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CARBON EMISSIONS IN SUPPLY CHAINS: THE CASE OF TWO-PRODUCT JOINT REPLENISHMENT

by
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AN ABSTRACT OF THE THESIS OF

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This thesis studies the impact of operations management, namely supply chain management, on carbon emissions. Recent literature has demonstrated how classical inventory management models, such as the economic order quantity (EOQ) model, can be amended to allow jointly reducing operational costs and carbon emissions. However, most of this literature is concerned with single-product inventory management models, with little attention paid to realistic supply chain contexts involving several products and locations. Along the line of studying carbon emissions in supply chains, this thesis analyzes an inventory management model with two products replenished jointly over a common cycle in a framework following the assumptions of the classic EOQ model. This is a typical practical situation, when, for example, one truck is used to deliver multiple products from a supplier to a retailer. The research objective is to identify the conditions under which ordering multiple products jointly is “better” than ordering them individually with respect to costs and emissions. Another objective is to analyze carbon control policies that offer a good balance between costs and emissions.

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CHAPTER I

INTRODUCTION

Global warming is a worldwide threat. The Intergovernmental Panel on Climate Change (IPCC) estimated that the Earth's temperature will increase by 1.8-4 °C by the end of this century (Solomon, et al., 2007). Global warming is mainly caused by increased greenhouse gas (GHG) emissions, such as carbon emissions. GHG emissions must be reduced by 50% (based on 1990 levels) by 2050 in order to have a 50% chance in limiting the increase in temperature to 2°C (Meinshausen, et al., 2009). To reduce the effect of global warming, legislations and regulations that reduce carbon emissions are being enacted by the United Nations; the European Union and other countries. The Kyoto protocol (United Nations, 1998) and the European Union's (EU) Emission Trading Scheme (ETS) are examples of these efforts. The European Union's ETS covers for example 46% of total EU CO_2 emissions (Wagner, 2004). As a response, many firms are investing in new technologies that are more environmentally friendly, using more environmentally friendly raw materials, and are focusing on waste management, reverse logistics, network design, green manufacturing and green remanufacturing. However these approaches take time to be implemented and require the investment of large amounts of money. Instead, firms can meet the requirements through operational adjustments. Operational adjustments can be analyzed by amending classical models, such as those on inventory management, to account for emissions.

There is much interest in amending inventory models to account for green considerations. However, a limited number of studies analyze multi-item inventory systems with green considerations, opposed to the number of single-item studies. In addition, to the author's knowledge, not a study was conducted comparing the effects of joint ordering and disjoint ordering on costs and emissions. The purpose of the thesis is to study a two-item inventory model under different carbon policies and to determine the cases where joint ordering benefits more than disjoint ordering with respect to costs and emissions (i) if no policy is applied and (ii) under each of different applied policies.

To meet the research objectives, this thesis will base on the classical EOQ model to first formulate models for the cases where two products are replenished jointly and disjointly if no policy is applied, the thesis will second compare the costs and emissions in both cases to determine the conditions under which replenishing jointly is preferred over replenishing disjointly, the thesis will third repeat the process under different applied carbon policies. The results show that joint replenishment does not always save on costs and emissions; it even increases them under certain conditions. The results also show that applying carbon control policies can reduce emissions significantly for small increases in costs in the case of joint replenishment.

The remainder of his thesis is organized as follows. In Chapter II, we survey the related literature. In Chapter III, we present our assumptions and formulate models for joint ordering and disjoint ordering if no policy is applied. In Chapter IV, we compare joint ordering and disjoint ordering with respect to costs and emissions and we discuss our results. In Chapter V, we modify the models to account for three carbon policies, carbon tax, carbon cap-and-trade, and strict carbon cap and we compare joint and disjoint ordering under these policies. Finally we conclude in Chapter VI.

CHAPTER II

LITERATURE SURVEY

This chapter presents a survey of relevant literature. Section II. A of the chapter covers works on single-item inventory models. Then, Section II. B covers works on multi-item inventory models.

A. Single-Item Models

Many scholars are studying the impact of environmental policies and environmental considerations on inventory models with the aim of reducing GHG emissions. Scholars mostly consider single-item inventory models and modify them to account for emissions. The methods commonly used in altering inventory models are (i) reformulating the model to consider environmental policies, (ii) associating costs with environmental emissions, and (iii) considering emissions in the objective function.

Reformulating inventory models under environmental policies reflects the regulations and policies enacted by many countries in their effort to reduce emissions. The most considered regulations are strict carbon cap, carbon tax, and carbon cap-and-trade. Strict carbon cap policies set a ceiling on emissions that firms cannot cross. Carbon tax policies impose a tax on firms per unit of carbon emission. Carbon cap-and-trade policies set a ceiling on emissions but allow firms to buy and sell carbon allowances.

Chen et al. (2013) use the EOQ model to prove analytically that operational decisions alone can lead to significant reduction in carbon emissions without causing significant increase in costs. The authors extend the EOQ model to consider 1) strict cap regulations, 2) carbon tax regulations, 3) cap-and-offset regulations where emissions are taxed if they only exceed a certain threshold.

Hua et al. (2011) modify the classical EOQ model to account for the cap-and-trade policy. They find that optimal order size lies between the order quantity minimizing emissions and that minimizing costs. They also find that reducing emission generally leads to an increase in cost, but the retailer can reduce both emissions and costs under some conditions.

Similarly, Toptal et al. (2014) extend the EOQ model to consider strict cap, carbon tax, and carbon cap-and-trade policies. However, besides determining the order sizes, they analyze investment in emission reduction technologies. Their results show that the investment option can help the retailer reduce costs under the three policies. But emissions do not decrease under strict cap policy.

Benjaafar et al. (2013) extend the lot sizing problem for single and multiple firms accounting for different carbon policies. Based on numerical examples, reducing carbon emissions is possible without significantly increasing costs.

Song and Leng (2012) extend the newsvendor model to consider carbon policies and draw useful managerial insights. Specifically they specify conditions where firms can increase profit and decrease emissions under cap-and-trade policy. They also argue that the state should tax firms differently depending on the profitability of the product they sell.

Other than reformulating the model under carbon policies, many scholars associate costs to emissions in their effort to develop more environmentally aware models. For example, Bonney and Jaber (2011) include environmental costs in their extension of the EOQ model. They argue that such models must use non costs metrics and must consider all the logistics chain. Wahab et al. (2011) associate costs to CO_2 emissions from transportation in an extension of the EOQ model. They classify emission costs into fixed and variable costs. Fixed costs depend on fuel efficiency, emissions per gallon, and distance. Variable costs depend on the weight of shipments. They study the impact of emissions costs in a scenario where the vendor and buyer are in different countries. Battini et al. (2014) analyze the traditional EOQ model with additional costs related to transportation and obsolescence costs and emissions.

Bouchery et al. (2012) analyze a multi-objective (cost and emission) EOQ model. They identify a set of efficient frontier solutions. The decision maker selects a solution from this set based on his utility function. Bozorgi et al. (2014) not only include emission in the objective function, they also formulate nonlinear holding and transportations costs and emissions, which were either not modeled or considered as linear functions in the previous works. They argue that their model will result in fewer emissions compared to the model of Bouchery et al. (2012).

B. Multi-Item Models

All of the previously mentioned works consider single-item inventory model. However, in many cases, firms need to manage the inventory of multiple items and might order some products jointly to save on fixed costs. Joint replenishment models aim to determine the best grouping strategies that minimize costs. Works on joint replenishment problems (JRPs) began decades ago with the works of Starr and Miller (1962) and Shu (1971). Starr and Miller (1962) extend the dynamic lot sizing model under certainty to account for multiple items ordered jointly over a common cycle. Shu (1971) develops a set of criteria which can be used to determine an optimum order frequency for the item with the smallest demand and for the rest of the items that are ordered jointly within the EOQ model framework.

Modifying the joint replenishment problem to include environmental consideration leads to a resource constrained JRP (Schaefer & Konur, 2014). Contrary to classical JRPs, a limited number of studies covers resource constrained JRPs. Moon and Cha (2006) study the joint replenishment problem with budget constraint. Porras and Dekker (2006) develop a global optimization procedure to solve a constrained joint replenishment problem, based on a minimum order quantity for each product.

Works on JRPs with environmental considerations are scarce. Zhang and Xu (2013) study the production planning of a multi-product newsvendor problem under the cap-and-trade policy. In this problem the products share the same carbon cap. The authors present a solution method to determine the optimal ordering policy.

Schaefer and Konur (2014) include carbon cap constraint in JRP where each product is subject to the assumptions of the EOQ model. They use genetic algorithm methods on numerical examples to find optimal conditions. Results show a decrease in costs and an increase in emissions with increasing carbon cap. In addition, increasing carbon cap leads to a decrease in setup cost and an increase in holding costs, and leads to a decrease in set up emissions and an increase in emissions from holding.

Although I will cover in this work a two item inventory model under carbon policies similar to Schaefer and Konur (Schaefer & Konur, 2014), the presence of new analytical results and managerial insights and the comparison between joint and disjoint ordering distinguish my work from theirs.

CHAPTER III

PROBLEM FORMULATION UNDER NO POLICY

In this chapter we first list our assumptions in Section III. A, then we present our notations in Section III. B, then we formulate our model for the case where the products are replenished disjointly in Section III. C, and we present the joint replenishment model in Section III. D. In this chapter, we will not consider any carbon policy.

A. Assumptions

1. Assumptions of the EOQ model are valid for each product. Each product has a constant demand rate, production costs, and linear holding costs. If products are ordered individually, then each product has fixed costs. If products are jointly ordered, then they both share common fixed costs.

2. As introduced by Chen et al. (2013), each product has emission parameters, emissions from holding, emissions from production and fixed emissions.

3. In contrast with Chen et al. (2013), fixed emissions are related to fixed costs to account for savings in fixed emissions in joint.

4. Products have a common cycle if they are ordered jointly as recommended by Starr and Miller (1962) and as illustrated in the Figure 1. The source of Figure 1 is Salameh et al. (2014).

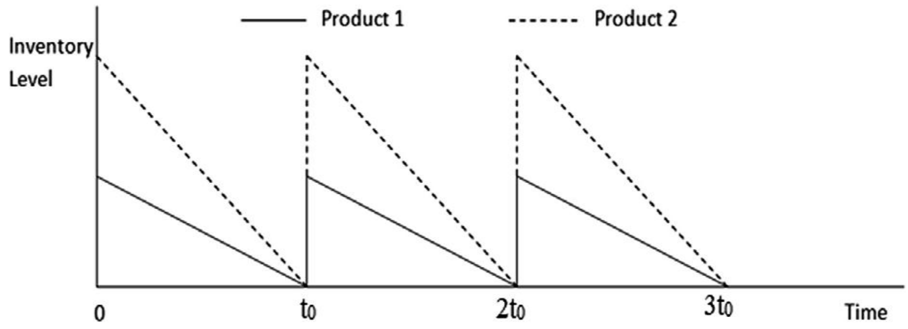

Figure 1 Behavior of inventory over time in joint replenishment case

B. Notation

A_i : Fixed costs associated with product *i*

h_i : Holding costs associated with product *i*

c_i : Unit price of product *i*

Q_i : Order size of product *i*

D_i : Demand rate of product *i*

$\hat{A}_i$: Fixed emissions associated with product *i*

$\hat{h}_i$: Emissions associated with holding of product *i*

$\hat{c}_i$: Emissions associated with the production of product *i*

A : Fixed costs for the joint ordering case

$\hat{A}$: Fixed emissions for the joint ordering case

m_i : Fixed emissions associated with product *i*

n_i : Emission factor related to product *i*

α : Proportion of product 1 in each order. $\alpha = \dfrac{D_1}{D_1 + D_2}$

$\hat{A}_i$ is related to m_i and n_i by $\hat{A}_i = m_i + n_i A_i$

Fixed emissions are assumed to have two parts, fixed emissions that are affected by joint replenishment of products, and fixed emissions that are not affected by joint replenishment of products. Emissions affected by joint replenishment mainly comprise emissions from transportation and from some machinery works. These emissions are related to fixed costs by a factor n_i covering emissions factors and including the proportion of fixed costs that are affected by joint replenishment. Emissions that are not affected by joint replenishment cover emissions from packaging, from some equipment set-up and other similar emissions. These emissions are expressed by m_i. In the special case where all fixed emissions are affected by joint replenishment, then the m_i's are zero and the n_i's have higher values.

In joint ordering, $\hat{A} = m_1 + m_2 + n_1 \alpha A + n_2 (1-\alpha) A$

Total fixed emissions in joint replenishment are the summation of the contributions of each product towards total fixed emissions. We assumed that if product 1 constitutes α of the total order quantity in joint replenishment, then product 1 is responsible of α of the total fixed costs in joint replenishment. Therefore the contribution of product 1 towards total fixed emissions in joint replenishment is m_1 in addition to the emission factor n_1 times the share αA of product 1 of the total fixed costs. Similarly, the contribution of product 2 towards total fixed emissions is $m_2 + n_2 (1-\alpha) A$.

C. Disjoint Ordering Problem

Suppose a retailer is managing the inventory of two items. The retailer might order the two products disjointly. In this case, he wants to solve the following model

$$\min_{Q_1, Q_2} \frac{A_1 D_1}{Q_1} + \frac{h_1 Q_1}{2} + \frac{A_2 D_2}{Q_2} + \frac{h_2 Q_2}{2} + c_1 D_1 + c_2 D_2$$

We are minimizing the total cost per unit for the two products. It is the sum of the total cost per unit of each product.

Minimum costs for this case occurs by ordering the economic order quantity for each product i.e.

$$Q_1^* = \sqrt{\frac{2A_1 D_1}{h_1}}$$

$$Q_2^* = \sqrt{\frac{2A_2 D_2}{h_2}}$$

Total costs per unit are $\sqrt{2A_1 D_1 h_1} + \sqrt{2A_2 D_2 h_2} + c_1 D_1 + c_2 D_2$

Total emissions per unit time are

$$\frac{(m_1 + n_1 A_1) D_1}{Q_1} + \frac{(m_2 + n_2 A_2) D_2}{Q_2} + \frac{\hat{h}_1 Q_1}{2} + \frac{\hat{h}_2 Q_2}{2} + \hat{c}_1 D_1 + \hat{c}_2 D_2 \qquad (1)$$

It has the same structure as the cost function. Therefore, the minimum costs occur by ordering the economic order quantity for each product with emission parameters instead.

$$\hat{Q}_1 = \sqrt{\frac{2(m_1 + n_1 A_1) D_1}{\hat{h}_1}}$$

$$\hat{Q}_2 = \sqrt{\frac{2(m_2 + n_2 A_2) D_2}{\hat{h}_2}} \qquad (2)$$

Therefore the order quantities that minimize costs do not necessarily minimize emissions.

D. Joint Ordering Problem

In case the retailer orders the two products jointly, the total cost per cycle will be

$$A+\frac{h_1Q_{1j}^2}{2}+\frac{h_2Q_{2j}^2}{2}+c_1Q_{1j}+c_2Q_{2j}\,,$$

with $\frac{Q_{1j}}{D_1}=\frac{Q_{2j}}{D_2}$, so $Q_{2j}=\frac{Q_{1j}D_2}{D_1}$

The above model is for total cost per cycle, total cost per unit time is

$$\frac{A+\frac{h_1Q_{1j}^2}{2}+\frac{h_2Q_{2j}^2}{2}+c_1Q_{1j}+c_2Q_{2j}}{Q_{1j}/D_1}$$

Replacing Q_{2j} by $\frac{Q_{1j}D_2}{D_1}$, the model becomes

$$\min_{Q_{1j}}\ \frac{AD_1}{Q_{1j}}+\frac{Q_{1j}(h_1+h_2\,{}^{D_2}\!/\!_{D_1})}{2}+c_1D_1+c_2D_2$$

Setting the derivative of the cost function to zero, the order quantities minimizing costs are

$$Q_{1j}{}^{*}=\sqrt{\frac{2AD_1}{h_1+h_2\,{}^{D_2}\!/\!_{D_1}}}$$

$$Q_{2j}{}^{*}=\sqrt{\frac{2AD_2}{h_2+h_1\,{}^{D_1}\!/\!_{D_2}}}$$

Moreover, $\frac{d^2TC}{dQ_{1j}^2}=\frac{2AD_1}{Q_{1j}^3}\geq 0$, so the total cost is convex in Q_{1j}

Total emissions per unit time are

$$\frac{\hat{A}D_1}{Q_{1j}}+\frac{Q_{1j}(\hat{h}_1+\hat{h}_2\,{}^{D_2}\!/\!_{D_1})}{2}+\hat{c}_1D_1+\hat{c}_2D_2 \qquad (3)$$

The carbon emission equation also resembles an EOQ model, so emissions can be minimized for the following quantities:

$$\hat{Q}_{1j} = \sqrt{\frac{2\hat{A}D_1}{\hat{h}_1 + \hat{h}_2 \, {}^{D_2}\!/\!_{D_1}}}$$
$$\hat{Q}_{2j} = \sqrt{\frac{2\hat{A}D_2}{\hat{h}_2 + \hat{h}_1 \, {}^{D_1}\!/\!_{D_2}}} \tag{4}$$

Again, the order quantities that minimize costs do not necessarily minimize emissions.

