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A General Solution to Mimetic Dust with Astrophysical and Cosmological Applications

by
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AN ABSTRACT OF THE THESIS OF

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The purpose of this thesis is to investigate solutions for the recently proposed model of dark matter, mimetic dark matter. The model, proposed by Chamseddine and Mukhanov in 2013[15], suggests a reformulation of Einstein's Theory of Gravitation through isolating the conformal degree of freedom in a covariant way. In order to perform that, the physical metric is written in terms of an auxiliary metric and a scalar field appearing through its first derivatives. As a result, the conformal degree of freedom becomes dynamic even in the absence of matter. The extra degree of freedom has been shown to imitate the potential motion of dust. This thesis provides a review of Einstein's Theory of Relativity in addition to astrophysical and cosmological applications. Furthermore, it examines the mimetic dust and provides particular and general solutions for specific behavior of this dust by specifying an arbitrary function, $f(r)$, related to the initial conditions.

The aim of this thesis is to prove that mimetic matter indeed behaves as dust. As expected for regular dust, this mimetic dust will undergo gravitational collapse. As for the cosmological implications, it will be shown that mimetic dust provides a mean of describing a spatially flat, "matter dominated" universe. Finally, further directions of research will be indicated.

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Chapter 1

Motivation

" After the untenability of the theory of action at a distance had thus been proved in the domain of electrodynamic, confidence in the correctness of Newton's action-at-a-distance theory of gravitation was shaken. One had to believe that Newton's Law of Gravity could not embrace the phenomena of gravitation in their entirely, any more than Coulomb's Law of electrostatics and magnetostatics embraced the totality of electromagnetic processes." Albert Einstein, 1913 addressing the Congress after Hertz verified Maxwell's suggestion that the effects of electromagnetism propagate at finite speeds. The first chapter of this thesis will be devoted to briefly describing the Theory of Relativity as well as Dark Matter. In addition, some notes on the proposed Mimetic Dark Matter will be highlighted in order to provoke the curiosity of the reader. Finally, an outline of this thesis will be presented.

1.1 Geometria

Geometry, the measurement of the earth, was and still is integrated in every aspect of the scientific journey of mankind. Since the ancient Greek whom studied planes and solid figures, Geometry has always been an interesting science. From Hippias who studied curves such as quadratix, to Euclid who wrote 13 books on geometry, to Archimedes who studied the spirals, till Nichomedes and the study of the conchoid, the relation between matter and space was the common theme as they all attempted to study points and their relation in space. Descartes, Henry More, Newton with his philosophical foundations upon geometrical principles, as well as Kante, where all interested in geometry. They were recently followed by Gauss, Lobachevsky and Poincaré. However, it was not until Riemann's theory of manifolds which introduced the study of curved space that scientists found new applications to mechanics and other principles. Riemannian Geometry was mastered by several. namely Christoffel, Schur, and Ricci-Curbastro. Even though the theory was adopted by many, the applications of this new non-Euclidean Geometry were not taken seriously until Albert Einstein based his entire Theory of General Relativity on it. It is this theory, in addition to Klein's introduction of groups and invariant transformations of the geometry, as well as Hermann Minkowski modification and connection of space and time, that made the Theory of Relativity possible. By 1905, Einstein had already formulated the relativistic electrodynamics of moving bodies, which was in itself an invariant linear transformation of the Lorentz group. Therefore, it can be seen that Einstein and the world owe it all to geometry, which upon it non-Newtonian physics - Special Relativity and General Relativity afterwards - was built upon.

1.2 The Desire for a Complete Theorem: General relativity

"That the special theory of relativity is only the first step of a necessary development became completely clear to me only in my efforts to represent gravitation in the framework of this theory", A. Einstein (Autobiographical Notes, 1949) [2].

Einstein regarded his Theory of Relativity as based on a simple fundamental idea; the relativity of all motion. That is, one can detect and measure the motion of a body relative to another but cannot assert any meaning to the absolute motion. Before General Relativity, Special Relativity attempted to make use of this idea but was restricted to uniform translatory motions, given that all gravitational effects are neglected. " The laws of physics must be such a nature that they apply to systems of reference in any kind of motion"[1]. The second postulate of Special Relativity was the constancy of speed of light. The speed of light 'c' is constant in all reference frames.

The General Theory of Relativity, however, deals with all kinds of motion and in the presence of any kind of gravitational effects. From here, two postulates have been made: The Principle of Covariance and the Principle of Equivalence. The Principle of Covariance states that the general laws of physics should be expressed independent of the choice of the coordinate system (or else this would assert some meaning to the absolute motion). The equivalence principle states the local equivalence between acceleration and gravity, and requires that a free falling observer must result in a co-moving local Minkowski space. From these two principles, Einstein then proceeded to state the relation of the four co-ordinates to measurement in space and time. He deduced that if a law of nature is expressed by equating all the components of a tensor to zero, it is generally covariant, and hence by examining the laws of the formation of tensors, we acquire the means of formulating generally covariant laws[1]. Mathematically speaking, the expression of the formula for an interval in a covariant language has a simple form:

$$ds^2 = g_{ik} dx^i dx^k \tag{1.1}$$

where the Latin indices i and k run over 0, 1, 2, 3 (while Greek indices run over 1,2 3) according to Einstein's summation convention.

1.3 From the Light to the Dark: Dark Matter

"An era is said to end when all its basic illusions are exhausted", Arthur Miller

The belief that the mass of the Universe was entirely due to that of the stars was undoubted in 1900. 100 years later, however it became well established that stars in addition to cold gases constitute only 1% of the mass of the Universe. The rest of the mass can be roughly estimated by $\sim 3\%$ hot baryons (matter made up of protons and neutrons) , $\sim 30\%$ cold dark matter, and $\sim 66\%$ dark energy. To completely understand this dark matter, physicists utilized different aspects of physics and astronomy. For example, statistical mechanics and thermodynamics were required to explain the creation of this dark matter during the hot expansion of the universe. On the other hand, particle physics was employed to generate possible candidates to this dark matter and to study its interactions with ordinary matter. In order to study dark matter on large scales, and to consider how the Universe could be made a laboratory to study this dark matter, General relativity, astrophysics and cosmology were embedded[3]. One of the first frontiers to suggest the existence of gravitational effects not contributed to baryon and visible matter was the Swiss astronomer Fritz Zwicky[4]. In 1933, Zwicky published one of the first observations concerning this matter where he observed large velocity dispersion of the members of the Coma galaxy cluster[5]. Unfortunately, Zwicky's paper did not get any feedback and interest from the physics society and up until the first half of the Twentieth Century till the late 1970's and 1980's where the cold matter, moving with non-relativistic velocities when structure formed in the Universe, paradigm appeared. "Today, extensive data from large scale galaxies, supernovas, microwave background radiation, enhanced the Standard Model of cosmology, where structure formed through gravitational amplification of small density perturbations with the help of cold dark matter"[5]. Finally, this introduction serves to highlight the importance of Dark matter. There will be a separate section devoted for the most popular candidates of DM.

Imagine not needing to utilize all the branches of physics to explain Dark Matter! What if there exists a way to unveil the darkness of this matter? It is the recent work of Chamseddine and Mukhanov on modifying the gravitational theory that enabled them to "mimic" the effects of "dark matter" without the need of matter! Amazing isn't it.

1.4 Why Read My Thesis: The Apocalypse

In the following chapters to come, I will elaborate more on General Relativity and Dark matter in two separate chapters. Chapter two will be devoted to discuss thoroughly General relativity, focusing on an interesting choice of reference system, the synchronous reference system. In chapter three, I will try to give a some insight on dark matter through listing the proposed candidates so far. In addition, chapter three will include all the information on the new proposed mimetic dark matter, equations and results. However, the applications that this work aims for are both astrophysical and cosmological. In chapter four, I will define the general gravitational collapse along with the accepted cosmological models. In chapter five, I will try to show that results using the modifications and constraints of mimetic dark matter do indeed reproduce the ones obtained previously through the usual gravitational field equations. Finally, some concluding remarks will be presented in chapter six, along with some thoughts on future work to be.

Chapter 2

General Relativity: Briefing

“ Either the well was very deep, or she fell very slowly, for she had plenty of time as she went down to look about her and to wonder what was going to happen next.” Lewis Carroll, Alice in Wonderland.

The aim of this chapter is to briefly restate the main postulates and equations of General Relativity as well along with their consequences. In addition an interesting choice of reference system, called the synchronous reference system, will be discussed in details as it will be the basis of the later chapters.

2.1 The Beginning

Before dwelling into the concepts and equations of General Relativity it would be wise enough to consider the early version of it: the theory of special relativity. Special relativity (SR) is based on the invariance of the speed of light, c, in all reference frames. Speed has units of space and time, so if there exists a constant quantity relating both, then it is obvious that space and time are joined together forming space time with a converging factor fortunately being c (which will be set equal to unity throughout our discussion)[6]. The Cartesian coordinates as given by space-time are chosen to be as follows:

$$\begin{aligned} x^0 &= t \\ x^1 &= x \\ x^2 &= y \\ x^3 &= z \end{aligned} \tag{2.1}$$

The x^μ are coordinates on the manifold in which SR in studied upon, the Minkowski spacetime. Minkowski spacetime is four dimensional, thus vectors are fixed and called four-vectors. The Minkowski metric η_{ik} is defined as:

$$\eta_{ik} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{2.2}$$

The dot product of two vectors in Minkowski space is defined as:

$$A \cdot B = \eta_{ik} A^i B^k = A^0 B^0 - A^1 B^1 - A^2 B^2 - A^3 B^3 \tag{2.3}$$

The space time interval is thus:

$$ds^2 = \eta_{ik} dx^i dx^k = dt^2 - dx^2 - dy^2 - dz^2 \tag{2.4}$$

Finally, one can define the proper time τ when a particle is fixed:

$$ds^2 = d\tau^2 \tag{2.5}$$

2.2 General Relativity: Postulates and Equations

2.2.1 Postulates

Before stating the vital equations of General relativity required for the development of this thesis, a recap of the basic postulates will be presented. The two postulates are brief yet of extreme importance. The first postulate is the principle of general covariance. It states that all physical laws are independent of the choice of a particular coordinate system. In other words, the equations expressing physical laws must be written in terms of tensors[7]. The second postulate is the principle of equivalence. Weinberg formulated this principle is the following statement: "At every space-time point in an arbitrary gravitational field it is possible to choose a "locally inertial coordinate system" such that, within a sufficiently small region of the point in question, the laws of nature take the same form as in unaccelerated Cartesian coordinate systems in the absence of gravitation"[18].

2.2.2 The Metric

"Space acts on matter, telling it how to move. In turn, matter reacts back on space, telling it how to curve" John Wheeler.[9]

The metric g_{ik} is a generalization of the Minkowski metric mentioned earlier. The metric holds in it all the characteristics of the manifold. When speaking about curvatures, one's imagination is limited since everything around us seems to be flat. However, one can make an analogy of the space-matter with a bowling ball resting on a trampoline. As the ball curves the trampoline, it forces all objects around it to be attracted to it. This is how matter curves space time.

