AMERICAN UNIVERSITY OF BEIRUT

Local Existence and Uniqueness of the Solution to the 2D Hasegawa−Mima Equation with Periodic Boundary Conditions

by

HAGOP KORK KARAKAZIAN

A thesis
submitted in partial fulfillment of the requirements
for the degree of Master of Science
to the Department of Mathematics
of the Faculty of Arts and Sciences
at the American University of Beirut

Beirut, Lebanon
February 2016

AMERICAN UNIVERSITY OF BEIRUT

Local Existence and Uniqueness of the Solution to the 2D Hasegawa–Mima Equation with Periodic Boundary Conditions

by

HAGOP KORK KARAKAZIAN

Approved by:

Prof. Nabil Nassif, Professor	Advisor
Mathematics	
Prof. Mohammad El Smaily, Assistant Professor	Member of Committee
Mathematics	
Prof. Bassam Shayya, Professor	Member of Committee
Mathematics	

Date of thesis defense: February 8, 2016

AMERICAN UNIVERSITY OF BEIRUT

THESIS, DISSERTATION, PROJECT RELEASE FORM

Student Name: Karakazian Hagop Kork

Last First Middle

○ Master's Thesis ○ Master's Project ○ Doctoral Dissertation

☐ I authorize the American University of Beirut to: (a) reproduce hard or electronic copies of my thesis, dissertation, or project; (b) include such copies in the archives and digital repositories of the University; and (c) make freely available such copies to third parties for research or educational purposes.

☐ I authorize the American University of Beirut, **three years after the date of submitting my thesis, dissertation, or project,** to: (a) reproduce hard or electronic copies of it; (b) include such copies in the archives and digital repositories of the University; and (c) make freely available such copies to third parties for research or educational purposes.

Signature Date

Acknowledgments

First and foremost, I give my praise and thankfulness to the One who spoke me into existence, and gave me all the means and abilities to produce this work.

I also give my deepest appreciation to my advisor, Prof. Nabil Nassif, who sparked in me the passion of studying PDEs both in numerical and theoretical context. Thank you for always being ready to smile, help, discuss and motivate.

I also thank my committee members: Prof. Bassam Shayya and Prof. Mohammad El Smaily for their encouragement.

Finally, I thank my parents, my sister and my fiancé for supporting me, not only throughout my studies at AUB, but also throughout their existence and uniqueness in my life.

Hagop Karakazian

An Abstract of the Thesis of

HAGOP KORK KARAKAZIAN for Master of Science
Major: Mathematics

Title: Local Existence and Uniqueness of the Solution to the 2D Hasegawa−Mima Equation with Periodic Boundary Conditions

Magnetic plasma confinement is one of the most promising ways in future energy production. To understand the phenomena related to energy production through plasma confinement, several mathematical models can be found in literature (see [1, 2, 3], for example), of which the simplest and powerful 2D turbulent system model is the Hasegawa-Mima equation, which describes the time evolution of drift wave. Although it was originally derived by Akira Hasegawa and Kunioki Mima in [2], it can be extended [4, 5] and put as

$$(\Delta - I)u_t + \{u, \Delta u\} + ku_y = 0 \tag{1}$$

where $\{u, v\} = u_x v_y - u_y v_x$ is the Poisson bracket, $u(x, y, t)$ describes the electrostatic fluctuations, $k = \partial_x \ln \frac{n_0}{\omega_{ci}}$, where n_0 is the background particle density that depends only on the x-direction, and ω_{ci} is the ion cyclotron frequency that depends only on the initial magnetic field [6]. As a cultural note, equation (1) is also referred as the Charney-Hasegawa-Mima equation in geophysical context.

In this thesis, we give a new elementary construction of periodic Sobolev spaces on an open domain $\Omega = (0, L) \times (0, L)$ on which we prove the local existence and uniqueness of the solution to equation (1), coupled with necessary periodic boundary conditions. We do this via perturbing it into a semilinear abstract Cauchy problem and using analytic semigroup methods to obtain a local solution. Employing some *a priori* estimates, we obtain a local solution to the original problem. We finally establish its uniqueness, and comment on the existence of a global solution.

Contents

To my parents for their selfless love.

Chapter 1

Introduction

1.1 The Hasegawa-Mima Equation

The Hasegawa-Mima Equation is the simplest and powerful 2D turbulent system model originally derived by Akira Hasegawa and Kunioki Mima in 1978 (see [2]). However it can be extended [4, 5] and put as

$$(\Delta - I)u_t + \{u, \Delta u\} + ku_y = 0 \tag{1.1}$$

where $\{u, v\} = u_x v_y - u_y v_x$ is the Poisson bracket, $u(x, y, t)$ describes the electrostatic fluctuations, $k = \partial_x \ln \frac{n_0}{\omega_{ci}}$, where n_0 is the background particle density that depends only on the x-direction, and ω_{ci} is the ion cyclotron frequency that depends only on the initial magnetic field [6]. For a detailed derivation of (1.1), we refer the reader to [5].

Throughout this thesis, we consider an open bounded square domain $\Omega = (0, L) \times (0, L)$ with boundary $\Gamma = \partial\Omega$, and develop necessary results to establish the local existence and uniqueness of the solution to the two-dimensional Periodic Hasegawa-Mima problem

$$\text{(HM)} \begin{cases} (\Delta - I)u_t + \{u, \Delta u\} + ku_y = 0 & \text{on } \Omega \times (0, T] \\ u(x, y, 0) = u_0(x, y) & \text{on } \Omega \\ u \text{ satisfies some periodic boundary conditions} & \text{on } \Gamma \end{cases} \tag{1.2}$$

for a given initial condition u_0. For example, one can impose the following periodic boundary conditions of zeroth and first order

$$
\text{(PBC)} \begin{cases} u(0,y) = u(L,y) \text{ and } u_x(0,y) = u_x(L,y) & \forall y \in (0,L) \\ u(x,0) = u(x,L) \text{ and } u_y(x,0) = u_y(x,L) & \forall x \in (0,L) \end{cases} \tag{1.3}
$$

In fact, we will show that for a smooth enough given initial condition u_0 (in $H^4_P(\Omega)$), the problem (HM) with (PBC) above has a unique local $C^{\infty,2}$ solution on $(0,T^*)\times\Omega$ where $T^* > 0$ is a temporal value depending only on u_0 (see Corollary 5.14).

