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SUPPLY CHAIN MANAGEMENT WITH A MIXTURE OF CONTINUOUS AND LUMPY DEMAND: THE CASE OF THE EPQ MODEL

by
RAWAN RAMI KORDAB

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RAWAN RAMI KORDAB

Approved by:

Dr. Moueen Salameh, Professor — Advisor
Industrial Engineering and Management

Dr. Ali Yassine, Professor — Member of Committee
Industrial Engineering and Management

Dr. Bacel Maddah, Associate Professor — Member of Committee
Industrial Engineering and Management

Date of thesis defense: July 14, 2015

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AN ABSTRACT OF THE THESIS OF

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When it comes to inventory management in supply chains, most of the literature considers models with continuous demand, where products are demanded one unit at a time. However some firms experience multiple demand streams. These demand streams might differ in types and rates.

Our paper considers a firm adopting the EPQ model under a lumpy demand stream of known magnitude occurring at regular time intervals, together with the conventional uniform demand stream. This can be the case of a manufacturer selling in discrete shipments to downstream retailers and selling directly to customers through an outlet store.

We derive expressions for the optimal production quantity and the minimum total cost of the system. The expression of the optimal production quantity is similar to that of the classical EPQ model with the weighted average of the discrete and continuous demand rates replacing the classical uniform demand rates.

Furthermore, we present a one-vendor-one buyer supply chain application to investigate the effect of mixed demand streams on the supply chain performance. We demonstrate that smoothing the lumpy demand with continuous demand allow a better match of supply and demand, and increases the benefits of supply chain integration.

CONTENTS

ILLUSTRATIONS

TABLES

Table Page

CHAPTER 1

INTRODUCTION

When it comes to inventory management in supply chains, most of the literature considers models with continuous demand, where products are demanded one unit at a time, in a uniform fashion sometimes, or at random at other times. A limited stream of literature also considers lumpy demand, where demand arrives in batches at discrete times, with a typical application to supply chains which are vertically integrated, where, for example, one supplier receives lumpy orders from a retailer.

To our knowledge, the case of a supply chain with a mixture of continuous and lumpy demand has not been studied in the literature, despite the practical relevance of such demand patterns. For example, a growing trend in the United States and other Western countries is for manufacturers to have outlet stores, where price-sensitive customers can drive few hours far from the city to buy discounted products (see for example, Coughlan et al. 2004, Coughlan et al. 2005 and Park et al. 2006). This is typical for clothing manufacturers. As such, these manufacturers face lumpy demand from retailers' orders, and continuous demand from the outlet store customers. Moreover, many manufacturers and suppliers, not having outlet stores, establish channels for selling directly to customers, typically close to their operations.

With these motivations, this paper develops what seems to be the first model on inventory management with a mixture of continuous and lumpy demand. Specifically, we consider the classical economic production quantity (EPQ) model under a lumpy demand stream of known magnitude occurring at regular time intervals, together with the conventional uniform demand stream. For this extended EPQ model, we first establish the structure of the optimal production policy. This involves a duration of the production cycle which is a whole-integer multiple of the time of recurrence of discrete demand, with the conventional order-at-zero-inventory approach. We then develop a closed-form expression for the optimal production quantity. Interestingly, this quantity has the familiar form of the classical EPQ, with the time-average of lumpy and uniform demands replacing the continuous rate.

To gain further insights, we extend our model vertically and consider the manufacturer's cost structure, and we develop the jointly optimal order quantities of the manufacturer and retailer in the integrated system. This leads to useful results on the effect of mixed demand streams on supply chains. The most notable insight we obtain, by comparison to systems with no continuous demand, is that considering a mixture of lumpy and uniform demand enhances the efficiency of the supply chain. That is, the relative cost savings from vertical integration are higher under the demand mixture. We believe that this is due to continuous demand "smoothing" the capacity of the manufacture, allowing a better match of supply and demand.

This paper is organized as follows. Section 2 briefly reviews the related literature. Section 3 presents our model, assumption and analytical results on the structure of the optimal ordering policy. Section 4, presents a supply chain integration application with a supplier facing lumpy demand from a retailer following the EOQ model, along with a stream of continuous demand. Section 5 presents numerical examples with analytical insights. Section 6, states the conclusion.

CHAPTER 2

LITERATURE REVIEW

Our work contributes to two streams of literature. One is on inventory models with multiple demand streams. The other is related to models with lumpy demand where demand arrives in batches at discrete times. In Sections 2.1 and 2.2 respectively, we briefly review the works related to these two areas of the literature.

2.1. Multiple Demand Streams

The case where a firm faces multiple demand streams is studied in the literature. A growing trend in the US and other Western countries is for manufacturers to have outlet stores. These manufacturers face a mixture of lumpy demand from retailers' orders, and continuous demand from the outlet store customers. Originally, the purpose of outlet stores was to liquidate excess inventory; however, modern outlet stores evolved into a considerably different format and nowadays some offer complete lines of merchandise (Ngwe, 2014 and Coughlan et al, 2005). Coughlan et al. (2004) explain that outlets expand market coverage through serving price-sensitive customers who are usually less service-sensitive and they have more time to travel to outlet stores. Hence, outlets are considered a tool for market segmentation. Moreover, studies show that selling through outlet stores increases the manufacturer's earnings, (e.g. Park et al, 2006 and Coughlan et al, 2004). It is proposed by Coughlan et al. (2005) that in metropolitan areas outlet stores will convert into traditional malls. As such Ngwe (2014) states that improving inventory management does not appear to be a regular purpose of outlet stores. However we show in this research that manufacturers can use the continuous demand from outlet stores to better utilize their capacity. That is, the outlet stores' continuous demand smoothes the lumpy demand from retailers.

Several papers discuss inventory systems where multiple demand streams exist. For example Maddah et al. (2009) consider a system where items are produced in two different qualities each having its own demand and cost structure.