2.2.3 Mathematics and Equations

We recall that in the usual Galilean coordinates one finds that the differentials dA_i of a vector A_i also form a vector, whereas the derivatives $\partial A_i/\partial x^k$ of the components of a vector with respect to the coordinates yield a tensor[8]. Unfortunately, this is not the case in curvilinear coordinates since dA_i is not a vector and $\partial A_i/\partial x^k$ is not a tensor. The difference can be made obvious if we conduct a parallel translation of a vector. In the usual Galilean system of coordinates the components of vector do not change. However, in curvilinear coordinates the components of a vector will change under translation. Thus, the usual differentiation cannot be utilized and a "Covariant Differentiation" is needed and defined by:

$$DA^i = dA^i - \delta A^i \tag{2.6}$$

where

$$\delta A^i = -\Gamma^i_{\lambda l} A^k dx^l \tag{2.7}$$

The quantity $\Gamma^i_{\lambda l}$ is called affine connections or Christoffel symbols. Affine connections can be related to the metric tensor (which was th Minkowski tensor η in SR, now it's $g_{\mu\nu}$ in GR) in the following manner:

$$\Gamma^i_{kl} = \frac{1}{2} g^{im} \left(\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m} \right) \tag{2.8}$$

In addition to the stated property of the affine connection, it is used to calculate the Ricci tensor which is responsible for the calculation of the curvature. It is reasonable, after studying how translation along a curved space differs from that of flat space, to discuss the motion in a gravitational field (curved space). The motion of a particle is determined in GR from the principle of least action.

It was realized in SR, however, it can be applied in GR since the difference is embedded in the ds^2 expression. According to the principle of least action:

$$\delta S = -mc\delta \int ds = 0 \tag{2.9}$$

Hence, in GR, that is in the presence of a gravitational field, a particle moves along extremal, called the geodesic line. It is nevertheless not a straight-line and its spatial motion is not uniform nor rectilinear. In comparison to special relativity regarding the free motion of a particle, the same can be applied to GR but using covariant derivatives and affine connections.

$$Du^i = 0 \tag{2.10}$$

and using the definition of covariant derivative, D, we obtain

$$du^i + \Gamma^i_{kl} u^k dx^l = 0$$

Hence, we obtain the desired equation of motion after dividing by ds^2

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{kl} \frac{dx^k}{ds} \frac{dx^l}{ds} = 0 \tag{2.11}$$

If we compare the obtained EOM with the classical one, it might be inferred that the motion of the particle in a gravitational field is governed by the Γ^i_{kl}. On the other hand, the first term d^2x^i/ds^2 might be regarded as the 4-acceleration of the particle.

Let us now define the four-momentum of a particle in a gravitational field

$$p^i = mcu^i \tag{2.12}$$

Squaring both sides and substituting $-\partial S/\partial x^i$ for p_i (where S is the action in eq. 2.9) we find:

$$g^{ik}\frac{\partial S}{\partial x^i}\frac{\partial S}{\partial x^k} - m^2c^2 = 0 \tag{2.13}$$

This is the known Hamilton-Jacobi equation for a particle in a gravitational field. The resemblance of this equation will occur while studying the mimetic dark matter.

As mentioned earlier, the metric contain the information regarding the curvature of the manifold. Since the affine connections are not tensors and thus can be zero or non-zero depending on a mere choice of coordinates, the Riemann curvature tensor contains all the information needed. The Riemann tensor is a four-component tensor and is somehow deduced in the same fashion as the Christoffel's symbols. A parallel transport around a closed loop reveals that there exists a tensor that deduces the curvature. This fourth rank tensor is given in terms of the connection coefficients as:

$$R^i_{klm} = \frac{\partial \Gamma^i_{km}}{\partial x^l} - \frac{\partial \Gamma^i_{kl}}{\partial x^m} + \Gamma^i_{nl}\Gamma^n_{km} - \Gamma^i_{nm}\Gamma^n_{kl} \tag{2.14}$$

There are two important contractions of this tensor which are beneficial, the Ricci Tensor and the Ricci scalar. However, because curvatures are the core of this theory it would be useful if some properties of this Riemann tensor are stated. For example, to explore the symmetry properties, one can change the mixed tensor into a completely covariant one such that:

$$R_{iklm} = g_{in} R^{n}_{klm}$$

$$R_{iklm} = \frac{1}{2}\left(\frac{\partial^2 g_{im}}{\partial x^k \partial x^l} + \frac{\partial^2 g_{kl}}{\partial x^i \partial x^m} - \frac{\partial^2 g_{il}}{\partial x^k \partial x^m} - \frac{\partial^2 g_{km}}{\partial x^i \partial x^l}\right) + g_{np}(\Gamma^{n}_{kl}\Gamma^{p}_{im} - \Gamma^{n}_{km}\Gamma^{p}_{il}) \quad (2.15)$$

It is simple to observe that the tensor is antisymmetric in the index pairs (i, k) and (l, m) while being symmetric under the interchange of the two pairs with each other. In fact, only 20 components out of the 256 are in independent. Thus, we can write some properties:

1- For the antisymmetric:

$$R_{iklm} = R_{kilm} = R_{ikml} \quad (2.16)$$

2- For the symmetric:

$$R_{iklm} = R_{lmik} \quad (2.17)$$

3- The cyclic sum of the components of R_{iklm} is zero:

$$R_{iklm} + R_{imkl} + R_{ilmk} = 0 \quad (2.18)$$

4- The Bianchi identity:

$$\frac{\partial R^n_{ikl}}{\partial x^m} + \frac{\partial R^n_{imk}}{\partial x^l} + \frac{\partial R^n_{ilm}}{\partial x^k} = 0 \tag{2.19}$$

We now turn our sight to the contractions of the Riemann Tensor:

1- The Ricci Tensor; a symmetric tensor

$$R_{ik} = g^{lm} R_{limk} = R^l_{ilk} \tag{2.20}$$

$$R_{ik} = \frac{\partial \Gamma^l_{ik}}{\partial x^l} - \frac{\partial \Gamma^l_{il}}{\partial x^k} + \Gamma^l_{ik}\Gamma^m_{lm} - \Gamma^m_{il}\Gamma^l_{km} = R_{ki} \tag{2.21}$$

2- The Ricci Scalar or scalar curvature:

$$R = g^{ik} R_{ik} = g^{il} g^{km} R_{iklm} \tag{2.22}$$

2.2.4 Einstein's Equations

The final subsection of this section will be devoted to Einstein's equations of the gravitational field. The equations are obtained in a standard fashion by varying the total action, that is, from the principle of least action.

Two actions will be varied. The action S_g related to the gravitational field, and the action S_m related to matter. Hence, the aim is to find $\delta(S_g + S_m) = 0$.

Start by varying the the gravitational action $S_g = \int R\sqrt{-g}d\Omega$.

$$\delta \int R\sqrt{-g}d\Omega = \delta \int g^{ik} R_{ik}\sqrt{-g}d\Omega$$

$$= \int (R_{ik}\sqrt{-g}\delta g^{ik} + R_{ik}g^{ik}\delta\sqrt{-g} + g^{ik}\sqrt{-g}\delta R_{ik})d\Omega$$

Using the fact that

$$dg = gg^{ik}dg_{ik} = -gg_{ik}dg^{ik}$$

so,

$$\delta\sqrt{-g} = -\frac{1}{2\sqrt{-g}} = -\frac{1}{2}\sqrt{-g}g_{ik}\delta g^{ik}$$

Thus,

$$\delta \int R\sqrt{-g}d\Omega = \int (R_{ik} - \frac{1}{2}g_{ik}R)\delta g^{ik}\sqrt{-g}d\Omega + \int g^{ik}\delta R_{ik}\sqrt{-g}d\Omega \qquad (2.23)$$

Using a locally geodesic system of coordinates where the affine connections vanish and the first derivatives of the metric are zero we can obtain:

$$g^{ik}\delta R_{ik} = g^{ik}\{\frac{\partial\delta}{\partial x^l}\Gamma^l_{ik} - \frac{\partial\delta}{\partial x^k}\delta\Gamma^l_{il}\}$$

Now, $\int g^{ik}\delta R_{ik}\sqrt{-g}d\Omega$ can be transformed by Gauss's law into an integral over the hyperspace surrounding the four-volume. Knowing that the variation vanishes at the integration limits, this term drops and hence we are left with:

$$\delta S_g = -\frac{1}{16\pi k}\int (R_{ik} - \frac{1}{2}g_{ik}R)\delta g^{ik}\sqrt{-g}d\Omega \qquad (2.24)$$

The variation of the matter action is straight forward and is given by:

$$\delta S_m = \frac{1}{2}\int T_{ik}\delta g^{ik}\sqrt{-g}d\Omega \qquad (2.25)$$

where T_{ik} is the energy momentum tensor of the matter.

Now that we have defined both actions, from the least action principle applied to $\delta(S_g + S_m) = 0$ we obtain:

$$-\frac{1}{16\pi k}\int (R_{ik} - \frac{1}{2}g_{ik}R - 8\pi k T_{ik})\delta g^{ik}\sqrt{-g}d\Omega = 0 \tag{2.26}$$

$$R_{ik} - \frac{1}{2}g_{ik}R - 8\pi k T_{ik} = 0 \tag{2.27}$$

$$R_{ik} - \frac{1}{2}g_{ik}R = 8\pi k T_{ik} \tag{2.28}$$

Define G_{ik} as the Einstein Tensor such that:

$$G_{ik} = R_{ik} - \frac{1}{2}g_{ik}R \tag{2.29}$$

The Einstein Equation possesses some characteristics:

1- By definition, G_{ik} is a tensor.

2- Assuming that the gravitational field equations are uniform in scale, so that only N=2 derivatives of the metric are allowed, G_{ik} either contains linear second derivatives or quadratic first derivatives of the metric.

3- T_{ik} is symmetric, so must be G_{ik}

4- Since T_{ik} is a conserved quantity, then the covariant differentiation $G^i_{k;m} = 0$

.

2.3 The Synchronous

The arbitrariness of the choice of space-time coordinate system to define the position and time of a body raises the question of how actual distances and time intervals can be determined. We will first relate the proper time, τ, to the time coordinate x^0. Then, we will determine the quantity dl of the spatial distance.

Consider two infinitesimally separated events occurring at the same point in space. The differentials of space will be zero ($dx^1 = dx^2 = dx^3 = 0$) and our interval between the two events will be given by:

$$ds^2 = d\tau^2$$

$$ds^2 = g_{ik}dx^i dx^k = g_{00}(dx^0)^2$$

Hence,

$$d\tau = \sqrt{g_{00}}dx^0 \tag{2.30}$$

So the proper time related to the time between any two events occurring at the same spatial point, that is, relation that determines the actual time of a change of the coordinate x^0, is given by:

$$\tau = \int \sqrt{g_{00}}dx^0 \tag{2.31}$$

The second task is to determine the actual distance. It is accustomed in SR to define dl as the interval separating two infinitesimal events occurring at the same moment. Unfortunately, it is not possible in GR to determine dl by setting $dx^0 = 0$ in ds due to the fact that in the presence of a gravitational field the proper time at different points in space has different dependence in the coordinate x^0.

In order to find dl we will imagine a small hypothetical experiment to find the conditions for obtaining dl. Suppose that a light signals arrives at a point A, with coordinates x^α, from an infinitesimally close point B, with coordinates $x^\alpha + dx^\alpha$, and then goes back The observed time at point B for this round trip is twice the distance between the two points.

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta + 2g_{0\alpha}dx^0dx^\alpha + g_{00}(dx^0)^2 \tag{2.32}$$

Note that we sum over Greek indices from 1 to 3.

