anisotropies can be attributed to what is called the Sachs-Wolfe effect: At about $10^{11} - 10^{12}s$ after the Big Bang, matter started to decouple from radiation, helium recombined and became neutral, which allowed photons to travel freely. This is the *last scattering surface*, it's the set of spatial positions at which matter decoupled from radiation. This inhomogeneous distribution of matter at the time, created deep gravitational potentials on the surface of last scattering, which photons would have to travel in. The result is a peak in the anisotropy spectrum on the angular scale corresponding to the apparent size of the sound horizon at recombinations (the sound horizon at recombination is the boundary of the last scattering surface). By expanding the temperature variations in terms of spherical harmonics [16]:

$$\frac{\delta T}{T_0} = \sum^{l,m} a_{lm} Y_{lm}(\theta, \phi) \tag{3.21}$$

one can show that the peak occurs at a multipole [21] $l = 220/\sqrt{1 - \Omega_k}$ where Ω_k is the curvature parameter expressed as a fraction of the critical energy density. Ob-

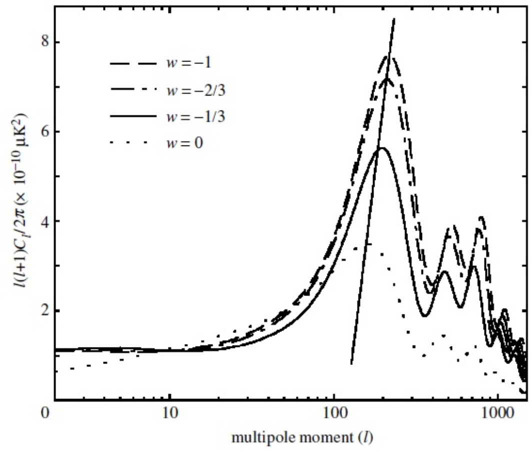

Figure 3.3: the power spectrum vs. multipole expansion, it shows that the anisotropy peaks occurs at a multipole expansion approximately equals 200, which implies that the universe must have 0 curvature

servations have shown that the peak occurs at about $l \approx 220$ [22], which means that the curvature parameter is $\Omega_k \approx 0$. This shows that we are living in a spatially flat universe! And finally, another surprising effect that was observed is the accelerated expansion of the universe, as mentioned in the introduction. These three issues give us a low density, spatially flat, accelerating universe.

The first solution to this problem came by introducing a cosmological constant with a negative pressure effect, to Einstein's equations of motion:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = T_{\mu\nu}. \tag{3.22}$$

However, introducing such a term raises yet another issue: Where did it come from? Is it some type of quantum gravitational effects? In order for the

constant energy density to be dominant today, it must have contributed a negligible amount to the total energy density at previous times(mainly during recombinations, reheating and nucleosynthesis). In order to account for such a fact, one has to think of some dynamical process which leads to such a consequence, a scalar field: Quintessence. Quintessence is a time varying, spatially inhomogeneous scalar field and has a negative pressure contribution. Some models suggest that it is characterized by an equation of state ($p = w\rho$), with $0 \geq w \geq -1$, while the cosmological constant has $w = -1$. The same equations that were used to explain the dynamics of the "inflaton" field can be used as well for quintessence, with the simple substitution of ϕ to Q, and change in the boundary conditions. However, in addition to these equations, one has to look into how perturbations in this field will affect the spectrum of the CMB. We will not dwell into the detailed calculations of cosmological perturbation theory, due to the lack of "space and time", in addition to the fact that one must have Gandhi's patience in order to write the derivation using Latex. From here, we will just consider the main points in perturbation theory that will lead to a good understanding of the physics behind it.

The basic idea of cosmological perturbations goes as follows: start with a metric $g_{\mu\nu}$, make the following transformation: $g_{\mu\nu} \rightarrow g_{\mu\nu} + \delta g_{\mu\nu}$, where $\delta g_{\mu\nu}$ is a small change in the metric (aka perturbation), then study the equations of evolution of these perturbations. Next, perturb the matter content: $T_{\mu\nu} \rightarrow T_{\mu\nu} + \delta T_{\mu\nu}$, and study the evolution of the matter perturbations, using concervation of the stress energy tensor, $\nabla_\mu T^{\mu\nu} = 0$. For the purpose of Quintessence, i.e a scalar field (and also for inflation, however the difference between the two comes in the parameters that are used in the perturbed metric), start with the following metric:

$$ds^2 = a^2[(1+2\phi)d\eta^2 + 2B_{,i}\, dx^i d\eta - ((1-2\psi)\delta_{ij} - 2E_{,ij}\,)dx^i dx^j] \tag{3.23}$$

where B, E, ϕ and ψ are scalar fields that are used to characterize vector, tensor and scalar perturbations, respectively. In addition, we need to perturb the scalar field(this means perturbing the energy-momentum tensor) $Q \rightarrow Q_0 + \delta Q(\mathbf{x}, \eta)$. Substituting all of this into the Klein-Gordon equation, we get:

$$Q_0'' + 2HQ_0' + a^2 V_{,\phi} = 0 \tag{3.24}$$

$$\delta Q''+2H\delta Q'-\Delta(\delta Q-Q0'(B-E'))+a^2V_{,QQ}\,\delta Q-Q_0'(3\psi+\phi)'+2a^2V_{,Q}\,\phi=0 \quad (3.25)$$

here prime denotes derivative with respect to conformal time η. Now let:

$$\Phi\equiv\phi-\frac{1}{2}[a(B-E')]'; \quad \Psi\equiv\psi+\frac{a'}{a}(B-E') \qquad (3.26)$$

these quantities are gauge invariant(see [16], page 293 for details), and they allow us to know whether these inhomogeneities are coming from the coordinate system chosen or whether they are real physical inhomogeneities. Moreover, if we define:

$$\delta\bar{Q}\equiv\delta Q-Q_0'(B-E')$$

, then:

$$\delta\bar{Q}''+2H\delta\bar{Q}'-\Delta\delta\bar{Q}+a^2V_{,QQ}\,\delta\bar{Q}-Q_0'(3\Psi+\Phi)'+2a^2V_{,Q}\,\Phi=0 \qquad (3.27)$$

To solve such an equation, one may take the two limiting cases: perturbations with physical wavelength λ_{ph} smaller than the curvature scale H^{-1}, and for long wavelength perturbations. We will begin with the short wavelength perturbations. First, we do a Fourier transform is space:

$$\delta Q=\frac{1}{(2\pi)^3}\int\delta Q_k(t)\exp{(k.x)}d^3x \qquad (3.28)$$

If $\lambda_{ph}<<1/H$ (or if $k>Ha$ where k is the comoving wave number), and assuming a slow role behavior(there's nothing special about slow role approximation for inflaton. This approximation means that the energy density of the field is dominated by the potential energy and not the kinetic), we get from (3.27):

$$\bar{Q_k}'' + 2H\bar{Q_k}' + k^2\bar{Q_k} \simeq 0. \tag{3.29}$$

Let $\bar{Q_k} = u_k/a$, and assuming that $k|\eta| >> 1$, we get

$$\delta\bar{Q_k} \simeq \frac{C_k}{a} \exp \pm ik\eta \tag{3.30}$$

with C_k being a constant of integration set by initial conditions.