In addition, several papers in the literature consider two demand streams arising from make-to-order (MTO) and make-to-stock (MTS) production modes. In particular, Cattani et al. (2010) consider a firm that produces custom MTO products, as demanded in each period, and then it fills in or "spackles" the production schedule with MTS items to restock inventory. Similar to Cattani et al. (2010), we use continuous demand to smooth (spackle) discrete demand and we demonstrate the benefits this approach in better matching supply and demand. Several other papers tackle the problem of MTO and MTS systems and consider hybrid systems to jointly manage these operations, e.g., Adan and Vander Wal (1998), Chang et al. (2003), Wu et al. (2008) and Zhang et al. (2013). These studies are relevant to our work where discrete demand is spackled with continuous demand in an integrated supply chain model. We prove that the mixture of lumpy and uniform demand enhances the efficiency of the supply chain.

2.2. Lumpy Demand

The classical inventory literature assumes demand to be continuous. However, in a real life vendor-buyer integrated supply chain environment, lumpy demand is common. Therefore, several papers consider the case of lumpy demand (e.g., Chui et al. 2011, Hsu, 2000 and Yuan et al., 2011).

Goyal (1977) was the first to study an integrated single vendor-single buyer problem. He considers a problem where the production rate is infinite, the shipment sizes are equal and the production batch size is a multiple of the shipment size. Similar to our approach, Goyal (1977) considers that the buyer follows the EOQ model and formulates the system's total cost expression. However, Goyal (1977) did not propose a technique to find the optimal number of shipments. Benerjee (1986) assumes a lot-for-lot production and shipment policy with finite production rate and he discusses price discounts schemes the vendor can offer the buyer as a compensation for adopting the joint economic lot size. Goyal et al. (1988) combine the previous studies and develops a model with a finite production rate at the vendor with a lot size that is a multiple of the buyer shipment size. Goyal et al. (1988) assume that no shipments are allowed before the production of the entire lot is complete. Lu (1995) relaxes this assumption and proposes that shipping can start as soon as enough inventory is accumulated to satisfy the first shipment. Our model in this paper adopts Lu's approach.

Goyal (1995) and Hill (1997) propose an amendment to Lu's work, cases where the shipment sizes are not equal, but rather increase gradually. Subsequently, Hill (1999) formally proved that a policy with increasing shipment sizes is optimal and proposed and approach to find this optimal policy. Finally Goyal (2000), proposed a simple heuristic that allows obtaining a near optimal policy which performs well compared to the optimal policy in Hill (1999). In this paper, we are concerned with external lumpy demand which typically arises in equal shipments. As such, we adopt a model similar to that of Lu (1995).

Other papers that consider the lumpy demand in a supply chain context are those by Golhar et al. (1992) and Sarker et al. (1996). An interesting result in both papers is that the expression of the optimal production quantity is independent of the demand type and is similar to that of the classical EPQ model with the average of the lumpy demand per unit time replacing the continuous demand rate. A similar result in obtained in this paper with the time average of demand consisting of both continuous and discrete streams.

To our knowledge, the case of a supply chain with a mixture of continuous and lumpy demand has not been studied in the literature. This paper develops what seems to be the first model on inventory management with a mixture of continuous and lumpy demand.

CHAPTER 3

THE MODEL

Consider a manufacturer that has to meet a scheduled retailer's discrete demand while satisfying the uniform demand of regular customers. Specifically the manufacturer utilizes a variant of the classical economic production quantity (EPQ) model with a production rate α per unit time, a uniform demand β per unit time *and* lumpy demand stream of d units every t units of time (e.g., t = 1/12 year, i.e. one month) as shown in Figure 1. We also assume an inventory holding cost at h \$/unit/year and a fixed production cost of K \$/order. All the notations used are shown in Table 1.

For this extended EPQ model, we first establish the structure of the optimal production policy. This involves a duration of the production cycle which is a multiple of the time of occurrence of discrete demand, with the conventional order-at-zero-inventory approach. We then develop a closed-form expression for the optimal production quantity Q^*, where $Q^* = n(d + \beta t)$ must be satisfied, and n is a positive integer.

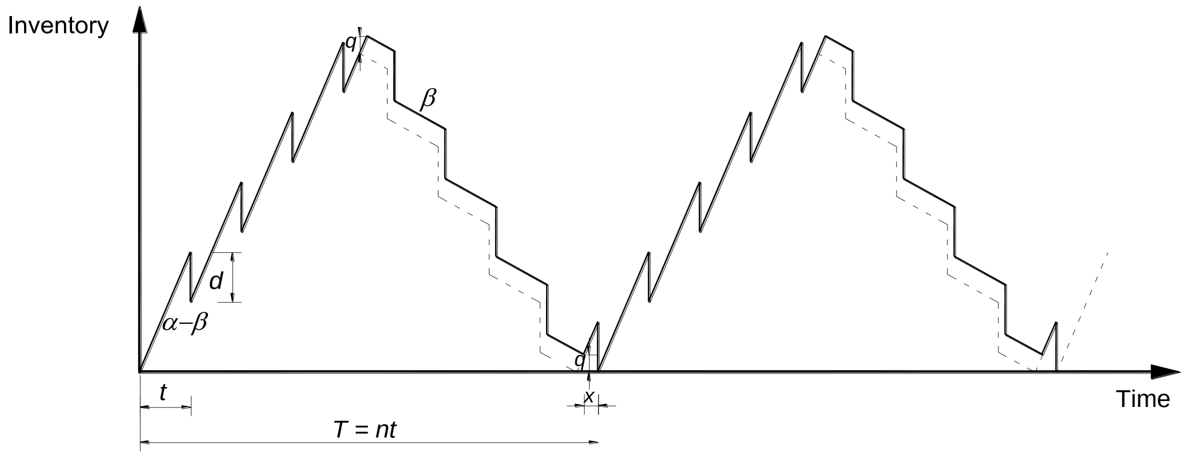

Figure 1. Inventory level over time

Q	Maximum production level
$Q*$	Optimal Production level
K	Production setup cost
α	Constant production rate per unit time
β	Constant continuous demand rate per unit time
d	Discrete demand per unit time
h	Holding cost per unit per unit time
TCU	Total cost per unit time
t	Time between two consecutive discrete demands
n	Number of discrete demands in one complete cycle

Table 1. Notations

Theorem 1 establishes the structure of the optimal production policy.