CHAPTER IV

JOINT REPLENISHMENT EFFECTIVENESS UNDER NO POLICY

In this chapter we present an analysis aimed at identifying conditions that make joint replenishment favorable for reducing costs in Section IV.A and for reducing emissions in Section IV.B.

A. Cost Analysis of Joint Replenishment Effectiveness

Lemma 1 *Joint replenishment saves on costs if and only if* $r \le \dfrac{1}{1+c\Delta t^2}$ *,*

where $r = \dfrac{A}{A_1 + A_2}$ *,* $c = \dfrac{1}{\left(\sqrt{\dfrac{2A_1}{D_2 h_2}} + \sqrt{\dfrac{2A_2}{D_1 h_1}}\right)^2}$ *, and* $\Delta t = t_2 - t_1$ *,*

with $t_1 = \sqrt{\dfrac{2A_1}{D_1 h_1}}$ *and* $t_2 = \sqrt{\dfrac{2A_2}{D_2 h_2}}$ *being the order cycles lengths of the two products under the individual EOQ models.*

Proof. See Appendix I.

Lemma 2 *The number of cases where joint replenishment saves on costs decreases the farther the two order cycles lengths are from each other.*

Proof. See Appendix II.

Lemma 1 indicates that there is a certain threshold for fixed costs in joint replenishment beyond which disjoint ordering is more beneficial in terms of costs. This threshold depends on the closeness of the disjoint order cycles of the two products. Lemma 2 indicates that costs savings depends on the homogeneity of the products.

B. Emission Analysis of Joint Replenishment Effectiveness

Lemma 3 *Joint replenishment saves on emissions if and only if r is bounded by two values r_1 and r_2, where* $r_1 = (\frac{-b+\sqrt{b^2+4c}}{-2})^2$, $r_2 = (\frac{-b-\sqrt{b^2+4c}}{-2})^2$,

$$b = \frac{(P_1\sqrt{\frac{A_1}{D_2h_2}} + P_2\sqrt{\frac{A_2}{D_1h_1}})}{P_3\sqrt{(\sqrt{\frac{A_1}{D_2h_2}} + \sqrt{\frac{A_2}{D_1h_1}})^2 + \frac{1}{2}\Delta t^2}},\ c = -\frac{P_4}{P_3},\ P_1 = \frac{m_1}{A_1} + n_1 + \frac{\hat{h}_1}{h_1}, P_2 = \frac{m_2}{A_2} + n_2 + \frac{\hat{h}_2}{h_2},$$

$$P_3 = n_1\alpha + n_2(1-\alpha) + \frac{(\hat{h}_1 + \hat{h}_2\, {}^{D_2}\!/\!_{D_1})}{(h_1 + h_2\, {}^{D_2}\!/\!_{D_1})},\ \textit{and}\ P_4 = \frac{m_1+m_2}{A_1+A_2}.$$

Proof . See appendix III.

Lemma 3 suggests that joint ordering saves on emissions if the joint fixed costs are between two boundaries. These boundaries also depend on the closeness of the disjoint order cycles of the two products.

C. Discussion

1. Insights from lemmas 1,2 and 3

Joint replenishment does not necessarily save on both costs and emissions. In fact four cases are possible as shown in Figure 2 drawn from a numerical example. Figure 2 shows the variations of the cost threshold ratio *r**, and the two emission boundaries r_1 and r_2 as functions of $\Delta t = t_1 - t_2$.

Different values of Δt were obtained by fixing the order cycle of one product and varying the order cycle of the second product. This can be done by varying A_2, the fixed costs related to the second product as the order cycle depends on A_2 via the relation $t_2 = \sqrt{\frac{2A_2}{D_2h_2}}$. Different values of Δt can be obtained as well by changing D_2 or h_2.

Figure 2 shows that when the ratio of joint fixed costs on disjoint fixed costs ratio, r, is below r^*, then joint replenishment saves on costs, while joint replenishment saves on emissions if r is between r_1 and r_2.

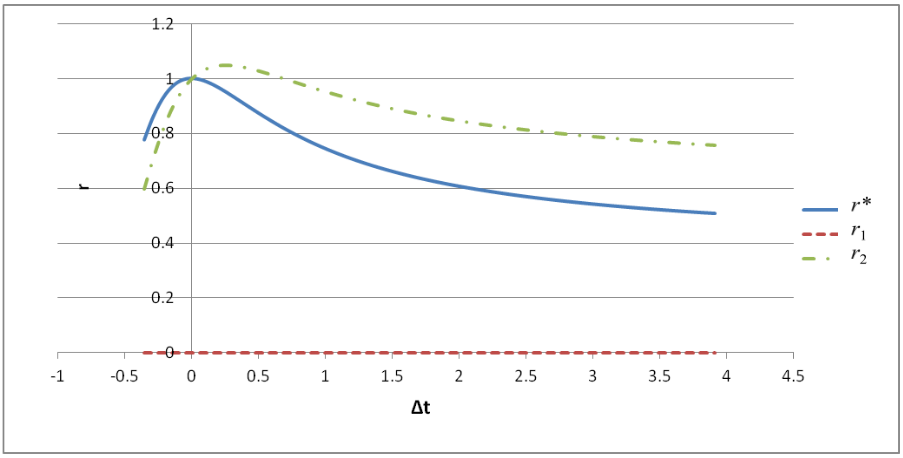

Figure 2 Variations of r^*, r_1, and r_2

In comparing joint and disjoint ordering, four cases are possible: joint replenishment saves on both costs and emissions, joint replenishment saves on emissions but not on costs, joint replenishment saves on costs but not on emissions, and joint replenishment saves neither on costs nor on emissions. The four cases are mapped to the four regions shown in Figure 2. Benefits from joint ordering do not depend only on savings in fixed costs, but also depends on the homogeneity of the products. The closer the two cycles are, the larger the range of r where joint ordering saves on costs. The farther the two cycles are, the stricter the range of r where joint ordering can save costs. However, other parameters also impact the benefits of joint ordering, as explained next.

2. Impact of other parameters

To assess the impact of other parameters, a base case was taken and each parameter was changed by -50%, -25%, 25% and 50% respectively. The base case was taken so that the cost and emission functions are identical. The different values of Δt were obtained by varying A_2. Figure 3 shows the resulting r^*, r_1, and r_2. Table 1 shows the input parameters.

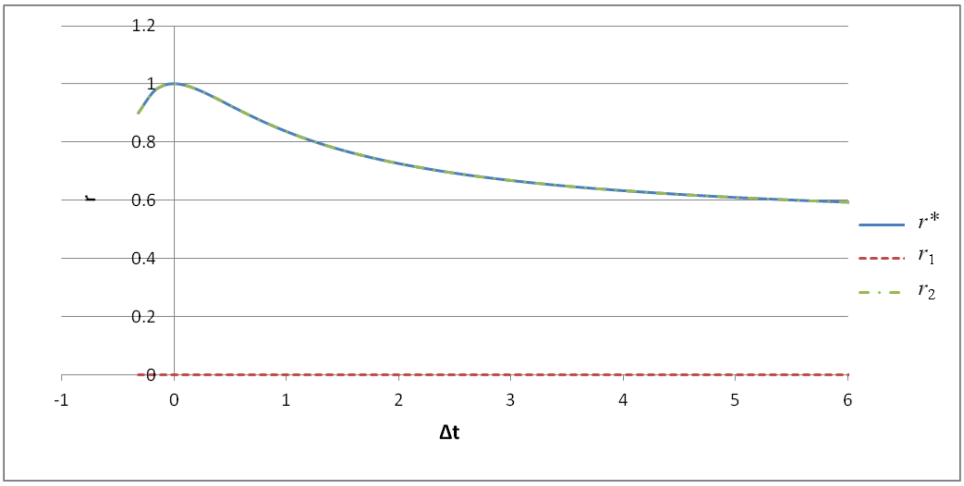

Figure 3 Behavior of r^*, r_1, and r_2 for base case

Table 1 Cost and emission parameters of base case

Parameter	Value	Parameter	Value
m_1	0	A_1	10
m_2	0	A_2	Variable
n_1	1	h_1	1
n_2	1	h_2	1
$\hat{h}_1$	1	D_1	50
$\hat{h}_2$	1	D_2	50

Since the cost and emission functions are identical, r^* and r_2 must also be identical while r_1 is always zero. Therefore there are only two regions, one below the curves where joint replenishment saves on both costs and emissions, and one above the curves where joint replenishment saves neither on costs and emissions.

Increasing n_1 by 50% leads to an upward shift of r_2 for small Δt, and a downward shift of r_2 for large Δt as shown in Figure 4.

Thus increasing n_1 restricts the area where joint replenishment saves on both costs and emissions for large Δt.

Table 2 summarizes the findings after increasing other emission parameters. Some of the respective figures are shown in Appendix IV. Changing the emission parameters affects the aforementioned b and Δ. In turn these changes will be reflected in r_1 and r_2. Note that the findings in Table 2 cannot be generalized, as r_1 and r_2 depends collectively on all parameters.

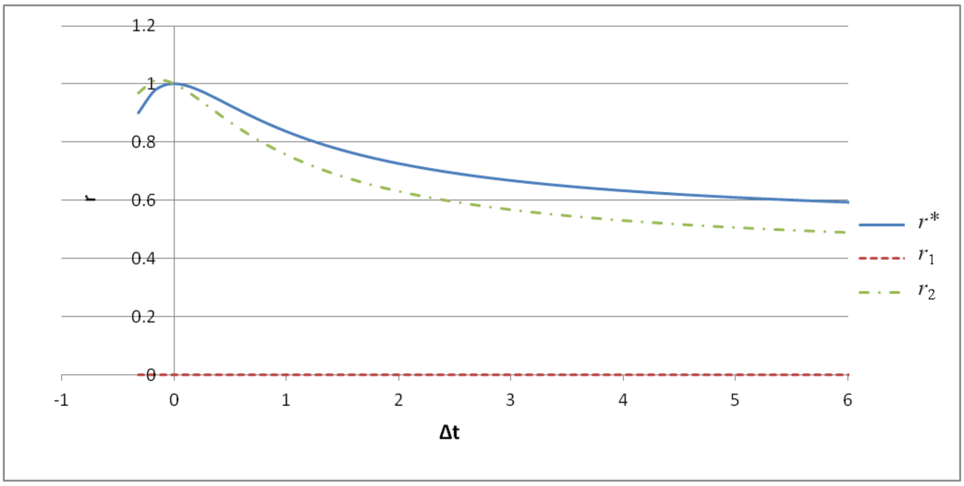

Figure 4 Behavior of r^*, r_1, and r_2 after increasing n_1

Table 2 Effects of increasing base case emission parameters

Parameter	Effect of Increasing Parameter
m_1	Expansion to the area where joint replenishment saves on emissions for large Δt. However it leads to a restriction of the area joint replenishment saves both emissions and costs for small Δt.
m_2	Reduction in the area where joint replenishment saves both on emissions and costs.
n_1	Restriction to the area where joint replenishment saves on both costs and emissions for large Δt.
n_2	Expansion to the area where joint replenishment saves on emissions. The area where joint replenishment saves on both remains the same.
$\hat{h}_1$	Restriction to the area where joint replenishment saves on both costs and emissions for large Δt.
$\hat{h}_2$	Expansion to the area where joint replenishment saves on emissions. The area where joint replenishment saves on both is the same.

3. *Interpretation of the effects of changes in emission parameters*

To interpret the effects of changes in emission parameters, the variations of order quantities Q_1^*, Q_2^*, Q_{1j}^*, Q_{2j}^*, $\hat{Q}_1$, $\hat{Q}_2$, $\hat{Q}_{1j}$ and $\hat{Q}_{2j}$ are shown in Figure 5 for the base case and in Figure 6 after increasing n_1 by 50%.

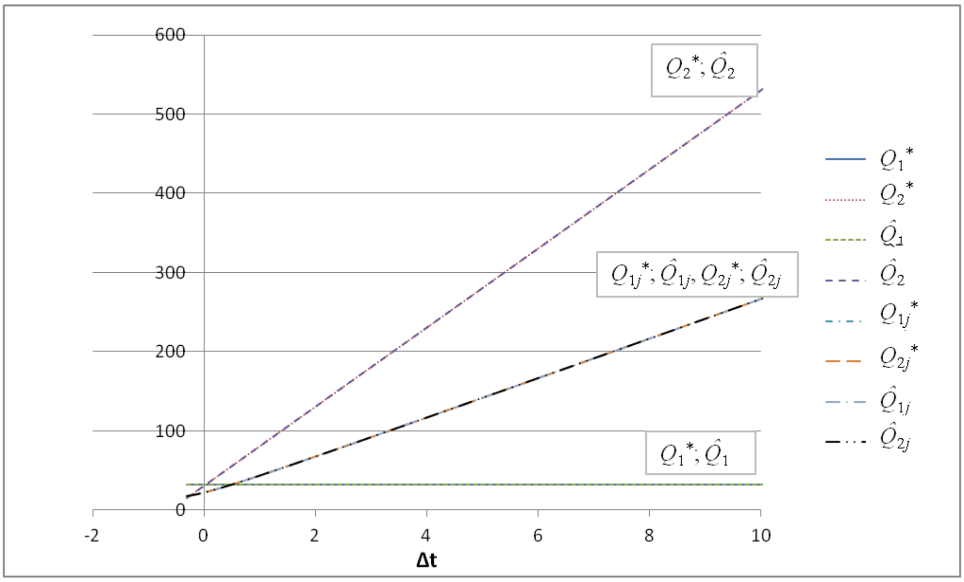

Figure 5 Disjoint and joint order quantities minimizing costs and emissions for the base case

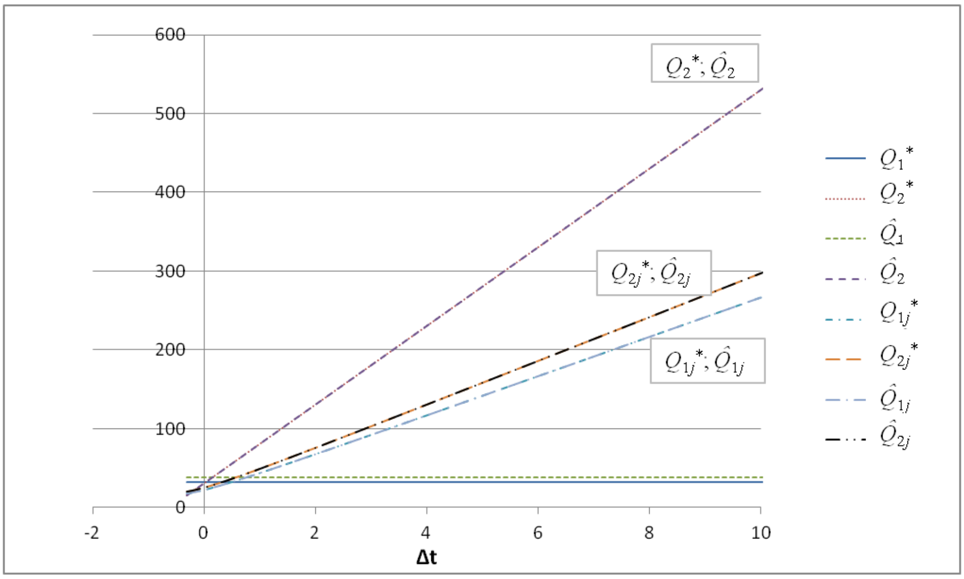

Figure 6 Disjoint and joint order quantities minimizing costs and emissions after increasing n_1 by 50%

Figure 6 shows that after increasing n_1, the deviations of $Q_{1,j}^*$ and $Q_{2,j}^*$ from $\hat{Q}_{1,j}$ and $\hat{Q}_{2,j}$ were larger than the deviations of Q_1^* and Q_2^* from $\hat{Q}_1$ and $\hat{Q}_2$ for positive Δt. And it is the other way around for negative Δt. This shows that after increasing n_1, the difference between joint order quantities and the minimum emissions order quantities is larger (smaller) than the difference between disjoint order quantities and the minimum emissions order quantities for positive (negative) Δt. This can explain the fact that increasing n_1 restricts the area where joint replenishment saves on emissions for positive Δt since increasing n_1 benefits the disjoint scenario and not the joint scenario. The same reasoning can be applied to explain the effects of increasing other parameters.