Our two main existence results are Theorem 5.13 and Corollary 5.14, followed by the uniqueness result Theorem 5.16.

1.2 Known Results and Issues in Literature

There is no reliable paper regarding the existence and uniqueness of the solution to the two-dimensional Hasegawa-Mima equation with periodic boundary conditions. However, several papers address the topic on the whole plane $\mathbb{R}^2$, from which [6] by Boling Guo and Yongqian Han, and [7] by Lionel Paumond, played a crucial role in writing this thesis.

Both papers perturb the Hasegawa-Mima equation with a linear operator and use the theory of analytic semigroups, but with opposing methods. [6] formulates the perturbed Hasegawa-Mima equation as a straight Banach fixed point problem as follows.

Problem 1.1. Consider a non-linear operator G on the metric space

$$
\begin{aligned}
Y = \{w \in C([0,T]; H^m(\mathbb{R}^2)) \ : \ t^\alpha w(t) \in C^\alpha((0,T]; H^m(\mathbb{R}^2)),\ 0<\alpha<1,\ w(0) = w_0 \\
\|w\|_{C([0,T];H^m)} + [t^\alpha w]_{C^\alpha((0,T];H^m)} \le \rho,\ m \ge 2\}
\end{aligned}
$$

where $\rho > 0$ is to be determined, and define $G(w) = v$, where v is the solution of the PDE

$$
v_t - \lambda\Delta v = -\left\{(I-\Delta)^{-1}w, w\right\} - k(I-\Delta)^{-1}w_y
$$

Show that G has a fixed point.

The issue with this formulation is the continuity of G with respect to its metric, simply because the Poisson bracket decreases spacial regularity by 3.

On the other hand, [7] formulates it as a semilinear abstract Cauchy problem and uses fractional powers of the perturbing operator to resolve the continuity issue found above. It proves local existence and uniqueness for $u_0 \in H^4(\mathbb{R}^2)$, whose global existence still remains open. It also proves global existence of a weak solution for $u_0 \in H^2(\mathbb{R}^2)$, whose uniqueness still remains open.

1.3 Outline of the Thesis

In chapter 2, we will define the two-dimensional periodic boundary conditions of arbitrary order, give a new elementary construction of two-dimensional Periodic Sobolev space $H_P^m(\Omega)$, and study its properties.

In chapter 3, we will consider the strongly elliptic operator $A_\lambda = \lambda\Delta(\Delta - I)$ on $H_P^5(\Omega)$, with real $\lambda > 0$. We will first show that the negation of $A_\lambda + \eta_0 I$ is the infinitesimal generator of a uniformly bounded analytic semigroup of operators on $H_P^1(\Omega)$, where $\eta_0 > \lambda$. And then conclude that the negation of A_λ is the infinitesimal generator of an analytic semigroup of operators on $H_P^1(\Omega)$. We will finally conjecture that this is true for every strongly elliptic operator on $H_P^{2m+1}(\Omega)$.

In chapter 4, we will consider an equivalent version of the Hasegawa-Mima equation and perturb it into a semilinear abstract Cauchy problem. Then we will follow the analytic semigroup methods found in [7] and [8] to establish the local existence of a solution. We will also comment on the global existence.

Finally in chapter 5, we will prove some *a priori* estimates and use them to obtain a local solution to the Hasegawa-Mima equation. We will then establish the uniqueness of the local solution.

For convenience, we give preliminarily results from functional analysis and analytic semigroup theory in the Appendices A and B.

Chapter 2

The Periodic Sobolev Space $H^m_P(\Omega)$

Despite the fact that we have a rich literature on PDEs with Dirichlet and Neumann boundary conditions (see for example [9, 10]), there is the lack of results on PDEs with periodic boundary conditions. It could be that the way of imposing periodic boundary conditions has been done simply through using the Sobolev space $W^{m,p}(\mathbb{T}^N)$ defined on the N-dimensional torus $\mathbb{T}^N = \mathbb{R}^N/\mathbb{Z}^N$. On the other hand, [11, p.149] defines the periodic Sobolev space $H^m_p(Q^N)$ of order m as the $\|\cdot\|_m$ completion of the space of infinitely differentiable functions restricted to the N-dimensional cube $Q^N = [0, L]^N$ which are L-periodic in each direction

$$u(x + Le_j) = u(x) \quad j = 1, \cdots N$$

where e_j a unit vector in the j^{th} component. However it fails to indicate whether weak derivatives are also L-periodic.

In this chapter, we define the two-dimensional periodic boundary conditions of arbitrary order, give a new elementary construction of two-dimensional Periodic Sobolev space $H^m_P(\Omega)$, and study its properties. This space turns out to be slightly different than the ones mentioned above. For example, unlike $W^{m,p}(\mathbb{T}^2)$, it doesn't assume m^{th} order periodic boundary conditions, and possesses Poincaré inequality; and unlike $H^m_p(Q^2)$, weak derivatives satisfy periodic boundary conditions of order up to $m - 1$. Following our construction, one can define $H^m_P((0, L)^N)$ and $W^{m,p}_P((0, L)^N)$.

2.1 The Periodic Boundary Conditions

A fundamental reason one might want to impose periodic boundary conditions is that they physically model the interaction of a wave (in our case, plasma drift waves) with the boundary of its medium. We define them formally as follows.

Definition 2.1. We say that a real-valued function u satisfies the periodic boundary conditions

PBC^0 of order 0 if and only if

$$(\text{PBC}^0)\begin{cases}(\text{PBC}^0_x) & u(0,y)=u(L,y) \quad \forall y\in(0,L)\\ (\text{PBC}^0_y) & u(x,0)=u(x,L) \quad \forall x\in(0,L)\end{cases}$$

Also, if k is a positive integer, we say that a real-valued function u satisfies the periodic boundary conditions PBC^k of order k if and only if

$$(\text{PBC}^k)\begin{cases}(\text{PBC}^k_x) & \partial^k_x u(0,y)=\partial^k_x u(L,y) \quad \forall y\in(0,L)\\ (\text{PBC}^k_y) & \partial^k_y u(x,0)=\partial^k_y u(x,L) \quad \forall x\in(0,L)\end{cases}$$

or equivalently, $\partial^k_x u$ satisfies PBC^0_x and $\partial^k_y u$ satisfies PBC^0_y, where ∂_x and ∂_y denote the differential operators $\dfrac{\partial}{\partial x}$ and $\dfrac{\partial}{\partial y}$, respectively, for short.

