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# INVARIANT METRICS AND COMPLEX GEODESICS IN SEVERAL COMPLEX VARIABLES

by

PRESKELLA MOUNIR MRAD

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PRESKELLA MOUNIR MRAD

Approved by:

Florian Bertrand, Assistant Professor
Math Department, AUB
Advisor

Faruk Abi-Khuzam, Professor
Faruk Abi · Khuzam
Math Department, AUB
Member of Committee

Giuseppe Della Sala, Assistant Professor
Giuseppe Della Sala
Math Department, AUB
Member of Committee

Date of thesis defense: [December 17, 2015]

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# An Abstract of the Thesis of

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## AN ABSTRACT OF THE THESIS OF

Abstract: Biholomorhically invariant metrics are important tools in Several Complex Variables which generalize Hermitian metrics and are particularly adapted for the study of holomorphic maps and function spaces. The Carathéodory metric and the Kobayashi metric are instances of such metrics. The main goal of this thesis is to study the elementary properties of geodesics related to these two metrics.

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# Appendix A

# Introduction

In one complex variable, the celebrated Riemann Mapping Theorem states that a nonempty proper and simply connected domain of $\mathbb{C}$ is biholomorphically equivalent to the unit disk $\Delta$. However, such a classification result does not hold in higher dimension. Indeed, In 1907 Poincaré observed that the unit ball $\mathbb{B}$ and the polydisk $\Delta \times \cdots \times \Delta$ are not biholomorphically invariant in $\mathbb{C}^n$ for $n \geq 2$. The classification of domains with boundary in Several Complex Variables is a difficult question and one needs to introduce biholomorphic invariants that capture the geometry of the domains. In this vein, biholomorphically invariant metrics appear to be an important tool in Several Complex Variables which generalize the concept of Hermitian metrics. The Carathéodory and Kobayashi pseudometrics are instances of such metrics; both generalize the Poincaré metric and play an important role in the classification of domains and are particularly adapted for the study of holomorphic maps. The present thesis is dedicated to the study of these two pseudometrics and related geodesics.

The thesis is organized as follows. We begin by recalling Schwarz Lemma which is the starting point of the theory of invariant metrics. This leads us to define on the unit disk a metric invariant under automorphisms, namely the Poincaré metric. From a metric viewpoint, the Schwarz Lemma states that Poincaré metric decreases under the action of holomorphic maps and that automorphisms are isometries. An important question arises of whether the Poincaré metric can be generalized in Several Complex Variables. This motivates us to introduce invariant pseudometrics and pseudodistances. We focus our study on the Carathéodory and the Kobayashi pseudodistances and pseudometrics whose invariance under biholomorphims follows directly from their definitions. We show that in the case of the unit disk $\Delta \subset \mathbb{C}$, the Carathéodory, Kobayashi and Poincaré distances coincide. However, we note that in general, these pseudodistances may degenerate and we focus our attention on the hyperbolic case, where both of them are actually (positive definite) distances. The main objects of our study are complex geodesics issued from these distances and metrics. The existence and unicity of complex geodesics is in general a difficult problem. We first study the main properties of geodesics. We then present an application of complex geodesics to a mapping problem. More precisely, given two domains $D_1$ and $D_2$ in $\mathbb{C}^n$ satisfying certain geometric assumptions the only holomorphic maps preserving the relative Carathéodory or Kobayashi distances between them are linear. This result may be viewed as a generalization of Schwarz Lemma and Cartan uniqueness Theorem.

# Appendix B

# Invariant Metrics

In the previous 40 years, invariant metrics have proved to be essential tools in the study of function theory and geometry of several complex variables. We present in this section a survey of the theory of invariant metrics. For a complete expository see [6].

## B.1 The Poincaré Disk

In the late nineteenth century, Henri Poincaré presented the very unique idea of equipping the unit disk $\Delta = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ in the complex plane $\mathbb{C}$ with a metric that is invariant under automorphisms of $\Delta$. The construction of this invariant metric comes naturally from the Schwarz Lemma. The way that an invariant metric is thereby constructed is closely identified to the broader Pick-Ahlfors Lemma B.1.2. We take this opportunity to review those ideas.

**Lemma B.1.1** (Schwarz Lemma [4]).
*Let* $f : \Delta \to \Delta$ *be holomorphic and assume* $f(0) = 0$.
*Then* $|f(z)| \leqslant |z|\ \forall z \in \Delta$ *and* $|f'(0)| \leqslant 1$.
*Moreover,* $|f(z)| = |z|\ \forall z \in \Delta$ *or* $|f'(0)| = 1$ *if and only if* $f$ *is a rotation where* $f(z) = e^{i\theta}$

*for some $\theta \in \mathbb{R}$.*

**Lemma B.1.2** (Pick-Ahlfors Lemma [4])**.**

*Let $f : \Delta \to \Delta$ be holomorphic and take $\zeta$ and $\zeta'$ in $\Delta$. Then we have:*

$$\left|\frac{f(\zeta) - f(\zeta')}{1 - \overline{f(\zeta)}f(\zeta')}\right| \leqslant \left|\frac{\zeta - \zeta'}{1 - \bar{\zeta}\zeta'}\right| \quad \textit{and} \quad \frac{|f'(\zeta)|}{1 - |f(\zeta)|^2} \leqslant \frac{1}{1 - |\zeta|^2}$$

The latter inequality motivates the following definition:

**Definition B.1.3.** *The infinitesimal Poincaré metric on $\Delta$ is the function $k$ from $\Delta \times \mathbb{C}$ to $\mathbb{R}_+$ given by*

$$(\zeta, v) \mapsto k(\zeta, v) = \frac{|v|}{1 - |\zeta|^2}$$

*where $\zeta$ is a point in $\Delta$ and $v$ is a vector in $\mathbb{C}$.*

One of the most important analytical properties satisfied by the Poincaré metric is the decreasing property which states that when $f$ is a holomorphic map from $\Delta$ to $\Delta$ then $k(f(\zeta), f'(\zeta)v) \leqslant k(\zeta, v)$ for any $\zeta \in \Delta$ and $v \in \mathbb{C}$. This is directly ensured by Pick-Ahlfors Lemma B.1.2.

This metric is also invariant under automorphisms of $\Delta$, meaning that when $f$ is an automorphism of $\Delta$, we have $k(f(p), f'(p)v) = k(p, v)$ for any $\zeta \in \Delta$ and $v \in \mathbb{C}$.

