AMERICAN UNIVERSITY OF BEIRUT

GROMOV'S NON-SQUEEZING THEOREM AND PSEUDOHOLOMORPHIC DISCS

by
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AN ABSTRACT OF THE THESIS OF

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Abstract:

In order to understand the geometry of a given symplectic manifold (M, ω), one can study how elementary geometric subsets of M, such as balls, are transformed by symplectomorphisms, i.e. diffeomorphisms preserving the symplectic structure ω. Although such diffeomorphisms necessarily preserve the volume, M.Gromov proved in 1985 that symplectomorphisms behave in a more rigid way than volume preserving maps by establishing his celebrated non-squeezing theorem; roughly speaking, one cannot deform symplectomorphically a ball to a thin ball in order to squeeze it in a cylinder. Very recently, A. Sukhov and A. Tumanov in [13] gave an elegant and self-contained proof of Gromov's non-squeezing theorem based on the theory of attached pseudoholomorphic discs. The main goal of the proposed Master thesis is to study their approach.

Contents

Introduction

Symplectic geometry is an important branch of differential geometry and topology, carrying precious global information on the geometry of the ambient manifold; its development goes back to the Hamiltonian formulation of classical mechanics systems such as the planetary system. According to Darboux's theorem, symplectic manifolds are all locally equivalent and therefore there is no local invariants in Symplectic Geometry. The absence of local invariant gives rise to an infinite dimensional group of symplectomorphisms, i.e. diffeomorphisms preserving a given symplectic structure. Understanding the dynamical and rigidity properties of the group of symplectomorphisms, such as its relative size in the group of volume preserving diffeomorphisms, has attracted lots of attention in the development of Symplectic Geometry.

In 1985, M.Gromov [6] proved his important non-squeezing theorem, stating that symplectomorphisms behave in a more rigid way than volume preserving maps; more precisely, one cannot deform symplectomorphically a large ball in order to squeeze it in a thin cylinder. His approach was based on the interplay between Symplectic Geometry and Almost Complex Geometry and more precisely on the method of pseudoholomorphic curves.

Very recently, A. Sukhov and A. Tumanov [13], gave a new original and short proof of Gromov's non-squeezing theorem. Their methods relies on a new construction of J-homolomorphic discs attached cylinder with triangular base. Such discs are solutions of the classical Beltrami equation and the idea of attaching a J-homolomorphic disc to a cylinder is a boundary value problem in Partial Differential Equations. Their main idea was to consider triangular cylinders instead of circular ones in order to make use of linear boundary value conditions.

The present thesis is organized as follows. In Chapter 1 we cover the necessary preliminaries. In particular we recall the basic facts of Almost Complex Geometry and Symplectic Geometry. Chapter 2 is devoted to the classical and modified Cauchy-Green operators which are the main tools in the study of the Beltrami equation. In the Chapter 3 we construct a J-holomorphic disc attached to a triangular cylinder following A. Sukhov and A. Tumanov. We also suggest a simplified construction based on the Banach fixed point theorem. Finally, in Chapter 4 we state and prove Gromov's non-squeezing theorem using the J-holomorphic disc previously constructed.

Chapter 1

Preliminaries

We start this chapter by defining some notations that will be used in this thesis.

- In $\mathbb{R}^{2n}$, every point z is represented by the coordinates $(x_1, y_1, ..., x_n, y_n)$. $\mathbb{R}^{2n}$ is identified with $\mathbb{C}^n$ in which the variable will be denoted by $z = (z_1, ..., z_n)$ where each $z_j = x_j + iy_j$.
- In $\mathbb{C}$, the unit disc will be denoted by $\mathbb{D}$=$\{\zeta \in \mathbb{C}; \zeta\overline{\zeta} = 1\}$.
- We will denote the identity map by $I_d : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ for any n.

1.1 Function Spaces: Defintions and Notations

In this section we intend to define some spaces that we will use in the coming sections. In what follows, $k \in \mathbb{N}$ and $\alpha, p \in \mathbb{R}$ with $0 < \alpha < 1$.

- We denote by $L^p(\mathbb{D})$, the space of measurable functions $f : \overline{\mathbb{D}} \to \mathbb{C}$ such that

$$\iint_{\overline{\mathbb{D}}} |f(z)|^p dxdy < \infty.$$

  The L^p norm of a function $f \in L^p(\mathbb{D})$ is defined to be

$$\|f\|_p := \left(\iint_{\overline{\mathbb{D}}} |f(z)|^p dxdy\right)^{1/p}.$$

- We denote by $W^{k,p}(\mathbb{D})$ the Sobolev space of functions f on $\mathbb{D}$ whose derivatives to order k are in $L^p(\mathbb{D})$. We define the $W^{k,p}$-norm of a function $f \in W^{k,p}(\mathbb{D})$ by

$$\|f\|_{W^{k,p}} := \left(\sum_{j<k} \|D^j f\|_p^p\right)^{1/p}$$

  We will be only dealing with $W^{1,p}(\mathbb{D})$, namely the Sobolev space of functions f on $\mathbb{D}$ whose first derivative is in $L^p(\mathbb{D})$.

- We denote by $C^0(\mathbb{D})$ the space of all continuous functions
- We denote by $C^k(\mathbb{D})$ to be the space of functions $f : \overline{\mathbb{D}} \to \mathbb{C}$ differentiable up to order k, whose k^{th} derivatives are continuous. More precisely, a function f is said to be of class C^k if $f^{(1)}, f^{(2)}, f^{(3)} ..., f^k$ exist and are continuous. We also define $C^\infty(\mathbb{D})$ to be the space of infinitely differentiable functions.
- We denote by $C^{k,\alpha}(\mathbb{D})$ the space of functions $f : \overline{\mathbb{D}} \to \mathbb{C}$ differentiable up to order k, whose partial derivatives of order k satisfy the following Hölder condition:

  $$|D^k f(\zeta) - D^k f(\omega)| \leq C|\zeta - \omega|^\alpha$$

  for some positive constant.
- Finally, we denote by $C_0^k(\mathbb{D})$, $k \in \mathbb{N} \cup \{\infty\}$, the set of functions f of class $\mathcal{C}^k$ on $\mathbb{D}$ with compact support. A function is said to be of compact support if it is equal to zero outside a compact set.

1.2 Almost Complex Geometry

Definition 1.2.1. *An almost complex structure J on $\mathbb{R}^{2n}$, is a continuous map $J : \mathbb{R}^{2n} \to End(\mathbb{R}^{2n})$, which associates to every point $z \in \mathbb{R}^{2n}$ a linear isomorphism satisfying $J(z)^2 = -I_d$.*

Example 1.2.2. *In $\mathbb{R}^2$, $J_{st} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = i$, and $(\mathbb{R}^2, J_{st}) \simeq (\mathbb{C}, i)$. We will denote by i the standard structure in $\mathbb{R}^2$, known as the rotation matrix. One can show that $J_{st}^2 = -I_d$:*

$$J_{st}^2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = -Id.$$

More generally, almost complex structures in $\mathbb{R}^2$ are of the form

$$J(z) = \begin{bmatrix} a(z) & -\frac{1+a(z)^2}{c(z)} \\ c(z) & -a(z) \end{bmatrix}. \tag{1.1}$$

Indeed, if

$$J(z)^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

then,

$$a^2 + bc = -1$$

$$ab + bd = 0$$

$$ac + cd = 0$$

$$bc + d^2 = -1.$$

By solving this system of equations, we get the form (1.1), where $a(z)$ and $c(z)$ are continuous functions in $\mathbb{R}^2$, $c(z) \neq 0$.

Example 1.2.3. *In $\mathbb{R}^{2n}$, J_{st} the standard complex structure is represented by the block $2n \times 2n$ diagonal matrix:*

$$J_{st} = \begin{bmatrix} 0 & -1 & \cdots & \cdots & 0 & 0 \\ 1 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & \ddots & & 0 & 0 \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & -1 \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix}.$$

Note that:

$$J_{st}^2 = -I$$
$$J_{st}^{-1} = -J_{st}.$$

Definition 1.2.4. *Let $\mathbb{R}^{2m}$ and $\mathbb{R}^{2n}$ be endowed with two almost complex structures J' and J respectively, and let D' and D be two subsets of $\mathbb{R}^{2m}$ and $\mathbb{R}^{2n}$ respectively. A C^1 map $f : D' \to D$ is called (J', J)-holomorphic if it satisfies the Cauchy-Riemann equation:*

$$df \circ J' = J(f) \circ df \tag{1.2}$$

Example 1.2.5. *Let $D' \subseteq (\mathbb{R}^{2m}, J_{st})$, $D \subseteq (\mathbb{R}^{2n}, J_{st})$, then f is (J_{st}, J_{st}) holomorphic if and only if f is holomorphic in the usual case, namely each of components of f is holomorphic.*

Definition 1.2.6. *For $D' = \mathbb{D}$ and $J' = i$, we call the map f a J-holomorphic disc. In other words a J-holomrphic disc or pseudo-holomorphic disc is a (i, J) holomorphic map $u : \mathbb{D} \to D \subset \mathbb{R}^{2n}$.*

The J-holomorphy equation (1.2) for a J-holomorphic disc $u : \mathbb{D} \to \mathbb{C}^n$ can be written in the form:

$$\frac{\partial u}{\partial y} = J(u)\frac{\partial u}{\partial x}. \tag{1.3}$$

Indeed let $\frac{\partial u}{\partial x} = (\frac{\partial u_1}{\partial x}, \frac{\partial u_2}{\partial x}, ..., \frac{\partial u_{2n}}{\partial x})^t$ and let $\frac{\partial u}{\partial y} = (\frac{\partial u_1}{\partial y}, \frac{\partial u_2}{\partial y}, ..., \frac{\partial u_{2n}}{\partial y})^t$ and note that $\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \end{bmatrix}$ is a $2n \times 2$ matrix. The equation $du \circ i = J(u) \circ du$ can be written:

$$\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \end{bmatrix} \circ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = J(u) \circ \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial u}{\partial y} & -\frac{\partial u}{\partial x} \end{bmatrix} = J(u) \circ \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \end{bmatrix}.$$

So, we get $\frac{\partial u}{\partial y} = J(u)\frac{\partial u}{\partial x}$ and $-\frac{\partial u}{\partial x} = J(u)\frac{\partial u}{\partial y}$. This gives (1.3).

Remark 1.2.7. *A. Nijenhuis and W. Woolf proved in [11] the existence of small J-holomorphic discs. More precisely, if the structure J is of class $C^{k,\alpha}$ for $k > 0$, for any point $z \in \mathbb{R}^{2n}$ and any tangent vector X of z, there exists a J-holomorphic disc u of class $C^{k,\alpha}$ where $u(0)=z$ and $\frac{\partial u}{\partial x}(0) = \lambda X$ where $\lambda > 0$ is small enough.*

1.3 Symplectic Geometry

We start this section by discussing linear symplectic geometry. For more details, see the monography [8].

Definition 1.3.1. *Let V be a real vector space.*

(i) A bilinear form ω on a vector space V is a bilinear map $\omega : V \times V \to \mathbb{R}$, namely satisfying:

a- $\omega(X+Y,Z) = \omega(X,Z) + \omega(Y,Z)$
b- $\omega(X,Y+Z) = \omega(X,Y) + \omega(X,Z)$
c- $\omega(\lambda X,Y) = \omega(X,\lambda Y) = \lambda\omega(X,Y)$

for all $X, Y, Z \in V$ and $\lambda \in \mathbb{R}$.

(ii) A bilinear map is said to be skew-symmetric if $\omega(X,Y) = -\omega(Y,X)$ for all $X, Y \in V$.

Note that a bilinear map $\omega : V \times V \to \mathbb{R}$ induces a linear map $\omega^* : V \to V^*$ where V^* is the dual space of V. In case ω^* is an isomorphism, we say that ω is *non-degenerate*. In other words, ω is non-degenerate whenever the kernel $\ker\,\omega = \{X \in V \ / \ \omega(X,Y) = 0, \text{ for all } Y \in V\}$ is trivial.

Definition 1.3.2. *Let V be a real vector space.*

(i) A 2-form ω on V is a bilinear skew-symmetric map $\omega : V \times V \to \mathbb{R}$. In other words ω is an element of $V^* \wedge V^*$.

(ii) A linear symplectic form ω on V is a non-degenerate 2-form.