Setting $ds^2 = 0$ we obtain two solutions:

$$\begin{aligned} dx^{0(1)} &= \frac{1}{g_{00}}\{-g_{0\alpha}dx^\alpha - \sqrt{(g_{0\alpha}g_{0\beta} - g_{\alpha\beta}g_{00})dx^\alpha dx^\beta}\} \\ dx^{0(2)} &= \frac{1}{g_{00}}\{-g_{0\alpha}dx^\alpha + \sqrt{(g_{0\alpha}g_{0\beta} - g_{\alpha\beta}g_{00})dx^\alpha dx^\beta}\} \end{aligned} \tag{2.33}$$

The solutions correspond to the propagation of light from A to B. The total time interval between departure and arrival is given by $dx^{0(2)} - dx^{0(1)}$ and is equal to:

$$dx^{0(2)} - dx^{0(1)} = \frac{2}{g_{00}}\sqrt{(g_{0\alpha}g_{0\beta} - g_{\alpha\beta}g_{00})dx^\alpha dx^\beta} \tag{2.34}$$

Multiplying by $\sqrt{g_{00}}/2$ we obtain the dl as:

$$dl^2 = (-g_{\alpha\beta} + \frac{g_{0\alpha}g_{0\beta}}{g_{00}})dx^\alpha dx^\beta \tag{2.35}$$

Let

$$\gamma_{\alpha\beta} = (-g_{\alpha\beta} + \frac{g_{0\alpha}g_{0\beta}}{g_{00}}) \tag{2.36}$$

be the three-dimensional metric tensor determining the geometric properties of space. Thus,

$$dl^2 = \gamma_{\alpha\beta} dx^\alpha dx^\beta \tag{2.37}$$

It is unfortunate to say that since metric g_{ik} generally depends on x^0, then the space metric would change with time and hence the integration of dl^2 would be meaningless. It is only in the case where g_{ik} does not depend on time, that we can speak of defined distances.

We will now utilize the experiment considered earlier to check whether it is possible to synchronize clocks located at different points in space. This is known as the concept of simultaneity in the general theory of relativity. The exchange of light signals between two infinitesimally neighboring points serves as a mean to achieve this synchronization. So we go back to points A and B previously discussed and regard that as simultaneous with the moment x^0 at the point A that the reading of the clock at point B which is half way between the moments of departure and return of the signal to that point.

$$x^0 + \Delta x^0 = x^0 + \frac{1}{2}(dx^{0(1)} + dx^{0(2)})$$

Substituting equation (2.33) in the above equation we obtain:

$$\Delta x^0 = -\frac{g_{0\alpha} dx^\alpha}{g_{00}} \equiv g_\alpha dx^\alpha \tag{2.38}$$

The obtained relation enables us to synchronize clocks in any infinitesimal region in space. This synchronization, however, is only possible along any open curve. If we consider synchronization along a closed contour we will obtain $\Delta x^0 \neq 0$ and thus conclude that it is impossible to synchronize clocks over all space. This is only permissible if the components $g_{0\alpha} = 0$. It should be noted that the lack to synchronize all clocks is due to the arbitrariness of the reference frames and not a property of space-time itself. In any gravitational field it is always possible to choose the reference system so that $g_{0\alpha} = 0$ and make it possible to completely synchronize the clocks.

Now that we have established the condition for synchronizing clocks at different points in space, we are ready to define the synchronous reference frame. If a reference frame has its metric components $g_{0\alpha} = 0$ in addition to $g_{00} = 1$, that is, the proper time coincides with the $x^0 = t$ coordinate at each point in space, then it is said to be synchronous.
The interval element will have the form:

$$ds^2 = dt^2 - \gamma_{\alpha\beta} dx^\alpha dx^\beta \tag{2.39}$$

The synchronous reference frame has the following:
1- The time lines are geodesic in the four-space.
2- The four-vector $u^i = dx^i/ds$, which is tangential to the world line $x^\alpha =$ *constant* has components $u^\alpha = 0, u^0 = 1$.
3- u^i satisfies the geodesic equation $\frac{du^i}{ds} + \Gamma^i_{kl} u^k u^l = \Gamma^l_{00} = 0$

To construct a synchronous reference system geometrically in any space-time, choose the starting surface to be any space-like hyper-surface. This hyper-surface will have normals at each point having time-like direction[8]. If we construct the family of geodesic lines normal to this hyper-surface and choose them to be as the time coordinate lines and determine the time coordinate t as the length s of the geodesic line measured from the initial hyper-surface then we obtain a synchronous reference system.

Finally, it is worthy to mention that the transformation to a synchronous reference system can be done using the Hamilton-Jacobi equation (setting the mass to unity)

$$g^{ik}\frac{\partial S}{\partial x^i}\frac{\partial S}{\partial x^k} = 1 \tag{2.40}$$

since we utilize the fact that the trajectories of a particle in a gravitational field are just the geodesic lines.

The matter filling the space cannot in general be at rest relative to the synchronous system because under pressure matter moves along lines that are not geodesic whereas the world-line of a rest particle is a time-line and hence is geodesic in the synchronous reference. Fortunately, an exception arises. When the pressure is zero there exist "dust" particles which move along geodesic lines. From here, the condition for a synchronous reference frame does not contradict the condition that it be co-moving with matter.

We must highlight the fact that a singularity arises from the use of the synchronous reference system. However, since the singularity is not a physical one, that is, it is not a characteristic of space-time, and is merely related to the characteristic of the reference frame, this singularity disappears when we change (by simple means of coordinate transformation) to another non-synchronous reference frame[8].

Chapter 3

Mimetic Dark Matter

It is convenient before defining MDM, to introduce some of the basic aspects of the current models of DM. This would serve as a historical overview as well as a good comparison between the proposed model and the models found in the literature. Hence, this chapter will be divided into two sections, one pertaining to DM and the other to MDM. The DM section will only touch on some characteristics and candidates of DM, while in the MDM section, a detailed discussion of the model will be presented.

3.1 Dark Matter

We have already stated in the introduction that a fairly large amount of the constitutes of the Universe is attributed to DM. This section is devoted to the characteristics of DM as well as its most probable candidates.

3.1.1 Characteristics

It is well established now that dark matter cannot consist of baryons. This is attributed to two main arguments. First, if dark matter did indeed consist of baryonic material, then the cosmic microwave background (CMB) and cosmic web of structure would look radically different[11]. Secondly, the baryon-to-photon ratio of the Universe dictates the amount of line elements created during big-bang nucleosynthesis. In addition, the amount of deuterium and ^4He constrain give similar constraints on the baryon density in the Universe as those coming from the CMB observations. Upon this, a once accepted candidate, the Massive Compact Halo Object (MaCHO) class is cosmologically insignificant. Furthermore, dark matter cannot be made up of sub-kev-particles, that is, light particles. Since light particles are known to be relativistic at early times of the Universe, and hence can 'fly' out of small scale perturbations, then it is only possible that DM consists of light particles if they were created via phase transition in the early Universe (QCD Axions for example).

Dark matter can not possess any electromagnetic characteristics other than neutrality. This is due to the fact that if it did possess small electric or magnetic moment then it would couple to the photon-baryon fluid before recombination. If this recombination was to occur,, it would have altered the sub-degree-scale features of the CMB.

Finally, we address the issue of self-interaction. Since few is known about DM, there is no reason for physicists to exclude the idea that DM particles interact with each other or other new dark particles mediated by a new dark gauge bosons [12].

3.1.2 Candidates

There are several candidates for DM. In this section we are going to list briefly the most popular among them, along with a brief description of each.

1- The weakly-interacting massive particles (WIMPs):

Introduced by Steigman & Turner, the main feature of WIMPs is that they possess interactions near typical weak-force interactions (coupling $\sim 10^-2$) and masses near the weak scale ($\sim 100GeV$). Candidates of WIMPs include the super-symmetric neutralino (the lowest-mass eigenstate of the supersymmetric partners of neutral Standard Model gauge bosons)and the Kaluza-Klein photon [11].

2- Axions:

Postulated by the PecceiQuinn theory in 1977, the Axions emerge out of a solution to the strong-CP problem in particle physics. The axions are predicted to have very low mass ($\sim 10^{-6}eV$) and very low interaction cross sections. Axions have no electric charge, interact minimally with ordinary matter and will transform to and from photons in the presence of a magnetic field.

3- Gravitinos:

Being the supersymmetric partners of gravitons, the gravitino's mass can be of the order of few eV to few TeV. Their masses depends on how the supersymmetry is broken. Considering the light gravitinos as candidates for DM, one must tweak the Standard model of cosmology. For heavier gravitinos, one must assume that they do not interact with the Standard Model, and that they acquire WIMP like behavior, more precisely a superWIMP. Gravitinos are not as favoured as WIMPs as dark-matter candidates because of the difficulty of getting the abundance just right and because they are much harder to detect using conventional methods[19].

4- Sterile Neutrinos:

Also called inert neutrinos, Sterile neutrinos do not interact via any of the fundamental interactions of the Standard Model other than gravity. Regarding DM, sterile neutrinos are thought to be created in the early Universe in different manners. The creation process dictates its effects on smaller-scale structures of the universe. Sterile neutrinos have Yukawa interactions with ordinary leptons and Higgs bosons, where by the Higgs mechanism can mix with active neutrinos and thus have a small decay probability to an active neutrino or a photon. Finally, since no X-ray detection of this decay was observed, in addition to the observations of the small-scale Universe, the simplest model of the sterile neutrino (Dodelson-Widrow neutrinos) is abandoned as a candidate to DM.

3.2 Mimetic Dark Matter

3.2.1 Conformal Transformation and The Scindo

Consider a space-time (M, g_{ik}), where M is a smooth manifold of dimension n. and g_{ik} is a Lorentzian metric on M. If Ω is a smooth, strictly positive function then

$$\tilde{g}_{ik}(x) = \Omega^2(x) g_{ik}(x) \tag{3.1}$$

is a conformal transformation with Ω being the conformal factor.

Now suppose we partition the physical metric g_{ik} of Einstein's theory of gravity into a scalar field ϕ and an auxiliary metric $\tilde{g}_{ik}$ defined by:

$$g_{ik} = (\tilde{g}^{lm}\partial_l\phi\partial_m\phi)\tilde{g}_{ik} \equiv P\tilde{g}_{ik} \tag{3.2}$$

Performing a conformal transformation given by eq. (3.1) of the auxiliary metric $\tilde{g}_{ik}$ yields

$$\begin{aligned} \tilde{g}_{ik} &\rightarrow \Omega^2\tilde{g}_{ik} \\ \Rightarrow \quad \tilde{g}^{ik} &\rightarrow \Omega^{-2}\tilde{g}^{ik} \end{aligned} \tag{3.3}$$

Inserting eq. (3.3) into eq.(3.2) we obtain:

$$g_{ik} = (\tilde{g}^{lm}\partial_l\phi\partial_m\phi)\tilde{g}_{ik}$$

$$\Rightarrow \quad g_{ik} = (\Omega^{-2}\tilde{g}^{lm}\partial_l\phi\partial_m\phi)\Omega^2\tilde{g}_{ik}$$

$$\Rightarrow \quad g_{ik} = (\tilde{g}^{lm}\partial_l\phi\partial_m\phi)\tilde{g}_{ik}$$

So we can see that the physical metric is invariant if we perform a conformal transformation of the auxiliary metric. That is, $g_{ik} \longrightarrow g_{ik}$ when $\tilde{g}_{ik} \longrightarrow \Omega^2\tilde{g}_{ik}$

3.2.2 The Action and Equations of Motion

The fact that we bisected the physical metric into a scalar field and an auxiliary metric does not forbid us from performing the usual variation of the action, that is, applying the principle of least action. As done in section 2.2.4, we define S as the general action constituting of the "gravitational Lagrangian" and the "matter Lagrangian ($\mathcal{L}_m$)". However, the action will be in terms of the physical metric, which in return is in terms of the scalar field ϕ and the auxiliary metric $\tilde{g}_{ik}$:

$$S = -\frac{1}{2}\int d^4x\sqrt{-g(\tilde{g}_{ik},\phi)}\{R(g_{ik}(\tilde{g}_{ik},\phi)) + \mathcal{L}_m\} = \int dx^4\mathcal{L} \tag{3.4}$$

Since g_{ik} is invariant under the conformal transformation $\tilde{g}_{ik} \longrightarrow \Omega^2\tilde{g}_{ik}$ and since the above action is solely a function of the g_{ik} then the action is in itself invariant under the transformation.