Theorem 1. *The optimal production policy is characterized by*

(i) *An inventory cycle of length* nt , $n = 1,2,\ldots,$ *with the inventory level reaching zero at the end of this cycle.*

(ii) *A production period starting at a time* $\dfrac{d}{\alpha-\beta}$ *before the end of an inventory cycle, where the inventory level is 0, and extending into the next ordering cycle.*

Proof See Appendix A.

With the structure of the optimal production policy given by the dashed-line inventory in Figure 1, we proceed to determine the annual cost. In the following we develop the annual cost as a function of the number of demand cycles in an inventory cycle, n and the order quantity Q. Note that Q and n are closely related as it can be easily seen that

$$Q = n(d + \beta t). \qquad (1)$$

However, we derive the cost as function of Q and n to simplify the notation. Note that (1) holds in the limit when $\beta = 0$. In this case, where the model has lumpy demand only, it has been shown that the optimal policy satisfies $Q = nd$ (Chen et al. 2004).

The major challenge in deriving the annual cost is the holding cost per inventory cycle. Following an approach similar to that of Joglekar (1988), we derive the holding cost per cycle by subtracting the lumpy and continuous demand level from the production level, as illustrated in Figure 2. In Figure 2, x is the production time towards the end of the cycle,

$$x = \frac{d}{(\alpha-\beta)}. \qquad (2)$$

Then, the area under the production level curve from Figure 2 is

$$AP(Q,n) = \frac{(Q-\alpha x)^2}{2\alpha} + (Q-\alpha x)(nt - \frac{(Q-\alpha x)}{\alpha} - x) + \frac{(Q-\alpha x + Q)}{2}x.$$

Utilizing (2) and simplifying gives

$$AP(Q,n) = \frac{[Q(\alpha-\beta)-\alpha d]^2}{2\alpha(\alpha-\beta)^2} + \frac{[Q(\alpha-\beta)-\alpha d]}{\alpha-\beta}\left(nt - \frac{Q}{\alpha}\right) + \frac{[2Q(\alpha-\beta)-\alpha d]}{2(\alpha-\beta)^2}d\,.$$

Utilizing (1) and simplifying further gives

$$AP(Q) = \frac{[Q(\alpha-\beta)-\alpha d]^2}{2\alpha(\alpha-\beta)^2} + \frac{[Q(\alpha-\beta)-\alpha d]}{\alpha-\beta}\left(\frac{Q}{d+\beta t}t - \frac{Q}{\alpha}\right) + \frac{[2Q(\alpha-\beta)-\alpha d]}{2(\alpha-\beta)^2}d\,.$$

Then,

$$AP(Q) = Q^2\left(\frac{t}{d+\beta t} - \frac{1}{2\alpha}\right) + \frac{Qd}{\alpha-\beta}\left(1 - \frac{t\alpha}{d+\beta t}\right). \qquad (3)$$

In addition, the area under the lumpy demand curve in Figure 2 is

$$ALD(n) = \sum_{i=1}^{n-1} idt = \frac{n(n-1)dt}{2}\,.$$

Utilizing (1) and simplifying gives

$$ALD(Q) = \sum_{i=1}^{n-1} idt = \frac{Q(Q-d-\beta t)dt}{2(d+\beta t)^2}\,. \qquad (4)$$

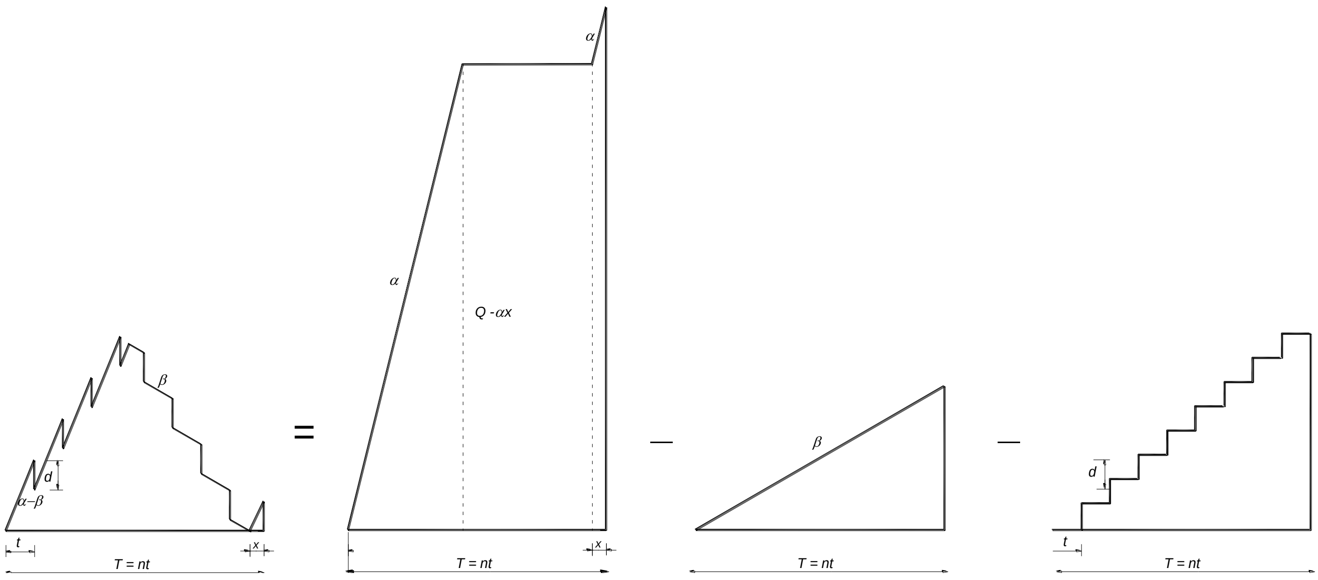

Figure 2. Holding cost per cycle

Finally, the area under the continuous demand curve in Figure 2 is $ACD(n) = \dfrac{\beta(nt)^2}{2}$, which utilizing (1) again is equivalent to

$$ACD(Q) = \frac{\beta t^2 Q^2}{2(d+\beta t)^2}. \qquad (5)$$

The total cost per unit time, *TCU(Q)*, is the total cost per cycle, $TC(Q)$, divided by the cycle duration, *T*.