Example 1.3.3. *A model example in $\mathbb{R}^{2n}$ with the coordinates $z_j = x_j + iy_j$, $j = 1, ..., n$, is the standard symplectic form*

$$\omega_{st} = \sum_{j=1}^{n} dx_j \wedge dy_j = \frac{i}{2}\sum_{j=1}^{n} dz_j \wedge d\overline{z}_j.$$

Here $dx_1, dx_2, ..., dx_n, dy_1, dy_2, ..., dy_n$ denotes the standard basis of the dual $(\mathbb{R}^{2n})^*$ respectively.

Definition 1.3.4. *An inner product on $\mathbb{R}^{2n}$ is a bilinear symmetric positive definite form $g : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}$.*

Recall that g is positive definite if $g(X,X) > 0$ for every non-zero vector X.

Example 1.3.5. *A model example in $\mathbb{R}^{2n}$ is the inner product of two vectors $X = (X_1, X_2, \cdots, X_{2n})$ and $Y = (Y_1, Y_2, \cdots, Y_{2n})$ defined by:*

$$g(X,Y) := \sum_{j=1}^{2n} X_j.Y_j.$$

We can now discuss the not necessarily linear symplectic geometry.

Definition 1.3.6. *A symplectic form on $\mathbb{R}^{2n}$ is a closed ($d\omega = 0$) non-degenerate exterior 2-form ω on $\mathbb{R}^{2n}$; i.e. ω is a smooth map that associates to each point $z \in \mathbb{R}^{2n}$ a non-degenerate (linear) 2-form ω_z.*

Example 1.3.7. *The standard symplectic form*

$$\omega_{st} := \sum_{j=1}^{n} dx_j \wedge dy_j = \frac{i}{2}\sum_{j=1}^{n} dz_j \wedge d\overline{z}_j$$

plays a major role in Symplectic Geometry. By Darboux theorem [8], any symplectic form on a given smooth real manifold is locally diffeomorphic to ω_{st}, i.e. can be expressed in local coordinates as ω_{st}. In particular this implies that there is no local invariants on a symplectic manifold. This is a great contrast with Riemannian Geometry where the curvature is a local invariant.

We will need the following convenient definition

Definition 1.3.8. *Let $\Phi : \mathbb{R}^{2n} \to \mathbb{R}^{2m}$ be a smooth map.*

(i) Let $f : \mathbb{R}^{2m} \to \mathbb{R}$ be a smooth function, we define the pullback of f by Φ by:

$$\Phi^* f = f \circ \Phi.$$

(ii) Let α be a differential k-form on $\mathbb{R}^{2m}$, we define the pullback of α by Φ by:

$$(\Phi^*\alpha)_z(X_1, X_2, ..., X_k) = \alpha_{\phi(z)}(D_z\Phi(X_1), D_z\Phi(X_2), ..., D_z\Phi(X_k)).$$

Note that $\Phi^*\alpha$ is a k-form on $\mathbb{R}^{2n}$.

Definition 1.3.9. *Let ω_1 and ω_2 be two symplectic forms on $\mathbb{R}^{2n}$. A smooth map $\Phi : (\mathbb{R}^{2n}, \omega_1) \to (\mathbb{R}^{2n}, \omega_2)$ is called a symplectomorphism if it satisfies*

$$\Phi^*(\omega_2) = \omega_1.$$

Definition 1.3.10. *A Riemannian metric on $\mathbb{R}^{2n}$ is a smoothly varying collection of inner products $z \to g_z$:*

$$g := \{g_z : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R} \text{ inner product}, z \in \mathbb{R}^{2n}\}.$$

Definition 1.3.11. *Let ω be a symplectic structure on $\mathbb{R}^{2n}$. An almost complex structure on $(\mathbb{R}^{2n}, \omega)$ is called ω-tamed if $\omega(X, JX) > 0$, for all $X \neq 0$. In such case $g(X, Y) := \omega(X, JY)$ defines a Riemannian metric.*

Example 1.3.12. *A model example is provided by $\mathbb{R}^{2n}$ endowed with the standard symplectic form ω_{st} and the standard complex structure J_{st}, $(\mathbb{R}^{2n}, \omega_{st}, J_{st})$.*

To check that J_{st} is ω_{st}-tamed, let us prove that for all non-zero vector $X = (X_1, Y_1, \cdots, X_n, Y_n) \in \mathbb{R}^{2n}$, $\omega_{st}(X, J_{st}X) > 0$. We first find the expression of $J_{st}X$.

$$J_{st}X = \begin{bmatrix} 0 & -1 & \cdots & \cdots & 0 & 0 \\ 1 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & \ddots & & 0 & 0 \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & -1 \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix} \times \begin{pmatrix} X_1 \\ Y_1 \\ X_2 \\ \vdots \\ X_n \\ Y_n \end{pmatrix} = \begin{pmatrix} -Y_1 \\ X_1 \\ -Y_2 \\ \vdots \\ -Y_n \\ X_n \end{pmatrix}.$$

It follows that,

$$\omega_{st}(X, J_{st}X) = X_1^2 + Y_1^2 + X_2^2 \cdots + X_n^2 + Y_n^2 > 0.$$

Finally, we need to define the symplectic area of a map:

Definition 1.3.13. *Let $u : \mathbb{D} \to \mathbb{R}^{2n}$ be a map, we define the symplectic area of u by:*

$$\mathrm{Area}(u) = \int_{\mathbb{D}} u^*\omega. \tag{1.4}$$

1.4 Local J-holomorphic Equations of Discs

Proposition 1.4.1. *Let J be an almost complex structure ω_{st}-tamed on $\mathbb{R}^{2n} = \mathbb{C}^n$. Equation (1.2) for a J-complex disc $u : \mathbb{D} \to \mathbb{C}^n$, $u : \zeta \mapsto u(\zeta)$ can be written in the form:*

$$u_{\overline{\zeta}} = A(u)\overline{u}_{\overline{\zeta}}, \tag{1.5}$$

where $\zeta = x + iy$ and

$$A(z)(X) = (J(z) + J_{st})^{-1} (J_{st} - J(z)) (\overline{X})$$

is a complex linear endomorphism for every $z \in \mathbb{C}^n$.

Proof. We set

$$\frac{\partial u}{\partial \zeta} = \frac{1}{2}(\frac{\partial u}{\partial x} - J_{st}\frac{\partial u}{\partial y}), \tag{1.6}$$

and

$$\frac{\partial u}{\partial \overline{\zeta}} = \frac{1}{2}(\frac{\partial u}{\partial x} + J_{st}\frac{\partial u}{\partial y}). \tag{1.7}$$

Recall that J_{st} is the standard complex structure in $\mathbb{R}^{2n}$. By adding (1.6) and (1.7), we get:

$$\frac{\partial u}{\partial \zeta} + \frac{\partial u}{\partial \overline{\zeta}} = \frac{\partial u}{\partial x}.$$

By subtracting (1.6) and (1.7), we get:

$$\frac{\partial u}{\partial \zeta} - \frac{\partial u}{\partial \overline{\zeta}} = -J_{st}\frac{\partial u}{\partial y}.$$

And also,

$$\frac{\partial u}{\partial y} = (-J_{st})^{-1}\left(\frac{\partial u}{\partial \zeta} - \frac{\partial u}{\partial \overline{\zeta}}\right) = J_{st}\left(\frac{\partial u}{\partial \zeta} - \frac{\partial u}{\partial \overline{\zeta}}\right).$$

By substituting in (1.3), we get:

$$J_{st}\left(\frac{\partial u}{\partial \zeta} - \frac{\partial u}{\partial \overline{\zeta}}\right) = J(u)\left(\frac{\partial u}{\partial \zeta} + \frac{\partial u}{\partial \overline{\zeta}}\right)$$

$$J_{st}\frac{\partial u}{\partial \zeta} - J_{st}\frac{\partial u}{\partial \overline{\zeta}} = J(u)\frac{\partial u}{\partial \zeta} - J(u)\frac{\partial u}{\partial \overline{\zeta}}$$

$$(-J_{st} - J(u))\frac{\partial u}{\partial \overline{\zeta}} = (J(u) - J_{st})\frac{\partial u}{\partial \zeta}$$

$$(J(u)+J_{st})\frac{\partial u}{\partial \bar{\zeta}} = (J_{st}-J(u))\frac{\partial u}{\partial \zeta}.$$

Having J tamed by ω_{st},

$$\omega_{st}(X,(J+J_{st})X) = \omega_{st}(X,JX) + \omega_{st}(X,J_{st}X) > 0.$$

So, $ker\,(J+J_{st}) = \{0\}$ and therefore $(J+J_{st})$ is invertible. We can then write

$$\frac{\partial u}{\partial \bar{\zeta}} = (J(u)+J_{st})^{-1}(J_{st}-J(u))\frac{\partial u}{\partial \zeta}.$$

This proves the proposition.

□

Remark 1.4.2. *(i) For all $z \in \mathbb{C}^n, A(z)$ can be considered as a $n \times n$ matrix with complex coefficients. Moreover, A has the same regularity as J.*

(ii) A is called the complex matrix of J, denoting the matrix representation of the complex anti-linear operator $(J_{\mathrm{st}}+J)^{-1}(J_{\mathrm{st}}-J)$ and can be uniquely determined by J.

(iii) $A(z) = 0$ if and only if $(J_{\mathrm{st}}+J(z))^{-1}(J_{\mathrm{st}}-J(z)) = 0$, that is $J(z) = J_{st}$.

(iv) In the Cauchy-Riemann Equation (1.5) the $n \times n$ matrix function A satisfies:

$$\|A(z)\| < 1 \tag{1.8}$$

for all $z \in \mathbb{C}^n$, where the matrix norm $\|.\|$ is induced by the Euclidean inner product, namely $\|A(z)\| = \max_{X\in\mathbb{C}^n\setminus\{0\}} \frac{|A(z)X|}{|X|}$.

Proposition 1.4.3. *A satisfies the following properties:*

(i) A is conjugate linear operator in the identification of $\mathbb{R}^{2n}$ of $\mathbb{C}^n$, i.e. $A(iX) = -iA(X)$. In other words, in real notations:

$$AJ_{st} = -J_{st}A.$$

(ii) Having the following equality $(J+Jst)A = (Jst-J)$ and given the conjugate linear operator A, the corresponding almost complex structure J is given by $J = J_{st}(1-A)(1+A)^{-1}$.

(iii) Define $J = J_{st}(I_d - A)(I_d + A)$. Then J is an almost complex structure, namely $J^2 = -I_d$.

Proof. (i) We have,

$$(J_{st}+J) \times Jst = -1 + JJ_{st} = J(J+J_{st})$$

Taking inverses we get:

$$J_{st}^{-1}(J_{st}+J)^{-1} = (J+J_{st})^{-1}J^{-1}$$

$$-J_{st}(J_{st}+J)^{-1} = -(J+J_{st})^{-1}J.$$

We also have:

$$J\left(J_{st}-J\right)=-\left(J_{st}-J\right)J_{st}.$$

So,

$$\begin{aligned} J_{st}A &= J_{st}\left(J+J_{st}\right)^{-1}\left(J_{st}-J\right) \\ &= \left(J+J_{st}\right)^{-1}J\left(J_{st}-J\right) \\ &= -\left(J+J_{st}\right)^{-1}\left(J_{st}-J\right)J_{st} \\ &= -AJ_{st}. \end{aligned}$$

(ii) Expanding the following equality, $(J+Jst)\,A=(Jst-J)$, we get:

$$JA+J_{st}A=J_{st}-J$$

$$J\left(A+I_d\right)=J_{st}\left(I_d-A\right)$$

$$J=J_{st}\left(I_d-A\right)\left(I_d+A\right)^{-1}.$$

This is defined when the operator norm of A satisfies $\|A\|<1$, in other words when J is close to J_{st}.