Given the Poincaré metric, we define the corresponding integrated length and distance:

**Definition B.1.4.** *Let $\gamma$ be a path of class $C^1$ from $[0, 1]$ to $\Delta$. We define the length of $\gamma$ to be*

$$l(\gamma) = \int_0^1 k\left(\gamma(t), \gamma'(t)\right) dt.$$

*The Poincaré distance between two points $\zeta$ and $\zeta'$ in $\Delta$ is*

$$\omega(\zeta, \zeta') = \inf\left\{l(\gamma), \gamma : [0, 1] \overset{C^1}{\longrightarrow} \Delta,\, \gamma(0) = \zeta,\, \gamma(1) = \zeta'\right\}$$

The important properties of $\omega$ are summarized in the following theorem (see [4]):

**Theorem B.1.5.**

1. *$(\Delta, \omega)$ is a metric space and induces the same topology as the one of the Euclidean distance.*

2. *$(\Delta, \omega)$ is complete.*

3. *Let $f : \Delta \to \Delta$ be holomorphic then for $\zeta \in \Delta$ and $v \in \mathbb{C}$, we have the decreasing property expressed by $\omega(f(\zeta), f'(\zeta)v) \leqslant \omega(\zeta, v)$.*

4. *For any automorphism $f$ of the unit disk, we have an isometry expressed by $\omega(f(\zeta), f'(\zeta)v) = \omega(\zeta, v)$ for $\zeta \in \Delta$ and $v \in \mathbb{C}$.*

An explicit formula for the Poincaré distance between $\zeta, \zeta' \in \Delta$ is given by

$$\omega(\zeta, \zeta') = \frac{1}{2} \ln \left( \frac{1 + |B_{\zeta'}(\zeta)|}{1 - |B_{\zeta'}(\zeta)|} \right)$$

where $B_{\zeta'}(\zeta) = \dfrac{\zeta - \zeta'}{1 - \overline{\zeta'}\zeta}$ is the Blaschke function or Möbius transformation.
Poincaré's construction of this metric is special to the disk. It is of our interest to equip any domain in higher dimensional complex space with a biholomorphically invariant metric. Examples of the generalizations of the Poincaré metric in higher dimensions are given by the Carathéodory pseudometric and the Kobayashi pseudometric. The present thesis is committed to the study of those two pseudometrics.

## B.2 The Carathéodory Pseudometric

In 1927, Constantin Carathédory [3] constructed an invariant pseudometric that now holds his name. We will prove that the Carathéodory pseudometric coincides with the Poincaré metric on the unit disk in $\mathbb{C}$ and is invariant under biholomorphic mappings. But first let us start by defining this pseudometric and giving some examples.

**Definition B.2.1.** *Given a domain $D$ in $\mathbb{C}^n$ and two points $p$ and $q$ in $D$, we define the Carathéodory pseudodistance between $p$ and $q$ as*

$$c_D(p,q) = \sup_f \omega\left(f(p), f(q)\right)$$

*where the least upper bound is taken over all holomorphic maps $f$ from $D$ into the unit disk $\Delta$ and $\omega(\zeta, \zeta')$ is the Poincaré distance between $\zeta$ and $\zeta' \in \Delta$.*

**Remark 1.** *The Carathéodory pseudodistance is not necessarily a distance.*
*For example on $\mathbb{C}$, we have $c_{\mathbb{C}}(p,q) = 0 \ \forall \ p, q \in \mathbb{C}$ since any holomorphic function $f : \mathbb{C} \to \Delta$ is constant by Liouville Theorem.*

We now introduce the Carathéodory pseudometric.

**Definition B.2.2.** *Let $D$ be a domain in $\mathbb{C}^n$, $p$ a point in $D$ and $v \in \mathbb{C}^n$ a tangent vector. The Carathéodory pseudometric is defined as*

$$\gamma_D(p,v) = \sup\left\{k\left(f(p), df(p)v\right) \ \textit{such that } f : D \to \Delta \textit{ is holomorphic}\right\}$$

*where the least upper bound extends over all holomorphic maps $f$ from $D$ into the unit disk $\Delta$.*

**Remark 2.** *If $v = \displaystyle\sum_{k=1}^{n} v_k \frac{\partial}{\partial z_k}$ then the differential is expressed by $df(v) = \displaystyle\sum_{k=1}^{n} \frac{\partial f}{\partial z_k} v_k$.*

**Remark 3.** *We can define the Carathéodory pseudometric by*

$$\gamma_D(p,v) = \sup_f \{|df(p)v| \text{ such that } f : D \to \Delta \text{ is holomorphic and } f(p) = 0\}.$$

*Indeed, let* $\alpha = \sup\{k\left(f(p), df(p)v\right)$ *such that* $f : D \to \Delta$ *is holomorphic*$\}$ *and let* $\beta = \sup\{|df(p)v|$ *such that* $f : D \to \Delta$ *is holomorphic and* $f(p) = 0\}$*. It is clear that* $\beta$ *is less than* $\alpha$*. Conversely, when assuming* $f(p) = a \neq 0$ *we can take* $g \circ f$ *where* $g(\zeta) = \dfrac{\zeta - a}{1 - \overline{a}\zeta}$ *which results in* $g \circ f(p) = 0$ *and we conclude by taking the supremum over* $g \circ f$*.*

A remarkable fact is that the Carathéodory pseudodistance(respectively pseudometric) coincides with the Poincaré distance(respectively metric) on the unit disk $\Delta$:

**Lemma B.2.3.** *In the case of the disk, we have* $c_\Delta(\zeta,\zeta') = \omega(\zeta,\zeta')$ *and* $\gamma_\Delta(\zeta,v) = k(\zeta,v)$*.*

*Proof: Let* $\zeta,\zeta' \in \Delta$*. We have* $\omega(\zeta,\zeta') \leqslant \sup \omega(f(\zeta), f(\zeta')) = c_\Delta(\zeta,\zeta')$ *since the identity map* $I : \Delta \to \Delta$ *is holomorphic. Moreover by the decreasing property of the Poincaré distance,* $\omega(f(\zeta), f(\zeta')) \leqslant \omega(\zeta,\zeta')$ *for any function* $f$*. So* $c_\Delta(\zeta,\zeta') \leqslant \omega(\zeta,\zeta')$*.*

*Let* $\alpha = \sup\{k\left(f(\zeta), f'(\zeta)v\right)$ *such that* $f : \Delta \to \Delta$ *is holomorphic*$\}$*. Let* $\zeta \in \Delta$ *and* $v \in \mathbb{C}$ *and* $I : \Delta \to \Delta$*, then* $k\left(f(\zeta), f'(\zeta)v\right) \leqslant k(\zeta,v)$*. This implies that* $\alpha \leqslant k(\zeta,v)$

One of the main properties of this pseudometric is the decreasing property under holomorphic maps.