When seeking a (classical) solution to a given PDE on an open bounded domain Ω, one usually works in a suitable Sobolev space $W^{m,p}(\Omega)$ or a subspace to see whether it has a weak solution. However when imposing boundary conditions on functions in $W^{m,p}(\Omega)$, one needs to make sure that they and their corresponding derivatives are defined on the boundary Γ. Thanks to the Trace Theorem ([10, p.258] and [9, p.315]) below, which allows us to do that.

Theorem 2.2 (Trace Theorem)**.** *Let $U\subset\mathbb{R}^N$ be a regular open set. Then there exists a bounded linear operator* $\text{Tr}: W^{1,p}(U)\longrightarrow L^p(\partial U)$*, namely the extension by density of the function* $C_0^\infty(\mathbb{R}^N)\longrightarrow L^p(\partial U)$ *by* $u\mapsto u|_{\partial U}$*, such that*

(i) $\text{Tr}(u)=u|_{\partial U}$ *if* $u\in W^{1,p}(U)\cap C(\overline{U})$

(ii) $\|\text{Tr}(u)\|_{L^p(\partial U)}\leq C\,\|u\|_{W^{1,p}(U)}$ *for each* $u\in W^{1,p}(U)$*, with the constant* C *depending only on* p *and* U*.*

Now if $u\in H^m(\Omega)=W^{m,2}(\Omega)$ with integer $m\geq 1$, then for every integer $k=0,1,\cdots m-1$, we have that $\partial^k_x u,\partial^k_y u\in H^1(\Omega)$ so that $\text{Tr}(\partial^k_x u),\text{Tr}(\partial^k_y u)\in L^2(\Gamma)$. Thus we make the following definition.

Definition 2.3. Let m and k be two integers such that $m\geq 1$ and $0\leq k\leq m-1$. We say that $u\in H^m(\Omega)$ satisfies the periodic boundary conditions PBC^k of order k if and only if $\text{Tr}(u)$ does almost everywhere.

2.2 The Periodic Sobolev Space $H_P^m(\Omega)$ and its Properties

Definition 2.4. Let $m \geq 1$ be an integer. Define the *Periodic Sobolev space of order* m as

$$H_P^m(\Omega) = \left\{ u \in H^m(\Omega) \ : \ \int_\Omega u \, d\mu = 0 \text{ and } u \text{ satisfies PBC}^k \ \forall k = 0, 1, \cdots m-1 \right\}$$

where μ is the Lebesgue measure, and

$$\begin{aligned} & u \text{ satisfies PBC}^k \ \forall k = 0, 1, \cdots m-1 \\ \iff & \text{Tr}(u) \text{ satisfies PBC}^k \text{ a.e. } \forall k = 0, 1, \cdots m-1 \\ \iff & \text{Tr}(\partial_x^k u) \text{ satisfies PBC}_x^0 \text{ and } \text{Tr}(\partial_y^k u) \text{ satisfies PBC}_y^0 \text{ a.e. } \forall k = 0, 1, \cdots m-1 \end{aligned}$$

For convenience, we let

$$H_P^0(\Omega) = \left\{ u \in L^2(\Omega) \ : \ \int_\Omega u \, d\mu = 0 \right\}$$

Remark. $H^m(\mathbb{T}^2)$, with the zero average condition, is a proper subset of $H_P^m(\Omega)$, as demonstrated by the example below.

Example 2.5. The function $u(x, y) = \sin \pi x - 2/\pi$ with zero average on $[0, 1] \times [0, 1]$ is in $H_P^1(\Omega)$ but not in $H^1(\mathbb{T}^2)$, because its x-derivative fails to be periodic on the boundary.

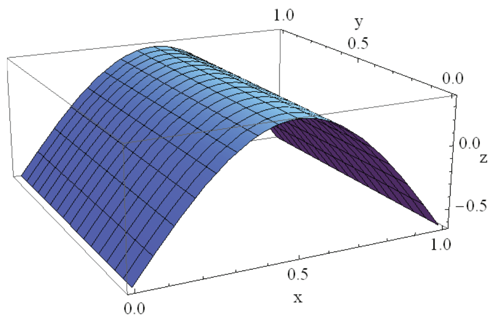

Figure 2.1: $u(x, y) = \sin \pi x - 2/\pi$ is in $H_P^1(\Omega)$ but not in $H^1(\mathbb{T}^2)$

Proposition 2.6. *For every integer $m \geq 0$, $H_P^m(\Omega)$ is a closed subspace of $H^m(\Omega)$, and so it is a Hilbert space.*

Proof. Let $u_1, u_2 \in H_P^m(\Omega)$, and $r_1, r_2 \in \mathbb{R}$. Then $r_1 u_1 + r_2 u_2 \in H^m(\Omega)$ because $H^m(\Omega)$ is a vectorspace. Observe that

$$\int_\Omega (r_1 u_1 + r_2 u_2) \, d\mu = r_1 \int_\Omega u_1 \, d\mu + r_2 \int_\Omega u_2 \, d\mu = 0$$

Now assume $m \geq 1$, then for each $k = 0, 1, \cdots, m-1$, using the linearity of Trace and differential operators, we have

$$\mathrm{Tr}(\partial_x^k(r_1u_1 + r_2u_2)) = r_1\mathrm{Tr}(\partial_x^k u_1) + r_2\mathrm{Tr}(\partial_x^k u_2)$$

which satisfies PBC_x^0 because $\mathrm{Tr}(\partial_x^k u_1)$ and $\mathrm{Tr}(\partial_x^k u_2)$ do. Similarly $\mathrm{Tr}(\partial_y^k(r_1u_1{+}r_2u_2))$ satisfies PBC_y^0. Therefore, $H_P^m(\Omega)$ is a subspace of $H^m(\Omega)$.