For long wavelength perturbations, it's more interesting to go back to the form of the Klein-Gordon equation in physical time rather than conformal time:

$$\ddot{Q}_0 + 3H\dot{Q}_0 + V_{,Q} = 0 \tag{3.31}$$

if we neglect the second derivative (which is valid in the slow role approximation), we get:

$$3H\dot{Q}_0 + V_{,Q} \simeq 0 \tag{3.32}$$

and

$$\delta\ddot{Q} + 3H\delta\dot{Q} - \Delta\delta Q + V_{,QQ}\,\delta Q - 4\dot{Q}_0\dot{\Phi} + 2V_{,Q}\,\Phi = 0 \tag{3.33}$$

an equation for $\dot{\Phi}$ can be achieved by considering the evolution of the stress-energy tensor perturbations: $\dot{\Phi} + H\Phi = 4\pi\dot{Q}_0\delta Q$. The solution of the equations will be:

$$\delta Q_k = A_k\frac{V_{,Q}}{V}; \quad \Phi_k = -\frac{1}{2}A_k(\frac{V_{,Q}}{V})^2 \tag{3.34}$$

with A_k set by boundary conditions(for example, when studying long wavelength perturbations during inflation, A_k is set by requiring that $\delta\phi_k$ has the minimum amplitude when the perturbations' wave number becomes comparable to the inverse of the Hubble radius. This is known as the moment of *crossing the horizon*). So these are the limiting cases in cosmological perturbation theory of a scalar field. The behavior of these perturbations(whether they are occurring in an inflationary, a quintessential or a quintessential-inflation phase) will depend on the boundary conditions employed. We will end this section by a short description of the multipole expansion that determines the anisotropy in the CMB spectrum.

Multipole expansion

A sky map of the CMB temperature fluctuations can be described in terms of an infinite sequence of correlation functions called *temperature autocorrelation function*:

$$C(\theta) \equiv \langle \frac{\delta T}{T_0}(\vec{l_1}) \frac{\delta T}{T_0}(\vec{l_2}) \rangle \tag{3.35}$$

where $\langle\rangle$ means averaging over all possible directions $\vec{l_1}, \vec{l_2}$. $\dfrac{\delta T}{T_0}$ are temperature fluctuations that arise from inhomogeneities in the radiation energy density. It can be shown to be:

$$\frac{\delta T}{T} = \frac{1}{4}(\delta_k + \frac{3i}{k^2}(k_m l^m)\delta_k') \tag{3.36}$$

where δ_k is the k^{th} Fourier mode of the radiation energy perturbation. Now, expand both the autocorrelation function and the temperature fluctuations in a series of Legendre polynomials and spherical harmonics, respectively, such as:

$$C(\theta) = \frac{1}{4\pi}\sum_{l=2}^{\infty}(2l+1)C_l P_l(cos(\theta)); \quad \frac{\delta T}{T_0} = \sum_{l,m} a_{lm} Y_{lm}(\theta, \phi) \tag{3.37}$$

Inserting (3.37) into (3.36) and (3.35), the result is :

$$l(l+1)C_l \simeq \frac{9|(\Phi_k^0)^2 k^3|}{100\pi} \tag{3.38}$$

where Φ_k^0 is the value of the k^{th} mode of gravitational perturbation at the surface of last scattering[16]. These are the multipole components for anisotropies on large angular scales (i.e $l << 200$), where the wavelength of the perturbations exceed the Hubble radius. For short angular scales, the derivation is much more evolved and messy, therefore we will just give the final result [16]:

$$\frac{l(l+1)C_l}{(l(l+1)C_l)_{lowl}} = \frac{100}{9}(O + N_1 + N_2 + N_3). \tag{3.39}$$

All the letters are different types of integrals. O represents contributions from the gravitational potential perturbations to the multipole moments, N_1 from baryonic matter, N_2 from dark matter and N_3 from quintessence, and $(l(l+1)C_l)_{lowl}$ is nothing but (3.38). We are not going to deal with the calculations of these integrals in this thesis, however they will be carried out later on in future works.

In this chapter we have reviewed the problem that led to the creation of the inflation model: the horizon problem. Then we've encountered the quintessence field and what problems it's suppose to solve. Finally, we reviewed briefly the theory of cosmological perturbations, and investigated the multipole moment that arises from temperature fluctuation in the CMB. In the Next chapter, we will witness the model presented by Chamseddine and Mukhanov in 2013.

Chapter 4

Mimetic Dark Matter and Cosmology

In 2013, Chamseddine and Mukhnoov proposed a new model to explain the behavior of what is called "Dark Matter". In this chapter, we will review this model that was introduced in [14], along with its extensions [15] [23]

4.1 The Model

Let us start by defining a physical metric $g_{\mu\nu}$ in terms of an auxiliary metric $\tilde{g}_{\mu\nu}$ and a scalar field ϕ in the following way:

$$g_{\mu\nu} = \tilde{g}_{\mu\nu}(\tilde{g}^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi) \equiv P\tilde{g}_{\mu\nu} \tag{4.1}$$

The metric written in this form is invariant under a conformal transformation of the auxiliary metric:

$$\tilde{g}_{\mu\nu} \to \Omega^2(x)\tilde{g}_{\mu\nu} \implies g_{\mu\nu} \to g_{\mu\nu}.$$

Note also that such a representation of the metric did not violate the fact that the metric is covariant under a general coordinate transformation. From here, one can construct the metric in the usual way, with the metric being now a function of both the auxiliary metric and the scalar field ϕ:

$$S = -\frac{1}{2}\int d^4x\sqrt{-g(\tilde{g}_{\mu\nu})}[R(g_{\mu\nu}(\tilde{g}_{\mu\nu},\phi)) + L_m] \tag{4.2}$$

with L_m being the Lagrangian of matter(by matter this means anything). Now varying the action, we get:

$$\delta S = \int d^4x \frac{\delta S}{\delta g_{\alpha\beta}}\delta g_{\alpha\beta}$$

$$= \int d^4x\delta(\sqrt{-g(\tilde{g}_{\mu\nu})})[R(g_{\mu\nu}(\tilde{g}_{\mu\nu},\phi))+L_m]+\int d^4x\sqrt{-g(\tilde{g}_{\mu\nu})}[\delta R_{\alpha\beta}g^{\alpha\beta}+\delta g^{\alpha\beta}R_{\alpha\beta}+\delta L_m] \tag{4.3}$$