**Proposition B.2.4.** *Let* $F : D \to D'$ *be holomorphic. Then we have*

$$c_{D'}(F(p), F(q)) \leqslant c_D(p,q) \text{ and } \gamma_{D'}(F(p), F(v)) \leqslant \gamma_D(p,v)$$

*This is known as the decreasing property of the Carathéodory pseudodistance and pseudometric.*

*Proof.* Let $F$ be a holomorophic map from $D$ to $D'$. And let $g$ be holomorphic from $D'$ to

$\Delta$. Then the map $g \circ F$ is holomorphic from $D$ to $\Delta$. We know that

$$c_D(p,q) = \sup_f \omega(f(p), f(q)) \geqslant \omega\left(g \circ F(p), g \circ F(q)\right).$$

Taking the supremum over all $g : D' \to \Delta$ yields $c_{D'}(F(p), F(q)) \leqslant c_D(p,q)$. ☐

**Remark 4.** *If $f : \Delta \to D$ is holomorphic then $c_D(f(\zeta), f(\zeta')) \leqslant \omega(\zeta, \zeta')$.*

An important consequence is the following:

**Corollary B.2.5.** *If $D \subset D' \subset \mathbb{C}^n$ then $c_{D'}(p,q) \leqslant c_D(p,q)$.*

*Proof.* This follows directly from Prosposition B.2.4 applied to the inclusion map $i : D \to D'$. ☐

As a main consequence we prove that the Carathéodory pseudodistance and pseudometric are invariant under biholomorphisms:

**Corollary B.2.6.** *If $F : D \to D'$ is biholomorphic then:*
*i) $F$ is an isometry in the Carathéodory metric: $\gamma_{D'}(F(p), F(v)) = \gamma_D(p,v)$ for $p \in D, v \in \mathbb{C}$.*
*ii) $F$ preserves distances: $c_{D'}(F(p), F(q)) = c_D(p,q)$ for every two points $p$ and $q$ in $D$.*

*Proof.* This is directly obtained from applying Proposition B.2.4 to $F$ and $F^{-1}$. ☐

The automorphisms of the unit ball (see [8]) act transitively on $\mathbb{B}$; in such case, we call the domain *homogeneous*. This is why it suffices to compute the Carathéodory pseudodistance (resp. pseudometric) at 0 and $p = (p_1, 0, ..., 0)$ (resp. at 0 and $v = (v_1, 0, ..., 0)$).

**Example B.2.7.** *We have $c_{\mathbb{B}}(0,p) = \omega(0, p_1)$ where $p = (p_1, 0, ..., 0)$.*

*Proof.* Let $||.|| : \mathbb{C}^n \to \mathbb{R}_+$ be a norm on $\mathbb{C}^n$, and $\mathbb{B}$ the unit ball for this norm. Take $z \in \mathbb{B}$, and define $\phi : \Delta \to \mathbb{B}$ by $\phi(\zeta) = \dfrac{\zeta}{||p||}p$. Then the decreasing property and Lemma B.2.3

yield in $c_{\mathbb{B}}(0,p) \leqslant \omega(0,||p||)$. On the other hand, taking the map $\pi$ from $\mathbb{B} \to \Delta$ defined by $\pi(p_1,...,p_n) = p_1$ then $c_\Delta(\pi(0),\pi(p)) \leqslant c_{\mathbb{B}}(0,p)$ resulting in $c_\Delta(0,p_1) \leqslant c_{\mathbb{B}}(0,p)$. And the result follows directly. □

**Example B.2.8.** $\gamma_{\mathbb{B}}(0,v) = k(0,||v||) = ||v||$

*Proof.* First, let's prove that $\gamma_{\mathbb{B}}(0,v) \leqslant ||v||$. Take $\varphi : \Delta \to \mathbb{B}$ defined by $\varphi(\zeta) = \dfrac{\zeta}{||v||}v$ where $\zeta \in \Delta$. By the decreasing property in Proposition B.2.4, we have $\gamma_{\mathbb{B}}(0,\varphi'(0)) \leqslant \gamma_\Delta(0,1)$. But $\gamma_\Delta(0,1) = k(0,1) = 1$, then $\gamma_{\mathbb{B}}\left(0,\dfrac{v}{||v||}\right) \leqslant 1$ and so $\gamma_{\mathbb{B}}(0,v) \leqslant ||v||$.
On the other hand, using unitary transformations, it suffices to prove $\gamma_{\mathbb{B}}(0,v) \geqslant ||v||$ for $v = (v_1,0,...,0)$ since $\gamma$ is invariant under biholomorphisms. Take $f : \mathbb{B} \to \Delta$ having $f(0,...,0) = 0$ and $f'(v_1,0,...,0) = v_1$. Then $\gamma_{\mathbb{B}}(0,v) = \sup\{k(f(0),df(p)v)\} \geqslant ||v||$. □

## B.3 The Kobayashi Pseudometric

In 1967, Shoshichi Kobayashi [6] introduced a pseudodistance $d_D$ on a domain $D$ of $\mathbb{C}^n$ using holomorphic disks. In 1971, Halsey Royden [10] introduced an infinitesimal pseudometric $K_D$ such that its integrated pseudodistance coincides with $d_D$. We will start by defining this pseudometric and then study some of its properties that we will need for our study. It can be thought of as the dual of the Carathéodory pseudometric but with holomorphic maps from the disk into the domain $D$ as opposed to holomorphic maps from $D$ to $\Delta$.

**Definition B.3.1.** *A holomorphic map $\varphi : \Delta \to D$ is called a holomorphic disk.*

**Definition B.3.2.** *Let $D$ be a domain in $\mathbb{C}^n$, $p$ a point in $D$ and $v \in \mathbb{C}^n$ a tangent vector. The Kobayashi pseudometric is defined as*

$$K_D(p,v) = \inf\left\{\frac{1}{r}, \varphi : \Delta \xrightarrow{holom} D, \varphi(0) = p, \varphi'(0) = rv\right\}$$

*where the greatest lower bound is taken over all holomorphic disks $\varphi$ in $D$.*

**Remark 5.** *The definition of the Kobayashi pseudometric can be given in the following form where the holomorphic disks map $\Delta_r = \{\zeta \in \mathbb{C} : |\zeta| < r\}$ to $D$:*

$$K_D(p, v) = \inf \left\{ \frac{1}{r}, \varphi : \Delta_r \xrightarrow{\textit{holom}} D, \varphi(0) = p, \varphi'(0) = v \right\}$$

*In other words, the Kobayashi pseudometric measures the size of holomorphic disks contained in $D$.*

*Proof.* Let $\alpha = \inf\limits_{\varphi} \left\{ \frac{1}{r}, \varphi : \Delta \xrightarrow{\text{holom}} D, \varphi(0) = p, \varphi'(0) = rv \right\}$ and let $\beta = \inf\limits_{\varphi} \left\{ \frac{1}{r}, \varphi : \Delta_r \xrightarrow{\text{holom}} D, \varphi(0) = p, \varphi'(0) = v \right\}$. First, in order to prove that $\beta$ is less than $\alpha$, we let $\varphi : \Delta \to D$ be a holomorphic disk such that $\varphi(0) = p$ and $\varphi'(0) = rv$. Consider $\tilde{\varphi} = \varphi \circ g : \Delta_r \to \Delta \to D$ where $g(\zeta) = \frac{\zeta}{r}$. Since $\tilde{\varphi}(0) = \varphi(0) = p$ and $\tilde{\varphi}'(0) = v$, then $\beta \leqslant \frac{1}{r}$ for all $\frac{1}{r}$ as in $\alpha$ concluding that $\beta$ is less than $\alpha$.