Now variation of the action, δS

$$\delta S = \int dx^4\frac{\delta\mathcal{L}}{\delta g_{ik}}\delta g_{ik}$$

$$\delta S = -\frac{1}{2}\int d^4x\delta\{\sqrt{-g(\tilde{g}_{ik},\phi)}(R(g_{ik}(\tilde{g}_{ik},\phi)) + \mathcal{L}_m)\}$$

The variation $\delta R_{ik}\sqrt{-g}$ vanishes as seen in section 2.2.4 and the remaining part will produce the same equations. That is,

$$\begin{aligned}\delta S = \int dx^4\frac{\delta\mathcal{L}}{\delta g_{ik}}\delta g_{ik} &= -\frac{1}{2}\int d^4x\sqrt{-g}\{R^{ik} - \frac{1}{2}g^{ik}R - T^{ik}\}\delta g_{ik}\\ &= -\frac{1}{2}\int d^4x\sqrt{-g}\{G^{ik} - T^{ik}\}\delta g_{ik} \end{aligned}\tag{3.5}$$

where G^{ik} is the Einstein tensor, and T^{ik} is the energy momentum tensor for the matter.

One realizes that there exists a difference between this variation and the standard one done previously. The variation of the physical metric δg_{ik} now depends on the variation of the auxiliary metric $\delta\tilde{g}_{ik}$ and that of the scalar field $\delta\phi$. Using eq. (3.2) $g_{ik} = (\tilde{g}^{lm}\partial_l\phi\partial_m\phi)\tilde{g}_{ik} \equiv P\tilde{g}_{ik}$ the variation of the metric becomes:

$$\delta g_{ik} = P\delta\tilde{g}_{ik} + \tilde{g}_{ik}\delta P$$

$$\begin{aligned}
\delta P &= \delta\{(\tilde{g}^{lm}\partial_l\phi\partial_m\phi)\} \\
&= \delta\tilde{g}^{ik}\partial_l\phi\partial_m\phi + \tilde{g}^{ik}\delta(\partial_l\phi)\partial_m\phi + \tilde{g}^{ik}\delta(\partial_m\phi)\partial_l\phi \\
&= \delta\tilde{g}^{ik}\partial_l\phi\partial_m\phi + 2\tilde{g}^{ik}\partial_l\delta\phi\partial_m\phi
\end{aligned}$$

Since the auxiliary metric processes the same "mathematical" properties as the physical metric, that is, it is a tensor, we can use the lowering and raising ($\tilde{g}_{ik}$ & $\tilde{g}^{ik}$) operators as we did before, then we can write:

$$\Rightarrow \quad \delta P = -\tilde{g}^{mi}\tilde{g}^{nk}\delta\tilde{g}_{ik}\partial_m\phi\partial_n\phi + 2\tilde{g}^{mn}\partial_m\delta\phi\partial_n\phi$$

$$\Rightarrow \quad \delta g_{ik} = P\delta\tilde{g}_{mn}(\delta_i^m\delta_k^n - g_{ik}g^{jm}g^{ln}\partial_j\phi\partial_l\phi) + 2g_{ik}\tilde{g}^{jl}\partial_j\delta\phi\partial_l\phi \tag{3.6}$$

Substituting eq.(3.6) into the action in eq.(3.5) we obtain the action as :

$$\begin{aligned}
\delta S = -\frac{1}{2}\int d^4x\sqrt{-g}(G^{ik} - T^{ik}) \\
\times P\delta\tilde{g}_{mn}(\delta_i^m\delta_k^n - g_{ik}g^{jm}g^{ln}\partial_j\phi\partial_l\phi) + 2g_{ik}\tilde{g}^{jl}\partial_j\delta\phi\partial_l\phi\}
\end{aligned} \tag{3.7}$$

Putting the principle of least action to good use we obtain the equations of motion as such:

$$(G^{ik} - T^{ik}) - (G - T)g^{im}g^{kn}\partial_m\phi\partial_n\phi = 0 \tag{3.8}$$

$$\frac{1}{\sqrt{-g}}\partial_j(\sqrt{-g}(G - T)g^{jl}\partial_l\phi) = \nabla_j((G - T)\partial^j\phi) = 0 \tag{3.9}$$

The ∇_j is the total derivative, or in tensor language, the covariant derivative with respect to the metric g_{ik}. One can observe that the auxiliary metric serves as a means to an end, for it does not appear explicitly in the EOM. It is only through the physical metric that the auxiliary metric appear implicitly. On the other hand, the scalar field ϕ happens to appear explicitly and hence plays an important role in our discussion.
The contravariant partner of eq.(3.2)can be written as $g^{ik} = P^{-1}\tilde{g}^{ik}$ and hence the scalar field satisfies the constraint equation:

$$g^{ik}\partial_i\phi\partial_k\phi = 1 \tag{3.10}$$

Examining equation (3.10), it overlaps with equation (2.13) described in chapter two, which is the Hamilton-Jacobi equation with the mass set to unity. Thus, if the action S is identified with ϕ then the field satisfies the H-J equation for a unit mass relativistic particle in a gravitational field.

If we take the trace of eq. (3.8)we obtain:

$$(G - T)(1 - g^{ik}\partial_i\phi\partial_k\phi) = 0 \tag{3.11}$$

Due to constraint equation (3.10) this equation is satisfied even when $G \neq T$. In addition the trace $G - T$ is determined by eq. (3.9) and eq.(3.10) and even in the absence of matter,that is, T^{ik} the equations for the gravitational field have nontrivial solutions for the conformal mode. Once the H-J equation is solved for ϕ then equation (3.9) determines $G - T$.

Einstein's gravity is the unique low energy effective theory of spin-2 massless field[13]. That is, the gravitational force is mediated by a massless spin-2 particle (with helicity =2). Massless spin 2 particles such as gravitons have two transverse degrees of freedom (with 3 degrees of freedom taken away due to their zero mass, due to gauge invariance). However, we can see now that in addition to the two transverse degrees of freedom, the gravitational field acquires an extra longitudinal degree of freedom shared by the scalar field ϕ and a conformal factor of the physical metric, yet is subjected to a constrain due to conformal invariance.

3.2.3 The Extra Degree of Freedom

In order to analyze the extra degree of freedom discussed in section 3.2.2, we rewrite the equation of motion using the following:

$$G^{ik} = T^{ik} + \tilde{T}^{ik} \tag{3.12}$$

where

$$\tilde{T}^{ik} = (G - T) g^{im} g^{kn} \partial_m \phi \partial_n \phi \tag{3.13}$$

The energy-momentum tensor for a perfect fluid, that is, a fluid that is isotropic in its rest frame, has the form:

$$
\begin{aligned}
T_{ik} &= (\varepsilon + p)u_i u_k + p g_{ik} \\
g^{ik} u_i u_k &= -1 \\
\Rightarrow \quad T^{ik} &= (\varepsilon + p)U^i U^k - p g^{ik}
\end{aligned}
$$

where ε is the energy density, p is the pressure and u^i is the 4-velocity. If we set $p = 0$ and make the following identifications:

$$
\varepsilon \equiv G - T, \quad u^i = g^{im}\partial_m \phi \tag{3.14}
$$

thus the two energy momentum tensors T^{ik} and $\tilde{T}^{ik}$ overlap.

Hence, the extra degree of freedom can imitate the potential motion of "dust" with energy $\varepsilon \equiv G - T$ and the scalar field plays the role of the velocity potential. In the absence of matter, that is, when $T = 0$,

$$
\begin{aligned}
G &= g^{ik} G_{ik} \\
&= g^{ik} R_{ik} - \frac{g^{ik} g_{ik}}{2} R \\
&= R - 2R = -R
\end{aligned}
$$

and which does not vanish for generic solutions.

Now the normalization of the velocity for vector is as stated $u^i u_k = 1$. If we apply this for the defined $u^i = g^{im}\partial_m \phi$, we obtain the same scalar field equation (3.10). Finally, the conservation law fot $\tilde{T}^{ik}$ gives

$$
\nabla_i \tilde{T}^i_k = 0
$$

$$\Rightarrow \quad \partial_k\phi\nabla_i((G-T)\partial^i\phi) + (G-T)\partial^i\nabla_i\partial^k\phi = 0 \tag{3.15}$$

Differentiating eq.(3.10) we obtain, $\nabla_k(g^{ik}\partial_i\phi\partial_k\phi) = 0$ and hence

$$\partial^i\phi\nabla_k\partial_i\phi = 0$$

$$\nabla_k\partial_i\phi = \nabla_i\partial_k\phi$$

and thus the conservation law for $\tilde{T}^{ik}$ leads to equation (3.9).

3.2.4 Finding a Solution

It was found suitable to work in the synchronous reference system described in Section 2.3 in order to find an explicit solution of the equation. Recalling that the metric takes the form

$$ds^2 = dt^2 - \gamma_{\alpha\beta}dx^\alpha dx^\beta$$

with $\gamma_{\alpha\beta}$ being the spatial metric. In addition taking the hypersurfaces of constant time to be the same as the hypersurface of constant ϕ, that is, $\phi(x^i) \equiv \tau$, we find that equation (3.10) is satisfied.

Moreover, equation (3.9) becomes:

$$\partial_0(\sqrt{det\gamma}(G-T)) = 0 \tag{3.16}$$

where $det\gamma$ is the determinant of the spatial metric $\gamma_{\alpha\beta}$

. Thus if we integrate once we obtain:

$$G - T = \frac{C(x^\mu)}{\sqrt{det\gamma}} \tag{3.17}$$

where $C(x^\mu)$ is a constant of integration depending on spatial coordinates.

Finally, if we take a particular case, the Friedman Universe, where

$$\gamma_{\alpha\beta} = a^2(\tau)\delta_{\alpha\beta} \tag{3.18}$$

where $\varepsilon = \frac{C}{a^3}$.

So the proposed model predicts dark matter without dark matter, which is imitated by an extra degree of freedom of the gravitational field. With respect to the gravitational interaction, this new mimetic dark matter behaves precisely in the same way as the usual dark matter (in particular, it is influenced by the gravitational instability), but it does not participate in any other interaction besides the gravitational one. The amount of this mimetic dark matter is determined by the constant of integration $C(x^\mu)$.

Chapter 4

Some Astrophysical and Cosmological Applications

In this chapter, some applications to astrophysics and cosmology will be stated. From the astrophysical point of view, gravitational collapse associated with mimetic dust will be briefly explained as well as for a general collapse. As for the cosmological applications, we will discuss the Friedmann-Robertson-Walker metric, mainly in flat space (Einstein-de Sitter), in addition to some cosmological models, namely the matter dominated universe.