The first term in the second integral vanishes (assuming there's no contribution at the boundaries) [17], and the variation of the action becomes:

$$\delta S = \int d^4x\sqrt{-g}(G^{\alpha\beta} - T^{\alpha\beta})\delta g_{\alpha\beta} \tag{4.4}$$

with $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta}$ being the Einstein tensor and $T_{\alpha\beta}$ is the energy-momentum tensor. The variation of the physical tensor is no longer the usual variation used, rather it is now a variation as a function of the auxiliary metric $\tilde{g}_{\alpha\beta}$ and the scalar field ϕ, and it's given by:

$$\delta g_{\alpha\beta} = P\delta\tilde{g}_{\alpha\beta} + \tilde{g}_{\alpha\beta}\delta P$$

$$\delta P = \delta(\tilde{g}^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi) = \delta\tilde{g}^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + 2\tilde{g}_{\alpha\beta}\partial_\alpha\phi\partial_\beta\delta\phi = -\tilde{g}^{\kappa\mu}\tilde{g}^{\lambda\nu}\delta\tilde{g}_{\mu\nu}\partial_\kappa\phi\partial_\lambda\phi + 2\tilde{g}^{\kappa\lambda}\partial_\kappa\delta\phi\partial_\lambda\phi$$

$$\Rightarrow \delta g_{\alpha\beta} = P\delta\tilde{g}_{\alpha\beta} + \tilde{g}_{\alpha\beta}(-\tilde{g}^{\kappa\mu}\tilde{g}^{\lambda\nu}\delta\tilde{g}_{\mu\nu}\partial_\kappa\phi\partial_\lambda\phi + 2\tilde{g}^{\kappa\lambda}\partial_\kappa\delta\phi\partial_\lambda\phi)$$

$$= P\delta\tilde{g}_{\mu\nu}\delta^\mu_\alpha\delta^\nu_\beta + \frac{1}{P}g_{\alpha\beta}(-P^2g^{\kappa\mu}g^{\lambda\nu}\delta\tilde{g}_{\mu\nu})\partial_\kappa\phi\partial_\lambda\phi + 2g_{\alpha\beta}\tilde{g}^{\kappa\lambda}\partial_\kappa\delta\phi\partial_\lambda\phi$$

$$\delta g_{\alpha\beta} = P\delta\tilde{g}_{\mu\nu}(\delta^\mu_\alpha\delta^\nu_\beta - g_{\alpha\beta}g^{\kappa\mu}g^{\lambda\nu}\partial_\kappa\phi\partial_\lambda\phi) + 2g_{\alpha\beta}\tilde{g}^{\kappa\lambda}\partial_\kappa\delta\phi\partial_\lambda\phi \tag{4.5}$$

where in the second line we have used the fact that $\delta \tilde{g}^{\alpha\beta} = -\tilde{g}^{\alpha\mu}\tilde{g}^{\beta\nu}\delta\tilde{g}_{\mu\nu}$, and in the 4^{th} line we have used the identity $\tilde{g}^{\alpha\beta} = Pg^{\alpha\beta}$, which is a direct consequence of (4.1). Plugging (4.5) into (4.4), we get:

$$\delta S = \int d^4x\sqrt{-g}(G^{\alpha\beta} - T^{\alpha\beta}) \times (P\delta\tilde{g}_{\mu\nu}(\delta^\mu_\alpha\delta^\nu_\beta - g_{\alpha\beta}g^{\kappa\mu}g^{\lambda\nu}\partial_\kappa\phi\partial_\lambda\phi) + 2g_{\alpha\beta}\tilde{g}^{\kappa\lambda}\partial_\kappa\delta\phi\partial_\lambda\phi) \tag{4.6}$$

the least action principle requires $\dfrac{1}{\sqrt{-g}}\dfrac{\delta S}{\delta\tilde{g}_{\mu\nu}} = 0$, which means

$$(G^{\mu\nu} - T^{\mu\nu}) - (G - T)g^{\mu\alpha}g^{\nu\beta}\partial_\alpha\phi\partial_\beta\phi = 0 \tag{4.7}$$

, while variation with respect to the scalar field becomes:

$$\frac{1}{\sqrt{-g}}\partial_\kappa(\sqrt{-g}(G - T)g^{\kappa\lambda}\partial_\lambda\phi) = \nabla_\kappa((G - T)\partial^\kappa\phi) = 0 \tag{4.8}$$

where ∇_κ denotes covariant derivative with respect to the physical metric $g_{\mu\nu}$. It is interesting to see that the auxiliary metric $\tilde{g}_{\mu\nu}$ doesn't appear anymore in the equations of motion, rather it is manifested by the physical metric and the scalar field ϕ. As was mentioned before, the form of (4.1) implies that $g^{\mu\nu} = \dfrac{1}{P}\tilde{g}^{\mu\nu}$, and thus the scalar field satisfies the constraint equation:

$$g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = 1 \tag{4.9}$$

If we take the trace of (4.7) the resultant equation would be:

$$(G - T)(1 - g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi) = 0 \tag{4.10}$$

This equation is satisfied identically, independently of what value $G - T$ has. Moreover, $G - T$ is determined by (4.7) and (4.10), and whether we have matter or not, the solution for the gravitational field has nontrivial solutions for the conformal mode($\partial\phi$). The field ϕ satisfies Hamilton-Jacobi like equation, for a relativistic particle of unit mass. After solving the equation for ϕ, one can determine $G-T$ from (4.8). The gravitational field, represented by the metric, has initially 10 degrees of freedom due to the symmetry under the interchange of the two indicies. Moreover, if we consider the symmetry of the metric under both Lorentz and gauge transformations, the independent components of the metric reduce to two, which are the degrees of freedom of the graviton. However, the field acquires now an extra degree of freedom manifested by the scalar field ϕ in the conformal factor of the physical metric. To see how this system, which is constrained by conformal invariance, can manifest this extra degree of freedom, let's write equation (4.7) in the following way:

$$G^{\mu\nu} = T^{\mu\nu} + \tilde{T}^{\mu\nu} \tag{4.11}$$

where

$$\tilde{T}^{\mu\nu} = (G-T)g^{\mu\alpha}g^{\nu\beta}\partial_\alpha\phi\partial_\beta\phi \tag{4.12}$$

comparing this equation to the energy momentum tensor of a perfect fluid:

$$T^{\mu\nu} = (\rho + p)U^\mu U^\nu - pg^{\mu\nu}$$

with ρ being, as usual, the energy density, p is the pressure and U^μ is the 4-velocity of the fluid that satisfies the normalization condition $U^\mu U_\mu = 1$. If we set $p = 0$, we can identify:

$$\rho = G - T; \quad U^\mu = g^{\mu\alpha}\partial_\alpha\phi \tag{4.13}$$

which means that the above stress-energy tensor is the same as $\tilde{T}^{\mu\nu}$. Thus, the extra degree of freedom "mimics" the motion of dust particles, with energy density $G-T$

and 4-velocity $g^{\mu\alpha}\partial_\alpha\phi$. In the absence of matter,

$$\rho = G = g^{\mu\nu}G_{\mu\nu} = g^{\mu\nu}R_{\mu\nu} - \frac{g^{\mu\nu}g_{\mu\nu}}{2}R = R - 2R = -R$$

Moreover, the amazing thing is that the normalization condition of the 4-velocity is built in the constraint equation (4.9). In addition, to check for conservation of $\tilde{T}^{\mu\nu}$, we note first that $\nabla_\nu(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi) = 0 \Rightarrow \partial^\mu\phi\nabla_\nu\partial_\mu\phi = 0$. Therefore,

$$\nabla_\mu\tilde{T}^{\mu\nu} = \partial^\nu\phi\nabla_\mu((G-T)\partial^\mu\phi) + (G-T)\partial^\mu\nabla_\mu\partial^\nu\phi = \partial^\nu\phi\nabla_\mu((G-T)\partial^\mu\phi) \equiv 0 \quad (4.14)$$

where the last equality follows from (4.8).