(iii) Note that:

$$\begin{aligned} J_{st}(I_d-A) &= J_{st}-J_{st}A \\ &= J_{st}+AJ_{st} \\ &= (I_d+A)J_{st}. \end{aligned}$$

We compute

$$\begin{aligned} J^2 &= J_{st}\left(I_d-A\right)\left(I_d+A\right)^{-1}J_{st}\left(I_d-A\right)\left(I_d+A\right)^{-1} \\ &= \left(I_d+A\right)J_{st}\left(I_d+A\right)^{-1}\left(I_d+A\right)J_{st}\left(I_d+A\right)^{-1} \\ &= \left(I_d+A\right)J_{st}J_{st}\left(I_d+A\right)^{-1} \\ &= \left(I_d+A\right)\left(-I_d\right)\left(I_d+A\right)^{-1} \\ &= -\left(I_d+A\right)\left(I_d+A\right)^{-1} \\ &= -I_d. \end{aligned}$$

Hence, $J^2=-I_d$. □

Lemma 1.4.4. *Let J be an almost complex structure on $\mathbb{C}^n$, then J is tamed by ω_{st} if and only if the complex matrix A of J satisfied the condition $\|A(z)\|<1$, for all $z\in\mathbb{C}^n$.*

Proof. The complex matrix A is defined by $A=(J_{\mathrm{st}}+J)^{-1}(J_{\mathrm{st}}-J)$, where $A(z)(X)=\left(J(z)+J_{st}\right)^{-1}\left(J_{st}-J(z)\right)(\overline{X})$. Our first goal will be proving that A is well defined and this is done by proving that if J is tamed by ω_{st} then the $det(J_{\mathrm{st}}+J)\neq 0$, and that will automatically lead to A being well-defined. Note that $\omega_{st}\left(X,\left(J+J_{st}\right)X\right)=\omega_{st}\left(X,JX\right)+\omega_{st}\left(X,J_{st}X\right)>0$. So, $ker\left(J+J_{st}\right)=\{0\}$ and therefore $\left(J+J_{st}\right)$ is invertible.

We have

$$\begin{aligned} A &= (J_{st} + J)^{-1}(J_{st} - J) \\ &= (J_{st}(I_d - J_{st}J)^{-1}(J_{st}(I_d + J_{st}J)) \\ &= (I_d - J_{st}J)^{-1}(J_{st})^{-1}J_{st}(I_d + J_{st}J) \\ &= (I_d - J_{st}J)^{-1}(I_d + J_{st}J) \end{aligned}$$

Now, having $A = (I_d - J_{st}J)^{-1}(I_d + J_{st}J) = (I_d + J_{st}J)(I_d - J_{st}J)^{-1}$. Now, having $\|A\| < 1$, then

$$|AX| < |X|$$

$$|(I_d + J_{st}J)(I_d - J_{st}J)^{-1} X| < |X|$$

$$|(I_d + J_{st}J) X| < |(I_d - J_{st}J) X|$$

$$\|(I_d + J_{st}J)\| < \|(I_d - J_{st}J)\|$$

Now,

$$|X - J_{st}JX|^2 - |X + J_{st}JX|^2 = \omega_{st}(X - J_{st}JX, J_{st}(X - J_{st}JX)) - \omega_{st}(X + J_{st}JX, J_{st}(X + J_{st}JX))$$

$$= \omega_{st}(X - J_{st}JX, J_{st}X) + \omega_{st}(X - J_{st}JX, JX) - \omega_{st}(X + J_{st}JX, J_{st}X) - \omega_{st}(X + J_{st}JX, -JX)$$

$$= \omega_{st}(X, J_{st}X) - \omega_{st}(J_{st}JX, J_{st}X) + \omega_{st}(X, JX) - \omega_{st}(J_{st}JX, JX) - \omega_{st}(X, J_{st}X)$$
$$-\omega_{st}(J_{st}JX, J_{st}X) - \omega_{st}(X, -JX) - \omega_{st}(J_{st}JX, -JX)$$

$$= \omega_{st}(X, J_{st}X) - \omega_{st}(J_{st}JX, J_{st}X) + \omega_{st}(X, JX) - \omega_{st}(J_{st}JX, JX) - \omega_{st}(X, J_{st}X)$$
$$-\omega_{st}(J_{st}JX, J_{st}X) + \omega_{st}(X, JX) + \omega_{st}(J_{st}JX, JX)$$

$$= -2\omega_{st}(J_{st}JX, J_{st}X) + 2\omega_{st}(X, JX)$$

$$= 2\omega_{st}(X, JX) - 2\omega_{st}(JX, X)$$

$$= 2\omega_{st}(X, JX) + 2\omega_{st}(X, JX)$$

$$= 4\omega_{st}(X, JX)$$

So, J is ω_{st} tamed if and only if $\omega_{st}\left(X, JX\right) > 0$
Now,

$$|X - J_{st}JX|^2 - |X + J_{st}JX|^2 > 0$$

$$\left(|X - J_{st}JX| + |X + J_{st}JX|\right)\left(|X - J_{st}JX| - |X + J_{st}JX|\right) > 0$$

$$|X - J_{st}JX| > |X + J_{st}JX|$$

$$\|A\| < 1$$

□

Chapter 2

Modified Cauchy-Green Operator

2.1 Notation

In this chapter we will denote by:

- Δ: the triangle $\Delta = \{\zeta \in \mathbb{C} : 0 < \operatorname{Im}\zeta < 1 - |\operatorname{Re}\zeta|\}$, which is bounded by the straight lines of equations $y = 1 + x$, $y = 1 - x$ and $y = 0$. The triangle Δ is an isosceles triangle with base of length equals to 2 and height equals to 1. The area of Δ is $\text{Area}(\Delta) = 1$.

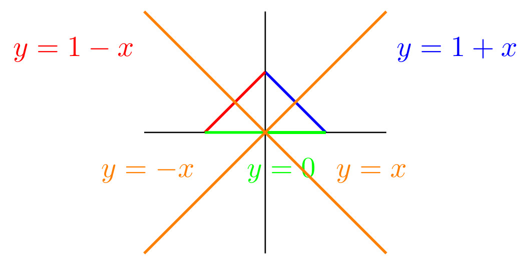

Figure 2.1: the triangle Δ

- Σ: the triangular cylinder $\Sigma = \Delta \times \mathbb{C}^{n-1}$ in $\mathbb{C}^n$ using the notation $z = (z_1, z_2, \cdots, z_n) = (z_1, z') \in \mathbb{C} \times \mathbb{C}^{n-1} = \mathbb{C}^n$ for representing a point in $\mathbb{C}^n$.

2.2 Construction of a Riemann Mapping for Δ

First we start introducing important theorems for our construction and then we will move forward in constructing the Riemann Mapping.

Theorem 2.2.1. ***Riemann Mapping Theorem [14]:*** *Let D be a non-empty proper simply connected domain in the complex plane $\mathbb{C}$, $D \subset \mathbb{C}$. Then there exist a biholomorphism $\omega = f(\zeta)$ from D onto the unit disc $\mathbb{D}$.*

A famous example is the Möbius function $\omega = f(\zeta) = \frac{i(1+\zeta)}{1-\zeta}$ which maps the unit disc $\mathbb{D}$ to the upper half plane $\mathbb{H} = \{\omega; Im(\omega) > 0\}$ and where the inverse function of f is $f^{-1}(\omega) = \frac{\omega-1}{\omega+i}$.

In order to find a biholomorphism from the unit disc $\mathbb{D}$ onto the triangle Δ, we will use the Möbius function f from the unit disc $\mathbb{D}$ onto the upper half plane $\mathbb{H}$ and then find a bihlomorphism from the upper half plane $\mathbb{H}$ to the triangle Δ using Schwartz Christofell Transformation.

Theorem 2.2.2. ***Schwartz Christofell Transformation [14]:*** *Let P be a polygon in the complex plane with vertices $p_1, p_2, \cdots, p_n$ and interior angles $\alpha_1, \alpha_2, \cdots, \alpha_n$, where $-\pi < \alpha_j < \pi$. There exist a biholomorphism f from the upper half plane $\mathbb{H}$ to the polygon P*

$$f(\zeta) = c_1 \int_{[0,\zeta]} ((\omega - x_1)^{1-\alpha_1/\pi}(\omega - x_2)^{1-\alpha_2/\pi} \cdots (\omega - x_n)^{1-\alpha_n/\pi})d\omega + c_2,$$

for some constants c_1, c_2.

Although the biholomorphism Φ from the unit disc to the isosceles triangle shown in figure (2.1) can not be expressed explicitly, one can in theory find it via Schwartz-Christofell Transformation. The biholomorphism $g : \mathbb{H} \to \Delta$ using Schwartz Christofell Transformation is given by:

$$\begin{aligned} g(t) &= c_1 \int_{[0,t]} \frac{d\omega}{(\omega - 1)^{3/4}(\omega + 1)^{3/4}} + c_2 \\ &= c_1 \int_{[0,\zeta]} \frac{d\omega}{(\omega^2 - 1)^{3/4}} + c_2. \end{aligned}$$

So,

$$\begin{aligned} \Phi(\zeta) = g \circ f(\zeta) &= c_1 \int_{[0,\zeta]} \frac{d\omega}{\left(\left(\frac{i+i\omega}{1-\omega}\right)^2 - 1\right)^{3/4}} + c_2 \\ &= c_1 \int_{[0,\zeta]} \frac{(1-\omega)^{-3/2} d\omega}{\left(-(1+\omega)^2 - (1-\omega)^2\right)^{3/4}} + c_2 \\ &= c_1 \int_{[0,\zeta]} \frac{(1-\omega)^{-3/2} d\omega}{(i^2(2 - 2\omega^2))^{3/4}} + c_2. \end{aligned}$$

Since we look for a biholomorphism satisfying the extra conditions $\Phi(1) = 1, \Phi(i) = i, \Phi(-i) = -i$, we have

$$\begin{aligned} \Phi(1) &= c_1 \int_{[0,1]} \frac{(1-\omega)^{-3/2} d\omega}{(i^2(2 - 2\omega^2))^{3/4}} + c_2 = 1 \\ \Phi(i) &= c_1 \int_{[0,i]} \frac{(1-\omega)^{-3/2} d\omega}{(i^2(2 - 2\omega^2))^{3/4}} + c_2 = i \\ \Phi(-i) &= c_1 \int_{[0,-i]} \frac{(1-\omega)^{-3/2} d\omega}{(i^2(2 - 2\omega^2))^{3/4}} + c_2 = -i. \end{aligned}$$

This is enough for our purpose.

2.3 Cauchy-Green Operator

Cauchy-Green operator was first introduced in Cauchy-Pompeiu's formula, which is a generalization of Cauchy's formula for non-holomorphic functions.

Theorem 2.3.1. ***Cauchy-Pompeiu Formula [14]:*** *Suppose D is a bounded domain with piecewise smooth boundary. If $g(\omega)$ is a smooth complex-valued function on $D \cup \partial D$, then*

$$g(\zeta) = \frac{1}{2\pi i}\int_{\partial D}\frac{g(\omega)}{\omega - \zeta}d\omega - \frac{1}{\pi}\iint_D \frac{\partial g(\omega)}{\partial \overline{\omega}}\frac{1}{\omega - \zeta}dxdy = Cg(\zeta) + T\frac{\partial g}{\partial \overline{\omega}}(\zeta)$$

for $\zeta \in D$.

Here C denotes the Cauchy transform

$$Cg(\zeta) := \frac{1}{2\pi i}\int_{\partial D}\frac{g(\omega)}{\omega - \zeta}d\omega,$$

and

$$T\frac{\partial g}{\partial \overline{\omega}}(\zeta) := \frac{1}{\pi}\iint_D \frac{\partial g(\omega)}{\partial \overline{\omega}}\frac{1}{\zeta - \omega}dxdy.$$

Cauchy-Pompeiu's formula differs from the Cauchy's formula by what is called the correction term $T\frac{\partial g}{\partial \overline{\omega}}$. Once can notice that if $g(\omega)$ was chosen to be a holomorphic function on the domain D, the second term would be equal to zero since $\frac{\partial g}{\partial \overline{\omega}} = 0$. This motivates the introduction of the Cauchy-Green operator T on a domain $\mathbb{D} \subseteq \mathbb{C}$:

$$T\Phi(\zeta) := \frac{1}{\pi}\iint_D \frac{\Phi(\omega)}{\zeta - \omega}dxdy,$$

where Φ is such that $T\Phi$ makes sense (see Proposition 2.3.4). Notice when g has compact support on D,

$$g(\zeta) = \frac{1}{\pi}\iint_D \frac{\partial g}{\partial \overline{\omega}}(\omega)\frac{1}{\zeta - \omega}dxdy = T\frac{\partial g}{\partial \overline{\omega}}(\zeta),$$

and

$$[\frac{1}{\pi\omega} * g](\zeta) = \frac{1}{\pi}\iint_D \frac{g(\omega)}{\zeta - \omega}dxdy.$$

Therefore $\frac{1}{\pi\omega}$ can be considered as a distribution. Moreover

$$\frac{\partial}{\partial \overline{\omega}}[\frac{1}{\pi\omega} * g] = (\frac{\partial}{\partial \overline{\omega}}\frac{1}{\pi\omega}) * g = \frac{1}{\pi\omega}\frac{\partial g}{\partial \overline{\omega}} = T\frac{\partial g}{\partial \overline{\omega}} = g.$$

We have proved the following proposition

Proposition 2.3.2. *We have*

(i) $\frac{\partial}{\partial \overline{\omega}}(\frac{1}{\pi\omega}) * g = g.$

(ii) $\frac{\partial}{\partial \overline{\omega}}[\frac{1}{\pi\omega} * g] = \frac{\partial}{\partial \overline{\omega}} Tg = g$, *which means that* $\frac{\partial}{\partial \overline{\omega}} \circ T = I_d$.