This chapter showed the presence of four "strategy regions", resulting from minimizing costs and studying the impact on emissions. However, certain policies might be enacted by regulations to reduce emissions. These regulations will affect the above regions as the optimal solutions might change.

CHAPTER V

POLICY ANALYSIS USING JRM

In this chapter we study the effects of carbon policies on order quantities, costs and emissions. In addition we study the effectiveness of joint ordering for some of the policies. Three policies are taken into considerations, carbon tax in Section V. A, carbon cap-and-trade in Section V. B, and strict carbon cap in Section V. C.

A. Carbon Tax

In this section, we study the impact of carbon tax policy on costs and emissions in the cases of joint ordering and disjoint ordering. In this policy, a financial penalty, a tax, is imposed per unit of carbon emitted. Let t is the penalty per unit of carbon emitted. We will first determine the optimal order quantities for disjoint and joint models, and then we will compare the two models.

1. Disjoint ordering problem

The model is as follows.

$$\min_{Q_1,Q_2} \frac{A_1D_1}{Q_1}+\frac{h_1Q_1}{2}+\frac{A_2D_2}{Q_2}+\frac{h_2Q_2}{2}+c_1D_1+c_2D_2+t\left(\frac{\hat{A}_1D_1}{Q_1}+\frac{\hat{A}_2D_2}{Q_2}+\frac{\hat{h}_1Q_1}{2}+\frac{\hat{h}_2Q_2}{2}+\hat{c}_1D_1+\hat{c}_2D_2\right)$$

Then it can be easily shown that the optimal order quantities of both products are given by

$$Q_{1,t}{}^{*}=\sqrt{\frac{2(A_1+t\hat{A}_1)D_1}{(h_1+t\hat{h}_1)}}$$

$$Q_{2,t}{}^{*}=\sqrt{\frac{2(A_2+t\hat{A}_2)D_2}{(h_2+t\hat{h}_2)}}$$

The corresponding total cost and emission per unit time are

$$TC_t^{*}=\sqrt{2(A_1+t\hat{A}_1)D_1(h_1+t\hat{h}_1)}+\sqrt{2(A_2+t\hat{A}_2)D_2(h_2+t\hat{h}_2)}+c_1D_1+c_2D_2+t(\hat{c}_1D_1+\hat{c}_2D_2)$$

$$E_t=\frac{(m_1+n_1A_1)D_1}{Q_1}+\frac{(m_2+n_2A_2)D_2}{Q_2}+\frac{\hat{h}_1Q_1}{2}+\frac{\hat{h}_2Q_2}{2}+\hat{c}_1D_1+\hat{c}_2D_2$$

Similar to the non-policy scenario in Section III. C, minimum emissions occur by ordering $\hat{Q}_1$ and $\hat{Q}_2$ in (2).

Therefore, the order quantities that minimize costs do not necessarily minimize emissions. But the presence of a tax will push the optimal order quantities towards the order quantities minimizing emissions.

2. *Joint ordering Problem*

In this case, utilizing the fact that $Q_{2j}=\frac{Q_{1j}D_2}{D_1}$ (see Section III. D), the model becomes $\min\limits_{Q_{1j}} \ \frac{AD_1}{Q_{1j}}+\frac{Q_{1j}(h_1+h_2\,{}^{D_2}\!/\!_{D_1})}{2}+c_1D_1+c_2D_2+t\left(\frac{\hat{A}D_1}{Q_{1j}}+\frac{Q_{1j}(\hat{h}_1+\hat{h}_2\,{}^{D_2}\!/\!_{D_1})}{2}+\hat{c}_1D_1+\hat{c}_2D_2\right)$

Then, the optimal order quantities are given by

$$Q_{1j,t}{}^{*}=\sqrt{\frac{2(A+t\hat{A})D_1}{h_1+h_2\,{}^{D_2}\!/\!_{D_1}+t(\hat{h}_1+\hat{h}_2\,{}^{D_2}\!/\!_{D_1})}}$$

$$Q_{2j,t}{}^{*}=\sqrt{\frac{2(A+t\hat{A})D_2}{h_2+h_1\,{}^{D_1}\!/\!_{D_2}+t(\hat{h}_2+\hat{h}_1\,{}^{D_1}\!/\!_{D_2})}}$$

The corresponding total cost per unit time is

$$\sqrt{2(A+t\hat{A})D_1(h+t\hat{h})}+c_1D_1+c_2D_2+t(\hat{c}_1D_1+\hat{c}_2D_2),$$

The total emissions per unit time and the order quantities minimizing emissions have the same expression as (3) and (4). Again, the order quantities that minimize costs do not necessarily minimize emissions.

Let us note

$$\hat{h}=\hat{h}_1+\hat{h}_2\,{}^{D_2}\!/_{D_1}$$
$$h=h_1+h_2\,{}^{D_2}\!/_{D_1}$$

The carbon tax problem under joint ordering is similar to the one developed by Chen et al. (2013). So their results are applicable to this model.

In particular:

- The relative reductions in emissions $\delta E'$ and relative increases in direct costs $\delta Z_t'$ are positive and strictly increasing with tax *t*. with $\delta Z_t' < \delta E'$ and both converging to $\frac{(1-\sqrt{\alpha})^2}{1+\alpha}$ with $\alpha=\frac{\hat{A}/\hat{h}}{A/h}$

- $\delta Z_t'$ and $\delta E'$ are both increasing for α >1 and decreasing for α <1.

3. *Cost Analysis of Joint Replenishment Effectiveness*

Lemma 4 *Joint replenishment saves on costs if and only if*

$$r\le\frac{\frac{1}{1+c\Delta t^2}(A_1+A_2+t\hat{A}_1+t\hat{A}_2)-tm_1-tm_2}{\left(1+tn_1\alpha+tn_2(1-\alpha)\right)}$$, where $\Delta t=t_2-t_1$, and

$$c=\frac{1}{\left(\sqrt{\frac{2(A_1+t\hat{A}_1)}{D_2(h_2+t\hat{h}_2)}}+\sqrt{\frac{2(A_2+t\hat{A}_2)}{D_1(h_1+t\hat{h}_1)}}\right)^2}$$, with $t_1=\sqrt{\frac{2(A_1+t\hat{A}_1)}{D_1(h_1+t\hat{h}_1)}}$ and $t_2=\sqrt{\frac{2(A_2+t\hat{A}_2)}{D_2(h_2+t\hat{h}_2)}}$

Proof. See Appendix V.

Similar to the non-policy scenario in Section IV. A, Lemma 4 indicates the presence of a certain threshold for joint fixed costs that depends on the closeness of the disjoint order cycles. Beyond that threshold, joint replenishment is not beneficial in terms of costs.

4. Emission Analysis of Joint Replenishment Effectiveness

Lemma 5 *Joint replenishment saves on emissions if and only* $r_1 \le r \le r_2$*, where*

$$r_1 = \frac{u_1(A_1 + A_2 + t\hat{A}_1 + t\hat{A}_2) - tm_1 - tm_2}{(1 + tn_1\alpha + tn_2(1-\alpha))},\ r_2 = \frac{u_2(A_1 + A_2 + t\hat{A}_1 + t\hat{A}_2) - tm_1 - tm_2}{(1 + tn_1\alpha + tn_2(1-\alpha))},\ \textit{with}$$

$$u_1 = \left(\frac{-b + \sqrt{b^2 + 4c}}{-2}\right)^2,\ u_2 = \left(\frac{-b - \sqrt{b^2 + 4c}}{-2}\right)^2,\ b = \frac{P_1\sqrt{\dfrac{A_1 + t\hat{A}_1}{D_2(h_2 + t\hat{h}_2)}} + P_2\sqrt{\dfrac{A_2 + t\hat{A}_2}{D_1(h_1 + t\hat{h}_1)}}}{P_3\sqrt{\left(\sqrt{\dfrac{A_1 + t\hat{A}_1}{D_2(h_2 + t\hat{h}_2)}} + \sqrt{\dfrac{A_2 + t\hat{A}_2}{D_1(h_1 + t\hat{h}_1)}}\right)^2 + \dfrac{1}{2}\Delta t^2}},$$

$$c = -\frac{P_4}{P_3},\ P_1 = \frac{\hat{A}_1}{A_1 + t\hat{A}_1} + \frac{\hat{h}_1}{(h_1 + t\hat{h}_1)},\ P_2 = \frac{\hat{A}_2}{A_2 + t\hat{A}_2} + \frac{\hat{h}_2}{h_2 + t\hat{h}_2},$$

$$P_3 = \frac{(\hat{h}_1 + \hat{h}_2\, {}^{D_2}\!/\!_{D_1})}{h_1 + h_2\, {}^{D_2}\!/\!_{D_1} + t\hat{h}_1 + t\hat{h}_2\, {}^{D_2}\!/\!_{D_1}} + \frac{(n_1\alpha + n_2(1-\alpha))}{1 + tn_1\alpha + tn_2(1-\alpha)},\ \textit{and}$$

$$P_4 = \frac{m_1 + m_2 - t(m_1 + m_2)\dfrac{(n_1\alpha + n_2(1-\alpha))}{(1 + tn_1\alpha + tn_2(1-\alpha))}}{(A_1 + A_2 + t\hat{A}_1 + t\hat{A}_2)}$$

Proof. See Appendix VI.

Similar to the non-policy scenario in Section IV. B, Lemma 5 indicates that joint ordering saves on emissions if the joint fixed costs are between two boundaries that depend on the closeness of the two cycles.

Lemma 6 *As t goes to infinity, joint replenishment either saves on both costs and emissions or it does not save on both.*

Proof. See Appendix VII

The following numerical example shows illustrates lemmas 4, 5 and 6.

5. Numerical example 1

Suppose a retailer is managing the inventory of two products. The retailer might order the products individually or jointly. Table 3 presents the costs and emissions parameters. The tax rate is initially zero. Depending on Δt and on the ratio r of fixed costs in joint ordering to fixed costs in disjoint ordering, four cases are possible as discussed in Section IV. C. 1 and as shown in figure 7a.

Table 3 Cost and emission parameters for numerical example 1

Parameter	Value	Parameter	Value
m_1	0	A_1	10
m_2	0	A_2	Variable
n_1	1.5	h_1	1
n_2	1	h_2	1
$\hat{h}_1$	1	D_1	50
$\hat{h}_2$	1	D_2	50
t	0		

Suppose the state imposed a tax of $1 per unit emissions, the new curves and regions are shown in figure 7b. Imposing the tax led to moving both curves closer to each other's. The cost curve shifted downward, the emission curve shifted upward for large Δt. Therefore the region where joint replenishment saves on costs becomes tighter, while the region where joint replenishment saves on emission becomes larger.

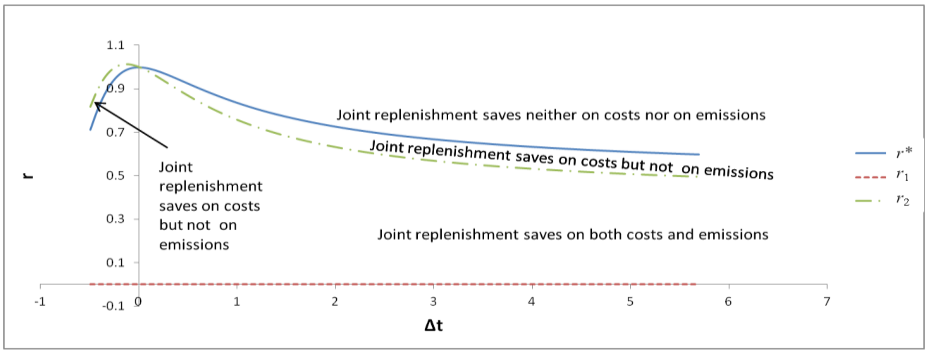

Figure 7a Behavior of r^*, r_1, and r_2 for numerical example 1, tax= 0

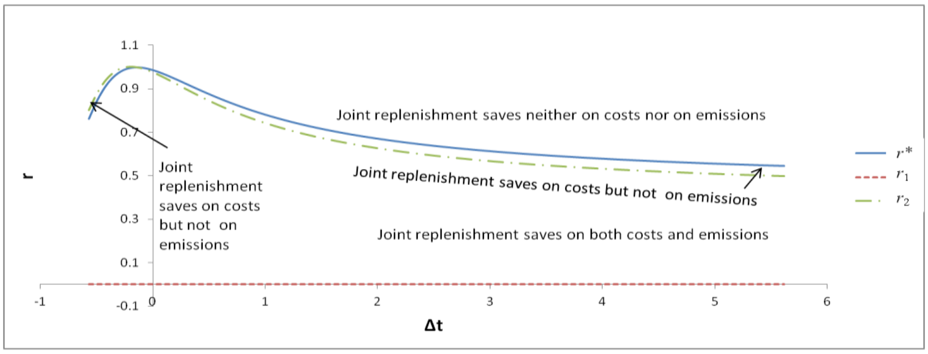

Figure 7b Behavior of r^*, r_1, and r_2, t=1

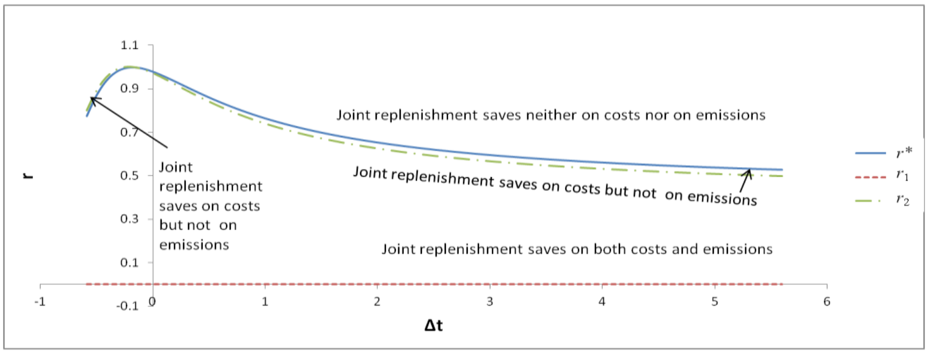

Figure 7c Behavior of r^*, r_1, and r_2, t=2

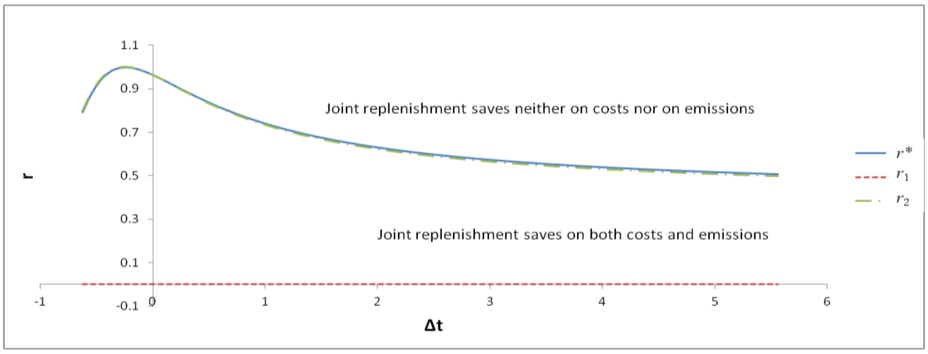

Figure 7d Behavior of r^*, r_1, and r_2, t=10

This leads to restricting the regions where joint replenishment saves either on costs or on emissions. If the tax rate is increased to 2 per unit emissions, the regions where joint replenishment saves either on costs or on emissions become more restricted as shown in figure 7c. Further increasing the tax rate will lead to the disappearance of these regions as shown in figure 7d.

Therefore, in the presence of carbon tax, the retailer is pushed toward choosing the ordering strategy that balances both costs and emissions.

B. Cap and trade model

A firm is allocated a limit or cap C on carbon emissions. If its amount of carbon emissions exceeds the carbon cap, it can buy the right to emit extra carbon from the carbon trading market. Otherwise, it can sell its surplus carbon credit. Denote by X the quantity sold or bought and denote by p is the price of carbon unit. Note that X is positive when selling X units of carbon credit and is negative in the case of purchasing $|X|$ units of carbon credit. In this section we will first develop the models for disjoint and joint ordering, and then we will compare the models to determine the conditions under which joint ordering is beneficial with respect to costs and emissions.