4.1 Singularities and Gravitational Collapse

It was due to GR that physicists could attempt a quantitative discussion of the cosmological problem. Although Einstein did not obtain satisfying results after applying his gravitational field equations to study the homogeneity and isotropy of the universe, he proposed to modify them by adding a constant called, the cosmological constant. However, Alexander Friedmann, retrieved time-dependent cosmological solutions to the initial field equations. These solutions described an expanding, or contracting universe. The Friedmann solution has a distinct feature: it predicts a geometrical singularity in the past. This means that the expansion started in an explosion-like manner at a certain moment where the three-dimensional space was just a point. The Friedmann singularity might be regarded as a cosmological one. On the other hand, there exist non-cosmological singularities in the solutions representing limited physical systems.

A singularity can be simply defined as a point where the laws of physics (as we know them till now) break down. For this reason, we will consider the simplest way to arrive to a non-cosmological singularity. The fate of an isolated star is a clear example of a singularity. First proposed by Oppenheimer and Snyder, the fate of a sufficiently heavy star when it has consumed all its thermonuclear sources of energy is a collapse. This is due to the fact that a state of equilibrium cannot exist if the star's mass is sufficiently large and will hence contract under its own gravitational field till the matter reaches the center. Upon reaching the center, a gravitational collapse will occur leading to the formation of a geometrical singularity. It is labeled as a geometrical singularity since the curvature is infinite and the metric degenerate. Upon this, it would be impossible to determine the Christoffel's symbols and all the quantities that follow. Hence, we are not capable of formulating a mathematical description for the equations of motion at the singularity.

The difference between this singularity and the cosmological singularity is that in the latter, the whole universe shrinks into one point, while in the former there is a singular half-worldline of the 4-D space where all points outside this half-world line behave perfectly regular. A detailed discussion will be as follows. Consider a spherically symmetric star (or sphere of dust). Outside the sphere nothing strange happens and we have the usual Schwarzschild solution for vacuum. After the collapse, the Schwarzschild solution still holds for $r > r_g$ where r_g is the Schwarzschild radius. However, approaching this radius, the situation changes. Inside this sphere, the light cones are inclined inwards towards the singularity, and therefore anything arriving at the sphere will fall into the singularity. This sphere has been denoted by the term horizon, more precisely the event horizon. Consequently, the collapsing matter, star, dust, or mimetic matter, being inside the sphere will be unable to send any signals outwards and will appear *black* to an outside observe . From here, the term *black holes* has been proposed to stress the fact that anything can cross the horizon inwards but nothing can escape it.[16]

4.2 Cosmology

4.2.1 The Copernican Principle

The Copernican principle states that the Universe is no special entity and is the same everywhere. It is related to two basic mathematical properties a manifold might possess: isotropy and homogeneity. Isotropy applies at some specific point in space where the universe is the same in all directions. Homogeneity on the other hand, refers to the fact that the metric is the same throughout the space, that is, the curvature of any two points at a given time t is the same[6]. Upon observing distant galaxies however, they appear to be moving away from us. This indicates that the Universe is not static but changing with time. Hence, a construction of cosmological model should be based on the assumptions of homogeneity and isotropy of space but not of time. In GR language, this means that we can slice the Universe in space-like manner such that each slice is homogeneous and isotropic. Upon this, we consider our spacetime to be $\boldsymbol{R} \times \Sigma$ $\boldsymbol{R}$ denoting the time direction and Σ denoting the homogeneous and isotropic three-manifold. Therefore, we can consider our metric to be of the form:

$$ds^2 = dt^2 - a^2(t)\gamma_{\alpha\beta}(u)du^\alpha du^\beta \tag{4.1}$$

where t is the time-like coordinate, and u^α are the coordinates on Σ. In addition, $\gamma_{\alpha\beta}$ is the maximally symmetric metric on Σ. The function $a(t)$ is known as the scale factor, and it tells us the relative sizes of the spatial surfaces at a moment t. Finally we note that the metric is in comoving frame, the synchronous reference system. Only a comoving observer will think that the universe looks isotropic; in fact on Earth we are not quite comoving, and as a result we see a dipole anisotropy in the cosmic microwave background as a result of the conventional Doppler effect[6].

4.2.2 FRW Metric and Cosmological Models

As shown in the above section, the general metric for describing the universe can be considered to be of the form (4.1). After solving for $\gamma_{\alpha\beta}$, the maximally symmetric tensor, we obtain the following metric:

$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\big(d\theta^2 + sin^2\theta d\phi^2\big)\right] \tag{4.2}$$

where the constant k can be -1, 0, or $+1$.

The case for $k = -1$ is known as an ***open*** universe, in which the preferred three-surfaces are "three-hyperboloids" ; $k = 0$ is a ***flat*** universe, in which the preferred three-surfaces are flat space; and $k = +1$ is a ***closed*** universe, in which the preferred three-surfaces are three-spheres.

The FRW metric is the only possible homogeneous isotropic metric, so it is left to calculate the scale factor through Einstein's equations. However, using the simplest case, vacuum, would result in the Minkowski space. Hence, we introduce some energy and momentum to extract some information through choosing a perfect fluid with pressure p and density ρ. The end result would be the well-known equations: the Friedmann equations.

$$\begin{aligned} \left(\frac{\dot{a}}{a}\right)^2 &= \frac{8\pi G}{3}\rho - \frac{k}{a^2} \\ \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}\left(\rho + 3p\right) \end{aligned} \tag{4.3}$$

Since the Friedmann equations govern the evolution of RW metrics, one often speaks of Friedman-Robertson-Walker (FRW) cosmology.

The expansion rate of the universe is measured by the Hubble parameter:

$$H \equiv \frac{\dot{a}}{a} \tag{4.4}$$

and the change of this quantity with time is parameterized by the deceleration parameter:

$$q \equiv -\frac{\ddot{a}a}{\dot{a}^2} = -\left(1 + \frac{\dot{H}}{H^2}\right) \tag{4.5}$$

In order to predict the evolution of the Universe, we need a relation between the density ρ and the pressure p. This relationship is known as the equation of state. We will consider two cosmological models (for flat space, for which, $k = 0$):[17]

1- Matter dominated universe.

2- Radiation dominated universe.

$1-$ *Matter dominated* : Here matter refers to type a of material that exerts negligible pressure, that is, dust. Starting with the fluid equation:

$$\begin{aligned} \dot{\rho} + 3\frac{\dot{a}}{a}\rho &= 0 \\ \Rightarrow \quad \frac{d}{dt}(\rho a^3) &= 0 \end{aligned} \tag{4.6}$$

ρa^3 is a constant in time, thus

In a matter dominated universe, the energy density decreases as the volume increases, so

$$\rho \propto \frac{1}{a^3} \tag{4.7}$$

and we can write:

$$a(t) \propto t^{\frac{2}{3}} \tag{4.8}$$

In this solution, the Universe expands forever, but the rate of expansion $H(t)$ decreases with time

$$H \equiv \frac{\dot{a}}{a} = \frac{2}{3t} \tag{4.9}$$

$2-$ *Radiation dominated* : Particles of light move, at the speed of light, hence, their kinetic energy leads to a pressure force. This force is called the

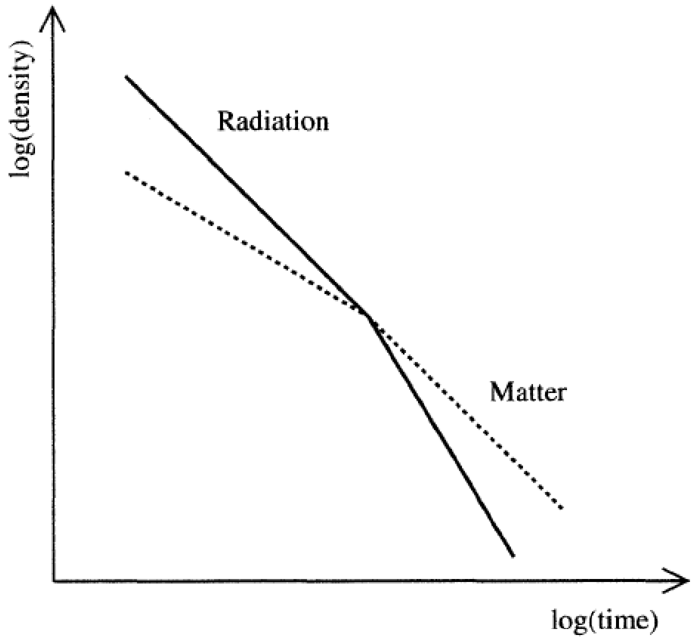

Figure 4.1: A schematic illustration of the evolution of a Universe containing radiation and matter. Once matter comes to dominate the expansion rate speeds up, so the densities fall more quickly with time[17].

radiation pressure, which using the standard theory of radiation can be shown to be $p = \frac{1}{3}\rho$.

by the same means and calculations done in the matter dominated part, we get:

$$\rho \propto \frac{1}{a^4}$$

$$\text{and consequently:} \tag{4.10}$$

$$a(t) \propto t^{\frac{1}{2}}$$

Finally, i n a flat universe, the density is equal to the critical density, given by:

$$\rho_{\text{crtical}} = \frac{3H^2}{8\pi G} \tag{4.11}$$

Chapter 5

Solutions to Mimetic Dust

We saw in chapter three, that MDM possesses an extra degree of freedom which can imitate the potential motion of dust. In addition, it was found suitable to work in the synchronous reference frame. Based on this, this chapter will be devoted first to a treatment of out equations in the synchronous frame and second to the examination of the solution obtained as an application to astrophysics and cosmology.

5.1 The Synchronous System

5.1.1 The Metric

We have seen earlier that the conditions for obtaining the synchronous reference system are:

$$g_{0\alpha} = 0 \qquad ; \qquad g_{00} = 1 \tag{5.1}$$

such that the interval will be:

$$ds^2 = (dx^0)^2 - \gamma_{\alpha\beta} dx^\alpha dx^\beta \tag{5.2}$$

In addition, we will assume spherically central symmetry such that our interval will be:

$$ds^2 = dt^2 - e^{\lambda(t,r)} dr^2 - e^{\mu(t,r)} (d\theta^2 + \sin^2\theta d\phi^2) \tag{5.3}$$

$\Rightarrow$ our metric will have the form:

$$g_{ik} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -e^{\lambda(t,r)} & 0 & 0 \\ 0 & 0 & -e^{\mu(t,r)} & 0 \\ 0 & 0 & 0 & -e^{\mu(t,r)} \sin^2\theta \end{bmatrix} \tag{5.4}$$

and in contravariant notation:

$$g_{ik} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -e^{-\lambda(t,r)} & 0 & 0 \\ 0 & 0 & -e^{-\mu(t,r)} & 0 \\ 0 & 0 & 0 & -e^{-\mu(t,r)} \csc^2\theta \end{bmatrix} \tag{5.5}$$