Where $TC(Q) = K + h\left[AP(Q) + ALD(Q) + ACD(Q)\right]$ and $T = nt = \dfrac{Qt}{(d+\beta t)}$.

Therefore,

$$TCU(Q) = \frac{\{K + h[AP(Q) - ALD(Q) - ACD(Q)]\}(d+\beta t)}{Qt}.$$

Substituting *AP*(*Q*), *ALD*(*Q*) and *ACD*(*Q*) by their expressions in (3), (4) and (5) respectively, we obtain

$$TCU(Q) = \frac{K(d+\beta t)}{Qt} + \frac{h(d+\beta t)}{Qt}\left[Q^2\left(\frac{t}{d+\beta t} - \frac{1}{2\alpha}\right) + \frac{dQ}{(\alpha-\beta)}\left(1 - \frac{t\alpha}{d+\beta t}\right)\right]$$
$$- \frac{hQ\beta t}{2(d+\beta t)} - \frac{hd(Q-d-\beta t)}{2(d+\beta t)}.$$

Simplifying gives

$$TCU(Q) = \frac{K(d+\beta t)}{Qt} + \frac{hQ}{2}\left(1 - \frac{d+\beta t}{\alpha t}\right) - \frac{hd}{2} + \frac{hd^2}{t(\alpha-\beta)}.$$

where $\alpha > (d+\beta t)$.

The first-order optimality conditions imply that

$$\frac{dTCU(Q)}{dQ} = \frac{-K(d+\beta t)}{tQ^2} + \frac{h}{2}\left(1 - \frac{d+\beta t}{\alpha t}\right) = 0,$$

which implies that

$$Q = \sqrt{\frac{2K(d+\beta t)/t}{h\left(1 - \frac{(d+\beta t)/t}{\alpha}\right)}} = \sqrt{\frac{2K\beta'}{h\left(1 - \frac{\beta'}{\alpha}\right)}}\ , \qquad (6)$$

where $t \geq \frac{d}{\alpha-\beta}$, and $\beta' = \frac{(d+\beta t)}{t}$ is the demand rate per unit time.

This leads to

$$TCU(Q) = \sqrt{2hK\beta'(1 - \frac{\beta'}{\alpha})} - \frac{hd}{2} + \frac{hd^2}{t(\alpha-\beta)}. \qquad (7)$$

However we should check for convexity of the function $TCU(Q)$,

$$\frac{\partial^2 TCU(Q)}{\partial Q^2} = \frac{2K(d+\beta t)}{tQ^3} \geq 0$$, where Q is greater than zero.

Thus the function is convex, and it admits a minimum at Q .

From (1) we know that the corresponding number of shipments

$$n = \frac{Q}{(d+\beta t)}. \qquad (8)$$

Yet for optimality, *n* should be an integer. Therefore,

$$n^* = \arg\min_{\lfloor n \rfloor, \lceil n \rceil}(TCU(\lfloor n \rfloor(d+\beta t)), TCU(\lceil n \rceil(d+\beta t)) \qquad (9)$$

Finally, the optimal production quantity is

$$Q^* = n^*(d+\beta t). \qquad (10)$$

This model leads to the following results at special cases

For d = 0,

$$Q = \sqrt{\frac{2K\beta}{h(1-\frac{\beta}{\alpha})}}.$$

It is identical to the optimal production quantity of the classical (EPQ) where continuous demand is the only demand present in the system.

As $t \rightarrow \infty$,

$$Q \rightarrow \sqrt{\frac{2K\beta}{h(1-\frac{\beta}{\alpha})}}.$$

This converges to the optimal production quantity of the classical EPQ. This is not surprising since in such a case the manufacturer will not be concerned by satisfying the discrete demand which will be in the very far future. The manufacturer's main goal is meeting continuous demand in this case, hence, his model converges to the EPQ.

For $\beta = 0$,

$$Q = \sqrt{\frac{2Kd/t}{h(1-\frac{d}{\alpha t})}}.$$

This optimal production quantity expression is in the EPQ form. However with the discrete demand per unit time $\frac{d}{t}$ replacing β, which is the only demand to be satisfied in this case.

CHAPTER 4

SUPPLY CHAIN INTEGRATION APPLICATION

As presented in Figure 3, we consider a supply chain between a supplier and a buyer. The supplier follows the EPQ with lumpy demand system that we analysed in Section 3. The buyer orders an amount q periodically according to the classical EOQ model. We use a similar system to that in the paper by Lu (1995).

The buyer orders q every $\frac{q}{D}$ units of time. The buyer's cost follows the EOQ model while the suppliers cost follows our model in Section 3 with $d = q$ and $t = \frac{q}{D}$.

Then, the total cost of the joint system is

$$TCU_J(Q,q) = TCU_{supplier}(Q,q) + TCU_{buyer}(q)$$

where,

$$TCU_{supplier}(Q,q) = \frac{K_S(D+\beta)}{Q} + \frac{h_S Q}{2}\left(1 - \frac{D+\beta}{\alpha}\right) - \frac{h_S q}{2} + \frac{h_S q D}{(\alpha-\beta)}$$