Under disjoint ordering, the model is as follows,

$$\min_{Q_1,Q_2} \frac{A_1 D_1}{Q_1} + \frac{h_1 Q_1}{2} + \frac{A_2 D_2}{Q_2} + \frac{h_2 Q_2}{2} + c_1 D_1 + c_2 D_2 - pX$$

subject to $\frac{\hat{A}_1 D_1}{Q_1} + \frac{\hat{A}_2 D_2}{Q_2} + \frac{\hat{h}_1 Q_1}{2} + \frac{\hat{h}_2 Q_2}{2} + \hat{c}_1 D_1 + \hat{c}_2 D_2 + X = C$ (5)

From (5) $X = C - (\frac{\hat{A}_1 D_1}{Q_1} + \frac{\hat{A}_2 D_2}{Q_2} + \frac{\hat{h}_1 Q_1}{2} + \frac{\hat{h}_2 Q_2}{2} + \hat{c}_1 D_1 + \hat{c}_2 D_2)$

The model becomes

$$\min_{Q_1,Q_2} \frac{A_1D_1}{Q_1}+\frac{h_1Q_1}{2}+\frac{A_2D_2}{Q_2}+\frac{h_2Q_2}{2}+c_1D_1+c_2D_2+p(\frac{\hat{A}_1D_1}{Q_1}+\frac{\hat{A}_2D_2}{Q_2}+\frac{\hat{h}_1Q_1}{2}+\frac{\hat{h}_2Q_2}{2}+\hat{c}_1D_1+\hat{c}_2D_2)-pC$$

The emissions per unit time are

$$\frac{(m_1+n_1A_1)D_1}{Q_1}+\frac{(m_2+n_2A_2)D_2}{Q_2}+\frac{\hat{h}_1Q_1}{2}+\frac{\hat{h}_2Q_2}{2}+\hat{c}_1D_1+\hat{c}_2D_2$$

The disjoint ordering model under cap-and-trade policy is identical to the model under tax policy where the carbon price acts as the carbon tax. The difference is an extra constant cost term ($-pC$) in the cap-and-trade model that does not impact the size of the order quantities. Therefore the cap-and-trade policy under disjoint ordering is equivalent to a carbon tax policy with *t*=*p*.

Under joint ordering, the model is as follows (Noting that $Q_{2j}=\frac{Q_{1j}D_2}{D_1}$ and letting $\hat{h}=\hat{h}_1+\hat{h}_2\,{}^{D_2}\!/_{D_1}$ and $h=h_1+h_2\,{}^{D_2}\!/_{D_1}$)

$$\underset{Q_{1j}}{Min}\ \frac{AD_1}{Q_{1j}}+\frac{hQ_{1j}}{2}+c_1D_1+c_2D_2-pX$$

Subject to $\frac{\hat{A}D_1}{Q_{1j}}+\frac{\hat{h}Q_{1j}}{2}+\hat{c}_1D_1+\hat{c}_2D_2+X=C$ (6)

From (6), $X=C-\frac{\hat{A}D_1}{Q_{1j}}-\frac{\hat{h}Q_{1j}}{2}-\hat{c}_1D_1-\hat{c}_2D_2$

The model becomes

$$\min_{Q_{1j}} \frac{AD_1}{Q_{1j}}+\frac{hQ_{1j}}{2}+c_1D_1+c_2D_2+p(\frac{\hat{A}D_1}{Q_{1j}}+\frac{\hat{h}Q_{1j}}{2}+\hat{c}_1D_1+\hat{c}_2D_2)-pC$$

Total emissions per unit time have the same expression as (3).

This model is also identical to the carbon tax model with t=p. The difference is the same extra constant term ($-pC$) in the cap-and-trade model.

In conclusion, the cap-and-trade policy is equivalent to the carbon tax policy, and its analysis can be done similar to Section V. A.

In addition, the model under joint ordering is identical to the model developed by developed by (Hua, Cheng, & Wang, 2011), therefore their results hold, in particular:

1. The retailer is induced to reduce emissions under cap-and-trade policy.
2. The retailer can both reduce costs and emissions.
3. Retailer's emissions and the quantity ordered do no depend on the carbon cap.

C. Strict cap policy

In this policy, each company is given a ceiling C on its emissions that cannot be exceeded. The ceiling, or carbon cap, is based on the nature of the company. In this case, the cost function is the same as the non-policy model in Chapter III), but a constraint on emissions is added.

In the following, we formulate the models for disjoint in Section V. C. 1 and joint ordering in Section V. C. 2, then we draw some insights from analyzing the joint model and we present numerical examples in Section V. C. 3 and V. C. 4 showing the effect of strict carbon cap policies on the effectiveness of joint and disjoint ordering regarding costs and emissions.

1. Disjoint Ordering Problem

In this case, the model is as follows.

$$\min_{Q_1,Q_2} \frac{A_1D_1}{Q_1}+\frac{h_1Q_1}{2}+\frac{A_2D_2}{Q_2}+\frac{h_2Q_2}{2}+c_1D_1+c_2D_2$$

Subject to $\frac{(m_1+n_1A_1)D_1}{Q_1}+\frac{(m_2+n_2A_2)D_2}{Q_2}+\frac{\hat{h}_1Q_1}{2}+\frac{\hat{h}_2Q_2}{2}+\hat{c}_1D_1+\hat{c}_2D_2 \le C$

This model could not be solved analytically. Nevertheless, both the objective function and the constraint are convex in Q_1 and Q_2. In fact the second partial derivatives of the objective function with respect to Q_1 and Q_2 are respectively $\frac{2A_1D_1}{Q_1^{\ 3}}$ and $\frac{2A_2D_2}{Q_2^{\ 3}}$ which are positive terms. The second partial derivatives of the constraint with respect to Q_1 and Q_2 are respectively $\frac{2\hat{A}_1D_1}{Q_1^{\ 3}}$ and $\frac{2\hat{A}_2D_2}{Q_2^{\ 3}}$ which are also positive terms. Therefore the model has a solution if the cap is not smaller than the minimum emissions per unit time $TE_{\min}=\sqrt{2\hat{A}_1D_1\hat{h}_1}+\sqrt{2\hat{A}_2D_2\hat{h}_2}+\hat{c}_1D_1+\hat{c}_2D_2$.

2. Joint ordering Problem

In this case, the model becomes

$$\min_{Q_{1j}} \frac{AD_1}{Q_{1j}}+\frac{Q_{1j}h}{2}+c_1D_1+c_2D_2$$

subject to $\frac{\hat{A}D_1}{Q_{1j}}+\frac{Q_{1j}\hat{h}}{2}+\hat{c}_1D_1+\hat{c}_2D_2 \le C$ (7)

If the constraint (7) is not binding, then the optimal order quantities are

$Q_{1j,c}{}^{*}=Q_{1j}{}^{*}=\sqrt{\frac{2AD_1}{h_1+h_2\,{}^{D_2}\!/_{D_1}}}$ and $Q_{2j,c}{}^{*}=Q_{2j}{}^{*}=\sqrt{\frac{2AD_2}{h_2+h_1\,{}^{D_1}\!/_{D_2}}}$

If the constraint is binding, then

$$\frac{\hat{A}D_1}{Q_{1j}}+\frac{Q_{1j}\hat{h}}{2}=C-\hat{c}_1D_1-\hat{c}_2D_2$$

Let $\hat{C}=C-\hat{c}_1D_1-\hat{c}_2D_2$,

The equation can be written as $\frac{\hat{A}D_1}{Q_{1j}}+\frac{Q_{1j}\hat{h}}{2}=\hat{C}$, This equations has two solution if

$4\hat{C}^2-8\hat{A}\hat{h}D_1\geq 0$, i.e. when $C\geq\sqrt{2\hat{A}\hat{h}D_1}+\hat{c}_1D_1+\hat{c}_2D_2$, in other words when the carbon cap is bigger than the minimum possible carbon emissions.

The two solutions are,

$$Q_{1,1}=\frac{\hat{C}-\sqrt{\hat{C}^2-2\hat{A}\hat{h}D_1}}{\hat{h}}$$
$$Q_{1,2}=\frac{\hat{C}+\sqrt{\hat{C}^2-2\hat{A}\hat{h}D_1}}{\hat{h}}$$

In conclusion, for the joint case, $Q_{1,j,c}{}^*=\begin{cases}Q_{1,j}{}^* \text{ if } Q_{1,1}\leq Q^*_{1,j}\leq Q_{1,2}\\ Q_{1,1} \text{ if } Q^*_{1,j}\leq Q_{1,1}\\ Q_{1,2} \text{ if } Q^*_{1,j}\geq Q_{1,2}\end{cases}$

The joint model turns out to be identical to the model in (Chen, Benjaafar, & Elomri, 2013), with minor changes in the parameters. Therefore the results of (Chen, Benjaafar, & Elomri, 2013) hold, in particular:

1. If $C>TE_{min}$, emission is linearly non-decreasing in C while cost is non-increasing and convex in C

2. if $\frac{\hat{A}}{\hat{h}}\geq\frac{A}{h}$, then increasing the order quantity to $Q_{1,1}$ will reduce emissions

3. if $\frac{\hat{A}}{\hat{h}}\leq\frac{A}{h}$, then decreasing the order quantity to $Q_{1,2}$ will reduce emissions.

4. A large deviation in the order quantity (up to a limit if decreasing) will lead to a small increase in costs.

3. Numerical Example 2

In this example, we will compare the effect of strict carbon cap policy on the effectiveness of joint and disjoint ordering with respect to costs and emissions. The parameter values are given in Table 4. Figure 8 plots r^*, r_1 and r_2 when no policy is applied. The values are obtained from the relations given in Sections III.1 and III.2. Figure 9 plots r^*, r_1 and r_2 if a strict carbon cap policy of 560 is applied. The values are found by setting the ratio r so that the costs (emissions) with joint replenishment equal the costs (emissions) with disjoint ordering.

Table 4 Cost and emission parameters of numerical example 2

Parameter	Value	Parameter	Value
m_1	15	A_1	50
m_2	12	A_2	Variable
n_1	1	h_1	0.5
n_2	1	h_2	0.2
$\hat{h}_1$	2	D_1	50
$\hat{h}_2$	2	D_2	40
c_1	5	c_2	2

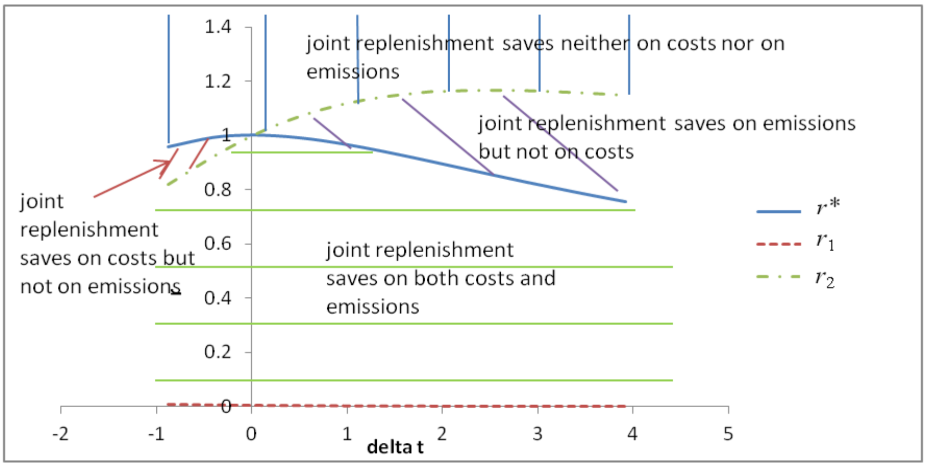

Figure 8 Behavior of r^*, r_1 and r_2 if no policy is applied

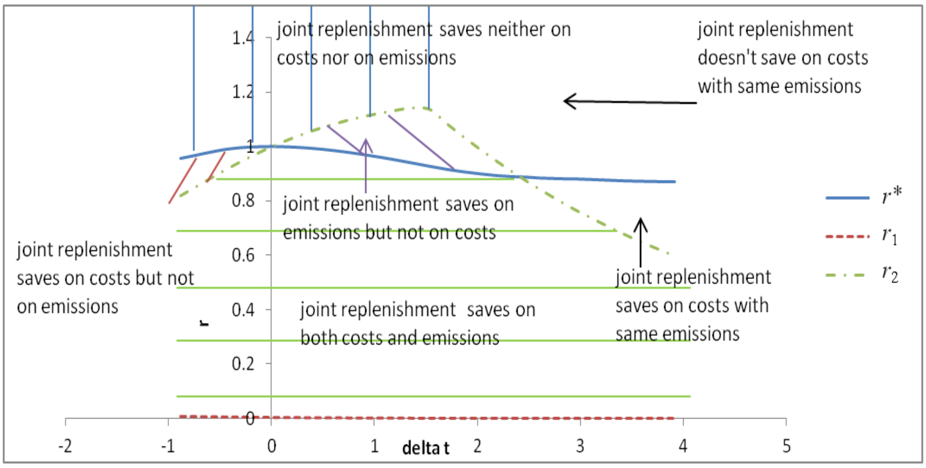

Figure 9 Behavior of r^*, r_1 and r_2 under strict carbon cap policy

<u>Observations</u>

Observing the graphs we can draw the following remarks

1. Joint replenishment saves on costs if r is below a certain threshold r^* under strict carbon cap policy.

2. Joint replenishment saves on emissions if r is between two boundaries r_1 and r_2 under strict carbon cap policy.

3. Carbon Cap policy limits the impact of joint and disjoint ordering on emissions.

The first two observations are common to all policies. The third observation is what differentiates the carbon cap policy from other policies. This is due to the fact that there is the constraint is binding in both joint and disjoint models, so the maximum emissions one can get is the set cap in both models. Therefore, whatever ordering policy the retailer is choosing, emissions will be within the cap.

4. Numerical example 3

In this subsection we will consider a mini case study where we compare the effect of three carbon policies on a small shop

Suppose a small shop is ordering two products from one manufacturer located at a distance of 100 Km. If products are ordered individually, the quantities of each product are carried by a 12’ truck having a fuel economy of 12 mpg. If the products are ordered jointly, the order quantities of the two products are carried by a 15’ truck having a fuel economy of 10 mpg. The driver is paid $0.35 per mile (Truck Driving Per Mile Salary, 2014), each gallon costs $2.814 (Gasoline and Diesel Fuel Update, 2015) and each liter of fuel emits 2.61 Kg of (CO_2) (Ubeda, F.J.Arcelus, & J.Faulin, 2011). From these given, transportation costs are $73 for each product if they are ordered individually, and are $79 if the two products are ordered jointly. Table 5 shows the costs and emissions parameters for each product. We assumed that product 2 needs to be stored at low temperature in contrast with Product 1 so it has higher emissions from holding.

Table 5 Cost and emission parameters for numerical example 3

Parameter	Product 1	Product 2
Unit Cost ($)	7	11
Holding Cost ($)	1.5	2.5
Fixed Packaging Costs ($)	15	20
Transportation Costs ($)	73	73
Total fixed costs in disjoint scenario ($)	15+73=88	93
Emissions from holding (Kg CO_2/item per year)	4.5	13.6
m_i :Fixed emission from packaging (Kg CO_2)	0.25	0.25
n_i	1.17	1.1
Emission from purchasing (Kg CO_2/item)	0	0
Annual Demand Rate	580	210

In table 5, A_1 and n_1 were found as follows

$A_1 = 2 * \dfrac{100km}{1.6\, {}^{km}\!/_{mile} * 12\, {}^{miles}\!/_{gal}} * {}^{\$2.814}\!/_{gal} + 2 * \dfrac{100km * {}^{\$0.35}\!/_{mile}}{1.6\, {}^{km}\!/_{mile}}$, transportation costs in the joint case are found similarly.

$n_1 = \dfrac{\$73}{\$73 + \$45} * \dfrac{2 * \dfrac{100km * 3.785\, {}^{l}\!/_{gal}}{1.6\, {}^{km}\!/_{miles} * 12\, {}^{miles}\!/_{gal}} * 2.61\, {}^{kg\ CO_2}\!/_{l}}{\$73}$, where the first part is the share of transportation costs to total fixed costs, and the second part is the emission per dollar transportation cost.

In joint replenishment the total fixed costs are \$15 + \$20 + \$79 = \$114.

Optimal order quantities, costs and emissions are calculated for the joint and disjoint scenarios under different carbon policies as shown in table 6.