Conversely, to prove that $\alpha$ is less than $\beta$, let $\frac{1}{r}$ be as in $\beta$ then we have $\varphi$ from $\Delta_r$ to $D$ such that $\varphi(0) = p$ and $\varphi'(0) = v$. Let $\psi(\zeta) = \varphi(r\zeta) : \Delta \to D$. This holomorphic disk satisfies $\psi(0) = \varphi(0) = p$ and $\psi'(0) = \varphi'(0)r = rv$. Then $\frac{1}{r}$ is as in $\alpha$ and so $\alpha \leqslant \frac{1}{r}$ for all $r$ concluding that $\alpha$ is less than $\beta$. ☐

We will use either definition as the need emerges. We now introduce the integrated Kobayashi pseudodistance:

**Definition B.3.3.** *Let $D$ be a domain in $\mathbb{C}^n$. We define the Kobayashi length to be*

$$l(\gamma) = \int_0^1 K_D\left(\gamma(t), \gamma'(t)\right) dt$$

*where $\gamma : [0,1] \xrightarrow{C^1} D$ and $\gamma(0) = p$ and $\gamma(1) = p'$. We then define the Kobayashi pseudodistance* *between two points $p$ and $p'$ in $D$ to be*

$$d_D(p,p') = \inf_{\gamma} l(\gamma) = \inf_{\gamma} \int_0^1 K_D\left(\gamma(t), \gamma'(t)\right) dt$$

*where the greatest lower bound is taken over all smooth paths $\gamma$ connecting $p$ and $p'$ in $D$ where $\gamma(0) = p$ and $\gamma(1) = p'$.*

**Remark 6.** *It was observed by Royden [10] that the $K_D$ is upper semi-continuous. This legitimates the definition of the length.*

**Remark 7.** *The Kobayashi pseudo-distance is not necessarily a distance. For example on $\mathbb{C}$, we have $d_{\mathbb{C}}(p,q) = 0$ since $K_{\mathbb{C}}(p,v) = 0$ $\forall p \in \mathbb{C}$ and $v \in \mathbb{C}^n$.*
*Indeed take $\varphi : \Delta_r \to \mathbb{C}$ s.t. $\varphi(\zeta) = p + \zeta v$. We have $\varphi(0) = p$ and $\varphi'(0) = v$.*
$K_{\mathbb{C}}(p,v) = \inf_{\varphi} \left\{ \frac{1}{r}, \varphi : \Delta_r \xrightarrow{holom} \mathbb{C}, \varphi(0) = p, \varphi'(0) = v \right\} \leqslant \frac{1}{r}, \forall r > 0.$
*Then $0 \leqslant K_{\mathbb{C}}(p,v) \leqslant \frac{1}{r}$, $\forall r > 0$. And thus $K_{\mathbb{C}}(p,v) = 0$ $\forall p$ and $v$.*

Similarly to the Carathéodory pseudometric and pseudodistance, we have:

**Lemma B.3.4.** *In the case of the disk, we have $K_{\Delta}(p,v) = k(p,v)$ and $d_{\Delta}(p,p') = \omega(p,p')$ for all $p, p' \in \Delta$ and $v \in \mathbb{C}^n$.*

*Proof.* Let $v \in \mathbb{C} \setminus \{0\}$ and let $\varphi : \Delta \to \Delta$ be holomorphic such that $\varphi(0) = 0$ and $\varphi'(0) = rv$. By Schwarz Lemma B.1.1, we know that $|\varphi'(0)| \leqslant 1$; resulting in $\frac{1}{r} \geqslant |v|$ and so $K_{\Delta}(0,v) \geqslant |v|$. Consider $\varphi_0 : \Delta \to \Delta$ such that $\varphi_0(\zeta) = \frac{\zeta v}{|v|}$. In this case, $\varphi_0(0) = 0$ and $\varphi_0'(0) = \frac{v}{|v|}$. Thus we have, $K_{\Delta}(0,v) \leqslant |v|$. We conclude that $K_{\Delta}(0,v) = k(0,v)$ .
Since the Poincaré metric and the Kobayashi pseudometric are invariant under automorphisms of the unit disk (by corollary B.3.9), we have $K_{\Delta}(p,v) = K_{\Delta}(0, B_p'(p)v) = k(0, B_p'(p)v) = k(p,v)$. where $B_p(\zeta) = \dfrac{\zeta - p}{1 - \bar{p}\zeta}$.

The result for the distance follows from the fact that the Poincaré distance and the Kobayashi distance are both integrated distances. □

One of the main property of the Kobayashi pseudometric is the decreasing property under holomorphic maps.

**Lemma B.3.5.** *Let $F$ be holomorphic from $D$ to $D'$, then we have*

$$K_{D'}(F(p), dF(p)v) \leqslant K_D(p, v)$$

*This is known as the decreasing property of the Kobayashi pseudometric.*

*Proof.* Let $\alpha = \inf\left\{\frac{1}{r}, \varphi : \Delta \xrightarrow{holom} D, \varphi(0) = p, \varphi'(0) = rv\right\} = K_D(p, v)$ and let $\beta = \inf\left\{\frac{1}{r}, \varphi : \Delta \xrightarrow{holom} D', \varphi(0) = F(p), \varphi'(0) = F'(p)rv\right\} = K_{D'}(F(p), F'(p)v)$. Let $\frac{1}{r}$ be as in $\alpha$. Then there exists a holomorphic disk $\phi : \Delta \to D$ such that $\phi(0) = p$ and $\phi'(0) = rv$. Define the holomorphic disk $\varphi$ to be $F \circ \phi : \Delta \to D'$.
We have $\varphi(0) = F \circ \phi(0) = F(p)$ and $\varphi'(0) = (F \circ \phi)'(0) = F'(p)rv$. So $\frac{1}{r} \in B$ where

$$B = \left\{\frac{1}{r}, \varphi : \Delta \xrightarrow{\text{holom}} D', \varphi(0) = F(p), \varphi'(0) = F'(p)rv\right\}.$$

We conclude that $\beta \leqslant \frac{1}{r}$ for all $r$. This implies that $K_{D'}(F(p), dF(p)v) \leqslant K_D(p, v)$. ☐

It follows

**Proposition B.3.6.** *Let $F : D \to D'$ be holomorphic. Then we have*

$$d_{D'}(F(p), F(p')) \leqslant d_D(p, p')$$

*This is known as the decreasing property of the Kobayashi pseudodistance.*

*Proof.* Let $F : D \xrightarrow{\text{holom}} D'$ then (by Lemma B.3.5) we have
$K_{D'}(F(p), F(p')) \leqslant K_D(p, v)$. Let $\gamma : [0, 1] \xrightarrow{C^1} D$ s.t. $\gamma(0) = p$ and $\gamma(1) = v$ and integrate from 0 to 1 with respect to $t$ to get:

$$\int_0^1 K_{D'}\left(F\left(\gamma(t)\right), F'\left(\gamma(t)\right)\gamma'(t)\right) \leqslant \int_0^1 K_D\left(\gamma(t), \gamma'(t)\right).$$