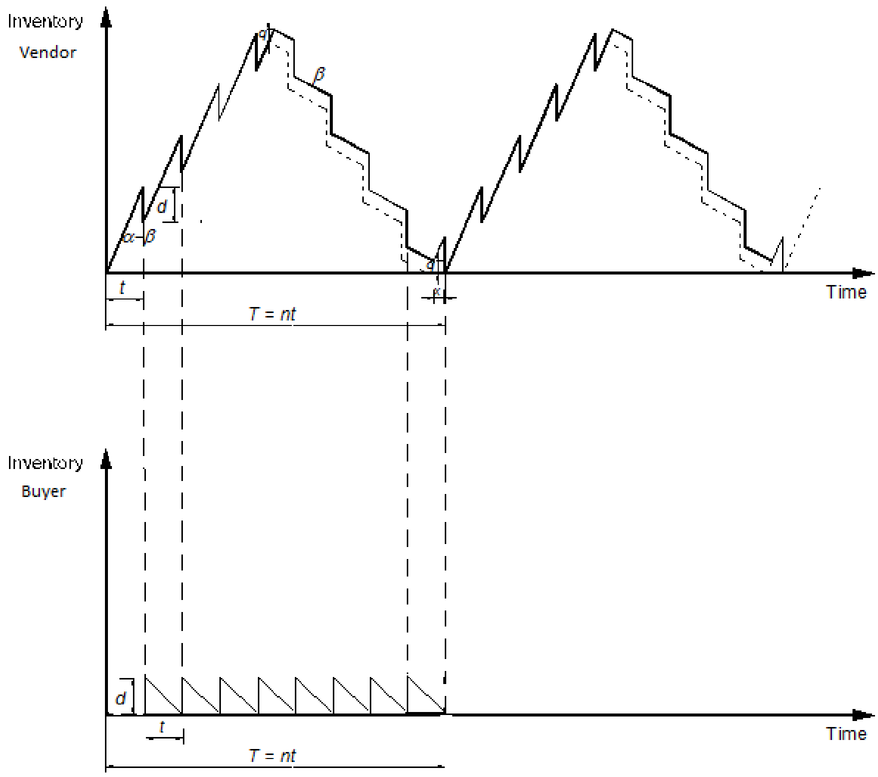

Figure 3. Joint supplier-retailer system

and,

$$TCU_{buyer}(q) = \frac{K_b D}{q} + \frac{h_b q}{2}$$

Then, since from Theorem 1, at optimality, $Q = n(q + \beta \frac{q}{D})$, the joint cost can be written as

$$TCU_J(q,n) = \frac{K_s D}{nq} + \frac{h_s nq(D+\beta)}{2D} - \frac{h_s nq(D+\beta)^2}{2D\alpha} + \frac{h_s Dq}{(\alpha-\beta)} - \frac{h_s q}{2} + \frac{K_b D}{q} + \frac{h_b q}{2}$$

In the following we analyze the convexity of $TCU_J(q,n)$.

Theorem 2. *The cost $TCU_J(q,n)$ is jointly convex in q and n, when n is treated as a continuous variable.* **Proof** See Appendix B.

Since n is an integer, we find it easier to find the optimal policy for this supply chain sequentially as given in theorem 3.

Theorem 3. *For a given number of shipments n, the optimal buyer's order quantity is*

$$q^*(n) = \sqrt{\frac{2D(nK_b + K_S)}{n\left[nh_S(1+\frac{\beta}{D})(1-\frac{D+\beta}{\alpha}) + h_b - h_S + \frac{2h_S D}{(\alpha-\beta)}\right]}} \tag{11}$$

Furthermore, the function $TCU_J(n) = \min_q(TCU_J(q,n)) = TCU_J(q^*(n), n)$ *is quasi convex in n, when n I treated as a continuous variable.*

Proof See Appendix C.

Following theorem 3, let $n = \min_n TCU_J(n)$, be the continuous value of n that minimize cost. Then, the optimal integer value of n is $n^* = \arg\min TCU_J(\lfloor n \rfloor, \lceil n \rceil)$, which follows from the quasiconvexity property in theorem 3. Once the optimal number of shipments, n^*, is found, the optimal buyer order quantity is found as $q^* = q^*(n^*)$, from (11). Then, the optimal supplier order quantity is found as $Q^* = n^*(q^* + \beta\frac{q^*}{D})$, from (1).

Finally, we discuss the case where the buyer dictates his optimal EOQ on the supply chain.

Then, the buyer's optimal ordering quantity is

$$q_b^* = \sqrt{\frac{2K_b D}{h_b}}.$$

This leads to the supplier's production quantity

$$Q = \sqrt{\frac{2K_s(D+\beta)}{h_s(1-\frac{D+\beta}{\alpha})}}, \tag{12}$$

and the number of shipments $n = \frac{Q}{q_b^* + \beta \frac{q_b^*}{D}}$.

However, the number of shipments should be an integer. Therefore, the optimal number of shipments is

$$n^* = \arg\min_{\lfloor n \rfloor, \lceil n \rceil}(TCU_J(\lfloor n \rfloor(q_b^* + \beta \frac{q_b^*}{D})), TCU_J(\lceil n \rceil(q_b^* + \beta \frac{q_b^*}{D}))$$

and the optimal production quantity

$$Q^* = n^*(q_b^* + \beta \frac{q_b^*}{D}).$$

Note finally that the order quantity in (12) is interesting as it follows the exact same structure as the EPQ formula with “total” demand rate $D+\beta$.

CHAPTER 5

NUMERICAL EXAMPLES AND SENSITIVITY ANALYSIS

In this section we present numerical examples. The example in Section 5.1 is related to the base model, then in Section 5.2 we present a joint supply chain application where the supplier and buyer cooperate to achieve minimum cost for joint system.

5.1. Base model

Consider a manufacturing process where the set up cost K is \$100 per cycle, and the holding cost, h is \$5 per unit per year. The production rate α is 150,000 per year and there are two types of demand; discrete demand of magnitude 1500 units ordered every time t = 1 week, and continuous demand β at a rate 60,000 per year.

The solution of the production quantity in (6) is Q = 8306.6 units. The corresponding number of shipments is given from (8), $n = \dfrac{Q}{(d + \beta t)}$ = 3.13. Then the optimal number of shipment is given from (9) as n^* = 3, and the corresponding optimal production quantity is found from (10) as Q^* = 7961.5 units, with a total annual cost equal to \$6075.6.

For further insights Figures 4-7 present some sensitivity analysis on the model. When we fix all variables and change the magnitude of the lumpy demand only, we can see from Figure 4 and Figure 6 the behavior of the optimal production quantity and the number of shipments per cycle respectively. They both increase as the magnitude of lumpy demand increases.