Table 6 Impact of different carbon policies

	No policy		Carbon Tax Policy		Strict Cap Policy		Carbon Cap and Trade	
Ordering Strategy	Disjoint	Joint	Disjoint	Joint	Disjoint	Joint	Disjoint	Joint
Q_1	261	234	254	226	248	234	254	226
Q_2	125	85	118	82	111	85	118	82
Total Costs	7073.8	6934.1	7128.2	6976.5	7076.5	6934.1	7075.7	6924
Total Emissions	1838.9	1430.8	1792.1	1404.1	1748.3	1430.8	1792.1	1404.1
r	x	0.629	x	0.629	x	0.629	x	0.629
r*	x	0.98	x	0.98	x	0.98	x	0.98
r_1	x	0	x	0	x	0	x	0
r_2	x	1.04	x	1.03	x	0.94	x	1.03
Change in costs	x	x	0.8%	0.6%	0.04%	0.00%	0.03%	-0.15%
Change in emissions	x	x	-2.5%	-1.9%	-4.9%	0.0%	-2.5%	-1.9%
Change in costs without variable costs	x	x	7.7%	7.5%	0.4%	0.00%	0.27%	-1.8%

Carbon tax is not implemented in the US, so we used the $30 per metric ton of CO_2 tax applied in British Columbia (P.F., 2014). For the strict cap policy, we set a cap of 1750 kg/ CO_2 per year. In this numerical example, the ratio of joint fixed costs to disjoint fixed costs is below r*, and between r_1 and r_2 for the three policies. Thus joint replenishment must saves on both costs and emissions as shown in table 6. Moreover, applying carbon tax and strict cap policies leads to a small increase in total costs but reduces emissions significantly. For example, under the carbon tax policy, the cost increased by 0.8% and 0.6% over the no-policy model under disjoint and joint replenishment, while emissions decreased by 2.5% and 1.9%. It is worth mentioning that such savings in emissions are considered significant given the flat emission cost structure in the EOQ framework that we adopt. The smallness of the costs increase is partly done to the presence of significant variable costs $c_i D_i$ which are independent of the order quantity.

As for policy effectiveness, strict carbon policy shows the best results in the disjoint case. It reduces emissions the most, and increases costs the least. However in the joint case, it did not have any impact because the emissions level was already below the cap. Under the cap-and-trade policy, a decrease in costs and emissions was achieved; however the surplus of carbon credit is sold to a firm who wants to use it, limiting hence the benefits from the shown reduction in emissions.

CHAPTER VI

CONCLUSION

There is a growing interest in green supply chains with the main goal of reducing carbon emissions. Ordering products jointly might seem a way to reduce carbon emissions. However, it is not always the case as ordering products jointly might increase emissions in some cases, as we demonstrate in this thesis, for the case of two products.

This thesis also identifies (quantitative) conditions on the cost and emission parameters that favor joint replenishment, which allows a useful graphical sensitivity analysis in the form of "strategy regions." Joint replenishment saves on costs if the ratio of fixed costs in joint case to those in the disjoint case is smaller than a certain threshold, and joint replenishment saves on emissions if this ratio is between two boundaries.

This thesis also studies the impact of three carbon policies: Carbon tax, strict carbon cap, and carbon cap and trade on costs and emissions under ordering. It is found that these policies strike a good balance between costs and emissions allowing to reduce emissions with limited additional cost.

Future work could include further analysis of some of the carbon policies under disjoint and joint replenishment. In particular, the carbon cap policies for which our study was mostly numerical deserves further attention in future work. In addition, analyzing the effect of joint replenishment of multiple (>2) products on emissions is an important area for future research, as our study considers only two products.

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APPENDIX I

PROOF OF LEMMA 1

Let $\mathrm{A} = r(A_1 + A_2)$, letting $\mathrm{C}_{disjoint} \geq \mathrm{C}_{joint}$ implies that

$$\sqrt{2A_1D_1h_1} + \sqrt{2A_2D_2h_2} + c_1D_1 + c_2D_2 \geq \sqrt{2AD_1(h_1 + h_2 \, {}^{D_2}\!/_{D_1})} + c_1D_1 + c_2D_2 \, .$$

It can be shown that $\sqrt{r} \leq \dfrac{\sqrt{2A_1D_1h_1} + \sqrt{2A_2D_2h_2}}{\sqrt{2(A_1 + A_2)(D_1h_1 + D_2h_2)}}$, i.e.

$$r \leq \frac{1}{1 + \left(\dfrac{\sqrt{2A_2D_1h_1} - \sqrt{2A_1D_2h_2}}{\sqrt{2A_1D_1h_1} + \sqrt{2A_2D_2h_2}}\right)^2} = \frac{1}{1 + \left(\dfrac{\sqrt{\dfrac{2A_2}{D_2h_2}} - \sqrt{\dfrac{2A_1}{D_1h_1}}}{\sqrt{\dfrac{2A_1}{D_2h_2}} + \sqrt{\dfrac{2A_2}{D_1h_1}}}\right)^2} \, .$$

Note $c = \left(\dfrac{1}{\sqrt{\dfrac{2A_1}{D_2h_2}} + \sqrt{\dfrac{2A_2}{D_1h_1}}}\right)^2$, knowing that $t_1 = \sqrt{\dfrac{2A_1}{D_1h_1}}, t_2 = \sqrt{\dfrac{2A_2}{D_2h_2}}, \Delta t = t_1 - t_2$,

where t_1 and t_2 are the respective disjoint order cycles for products 1 and 2, the above inequality can be written as $r \leq \dfrac{1}{1 + c\Delta t^2}$

APPENDIX II

PROOF OF LEMMA 2

To prove that the region where joint replenishment saves on costs becomes stricter the farther the two order cycles lengths are from each other, we must prove that $r*$ is decreasing as Δt gets larger and larger.

Δt increases by increasing A_2, decreasing D_2, decreasing h_2, decreasing A_1, increasing D_1, and increasing h_1.

Recall that $r* = \dfrac{(\sqrt{2A_1D_1h_1} + \sqrt{2A_2D_2h_2})^2}{2(A_1 + A_2)(D_1h_1 + D_2h_2)}$

1-Changing A_2

Let us denote by $(r_{A_2}*)'$ the derivative of $r*$ with respect to A_2, we can show that $(r_{A_2}*)' = \dfrac{2*(D_1h_1 + D_2h_2)*(\sqrt{2A_1D_1h_1} + \sqrt{2A_2D_2h_2})}{\left(2(A_1 + A_2)(D_1h_1 + D_2h_2)\right)^2} * (2A_1 \dfrac{\sqrt{D_2h_2}}{\sqrt{2A_2}} - \sqrt{2A_1D_1h_1})$ which is negative for $A_2 \geq \dfrac{A_1D_2h_2}{D_1h_1}$ and positive otherwise which means that $r*$ is decreasing as A_2 gets larger and larger, therefore decreasing as Δt gets larger and larger.

2-Changing D_2

Let us note $(r_{D_2}*)'$ the derivative of $r*$ with respect to D_2. We can show that $(r_{D_2}*)' = \dfrac{2*(A_1 + A_2)*(\sqrt{2A_1D_1h_1} + \sqrt{2A_2D_2h_2})}{\left(2(A_1 + A_2)(D_1h_1 + D_2h_2)\right)^2} * (2D_1h_1 \dfrac{\sqrt{A_2h_2}}{\sqrt{2D_2}} - h_2\sqrt{2A_1D_1h_1})$ which is positive for $D_2 \leq \dfrac{A_2D_1h_1}{A_1h_2}$ and negative otherwise. Therefore $r*$ increases as D_2 increases for small D_2. Hence $r*$ decreases with decreasing D_2 for small D_2, i.e. $r*$ decreases as Δt gets larger and larger.

3-Changing h_2

Let us note $(r_{h_2}*)'$ the derivative of $r*$ with respect to h_2. We can show that

$(r_{h_2}*)' = \frac{2*(A_1+A_2)*(\sqrt{2A_1D_1h_1}+\sqrt{2A_2D_2h_2})}{\left(2(A_1+A_2)(D_1h_1+D_2h_2)\right)^2}*(2D_1h_1\frac{\sqrt{A_2D_2}}{\sqrt{2h_2}} - D_2\sqrt{2A_1D_1h_1})$ which is positive for $h_2 \leq \frac{A_2D_1h_1}{A_1D_2}$ and negative otherwise. Therefore $r*$ increases as h_2 increases for small h_2. Hence $r*$ decreases with decreasing h_2 for small h_2, i.e. $r*$ decreases as Δt gets larger and larger.

Using similar arguments, we can show that $r*$ decreases as A_1 decreases for small A_1, and $r*$ increases as D_1 and h_1 increase for large D_1 and h_1.

APPENDIX III

PROOF OF LEMMA 3

Letting $\mathrm{E}_{disjoint} \geq \mathrm{E}_{joint}$ implies

$$\frac{(m_1 + n_1 A_1)D_1}{Q_1} + \frac{(m_2 + n_2 A_2)D_2}{Q_2} + \frac{\hat{h}_1 Q_1}{2} + \frac{\hat{h}_2 Q_2}{2} \geq \frac{(m_1 + m_2)D_1}{Q_{1j}} + \frac{(n_1\alpha + n_2(1-\alpha))AD_1}{Q_{1j}} + \frac{Q_{1j}(\hat{h}_1 + \hat{h}_2 {}^{D_2}\!/\!_{D_1})}{2} \cdot$$

Knowing that

$$Q_1 = \sqrt{\frac{2A_1D_1}{h_1}}, Q_2 = \sqrt{\frac{2A_2D_2}{h_2}}, Q_{1j} = \sqrt{\frac{2AD_1}{h_1 + h_2 {}^{D_2}\!/\!_{D_1}}}$$

The above inequality can be written as

$$(\frac{m_1}{A_1} + n_1 + \frac{\hat{h}_1}{h_1})\sqrt{\frac{A_1D_1h_1}{2}} + (\frac{m_2}{A_2} + n_2 + \frac{\hat{h}_2}{h_2})\sqrt{\frac{A_2D_2h_2}{2}} \geq (m_1 + m_2)\sqrt{\frac{h_1D_1 + h_2D_2}{2A}} + (n_1\alpha + n_2(1-\alpha) + \frac{(\hat{h}_1 + \hat{h}_2 {}^{D_2}\!/\!_{D_1})}{(h_1 + h_2 {}^{D_2}\!/\!_{D_1})})\sqrt{\frac{A(h_1D_1 + h_2D_2)}{2}} \cdot$$

We get,

$$(\frac{m_1}{A_1} + n_1 + \frac{\hat{h}_1}{h_1})\sqrt{\frac{A_1D_1h_1}{2}} + (\frac{m_2}{A_2} + n_2 + \frac{\hat{h}_2}{h_2})\sqrt{\frac{A_2D_2h_2}{2}} \geq \frac{(m_1 + m_2)}{\sqrt{r}}\sqrt{\frac{h_1D_1 + h_2D_2}{2(A_1 + A_2)}} + (n_1\alpha + n_2(1-\alpha) + \frac{(\hat{h}_1 + \hat{h}_2 {}^{D_2}\!/\!_{D_1})}{(h_1 + h_2 {}^{D_2}\!/\!_{D_1})})\sqrt{r}\sqrt{\frac{(A_1 + A_2)(h_1D_1 + h_2D_2)}{2}}$$

Let

$$P_1 = \frac{m_1}{A_1} + n_1 + \frac{\hat{h}_1}{h_1}$$

$$P_2 = \frac{m_2}{A_2} + n_2 + \frac{\hat{h}_2}{h_2}$$

$$P_3 = n_1\alpha + n_2(1-\alpha) + \frac{(\hat{h}_1 + \hat{h}_2 {}^{D_2}\!/\!_{D_1})}{(h_1 + h_2 {}^{D_2}\!/\!_{D_1})}$$

$$P_4 = \frac{m_1 + m_2}{A_1 + A_2}$$

The above inequality can be reduced to

$$\sqrt{r}\,\frac{(P_1\sqrt{\frac{A_1}{D_2h_2}} + P_2\sqrt{\frac{A_2}{D_1h_1}})}{P_3\sqrt{(\sqrt{\frac{A_1}{D_2h_2}} + \sqrt{\frac{A_2}{D_1h_1}})^2 + \frac{1}{2}\Delta t^2}} - \frac{P_4}{P_3} - (\sqrt{r})^2) \ \geq 0$$

Let us consider the equation $\left(\sqrt{r}\,\dfrac{(P_1\sqrt{\dfrac{A_1}{D_2h_2}}+P_2\sqrt{\dfrac{A_2}{D_1h_1}})}{P_3\sqrt{(\sqrt{\dfrac{A_1}{D_2h_2}}+\sqrt{\dfrac{A_2}{D_1h_1}})^2+\dfrac{1}{2}\Delta t^2}}-\dfrac{P_4}{P_3}-(\sqrt{r})^2\right)=0$.

It is a second degree equation, it has two roots if $\left(\dfrac{(P_1\sqrt{\dfrac{A_1}{D_2h_2}}+P_2\sqrt{\dfrac{A_2}{D_1h_1}})}{P_3\sqrt{(\sqrt{\dfrac{A_1}{D_2h_2}}+\sqrt{\dfrac{A_2}{D_1h_1}})^2+\dfrac{1}{2}\Delta t^2}}\right)^2-4\dfrac{P_4}{P_3}\geq 0$,

if we call $\Delta=\left(\dfrac{(P_1\sqrt{\dfrac{A_1}{D_2h_2}}+P_2\sqrt{\dfrac{A_2}{D_1h_1}})}{P_3\sqrt{(\sqrt{\dfrac{A_1}{D_2h_2}}+\sqrt{\dfrac{A_2}{D_1h_1}})^2+\dfrac{1}{2}\Delta t^2}}\right)^2-4\dfrac{P_4}{P_3}$,

$b=\dfrac{(P_1\sqrt{\dfrac{A_1}{D_2h_2}}+P_2\sqrt{\dfrac{A_2}{D_1h_1}})}{P_3\sqrt{(\sqrt{\dfrac{A_1}{D_2h_2}}+\sqrt{\dfrac{A_2}{D_1h_1}})^2+\dfrac{1}{2}\Delta t^2}}$, and $c=-\dfrac{P_4}{P_3}$

Then if $\Delta>0$

$\sqrt{r_1}=\dfrac{-b+\sqrt{\Delta}}{-2}=\dfrac{-b+\sqrt{b^2+4c}}{-2}\geq 0$ and $\sqrt{r_2}=\dfrac{-b-\sqrt{\Delta}}{-2}\geq 0$

Thus, $r_1=(\dfrac{-b+\sqrt{b^2+4c}}{-2})^2$ and $r_2=(\dfrac{-b-\sqrt{b^2+4c}}{-2})^2$.

The expression

$$(\sqrt{r}\,\frac{(P_1\sqrt{\frac{A_1}{D_2h_2}}+P_2\sqrt{\frac{A_2}{D_1h_1}})}{P_3\sqrt{(\sqrt{\frac{A_1}{D_2h_2}}+\sqrt{\frac{A_2}{D_1h_1}})^2+\frac{1}{2}\Delta t^2}}-\frac{P_4}{P_3}-(\sqrt{r})^2)$$ increases then decreases with respect to $\sqrt{r}$, therefore $(\sqrt{r}\,\frac{(P_1\sqrt{\frac{A_1}{D_2h_2}}+P_2\sqrt{\frac{A_2}{D_1h_1}})}{P_3\sqrt{(\sqrt{\frac{A_1}{D_2h_2}}+\sqrt{\frac{A_2}{D_1h_1}})^2+\frac{1}{2}\Delta t^2}}-\frac{P_4}{P_3}-(\sqrt{r})^2)$ is positive for $\sqrt{r_1}\le\sqrt{r}\le\sqrt{r_2}$, i.e. for $r_1\le r\le r_2$

If $\Delta\le 0$, then joint ordering can never have fewer emissions than disjoint ordering.

APPENDIX IV

IMPACT OF OTHER PARAMETERS

In this appendix, we plot the graphs showing the effects of changing the emission parameters of Product 1 only. Changing the same parameters for the other product has the exact opposite effect.