Now, let's consider what effect will this model have if we use a metric in the synchronous coordinate system:

$$ds^2 = d\tau^2 - \gamma_{ij}dx^i dx^j. \quad (4.15)$$

One can see that when the scalar field is identified with proper time

$$\phi(x^\mu) \equiv \tau \quad (4.16)$$

, then the constrained equation (4.9) is satisfied. This implies that (4.8) becomes

$$\partial_0(\sqrt{det\gamma}(G - T)) = 0 \quad (4.17)$$

Now that the model is described, we will look at extensions of the model, beginning with a different way of interpreting the conformal isolation in the physical metric, and then to look for dynamical solutions.

4.2 Extensions of The Model

Another way of reproducing the above results was presented by A. Golovnev in [23]. This is done by first introducing a set of Lagrange multipliers $\lambda^{\mu\nu}$, and write the action as (neglecting normal matter contribution):

$$S = -\int d^4x (R + \lambda^{\mu\nu}(g_{\mu\nu} - \tilde{g}_{\mu\nu}\tilde{g}^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi))\sqrt{-g} \tag{4.18}$$

which implies:

$$\delta S = -\int d^4x \delta\sqrt{-g}(R + \lambda^{\mu\nu}(\tilde{g}_{\mu\nu} - \tilde{g}_{\mu\nu}\tilde{g}^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi)$$

$$-\int d^4x\sqrt{-g}[\delta R + \delta\lambda^{\mu\nu}(g_{\mu\nu} - \tilde{g}_{\mu\nu}\tilde{g}^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi) + \lambda^{\mu\nu}(\delta g_{\mu\nu} - \delta\tilde{g}_{\mu\nu}\tilde{g}^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - \tilde{g}_{\mu\nu}\delta\tilde{g}^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi) \tag{4.19}$$

$$-2\tilde{g}_{\mu\nu}\tilde{g}^{\alpha\beta}\partial_\alpha\delta\phi\partial_\beta\phi]$$

From (4.19) we see that by varying the action with respect to λ, we get (4.1). Moreover, by variation with respect to the scalar field (we use the chain rule to write the term with $\delta\phi$ in the form $\nabla_\mu(\lambda\partial^\mu\phi)$ and the fact that total derivatives vanish at the boundary of integration) we get:

$$\nabla_\mu(\lambda\partial^\mu\phi) = 0 \tag{4.20}$$

where $\lambda = g_{\mu\nu}\lambda^{\mu\nu}$. With the help of (4.1) inserted in (4.7), we get

$$G_{\mu\nu} = -\lambda_{\mu\nu},$$

then (4.20) becomes (4.8). Equation (4.18) can be written in the following way:

$$S = -\int d^4x (R + \lambda(1 - g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi))\sqrt{-g} \tag{4.21}$$

Later, Chamseddine, Mukhanov and Vikman used this way of expressing the model, and they added another Lagrange multiplier to get:

$$S = \int d^4x\sqrt{-g}[\frac{-1}{2}R + L_m + \lambda(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi) + \tilde{\lambda}(\nabla_\mu V^\mu - 1)] \tag{4.22}$$

in this case, the first constraint results in having *mimetic dark matter*, while the second constraint results in *mimetic cosmological constant.* Now, varying this action with respect to the metric, we get:

$$G_{\mu\nu} - T_{\mu\nu} + 2\lambda\partial_\mu\phi\partial_\nu\phi + g_{\mu\nu}\tilde{\lambda} = 0 \tag{4.23}$$

while varying it with respect to the vector field V^μ, we obtain:

$$\partial_\mu\tilde{\lambda} = 0 \tag{4.24}$$

The last equation shows that $\tilde{\lambda} = \Lambda$, which means that the cosmological constant appears just like a constant of integration. Substituting this into (4.23) and taking its trace, we obtain the following relation for λ:

$$\lambda = \frac{-1}{2}(G - T + 4\Lambda) \tag{4.25}$$

so finally we obtain the equation of motion to be:

$$(G_{\mu\nu} - T_{\mu\nu}) - (G - T)\partial_\mu\phi\partial_\nu\phi + (g_{\mu\nu} - 4\partial_\mu\phi\partial_\nu\phi)\Lambda = 0 \tag{4.26}$$

Thus, it is shown that both Dark Matter and "Dark Energy" can arise from the modification of GR. To make the model more interesting, one has to see what type of dynamics can be generated by this scalar field, and this is done by introducing a potential for this field. This is a purely legitimate step in variational calculus.

4.2.1 Scalar potential for the Scalar field

The sacred action now is;

$$S = \int d^4x\sqrt{-g}[-\frac{1}{2}R + \lambda(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 1) - V(\phi) + L_m] \tag{4.27}$$

Variation of this action with respect to λ gives the constraint equation, $g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = 1$, while variation with respect to the metric $g^{\mu\nu}$ gives:

$$G_{\mu\nu} - T_{\mu\nu} - 2\lambda\partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}V = 0 \tag{4.28}$$

with the usual definition of each of the terms in this equation. Taking the trace of (4.28), we get the following relation for λ:

$$\lambda = \frac{1}{2}(G - T - 4V) \tag{4.29}$$

which means,

$$G_{\mu\nu} = (G - T - 4V)\partial_\mu\phi\partial_\nu\phi + g_{\mu\nu}V(\phi) + T_{\mu\nu} \tag{4.30}$$

Equations (4.30) and (4.1) are the equivalents of Einstein's equations of motion, with an extra longitudinal degree of freedom coming from the gradient of the scalar field ϕ. It is important to mention, however, that this degree of freedom cannot be attributed entirely to the scalar field ϕ alone, since this field satisfies first order Hamilton-Jacobi like type of equations, which means that it is not dynamical. That's why the potential of the scalar field, $V(\phi)$ was introduced, to make this extra degree of freedom dynamical.