A major consequence is that the Cauchy-Green operator solves the $\overline{\partial}$ problem for compactly supported smooth function.

Proposition 2.3.3. *Let* $g \in C_0^1(D)$ *where* $D \subset \mathbb{C}$. *Then there exists a function* u *such that* $\frac{\partial u}{\partial \overline{\omega}} = g$. *Solutions* u *are of the form* $Tg + f$ *where* f *is holomorphic on* D.

Proof. Set $u := Tg + f$, where f is a holomorphic function over D and let us check that u solves the $\overline{\partial}$ problem. First let us find the partial derivative with respect to $\overline{\omega}$, we get:

$$\frac{\partial u}{\partial \overline{\omega}} = \frac{\partial Tg}{\partial \overline{\omega}} + \frac{\partial f}{\partial \overline{\omega}} = g$$

since $\frac{\partial}{\partial \overline{\omega}} \circ T = I_d$ and $\frac{\partial f}{\partial \overline{\omega}} = 0$, f being holomorphic function. Therefore, u solves the requested equation.

Note that in order to insure unicity, boundary conditions need to be added; e.g. $u|_D = \Phi$. Indeed it follows by Cauchy's formula

$$f(\zeta) = \frac{1}{\pi} \int_{\partial D} \frac{f(\omega)}{\omega - \zeta} d\omega = C\Phi(\zeta)$$

that the unique solution to the $\overline{\partial}$ problem with the boundary condition $u|_{\partial D} = \Phi$ is given by $u = Tg + C\Phi$. □

In the next proposition we recall some classical facts about T:

Proposition 2.3.4. *The Cauchy-Green operator* T *satisfies*

(i) $T : L^p(\mathbb{D}) \to W^{1,p}(\mathbb{D})$ *is bounded for* $p > 2$.

(ii) $\frac{\partial}{\partial \overline{\zeta}} Tf = f$ *as Sobolev's derivative. In other words,* T *solves the* $\overline{\partial}$ *problem in the unit disc.*

(iii) Tf *is holomorphic on* $\mathbb{C} \setminus \overline{\mathbb{D}}$.

The proof of this classical proposition can be found in the monography [15] for instance. For seek of completeness we prove the third point.

Proof. (iii) Let us prove that Tf is holomorphic on $\mathbb{C} \setminus \overline{\mathbb{D}}$ using the Cauchy-Riemann equation.

$$\frac{\partial}{\partial \overline{\zeta}} Tf(\zeta) = \frac{1}{\pi} \frac{\partial}{\partial \overline{\zeta}} \iint_{\mathbb{C} \setminus \overline{\mathbb{D}}} \frac{f(\omega)}{\zeta - \omega} dxdy.$$

Since $\frac{f(\omega)}{\zeta - \omega}$ is integrable over $\mathbb{C} \setminus \overline{\mathbb{D}}$, then we can interchange $\frac{\partial}{\partial \overline{\omega}}$ with the integral, we get:

$$\frac{\partial}{\partial \overline{\zeta}} Tf(\zeta) = \frac{1}{\pi} \iint_{\mathbb{C} \setminus \overline{\mathbb{D}}} \frac{\partial}{\partial \overline{\zeta}} \frac{f(\omega)}{\zeta - \omega} dxdy.$$

$$= \frac{1}{\pi} \iint_{\mathbb{C} \setminus \overline{\mathbb{D}}} 0 dxdy = 0$$

$\frac{f(\omega)}{\zeta - \omega}$ being holomorphic in ζ over $\mathbb{C} \setminus \overline{\mathbb{D}}$. So Tf is holomorphic over $\mathbb{C} \setminus \overline{\mathbb{D}}$. □

2.4 Modified Cauchy-Green Operator

Introduce the functions

$$R(\zeta) = e^{3\pi i/4}(\zeta - 1)^{1/4}(\zeta + 1)^{1/4}(\zeta - i)^{1/2},$$

and

$$X(\zeta) = R(\zeta)/\sqrt{\zeta}$$

choosing the branch of R continuous in $\overline{\mathbb{D}}$ with $R(0) = e^{3\pi i/4}$. We are interested in the function X only on the circle $b\mathbb{D}$. For definiteness, we choose the branch of $\sqrt{\zeta}$ continuous in $\mathbb{C}$ with deleted positive real line, with $\sqrt{-1} = i$, although we do not care about the sign of X.

Proposition 2.4.1. *$(X(\zeta))^4$ is pure real for $\zeta \in b\mathbb{D}$.*

Proof.

$$\begin{aligned}
(X(\zeta))^4 &= (R(\zeta)/\sqrt{\zeta})^4 \\
&= (e^{3\pi i/4}(\zeta - 1)^{1/4}(\zeta + 1)^{1/4}(\zeta - i)^{1/2})^4/\zeta^2 \\
&= e^{3\pi i}(\zeta - 1)(\zeta + 1)(\zeta - i)^2/\zeta^2 \\
&= -1(\zeta^2 - 1)(\zeta^2 - 2i\zeta - 1)/\zeta^2 \\
&= -1(\zeta^4 - 2i\zeta^3 - 2\zeta^2 + 2i\zeta + 1)/\zeta^2 \\
&= -1(\zeta^2 - 2i\zeta - 2 + 2i\zeta^{-1} + \zeta^{-2}) \\
&= -1(\zeta^2 + \overline{\zeta^2} - 2i(\zeta - \overline{\zeta}) - 2) \\
&= 2Re\zeta^2 - 2 + 2Im(\zeta)
\end{aligned}$$

Hence, $X(\zeta))^4$ is pure real. □

The term $X(\zeta)^4$ could be alternatively written as:

$$\begin{aligned}
X(\zeta)^4 =& e^{3\pi i}(cos(2\theta) + isin(2\theta) + 2sin(\theta) - 2icos(\theta) - 2 + 2sin(\theta) \\
&+ 2icos(\theta) + cos(2\theta) - isin(2\theta)) + e^{3\pi i}(2cos(2\theta) - 2 + 4sin(\theta)).
\end{aligned} \tag{2.1}$$

We claim that since $X(\zeta)$ is continuous on the boundary of the disc $b\mathbb{D}$, and since each of 1, -1 and i are the roots of $X(\zeta) = 0$, then $\arg X$ is constant over each of the intervals $\gamma_1 = \{e^{i\theta} : 0 < \theta < \pi/2\}$, $\gamma_2 = \{e^{i\theta} : \pi/2 < \theta < \pi\}$, $\gamma_3 = \{e^{i\theta} : \pi < \theta < 2\pi\}$ and is equal to $3\pi/4$, $\pi/4$ and 0 respectively, bounded by these roots whose respective arguments are $\pi/2$, π and 2π. Since we have proved that $X(\zeta)^4$ is pure real number of argument 3π, the fourth root of this complex number has argument equal to 0, $\pi/4$, $3\pi/4$ or $2\pi/4$.

To be able to prove this claim, we intend to choose a complex number of argument lying in each of the intervals γ_1, γ_2, γ_3. First, let us choose the value of the argument to be $\theta = \pi/4 \in \gamma_1$. Substituting in (2.1), we get,

$$X(\zeta)^4 = e^{3\pi i}(2cos(\pi/2) - 2 + 4sin(\pi/4)) = e^{3\pi i}(0 - 2 + 2\sqrt{2}) = e^{3\pi i}(2\sqrt{2} - 2),$$

and hence $\arg X(\zeta) = 3\pi/4$. Second, let us choose the value of the argument to be $\theta = 3\pi/4 \in \gamma_2$. By substituting in (2.1)

$$X(\zeta)^4 = e^{3\pi i}(2cos(3\pi/2) - 2 + 4sin(3\pi/4)) = e^{3\pi i}(-2 + 2\sqrt{2}) = e^{3\pi i + \pi}(2\sqrt{2} - 2) = e^{\pi i}(2\sqrt{2} - 2)$$

and so $\arg X(\zeta) = \pi/4$. Finally, let us choose the value of the argument to be $\theta = 3\pi/2 \in \gamma_3$. Once again by substituting in (2.1)

$$X(\zeta)^4 = e^{3\pi i}(2cos(3\pi/2) - 2 + 4sin(3\pi/2)) = e^{3\pi i}(-2 - 2 + 4) = e^{3\pi i}(0) = e^{0i}$$

and hence $\arg X(\zeta) = 0$.

Another property of X is:

Proposition 2.4.2. *The function X satisfies the boundary conditions*

$$\begin{cases} \operatorname{Im}(1+i)X(\zeta) &= 0, \quad \zeta \in \gamma_1, \\ \operatorname{Im}(1-i)X(\zeta) &= 0, \quad \zeta \in \gamma_2, \\ \operatorname{Im} X(\zeta) &= 0, \quad \zeta \in \gamma_3, \end{cases} \tag{2.2}$$

which represent the lines passing through the origin 0 parallel to the sides of the triangle Δ.

Proof. For $\zeta \in \gamma_1$,

$$(1+i)X(\zeta) = \sqrt{2}e^{i\pi/4}(re^{3i\pi/4)} = r'e^{i\pi},$$

for $\zeta \in \gamma_2$,

$$(1-i)X(\zeta) = \sqrt{2}e^{i3\pi/4}(re^{i\pi/4)} = r'e^{i\pi},$$

and for $\zeta \in \gamma_3$,

$$X(\zeta) = r'e^{i0}$$

Each of these numbers is pure real number, and therefore imaginary parts are equal to zero in each of the above mentioned cases. □

We will modify the classical Cauchy-Green operator

$$Tf(\zeta) = \frac{1}{2\pi i}\int_{\mathbb{D}} \frac{f(\omega)}{\omega - \zeta} d\omega \wedge d\overline{\omega}.$$

Here, we have used the fact that $-\frac{1}{2\pi i} d\omega \wedge d\overline{\omega} = \frac{1}{\pi} dxdy$. For a function Q defined on $\mathbb{D}$, define the modified Cauchy-Green operator using the weight Q:

$$\begin{aligned} T_Q f(\zeta) :=& Q(\zeta)\left(T(f/Q)(\zeta) + \zeta^{-1}\overline{T(f/Q)(1/\overline{\zeta})}\right) \\ =& Q(\zeta)\int_{\mathbb{D}}\left(\frac{f(\omega)}{Q(\omega)(\omega - \zeta)} + \frac{\overline{f(\omega)}}{\overline{Q(\omega)}(\overline{\omega}\zeta - 1)}\right)\frac{d\omega \wedge d\overline{\omega}}{2\pi i}. \end{aligned}$$

We will consider only two special special weights, namely $T_1 = T_Q$ with $Q = \zeta - 1$ and $T_2 = T_Q$ with $Q = R$. The operator T_1 was first introduced by Vekua in [15], whereas operators similar to

T_2 apparently were first introduced by Antoncev and Monakhov in [1] for applications related to gas dynamics.