Base scenario

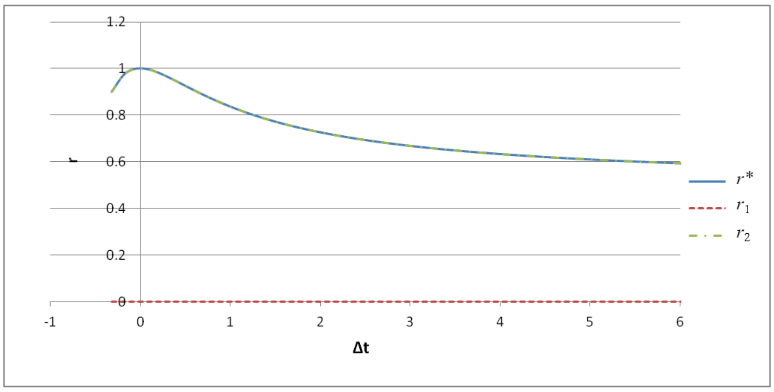

Figure 10 Behavior of r^*, r_1, and r_2 for base case

Changing n_1

i) By -50%

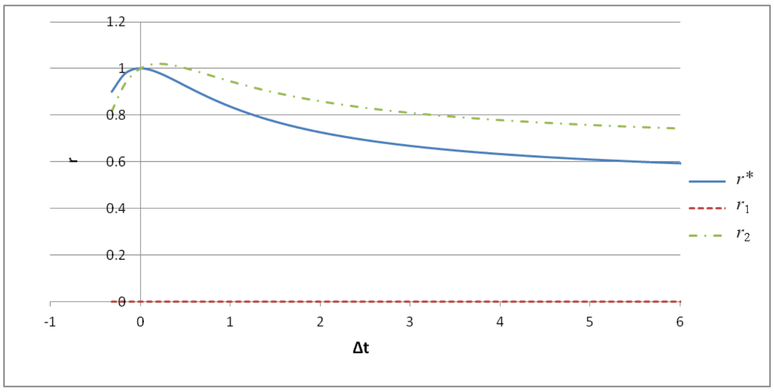

Figure 11a Changing n_1 by -50%

ii) By -25%

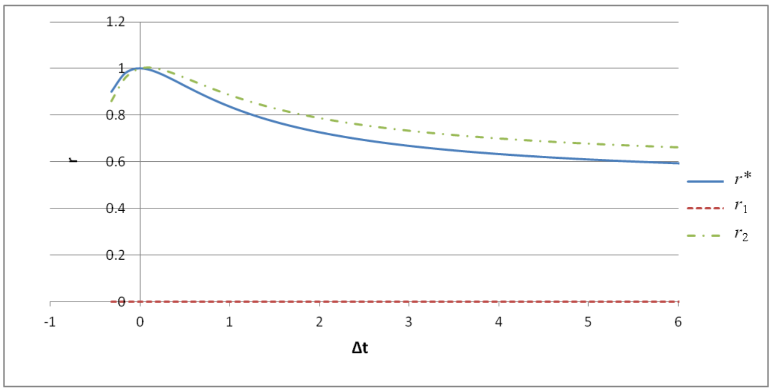

Figure 11b Changing n_1 by -25%

iii) By 25%

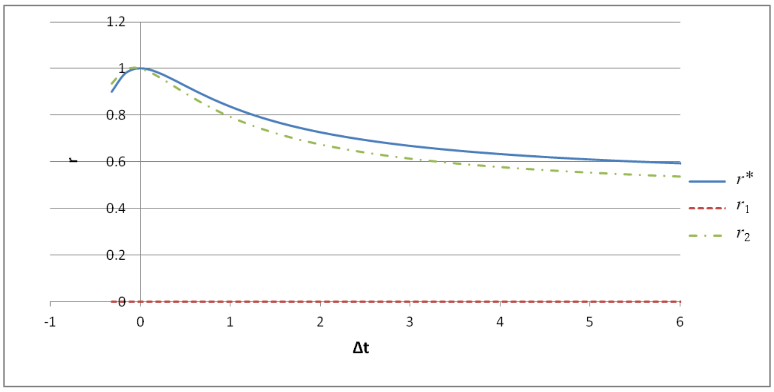

Figure 11c Changing n_1 by 25%

iv) By 50%

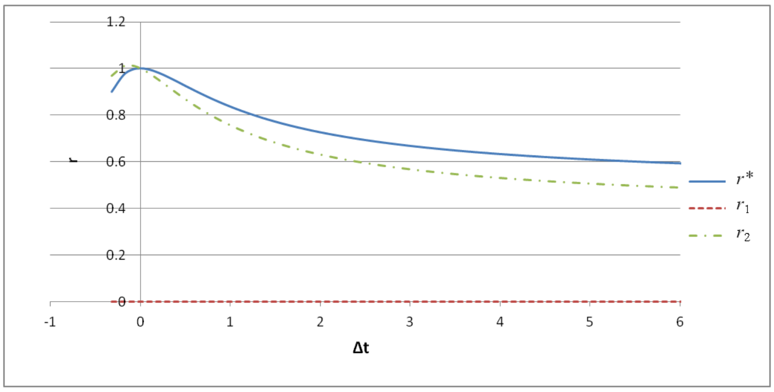

Figure 11d Changing n_1 by 50%

Changing $\hat{h}_1$

i) By -50%

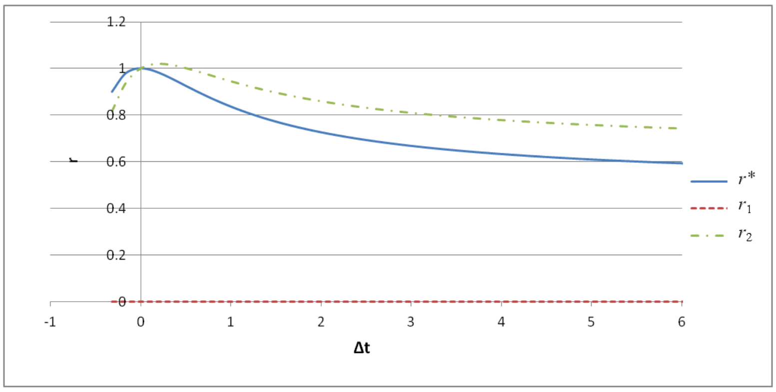

Figure 12a Changing $\hat{h}_1$ by -50%

ii) By -25%

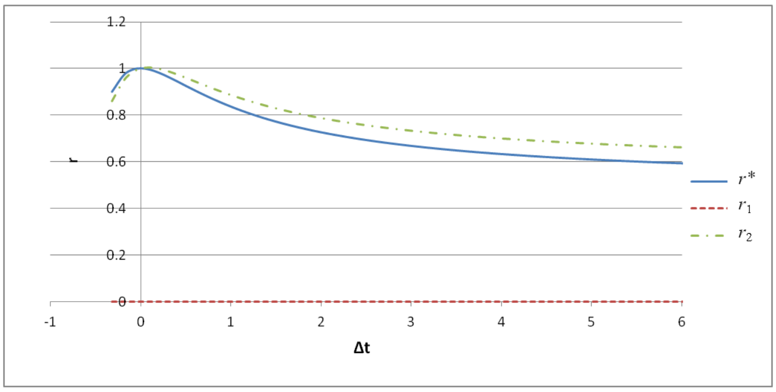

Figure 12b Changing $\hat{h}_1$ by -25%

iii) By 25%

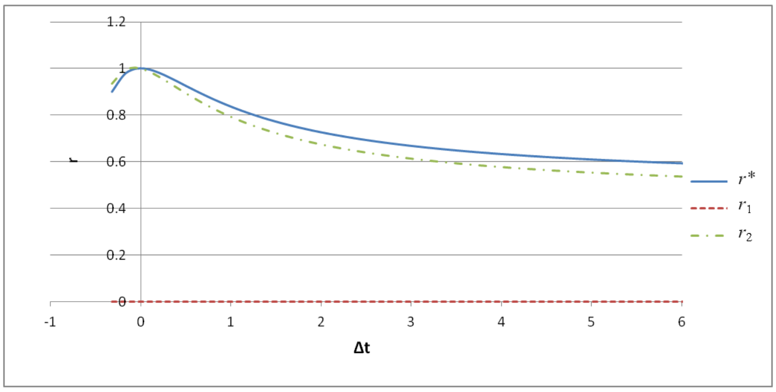

Figure 12c Changing $\hat{h}_1$ by 25%

iv) By 50%

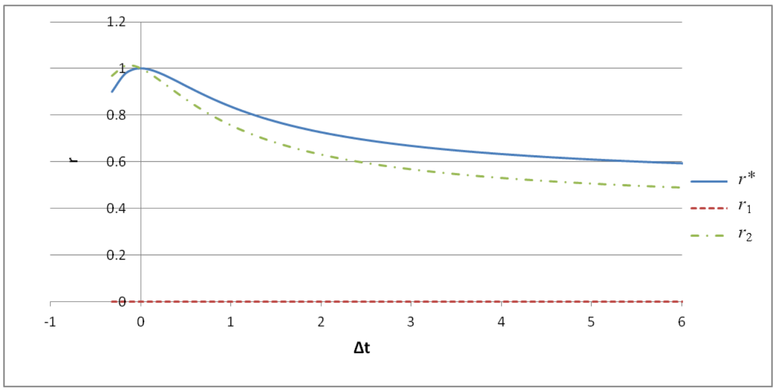

Figure 12d Changing $\hat{h}_1$ by 50%

New base case (m_1 is set to 10)

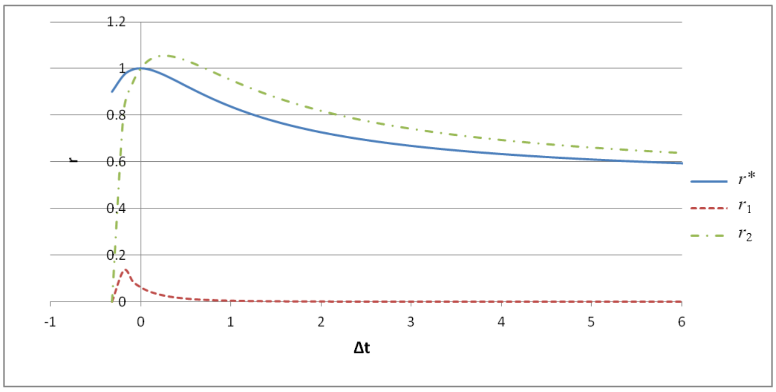

Figure 13a Base case for m_1

Changing m_1

i) By -50%

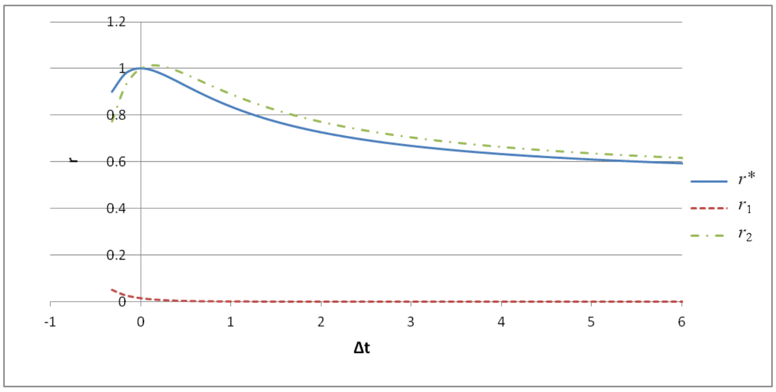

Figure 13b Changing m_1 by -50%

ii) By -25%

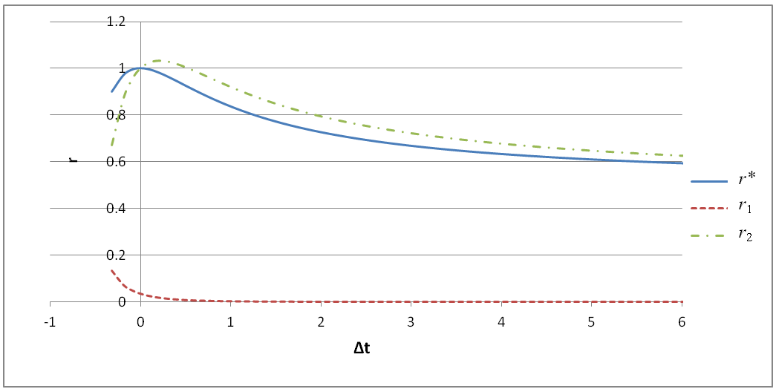

Figure 13c Changing m_1 by -25%

iii) By 25%

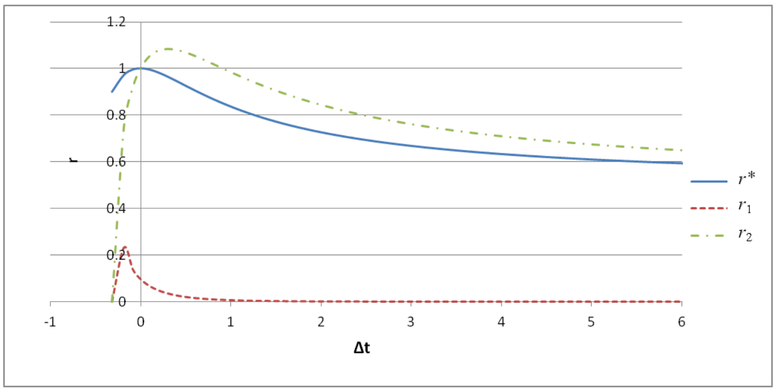

Figure 13d Changing m_1 by 25%

iv) By 50%

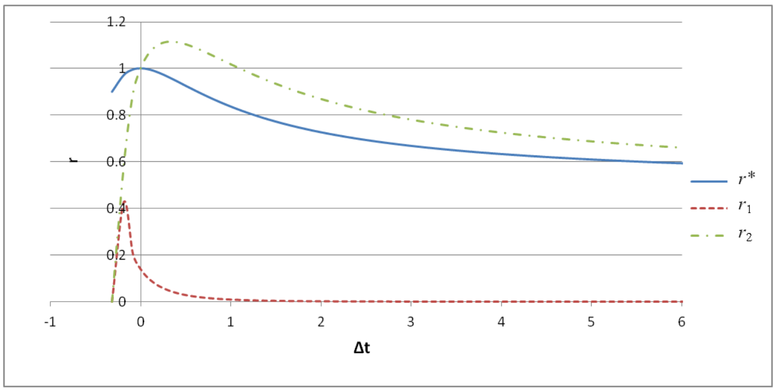

Figure 13e Changing m_1 by50%

APPENDIX V

PROOF OF LEMMA 4

Let $\mathrm{A} = r(A_1 + A_2)$, letting $\mathrm{C}_{disjoint} \geq \mathrm{C}_{joint}$ implies

$$\sqrt{2(A_1 + t\hat{A}_1)D_1(h_1 + t\hat{h}_1)} + \sqrt{2(A_2 + t\hat{A}_2)D_2(h_2 + t\hat{h}_2)} + c_1D_1 + c_2D_2 + t(\hat{c}_1D_1 + \hat{c}_2D_2)$$
$$\geq \sqrt{2(A + t(m_1 + m_2 + n_1\alpha A + n_2(1-\alpha)A)D_1(h_1 + h_2\, {}^{D_2}\!/_{D_1} + t\hat{h}_1 + \mathrm{t}\,\hat{h}_2\, {}^{D_2}\!/_{D_1})} + c_1D_1 + c_2D_2 + t(\hat{c}_1D_1 + \hat{c}_2D_2)$$

Let $u = \dfrac{A + t(m_1 + m_2 + n_1\alpha A + n_2(1-\alpha)A)}{A_1 + A_2 + t\hat{A}_1 + t\hat{A}_2}$, the above inequality becomes

$$\sqrt{2(A_1 + t\hat{A}_1)D_1(h_1 + t\hat{h}_1)} + \sqrt{2(A_2 + t\hat{A}_2)D_2(h_2 + t\hat{h}_2)}$$
$$\geq \sqrt{2u(A_1 + A_2 + t\hat{A}_1 + t\hat{A}_2)(h_1D_1 + h_2D_2 + t(\hat{h}_1D_1 + \hat{h}_2D_2))}$$

$$\Rightarrow \sqrt{u} \leq \frac{\sqrt{2(A_1 + t\hat{A}_1)D_1(h_1 + t\hat{h}_1)} + \sqrt{2(A_2 + t\hat{A}_2)D_2(h_2 + t\hat{h}_2)}}{\sqrt{2(A_1 + A_2 + t\hat{A}_1 + t\hat{A}_2)(h_1D_1 + h_2D_2 + t(\hat{h}_1D_1 + \hat{h}_2D_2))}}$$

$$\Rightarrow u \leq \frac{\left(\sqrt{(A_1 + t\hat{A}_1)D_1(h_1 + t\hat{h}_1)} + \sqrt{(A_2 + t\hat{A}_2)D_2(h_2 + t\hat{h}_2)}\right)^2}{(A_1 + A_2 + t\hat{A}_1 + t\hat{A}_2)\left(h_1D_1 + h_2D_2 + t(\hat{h}_1D_1 + \hat{h}_2D_2)\right)}$$

$$\Rightarrow u \leq \frac{(\sqrt{(A_1 + t\hat{A}_1)D_1(h_1 + t\hat{h}_1)} + \sqrt{(A_2 + t\hat{A}_2)D_2(h_2 + t\hat{h}_2)})^2}{(A_1 + t\hat{A}_1)D_1(h_1 + t\hat{h}_1) + (A_2 + t\hat{A}_2)D_2(h_2 + t\hat{h}_2) + (A_1 + t\hat{A}_1)D_2(h_2 + t\hat{h}_2) + (A_2 + t\hat{A}_2)D_1(h_1 + t\hat{h}_1)}$$

$$\Rightarrow u \leq \frac{\left(\sqrt{(A_1 + t\hat{A}_1)D_1(h_1 + t\hat{h}_1)} + \sqrt{(A_2 + t\hat{A}_2)D_2(h_2 + t\hat{h}_2)}\right)^2}{\left(\sqrt{(A_1 + t\hat{A}_1)D_1(h_1 + t\hat{h}_1)} + \sqrt{(A_2 + t\hat{A}_2)D_2(h_2 + t\hat{h}_2)}\right)^2 + \left(\sqrt{(A_2 + t\hat{A}_2)D_1(h_1 + t\hat{h}_1)} - \sqrt{(A_1 + t\hat{A}_1)D_2(h_2 + t\hat{h}_2)}\right)^2}$$