This implies that $l(F \circ \gamma) \leqslant l(\gamma),\ \forall \gamma$. This results in the following:

$d_{D'}(F(p), F(p')) \leqslant l(F \circ \gamma) \leqslant l(\gamma),\ \forall \gamma$ implying that $d_{D'}(F(p), F(p'))$ is a lower bound for $L(\gamma)$. But $d_D(p, p')$ is the greatest lower bound for $L(\gamma)$ thus $d_{D'}(F(p), F(p')) \leqslant d_D(p, p')$. □

**Corollary B.3.7.** *If $D \subset D'$ then we have $d_{D'} \leqslant d_D$ and $K_{D'} \leqslant K_D$.*

*Proof.* Apply the previous proposition to the inclusion map $i : D \to D'$. □

**Example B.3.8.** $d_{\mathbb{C}^n} = 0$

*Indeed $F : \mathbb{C} \to \mathbb{C} \times \{0\}$ is biholomorphic where $\zeta \to (\zeta, 0, ..., 0)$.*

*Then $d_{\mathbb{C}\times\{0\}}((z_1, 0, ..., 0), (z_2, 0..., 0)) = d_{\mathbb{C}}(z_1, z_2) = 0$. Since $\mathbb{C} \times \{0\} \subset \mathbb{C}^n$ thus by the decreasing property (corollary B.3.9), we obtain $d_{\mathbb{C}^n} \leqslant d_{\mathbb{C}\times\{0\}} = 0$.*

We now prove the invariance of the Kobayashi pseudo metric and pseudo distance under biholomorphisms.

**Corollary B.3.9.**

*i) If $F : D \to D'$ is a biholomorphism then $K_{D'}(F(p), dF(p)v) = K_D(p, v)$.*

*ii) If $F : D \to D'$ is a biholomorphism then $F$ is an isometry for the corresponding pseudodistances; i.e. $d_{D'}(F(p), F(p')) = d_D(p, p')$.*

*Proof.* Apply the previous proposition to $F^{-1} : D' \to D$ which is holomorphic to get

$d_D\left(F^{-1}(F(p)), F^{-1}(F(p'))\right) \leqslant d_{D'}\left(F(p), F(p')\right)$. And so $d_D(p, p') \leqslant d_{D'}\left(F(p), F(p')\right)$. Thus we have equality of distances. The same holds for the pseudo metric. □

Little is known about explicitly calculating the Kobayashi metric. For special domains such as the disk, the automorphism group is a powerful tool for obtaining an explicit formula. Let us now, just for illustrative purposes, calculate the Kobayashi metric on the unit ball. Once more we will use the fact that the group of automorphisms act transitively on $\mathbb{B}$.

**Example B.3.10.** *We have* $K_{\mathbb{B}}(0,v)=||v||$.

*Proof.* First, let's prove that $K_{\mathbb{B}}(0,v)\leqslant ||v||$. Take $\varphi:\Delta\to\mathbb{B}$ defined by $\varphi(\zeta)=\zeta v/||v||$ where $\zeta\in\Delta$. Then $\varphi(0)=0$ and $\varphi'(0)=\dfrac{v}{||v||}$. Since $\varphi'(\zeta)=\dfrac{v}{||v||}$, this shows that $K_{\mathbb{B}}(0,v)\leqslant ||v||$ by the definition of $K_{\mathbb{B}}$.
Second, since $K$ is invariant under biholomorphism, we prove $K_{\mathbb{B}}(0,v)\geqslant ||v||$ for $v=(v_1,0,...,0)$. Take $\varphi:\Delta\to\mathbb{B}$ having $\varphi(0)=0$ and $\varphi'(0)=(rv_1,0,...,0)$. We choose $\varphi_1:\Delta\to\Delta$ and we apply (Schwarz Lemma B.1.1) to get $|rv_1|\leqslant 1$; i.e. $\frac{1}{r}\geqslant |v|$. Thus $K_{\mathbb{B}}(0,v)\geqslant ||v||$. □

# B.4 Further Properties

## B.4.1 Comparison Between the Two Pseudometrics

We have seen that the Kobayashi pseudometric and the Carathéodory pseudometric coincide in many cases such as in $\mathbb{C}$, $\Delta$ and $\mathbb{B}$. But in general, we have:

**Proposition B.4.1.** *For $D$ a domain in $\mathbb{C}^n$, $p\in D$ and $v\in\mathbb{C}^n$*

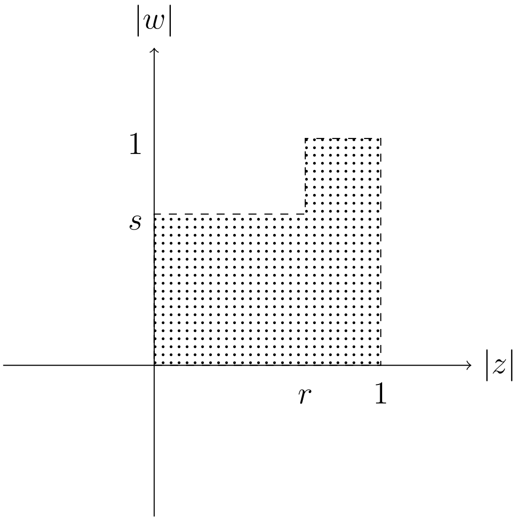
$$\gamma_D(p,v)\leqslant K_D(p,v)\ \textit{and}\ c_D\leqslant d_D$$

*Proof.* Let $f:D\to\Delta$ be holomorphic then by the decreasing property of K (lemma B.3.5), we have $k\left(f(p),df(p)v\right)\leqslant K_D(p,v)$ .Then $\sup\{k(f(p),df(p)v)\}\leqslant K_D(p,v)$ and thus we obtain $\gamma_D(p,v)\leqslant K_D(p,v)$. □

An example where these two pseudometrics don't coincide is given by the following:

**Example B.4.2.** *Consider the Hartogs domain*

$$D=\Delta\times\Delta\setminus\big\{(z,w)\in\mathbb{C}^2\ \textit{such that}\ |z|\leqslant r\ \textit{and}\ |w|\geqslant s\big\}.$$



It follows from Barth [2] that the Carathéodory and Kobayashi distances do not coincide on $D$. Also note that by Hartog's Theorem, any holomorphic function $f : D \to \Delta$ extends to the bidisk $\tilde{f} : \Delta \times \Delta \to \Delta$. It follows that $c_D$ is the restriction of $c_{\Delta\times\Delta}$ to $D$.