We also define formal derivatives $S_j f(\zeta) := \frac{\partial}{\partial\zeta} T_j f(\zeta)$ as integrals in the sense of the Cauchy principal value. That is to say;

$$S_j f(\zeta) = \lim_{\epsilon\to 0}\left(Q'(\zeta)\int_{|\omega-\zeta|}\left(\frac{f(\omega)}{Q(\omega)(\omega-\zeta)} + \frac{\overline{f(\omega)}}{\overline{Q(\omega)}(\overline{\omega}\zeta-1)}\right)\frac{d\omega\wedge d\overline{\omega}}{2\pi i}\right.$$
$$\left. + Q(\zeta)\int_{|\omega-\zeta|}\left(\frac{f(\omega)}{Q(\omega)(\omega-\zeta)^2} + \frac{-\overline{\omega}\overline{f(\omega)}}{\overline{Q(\omega)}(\overline{\omega}\zeta-1)^2}\right)\frac{d\omega\wedge d\overline{\omega}}{2\pi i}\right)$$

The modified Cauchy-Green operators satisfy the following properties:

Proposition 2.4.3. *(i) Each $S_j : L^p(\mathbb{D}) \to L^p(\mathbb{D})$, $j = 1, 2$, is a bounded linear operator for $p_1 < p < p_2$. Here for S_1 one has $p_1 = 1$ and $p_2 = \infty$ and for S_2 one has $p_1 = 4/3$ and $p_2 = 8/3$. For $2 < p < p_2$, one has $S_j f(\zeta) = \frac{\partial}{\partial\zeta} T_j f(\zeta)$ as Sobolev's derivatives.*

(ii) Each $T_j : L^p(\mathbb{D}) \to W^{1,p}(\mathbb{D})$, $j = 1, 2$, is a bounded linear operator for $2 < p < p_2$. In particular, $T_j : L^p(\mathbb{D}) \to L^\infty(\mathbb{D})$ is a compact operator. For $f \in L^p(\mathbb{D})$, $2 < p < p_2$, one has $\frac{\partial}{\partial\overline{\zeta}} T_j f = f$ on $\mathbb{D}$ as Sobolev's derivative.

(iii) For every $f \in L^p(\mathbb{D})$, $p > 2$, the function $T_1 f$ satisfies $\operatorname{Re} T_1 f|_{b\mathbb{D}} = 0$ whereas $T_2 f$ satisfies the same boundary conditions (2.2) as X.

(iv) Each $S_j : L^2(\mathbb{D}) \to L^2(\mathbb{D})$, $j = 1, 2$, is an isometry.

(v) The function $p \mapsto \|S_j\|_{L^p}$ approaches $\|S_j\|_{L^2} = 1$ as $p \searrow 2$.

Due to the technicality of the proof, we will prove only points iii) and iv) and refer to [15, 1] for the other points

Proof. *(iii)* We will first start by proving that $T_1 f(\zeta)$ satisfies $\operatorname{Re} T_1 f|_{b\mathbb{D}} = 0$.

$$\begin{aligned}
T_1 f(\zeta) &= (\zeta-1)\int_{\mathbb{D}}\left(\frac{f(\omega)}{(\zeta-1)(\omega-\zeta)} + \frac{\overline{f(\omega)}}{\overline{(\zeta-1)}\,\overline{(\omega\overline{\zeta}-1)}}\right)\frac{d\omega\wedge d\overline{\omega}}{2\pi i}\\
&= \int_{\mathbb{D}}\frac{f(\omega)}{(\omega-\zeta)}\frac{d\omega\wedge d\overline{\omega}}{2\pi i} + \zeta - 1\int_{\mathbb{D}}\frac{\overline{f(\omega)}}{\overline{(\zeta-1)}\,\overline{(\omega\overline{\zeta}-1)}}\frac{d\omega\wedge d\overline{\omega}}{2\pi i}\\
&= Tf(\zeta) + \frac{\zeta-1}{\zeta(\overline{\zeta}-1)}\int_{\mathbb{D}}\frac{\overline{f(\omega)}}{(\overline{\omega}-1/\zeta)}\frac{d\omega\wedge d\overline{\omega}}{2\pi i}\\
&= Tf(\zeta) + \frac{\zeta-1}{\zeta\overline{\zeta}-\zeta}\int_{\mathbb{D}}\frac{\overline{f(\omega)}}{(\overline{\omega}-1/\zeta)}\frac{d\omega\wedge d\overline{\omega}}{2\pi i}\\
&= Tf(\zeta) + \frac{\zeta-1}{1-\zeta}\int_{\mathbb{D}}\frac{\overline{f(\omega)}}{(\overline{\omega}-1/\zeta)}\frac{d\omega\wedge d\overline{\omega}}{2\pi i}\\
&= Tf(\zeta) - \overline{Tf(1/\overline{\zeta})}.
\end{aligned}$$

We need to show that $\overline{ReT_1 f}|_{bD} = 0$. Note that since $\zeta \in bD$ then $|\zeta| = 1$ and $\zeta = e^{i\theta}$, so $1/\overline{\zeta} = e^{i\theta}$. So, $T_1 f(\zeta) = Tf(\zeta) - \overline{Tf(\zeta)}$ is a pure imaginary number and hence $\mathrm{Re}\, T_1 f|_{b\mathbb{D}} = 0$.

Now let us prove that $T_2 f$ satisfies the same boundary conditions (2.2) as X,

$$\begin{aligned} T_2 f(\zeta) &= R(\zeta)(T(f/R)(\zeta) + \zeta^{-1}\overline{T(f/R)(1/\overline{\zeta})}) \\ &= R(\zeta)(T(f/R)(\zeta) + \zeta^{-1}\overline{\int_D \frac{f(\omega)}{R(\omega)(\omega - 1/\overline{\zeta})} \frac{d\omega \wedge d\overline{\omega}}{2\pi i}} \\ &= R(\zeta)(T(f/R)(\zeta) + 1/\zeta \int_D \frac{\overline{f(\omega)}}{\overline{R(\omega)}(\overline{\omega} - \overline{\zeta})} \frac{d\omega \wedge d\overline{\omega}}{2\pi i}. \end{aligned}$$

On the other hand, we know and since $|\zeta| = 1$ then$\sqrt{\zeta}/\zeta = \overline{\sqrt{\zeta}}$. So,

$$\begin{aligned} T_2 f(\zeta) &= R(\zeta)(T(f/R)(\zeta) + 1/\zeta\overline{(T(f/R)(\zeta)} \\ &= (R(\zeta)/\sqrt{\zeta}) \left(\sqrt{\zeta}T(f/R)(\zeta) + \overline{\sqrt{\zeta}T(f/R)(\zeta)}\right) \end{aligned}$$

and since $T(f/R)(\zeta) + 1/\zeta\overline{T(f/R)(\zeta)}$ is pure real, then $T_2 f(\zeta)$ and $(R(\zeta)/\sqrt{\zeta})$ have the same argument. But we have defined $(R(\zeta)/\sqrt{\zeta}) = X(\zeta)$ and therefore $T_2 f(\zeta)$ and $X(\zeta)$ have the same boundary conditions.

(iv) Let $f \in C_0^\infty(\mathbb{D})$. Since $T_j f(b\mathbb{D})$ lies on finitely many lines, then $\mathrm{Area}(T_j f) = 0$. In other words, for $j = 1$, we have $\int_{b\mathbb{D}} T_1 f d\overline{T_1 f} = -\int_{b\mathbb{D}} \overline{T_1 f} d\overline{T_1 f} = 0$ using the properties of T_1, and similarly for $j = 2$. Now, by Stokes' formula

$$\begin{aligned} 0 = (i/2) \int_{b\mathbb{D}} T_j f d\overline{T_j f} &= (i/2) \int_{\mathbb{D}} dT_j f \wedge d\overline{T_j f} \\ &= (i/2) \int_{\mathbb{D}} (\frac{\partial T_j f}{\partial \zeta} d\zeta + \frac{\partial T_j f}{\partial \overline{\zeta}} d\overline{\zeta}) \wedge (\overline{\frac{\partial T_j f}{\partial \zeta} d\zeta + \frac{\partial T_j f}{\partial \overline{\zeta}} d\overline{\zeta}}) \\ &= (i/2) \int_{\mathbb{D}} (S_j f d\zeta + f d\overline{\zeta}) \wedge (\overline{S_j f} d\overline{\zeta} + \overline{f} d\zeta) \\ &= (i/2) \int_{\mathbb{D}} \left(|S_j f|^2 d\zeta \wedge d\overline{\zeta} + S_j f \overline{f} d\zeta \wedge d\zeta + f \overline{S_j f} d\overline{\zeta} \wedge d\overline{\zeta} + |f|^2 d\overline{\zeta} \wedge d\zeta\right) \\ &= (i/2) \int_{\mathbb{D}} |S_j f|^2 d\zeta \wedge d\overline{\zeta} - (i/2) \int_{\mathbb{D}} |f|^2 d\zeta \wedge d\overline{\zeta}. \end{aligned}$$

Hence,

$$\int_{\mathbb{D}} |S_j f|^2 d\zeta \wedge d\overline{\zeta} = \int_{\mathbb{D}} |f|^2 d\zeta \wedge d\overline{\zeta}.$$

Therefore,

$$\|S_j f\|_{L^2(\mathbb{D})} = \|f\|_{L^2(\mathbb{D})}.$$

The above cancellations were due to the fact that $d\zeta \wedge d\zeta = 0$, $d\overline{\zeta} \wedge d\overline{\zeta} = 0$, $\dfrac{\partial T_j}{\partial \overline{\zeta}} = I_d$, $\dfrac{\partial T_j}{\partial \zeta} = S_j$.

Since $C_0^\infty(\mathbb{D})$ is dense in $L^p(\mathbb{D})$, the equality holds for all $f \in L^2(\mathbb{D})$. □

To prove (iv), we have used Stoke's Theorem.

Theorem 2.4.4. ***Stoke's Theorem [5]:*** *Let $D \subset \mathbb{R}^{2n}$ be a domain with piecewise C^1 boundary except on finite number of points. Let α be a differential form of degree k. Suppose that $d\alpha$ is Lebesgue integrable over D then $\int_{\partial D} \alpha = \int_D d\alpha$.*

Chapter 3

Attaching a J-Holomorphic Disc to a Triangular Cylinder

3.1 Fixed Point Theorems

In this section we recall some useful theorems that will be used in our proofs.

Definition 3.1.1. *In a metric space (V, d); a map $f : V \to V$ is called a contraction map on V if there exists $0 < \alpha < 1$, such that $d(f(x), f(y)) \leq \alpha d(x, y)$ for all $x, y \in V$.*

Theorem 3.1.2. ***Contraction Principle [3]:*** *Let $f : V \to V$ be a contraction map defined on a non-empty complete metric space (V, d). Then f admits a unique fixed point z^0, i.e. $f(z^0) = z^0$.*

Theorem 3.1.3. ***Schauder Fixed Point Theorem [12]:*** *Let K be a non-empty, compact and convex subset of $\mathbb{R}^{2n}$. Let $f : K \to K$ be a continuous map. Then f has a fixed point z^0 in K, i.e. $f(z^0) = z^0$.*

3.2 Main Theorem

Theorem 3.2.1. *Let A be a continuous $n \times n$ complex matrix map defined on $\mathbb{C}^n$ vanishing on $\mathbb{C}^n \setminus \Sigma$. Suppose there is a constant $0 < a < 1$ such that*

$$\|A(z)\| \leq a \tag{3.1}$$

for all $z \in \Sigma$. Then there exists $p > 2$ such that for every point $(z^0, w^0) \in \Delta \times \mathbb{C}^{n-1} = \Sigma$ there is a solution $u = (u_1, w) \in W^{1,p}(\mathbb{D})$ of (1.5) such that:

- *(i)* $u(\overline{\mathbb{D}}) \subset \overline{\Sigma}$
- *(ii)* $(z^0, w^0) \in u(\mathbb{D})$,
- *(iii)* $\text{Area}(u) = 1$,
- *(iv)* $u(b\mathbb{D}) \subset b\Sigma = (b\Delta) \times \mathbb{C}^{n-1}$.

We first construct a candidate disc u satisfying (1.5), we will check afterwards that the constructed disc satisfies *(i)*, *(ii)*, *(iii)* and *(iv)* of Theorem 3.2.1.

3.2.2 Construction of a J-Holomorphic Disc

We look for a solution $u = (u_1, w) : \mathbb{D} \to \mathbb{C}^n$ of the Beltrami equation (1.5), of the form:

$$\begin{cases} u_1 &= T_2 g + \Phi, \\ w &= T_1 h - T_1 h(\tau) + w^0. \end{cases} \tag{3.2}$$

for some $\tau \in \mathbb{D}$ and for some $(g, h) \in L^p(\mathbb{D}, \mathbb{C} \times \mathbb{C}^{n-1})$ to be defined later. Recall that Φ is the Riemann mapping from Section 2.2.3. Notice that by substituting τ in the second equation, we have:

$$w(\tau) = T_1 h(\tau) - T_1 h(\tau) + w^0 = w^0.$$

The suggested form (3.2) ensures that u_1 satisfies the requested boundary conditions (by proposition 2.2.4, part iii)), in other words it takes the boundary of the disc $b\mathbb{D}$ to the boundary of the triangle $b\Delta$. The boundary conditions of w need not to be specified since each w_j component takes the boundary of the disc to a line $Rew_j = constant$ (Proposition 2.2.4 part iii) $\operatorname{Re} T_1 h|_{b\mathbb{D}} = 0$, so the w component will not have any effect on the area of the disc.