Now let $\{u_n\}$ be a sequence in $H_P^m(\Omega)$ such that $u_n \to u$ in $H^m(\Omega)$. Observe that by Cauchy-Schwarz inequality

$$\begin{aligned}\left|\int_\Omega u\ d\mu\right| &= \left|\int_\Omega u\ d\mu - \int_\Omega u_n\ d\mu\right| = \left|\int_\Omega (u - u_n)\ d\mu\right| \leq \int_\Omega |u - u_n|\ d\mu \\ &\leq \|u - u_n\| \cdot \|1\| = \sqrt{\mu(\Omega)}\, \|u - u_n\| \leq L\, \|u - u_n\|_m \to 0 \quad \text{as } n \to \infty\end{aligned}$$

so that $\displaystyle\int_\Omega u\ d\mu = 0$. Now for each $k = 0, 1, \cdots, m-1$. By the Trace Theorem, there exists a constant $C > 0$ such that

$$\begin{aligned}\left\|\mathrm{Tr}(\partial_x^k u_n) - \mathrm{Tr}(\partial_x^k u)\right\| &= \left\|\mathrm{Tr}(\partial_x^k u_n - \partial_x^k u)\right\| \\ &\leq C\left\|\partial_x^k u_n - \partial_x^k u\right\|_1 \leq C\, \|u - u_n\|_m \to 0 \quad \text{as } n \to \infty\end{aligned}$$

so that $\mathrm{Tr}(\partial_x^k u_n) \to \mathrm{Tr}(\partial_x^k u)$ in $L^2(\Gamma)$, as $n \to \infty$. Hence there exists a subsequence $\{u_{n_l}\}$ in $H_P^m(\Omega)$ such that $\mathrm{Tr}(\partial_x^k u_{n_l}) \to \mathrm{Tr}(\partial_x^k u)$ pointwise a.e, as $l \to \infty$, from which we immediately conclude that $\mathrm{Tr}(\partial_x^k u)$ satisfies PBC_x^0 a.e. Similarly, $\mathrm{Tr}(\partial_y^k u)$ satisfies PBC_y^0 a.e. Therefore, $u \in H_P^m(\Omega)$, so that $H_P^m(\Omega)$ is a closed in $H^1(\Omega)$. □

Now we prove some fundamental properties of $H_P^m(\Omega)$. For that purpose, we first recall the following from [9, p.312].

Theorem 2.7 (Poincaré-Wirtinger's Inequality)**.** *Let Ω be a connected open set of class C^1 and let $1 \leq p \leq \infty$. Then there exists a constant $C > 0$ such that*

$$\|u - \widetilde{u}\|_{L^p} \leq C\, \|\nabla u\|_{L^p} \quad \forall u \in W^{1,p}(\Omega)$$

where $\widetilde{u} = \dfrac{1}{\mu(\Omega)}\displaystyle\int_\Omega u\ d\mu$ is the average value of u on Ω with respect to measure μ.

Corollary 2.8 (Poincaré Inequality for Periodic Sobolev spaces)**.** *There exists a constant $C > 0$ such that*

$$\|u\| \leq C\, \|\nabla u\| \quad \forall u \in H_P^1(\Omega)$$

Let $\alpha = (a, b)$ be a (2-dimensional) multi-index of length $|\alpha| = a + b$. We will denote by D^α the differential operator $\partial_x^a \partial_y^b$.

Theorem 2.9 ($H_P^m(\Omega)$ is a Banach Algebra for $m \geq 2$)**.** *Let $m \geq 2$ be an integer. If $u, v \in H_P^m(\Omega)$, then $uv \in H_P^m(\Omega)$.*

Proof. Let $m \geq 2$ be an integer, and $u, v \in H_P^m(\Omega)$. Observe that $\Omega = (0, L) \times (0, L)$ satisfies the *cone condition* in Appendix A because if we let define $\mathcal{C}$ in the xy-plane with the polar coordinates

$$\mathcal{C} = \{(r, \theta) \ : \ 0 \leq r \leq L/4, \ 0 < \theta < \pi/8\} \tag{2.1}$$

then clearly each point $p \in \Omega$ is the vertex of some finite cone $\mathcal{C}_p$ contained in Ω that is congruent to $\mathcal{C}$. Thus by Theorem A.8, $uv \in H^m(\Omega)$. Now to show that $D^\alpha(uv)$ satisfies PBC^0 for every multi-index $|\alpha| \leq m - 1$, write

$$D^\alpha(uv) = \sum_{\beta_1 + \beta_2 = \alpha} D^{\beta_1} u D^{\beta_2} v$$

where we have used Proposition A.7 recursively, and observe that the terms $D^{\beta_1}u$ and $D^{\beta_2}v$ satisfy PBC^0 for $|\beta_1| \leq m-1$ and $|\beta_2| \leq m-1$, respectively. Hence $D^\alpha(uv)$ for $\alpha \leq m-1$. $\square$

2.3 Density of $C_P^\infty(\Omega)$ in $H_P^m(\Omega)$

Let $U \subset \mathbb{R}^N$ be an open domain. Denote by $C^m(U)$, the Banach space of m-times continuously differentiable functions restricted to U (note that $C^m(U) \subset C(\overline{U})$), and set $C^\infty(U) = \bigcap_{m \geq 0} C^m(U)$. Define

$$C_P^m(U) = H_P^m(U) \cap C^m(U) \text{ and set } C_P^\infty(U) = \bigcap_{m \geq 0} C_P^m(U) \tag{2.2}$$

The aim of this section is to prove that $C_P^\infty(\Omega)$ is dense in $H_P^m(\Omega)$ using the double Fourier series.

If $u \in L^2(\Omega)$, we can write its double Fourier series as

$$u(x, y) = \sum_{n,m=0}^{\infty} F_{n,m} e^{i\pi(nx+my)/L} \tag{2.3}$$

or equivalently,

$$
\begin{aligned}
u(x,y) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} & A_{n,m}\sin\frac{\pi nx}{L}\sin\frac{\pi my}{L} + B_{n,m}\cos\frac{\pi nx}{L}\sin\frac{\pi my}{L} \\
& + C_{n,m}\sin\frac{\pi nx}{L}\cos\frac{\pi my}{L} + D_{n,m}\cos\frac{\pi nx}{L}\cos\frac{\pi my}{L}
\end{aligned}
\tag{2.4}
$$

where

$$
\begin{aligned}
A_{n,m} &= \frac{4}{L^2}\int_{\Omega} u(x,y)\sin\frac{\pi nx}{L}\sin\frac{\pi my}{L}\,d\mu \\
B_{n,m} &= \frac{4}{L^2}\int_{\Omega} u(x,y)\cos\frac{\pi nx}{L}\sin\frac{\pi my}{L}\,d\mu \\
C_{n,m} &= \frac{4}{L^2}\int_{\Omega} u(x,y)\sin\frac{\pi nx}{L}\cos\frac{\pi my}{L}\,d\mu \\
D_{n,m} &= \frac{4}{L^2}\int_{\Omega} u(x,y)\cos\frac{\pi nx}{L}\cos\frac{\pi my}{L}\,d\mu
\end{aligned}
\tag{2.5}
$$

Remark. The Fourier series of a function $u \in H^m(\Omega)$ converges in $\|\cdot\|_m$. The only mention we could find about this claim in literature is [12, p.558].