Dividing the numerator and the denominator by $D_1D_2(h_1 + t\hat{h}_1)(h_2 + t\hat{h}_2)$ yields to the following inequality

$$u \le \frac{\left(\sqrt{\frac{A_1+t\hat{A}_1}{D_2(h_2+t\hat{h}_2)}}+\sqrt{\frac{A_2+t\hat{A}_2}{D_1(h_1+t\hat{h}_1)}}\right)^2}{\left(\sqrt{\frac{A_1+t\hat{A}_1}{D_2(h_2+t\hat{h}_2)}}+\sqrt{\frac{A_2+t\hat{A}_2}{D_1(h_1+t\hat{h}_1)}}\right)^2+\left(\sqrt{\frac{A_2+t\hat{A}_2}{D_2(h_2+t\hat{h}_2)}}-\sqrt{\frac{A_1+t\hat{A}_1}{D_1(h_1+t\hat{h}_1)}}\right)^2}$$

$$\Rightarrow u \le \frac{1}{1+\left(\frac{\sqrt{\frac{A_2+t\hat{A}_2}{D_2(h_2+t\hat{h}_2)}}-\sqrt{\frac{A_1+t\hat{A}_1}{D_1(h_1+t\hat{h}_1)}}}{\sqrt{\frac{A_1+t\hat{A}_1}{D_2(h_2+t\hat{h}_2)}}+\sqrt{\frac{A_2+t\hat{A}_2}{D_1(h_1+t\hat{h}_1)}}}\right)^2}$$

$$u \le \frac{1}{1+\left(\frac{\sqrt{\frac{2(A_2+t\hat{A}_2)}{D_2(h_2+t\hat{h}_2)}}-\sqrt{\frac{2(A_1+t\hat{A}_1)}{D_1(h_1+t\hat{h}_1)}}}{\sqrt{\frac{2(A_1+t\hat{A}_1)}{D_2(h_2+t\hat{h}_2)}}+\sqrt{\frac{2(A_2+t\hat{A}_2)}{D_1(h_1+t\hat{h}_1)}}}\right)^2}$$

Knowing that $t_1=\sqrt{\frac{2(A_1+t\hat{A}_1)}{D_1(h_1+t\hat{h}_1)}}$ and $t_2=\sqrt{\frac{2(A_2+t\hat{A}_2)}{D_2(h_2+t\hat{h}_2)}}$, this last inequality can be written as $u \le \frac{1}{1+\left(\frac{t_2-t_1}{\sqrt{\frac{2(A_1+t\hat{A}_1)}{D_2(h_2+t\hat{h}_2)}}+\sqrt{\frac{2(A_2+t\hat{A}_2)}{D_1(h_1+t\hat{h}_1)}}}\right)^2}$

$\Rightarrow u \le \frac{1}{1+c\Delta t^2}$ with $c=\left(\frac{1}{\sqrt{\frac{2(A_1+t\hat{A}_1)}{D_2(h_2+t\hat{h}_2)}}+\sqrt{\frac{2(A_2+t\hat{A}_2)}{D_1(h_1+t\hat{h}_1)}}}\right)^2$

$$u=\frac{A+t(m_1+m_2+n_1\alpha A+n_2(1-\alpha)A)}{A_1+A_2+t\hat{A}_1+t\hat{A}_2}$$

$$\Rightarrow u=\frac{r(A_1+A_2)+t(m_1+m_2+n_1\alpha r(A_1+A_2)+n_2(1-\alpha)r(A_1+A_2))}{A_1+A_2+t\hat{A}_1+t\hat{A}_2}$$

$$\Rightarrow r=\frac{u(A_1+A_2+t\hat{A}_1+t\hat{A}_2)-tm_1-tm_2}{(A_1+A_2)(1+tn_1\alpha+tn_2(1-\alpha))}$$

APPENDIX VI

PROOF OF LEMMA 5

Letting $\mathrm{E}_{disjoint} \geq \mathrm{E}_{joint}$ implies

$$\frac{\hat{A}_1 D_1}{Q_1} + \frac{\hat{A}_2 D_2}{Q_2} + \frac{\hat{h}_1 Q_1}{2} + \frac{\hat{h}_2 Q_2}{2} \geq \frac{\hat{A}\mathrm{D}_1}{Q_{1,j}} + \frac{Q_{1,j}(\hat{h}_1 + \hat{h}_2\,{}^{D_2}\!/_{D_1})}{2}$$

with $Q_1 = \sqrt{\dfrac{2(A_1 + t\hat{A}_1)D_1}{(h_1 + t\hat{h}_1)}}$, $Q_2 = \sqrt{\dfrac{2(A_2 + t\hat{A}_2)D_2}{(h_2 + t\hat{h}_2)}}$, and

$Q_{1,j} = \sqrt{\dfrac{2(A + t(m_1 + m_2 + n_1\alpha A + n_2(1-\alpha)A))D_1}{h_1 + h_2\,{}^{D_2}\!/_{D_1} + t\hat{h}_1 + \mathrm{t}\,\hat{h}_2\,{}^{D_2}\!/_{D_1}}}$, so the above inequality becomes

$$\frac{\hat{A}_1 D_1}{\sqrt{\dfrac{2(A_1 + t\hat{A}_1)D_1}{(h_1 + t\hat{h}_1)}}} + \frac{\hat{h}_1}{2}\sqrt{\frac{2(A_1 + t\hat{A}_1)D_1}{(h_1 + t\hat{h}_1)}} + \frac{\hat{A}_2 D_2}{\sqrt{\dfrac{2(A_2 + t\hat{A}_2)D_2}{(h_2 + t\hat{h}_2)}}} + \frac{\hat{h}_2}{2}\sqrt{\frac{2(A_2 + t\hat{A}_2)D_2}{(h_2 + t\hat{h}_2)}}$$

$$\geq \frac{(m_1 + m_2 + n_1\alpha A + n_2(1-\alpha)A)D_1}{\sqrt{\dfrac{2(A + t(m_1 + m_2 + n_1\alpha A + n_2(1-\alpha)A))D_1}{h_1 + h_2\,{}^{D_2}\!/_{D_1} + t\hat{h}_1 + \mathrm{t}\,\hat{h}_2\,{}^{D_2}\!/_{D_1}}}} + \frac{(\hat{h}_1 + \hat{h}_2\,{}^{D_2}\!/_{D_1})}{2}\sqrt{\frac{2(A + t(m_1 + m_2 + n_1\alpha A + n_2(1-\alpha)A))D_1}{h_1 + h_2\,{}^{D_2}\!/_{D_1} + t\hat{h}_1 + \mathrm{t}\,\hat{h}_2\,{}^{D_2}\!/_{D_1}}}$$

$$\Rightarrow \frac{\hat{A}_1}{A_1 + t\hat{A}_1}\frac{(A_1 + t\hat{A}_1)D_1}{\sqrt{\dfrac{2(A_1 + t\hat{A}_1)D_1}{(h_1 + t\hat{h}_1)}}} + \frac{\hat{h}_1}{(h_1 + t\hat{h}_1)}\frac{(h_1 + t\hat{h}_1)}{2}\sqrt{\frac{2(A_1 + t\hat{A}_1)D_1}{(h_1 + t\hat{h}_1)}} + \frac{\hat{A}_2}{A_2 + t\hat{A}_2}\frac{A_2 + t\hat{A}_2 D_2}{\sqrt{\dfrac{2(A_2 + t\hat{A}_2)D_2}{(h_2 + t\hat{h}_2)}}} + \frac{\hat{h}_2}{(h_2 + t\hat{h}_2)}\frac{(h_2 + t\hat{h}_2)}{2}\sqrt{\frac{2(A_2 + t\hat{A}_2)D_2}{(h_2 + t\hat{h}_2)}}$$

$$\geq \frac{(m_1 + m_2)D_1}{\sqrt{\dfrac{2(A + t(m_1 + m_2 + n_1\alpha A + n_2(1-\alpha)A))D_1}{h_1 + h_2\,{}^{D_2}\!/_{D_1} + t\hat{h}_1 + \mathrm{t}\,\hat{h}_2\,{}^{D_2}\!/_{D_1}}}}$$

$$+ \frac{(n_1\alpha + n_2(1-\alpha))A}{A + t(m_1 + m_2 + n_1\alpha A + n_2(1-\alpha)A)}\frac{A + t(m_1 + m_2 + n_1\alpha A + n_2(1-\alpha)A)D_1}{\sqrt{\dfrac{2(A + t(m_1 + m_2 + n_1\alpha A + n_2(1-\alpha)A))D_1}{h_1 + h_2\,{}^{D_2}\!/_{D_1} + t\hat{h}_1 + \mathrm{t}\,\hat{h}_2\,{}^{D_2}\!/_{D_1}}}}$$

$$+ \frac{(\hat{h}_1 + \hat{h}_2\,{}^{D_2}\!/_{D_1})}{h_1 + h_2\,{}^{D_2}\!/_{D_1} + t\hat{h}_1 + \mathrm{t}\,\hat{h}_2\,{}^{D_2}\!/_{D_1}}\frac{h_1 + h_2\,{}^{D_2}\!/_{D_1} + t\hat{h}_1 + \mathrm{t}\,\hat{h}_2\,{}^{D_2}\!/_{D_1}}{2}\sqrt{\frac{2(A + t(m_1 + m_2 + n_1\alpha A + n_2(1-\alpha)A))D_1}{h_1 + h_2\,{}^{D_2}\!/_{D_1} + t\hat{h}_1 + \mathrm{t}\,\hat{h}_2\,{}^{D_2}\!/_{D_1}}}$$

Let $u = \dfrac{A + t(m_1 + m_2 + n_1\alpha A + n_2(1-\alpha)A)}{A_1 + A_2 + t\hat{A}_1 + t\hat{A}_2}$, the above inequality can be written

as

$$(\frac{\hat{A}_1}{A_1+t\hat{A}_1}+\frac{\hat{h}_1}{h_1+t\hat{h}_1})\sqrt{\frac{(A_1+t\hat{A}_1)D_1(h_1+t\hat{h}_1)}{2}}+(\frac{\hat{A}_2}{A_2+t\hat{A}_2}+\frac{\hat{h}_2}{h_2+t\hat{h}_2})\sqrt{\frac{(A_2+t\hat{A}_2)D_2(h_2+t\hat{h}_2)}{2}}$$

$$\geq (m_1+m_2)\sqrt{\frac{h_1D_1+h_2D_2+t(\hat{h}_1D_1+\hat{h}_2D_2)}{2u(A_1+A_2+t\hat{A}_1+t\hat{A}_2)}}$$

$$+\frac{(n_1\alpha+n_2(1-\alpha))(u(A_1+A_2+t\hat{A}_1+t\hat{A}_2)-t(m_1+m_2))}{u(A_1+A_2+t\hat{A}_1+t\hat{A}_2)(1+tn_1\alpha+tn_2(1-\alpha))}\sqrt{\frac{(h_1D_1+h_2D_2+t(\hat{h}_1D_1+\hat{h}_2D_2))(A+t(m_1+m_2+n_1\alpha A+n_2(1-\alpha)A))D_1}{2}}$$

$$+\frac{(\hat{h}_1+\hat{h}_2\,{}^{D_2}\!/_{D_1})}{h_1+h_2\,{}^{D_2}\!/_{D_1}+t\hat{h}_1+\mathrm{t}\,\hat{h}_2\,{}^{D_2}\!/_{D_1}}\sqrt{\frac{(h_1D_1+h_2D_2+t(\hat{h}_1D_1+\hat{h}_2D_2))(A+t(m_1+m_2+n_1\alpha A+n_2(1-\alpha)A))D_1}{2}}$$

$$\Rightarrow (\frac{\hat{A}_1}{A_1+t\hat{A}_1}+\frac{\hat{h}_1}{h_1+t\hat{h}_1})\sqrt{(A_1+t\hat{A}_1)D_1(h_1+t\hat{h}_1)}+(\frac{\hat{A}_2}{A_2+t\hat{A}_2}+\frac{\hat{h}_2}{h_2+t\hat{h}_2})\sqrt{(A_2+t\hat{A}_2)D_2(h_2+t\hat{h}_2)}$$

$$\geq (m_1+m_2)\sqrt{\frac{h_1D_1+h_2D_2+t(\hat{h}_1D_1+\hat{h}_2D_2)}{u(A_1+A_2+t\hat{A}_1+t\hat{A}_2)}}$$

$$+(\frac{(\hat{h}_1+\hat{h}_2\,{}^{D_2}\!/_{D_1})}{h_1+h_2\,{}^{D_2}\!/_{D_1}+t\hat{h}_1+\mathrm{t}\,\hat{h}_2\,{}^{D_2}\!/_{D_1}}+\frac{(n_1\alpha+n_2(1-\alpha))}{1+tn_1\alpha+tn_2(1-\alpha)})\sqrt{u(A_1+A_2+t\hat{A}_1+t\hat{A}_2)(h_1D_1+h_2D_2+t(\hat{h}_1D_1+\hat{h}_2D_2))}$$

$$-\frac{(n_1\alpha+n_2(1-\alpha))(t(m_1+m_2))}{u(A_1+A_2+t\hat{A}_1+t\hat{A}_2)(1+tn_1\alpha+tn_2(1-\alpha))}\sqrt{u(A_1+A_2+t\hat{A}_1+t\hat{A}_2)(h_1D_1+h_2D_2+t(\hat{h}_1D_1+\hat{h}_2D_2))}$$

Let $P_1=\dfrac{\hat{A}_1}{A_1+t\hat{A}_1}+\dfrac{\hat{h}_1}{(h_1+t\hat{h}_1)}$,

$$P_2=\frac{\hat{A}_2}{A_2+t\hat{A}_2}+\frac{\hat{h}_2}{h_2+t\hat{h}_2},$$

$$P_3=\frac{(\hat{h}_1+\hat{h}_2\,{}^{D_2}\!/_{D_1})}{h_1+h_2\,{}^{D_2}\!/_{D_1}+t\hat{h}_1+\mathrm{t}\,\hat{h}_2\,{}^{D_2}\!/_{D_1}}+\frac{(n_1\alpha+n_2(1-\alpha))}{1+tn_1\alpha+tn_2(1-\alpha)}$$

After dividing the two terms in the above inequality by

$P_3\sqrt{(A_1+A_2+t\hat{A}_1+t\hat{A}_2)(h_1D_1+h_2D_2+t(\hat{h}_1D_1+\hat{h}_2D_2))}$, we get

$$\frac{P_1\sqrt{(A_1+t\hat{A}_1)D_1(h_1+t\hat{h}_1)}+P_2\sqrt{(A_2+t\hat{A}_2)D_2(h_2+t\hat{h}_2)}}{P_3\sqrt{(A_1+A_2+t\hat{A}_1+t\hat{A}_2)(h_1D_1+h_2D_2+t(\hat{h}_1D_1+\hat{h}_2D_2))}}\geq\frac{m_1+m_2-t(m_1+m_2)\dfrac{(n_1\alpha+n_2(1-\alpha))}{(1+tn_1\alpha+tn_2(1-\alpha))}}{P_3(A_1+A_2+t\hat{A}_1+t\hat{A}_2)\sqrt{u}}+\sqrt{u}$$

Let $P_4=\dfrac{m_1+m_2-t(m_1+m_2)\dfrac{(n_1\alpha+n_2(1-\alpha))}{(1+tn_1\alpha+tn_2(1-\alpha))}}{(A_1+A_2+t\hat{A}_1+t\hat{A}_2)}$, we get

$$\frac{P_1\sqrt{(A_1+t\hat{A}_1)D_1(h_1+t\hat{h}_1)}+P_2\sqrt{(A_2+t\hat{A}_2)D_2(h_2+t\hat{h}_2)}}{P_3\sqrt{(A_1+A_2+t\hat{A}_1+t\hat{A}_2)(h_1D_1+h_2D_2+t(\hat{h}_1D_1+\hat{h}_2D_2))}}\geq\frac{P_4}{P_3\sqrt{u}}+\sqrt{u}$$

$$\Rightarrow \frac{\sqrt{u} * \left(P_1\sqrt{(A_1+t\hat{A}_1)D_1(h_1+t\hat{h}_1)} + P_2\sqrt{(A_2+t\hat{A}_2)D_2(h_2+t\hat{h}_2)} \right)}{P_3\sqrt{(A_1+t\hat{A}_1)D_1(h_1+t\hat{h}_1)+(A_2+t\hat{A}_2)D_2(h_2+t\hat{h}_2)+(A_1+t\hat{A}_1)D_2(h_2+t\hat{h}_2)+(A_2+t\hat{A}_2)D_1(h_1+t\hat{h}_1)}} \geq \frac{P_4}{P_3} + (\sqrt{u})^2$$

$$\Rightarrow \frac{\sqrt{u} * \left(P_1\sqrt{(A_1+t\hat{A}_1)D_1(h_1+t\hat{h}_1)} + P_2\sqrt{(A_2+t\hat{A}_2)D_2(h_2+t\hat{h}_2)} \right)}{P_3\sqrt{(\sqrt{(A_1+t\hat{A}_1)D_1(h_1+t\hat{h}_1)}+\sqrt{(A_2+t\hat{A}_2)D_2(h_2+t\hat{h}_2)})^2+(\sqrt{(A_2+t\hat{A}_2)D_1(h_1+t\hat{h}_1)}-\sqrt{(A_1+t\hat{A}_1)D_2(h_2+t\hat{h}_2)})^2}} \geq \frac{P_4}{P_3} + (\sqrt{u})^2$$