In order to find a solution $u = (u_1, w)$ of (1.5) of the form (3.2), let us differentiate (3.2) with respect to ζ and $\overline{\zeta}$. We first find the partial derivative with respect to ζ

$$\frac{\partial u_1}{\partial \zeta} = \frac{\partial T_2 g}{\partial \zeta} + \frac{\partial \Phi}{\partial \zeta} = S_2 g + \Phi'$$

and then the partial derivative with respect to $\overline{\zeta}$

$$\frac{\partial u_1}{\partial \overline{\zeta}} = \frac{\partial T_2 g}{\partial \overline{\zeta}} + \frac{\partial \Phi}{\partial \overline{\zeta}} = g$$

since Φ is a holomorphic function and since $\frac{\partial}{\partial \overline{\zeta}} \circ T_2 = I_d$. As for w,

$$\frac{\partial w}{\partial \zeta} = \frac{\partial T_1 h}{\partial \zeta} - \frac{\partial T_1 h}{\partial \zeta}(\tau) + \frac{\partial w^0}{\partial \zeta} = S_1 h$$

since $T_1 h(\tau)$ and w^0 are constants, and,

$$\frac{\partial w}{\partial \overline{\zeta}} = \frac{\partial T_1 h}{\partial \overline{\zeta}} - \frac{\partial T_1 h}{\partial \overline{\zeta}}(\tau) + \frac{\partial w^0}{\partial \overline{\zeta}} = h$$

since $\frac{\partial}{\partial \overline{\zeta}} \circ T_1 = I_d$.

Therefore the Beltrami equation (1.5) for $u = (u_1, w)$ of the form (3.2), turns into a singular integral equation

$$\begin{pmatrix} g \\ h \end{pmatrix} = A(u) \begin{pmatrix} \overline{S_2 g} + \overline{\Phi'} \\ \overline{S_1 h} \end{pmatrix}. \tag{3.3}$$

To solve this equation, fix $u : \mathbb{D} \to \mathbb{C}^n$ and introduce the map

$$L : L^p(\mathbb{D}, \mathbb{C}^n) \to L^p(\mathbb{D}, \mathbb{C}^n)$$

defined by,

$$\begin{pmatrix} g \\ h \end{pmatrix} \mapsto A(u) \begin{pmatrix} \overline{S_2 g} + \overline{\Phi'} \\ \overline{S_1 h} \end{pmatrix}. \tag{3.4}$$

Note that a fixed point of L, is precisely a solution of (3.3). We first prove that L is well defined, namely that $L(g,h) \in L^p(\mathbb{D}, \mathbb{C}^n)$ for any $(g,h) \in L^p(\mathbb{D}, \mathbb{C}^n)$. Let $(g,h) \in L^p(\mathbb{D}, \mathbb{C}^n)$, we compute: $\overline{S_2 g} + \overline{\Phi'}$, which is in $L^p(\mathbb{D}, \mathbb{C}^n)$ having Φ a function in $W^{1,p}$, than $\Phi' \in L^p(\mathbb{D}, \mathbb{C}^n)$, S_j being defined from $L^p(\mathbb{D}, \mathbb{D})$, so $S_2 g + \overline{\Phi'} \in L^p(\mathbb{D}, \mathbb{C}^n)$ and therefore $\overline{S_2 g} + \overline{\Phi'} \in L^p(\mathbb{D}, \mathbb{C}^n)$. Similarly, for $S_1 h$. Hence $L(g,h) \in L^p(\mathbb{D}, \mathbb{C}^n)$ and as a result L is well-defined.

Now, we prove that the function L is a contraction map. Let $(g,h), (g',h') \in L^P(\mathbb{D}, \mathbb{C}^n)$, we compute

$$\begin{aligned} L(g,h) - L(g',h') &= A(u) \begin{pmatrix} \overline{S_2 g} + \overline{\Phi'} \\ \overline{S_1 h} \end{pmatrix} - A(u) \begin{pmatrix} \overline{S_2 g'} + \overline{\Phi'} \\ \overline{S_1 h'} \end{pmatrix} \\ &= A(u) \left(\begin{pmatrix} \overline{S_2 g} + \overline{\Phi'} \\ \overline{S_1 h} \end{pmatrix} - \begin{pmatrix} \overline{S_2 g'} + \overline{\Phi'} \\ \overline{S_1 h'} \end{pmatrix} \right) \\ &= A(u) \begin{pmatrix} \overline{S_2 (g-g')} \\ \overline{S_1 (h-h')} \end{pmatrix}. \end{aligned}$$

Taking norms, we get,

$$\|L(g,h) - L(g',h')\|_p = \left\| A(u) \begin{pmatrix} \overline{S_2 (g-g')} \\ \overline{S_1 (h-h')} \end{pmatrix} \right\|_p.$$

By (3.1), we have

$$\begin{aligned} \|L(g,h) - L(g',h')\|_p &\leq a \left\| \begin{pmatrix} \overline{S_2 (g-g')} \\ \overline{S_1 (h-h')} \end{pmatrix} \right\|_p \\ &= a \|\overline{S_2 (g-g')}, \overline{S_1 (h-h')}\|_p \\ &= a (\|\overline{S_2 (g-g')}\|_p + \|\overline{S_1 (h-h')}\|_p) \\ &\leq a (\|S_2\|_p \|\overline{(g-g')}\|_p + \|S_1\|_p \|\overline{(h-h')}\|_p). \end{aligned}$$

Let s $= \max(\|S_1\|_p, \|S_2\|_p)$, we get;

$$\begin{aligned} \|L(g,h) - L(g',h')\|_p &\leq as \left\| \begin{pmatrix} \overline{g-g'} \\ \overline{h-h'} \end{pmatrix} \right\|_p \\ &= as \left\| \overline{\begin{pmatrix} g-g' \\ h-h' \end{pmatrix}} \right\|_p \\ &= as \left\| \begin{pmatrix} g-g' \\ h-h' \end{pmatrix} \right\|_p \\ &= as \|(g,h) - (g',h')\|_p \end{aligned}$$

and hence,

$$\|L(g,h) - L(g',h')\|_p \leq as \|(g,h) - (g',h')\|_p.$$

By proposition 3.4.3(v), we know that $\|S_j\|$ approaches 1 as p approaches 2. Decreasing the constant $a < 1$ in (3.1) if necessary, we get $as < 1$. Thus L is a contraction. By theorem 4.1.2, this gives us for every fixed $u = (u_1, w)$, a unique fixed point $v = (g, h) \in \mathbb{D}$ and therefore a solution of (3.3). We have therefore constructed a map that associates to each disc u an element $(g, h) \in L^P(\mathbb{D}, \mathbb{C}^n)$ solution of (3.3). the strategy is to use this map to construct the desired disc u.

Notice that the unique solution $v = (g, h)$ satisfies the following inequality:

$$\begin{aligned} \|v\|_p &\leq a\left(s\|v\|_p + \|\Phi'\|_p\right) \\ \|v\|_p - as\|v\|_p &\leq a\|\Phi'\|_p \\ \|v\|_p &\leq \frac{a\|\Phi'\|_p}{1 - as} = M_1. \end{aligned}$$

This implies that by (3.2),

$$\|u_1\|_p \leq \|T_2 g\|_p + \|\Phi\|_p \leq cM_1 + \|\Phi\|_p$$

since the operator T_2 is bounded by Proposition 3.4.3 *(iii)*. Similarly,

$$\|w\|_p \leq \|T_1(h - h(\tau))\|_p + \|w^0\|_p \leq cM_1 + w^0.$$

So, $\|u_1\|_\infty \leq M$ and $\|w\|_\infty \leq M$ where M is a constant depending on M_1 and w^0 and where the L_∞-norm $\|.\|_{L^\infty} = \sup\limits_{0<p<\infty} \|.\|_p$.

Next, we define a continuous map $\Psi : \mathbb{C} \to \overline{\mathbb{D}}$

$$\Psi(\zeta) = \begin{cases} \Phi^{-1}(\zeta), & \zeta \in \overline{\Delta}, \\ \\ \Phi^{-1}(b\Delta \cap [z^0, \zeta]), & \zeta \in \mathbb{C} \setminus \overline{\Delta}, \end{cases}$$

where $[z^0, \zeta]$ is the line segment joining z^0 to ζ, and the intersection of this line segment with the sides of the triangle previously defined $b\Delta \cap [z^0, \zeta]$ is a single point.

We define the following balls $E_{u_1} = \{u_1 \in L^\infty(\mathbb{D}) : \|u_1\|_{L^\infty} \leq M\}$ and $E_w = \{w \in L^\infty(\mathbb{D}) : \|w\|_{L^\infty} \leq M\}$ and the set $E = E_{u_1} \times E_w \times \overline{\mathbb{D}}$. We define a new map

$$F : (E, \|.\|_\infty) \to (E, \|.\|_\infty),$$

by $F(u_1, w, \tau) = (\tilde{u_1}, \tilde{w}, \tilde{\tau})$ with

$$\begin{aligned} \tilde{u_1} =& T_2 g + \Phi, \\ \tilde{w} =& T_1 h - T_1 h(\tau) + w_0, \\ \tilde{\tau} =& \Psi(z^0 - T_2 g(\tau)). \end{aligned}$$

where (g, h) is the unique solution of (3.3). Since A is continuous, we get that the map F a continuous. The set E is convex and the operators $T_j : L^p(\mathbb{D}) \to L^\infty(\mathbb{D})$ are compact. By applying Schauder's principle, we assure that the map F has a fixed point (u_1, w, τ) satisfying (3.2), (3.3) where $\tau = \Psi(z^0 - T_2 g(\tau))$.

3.2.3 Properties of the Constructed Disc

The disc $u = (u_1, w) \in W^{1,p}(\mathbb{D})$ that we have constructed satisfies the Cauchy-Riemann equations (1.5) and $w(\tau) = w^0$. In order to finish proving Theorem 3.2.1, it remains to prove $u(\overline{\mathbb{D}}) \subset \overline{\Sigma}$, $(z^0, w^0) \in u(\mathbb{D})$, Area$(u) = 1$, and $u(b\mathbb{D}) \subset b\Sigma = (b\Delta) \times \mathbb{C}^{n-1}$.

Lemma 3.2.4. *$\tau \in \mathbb{D}$ and $u_1(\tau) = z^0$.*

Proof. We will prove this lemma by contradiction. Suppose that $\tau \notin \mathbb{D}$ in particular suppose that $\tau \in b\mathbb{D}$, then $\Phi(\tau) \notin \Delta$ since Φ is a biholomorphism, so it takes the interior of the disc $\mathbb{D}$ to the interior of the triangle Δ and the boundary of the disc $\mathbb{D}$ to the boundary of the triangle Δ. Now, by the definition of Ψ,

$$\Phi(\tau) = \Phi(\Psi(z^0 - T_2g(\tau)) = \Phi(\Phi^{-1}(z^0 - T_2g(\tau)) = z^0 - T_2g(\tau) \notin \Delta.$$

Denote $q := T_2g(\tau) = u_1(\tau) - \Phi(\tau)$. Having $z^0 - T_2g(\tau) \notin \Delta$, then $q \neq 0$. Indeed, suppose that $q := u_1(\tau) - \Phi(\tau) = 0$, then $z^0 - T_2g(\tau) = z^0 - q = z^0 \notin \Delta$, which contradicts the fact that $z^0 \in \Delta$. Using the definition of Ψ, we have

$$\Phi(\tau) = \Phi((\Psi(b\Delta \cap [z^0, z^0 - q] = \Phi((\Phi^{-1}(b\Delta \cap [z^0, z^0 - q] = b\Delta \cap [z^0, z^0 - q].$$

z^0 being chosen arbitrarily, without loss of generality, one can assure $z^0 = i/2$, and we will also suppose $\tau \in \overline{\gamma}_1$. Then $\Phi(\tau) \in [1, i]$. And since $\operatorname{Im}(1+i)X(\zeta) = 0$ the number $(1+i)T_2g(\zeta)$ is a pure real complex number, say r. So,

$$\begin{aligned}(1+i)T_2g(\zeta) &= r\\ T_2g(\zeta) &= \frac{r}{(1+i)}\\ T_2g(\zeta) &= \frac{r(1-i)}{2}.\end{aligned}$$

This means that $T_2g(\zeta)$ is a real multiple of $1-i$ and so is q. So, Ψ, $\Phi(\tau) = [1, i] \cap [z^0, z^0 - q]$, and by supposing that $z^0 = i/2$, we notice that and since $i/2 - q = i/2 - r(1-i) = 3/2i - 1$ the slope of the segment $[i/2, i/2 - q] = \frac{-r}{r} = -1$. On the other hand, the slope of the straight line $[1, i]$ is $\frac{1}{-1} = -1$ and hence the two straight lines are parallel and no point of intersection exists.