Lemma 2.10. *If $u \in H^1_P(\Omega)$, then both n and m in its Fourier series (2.4) are even. That is, Fourier coefficients associated to odd n or odd m must be zero.*

Proof. Since $u(0,y) = u(L,y)$ for every $y \in (0,L)$, then we have

$$
\begin{aligned}
&\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[B_{n,m}\sin\frac{\pi my}{L} + D_{n,m}\cos\frac{\pi my}{L}\right] \\
&\qquad = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[B_{n,m}(-1)^n\sin\frac{\pi my}{L} + D_{n,m}(-1)^n\cos\frac{\pi my}{L}\right]
\end{aligned}
$$

so that by the orthogonality of $\sin\frac{\pi my}{L}$ and $\cos\frac{\pi my}{L}$ on $(0,L)$, we have that $(-1)^n = 1$. Thus n must be even.

Similarly, imposing $u(x,0) = u(x,L)$ for every $x \in (0,L)$, we get that m must be even. □

Theorem 2.11. *For $m \geq 1$, $C^\infty_P(\Omega)$ is dense in $H^m_P(\Omega)$ under the topology of $\|\cdot\|_m$.*

Proof. Let $u \in H^m_P(\Omega)$, and consider its double Fourier series given by (2.4). Now since $u \in H^1_P(\Omega)$, then by Lemma 2.10, both n and m in (2.4) must be even, so that all partial Fourier sums must be in $C^\infty_P(\Omega)$. Thus the result follows. □

The following is a useful consequence.

Theorem 2.12. *Let $u \in H_P^1(\Omega)$ be such that*

$$\int_\Omega uf \, d\mu = 0 \quad \forall f \in C_P^\infty(\Omega) \tag{2.6}$$

Then $u = 0$ a.e. on Ω.

Proof. By the density of $C_P^\infty(\Omega)$ in $H_P^1(\Omega)$, there exists a sequence $\{u_n\} \subset C_P^\infty(\Omega)$ such that $u_n \to u$ in $L^2(\Omega)$. Now

$$\|u\|^2 = \left| \int_\Omega uu_n \, d\mu - \int_\Omega u^2 \, d\mu \right| \leq \int_\Omega |u| \cdot |u_n - u| \, d\mu \leq \|u\| \cdot \|u_n - u\| \to 0$$

so that $u = 0$ a.e. on Ω. □

2.4 Spatial Consequences

Proposition 2.13. *If $u \in C_P^m(\Omega)$ where $m \geq 1$, then*

$$D^\alpha u \text{ satisfies PBC}^0 \text{ whenever } |\alpha| \leq m - 1 \tag{2.7}$$

We will refer (2.7) as the spacial consequences of PBCs.

Proof. Let $u \in C_P^m(\Omega)$. It suffices to show $D^{(a,b)}u$, if exists, satisfies PBC^0 whenever $0 \leq a, b \leq m - 1$. First, observe that

(i) $\partial_x^a u$ satisfies PBC^0 whenever $0 \leq a \leq m - 1$:

By definition, $\partial_x^a u$ satisfies PBC$_x^0$ whenever $0 \leq a \leq m - 1$. Moreover, u satisfies PBC$_y^0$, which serves as the base case of proving $\partial_x^a u(x, 0) = \partial_x^a u(x, L)$ iteratively for $1 \leq a \leq m - 1$ as follows

$$\begin{aligned}
\partial_x^a u(x, 0) &= \lim_{h \to 0} \frac{\partial_x^{a-1} u(x + h, 0) - \partial_x^{a-1} u(x, 0)}{h} \\
&= \lim_{h \to 0} \frac{\partial_x^{a-1} u(x + h, L) - \partial_x^{a-1} u(x, L)}{h} = \partial_x^a u(x, L) \quad \forall x \in (0, L)
\end{aligned}$$

Hence $\partial_x^a u$ satisfies PBC$_y^0$ whenever $0 \leq a \leq m - 1$.

(ii) $\partial_y^b u$ satisfies PBC^0 whenever $0 \leq b \leq m - 1$:

The proof is similar to that of (i).

We proceed by nested induction as follows, where $x, y \in (0, L)$.

1. *base case*: $b = 1$

   (a) *base case*: $a = 0$

   Follows from (ii) above.

   (b) *inner inductive step*: Assume that $D^{(a,1)}u$ satisfies PBC^0 for some $0 \leq a \leq m-2$. Then

$$D^{(a+1,1)}u(x,0) = \lim_{h\to 0} \frac{D^{(a,1)}u(x+h,0) - D^{(a,1)}u(x,0)}{h}$$
$$\overset{\text{asmp}}{=} \lim_{h\to 0} \frac{D^{(a,1)}u(x+h,L) - D^{(a,1)}u(x,L)}{h} = D^{(a+1,1)}u(x,L)$$

$$D^{(a+1,1)}u(0,y) = \lim_{h\to 0} \frac{D^{(a+1,0)}u(0,y+h) - D^{(a+1,0)}u(0,y)}{h}$$
$$\overset{(i)}{=} \lim_{h\to 0} \frac{D^{(a+1,0)}u(L,y+h) - D^{(a+1,0)}u(L,y)}{h} = D^{(a+1,1)}u(L,y)$$

   Hence $D^{(a+1,1)}u$ satisfies PBC^0.

   Therefore $D^{(a,1)}u$ satisfies PBC^0 for all $0 \leq a \leq m-1$.