5.1.2 The Christoffel Symbols

Recall that the Christoffel Symbols (Affine Connections) play a major role in determining the curvature of the manifold. Hence, we calculate them for the given metric above using the following equation:

$$\Gamma^i_{kl} = \frac{1}{2} g^{im} \left(\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m} \right)$$

$$\Gamma^0_{00} = 0 \quad ; \quad \Gamma^1_{00} = 0 \quad ; \quad \Gamma^2_{00} = 0 \quad ; \quad \Gamma^3_{00} = 0$$

$$\Gamma^0_{10} = 0 \quad ; \quad \Gamma^1_{01} = \frac{\dot{\lambda}}{2} \quad ; \quad \Gamma^2_{01} = 0 \quad ; \quad \Gamma^3_{01} = 0$$

$$\Gamma^0_{20} = 0 \quad ; \quad \Gamma^1_{02} = 0 \quad ; \quad \Gamma^2_{02} = \frac{\dot{\mu}}{2} \quad ; \quad \Gamma^3_{02} = 0$$

$$\Gamma^0_{30} = 0 \quad ; \quad \Gamma^1_{03} = 0 \quad ; \quad \Gamma^2_{03} = 0 \quad ; \quad \Gamma^3_{03} = \frac{\dot{\mu}}{2}$$

$$\Gamma^0_{11} = \quad ; \quad \Gamma^1_{11} = \frac{\lambda'}{2} \quad ; \quad \Gamma^2_{11} = 0 \quad ; \quad \Gamma^3_{11} = 0$$

$$\Gamma^0_{12} = 0 \quad ; \quad \Gamma^1_{12} = 0 \quad ; \quad \Gamma^2_{12} = \frac{\mu'}{2} \quad ; \quad \Gamma^3_{12} = 0$$

$$\Gamma^0_{13} = 0 \quad ; \quad \Gamma^1_{13} = 0 \quad ; \quad \Gamma^2_{13} = 0 \quad ; \quad \Gamma^3_{13} = \frac{\mu'}{2}$$

$$\Gamma^0_{22} = \frac{\dot{\mu}}{2} e^{\mu} \quad ; \quad \Gamma^1_{22} = -\frac{\mu'}{2} e^{\mu - \lambda} \quad ; \quad \Gamma^2_{22} = 0 \quad ; \quad \Gamma^3_{22} = 0$$

$$\Gamma^0_{23} = 0 \quad ; \quad \Gamma^1_{23} = 0 \quad ; \quad \Gamma^2_{23} = 0 \quad ; \quad \Gamma^3_{23} = \cot\theta$$

$$\Gamma^0_{33} = \frac{\dot{\mu}}{2} e^{\mu} \sin^2\theta; \quad \Gamma^1_{33} = -\frac{\mu'}{2} e^{\mu-\lambda} \sin^2\theta \quad ; \quad \Gamma^2_{33} = -\sin\theta\cos\theta \quad ; \quad \Gamma^3_{33} = 0$$

Note: Throughout the calculations, dots will represent partial derivatives with respect to t, while accent (primes) represents partial derivatives with respect to r.

5.1.3 Ricci Tensor

The Ricci tensor defined in chapter two tends to determine the curvature of space-time. It is given by equation (2.21)

$$R_{ik} = \frac{\partial \Gamma^l_{ik}}{\partial x^l} - \frac{\partial \Gamma^l_{il}}{\partial x^k} + \Gamma^l_{ik}\Gamma^m_{lm} - \Gamma^m_{il}\Gamma^l_{km} = R_{ki} \tag{5.6}$$

The Ricci tensor was calculated and found to be:

$$\begin{aligned} R_{00} &= \frac{1}{4}\{-\dot{\lambda}^2 - 2(\dot{\mu}^2 + \ddot{\lambda} + 2\ddot{\mu})\} \\ R_{00} &= -\frac{1}{2}(\ddot{\lambda} + \dot{\lambda}^2) - (\ddot{\mu} + \frac{1}{2}\dot{\mu}^2) \end{aligned} \tag{5.7}$$

$$R_{01} = \frac{1}{2}\{\mu'(\dot{\lambda} - \dot{\mu}) - 2\dot{\mu}'\} \tag{5.8}$$

$$R_{11} = \frac{1}{4}\{2(\lambda'\mu' - \mu'^2 - 2\mu'' + 2\dot{\lambda}\dot{\mu}e^{\lambda}) + e^{\lambda}(\dot{\lambda}^2 + \ddot{\lambda})\} \tag{5.9}$$

$$R_{22} = \frac{1}{4}e^{-\lambda}\{4e^{\lambda} + e^{\mu}\lambda'\mu' - 2e^{\mu}\mu'^2 - 2e^{\mu}\mu'' +$$
$$e^{\lambda+\mu}\dot{\lambda}\dot{\mu} + 2e^{\lambda+\mu}\dot{\mu}^2 + 2e^{\lambda+\mu}\ddot{\mu}\} \quad (5.10)$$

$$R_{33} = \frac{1}{4}e^{-\lambda}\sin^2\theta\{4e^{\lambda} + e^{\mu}\lambda'\mu' - 2e^{\mu}\mu'^2 - 2e^{\mu}\mu'' +$$
$$e^{\lambda+\mu}\dot{\lambda}\dot{\mu} + 2e^{\lambda+\mu}\dot{\mu}^2 + 2e^{\lambda+\mu}\ddot{\mu}\} \quad (5.11)$$

Finally, we calculate the Ricci Scalar defined by:

$$R = g^{ik}R_{ik} = g^{il}g^{km}R_{iklm} \quad (5.12)$$

$$R = \frac{1}{2}\{-4e^{-\mu} - 2e^{-\lambda}\lambda'\mu' + 3e^{-\lambda}\mu'^2 + 4e^{-\lambda}\mu'' - \dot{\lambda}^2 -$$
$$2\dot{\lambda}\dot{\mu} - 3\dot{\mu}^2 - 2\ddot{\lambda} - 4\ddot{\mu}\} \quad (5.13)$$

5.1.4 Einstein's Equations

Einstein's equations are calculated using equations (2.29) and are given by:

$$G_{ik} = R_{ik} - \frac{1}{2}g_{ik}R$$

$$G_{00} = \frac{1}{4}\{4e^{-\lambda} + 2e^{-\lambda}\lambda'\mu' - 3e^{-\lambda}\mu'^2 - 4e^{-\lambda}\mu'' + 2\dot{\lambda}\dot{\mu} + \dot{\mu}^2\} \tag{5.14}$$

$$G_{10} = \frac{1}{2}\{\mu'(\dot{\lambda} - \dot{\mu}) - 2\dot{\mu}'\} \tag{5.15}$$

$$G_{11} = \frac{1}{4}\{\mu'^2 - e^{\lambda-\mu}(4 + 3e^{\mu}\dot{\mu}^2 + 4e^{\mu}\ddot{\mu})\} \tag{5.16}$$

$$G_{22} = -\frac{1}{4}e^{\mu-\lambda}\{\lambda'\mu' - \mu'^2 - 2\mu'' + e^{\lambda}\dot{\lambda}^2 + e^{\lambda}\dot{\lambda}\dot{\mu} + e^{\lambda}\dot{\mu}^2 + 2e^{\lambda}\ddot{\lambda} + 2e^{\lambda}\ddot{\mu}\} \tag{5.17}$$

$$G_{33} = -\frac{1}{4}\sin^2\theta e^{\mu-\lambda}\{\lambda'\mu' - \mu'^2 - 2\mu'' + e^{\lambda}\dot{\lambda}^2 + e^{\lambda}\dot{\lambda}\dot{\mu} + e^{\lambda}\dot{\mu}^2 + 2e^{\lambda}\ddot{\lambda} + 2e^{\lambda}\ddot{\mu}\} \quad (5.18)$$

5.2 The Dust

We have seen in section 3.2.3 that the extra degree of freedom may imitate the potential motion of dust. Dust is a special case of a perfect fluid which has zero pressure. Hence, the energy-momentum tensor has the form:

$$T_{ik} = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{5.19}$$

We know that the Einstein's equations are related to energy momentum tensor by:

$$G_{ik} = 8\pi T_{ik}$$

Hence,

$$\begin{aligned} G_{11} = T_{11} &= 0 \\ G_{22} = G_{33} = T_{22} &= 0 \\ G_{10} = T_{10} &= 0 \\ G_{00} = T_{00} &= \rho \end{aligned} \tag{5.20}$$

One would solve these equations and obtain solutions for the metric coefficients.

However, we are going to study the case of no matter, by setting $T = 0$ in eq. (3.16) and thereby neglecting matter. Then we are solving for G_{00} corresponding to pure "mimetic dark matter" and no matter. Thus the last equation in (5.20) does not hold anymore and G_{00} is empty.

Now, using the fact that:

$$G = g^{ik} G_{ik}$$

$$G = g^{00} G_{00} \quad \Rightarrow \quad G = G_{00} \tag{5.21}$$

and consequently, G is also empty. However, this empty equation will be compensated by eq. (3.17) with $T = 0$. So, the aim now is to solve the following system of equations

$$\begin{aligned} G_{11} &= 0 \\ G_{22} = G_{33} &= 0 \\ G_{10} &= 0 \\ \partial_0(\sqrt{det\gamma} G) = 0 \quad \Leftrightarrow \quad (\sqrt{det\gamma} G) &= J(r) \end{aligned} \tag{5.22}$$

where $J(r)$ is a constant of integration (denoted $C(x^i)$ in eq.(3.17)), and study how mimetic matter will behave in the synchronous gauge.

5.3 System of Equations

It was found suitable to start with the G_{01} equation, since it yields a relation between the two metric coefficients e^{λ} and e^{μ}.

$$G_{01} = 0$$

So,

$$\Rightarrow \quad \frac{1}{2}\{\mu'(\dot{\lambda}-\dot{\mu})-2\dot{\mu}'\}=0$$

$$\Rightarrow \quad \{2\ln\mu'+\mu-\lambda\}^{\bullet}=0$$

$$\Rightarrow \quad 2\ln\mu'+\mu-\lambda=A(r)$$

$$\Rightarrow \quad \mu'^2=4(e^{-\mu}\cdot e^{\lambda}\cdot f^2(r))$$

$$\Rightarrow \quad e^{\lambda}=\frac{e^{\mu}\mu'^2}{4f^2(r)} \tag{5.23}$$

where $f^2(r)$ is an integration factor depending only on r, and the choice of the combination $4f^2(r)$ is to serve later as a simplification in our equations[10].

Now, the G_{00} and G_{11} equations will have the form:

$$G_{00}=e^{-\mu}\{1-f^2-\frac{1}{\mu'}ff'\}+\frac{3}{4}\dot{\mu}^2+\frac{\dot{\mu}\dot{\mu}'}{\mu'}$$

$$G_{11}=\frac{1}{4}\{\mu'^2-\frac{\mu'^2}{4f^2}(4+3e^{\mu}\dot{\mu}^2+4e^{\mu}\ddot{\mu})\}$$

We have seen that the $G_{\alpha\beta}=0$ and that the G_{00} equation is empty and equal to G.

Thus, we can write our equations:

$$G_{00}=e^{-\mu}\{1-f^2-\frac{1}{\mu'}ff'\}+\frac{3}{4}\dot{\mu}^2+\frac{\dot{\mu}\dot{\mu}'}{\mu'}=G \tag{5.24}$$

$$G_{11} = \frac{1}{4}\{\mu'^2 - \frac{\mu'^2}{4f^2}(4 + 3e^{\mu}\dot{\mu}^2 + 4e^{\mu}\ddot{\mu})\} = 0 \tag{5.25}$$

We turn now to study the G_{11} equation, since $G_{11} = 0$ then,

$$G_{11} = (1 - f^2) + e^{\mu}(\ddot{\mu} + \frac{3}{4}\dot{\mu}^2) = 0 \tag{5.26}$$

We now write equation (5.22) as a total derivative with respect to time such that:

$$\{\frac{e^{\frac{3}{2}}}{2}\dot{\mu}^2 + 2e^{\frac{\mu}{2}}(1 - f^2)\}^{\bullet} = 0$$

$$\Rightarrow \quad \frac{e^{\frac{3}{2}}}{2}\dot{\mu}^2 + 2e^{\frac{\mu}{2}}(1 - f^2) = B(r) \tag{5.27}$$

where $B(r)$ is a second integration factor depending only on r.