So our supposition that $\tau \notin \mathbb{D}$ is not right and therefore $\tau \in \mathbb{D}$. Then, $\tau = \Psi(z^0 - T_2g(\tau)) = \Phi^{-1}(z^0 - T_2g(\tau))$. Therefore, $\Phi(\tau) = z^0 - T_2g(\tau) = z^0 - u_1(\tau) + \Phi(\tau)$, and hence $u_1(\tau) = z^0$. □

Lemma 3.2.5. *The map u_1 satisfies $u_1(\overline{\mathbb{D}}) \subset \overline{\Delta}$, $u_1(b\mathbb{D}) \subset b\Delta$, and $\deg u_1 = 1$; here $\deg u_1$ denotes the degree of the map $u_1|_{b\mathbb{D}} : b\mathbb{D} \to b\Delta$. In particular, u satisfies $u(b\mathbb{D}) \subset b\Sigma = (b\Delta) \times \mathbb{C}^{n-1}$.*

Proof. Once again will prove the lemma by contradiction. Define $G = \{\zeta \in \mathbb{D} : u_1(\zeta) \notin \overline{\Delta}\}$ and suppose $G \neq \emptyset$. Since u_1 is continuous, and $\overline{\Delta}$ is closed then its complement $\overline{\Delta}^c$ is open and hence G is open (as inverse image of an open set by a continuous map). Let G_1 be a non-empty connected component of G. Then $u_1(bG_1) \subset \overline{\Delta}$. Since we notice that if $u_1(bG_1) \not\subset \overline{\Delta}$ then G_1 will be G itself. Since $A = 0$ on $\mathbb{C}^n \setminus \Sigma$ and since $u_{\overline{\zeta}} = A(u)\overline{u}_{\overline{\zeta}}$, then $\frac{\partial u}{\partial \overline{\zeta}} = 0$. And by the Cauchy-Riemann equation in $\mathbb{R}^2$, we conclude that u_1 is holomorphic on G_1. But then the set $u_1(G_1)$ has the farthest point from $\overline{\Delta}$ (when $\frac{\partial g}{\partial \zeta} = 0$ this point represents an extrema of the function, which violates the maximum principle stating that if we have a continuous holomorphic function u on a closed subset $\overline{D}$ of the domain, then the maximum value of $|g|$ on $\overline{D}$ (which always exists) occurs on the boundary of D). Therefore our supposition on G is not true and so $G = \emptyset$, $u_1(\mathbb{D}) \subset \overline{\Delta}$, and by continuity $u_1(\overline{\mathbb{D}}) \subset \overline{\Delta}$.

Having constructed Φ under the conditions that $\Phi(\pm 1) = \pm 1$ and $\Phi(i) = i$ and since

$$\begin{cases} \operatorname{Im}(1+i)T_2 g(\zeta) = 0, \quad \zeta \in \gamma_1, \\ \operatorname{Im}(1-i)T_2 g(\zeta) = 0, \quad \zeta \in \gamma_2, \\ \operatorname{Im} T_2 g(\zeta) = 0, \quad \zeta \in \gamma_3, \end{cases} \tag{3.5}$$

the map $u_1 = T_2 g + \Phi$ takes the arcs γ_j, $j = 1, 2, 3$, to the lines containing the corresponding sides of the triangle Δ. But we have proved that $u_1(\overline{\mathbb{D}}) \subset \overline{\Delta}$, so the the images $u_1(\gamma_j)$, $j = 1, 2, 3$, are exactly the sides of Δ. Hence $u_1(b\mathbb{D}) \subset b\Delta$ indicating that the domain wraps one time around the range under the considered mapping that is $\deg u_1 = 1$. □

Lemma 3.2.6. Area$(u) = 1$.

Proof. By Stokes' formula

$$Area(u) = \int_{\mathbb{D}} u^*\omega = \frac{i}{2}\int_{\mathbb{D}} \left(du_1 \wedge d\overline{u}_1 + \sum_{j=1}^{n-1} dw_j \wedge d\overline{w_j} \right) = \frac{i}{2}\int_{b\mathbb{D}} u_1\, d\overline{u}_1 + \sum_{j=1}^{n-1} \frac{i}{2}\int_{b\mathbb{D}} w_j\, d\overline{w_j}.$$

Since $u_1(b\mathbb{D}) \subset b\Delta$, and $\deg u_1 = 1$, then $(i/2)\int_{b\mathbb{D}} u_1\, d\overline{u}_1 = \text{Area}(\Delta) = 1$. Since $w_j = T_1 \frac{\partial w_j}{\partial \overline{\zeta}} - T_1 \frac{\partial w_j}{\partial \overline{\zeta}}(\tau) + w_j^0$ then $\operatorname{Re} w_j = \operatorname{Re} T_1 \frac{\partial w}{\partial \overline{\zeta}} - \operatorname{Re} T_1 \frac{\partial w}{\partial \overline{\zeta}}(\tau) + \operatorname{Re} w^0$ on $b\Delta$. By Proposition 2.4.3 *(iii)* if follows that $\operatorname{Re} w_j = \operatorname{Re} w_j^0$ and is constant on $b\Delta$, therefore $\int_{b\mathbb{D}} w_j\, d\overline{w_j} = 0$. Hence Area$(u) = 1$ as desired. □

3.3 Alternative Construction

In case the structure A is of class C^1, a more natural way to construct the disc in Theorem 3.2.1 is to use the contraction principle directly. However it is quite unclear how to preserve the initial condition $u(\tau) = (z_0, w_0)$. This point is crucial and the way τ was constructed in the Proof of Theorem 3.2.1 is important in order to obtain the property *(ii)* of Theorem 3.2.1.

Therefore instead of proposing a simpler proof of Theorem 3.2.1 under the assumption that A is C^1, we suggest an alternative construction of a holomorphic disc. Instead of using the Schauder fixed point argument to construct the disc, we will apply the contraction principle directly. Let us emphasize that the disc that we will construct satisfies conditions *(i)*, *(iii)* and *(iv)* of Theorem 3.2.1. Trying to simplify the proof, the usual solution that comes to someones mind is the use of smooth functions, i.e. functions of class C^1. But the Riemann mapping Φ is merely $W^{1,p}$, and so the C^1-norm can not be applied to Φ from one side and on the other side, we have no idea about the boundedness of the operators T_j and S_j in the C^1- norm. That is why one can notice that in what follows we will be dealing with a solution in $W^{1,p}$-norm.

Proof. Alternative construction. We denote by l_1 the first row of A and by A_1 the remaining $n-1\times n$ matrix after deleting the first row.

$$A = \begin{bmatrix} l_1 \\ \begin{bmatrix} \\ A_1 \\ \\ \end{bmatrix} \end{bmatrix}$$

Next, we introduce the map

$$L : W^{1,p}(\mathbb{D}, \mathbb{C}^n) \to W^{1,p}(\mathbb{D}, \mathbb{C}^n)$$

defined by

$$L(u) = L(u_1, w) = (L_1(u_1, w), L'(u_1, w))$$

with

$$\begin{cases} L_1(u_1, w) &= \qquad T_2 l_1(u)\overline{u_1}_{\overline{\zeta}} + \Phi, \\ \\ L'(u_1, w) &= T_1 A_1(u)\overline{w}_{\overline{\zeta}} - T_1 A_1(u)\overline{w}_{\overline{\zeta}}(\tau) + w^0. \end{cases} \tag{3.6}$$

One can check that each of the components of the suggested solution satisfy the requested Cauchy-Riemann Equation.

First of all we will check whether the suggested solution is well-defined, namely if $(u_1, w) \in W^{1,p}(\mathbb{D}, \mathbb{C}^n)$ then $L(u_1, w) \in W^{1,p}(\mathbb{D}, \mathbb{C}^n)$. Knowing that $(u_1, w) \in W^{1,p}(\mathbb{D}, \mathbb{C}^n)$, then each of $\overline{u_1}_{\overline{\zeta}}$ and $\overline{w}_{\overline{\zeta}}$ being the partial derivative with respect to $\overline{\zeta}$ will be in $W^{0,p}(\mathbb{D}, \mathbb{C}^n) = L^p(\mathbb{D}, \mathbb{C}^n)$. Applying the T_2 and T_1 transforms on $l_1(u)\overline{u_1}_{\overline{\zeta}}$ and $A_1(u)\overline{w}_{\overline{\zeta}}$ respectively we will get that each of $T_2 l_1(u)\overline{u_1}_{\overline{\zeta}}$ and $T_1 A_1(u)\overline{w}_{\overline{\zeta}}$ are in $W^{1,p}(\mathbb{D}, \mathbb{C}^n)$.

Now, notice that a fixed point of L, is precisely a solution of (1.5). Planning to prove that $L = (L, L')$ is a contraction map, we will prove that L_1 is a contraction map itself. The proof for L' is similar. Let $(u_1, w), (u_1', w') \in W^1(\mathbb{D}, \mathbb{C}^n)$, we compute:

$$L_1(u_1, w) - L_1(u_1', w') = T_2 l_1(u)\overline{u_1}_{\overline{\zeta}} - T_2 l_1(u')\overline{u_1'}_{\overline{\zeta}}.$$

Taking norm

$$\begin{aligned} \|L_1(u_1, w) - L_1(u_1', w')\|_{W^{1,p}} &= \|(T_2 l_1(u)\overline{u_1}_{\overline{\zeta}} - T_2 l_1(u')\overline{u_1'}_{\overline{\zeta}}\|_{W^{1,p}} \\ &= \|T_2 l_1(u)\overline{u_1}_{\overline{\zeta}} + T_2 l_1(u)\overline{u_1'}_{\overline{\zeta}} - T_2 l_1(u)\overline{u_1'}_{\overline{\zeta}} - T_2 l_1(u')\overline{u_1'}_{\overline{\zeta}}\|_{W^{1,p}} \\ &\le \|T_2 l_1(u)\overline{u_1}_{\overline{\zeta}} - T_2 l_1(u)\overline{u_1'}_{\overline{\zeta}}\|_{W^{1,p}} + \|T_2 l_1(u)\overline{u_1}_{\overline{\zeta}} - T_2 l_1(u')\overline{u_1'}_{\overline{\zeta}}\|_{W^{1,p}} \\ &\le \|T_2 l_1(u)(\overline{u_1}_{\overline{\zeta}} - \overline{u_1'}_{\overline{\zeta}})\|_{W^{1,p}} + \|(T_2 l_1(u) - T_2 l_1(u'))\overline{u_1}_{\overline{\zeta}}\|_{W^{1,p}} \end{aligned}$$

Let us try to deal with each term alone:

$$\begin{aligned} \|T_2 l_1(u)(\overline{u_1}_{\overline{\zeta}} - \overline{u_1'}_{\overline{\zeta}})\|_{W^{1,p}} &\le C\|l_1(u)(\overline{u_1}_{\overline{\zeta}} - \overline{u_1'}_{\overline{\zeta}})\|_{L^p} \\ &\le C\|l_1(u)\|\|\overline{u_1}_{\overline{\zeta}} - \overline{u_1'}_{\overline{\zeta}}\|_{L^p} \\ &\le C\|A(u)\|\|\overline{u_1}_{\overline{\zeta}} - \overline{u_1'}_{\overline{\zeta}}\|_{L^p} \\ &\le C\|A(u)\|\|u_1 - u_1'\|_{W^{1,p}}. \end{aligned}$$

On the other hand,

$$\begin{aligned}\|(T_2 l_1(u) - T_2 l_1(u'))\overline{u_1}_{\overline{\zeta}}\|_{W^{1,p}} &\leq C\|(l_1(u) - l_1(u'))\overline{u_1}_{\overline{\zeta}}\|_{L^p} \\ &= C\left(\int |(l_1(u) - l_1(u'))|^p |\overline{u_1}_{\overline{\zeta}}|^p d\zeta\right)^{1/p}\end{aligned}$$

Now, using the Mean Value Theorem, we have:

$$\|l_1(u) - l_1(u'))\|_{C^0} \leq \|l_1\|_{C^1}\|u - u'\|_{C^0} \leq \|A\|_{C^1}\|u - u'\|_{C^0}.$$

So,

$$\begin{aligned}\|(T_2 l_1(u) - T_2 l_1(u'))\overline{u_1}_{\overline{\zeta}}\|_{W^{1,p}} &\leq C\|A\|_{C^1}\left(\int |u - u'|^p_{C^0} |\overline{u_1}_{\overline{\zeta}}|^p d\zeta\right)^{1/p} \\ &\leq C\|A\|_{C^1}\|u - u'\|_{W^{1,p}}.\end{aligned}$$

Following the same strategy we prove that:

$$\|T_1 A_1(u)\overline{w}_{\overline{\zeta}} - T_1 A_1(u')\overline{w}_{\overline{\zeta}})\|_{W^{1,p}} \leq \|A\|_{C^1}\|u - u'\|_{W^{1,p}}$$

Finally we get that:

$$\|L(u) - L(u')\|_{W^{1,p}} \leq C\|A\|_{C^1}\|u - u'\|_{W^{1,p}}$$

Since the $\mathcal{C}^1$-norm of A is sufficiently small, this proves that L is a contraction. By theorem 4.1.2, this gives us a unique fixed point $u = (u_1, w) \in \mathbb{D}$ and therefore a solution of (1.5).