### B.4.2 Hyperbolic Domains

We have seen that on specific domains such as $\mathbb{C}$ the Kobayashi pseudodistance degenerates, whereas it turns out to be a distance on other domains such as $\mathbb{B}$. This induces the following definition:

**Definition B.4.3.** *We say that a domain $D$ in $\mathbb{C}$ is Kobayashi hyperbolic if the Kobayashi pseudodistance $d_D$ is a distance.*

**Lemma B.4.4.** *For $D \subset \mathbb{C}^n$ bounded, we have $D$ is Kobayashi hyperbolic.*

**Definition B.4.5.** *We say that a domain $D$ is Brody hyperbolic if there are no non-constant entire maps in $D$; that is, all holomorphic maps $f$ must be constants.*

**Example B.4.6.** *1. For $D \subset \mathbb{C}$ bounded, $D$ is Brody hyperbolic by Liouville Theorem.*

*2. According to the Little Picard Theorem* $\mathbb{C} \setminus \{0,1\}$ *is Brody hyperbolic.*

We have the following propostion:

**Proposition B.4.7.**

*i) If $D$ is Kobayashi hyperbolic, then $D$ is Brody hyperbolic.*

*ii) For compact $D$, Brody hyperbolicity is equivalent to Kobayashi hyperbolicity.*

*Proof.* i) If $D$ is not Brody hyperbolic then we have a non-constant entire map $\varphi : \mathbb{C} \to D$. By the decreasing property, $D$ is not hyperbolic.

ii) See [6]. ☐

**Example B.4.8.** *i)* $\mathbb{C}$ *is not Kobayashi hyperbolic since* $\mathbb{C}$ *is not Brody hyperbolic. This is shown by observing that* $\mathbb{C}$ *contains entire functions from* $\zeta$ *to* $\zeta$.

*ii)* $\mathbb{C} \setminus \{0\}$ *is not Kobayashi hyperbolic since it contains the holomorphic function from* $\mathbb{C}$ *to* $\mathbb{C} \setminus \{0\}$ *where* $\zeta$ *goes to* $e^{\zeta}$.

*iii)* $\mathbb{C} \setminus \{0,1\}$ *is Kobayashi hyperbolic.*

# Appendix C

# Complex Geodesics

## C.1 Main definitions

Let $D \subset \mathbb{C}^n$ be a domain. Assume that $c_D$ and $d_D$ are distances and let $\varphi : \Delta \to D$ be a holomorphic disk.

**Definition C.1.1.** *$\varphi$ is an infinitesimal K-extremal map (respectively infinitesimal C-extremal map) for $p \in D$ and $v \in \mathbb{C}^n$ if there exist $\zeta \in \Delta$ and $v_0 \in \mathbb{C}$ such that $\varphi(\zeta) = p$, $\varphi'(\zeta)v_0 = v$ and*

$$K_D(p, v) = k(\zeta, v_0)$$

$$(\textit{respectively } \gamma_D(p, v) = k(\zeta, v_0)).$$

**Definition C.1.2.** *$\varphi$ is an infinitesimal K-complex geodesic (respectively infinitesimal C-complex geodesic) if $\varphi$ is an infinitesimal K-extremal map (respectively C-extremal map) for $\varphi(\zeta)$, $\varphi'(\zeta)v_0$ for all $\zeta \in \Delta$ and $v_0 \in \mathbb{C}$.*

**Definition C.1.3.** *$\varphi$ is a K-extremal map (respectively C-extremal) for $p, q \in D$ if there exist $\zeta, \zeta' \in \Delta$ such that*

$$d_D(\varphi(\zeta), \varphi(\zeta')) = \omega(\zeta, \zeta')$$

$$(\textit{respectively } c_D(\varphi(\zeta), \varphi(\zeta')) = \omega(\zeta, \zeta')).$$

**Definition C.1.4.** *$\varphi$ is a K-complex geodesic (respectively C-complex geodesic) if $\varphi$ is a K-extremal map (respectively C-extremal map) for $\varphi(\zeta), \varphi(\zeta')$ for all $\zeta, \zeta' \in \Delta$, that is if $\varphi$ is an isometry for the relative Kobayashi distances (respectively Carathéodory distances).*

A consequence of Proposition B.4.1 and the decreasing properties is:

**Lemma C.1.5.** *1. If $\varphi$ is C-complex geodesic then $\varphi$ is K-complex geodesic.*

*2. If $\varphi$ is infinitesimal C-extremal map then $\varphi$ is infinitesimal K-extremal map.*

*3. If $\varphi$ is a C-extremal map then $\varphi$ is a K-extremal map.*

*Proof.* 1. Let $\varphi$ be a C-complex geodesic then for all $\zeta_0, \zeta \in \Delta$, $c_D(\varphi(\zeta_0), \varphi(\zeta)) = \omega(\zeta_0, \zeta)$. But from Proposition B.4.1 and the decreasing property of the Kobayashi pseudometric we know that $c_D(\varphi(\zeta_0), \varphi(\zeta)) \leqslant d_D(\varphi(\zeta_0), \varphi(\zeta)) \leqslant \omega(\zeta_0, \zeta)$. And the result is

$$d_D(\varphi(\zeta_0), \varphi(\zeta)) = \omega(\zeta_0, \zeta)$$

for all $\zeta_0, \zeta \in \Delta$ meaning that $\varphi$ is K-complex geodesic.

2. Since $\varphi$ is a holomorphic disk then for $p \in D$ and $v \in \mathbb{C}^n$ we have $K_D(p, v) \leqslant k(\zeta, v_0)$ by Lemma B.3.5. But $\varphi$ is an infinitesimal C-extremal map which results in $\gamma_D(p, v) = k(\zeta, v_0) \leqslant K_D(p, v)$ by Proposition B.4.1. This proves that $\varphi$ is infinitesimal K-extremal map.

3. Let $\varphi$ be a C-extremal map then for $p, q \in D$, there exist $\zeta, \zeta' \in \Delta$ such that

$$\omega(\zeta, \zeta') = c_D(\varphi(\zeta), \varphi(\zeta')) \leqslant d_D(f(\zeta_0), \varphi(\zeta)) \leqslant \omega(\zeta, \zeta')$$

where we conclude that $\varphi$ is a K-extremal map.

□

## C.2 Study of Complex Geodesics

An important and difficult question is to characterize complex geodesics and study their existence and uniqueness. The main result is (see Proposition 3.2 [12] and Proposition 2.6.3 [1])

**Theorem C.2.1.** *If $\varphi$ is an infinitesimal C-extremal map then $\varphi$ is a C-complex geodesic.*

*Proof.* Let $\varphi$ be an infinitesimal C-extremal map for $p \in D$ and $v \in \mathbb{C}^n$ then there exist $\zeta_0 \in \Delta$ and $v_0 \in \mathbb{C}$ such that $\varphi(\zeta_0) = p$, $\varphi'(\zeta_0)v_0 = v$ and $\gamma_D(p, v) = k(\zeta_0, v_0)$.
We can assume $\zeta_0 = 0$ ; since otherwise, we compose $\varphi$ with the automorphism of $\Delta$ $\psi(\zeta) = \dfrac{\zeta_0 + \zeta}{1 + \overline{\zeta_0}\zeta}$ where $\varphi \circ \psi(0) = p$ and $(\varphi \circ \psi)'(0)v_0 = (1 - |\zeta_0|^2)v$ which gives

$$\gamma_D\left(\varphi(\zeta_0), (\varphi \circ \psi)'(0)v_0\right) = (1 - |\zeta_0|^2)\gamma_D(p, v) = |v_0|.$$