Now,

$$\frac{d}{dt}e^{\frac{\mu}{2}} = \frac{\dot{\mu}}{2}e^{\frac{\mu}{2}}$$

$$\frac{d}{dt}\mu = 2e^{\frac{-\mu}{2}}\frac{d}{dt}e^{\frac{\mu}{2}}$$

$$\Rightarrow \quad (\frac{d}{dt}e^{\frac{\mu}{2}})^2 + (1 - f^2) = \frac{B}{2e^{\frac{\mu}{2}}}$$

$$\Rightarrow \quad \frac{de^{\frac{\mu}{2}}}{\sqrt{f^2 + \frac{B}{2e^{\frac{\mu}{2}}} - 1}} = dt$$

and hence,

$$\int \frac{1}{\sqrt{f^2 + \frac{B}{2e^{\frac{\mu}{2}}} - 1}} de^{\frac{\mu}{2}} = t + H(r) \tag{5.28}$$

where $H(r)$ is a third integration factor depending only on r.

It useful to derive one last expression relating the Einstein' Scalar G, to an arbitrary function of r. This will be very interesting when compared to MDM. From equation (5.22):

$$f^2 = 1 + e^{\mu}(\ddot{\mu} + \frac{3}{4}\dot{\mu}^2)$$

$$\Rightarrow \quad \frac{2ff'}{\mu'} = e^{\mu}(\ddot{\mu} + \frac{3}{4}\dot{\mu}^2 + \frac{\ddot{\mu}'}{\mu'} + \frac{3}{2}\frac{\dot{\mu}\dot{\mu}'}{\mu'})$$

Substituting the values of f, f^2 and f' in equation (5.20) we obtain:

$$G = -3\ddot{\mu} - 2\frac{\ddot{\mu}'}{\mu'} - \frac{3}{2}\dot{\mu}^2 - 2\frac{\dot{\mu}\dot{\mu}'}{\mu'} \tag{5.29}$$

From equations (5.22) and (5.23) we find that:

$$e^{\frac{3}{2}\mu}\frac{\dot{\mu}^2}{2} + 2e^{\frac{\mu}{2}}(\ddot{\mu} + \frac{3}{4}\dot{\mu}^2) = B(r)$$

Taking the derivative with respect to r we obtain:

$$\frac{\partial B}{\partial r} = \frac{3}{4}\mu' e^{\frac{3}{2}\mu}\dot{\mu}^2 + e^{\frac{3}{2}\mu}\dot{\mu}\dot{\mu}' - 3\mu' e^{\frac{3}{2}\mu}(\ddot{\mu} + \frac{3}{4}\dot{\mu}^2) - 2e^{\frac{3}{2}\mu}(\ddot{\mu}' + \frac{3}{2}\dot{\mu}\dot{\mu}')$$

$$\frac{\partial B}{\partial r} = \mu' e^{\frac{3}{2}\mu}(\frac{3}{2}\dot{\mu}^2 - 2\frac{\dot{\mu}\dot{\mu}'}{\mu'} - 2\frac{\ddot{\mu}'}{\mu'} - 3\ddot{\mu})$$

However, the term in the parenthesis is nothing but G,

$$\Rightarrow \quad G = \frac{\partial B}{\partial r}\frac{e^{-\frac{3}{2}\mu}}{\mu'} \tag{5.30}$$

If we take a look at eq. (3.16), setting $T = 0$ and calculating $\det\gamma$

$$\begin{aligned}\det\gamma &= g_{11} \times g_{22} \times g_{33} \\ &= e^{\lambda} \times e^{\mu} \times e^{\mu} sin^2\theta\end{aligned}$$

Using the relation (5.23), we obtain,

$$\sqrt{\det\gamma} = \frac{\sin\theta}{2f} e^{\frac{3}{2}\mu}\mu' \tag{5.31}$$

Hence, eq. (3.16) becomes

$$G = 2fJ(r)\frac{e^{-\frac{3}{2}\mu}}{\mu'} \tag{5.32}$$

Comparing (5.30) and (5.32) we find that there is an obvious resemblance between the obtain G from MDM and that of Einstein's equations. A fact that gives hope that MDM might be able to explain several phenomena.

The condition (3.17) with $T = 0$ takes the form:

$$\sqrt{\det \gamma} G = \frac{\sin \theta}{2f} e^{\frac{3}{2}\mu} \mu' \{ e^{-\mu} \{ 1 - f^2 - \frac{1}{\mu'} f f' \} + \frac{3}{4} \dot{\mu}^2 + \frac{\dot{\mu} \dot{\mu}'}{\mu'} \} \tag{5.33}$$

5.4 $f = 1$ Parabolic Solution

Now we will consider the case where $f = 1$, eq.(5.26) becomes:

$$G_{11} = e^{\mu}(\ddot{\mu} + \frac{3}{4}\dot{\mu}^2) = 0$$

$$\Rightarrow \quad (\ddot{\mu} + \frac{3}{4}\dot{\mu}^2) = 0 \tag{5.34}$$

A solution for this equation is:

$$e^{\mu} = \{F(r)t + D(r)\}^{\frac{4}{3}} \tag{5.35}$$

where F and D are constants of integration depending only on r. This equation satisfies the equations including eq.(3.16) upon using the derivatives of the solution found in Appendix B.

$$\sqrt{\det \gamma} G = e^{\frac{3}{2}\mu}\{\mu' \frac{3}{4}\dot{\mu}^2 + \frac{\dot{\mu}\dot{\mu}'}{\mu'}\} = 2J \tag{5.36}$$

It was found useful to perform a simple change of variable where

$$e^{\mu(r,t)} = R(r,t)^2$$

We state the results using the same treatment of the above equations:

$$ds^2 = dt^2 - e^{\lambda(r,t)}dr^2 - R(r,t)^2\{d\theta^2 + sin^2\theta d\phi^2\} \tag{5.37}$$

$$
\begin{aligned}
G_{11} = -e^{-\lambda}r'^2 + 2r\ddot{r} + \dot{r}^2 + 1 = 0 \\
G_{22} = G_{33} = -\frac{e^{-\lambda}}{r}(2r'' - r'\lambda') + \frac{\dot{r}\dot{\lambda}}{r} + \ddot{\lambda} + \frac{\dot{\lambda}^2}{2} + \frac{2\ddot{r}}{r} = 0 \\
G_{00} = -\frac{e^{\lambda}}{r^2}(2rr'' + r'^2 - rr'\lambda') + \frac{1}{r^2}(r\dot{r}\dot{\lambda} + \dot{r}^2 + 1) = G \\
G_{02} = 2\dot{r}' - \dot{\lambda}r' = 0
\end{aligned}
\tag{5.38}
$$

The G_{10} equation now gives the relation:

$$
\begin{aligned}
e^{\lambda} = \frac{R'^2}{f^2} \\
\Rightarrow \quad \sqrt{det\gamma} = \frac{R^2R'}{f} \\
\Longrightarrow \quad G = \frac{fJ}{R^2R'}
\end{aligned}
\tag{5.39}
$$

where $f(r)$ is just a constant of integration depending only on r.

The metric then becomes:

$$
ds^2 = dt^2 - R'^2dr^2 - R^2\{d\theta^2 + sin^2\theta d\phi^2\} \tag{5.40}
$$

The solution for the case of parabolic model,that is, $f = 1$ is obtained by integrating the following equation (Using simple separation of variables):

$$
\dot{R} = \{\frac{F(r)}{2R} - (1 - f(r)^2)\} \tag{5.41}
$$

$$
\Rightarrow \quad R(r,t) = \frac{1}{2}(9F)^{\frac{1}{3}}(t + \alpha(r))^{\frac{2}{3}} \tag{5.42}
$$

where $\alpha(r)$ and $F(r)$ are just constants of integration depending only on r. We constrain the function $J(r)$ to be equal to $F'(r)$ hence, we can now write,

$$G = \frac{F'(r)}{R^2 R'} \tag{5.43}$$

Let us define now the quantity $C(r)$ as

$$C(r) = \int_0^r J dr \tag{5.44}$$

This quantity $C(r)$ will determine the "density" of MDM and is given as:

$$C(r) = F(r_0), \quad F(r = 0) = 0 \tag{5.45}$$

If F is constant then: $G = 0$ and this case would relate to the absence to any mimetic matter and thus we obtain:

$$\begin{aligned} R^2 &= \frac{1}{4}(9F)^{\frac{2}{3}}(t + \alpha(r))^{\frac{4}{3}} \\ e^{\lambda} &= \frac{\alpha'^2}{\{\frac{3}{(9F))^{2/3}}(t+\alpha)^{2/3}\}} \end{aligned} \tag{5.46}$$

Setting $F(r_0) = R_g$, where R_g is some gravitational radius, and $\alpha = -r$, we obtain the following metric:

$$ds^2 = dt^2 - \frac{dr^2}{\{\frac{3}{(9R_g))^{2/3}}(t-r)^{2/3}\}} - \frac{1}{4}(9R_g)^{\frac{2}{3}}(t-r)^{\frac{4}{3}}(d\theta^2 + sin^2\theta d\phi^2) \tag{5.47}$$

Thus we obtain the Schwarzschild solution in a comoving frame (synchronous system). We note that there is no singularity as t approaches r since $(t - r) = const$ for a given R_G. This singularity was thought to exists up until it was proven not to be physical by a simple change of the coordinate system used.

Now when $F(r)$ is not constant, we have seen in (5.42) that the determinant is equal to R^2R' for $f = 1$. Then in terms of the expressions of R and R' we have:

$$\begin{aligned} det\gamma &= R^2 \times R^2 \times R' \\ det\gamma &= \frac{1}{16}(9F)^2(t+\alpha)^2(\alpha')^2 \end{aligned} \tag{5.48}$$

Hence, according to eq. (3.17) the energy-density equation becomes:

$$G = \frac{2F'}{(9F)(t+\alpha)(\alpha')} \tag{5.49}$$

Thus, we see that when t approaches $-\alpha$, that is, when t approaches the moment $-\alpha$ corresponding to the arrival of the center $R = 0$, gravitational collapse will occur.

Now we consider a different scenario. Assume that there is a non-zero limit as $r \to 0$, and that there are no "empty" regions of space. Here we shall see that there is no gravitational collapse since the field will be evolving outwards and will mimic an Einstein-de Sitter universe (flat FRW universe). Let us choose a new radial coordinate $\bar{r}$ such that $F(r) = \frac{8}{9}\bar{r}^3$.

Now the solution (5.42) becomes

$$R(\bar{r}, t) = \bar{r}(t + \alpha(\bar{r}))^{\frac{2}{3}} \tag{5.50}$$

The metric becomes:

$$ds^2 = -(t+\alpha)^{\frac{4}{3}}\{A^2dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2)\} + dt^2 \tag{5.51}$$

where we adopted the new coordinate $\bar{r}$ and dropped the *bar*, and where A is given by:

$$A = \frac{3(t+\alpha) + 2r\alpha'}{3(t+\alpha)} \Rightarrow \quad A = 1 + \frac{2r\alpha'}{3(t+\alpha)} \tag{5.52}$$

The energy density G is:

$$G = \frac{4}{3}\frac{1}{(t+\alpha)^2 A} \tag{5.53}$$

There are two singularities associated with the energy density, one for $t+\alpha = 0$ and for $A = 0$. However, these two singularities cross at $r = 0$ and thus correspond to the same singularity.

For the case where $\alpha = 0$ we have:

$$ds^2 = dt^2 - t^{\frac{4}{3}}\{dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2)\} \tag{5.54}$$

which is nothing but the Einstein-de Sitter model.
Notice that the singularity now is at $t = 0$.