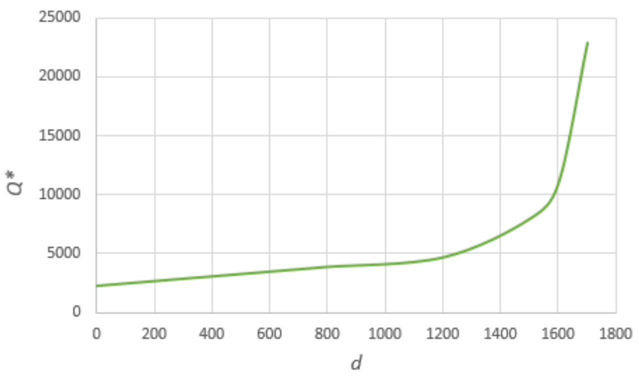

Figure 4. The optimal production quantity as a function of *d*

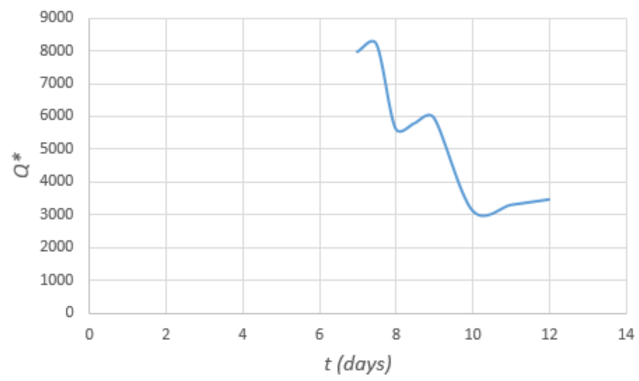

Figure 5. The optimal production quantity as a function of *t*

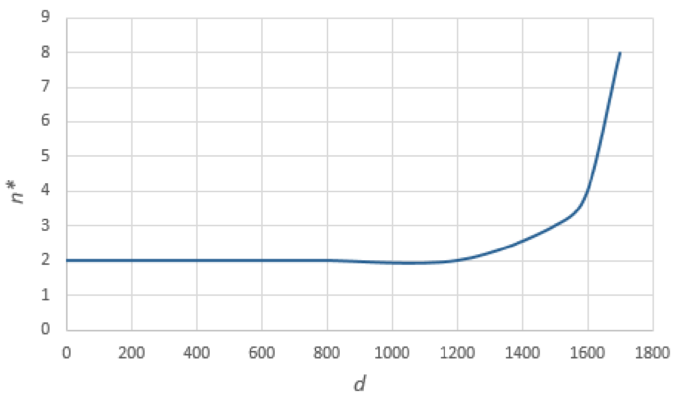

Figure 6. The optimal number of shipments as a function *d*

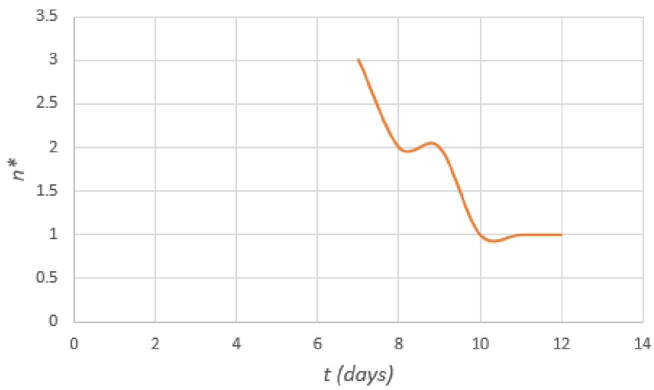

Figure 7. The optimal number of shipments as a function of *t*

Figure 5 and Figure 7 demonstrate the behavior of the optimal production quantity and the number of shipments per cycle respectively, as t increases. Both, n^* and Q^* decreases with the increase in time between shipments. We note that the fluctuation in Figure 5 is due to the rounding of n .

Moreover, in order to check the effect of having mixed demand streams on the system's profitability, we analyze the gross profit margin, $PM = \dfrac{(\frac{d}{t}+\beta)(p-c)-TCU(Q^*)}{(\frac{d}{t}+\beta)p}$, where p and c are the retail price and the unit cost of the product. For the base case, we use p= \$32 and c= \$25.

In the base case, PM=0.217. To gain insight into the effect of continuous demand on profitability, we analyze in Figure 4 the effect of changing β on profitability.

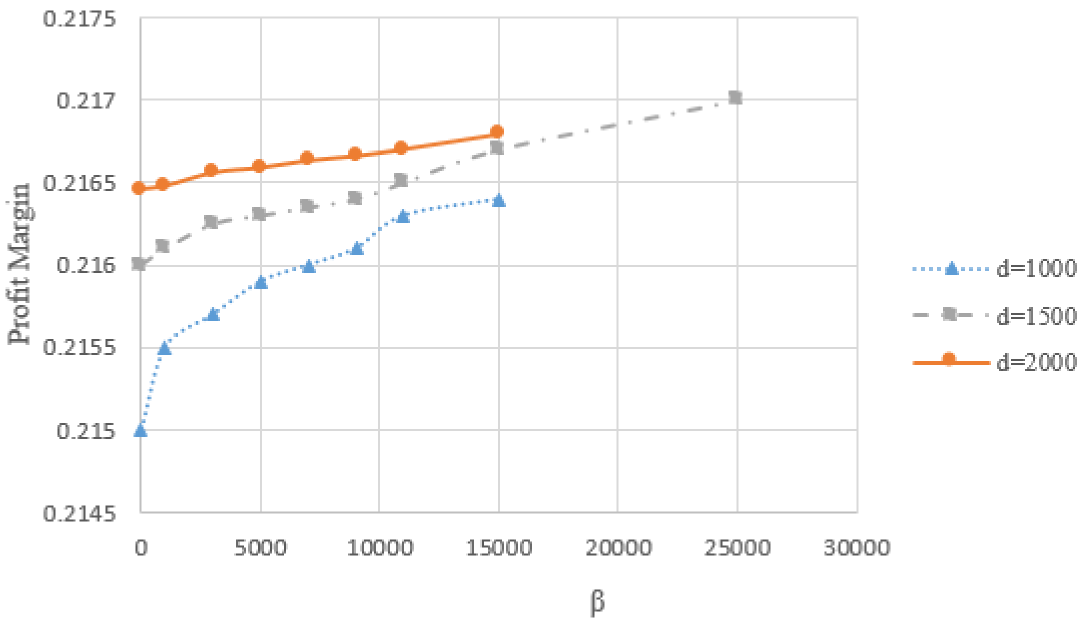

Figure 8. Effect of mixture demand streams on Profit Margin

Figure 8 illustrates the positive effect continuous demand has on the profit margin. As β increases, the profit margin increases. That is, having continuous demand is a way to smooth the demand and better match supply and demand. This strategy can be referred to as "Spackling, a term used by Cattani et al. (2002).