Now, taking the covariant derivative ∇^ν of (4.30), with the help of the Bianchi identity($\nabla^\nu G_{\mu\nu} = 0$) and conservation of stress-energy tensor, $\nabla^\nu T_{\mu\nu}$, we get:

$$\nabla^\nu[(G - T - 4V)\partial_\nu\phi\partial_\mu\phi + g_{\mu\nu}V(\phi)] = 0 \tag{4.31}$$

Knowing that, from the constraint equation:

$$\nabla^{\rho}(g^{\mu\nu}\partial_{\mu}\phi\partial_{\nu}\phi) = 0$$

$$\Rightarrow \nabla^{\rho}(g^{\mu\nu}\partial_{\mu}\phi\partial_{\nu}\phi) = 2g^{\mu\nu}(\nabla_{\mu}\partial^{\rho}\phi)\partial_{\nu}\phi = 0 \tag{4.32}$$

, this means that $\nabla_{\mu}\partial^{\rho}\phi = 0$, and therefore (4.31) simplifies to

$$\nabla^{\nu}((G - T - 4V)\partial_{\nu}\phi) = -V'(\phi) \tag{4.33}$$

Equation (4.30) can be compared to the stress-energy tensor of a perfect fluid, $T_{\mu\nu} = (\epsilon + p)U_{\mu}U_{\nu} - pg_{\mu\nu}$, and if we do the following match for the pressure:

$$\tilde{p} = -V \tag{4.34}$$

and for the energy density,

$$\tilde{\epsilon} = G - T - 3V \tag{4.35}$$

with $\partial_{\mu}\phi$ being the 4-velocity, and thus the constraint equation is nothing but the normalization condition for the 4-velocity of a perfect fluid. Since cosmology revolves around the use of the FRW metric, let's see what happens when we use it in this model(a flat FRW metric). So, for

$$ds^2 = dt^2 - a^2(t)\delta_{ij}dx^i dx^j \tag{4.36}$$

and let's assume that there's no ordinary matter, i.e $T^{\mu\nu} = 0$. Solving the constraint equation, we get a general solution:

$$\phi = \pm t + A \tag{4.37}$$

where A is an integration constant. Without loss of generality, A can be set to zero, and so the scalar field can be identified with time:

$$\phi = t \tag{4.38}$$

Now, G depends only on time (from the form of the metric), and V also depends only on time(from ϕ), therefore both the energy density (4.35) and the pressure (4.34) depend only on time. From equation (4.33), we get:

$$\nabla^0((\epsilon + p)\partial_0\phi) = \frac{1}{g}(\frac{d}{dt}(g(\tilde{\epsilon} - V)) = -V' \Rightarrow \frac{1}{a^3}\frac{d}{dt}(a^3(\tilde{\epsilon} - V)) = -\dot{V} \tag{4.39}$$

where g is the determinant of the metric. Integrating this equation, we get:

$$\tilde{\epsilon} = V - \frac{1}{a^3}\int a^3\dot{V}dt = \frac{3}{a^3}\int a^2 V da \tag{4.40}$$

while

$$\tilde{p} = -V \tag{4.41}$$

. Expanding the derivative of (4.39), we get:

$$\dot{\tilde{\epsilon}} = -3H(\tilde{\epsilon} + \tilde{p}) \tag{4.42}$$

where $H \equiv \dfrac{\dot{a}}{a}$ is the Hubble parameter. A constant of integration in (4.40) determines the amount of Dark Matter, which goes like a^{-3} (setting $\tilde{p} = 0$ in (4.42) and integrating). However, if V is different from 0, there's another contribution to mimetic matter that is entirely coming from V. In this case, an extra mimetic matter can represent a cosmological constant, without increasing the number of degrees of freedom compared to mimetic dust, since still both $\tilde{p}$ and $\tilde{\epsilon}$ are determined by the same potential V. From the 0-0 component of the modified Einstein equation (4.30), the Friedmann equation takes the form:

$$H^2 = \frac{1}{3}\tilde{\epsilon} = \frac{1}{a^3}\int a^2 V da \tag{4.43}$$

multiplying this by a^3 and differentiating with respect to time, we obtain:

$$2\dot{H} + 3H^2 = V(t) \tag{4.44}$$

to get rid of the non-linear terms, let $y = a^{3/2}$ then

$$H = \frac{2}{3}\frac{\dot{y}}{y}, \quad \dot{H} = \frac{2}{3}(\frac{\ddot{y}}{y} - (\frac{\dot{y}}{y})^2) \tag{4.45}$$

and hence, we finally get:

$$\ddot{y} - \frac{3}{4}V(t)y = 0 \tag{4.46}$$

This is the fundamental equation that will determine the fate of the universe, given a certain potential $V(t)$. Note that the fact that the scalar field is nothing but time has simplified the calculations drastically. Later on in the paper, the authors considered different types of potentials and studied the possible universes that would come out of such a potential. We will not go into describing these different models. Instead, we will jump directly into calculating quantum perturbations from such a metric, which will be used later on. One last note to be mentioned: in [15], the treatment of inflation or quintessence was not carried out in details, and it was not very clear how the behavior that resulted from eq. (4.46) represents the mentioned phenomena (at least not for quintessence). Therefore it is part of this thesis to elaborate more on these phenomena, in addition to merging them together.

4.2.2 Cosmological Perturbations in MDM

It has been shown in [15] that if one proceeds to calculate cosmological perturbations with the above described equations, one would get perturbations that are universal, i.e they apply to all perturbations irrespective of their wavelength. From there, these perturbations would be those of dust with a vanishing speed of sound, even if there was a V that is used. Therefore, quantum perturbations cannot be defined for MDM, and hence inflation would fail to explain how large scale structures came out of quantum perturbations. To fix this, one has either to add yet another scalar field, which will be like a curvaton field, or to modify the theory all together. The curvaton model was introduced by A.Linde and Mukhanov to explain non-Gaussian isothermal perturbation in the CMB's temperature spectrum [24]. However, this model turned out to be able to explain everything, yet predict nothing. As Mukhanov said during a lecture that he did at the Hebrew University of Jerusalem :"it was one of my biggest mistakes". Therefore, the best solution to get quantum perturbations from this model, is to fix the model. This is done by adding the following term to the action (4.27):

$$\frac{1}{2}\gamma(g^{\mu\nu}\nabla_\mu\nabla_\nu\phi)^2 \tag{4.47}$$

The only difference in the new equation of motion, in comparison to (4.46) is an additional factor multiplying the potential:

$$\ddot{y} - \frac{3}{4}\frac{2}{2-3\gamma}V = 0 \tag{4.48}$$

Now, consider the metric of a perturbed universe in the Newtonian gauge [16]:

$$ds^2 = (1+2\Phi)dt^2 - (1-2\Phi)a^2\delta_{ij}dx^i dx^j \tag{4.49}$$

with Φ being the Newtonian gravitational potential under the perturbation of the scalar field:

$$\phi \to t + \delta\phi. \tag{4.50}$$

The constraint equation implies:

$$\Phi = \delta\dot{\phi} \tag{4.51}$$

by looking at the linear perturbation of the 0-i component of the energy momentum tensor, the Einstein equation reduces to (after considering plane wave perturbations, $\delta\phi \propto \exp(ikx)$):

$$\delta\phi_k'' + (c_s^2 k^2 + \frac{a''}{a} - 2(\frac{a'}{a})^2)\delta\phi_k = 0 \tag{4.52}$$

where $c_s = \dfrac{\gamma}{2-3\gamma}$ is the speed of sound in mimetic matter, and prime denotes derivative with respect to conformal time. For short wavelength perturbations($\lambda_{ph} = a/k << c_s H^{-1}$), the perturbations in ϕ are:

$$\delta\phi_k \propto \exp \pm i c_s k \eta \tag{4.53}$$

while long wavelength perturbations yields:

$$\delta\phi = A\frac{1}{a}\int a^2 d\eta \tag{4.54}$$

To fix the amplitude of the quantum fluctuations, one has to identify the canonical quantization variable. This is done by expanding the action up to second order in perturbations, and to cut the story short, the canonically normalized variable is(for short wavelength perturbations):

$$v_k \sim \frac{\sqrt{\gamma}}{c_s} k \delta\phi_k \tag{4.55}$$

whose fluctuation in vacuum is:

$$\delta v_k \sim \frac{1}{\sqrt{w_k}} \sim \frac{1}{\sqrt{c_s k}} \tag{4.56}$$

and so finally,

$$\delta\phi_k \sim \sqrt{\frac{c_s}{\gamma}} k^{-3/2} \tag{4.57}$$

In this chapter, we have reviewed the model of "Mimetic Dark Matter" in it's different forms, and the equation of motion that governs the dynamics of the universe, mainly (4.46) or (4.48). And we ended up with looking at the quantum fluctuations that might result from this model. Now we will go on to examen how can a quintessential inflation model be built from MDM, and what sort of perturbations does it produce. This will be the topic of the next chapter

Chapter 5

Quintessential Inflation in Mimetic Dark Matter

In this chapter, we will consider a quintessential inflation model for MDM. We start by showing from where the inspiration of the model came. We do this using the normal cosmology described in Chapters 2 and 3, mainly the dynamics of a slow rolling field. We will then go onto considering the appropriate potential in MDM that would produce almost the same effect. We will finish by comparing with a model found in the literature[11], which has great similarity with our model

5.1 Inspiration from Slow Rolling Cosmology

Consider a slow-rolling scalar field, with the following potential:

$$V = e^{-\alpha\phi} \tag{5.1}$$

with $\phi = \ln(t)$ [7], thus we are using a power law potential. By solving eq. (3.19) for the scale factor, we get:

$$a = a_0 exp(\frac{\alpha}{3(2-\alpha)} t^{2-\alpha}). \tag{5.2}$$

The energy density of such a field would be:

$$\epsilon = \frac{1}{3M_{pl}^2} (\frac{\alpha}{t^{\alpha-1}})^2 \tag{5.3}$$

This model shows an exponential expansion of the universe, but with an energy density that goes like t^{-2} This density is that of radiation and matter (see fig.2.2). On the other hand, if we take the following potential:

$$V = \beta e^{-\phi} \tag{5.4}$$

the scale factor then is:

$$a = a_0 exp(\frac{1}{3}\beta t) \tag{5.5}$$

with the same definition for $\phi = \ln(t)$. Moreover, the energy density is now:

$$\epsilon = \frac{1}{3M_{pl}^2} \beta^2 \tag{5.6}$$

a constant energy density! From here, we see that to produce an energy density that first represents matter-radiation (i.e goes like t^{-2}) and reaches an asymptote, the potential must be some kind of combination between the two. Combining the two potentials together, while substituting the form of ϕ, we get a potential of the form:

$$V = At^{-\alpha} + Bt \tag{5.7}$$

So, let's try to see what physics will be produced from MDM if we use a potential of this form.

5.2 The Potential

Let's use the following potential in MDM:

$$V = \frac{2\alpha}{3}(1-\alpha)(t-t_0)^{-\alpha} + \frac{1}{3}[\alpha(t-t_0)^{-\alpha} + \beta]^2 \tag{5.8}$$

of course the exact form of the potential did not come down from heaven. After many (frustrating)trials and errors in the calculations, such a form seemed more reasonable.Moreover, the choice of the coefficients is made in such a way that no clustering of constants occur. t_0 is going to be the period at which inflation ends. plugging this potential in (4.46), we get:

$$\ddot{y} - [\frac{\alpha(1-\alpha)}{2}(t-t_0)^{-\alpha} - \frac{1}{4}[\alpha(t-t_0)^{1-\alpha} + \beta]^2]y = 0 \tag{5.9}$$

the solution of this equation will give us the scale factor to be[25]:

$$a = a_0 exp[\frac{\alpha}{3(2-\alpha)}(t-t_0)^{2-\alpha} + \frac{1}{3}\beta(t-t_0)] \tag{5.10}$$

and from (4.43), we get an energy density for the mimetic matter:

$$\tilde{\epsilon} = \frac{1}{3M_{pl}^2}[\frac{\alpha}{(t-t_0)^{\alpha-1}} + \beta]^2 \tag{5.11}$$

One can see that if α is very small, $\tilde{\epsilon} \propto t^{-2}$ at the beginning, that is near the end of inflation, and then as $t \to \infty$, $\tilde{\epsilon} \to \frac{1}{3M_{pl}^2}\beta^2$. So far, what we have is exactly the behavior we expect. What remains is fixing the parameters α andβ to produce the desired measurable quantities. However, there's still something wrong with this potential. First, the energy density and the scale factor might diverge, unless we have a good choice of the parameter α at the boundaries. Second, if $t < t_0$, and we have a fractional power in the energy density and the scale factor, we will get imaginary numbers. This is something definitely we don't want in real measurable quantities.

Therefore the solution will be as follows: we will separate the potential into two parts, one before inflation ($t \leq t_0$) and the other after inflation ($t \geq t_0$). We will then match these two values at $t = t_0$. This way, we will have the term $t_0 - t$ during inflation ($t < t_0$) and the term ($t - t_0$) after inflation. By doing this, we have solved the issue of having imaginary numbers. Now, concerning the divergence issue, we look at the form of the scale factor and the energy density in (5.10) and (5.11). To avoid divergences, during inflation, at $t = t_0$, $2 - \alpha$ must be positive, so must be $1 - \alpha$, therefore the solution to avoid divergence at $t = t_0$ as we approach it from the left, is to have:

$$\alpha < 1$$

Now, for $t > t_0$, keeping the same form of the potential, our concern is at ∞ now, since there we don't the energy density to diverge, rather we want it to be a very small number. Moreover, the scale factor should not diverge at $t = t_0$. Therefore, $2 - \alpha' > 0$ and $\alpha' - 1 > 0$. So for the post-inflation phase:

$$1 < \alpha' < 2$$

So the final result for the potential that would produce a quintessential inflation model in MDM is:

$$V = \begin{cases} \frac{2\epsilon}{3}(1-\epsilon)(t-t_0)^{-\epsilon} + \frac{1}{3}[\epsilon(t-t_0)^{-\epsilon} - \beta']^2, & t \geq t_0 \\ \\ \frac{2(2-\epsilon)}{3}(\epsilon-1)(t_0-t)^{\epsilon-2} + \frac{1}{3}[(2-\epsilon)(t_0-t)^{\epsilon-2} + \beta]^2, & t \leq t_0 \end{cases} \tag{5.12}$$

and so the corresponding scale factor is :

$$a = \begin{cases} a_0 exp[\frac{2-\epsilon}{3\epsilon}(t-t_0)^{\epsilon} - \frac{1}{2}\beta'(t-t_0)], & t \geq t_0 \\ \\ a_0 exp[\frac{\epsilon}{3(2-\epsilon)}(t_0-t)^{2-\epsilon} + \frac{1}{2}\beta(t_0-t)], & t \leq t_0 \end{cases} \tag{5.13}$$

while the energy density becomes (we will use ρ as the energy density instead of ϵ, to avoid confusion with the one used in the equations here):

$$\rho = \begin{cases} \frac{1}{3M_{pl}^2}[\frac{2-\epsilon}{(t-t_0)^{1-\epsilon}} - \beta']^2 & t \geq t_0 \\ \\ \frac{1}{3M_{pl}^2}[\frac{\epsilon}{(t_0-t)^{\epsilon-1}} + \beta]^2 & t \leq t_0 \end{cases} \tag{5.14}$$

where ϵ is an infinitesimal number and M_{pl} is the Planck Mass defined in Chapter 3. Now, to determine β, we have to use the number of e-folds of inflation. If inflation is to last for 60 e-folds, then:

$$N = \int_{t_i}^{t_0} H dt \equiv 60 \tag{5.15}$$

where t_i is the time at which inflation is supposed to have started. According to the model first presented by Guth, inflation should start at $t_i = 10^{-36}$ and end at

$t_0 = 10^{-32}$ [5]. Plugging in these numbers into (5.15), we get:

$$\beta \approx 6 \times 10^{32} \tag{5.16}$$

On the other hand, β' is determined by matching the value of the energy density at infinity to that of the cosmological constant [26]. This will result in

$$\beta' \approx \sqrt(3) \times 10^{-23} \tag{5.17}$$

Before continuing into checking the validity of the model, there's one last issue that needs to be tackled. It is apparent from the form of the energy density in (5.14) that it diverges at $t = t_0$. This might mean that there's a discontinuity in the energy density at the end of inflation. We can " approximately" solve this issue by looking at how much time it takes ρ to go from ∞ to the value of the field at $t = t_0$ if we are approaching it from the left(i.e using the expression of the energy density for $t \leq t_0$). If we plug in the value of β in ρ for $t \leq t_0$, we get the energy density at $t = t_0$ to be of the order of 10^{100}. Setting this value to be that of the field for $t \geq t_0$, and calculating the time interval, it turns out that it takes the energy density approximately $10^{-65}s$ to go from ∞ to 10^{100}. Since it is a very short period of time, then there's no real discontinuity in the energy density (approximately). These equations will result in the plots below for the scale factor and the energy density(we have used an $\epsilon = 0.01$) From the first plot, it is clear that the scale factor is increasing with $\ddot{a} > 0$, which implies it is an accelerated expansion of the universe. The expansion during inflation is much steeper than that after it, which is exactly what's needed. Furthermore, concerning the energy density plot, the graph shows a constant energy density, which is a characteristic of inflation as was pointed out in chapter 3. In addition, the energy density reaches an asymptote as $t \to \infty$, which is nothing but Quintessence. Moreover, to check whether the inflation parameters

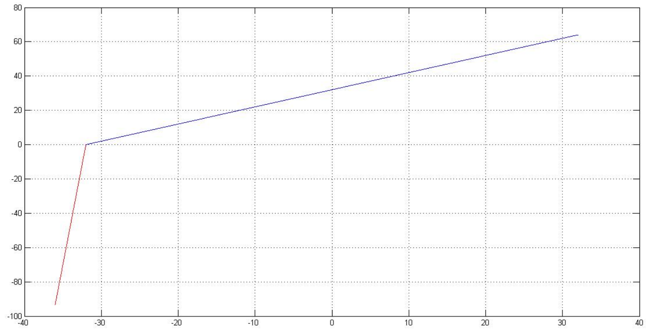

Figure 5.1: Plot of the logarithm of the scale factor as a function of the logarithm of time, for the two regimes: during inflation (red) and after inflation(blue). It is clear that there's a huge expansion in the universe during inflation, and a moderate one afterwards. The value of ϵ that's been used is 0.01

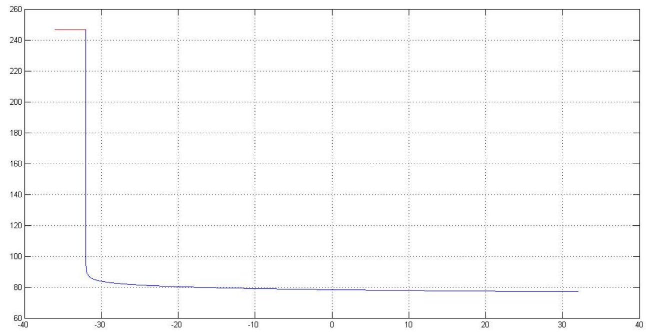

Figure 5.2: Plot of the logarithm of the energy density as a function of the logarithm of time for the two regimes

are satisfied, we use equations (3.11) and (3.13). With the above expressions of the scale factor, we get:

$$\epsilon \propto 10^{-35} << 1; \qquad \eta \sim 0 \tag{5.18}$$

which are consistent with the conditions for inflation.

Now we will discuss what type of perturbations does this model lead to. As was pointed out in Chapter 4, in MDM, the short wavelength perturbations are independent of the choice of the potential, the dependence appears only through the factor γ in the speed of sound in the mimetic matter (see equations (4.52), (4.54) and (4.56)). On the other hand, for long wavelength perturbations, equation (4.53), we do have a dependence on the choice of the potential, for it depends on the scale factor. The integral in the fluctuation is taken over the period of inflation, that is from $t_i = 10^{-36}$ to $t_0 = 10^{-32}$. At the end of inflation, the two terms in exp of eq. (5.13) die away. Therefore we can say that the integral is dominated by the lower limit, and since we have an exponential expansion, we can approximate the form of the scale factor to be $a \sim \exp(\beta(t - t_0))$. This will make the integral much easier to calculate. From here, we get:

$$\delta\phi = A\frac{1}{a}\int a^2 d\eta = \frac{A}{\beta} \simeq \frac{1}{H} \tag{5.19}$$

which corresponds to perturbations in an inflationary stage [15]. To get the factor A, we have to match the the value of the short wavelength perturbations to that of the long wavelength. The result is:

$$A \sim \sqrt{\frac{c_s}{\gamma}}\frac{H}{k^{3/2}} \tag{5.20}$$

with H being evaluated at $\eta \sim \dfrac{1}{c_s k}$, in agreement with [15]. Now, we will end this chapter by comparing this model to the one presented by Peebles and Vilenkin in

1998.