2. *outer inductive step*: Assume that $D^{(a,b)}u$ satisfies PBC^0 for some $1 \leq b \leq m-2$ and for all $0 \leq a \leq m-1$.

   (a) *base case*: $a = 0$

   Follows from (ii) above.

   (b) *inner inductive step*: Assume that $D^{(a,b+1)}u$ satisfies PBC^0 for some $0 \leq a \leq m-2$. Then

$$D^{(a+1,b+1)}u(x,0) = \lim_{h\to 0} \frac{D^{(a,b+1)}u(x+h,0) - D^{(a,b+1)}u(x,0)}{h}$$
$$\overset{\text{inner asmp}}{=} \lim_{h\to 0} \frac{D^{(a,b+1)}u(x+h,L) - D^{(a,b+1)}u(x,L)}{h} = D^{(a+1,b+1)}u(x,L)$$

$$D^{(a+1,b+1)}u(0,y) = \lim_{h\to 0} \frac{D^{(a+1,b)}u(0,y+h) - D^{(a+1,b)}u(0,y)}{h}$$
$$\overset{\text{outer asmp}}{=} \lim_{h\to 0} \frac{D^{(a+1,b)}u(L,y+h) - D^{(a+1,b)}u(L,y)}{h} = D^{(a+1,b+1)}u(L,y)$$

   Hence $D^{(a+1,b+1)}u$ satisfies PBC^0.

   Therefore $D^{(a,b+1)}u$ satisfies PBC^0 for all $0 \leq a \leq m-1$.

Therefore $D^{(a,b)}u$ satisfies PBC^0 for all $0 \leq a, b \leq m-1$. □

Corollary 2.14. *(2.7) holds for $u \in H_P^m(\Omega)$, where $m \geq 1$.*

Proof. Let $u \in H_P^m(\Omega)$ and consider a multi-index $|\alpha| \leq m-1$. Then by Theorem 2.11, there exists a sequence u_n in $C_P^m(\Omega)$ such that $u_n \to u$ in $\|\cdot\|_m$. Now by the Trace Theorem there exists a constant $C > 0$ so that

$$\|D^\alpha u_n|_\Gamma - \text{Tr}(D^\alpha u)\| \leq C \|D^\alpha u_n - D^\alpha u\|_1 \leq C \|u_n - u\|_m \to 0 \text{ as } n \to \infty$$

so that $D^\alpha u_n|_\Gamma \to \text{Tr}(D^\alpha u)$ in $L^2(\Omega)$. Hence there exists a subsequence $\{u_{n_l}\}$ in $C_P^m(\Omega)$ such that $D^\alpha u_{n_l}|_\Gamma \to \text{Tr}(D^\alpha u)$ pointwise a.e as $l \to \infty$, from which we immediately using Proposition 2.13 conclude that $D^\alpha u$ satisfies PBC^0 a.e. □

Corollary 2.15. *Let $I \subset \mathbb{R}$ be a temporal interval. Then (2.7) holds for every $u \in \mathcal{F}(I; H_P^m(\Omega))$, in the sense that it holds for $u(t)$ for all $t \in I$, where $m \geq 1$.*

At this point, we also mention other useful spacial consequences.

Proposition 2.16. *For every $u \in H_P^4(\Omega)$, we have $\|u\|_2^2 = \|u\|^2 + \|\nabla u\|^2 + \|\Delta u\|^2$.*

Proof. It suffices to show that $\|\Delta u\|^2 = \|u_{xx}\|^2 + 2\|u_{xy}\|^2 + \|u_{yy}\|^2$

$$\begin{aligned}\|\Delta u\|^2 &= \int_\Omega (u_{xx}^2 + 2u_{xx}u_{yy} + u_{yy}^2)\, d\mu \\ &= \|u_{xx}\|^2 + 2\int_\Omega u_{xx}u_{yy}\, d\mu + \|u_{yy}\|^2\end{aligned}$$

Now by Green's formula and periodic boundary conditions, we have

$$\begin{aligned}\|\Delta u\|^2 &= \|u_{xx}\|^2 + 2\cancelto{0}{\int_0^L [u_x u_{yy}]_0^L\, dy} - 2\int_\Omega u_x u_{yyx}\, d\mu + \|u_{yy}\|^2 \\ &= \|u_{xx}\|^2 - 2\cancelto{0}{\int_0^L [u_{yyx} u_{xy}]_0^L\, dx} + 2\int_\Omega u_{xy}^2\, d\mu + \|u_{yy}\|^2\end{aligned}$$

□

Proposition 2.17. *If $u \in H_P^1(\Omega)$, then both u_x and u_y have zero average value on Ω.*

Proof. By density of $C_P^\infty(\Omega)$ in $H_P^1(\Omega)$, consider a sequence $\{u_n\}$ in $C_P^\infty(\Omega)$ such that $u_n \to u$ in $H_P^1(\Omega)$, $u_{n_x} \to u_x$ in $L^2(\Omega)$, and $u_{n_y} \to u_y$ in $L^2(\Omega)$. Now for every $n \in \mathbb{N}$ we have

$$\int_\Omega u_{n_x}\, d\mu = \int_0^L \int_0^L u_{n_x}\, dxdy = \int_0^L \cancelto{0}{[u_n(L,y) - u_n(0,y)]}\, dy = 0$$

Now by Cauchy-Schwarz

$$\left|\int_\Omega u_x\, d\mu\right| = \left|\int_\Omega (u_x - u_{n_x})\, d\mu\right| \leq \int_\Omega |u_x - u_{n_x}|\, d\mu \leq \sqrt{\mu(\Omega)}\,\|u_x - u_{n_x}\| \to 0$$

as $n \to \infty$, where $\mu(\Omega) = L^2 < \infty$. Similarly, $\displaystyle\int_\Omega u_y \, d\mu = 0$. □

Remark. In the proof of Proposition 2.17, we didn't use the fact that u had zero average. Same is true in the following corollary.

Corollary 2.18. *If $u \in H_P^2(\Omega)$, then Δu has zero average.*

Proof. If $u \in H_P^2(\Omega)$, then $u_x, u_y \in H_P^1(\Omega)$, so that u_{xx} and u_{yy} have zero average. □

2.5 Integration Over the Boundary

One of the advantages of having periodic boundary conditions is that all integrals over the boundary vanish in the weak formulation(s). This is due to the following proposition.