5.2. Supply Chain Model

Consider one vendor-one buyer supply chain application. The production rate of the supplier is α = 150,000 per year and the continuous demand β equals 60,000 per year. The setup costs for the supplier and buyer are K_S = \$100 per cycle and K_b = \$30 per cycle respectively, and the holding costs are h_S = \$5 per unit per year h_b = \$6 per unit per year. The buyer has a continuous demand of D = 50,000 per year.

We use excel solver to get the minimum $TCU(n)$ = \$9850.7 for n = 2.7. We round n to its floor and ceiling integer values to find $TCU\left(\lfloor n \rfloor\right)$ = \$9968.8 and $TCU\left(\lceil n \rceil\right)$ = \$9861.6.

$TCU\left(\lceil n \rceil\right) < TCU\left(\lfloor n \rfloor\right)$. Then the minimum cost of the system is achieved when the number of shipments per cycle n^* = 3.

The time between shipments $t = \frac{q^*(n^*)}{D} = \frac{642.22}{50000}$ is almost equal to 5 days, and the cycle time $T = n^*t$ is 2 weeks.

CHAPTER 6

CONCLUSION

Our paper seems to be the first to address an EPQ model with a mixture of lumpy scheduled demand besides the conventional continuous demand. We establish the structure of the optimal production policy. Where the duration of the production cycle is a multiple of the time of recurrence of discrete demand, with the order-at-zero-inventory approach. We then develop a closed-form expression for the optimal production quantity. Interestingly, this quantity has the familiar form of the classical EPQ, with the time-average of lumpy and uniform demands replacing the continuous rate. To gain further insights, we develop the joint integrated system for the suppliers and the buyer. The supplier follows our base EPQ model, while the buyer follows a typical EOQ model. A notable observation appeared when comparing our system to systems with no continuous demand, which is that considering a mixture of lumpy and continuous demands enhances the efficiency of the supply chain. That is, the profit margin is higher under the demand mixture.

Our study is limited to a one supplier one buyer system with fixed interval of time separating two consecutive periodic orders. One extension for our work is to consider a multi-buyer system, where the buyers have different order quantities at distinct time intervals.

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APPENDIX A

Proof. Suppose the system starts by producing an arbitrary production quantity, Q , as shown in the solid-line inventory level of Figure 1. It can be easily seen that an optimal inventory cycle should end with a zero-inventory level. (In any other policy, where the inventory cycle starts and ends with some initial inventory level *I*, one can reduce the holding cost by producing *I* less units, i.e., $Q - I$.) In order for the cycle to end with a zero-inventory level, the production should restart when the inventory level reaches q at a time $x = \frac{d - q}{\alpha - \beta}$, as also shown in the solid-line inventory level of Figure 1. This leads to an inventory cycle of $T = nt$, *n* = 1, 2,... With the inventory level given by the solid line in Figure 1, one can also induce further reductions in the holding cost, over the same cycle length by producing an amount $Q - q$. This implies that inventory produced at the beginning of an inventory cycle will be consumed at a time $\frac{d}{\alpha - \beta}$ before the end of an inventory cycle, where the inventory level is 0, and production is restarted. This alternate (optimal) policy is illustrated in the dashed-line inventory level in Figure 1.

APPENDIX B

Proof that $TCU(q,n)$ *is convex in* q *and* n *:*

$$\text{Determinant}\begin{bmatrix} \frac{\partial^2 TCU(q,n)}{\partial q^2} & \frac{\partial^2 TCU(q,n)}{\partial q \partial n} \\ \frac{\partial^2 TCU(q,n)}{\partial n \partial q} & \frac{\partial^2 TCU(q,n)}{\partial n^2} \end{bmatrix}$$

$$=\left(\frac{\partial^2 TCU(q,n)}{\partial q^2}\right)\left(\frac{\partial^2 TCU(q,n)}{\partial n^2}\right)-\left(\frac{\partial^2 TCU(q,n)}{\partial q \partial n}\right)\left(\frac{\partial^2 TCU(q,n)}{\partial n \partial q}\right)$$

$$=\frac{4K_s D^2 (nK_b + K_s)}{n^4 q^4} - \left(\frac{DK_s}{n^2 q^2} + \frac{h_s}{2}(1+\frac{\beta}{D})(1-\frac{D+\beta}{\alpha})\right)^2$$

For the determinant to be positive:

$\frac{4K_s D^2 (nK_b + K_s)}{n^4 q^4}$ should be greater than $\left(\frac{DK_s}{n^2 q^2} + \frac{h_s}{2}(1+\frac{\beta}{D})(1-\frac{D+\beta}{\alpha})\right)^2$

Square root both sides and set nK_b = 0 to simplify the calculation, then;

$\frac{2K_s D}{n^2 q^2}$ should be greater than $\left(\frac{DK_s}{n^2 q^2} + \frac{h_s}{2}(1+\frac{\beta}{D})(1-\frac{D+\beta}{\alpha})\right)$

Thus,

$\frac{K_s D}{n^2 q^2}$ should be greater than $\left(\frac{h_s}{2}(1+\frac{\beta}{D})(1-\frac{D+\beta}{\alpha})\right)$

But, at optimality, $Q = nq(1+\frac{\beta}{D})$, then $nq = \frac{Q}{(1+\frac{\beta}{D})}$ where $Q^* = \sqrt{\frac{2K_s(\beta+D)}{h_s(1-\frac{(D+\beta)}{\alpha})}}$