5.3 Peebles-Velinkin Model

In 1998, P.J.E Peebles and A.Vilenkin presented a model of quintessential inflation, in which they used a potential that is similar to the one we used here. They have separated the potential into two time phases, one during inflation, and the other after it. The model was also used to show how current entropy and matter of the universe arise from gravitational coupling of the inflaton field, which is known as *gravitational particle production*. The ideas presented later on in the paper about radiation dominated decay of the inflaton are a good inspiration to develop the model in hand in future work. The potential used in the paper is:

$$V = \begin{cases} \lambda(\phi^4 + M^4); & \phi < 0 \\ \\ \dfrac{\lambda M^8}{\phi^4 + M^4}; & \phi \geq 0 \end{cases} \tag{5.21}$$

For $-\phi >> M$, chaotic inflation potential will result, and for $\phi >> M$, quintessence occurs. The values adopted for λ and M are: $\lambda = 1 \times 10^{-14}$ and $M = 8 \times 10^5$. The resultant energy density as a function of redshift is shown in figure 5.3. The difference between this model and the one we presented is that the calculations of the energy density and scale factor are carried out with the addition of matter contribution. This means that there will be a difference in the value of the Hubble constant at the end of inflation. In order to adjust this, we can couple the scalar field to matter fields, and with the appropriate choice of coupling constants, one can produce the same result. There are other models in the literature that uses tracker solutions of quintessence (these are scalar fields that decay rapidly to an equilibrium position, thereby avoiding the problem of fixing the initial conditions) which shows

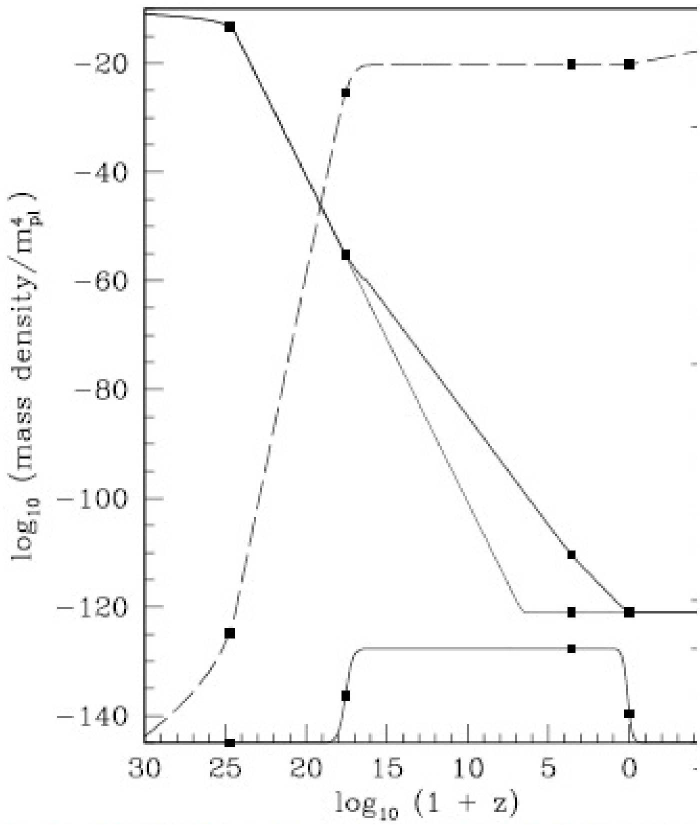

Figure 5.3: Plot of the logarithm of the energy density as a function of the logarithm of the redshift. This diagram shows similar results to those shown in Fig.5.2

similar behavior in the energy to the one shown here, as well as to that presented by Peebles and Vilenkin. [27]

Chapter 6

Conclusion and Future Work

In this thesis, a review of General Relativity and Cosmology was first introduced in Chapter 2. The different types of spaces were described quantitatively (namely flat, positively, and negatively curved spaces), with the result being given in (2.13) and (2.14). Furthermore, the FRW metric was introduced along with its impact on the motion of particles in the universe, whereby one has to take into account peculiar velocities when making measurements. Then, the dynamics and kinematics of a system, which are described by Einstein's equations of motion, have been explained briefly, and the Ricci tensor and scalar were presented for an FRW metric, which resulted in the Friedman equations. These equations tell us how the energy density of matter, radiation, or "Dark Energy" behave as a function of the scale factor. The summary is presented in Figure 2.2. In Chapter 3, the concept of "Inflation" and its dynamics have been laid out. Inflation is the mechanism by which the Hubble radius ($(aH)^{-1}$), decreases to the extent that the particle horizon of a system exceeds it. This results in having distant parts of the CMB causally connected, therefore explaining the Homogeneity of the CMB, and thus solving the horizon problem. Moreover, a detailed description of the dynamics of inflation from scalar field theory has been shown. Later, Quintessence was introduced, which is another scalar field that was initially proposed as an alternative to the cosmological constant and to

Dark Energy. This scalar field solved the flatness problem, the small anisotropies in the CMB, and the accelerated expansion of the universe after inflation. Finally, an outline on cosmological perturbations and the power spectrum of CMB were briefly presented. In Chapter 4, the mimetic dark matter model which was proposed by Chamseddine and Mukhanov in 2013 was detailed along with its extensions to cosmological applications. This model is based on splitting the physical metric into two parts as presented in (4.1). The first part is the conformal mode of the metric, represented by the derivative of the scalar field ϕ, and an auxiliary metric, which is like a cart that carries us through the calculations. The resultant equations of motion can explain the phenomenon of dark matter, without adding any normal matter from outside. Furthermore, the extension of this model, which is done by introducing a potential to the scalar field, was presented, and the resultant perturbations were described. In Chapter 5, the model was produced for the extension of MDM to a "Quintessential Inflation" model. The basic idea is to split the potential, which represents the dynamics of the scalar field ϕ, into two phases, as given by (5.12), one during inflation and the other afterward. This will result in a scale factor and energy densities that agree with both phases of the universe. Finally, the model was compared with the one presented by Peebles and Vilenkin in 1998, and was found compatible to it.

Worthy to note, this model still requires further development by examining the power spectrum with the corresponding multipole coefficients that must result in a peak at $l = 200$, which then agrees with experimental data. In addition, one can extend the model by coupling it to gauge fields at the end of inflation to see how will this affect the reheating process, and later on nucleosynthesis. Also, one can should try to understand, if such a model is accepted, what mechanism might lead to such a jump in the potential, hence in the scale factor and the energy density.

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