Proposition 2.19. *Let $I \subset \mathbb{R}$ be a temporal interval, and $f, g_1, g_2 \in \mathcal{F}(I; L^2(\Omega))$ be such that $f(t), g_1(t)$, and $g_2(t)$ all satisfy PBC^0 for all $t \in I$. Set $\vec{g} = (g_1, g_2)$, and let $\vec{\nu}$ be the unit outward-pointing normal vector to Γ. Then*

$$\int_\Gamma f\vec{g} \cdot \vec{\nu} \, ds = 0$$

Proof. Write $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$, where

$$\begin{cases} \Gamma_1 : [0,1] \longrightarrow \Gamma \text{ given by } s \mapsto (sL, 0) \text{ with } \vec{\nu} = (0,-1) \\ \Gamma_2 : [0,1] \longrightarrow \Gamma \text{ given by } s \mapsto (L, sL) \text{ with } \vec{\nu} = (1,0) \\ \Gamma_3 : [0,1] \longrightarrow \Gamma \text{ given by } s \mapsto (L-sL, L) \text{ with } \vec{\nu} = (0,1) \\ \Gamma_4 : [0,1] \longrightarrow \Gamma \text{ given by } s \mapsto (0, L-sL) \text{ with } \vec{\nu} = (-1,0) \end{cases}$$

Now for each $t \in I$, we have

$$\int_\Gamma f\vec{g} \cdot \vec{\nu} \, ds = -\int_{\Gamma_1} fg_2 \, ds + \int_{\Gamma_2} fg_1 \, ds + \int_{\Gamma_3} fg_2 \, ds - \int_{\Gamma_4} fg_1 \, ds$$

Using the parameterization of Γ_i, we get

$$\begin{aligned} \int_\Gamma f\vec{g} \cdot \vec{\nu} \, ds = &-\int_0^1 (fg_2)(sL, 0, t)L \, ds + \int_0^1 (fg_1)(L, sL, t)L \, ds \\ &+ \int_0^1 (fg_2)(L-sL, L, t)L \, ds - \int_0^1 (fg_1)(0, L-sL, t)L \, ds \end{aligned}$$

Substituting $x = sL$, $y = sL$, $x = L - sL$, and $y = L - sL$, respectively, we get

$$\begin{aligned} \int_\Gamma f\vec{g} \cdot \vec{\nu} \, ds = &-\int_0^L (fg_2)(x, 0, t) \, dx + \int_0^L (fg_1)(L, y, t) \, dy \\ &- \int_L^0 (fg_2)(x, L, t) \, dx + \int_L^0 (fg_1)(0, y, t) \, dy \end{aligned}$$

So that, by the periodicity of the integrands over Γ, the result follows. $\square$

Chapter 3

The Operator $A_\lambda = \lambda\Delta(\Delta - I)$

In the setting of Sobolev spaces whose elements are compactly supported, it has been shown that the negation of any strongly elliptic operator of even order is the infinitesimal generator of an analytic semigroup of operators on $L^2(\Omega)$ (see Theorem 7.2.7 of [8]). In this chapter, we conjecture a similar result in the setting of Periodic Sobolev spaces after establishing necessary results to show that the negation of the strongly elliptic operator $A_\lambda = \lambda\Delta(\Delta - I)$ on $H^5_P(\Omega)$, with real $\lambda > 0$, is the infinitesimal generator of an analytic semigroup of operators on $H^1_P(\Omega)$.

3.1 Strong Ellipticity

We recall the following definitions from [8, p.207-209].

An n-tuple of non-negative integers $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ is called a *multi-index* of length

$$|\alpha| = \sum_{i=1}^{n} \alpha_i$$

In the context of $\mathbb{R}^n$, we set

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \text{ for } x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$$

and

$$D^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}$$

Let $U \subset \mathbb{R}^n$ be a bounded open set with smooth boundary ∂U. Consider the differential operator

$$A(x, D) = \sum_{|\alpha| \leq 2m} a_\alpha(x) D^\alpha$$

of order $2m$, where the coefficients $a_\alpha(x)$ are sufficiently smooth complex-valued functions of $x \in \overline{U}$. The *principal part* $A'(x, D)$ of $A(x, D)$ is the operator

$$A'(x,D) = \sum_{|\alpha|=2m} a_\alpha(x) D^\alpha$$

Definition 3.1. The operator $A(x,D)$ is strongly elliptic iff there exists a constant $c > 0$ such that

$$\text{Re}\,\left[(-1)^m A'(x,\xi)\right] \geq c\,|\xi|^{2m}$$

for all $x \in \overline{U}$ and $\xi \in R^n$.

Proposition 3.2. *For every $\lambda > 0,\ A_\lambda$ is strongly elliptic.*

Proof. Observe that

$$A_\lambda = \lambda\left(\frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 y^2} + \frac{\partial^4}{\partial y^4} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)$$

is a differential operator of order 4 (i.e. $m = 2$), so that its principle part is

$$A'_\lambda = \lambda\left(\frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4}\right)$$

Now let $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ and set $c = \lambda/4 > 0$, so that $\lambda - c > c$. By the inequality between arithmetic and geometric means, we have

$$\frac{\xi_1^4 + \xi_2^4}{2} \geq \sqrt{\xi_1^4 \xi_2^4} = \xi_1^2 \xi_2^2$$

so that

$$\frac{\lambda - c}{2}(\xi_1^4 + \xi_2^4) \geq c\xi_1^2\xi_2^2$$

Now

$$\text{Re}\,\left[(-1)^2 A'(\xi)\right] = \lambda(\xi_1^4 + \xi_2^4) \geq c(\xi_1^4 + 2\xi_1^2\xi_2^2 + \xi_2^4) = c\,|\xi|^4$$

Therefore A_λ is strongly elliptic. □

Inspired from [8, p.209], we conjecture the following.