Given $\gamma_D\left(\varphi(0), \varphi'(0)\dfrac{v_0}{|v_0|}\right) = 1$, then there exists a sequence $h_\nu : D \to \Delta$ of holomorphic disks such that $h_\nu(\varphi(0)) = 0$ and

$$\lim_{\nu \to \infty} \left| dh_\nu(\varphi(0))\varphi'(0)\frac{v_0}{|v_0|} \right| = 1.$$

By Montel's Theorem, every family of holomorphic maps $\{h_\nu \circ \varphi\}$, where $h_\nu \circ \varphi$ is from $\Delta$ to $\Delta$, admits a subsequence $\{h_{\nu_j} \circ \varphi\}$ that converges normally on compacts of $\Delta$ to a map $g : \Delta \to \Delta$ having $g(0) = \lim\limits_{j \to \infty} (h_{\nu_j} \circ \varphi)(0) = \lim\limits_{\nu \to \infty} h_\nu(\varphi(0)) = 0$. By Weierstrass Theorem, we have

$$|g'(0)| = \lim_{j \to \infty} \left| dh_{\nu_j}(\varphi(0))\varphi'(0)\frac{v_0}{|v_0|} \right| = \lim_{j \to \infty} |d(h_\nu \circ \varphi(0))| = 1.$$

By Schwarz lemma, $g$ is a biholomorphism.

By the decreasing property, we have

$$c_D(\varphi(0), \varphi(\zeta)) \leqslant \omega(0, \zeta).$$

Applying the decreasing property on $h_{\upsilon_j}$ yields in

$$\omega(h_{\nu_j}(\varphi(0)), h_{\nu_j}(\varphi(0))) \leqslant \omega(0, \zeta).$$

As $j \to \infty$,

$$\omega(h_{\nu_j}(\varphi(0)), h_{\nu_j}(\varphi(0))) \to \omega(g(0), g(\zeta)).$$

But because $g$ is biholomorphic then by Theorem B.1.5 we have,

$$\omega(g(0), g(\zeta)) = \omega(0, \zeta).$$

Combining these inequalities results in

$$\omega(0, \zeta) = \omega(g(0), g(\zeta)) \leqslant c_D(\varphi(0), \varphi(\zeta)) \leqslant \omega(0, \zeta).$$

Therefore $c_D(\varphi(0), \varphi(\zeta)) = \omega(0, \zeta)$ for all $\zeta \in \Delta$ showing that $\varphi$ is a C-complex geodesic. □

The result also holds in the case of extremal maps (Proposition 3.3 [12] and Proposition 2.6.3 [1]]

**Theorem C.2.2.** *If $\varphi$ is a C-extremal map then $\varphi$ is a C-complex geodesic.*

*Proof.* Let $\varphi$ be a C-extremal map for $p, q \in D$ then there exist $\zeta_0, \zeta_1 \in \Delta$ such that $c_D(\varphi(\zeta_0), \varphi(\zeta_1)) = \omega(\zeta_0, \zeta_1)$. Then there exists a sequence of holomorphic functions $h_\nu : D \to \Delta$ such that

$$\lim_{\nu \to \infty} w(h_\nu(\varphi(\zeta_0)), h_\nu(\varphi(\zeta_1))) = w(\zeta_0, \zeta_1)$$

By Montel's Theorem, since $h_\nu \circ \varphi : \Delta \to \Delta$ is holomorphic for all $\nu$, then there exists a subsequence $\{h_{\nu_j} \circ \varphi\}$ normally convergent on compacts of $\Delta$ to a holomorphic $g : \Delta \to \Delta$ such that $\omega(g(\zeta_0), g(\zeta_1)) = \omega(\zeta_0, \zeta_1)$. Then by Ahlfors Pick Lemma, g is biholomorphic, and therefore for all $\zeta \in \Delta$ we have

$$c_D(\varphi(\zeta_0), \varphi(\zeta)) \leqslant \omega(\zeta_0, \zeta)$$

by remark 4. On the other hand, we have

$$\omega(h_{\nu_j}(\varphi(0)), h_{\nu_j}(\varphi(\zeta_0))) \leqslant \omega(\zeta_0, \zeta)$$

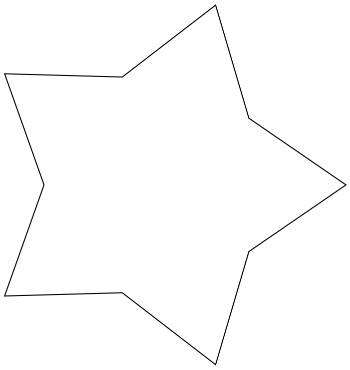
by the decreasing property on $h_{\nu_j}$. Therefore $c_D(\varphi(0), \varphi(\zeta)) = \omega(0, \zeta)$ for all $\zeta \in \Delta$ showing that $\varphi$ is a C-complex geodesic. $\square$

## C.3 A Mapping Problem

In this section we will study a rigidity mapping problem between two domains of $\mathbb{C}^n$. We first introduce some geometric notions.

**Definition C.3.1.** *A domain $D$ is called convex if for any two points $x, y \in D$ and $t$ in the interval $[0, 1]$, the segment joining $x$ and $y$ lies in $D$; meaning that, $(1-t)x + ty \in D$.*

**Example C.3.2.** *The domains $\Delta$, $\mathbb{B}$, and $\mathbb{C}^n$ are convex whereas the following star-shaped* *domain is not convex.*



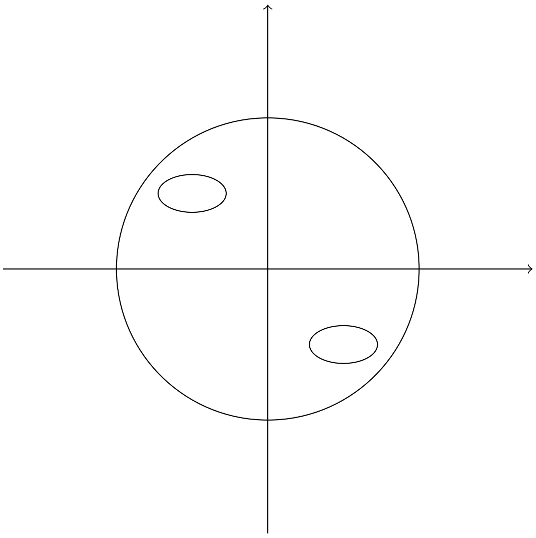
**Definition C.3.3.** *We say a point $x_0 \in \partial D$ is a complex extreme point of $\overline{D}$ if the only vector $y \in \mathbb{C}^n$ such that $x_0 + \Delta y \subset \overline{D}$ is $y = 0$; where $\Delta y = \{\zeta y$ such that $\zeta \in \Delta\}$.*

**Example C.3.4.** *All the boundary points on the disk ball $\mathbb{B}$ are extreme points whereas in the bidisk $\Delta \times \Delta \subset \mathbb{C}^2$, the vertices are extreme points while the point $(1, 0)$ is not an extreme point.*

**Definition C.3.5.** *A domain $D \in \mathbb{C}^n$ is called balanced if whenever $z \in D$ and $\lambda \in \mathbb{C}$ such that $|\lambda| \leqslant 1$ then $\lambda z \in D$.*

Note that balanced domains are symmetric with respect the origin.