□

Chapter 4

Gromov's Non-Squeezing Theorem

4.1 A Useful Lemma

Lemma 4.1.1. *Let*

$$\psi : G \subset R\mathbb{D} \times \mathbb{C}^{n-1} \to G' \subset \Sigma_R := \sqrt{\pi}R\Delta \times \mathbb{C}^{n-1}$$

be a diffeomorphism defined by $\psi(z_1, z') = (f(z_1), z')$, *where* f *is an area preserving map. Then* ψ *is a symplectomorphism, namely* $\psi^*\omega = \omega$.

Proof. Write

$$\psi(z_1, z') = (\psi_1(x_1, y_1), \psi_2(x_1, y_1), x_2, y_2, x_3, y_3, \cdots x_n, y_n).$$

Let $X = (X_1, Y_2, \cdots X_n, Y_n)$ and $Y = (X_1', Y_2', \cdots X_n', Y_n')$ be two vectors in $\mathbb{R}^{2n}$, we need to show that $\psi^*\omega(X,Y) = \omega(X,Y)$, in other words, we need to show that $\omega(D\psi X, D\psi Y) = \omega(X,Y)$. First of all let us compute the Jacobian matrix for ψ

$$D\psi = \begin{bmatrix} \frac{\partial\psi_1}{\partial x_1} & \frac{\partial\psi_1}{\partial y_1} & 0 & \cdots & 0 & 0 \\ \frac{\partial\psi_2}{\partial x_1} & \frac{\partial\psi_2}{\partial y_1} & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & 1 & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}.$$

So,

$$D\psi X = \begin{bmatrix} \frac{\partial\psi_1}{\partial x_1} & \frac{\partial\psi_1}{\partial y_1} & 0 & \cdots & 0 & 0 \\ \frac{\partial\psi_2}{\partial x_1} & \frac{\partial\psi_2}{\partial y_1} & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & 1 & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 1 \end{bmatrix} \begin{pmatrix} X_1 \\ Y_1 \\ X_2 \\ \vdots \\ X_n \\ Y_n \end{pmatrix} = \begin{pmatrix} \frac{\partial\psi_1}{\partial x_1}X_1 + \frac{\partial\psi_1}{\partial y_1}Y_1 \\ \frac{\partial\psi_2}{\partial x_1}X_1 + \frac{\partial\psi_2}{\partial y_1}Y_1 \\ X_2 \\ \vdots \\ X_n \\ Y_n \end{pmatrix},$$

and

$$D\psi v = \begin{pmatrix} \frac{\partial\psi_1}{\partial x_1}X_1' + \frac{\partial\psi_1}{\partial y_1}Y_1' \\ \frac{\partial\psi_2}{\partial x_1}X_1' + \frac{\partial\psi_2}{\partial y_1}Y_1' \\ X_2' \\ \vdots \\ X_n' \\ Y_n' \end{pmatrix}.$$

Now we compute $\omega(D\psi X, D\psi Y)$

$$\begin{aligned}
\omega(D\psi X, D\psi Y) =&(\frac{\partial\psi_1}{\partial x_1}X_1 + \frac{\partial\psi_1}{\partial y_1}Y_1)(\frac{\partial\psi_2}{\partial x_1}X_1' + \frac{\partial\psi_2}{\partial y_1}Y_1') \\
&- (\frac{\partial\psi_1}{\partial x_1}X_1' + \frac{\partial\psi_1}{\partial y_1}Y_1')(\frac{\partial\psi_2}{\partial x_1}X_1 + \frac{\partial\psi_2}{\partial y_1}Y_1) \\
&+ X_2Y_2' - X_2'Y_2 + \cdots + X_nY_n' - X_n'Y_n \\
=&\frac{\partial\psi_1}{\partial x_1}X_1\frac{\partial\psi_2}{\partial x_1}X_1' + \frac{\partial\psi_1}{\partial x_1}X_1\frac{\partial\psi_2}{\partial y_1}Y_1' + \frac{\partial\psi_1}{\partial y_1}Y_1\frac{\partial\psi_2}{\partial x_1}X_1' + \frac{\partial\psi_1}{\partial x_1}Y_1\frac{\partial\psi_2}{\partial y_1}Y_1' \\
&- \frac{\partial\psi_1}{\partial x_1}X_1'\frac{\partial\psi_2}{\partial x_1}X_1 - \frac{\partial\psi_1}{\partial x_1}X_1'\frac{\partial\psi_2}{\partial y_1}Y_1 - \frac{\partial\psi_1}{\partial y_1}Y_1'\frac{\partial\psi_2}{\partial x_1}X_1 - \frac{\partial\psi_1}{\partial y_1}Y_1'\frac{\partial\psi_2}{\partial y_1}Y_1 \\
&+ X_2Y_2' - X_2'Y_2 + \cdots + X_nY_n' - X_n'Y_n \\
=&\frac{\partial\psi_1}{\partial x_1}X_1\frac{\partial\psi_2}{\partial y_1}X_1' + \frac{\partial\psi_1}{\partial y_1}Y_1\frac{\partial\psi_2}{\partial x_1}X_1' \\
&- \frac{\partial\psi_1}{\partial x_1}X_1'\frac{\partial\psi_2}{\partial y_1}Y_1 - \frac{\partial\psi_1}{\partial y_1}Y_1'\frac{\partial\psi_2}{\partial x_1}X_1 \\
&+ X_2Y_2' - X_2'Y_2 + \cdots + X_nY_n' - X_n'Y_n \\
=&(\frac{\partial\psi_1}{\partial x_1}\frac{\partial\psi_2}{\partial y_1} - \frac{\partial\psi_1}{\partial y_1}\frac{\partial\psi_2}{\partial x_1})X_1Y_1' \\
&+ (\frac{\partial\psi_1}{\partial y_1}\frac{\partial\psi_2}{\partial x_1} - \frac{\partial\psi_1}{\partial x_1}\frac{\partial\psi_2}{\partial y_1})Y_1X_1' \\
&+ X_2Y_2' - X_2'Y_4 + \cdots + X_n'Y_n' - X_n'Y_n.
\end{aligned}$$

But since f is area preserving map then

$$\begin{cases} \frac{\partial\psi_1}{\partial x_1}\frac{\partial\psi_2}{\partial y_1} - \frac{\partial\psi_1}{\partial y_1}\frac{\partial\psi_2}{\partial x_1} = 1 \\ \\ \frac{\partial\psi_1}{\partial y_1}\frac{\partial\psi_2}{\partial x_1} - \frac{\partial\psi_1}{\partial x_1}\frac{\partial\psi_2}{\partial y_1} = 1 \end{cases}$$

and thus

$$\omega(D\psi X, D\psi Y) = X_1Y_1' + Y_1X_1' + X_2Y_2' - X_2'Y_4 + \cdots + X_n'Y_n' - X_n'Y_n.$$

On the other hand, $\omega(X, Y) = X_1Y_1' + Y_1X_1' + X_2Y_2' - X_2'Y_4 + \cdots + X_n'Y_n' - X_n'Y_n$ And therefore, the required result is attained. □

4.2 Gromov's Non-Squeezing Theorem

We denote by $\mathbb{B}^n$ the unit ball in $\mathbb{C}^n$.

Theorem 4.2.1. ***Gromov's Non-Squeezing Theorem*** *Let r, R be two positive real number, let $G \subset R\mathbb{D} \times \mathbb{C}^{n-1}$ be a domain. Let $\phi : r\mathbb{B}^n \to G$ be a symplectomorphism for ω, then $r \leq R$.*

Proof. We define a diffeomorphism

$$\psi : G \subset R\mathbb{D} \times \mathbb{C}^{n-1} \to G' \subset \Sigma_R := \sqrt{\pi} R\Delta \times \mathbb{C}^{n-1}$$

by $\psi(z_1, z') = (f(z_1), z')$, where f is an area preserving map. Due to Lemma 4.1.1, the map ψ is a symplectomorphism. Therefore the proof is reduced to considering $G \subseteq \Sigma_R$.

Since $\phi^*\omega = \omega$ then the almost complex structure $J := d\phi \circ J_{st} \circ d\phi^{-1}$ is tamed by ω, namely

$$\omega(X, JX) = \omega(X, d\phi \circ J_{st} \circ d\phi^{-1}X) = \omega(d\phi^{-1}X, J_{st} \circ d\phi^{-1}X) > 0,$$

since J_{st} is ω-tamed. Then the complex matrix $\tilde{A}$ of J satisfies $||\tilde{A}(z)|| < 1$ for $z \in G$ by Lemma 1.4.4.

Now consider a smooth cut-off function χ with support in G

$$\chi = \begin{cases} 1 & \text{on } \phi((r-\epsilon)\overline{\mathbb{B}}^n), \\ 0 & \text{on } \Sigma_R \setminus G, \end{cases}$$

and define a new matrix A by $A := \chi\tilde{A}$. Since J is continuous on G, so is A and therefore there exists a constant $a < 1$ such that $\|A(u)\| < a$. By theorem 3.2.1, there exists a solution u of (1.5) such that $\phi(0) \in u(\mathbb{D})$, $u(b\mathbb{D}) \subset b\Sigma_R$ and $\text{Area}(u) = \pi R^2$.

The curve $C = \phi^{-1}(u(\mathbb{D})) \cap (r-\epsilon)\mathbb{B}^n$ is a closed J_{st}-complex curve in $(r-\epsilon)\mathbb{B}^n$. More precisely $\phi^{-1} \circ u$ is a (standard) holomorphic disc. Indeed, since

$$du \circ i = J(u) \circ du = d\phi \circ J_{st} \circ d\phi^{-1} \circ du$$

we have

$$d\phi^{-1} \circ du \circ i = J_{st} \circ d\phi^{-1} \circ du.$$

Moreover $0 \in C$ and $\text{Area}(C) \leq \pi R^2$. On the other hand, by a classical result of P. Lelong (see [4] for instance), we have $\text{Area}(X) \geq \pi(r-\epsilon)^2$. Since ϵ is arbitrary $\pi R^2 \geq \pi r^2$ and finally, $r \leq R$ as desired. □

4.3 Analogy with Heisenberg Uncertainty Principle

In this thesis, we have introduced and proved Gromov's non-squeezing theorem. Hereby we give a small glance of the importance of this theorem. One of its important consequences lies in its application in Quantum Physics in particular Heisenberg Uncertainty Principle introduced in 1927, by the German physicist Werner Heisenberg stating that

$$\Delta P_j \Delta X j \geq \frac{1}{2}\overline{h},$$

where h is the Planck's constant, X_j is the position of the particle and P_j is its corresponding momentum. In other words, the Heisenberg Uncertainty Principle states that the precision in determining the position and the momentum of a certain particle are inversely proportional. After Gromov's non-squeezing theorem, the Uncertainty Principles and the classical and quantum physics in particular became easier to understand using classical physical settings. The non-Squeezing Theorem permitted the derivation of Heisenberg uncertainty principle in a way resembling the Schrödinger uncertainty principle.

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