Define Θ to be :

$$\begin{aligned} \Theta &= \frac{\partial}{\partial r}\log G \\ \Rightarrow \quad \Theta &= -\{\frac{A'}{A} + \frac{2\alpha'}{t+\alpha}\} \end{aligned} \tag{5.55}$$

We see that for arbitrary α, Θ tends to 0, as $t \to \infty$ for an arbitrary choice of initial conditions. We also note that at the singularity, Θ in infinite and we call the singularity point($t - \alpha$ or $A = 0$ or both) "The Big Bang". In addition, the energy-density, approaches the Einstein-de Sitter model as $t \to \infty$. As $t \to \infty$,

$Y \rightarrow 1$ and the metric can be written as:

$$ds^2 = dt^2 - t^{\frac{4}{3}}\{dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2)\} \tag{5.56}$$

Hence, we can conclude that for inhomogeneous expanding non-vanishing distribution of mimetic matter the parabolic case of MDM will evolve in the Einstein-de Sitter universe.

5.5 General Solution: f arbitrary function of r

For the purpose of finding the general solution of the system of equations (5.22), having the condition imposed by MDM satisfied, we will use the old coordinate system. That is, we will work with equations (5.14 − 5.18).

We have seen that from the $G_{01} = 0$ equation one finds the relation 5.23 . In addition, from the $G_{11} = 0$ equation, one finds the solution for $f = 1$ (5.35). In order to extend this solution for an arbitrary $f(r)$ we will do the following: using equation (3.17) for $T = 0$ we obtain:

$$\partial_0(\sqrt{\det\gamma}G) = \frac{1}{16\mu'}e^{-\mu}\sqrt{\frac{e^{3\mu}sin^2\theta}{f^2}}\mu'^2\{4\dot{\mu}(-4ff' - (1-f^2)\mu') - 8(-1+f^2)\dot{\mu}' +$$

$$e^{\mu}(9\dot{\mu}^3\mu' + 18\dot{\mu}^2\dot{\mu}' + 8\ddot{\mu}\dot{\mu}' + 4\dot{\mu}(3\ddot{\mu}\mu' + 2\ddot{\mu}')\} = 0 \quad (5.57)$$

The second line of equation (5.57) is identically zero when implementing the solution (5.35). In order to generalize the solution we demand:

$$4\dot{\mu}(-4ff' - (1-f^2)\mu') - 8(-1+f^2)\dot{\mu}' = 0 \quad (5.58)$$

Substituting the values for the derivatives of μ using Appendix B we obtain:

$$\frac{1}{(Ft+D)^2}\{\frac{16}{3}F(Ft+D)(-4ff') - \frac{4^3}{9}F(F't+D')(f^2-1) -$$
$$\frac{32}{3}F'(Ft+D)(f^2-1) + \frac{32}{3}F(F't+D')(f^2-1)\} = 0$$

Separating the time components and the spatial one we obtain:

$$\{\frac{16}{3}F^2(-4ff') - \frac{4^3}{9}FF'(f^2-1)\}t$$
$$+ \frac{16}{3}FD(-4ff') + \frac{32}{9}FD'(f^2-1) - \frac{32}{3}DF'(f^2-1) = 0$$

Thus, we obtain two equations:

$$\{\frac{16}{3}F^2(-4ff') - \frac{4^3}{9}FF'(f^2-1)\} = 0 \tag{5.59a}$$

$$\frac{16}{3}FD(-4ff') + \frac{32}{9}FD'(f^2-1) - \frac{32}{3}DF'(f^2-1) = 0 \tag{5.59b}$$

Substituting $(5.59a)$ in $(5.59b)$:

$$FD(-4ff') + \frac{2}{3}FD'(f^2-1) - 2DF'(f^2-1) = 0$$
$$\Rightarrow \quad F'D = FD' \tag{5.60}$$

Dropping the constants of integration: $F(r) = D(r)$

From equation (5.59):

$$Fff' + \frac{1}{3}F'(f^2-1) = 0$$
$$\ln(f^2-1)^{\frac{1}{2}} = \ln F^{\frac{-1}{3}}$$
$$\Rightarrow \quad F = (f^2-1)^{-\frac{3}{2}} \tag{5.61}$$
$$\Rightarrow \text{From 4.60} \quad D = (f^2-1)^{-\frac{3}{2}}$$

The e^μ coefficient now becomes:

$$
\begin{aligned}
& e^\mu = F^{\frac{4}{3}}(t+1)^{\frac{4}{3}} \\
\Rightarrow \quad & e^\mu = (f^2-1)^{-2}(t+1)^{\frac{4}{3}}
\end{aligned}
\tag{5.62}
$$

While the e^λ coefficient will have the form:

$$
\begin{aligned}
& e^\lambda = \frac{1}{4f^2} e^\mu \mu'^2 \\
& \text{Obtaining the value of } \mu' \text{ from Appendix B} \\
\Rightarrow \quad & e^\lambda = \frac{4f'^2}{(f^2-1)^4}(t+1)^{\frac{4}{3}}
\end{aligned}
\tag{5.63}
$$

The metric now becomes solely in terms of f and f' and is given by:

$$
ds^2 = dt^2 - (t+1)^{\frac{4}{3}}\left[\frac{4f'^2}{(f^2-1)^4}dr^2 + \frac{1}{(f^2-1)^2}d\Omega^2\right] \tag{5.64}
$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. We emphasize that $f \neq 1$ and is not a constant, but an arbitrary function of r.

We would like to note that F is related to the "amount" of mimetic dust and thus f plays the role of specifying this amount.

Let us consider the case where:

$$
f^2 = \frac{1}{r} + 1 \quad \Rightarrow \quad r^2 = \frac{1}{(f^2-1)^2} \tag{5.65}
$$

$$
f' = -\frac{1}{2}\frac{1}{\sqrt{\frac{1}{r}+1}r^2} \quad \Rightarrow \quad f'^2 = \frac{1}{4}\frac{1}{(\frac{1}{r}+1)r^4} \tag{5.66}
$$

hence the metric (5.64) becomes:

$$ds^2 = dt^2 - (t+1)^{\frac{4}{3}}\left[\frac{1}{\frac{1}{r}+1}dr^2 + r^2 d\Omega^2\right] \tag{5.67}$$

In the limit of $r >> 1$ we retrieve the well known flat Friedmann-Robertson-Walker metric. And thus we can write the metric as:

$$ds^2 = dt^2 - a(t)^2\left[dr^2 + r^2 d\Omega^2\right] \tag{5.68}$$

where $a(t) = (t+1)^{\frac{2}{3}}$.

From (5.68) It is shown that

$$a(t) \propto t^{\frac{2}{3}} \tag{5.69}$$

So we have a spatially flat, "matter dominated" universe[14].

We now analyze the Ricci scalar in terms off our solution: plugging the solution along with its relation to $f(r)$ into eq5.13 we obtain:

$$R = -\frac{4}{3(t+1)^2} \tag{5.70}$$

We know that

$$-R = 8\pi G T \tag{5.71}$$

and since $T = T_{00} = \rho$

$$-R = 8\pi G\rho$$

from the Friedmann equation we obtain

$$-R = 3H^2 \tag{5.72}$$

$$\Rightarrow \quad H = \sqrt{-\frac{R}{3}}$$

Thus substituting eq. (5.70) in (5.72):

$$H = \frac{2}{3}\frac{1}{t+1} \tag{5.73}$$

So we obtain the Hubble parameter $H(t)$ for a matter-dominated Universe.

Chapter 6

Conclusion and Future Work

This chapter of the thesis serves as a quick recap of what has been done. In addition, remarks concerning future work will be indicated. The first chapter was dedicated to serve as a motivation. It touches on the subjects to be presented throughout the thesis, mainly The General Theory of Relativity, Dark Matter, and their relation to produce the Mimetic Dark Matter model. In the second chapter, a brief yet condensed description of General Relativity is presented. It starts with the transition from Special Relativity to General Relativity. The postulates of GR are highlighted with a clear description of the metric and its derivatives. Following that, Einstein's equations are derived through the least action principle along with their relationship to the energy-momentum tensor. Finally, a co-moving frame of reference, called the synchronous reference system, is presented as it is the basis of our work.

Chapter three is the heart of the thesis. Although in the first section, a quantitative description of the usual dark matter is presented, the following sections contain the formalism of the recently proposed model: Mimetic Dark Matter.

Mimetic dark matter was proposed by Chamseddine and Mukhanov[15] as a result of a reformulation of Einstein's Theory of Gravity. Defining the physical metric in terms of an auxiliary metric and a scalar field, we can isolate the conformal degree of freedom in a covariant way. Upon that, the gravitational field acquire a longitudinal degree of freedom on top of the two associated to the graviton. The resulting equations of motion mimic a prefect pressure-less fluid, dust. In chapter four, we shed the light on the basic applications that we aimed to achieve through mimetic dark mater. The astrophysical application is related to the gravitational collapse of dust in addition to the formation of the event horizon and black holes. Singularities in General Relativity are defined and discussed briefly to highlight the importance of non-cosmological singularities. Furthermore, some cosmological applications were also stated. The FRW metric in addition to the cosmological models of the Universe are explained.

Finally, chapter five is considered the grand finale. Most of our contribution is in this chapter. We proved that mimetic matter, being a form of mimetic dust can undergo a gravitational collapse and resulting in a Shwarzschild sphere which at its boundary the event horizon is formed and inside it a black hole exists. Moreover, we proved that the arbitrary function $f(r)$ pertaining to the initial conditions and "amount of mimetic dust" gives a spatially flat matter dominated universe. Further directions of research include the possibility of retrieving the radiation dominated and cosmological constant dominated universe through the addition of a "potential" which would serve as an input pressure as well as the dominance in the late stages of the Universe.

Appendix A

Abbreviations

SR	Special Relativity
GR	General Relativity
DM	Dark Matter
MDM	Mimetic Dark Matter
c	Speed of Light
EOM	Equation of Motion
CMB	Cosmic Microwave Background

Appendix B

Derivatives of e^{μ}

$$e^{\mu} = (F(r)t + D(r)^{\frac{4}{3}} \quad \Rightarrow \quad \mu = \frac{4}{3} ln(Ft + D) \tag{B.1}$$

$$\dot{\mu} = \frac{4}{3} \frac{F}{Ft + D} \tag{B.2}$$

$$\ddot{\mu} = \frac{4}{3} \frac{F^2}{(Ft + D)^2} \tag{B.3}$$

$$\mu' = \frac{4}{3} \frac{F't + D'}{Ft + D} \tag{B.4}$$

$$\dot{\mu}' = \frac{4}{3} \frac{F'}{Ft + D} - \frac{4}{3} F \frac{(F't + D')}{(Ft + D)^2} \tag{B.5}$$

$$\mu'' = \frac{4}{3} \frac{F''t + D''}{Ft + D} - \frac{4}{3} \frac{(F't + D')^2}{(Ft + D)^2} \tag{B.6}$$

$$\dot{\mu}'' = \frac{4}{3}\frac{F''}{Ft+D} - \frac{8}{3}F'\frac{(F't+D')}{(Ft+D)^2} - \frac{4}{3}F\frac{(F''t+D'')}{(Ft+D)^2} + \frac{8}{3}F\frac{(F't+D')^2}{(Ft+D)^3} \tag{B.7}$$

$$\ddot{\mu}' = -\frac{8}{3}\frac{FF'}{(Ft+D)^2} + \frac{8}{3}F^2\frac{(F't+D')}{(Ft+D)^3} \tag{B.8}$$

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