Knowing that $t_1 = \sqrt{\frac{2(A_1+t\hat{A}_1)}{D_1(h_1+t\hat{h}_1)}}$, and $t_2 = \sqrt{\frac{2(A_2+t\hat{A}_2)}{D_2(h_2+t\hat{h}_2)}}$, we get

$$\sqrt{u}\,\frac{P_1\sqrt{\frac{A_1+t\hat{A}_1}{D_2(h_2+t\hat{h}_2)}}+P_2\sqrt{\frac{A_2+t\hat{A}_2}{D_1(h_1+t\hat{h}_1)}}}{P_3\sqrt{(\sqrt{\frac{A_1+t\hat{A}_1}{D_2(h_2+t\hat{h}_2)}}+\sqrt{\frac{A_2+t\hat{A}_2}{D_1(h_1+t\hat{h}_1)}})^2+\frac{1}{2}\Delta t^2}} - \frac{P_4}{P_3} - (\sqrt{u})^2 \geq 0 \text{ where } \Delta t = t_2 - t_1$$

$$\Rightarrow \sqrt{u}\,\frac{P_1\sqrt{\frac{A_1+t\hat{A}_1}{D_2(h_2+t\hat{h}_2)}}+P_2\sqrt{\frac{A_2+t\hat{A}_2}{D_1(h_1+t\hat{h}_1)}}}{P_3\sqrt{(\sqrt{\frac{A_1+t\hat{A}_1}{D_2(h_2+t\hat{h}_2)}}+\sqrt{\frac{A_2+t\hat{A}_2}{D_1(h_1+t\hat{h}_1)}})^2+(\sqrt{\frac{A_2+t\hat{A}_2}{D_2(h_2+t\hat{h}_2)}}-\sqrt{\frac{A_1+t\hat{A}_1}{D_1(h_1+t\hat{h}_1)}})^2}} \geq \frac{P_4}{P_3} + (\sqrt{u})^2$$

After dividing the numerator and denominator by $D_1D_2(h_1+t\hat{h}_1)(h_2+t\hat{h}_2)$, we get

$$\sqrt{u}\,\frac{P_1\sqrt{\frac{A_1+t\hat{A}_1}{D_2(h_2+t\hat{h}_2)}}+P_2\sqrt{\frac{A_2+t\hat{A}_2}{D_1(h_1+t\hat{h}_1)}}}{P_3\sqrt{(\sqrt{\frac{A_1+t\hat{A}_1}{D_2(h_2+t\hat{h}_2)}}+\sqrt{\frac{A_2+t\hat{A}_2}{D_1(h_1+t\hat{h}_1)}})^2+\frac{1}{2}\Delta t^2}} - \frac{P_4}{P_3} - (\sqrt{u})^2 \geq 0$$ The equation

$$\sqrt{u}\,\frac{P_1\sqrt{\frac{A_1+t\hat{A}_1}{D_2(h_2+t\hat{h}_2)}}+P_2\sqrt{\frac{A_2+t\hat{A}_2}{D_1(h_1+t\hat{h}_1)}}}{P_3\sqrt{(\sqrt{\frac{A_1+t\hat{A}_1}{D_2(h_2+t\hat{h}_2)}}+\sqrt{\frac{A_2+t\hat{A}_2}{D_1(h_1+t\hat{h}_1)}})^2+\frac{1}{2}\Delta t^2}} - \frac{P_4}{P_3} - (\sqrt{u})^2 = 0$$ is a second degree

equation, it has two roots if the discriminant

$$\Delta = \left(\frac{P_1\sqrt{\dfrac{A_1+t\hat{A}_1}{D_2(h_2+t\hat{h}_2)}}+P_2\sqrt{\dfrac{A_2+t\hat{A}_2}{D_1(h_1+t\hat{h}_1)}}}{P_3\sqrt{(\sqrt{\dfrac{A_1+t\hat{A}_1}{D_2(h_2+t\hat{h}_2)}}+\sqrt{\dfrac{A_2+t\hat{A}_2}{D_1(h_1+t\hat{h}_1)}})^2+\dfrac{1}{2}\Delta t^2}} \right)^2 - 4\frac{P_4}{P_3} \text{ is positive.}$$

If we note $b = \dfrac{P_1\sqrt{\dfrac{A_1+t\hat{A}_1}{D_2(h_2+t\hat{h}_2)}}+P_2\sqrt{\dfrac{A_2+t\hat{A}_2}{D_1(h_1+t\hat{h}_1)}}}{P_3\sqrt{(\sqrt{\dfrac{A_1+t\hat{A}_1}{D_2(h_2+t\hat{h}_2)}}+\sqrt{\dfrac{A_2+t\hat{A}_2}{D_1(h_1+t\hat{h}_1)}})^2+\dfrac{1}{2}\Delta t^2}}$ and $c = -\dfrac{P_4}{P_3}$, then

$$\sqrt{u_1} = \frac{-b+\sqrt{\Delta}}{-2} = \frac{-b+\sqrt{b^2+4c}}{-2} \geq 0 \text{ and } \sqrt{u_2} = \frac{-b-\sqrt{\Delta}}{-2} \geq 0$$

So $u_1 = (\dfrac{-b+\sqrt{b^2+4c}}{-2})^2$ and $u_2 = (\dfrac{-b-\sqrt{b^2+4c}}{-2})^2$

$$(\sqrt{u}\,\frac{P_1\sqrt{\dfrac{A_1+t\hat{A}_1}{D_2(h_2+t\hat{h}_2)}}+P_2\sqrt{\dfrac{A_2+t\hat{A}_2}{D_1(h_1+t\hat{h}_1)}}}{P_3\sqrt{(\sqrt{\dfrac{A_1+t\hat{A}_1}{D_2(h_2+t\hat{h}_2)}}+\sqrt{\dfrac{A_2+t\hat{A}_2}{D_1(h_1+t\hat{h}_1)}})^2+\dfrac{1}{2}\Delta t^2}} - \frac{P_4}{P_3} - (\sqrt{u})^2) \text{ increases then decreases}$$

with respect to $\sqrt{u}$, so $(\sqrt{u}\,\dfrac{P_1\sqrt{\dfrac{A_1+t\hat{A}_1}{D_2(h_2+t\hat{h}_2)}}+P_2\sqrt{\dfrac{A_2+t\hat{A}_2}{D_1(h_1+t\hat{h}_1)}}}{P_3\sqrt{(\sqrt{\dfrac{A_1+t\hat{A}_1}{D_2(h_2+t\hat{h}_2)}}+\sqrt{\dfrac{A_2+t\hat{A}_2}{D_1(h_1+t\hat{h}_1)}})^2+\dfrac{1}{2}\Delta t^2}} - \dfrac{P_4}{P_3} - (\sqrt{u})^2)$

is positive for, i.e. for $u_1 \leq u \leq u_2$

$$u = \frac{A+t(m_1+m_2+n_1\alpha A+n_2(1-\alpha)A)}{A_1+A_2+t\hat{A}_1+t\hat{A}_2} = \frac{r(A_1+A_2)(1+tn_1\alpha+tn_2(1-\alpha))+tm_1+tm_2}{A_1+A_2+t\hat{A}_1+t\hat{A}_2}$$

therefore $r = \dfrac{u(A_1+A_2+t\hat{A}_1+t\hat{A}_2)-tm_1-tm_2}{(A_1+A_2)(1+tn_1\alpha+tn_2(1-\alpha))}$.

$u_1 \le u \le u_2$ implies that $r_1 \le r \le r_2$, with $r_1 = \dfrac{u_1(A_1 + A_2 + t\hat{A}_1 + t\hat{A}_2) - tm_1 - tm_2}{(1 + tn_1\alpha + tn_2(1-\alpha))}$ and

$r_2 = \dfrac{u_2(A_1 + A_2 + t\hat{A}_1 + t\hat{A}_2) - tm_1 - tm_2}{(1 + tn_1\alpha + tn_2(1-\alpha))}$

If $\Delta \le 0$, then joint ordering can never have fewer emissions than disjoint ordering.

APPENDIX VII

PROOF OF LEMMA 6

To prove the existence of only two cases where joint replenishment either saves on both costs and emissions or does not save on neither costs nor emissions, we must prove that as t goes to infinity, r^* and r_2 are equal, and r_1 is less than or equal to zero.

Recall that $r_1 = \dfrac{u_1(A_1 + A_2 + t\hat{A}_1 + t\hat{A}_2) - tm_1 - tm_2}{(1 + tn_1\alpha + tn_2(1-\alpha))}$ and

$r_2 = \dfrac{u_2(A_1 + A_2 + t\hat{A}_1 + t\hat{A}_2) - tm_1 - tm_2}{(1 + tn_1\alpha + tn_2(1-\alpha))}$, with

$u_1 = (\dfrac{-b + \sqrt{b^2 + 4c}}{-2})^2$, and $u_2 = (\dfrac{-b - \sqrt{b^2 + 4c}}{-2})^2$.

From Appendix VI

$$b^2 = \frac{\left(\left(\frac{\hat{A}_1}{A_1 + t\hat{A}_1} + \frac{\hat{h}_1}{(h_1 + t\hat{h}_1)}\right)\sqrt{\frac{A_1 + t\hat{A}_1}{D_2(h_2 + t\hat{h}_2)}} + \left(\frac{\hat{A}_2}{A_2 + t\hat{A}_2} + \frac{\hat{h}_2}{h_2 + t\hat{h}_2}\right)\sqrt{\frac{A_2 + t\hat{A}_2}{D_1(h_1 + t\hat{h}_1)}}\right)^2}{\left(\frac{(\hat{h}_1 + \hat{h}_2 \,{}^{D_2}\!/_{D_1})}{h_1 + h_2 \,{}^{D_2}\!/_{D_1} + t\hat{h}_1 + t\hat{h}_2 \,{}^{D_2}\!/_{D_1}} + \frac{(n_1\alpha + n_2(1-\alpha))}{1 + tn_1\alpha + tn_2(1-\alpha)}\right)^2\left(\left(\sqrt{\frac{A_1 + t\hat{A}_1}{D_2(h_2 + t\hat{h}_2)}} + \sqrt{\frac{A_2 + t\hat{A}_2}{D_1(h_1 + t\hat{h}_1)}}\right)^2 + \left(\sqrt{\frac{A_2 + t\hat{A}_2}{D_2(h_2 + t\hat{h}_2)}} - \sqrt{\frac{A_1 + t\hat{A}_1}{D_1(h_1 + t\hat{h}_1)}}\right)^2\right)}$$

It can be easily shown that

$$\lim_{t\to\infty} b^2 = \frac{\frac{4}{t^2}\left(\sqrt{\frac{\hat{A}_1}{D_2\hat{h}_2}} + \sqrt{\frac{\hat{A}_2}{D_1\hat{h}_1}}\right)^2}{\frac{4}{t^2}\left(\left(\sqrt{\frac{\hat{A}_1}{D_2\hat{h}_2}} + \sqrt{\frac{\hat{A}_2}{D_1\hat{h}_1}}\right)^2 + \left(\sqrt{\frac{\hat{A}_2}{D_2\hat{h}_2}} - \sqrt{\frac{\hat{A}_1}{D_1\hat{h}_1}}\right)^2\right)} = \frac{1}{1 + \left(\frac{\sqrt{\frac{\hat{A}_2}{D_2\hat{h}_2}} - \sqrt{\frac{\hat{A}_1}{D_1\hat{h}_1}}}{\sqrt{\frac{\hat{A}_1}{D_2\hat{h}_2}} + \sqrt{\frac{\hat{A}_2}{D_1\hat{h}_1}}}\right)^2}$$

From Appendix VI

$$c = -\frac{P_4}{P_3} = -\frac{(m_1+m_2)\left(1 - t\,\frac{(n_1\alpha + n_2(1-\alpha))}{(1+tn_1\alpha + tn_2(1-\alpha))}\right)}{\left(\frac{(\hat{h}_1 + \hat{h}_2\, {}^{D_2}\!/\!_{D_1})}{h_1 + h_2\, {}^{D_2}\!/\!_{D_1} + t\hat{h}_1 + t\hat{h}_2\, {}^{D_2}\!/\!_{D_1}} + \frac{(n_1\alpha + n_2(1-\alpha))}{1 + tn_1\alpha + tn_2(1-\alpha)}\right)(A_1 + A_2 + t\hat{A}_1 + t\hat{A}_2)}$$

as t goes to infinity, we can verify that $\lim_{t\to\infty} c = \dfrac{(m_1+m_2)(1-1)}{\frac{2}{t}(\hat{A}_1 + \hat{A}_2)t} = 0$. Therefore,

$$\lim_{t\to\infty} u_2 = \left(\frac{-b - \sqrt{b^2}}{-2}\right)^2 = \lim_{t\to\infty} b^2 = \frac{1}{1 + \left(\dfrac{\sqrt{\frac{\hat{A}_2}{D_2\hat{h}_2}} - \sqrt{\frac{\hat{A}_1}{D_1\hat{h}_1}}}{\sqrt{\frac{\hat{A}_1}{D_2\hat{h}_2}} + \sqrt{\frac{\hat{A}_2}{D_1\hat{h}_1}}}\right)^2}$$

, and $\lim_{t\to\infty} u_1 = \left(\dfrac{-b + \sqrt{b^2}}{-2}\right)^2 = 0$

It can be easily shown that

$$\lim_{t\to\infty} r_2 = \frac{\dfrac{\hat{A}_1 + \hat{A}_2}{1 + \left(\dfrac{\sqrt{\frac{\hat{A}_1}{D_1\hat{h}_1}} - \sqrt{\frac{\hat{A}_2}{D_2\hat{h}_2}}}{\sqrt{\frac{\hat{A}_1}{D_2\hat{h}_2}} + \sqrt{\frac{\hat{A}_2}{D_1\hat{h}_1}}}\right)^2} - (m_1 + m_2)}{(n_1 * \alpha + n_2 * (1-\alpha)) * (A_1 + A_2)}$$

and that

$$\lim_{t\to\infty} r_1 = \frac{-(m_1+m_2)}{(n_1 * \alpha + n_2 * (1-\alpha)) * (A_1 + A_2)} \le 0 .$$

As for r^*, from Appendix V

$$r^* = \frac{\dfrac{1}{1+\left(\dfrac{\sqrt{\dfrac{2(A_2+t\hat{A}_2)}{D_2(h_2+t\hat{h}_2)}}-\sqrt{\dfrac{2(A_1+t\hat{A}_1)}{D_1(h_1+t\hat{h}_1)}}}{\sqrt{\dfrac{2(A_1+t\hat{A}_1)}{D_2(h_2+t\hat{h}_2)}}+\sqrt{\dfrac{2(A_2+t\hat{A}_2)}{D_1(h_1+t\hat{h}_1)}}}\right)^2}(A_1+A_2+t\hat{A}_1+t\hat{A}_2)-tm_1-tm_2}{\left(1+tn_1\alpha+tn_2(1-\alpha)\right)}$$

As t goes to infinity, we can easily show that

$$\lim_{t\to\infty} r^* = \frac{\dfrac{\hat{A}_1+\hat{A}_2}{1+\left(\dfrac{\sqrt{\dfrac{\hat{A}_1}{D_1\hat{h}_1}}-\sqrt{\dfrac{\hat{A}_2}{D_2\hat{h}_2}}}{\sqrt{\dfrac{\hat{A}_1}{D_2\hat{h}_2}}+\sqrt{\dfrac{\hat{A}_2}{D_1\hat{h}_1}}}\right)^2}-(m_1+m_2)}{(n_1*\alpha+n_2*(1-\alpha))*(A_1+A_2)} = \lim_{t\to\infty} r_2$$