**Example C.3.6.** *The following figure is not balanced.*



**Example C.3.7.** *In $\mathbb{C}^n$, balls centered at the origin are balanced domains.*

A beautiful result arises when we take specific geometric conditions on two domains $D_1$ and $D_2$ proving that any holomorphic map preserving the Carathéodory distances between them must be linear (Theorem 3 [12]).

**Theorem C.3.8.** *Let $D_1$ and $D_2$ be two bounded, convex, balanced open neighborhoods of* 0 *in $\mathbb{C}^n$ and let $F : D_1 \to D_2$ be holomorphic such that $F(0) = 0$ and*

$$c_{D_2}(0, F(x)) = c_{D_1}(0, x)$$

*for all $x \in D_1$. If every point of $\partial D_2$ is a complex extreme point of $\overline{D_2}$, then $F$ is the restriction to $D_1$ of a linear map from $\mathbb{C}^n$ to $\mathbb{C}^n$:*

$$F(x) = dF(0)x \quad \textit{for all } x \in D_1.$$

*Proof.* Let $D_1$ and $D_2$ be two bounded, convex, balanced open neighborhoods of 0 in $\mathbb{C}^n$. Let $u \in \partial D_1$ and consider the holomorphic disk $\varphi : \Delta \to D_1$ given by $\varphi(\zeta) = \zeta u$. Let $V$ be an open neighborhood of $u \in \partial D_1$. We need to find a sequence $\zeta_n \to 1$ such that $\varphi(\zeta_n) \to u$. Let $\zeta_n = 1 - \frac{1}{n}$ then $\varphi(1 - \frac{1}{n}) = (1 - \frac{1}{n})u \to u$ then for large $n$, $\varphi(1 - \frac{1}{n}) \in V \cap D_1$. And so $\varphi(\Delta_{1-\frac{1}{n}}) \subset D_1$.

Step 1: We prove that $\varphi$ is a C-complex geodesic. Let $\zeta$ be in $\Delta$. By the decreasing property of the Carathéodory metric (Proposition B.2.4), we have $c_{D_1}(0, \zeta u) \leqslant \omega(0, \zeta)$. Since $\Delta u \subset D_1$ we then consider the map $D_1 \to$ *spanu* to obtain

$$\omega(0, \zeta) \leqslant c_{\Delta u}(0, \zeta u) \leqslant c_{D_1}(0, \zeta u) \leqslant \omega(0, \zeta).$$

Thus $c_{D_1}(\varphi(0), \varphi(\zeta)) = \omega(0, \zeta)$ for all $\zeta \in \Delta$ meaning that $\varphi$ is a C-complex geodesic.

Step 2: We prove that $F \circ \varphi$ is a C-complex geodesic.

By assumption, we have $c_{D_2}(0, F(x)) = c_{D_1}(0, x)$ for all $x \in D_1$. Let $\zeta \in \Delta$ and $x = \zeta u \in D_1$. Knowing that $F(0) = 0$, we have for all $\zeta \in \Delta$,

$$c_{D_2}\left(F \circ \varphi(0), F \circ \varphi(\zeta)\right) = \omega(0, \zeta).$$

This shows that $F \circ \varphi$ is a C-complex geodesic.

Step 3: It follows from Vesentini [11] that $F \circ \varphi$ is linear.

Step 4: We conclude that $F$ is a linear map from $D_1$ to $\mathbb{C}^n$.

Let $F(x) = dF(0)x + P_2(x) + P_3(x) + ...$ be the Taylor power series expansion of $F$ around 0, where $P_2, P_3, ...$ are homogeneous polynomials of degrees $2, 3, ...$ in $\mathbb{C}^n$. Since $F$ is linear, then for all $x \in D_1$, we have $P_2(x) = 0, P_3(x) = 0, ...$ and we end up with

$$F(x) = dF(0)x \qquad \text{for all} \;\; x \in D_1.$$

Thus $F$ is a linear map from $D_1$ to $\mathbb{C}^n$. ☐

**Remark 8.** *In Theorem C.3.8, instead of assuming $c_{D_2}(0, F(x)) = c_{D_1}(0, x)$ for all $x \in D$, we can prove the linearity of $F$ by assuming that the Kobayashi distances are preserved:*

$$d_{D_2}(0, F(x)) = d_{D_1}(0, x) \quad \textit{for all } x \in D.$$

In the special case $D_1$ and $D_2$ are Euclidean balls in $\mathbb{C}^n$ we obtain a direct corollary (Lemma 4.1 [12]):

**Corollary C.3.9.** *Let $\mathbb{B}_1$ and $\mathbb{B}_2$ be the open unit balls for $\mathbb{C}^n$ and assume that every boundary point of $\mathbb{B}_2$ is a complex extreme point of $\overline{\mathbb{B}}_2$. If $F : \mathbb{B}_1 \to \mathbb{B}_2$ is holomorphic* *such that*

$$||F(x)|| = ||x|| \quad \textit{for all } x \in \mathbb{B}_1$$

*then $F$ must be linear: $F(x) = dF(0)x$ for all $x \in \mathbb{B}_1$.*

Theorem C.3.8 can be interpreted as the higher dimensional generalization of Schwarz Lemma; indeed the result follows directly from Theorem C.3.8 applied to $D_1 = D_2 = \Delta$. Theorem C.3.8 can also be interpreted as the metric version of Cartan's Uniqueness Theorem which we state and prove for the unit ball:

**Theorem C.3.10** (Cartan's Uniqueness Theorem)**.** *Let $F : \mathbb{B} \to \mathbb{B}$ be holomorphic. Assume that $F(0) = 0$ and $dF(0) = Id$.*

$$\textit{Then } F(z) = z \quad \textit{for all } z \in \mathbb{B}.$$

*Proof.* Consider $F : \mathbb{B} \to \mathbb{B}$ holomorphic such that $F(0) = 0$ and $dF(0) = Id$. Since $dF(0) = Id$ we have $\gamma_{\mathbb{B}}(0, dF(0)v) = \gamma_{\mathbb{B}}(0, v)$ for all $v \in \mathbb{C}^n$. By Theorem C.3.8 we know that $F$ is the restriction to $\mathbb{B}$ of a linear map from $\mathbb{C}^n$ into $\mathbb{C}^n$, namely $F(z) = dF(0)z$ for all $z \in \mathbb{B}$. □

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