So we have to check if $\frac{K_s D(1+\frac{\beta}{D})^2}{\left(\sqrt{\frac{2K_s(\beta+D)}{h_s(1-\frac{(D+\beta)}{\alpha})}}\right)}$ is greater than $\left(\frac{h_s}{2}(1+\frac{\beta}{D})(1-\frac{D+\beta}{\alpha})\right)$

with simplification we realize that the previous equation leads to,

$$\frac{h_s}{2}(1+\frac{\beta}{D})(1-\frac{(D+\beta)}{\alpha}) = \left(\frac{h_s}{2}(1+\frac{\beta}{D})(1-\frac{D+\beta}{\alpha})\right)$$

And since $nK_b \neq 0$ then the term $\frac{2D\sqrt{nK_b K_s}}{n^2 q^2}$ is added to the left hand side of the equality leading to the result:

Determinant $\begin{bmatrix} \frac{\partial^2 TCU(q,n)}{\partial q^2} & \frac{\partial^2 TCU(q,n)}{\partial q \partial n} \\ \frac{\partial^2 TCU(q,n)}{\partial n \partial q} & \frac{\partial^2 TCU(q,n)}{\partial n^2} \end{bmatrix}$ >0 for n >0 and q >0.

Therefore, $TCU(Q,n)$ is convex in q and n .

Appendix C

For a given number of shipments n, the optimal buyer's order quantity is

$$q^*_{(n)} = \sqrt{\frac{2D(nK_b + K_S)}{n\left[nh_S(1+\frac{\beta}{D})(1-\frac{D+\beta}{\alpha}) + h_b - h_S + \frac{2h_S D}{(\alpha-\beta)}\right]}}$$

Furthermore, the function $TCU_J(n) = \min_q(TCU_J(q,n)) = TCU_J(q^*(n),n)$ *is quasi convex in n.*

Proof

First,

$$\frac{dTCU_J(q,n)}{dq} = -\frac{K_S D}{nq^2} + \frac{nh_S(D+\beta)}{2D} - \frac{h_S n(D+\beta)^2}{2D\alpha} + \frac{h_S D}{\alpha-\beta} - \frac{h_S}{2} - \frac{K_b D}{q^2} + \frac{h_b}{2} = 0$$

$$\frac{K_s D}{nq^2} + \frac{K_b D}{q^2} = \frac{nh_s(D+\beta)}{2D} - \frac{h_s n(D+\beta)^2}{2D\alpha} + \frac{h_s D}{\alpha-\beta} - \frac{h_s}{2} + \frac{h_b}{2}$$

$$\frac{(nK_b + K_s)D}{nq^2} = \left(\frac{nh_s}{2D}(D+\beta)\right)\left(1-\frac{(D+\beta)}{\alpha}\right) + \frac{h_s D}{\alpha-\beta} - \frac{h_s}{2} + \frac{h_b}{2}$$

$$\frac{2(nK_b + K_s)(D+\beta)}{q^2} = \left(h_s n^2(1+\frac{\beta}{D})^2\right)\left(1-\frac{(D+\beta)}{\alpha}\right) + \frac{2h_s(D+\beta)n}{\alpha-\beta} - nh_s(1+\frac{\beta}{D}) + nh_b(1+\frac{\beta}{D})$$

Then

$$q^*_{(n)} = \sqrt{\frac{2D(nK_b + K_S)}{n\left[nh_S(1+\frac{\beta}{D})(1-\frac{D+\beta}{\alpha}) + h_b - h_S + \frac{2h_S D}{(\alpha-\beta)}\right]}}$$

Second,

Lemma 1. *Let f*: $\mathbf{R}^2 \to \mathbf{R}$, *be a real-valued, two-variable, convex, and differentiable function*. *Define* $v(y) = \min_{x\in\mathbf{R}} f(x,y)$. *Suppose v(y) is continuous, and admits a local minimum. Then, v(y) is quasiconvex*.

Proof. Let $x^*(y) = \arg\min_{x\in\mathbf{R}} f(x,y)$. By definition, $v(y) = f(x^*(y), y)$. Then,

$$\frac{\partial v(y)}{\partial y} = \left.\frac{\partial f(x,y)}{\partial x}\right|_{x=x^*(y)} \frac{\partial x^*(y)}{\partial y} + \left.\frac{\partial f(x,y)}{\partial y}\right|_{x=x^*(y)}. \tag{1}$$

The convexity of $f(x,y)$ implies that $f(x,y)$ is (marginally) convex in x and accordingly

$$\left.\frac{\partial f(x,y)}{\partial x}\right|_{x=x^*(y)} = 0,\ \forall y \in \mathbf{R}. \tag{2}$$

Utilizing (1) and (2),

$$\frac{\partial v(y)}{\partial y} = \left.\frac{\partial f(x,y)}{\partial y}\right|_{x=x^*(y)}. \tag{3}$$

Note that (3) is a well-known result commonly referred to as the envelope theorem (see, for example, Mas-Colell et al. 1995, pp. 964-965). We derive (3) here for completeness.

Utilizing (2) and (3), we show that a local minimum of $v(y)$ also defines a local minimum of $f(x,y)$. Suppose y_0 is a local minimum of $v(y)$, then $\left.\frac{\partial v(y)}{\partial y}\right|_{y=y_0} = 0$, and (2) and (3) imply that

$$\left.\frac{\partial f(x,y)}{\partial x}\right|_{x=x^*(y_0),\,y=y_0} = 0,$$
$$\left.\frac{\partial f(x,y)}{\partial y}\right|_{x=x^*(y_0),\,y=y_0} = 0.$$

Therefore, $(x^*(y_0), y_0)$ is a local minimum of $f(x,y)$.

Finally, by contradiction, assume $v(y)$ admits two distinct local minima y_0 and y_1. This implies that $(x^*(y_0), y_0)$ and $(x^*(y_1), y_1)$ are both local minima of $f(x,y)$. This contradicts the convexity of $f(x,y)$. It follows that $v(y)$ admits a unique local minimum, i.e., $v(y)$ is unimodal. A result in Bazaraa et al. (2006, p. 156) implies that a unimodal single-variable function is also quasiconvex.