Conjecture 3.3 (Gårding's Inequality for Periodic Sobolev spaces)**.** If $A(x,D)$ is a strongly elliptic operator of order $2m$, then there exist constants $c_0 > 0$ and $\eta_0 \geq 0$ such that for every $u \in H_P^{2m}(\Omega)$ we have

$$\text{Re}\,\langle Au, u\rangle \geq c_0\,\|u\|_m^2 - \eta_0\,\|u\|^2$$

Proposition 3.4. *Gårding's Inequality holds for A_λ.*

Proof.

$$\begin{aligned}\langle A_\lambda u, u\rangle &= \lambda \langle \Delta(\Delta u - u), u\rangle \\ &= \lambda \int_\Omega u\Delta^2 u \, d\mu - \lambda \int_\Omega u\Delta u \, d\mu\end{aligned} \tag{3.1}$$

Now by Green's formula, we have

$$\langle A_\lambda u, u\rangle = \lambda \int_\Gamma u\nabla(\Delta u)\cdot\vec{\nu}\, ds - \lambda \int_\Omega \nabla u\cdot\nabla(\Delta u)\, d\mu - \lambda \int_\Gamma u\nabla u\cdot\vec{\nu}\, ds + \lambda \int_\Omega (\nabla u)^2\, d\mu \tag{3.2}$$

where boundary terms vanish due to spacial consequences of PBCs. Now again applying Green's formula, we have

$$\langle A_\lambda u, u\rangle = \lambda \left\|\Delta u\right\|^2 - \lambda \int_\Gamma \Delta u\nabla u\cdot\vec{\nu}\, ds + \lambda \left\|\nabla u\right\|^2 \tag{3.3}$$

where boundary term vanishes due to spacial consequences of PBCs, so that by Proposition 2.16 we have

$$\langle A_\lambda u, u\rangle = \lambda \left\|\Delta u\right\|^2 + \lambda \left\|\nabla u\right\|^2 = \lambda \left\|u\right\|_2^2 - \lambda \left\|u\right\|^2 \tag{3.4}$$

Setting $c_0 = \lambda$ and $\eta_0 > \lambda$ completes the proof. □

3.2 Invertibility

The goal of this section is to show that $A_\lambda : H_P^m(\Omega) \longrightarrow H_P^{m-4}(\Omega)$ is invertible for every integer $m \geq 5$. For this purpose, we first recall two theorems.

Theorem 3.5. *[9, p.140][Lax-Milgram] Let $B(\cdot,\cdot)$ be a bicontinuous coercive bilinear form on a Hilbert space H. Then for every bounded linear functional F on H, there exists a unique $u \in H$ such that*

$$B(u, v) = F(v) \quad \forall v \in H$$

Proposition 3.6. *The operator $\Delta - I : H_P^3(\Omega) \longrightarrow H_P^1(\Omega)$ is invertible and*

$$\left\|u\right\|_3 \leq C \left\|(\Delta - I)u\right\|_1 \quad \forall u \in H_P^3(\Omega) \tag{3.5}$$

Proof. Equivalently, we show that $I - \Delta : H_P^3(\Omega) \longrightarrow H_P^1(\Omega)$ is invertible. Clearly, if $u \in H_P^3(\Omega)$, then $(I-\Delta)u \in H_P^1(\Omega)$, where we have used Corollary 2.18. Now let $f \in H_P^1(\Omega)$, and consider the following PDE

$$(\mathcal{P}_1) \begin{cases} u - \Delta u = f \\ u \text{ satisfies PBC}^0 \text{ and PBC}^1 \end{cases}$$

We aim to show that there exists a unique $u \in H_P^3(\Omega)$ such that $(I-\Delta)u = f$. Multiplying this last equation with $v \in H_P^1(\Omega)$ and integrating over Ω, we get

$$\int_\Omega uv \, d\mu - \int_\Omega \Delta uv \, d\mu = \int_\Omega fv \, d\mu$$

which by Green's formula, becomes

$$\int_\Omega uv \, d\mu - \int_\Gamma v\nabla u \cdot \vec{\nu} \, ds + \int_\Omega \nabla u \cdot \nabla v \, d\mu = \int_\Omega fv \, d\mu$$

where imposing PBC^1 for u and using Proposition 2.19, the boundary term vanishes, so that we get the weak formulation of $(\mathcal{P}_1)$

$$B(u,v) = \langle f, v\rangle \quad \forall v \in H_P^1(\Omega) \tag{3.6}$$

where

$$B(u,v) := \int_\Omega uv \, d\mu + \int_\Omega \nabla u \cdot \nabla v \, d\mu \tag{3.7}$$

Now observe that

(a) $B(\cdot,\cdot)$ is a bicontinuous bilinear form on $H_P^1(\Omega)$. To see this,

$$\begin{aligned} |B(u,w)| &\leq |\langle u, w\rangle| + |\langle \nabla u, \nabla w\rangle| \\ &\leq \|u\| \cdot \|w\| + \|\nabla u\| \cdot \|\nabla w\| \\ &\leq \|u\|_1 \cdot \|w\|_1 + \|u\|_1 \cdot \|w\|_1 \\ &\leq 2\,\|u\|_1 \cdot \|w\|_1 \end{aligned}$$

(b) $B(\cdot,\cdot)$ is a coercive on $H_P^1(\Omega)$. To see this,

$$B(u,u) = \langle u,u\rangle + \langle \nabla u, \nabla u\rangle = \|u\|^2 + \|\nabla u\|^2 = \|u\|_1^2$$

Now recall from Proposition 2.6 that $H_P^1(\Omega)$ is a Hilbert space. Now by the Lax-Milgram Theorem, there exists a unique $u \in H_P^1(\Omega)$ satisfying the weak formulation (3.6). With a usual elliptic regularity argument (see section 9.6 of [9], for example), it can be shown that $u \in H^3(\Omega)$ and $\|u\|_3 \leq C \|f\|_1$.

Now observe that since $C_P^\infty(\Omega) \subset H_P^1(\Omega)$, we have

$$\int_\Omega (u - \Delta u - f)v = 0 \quad \forall v \in C_P^\infty(\Omega)$$

so that by Theorem 2.12, $u - \Delta u = f$ a.e. on Ω, and so $u - \Delta u = f$ in $H_P^1(\Omega)$. To see $u \in H_P^3(\Omega)$, first observe that u being in $H_P^1(\Omega)$, $\int_\Omega u \, d\mu = 0$. Now since $\Delta u = u - f \in H_P^1(\Omega)$, then Δu satisfies PBC^0, namely (in the Trace Operator sense)

$$\begin{cases} u_{xx}(0,y) + u_{yy}(0,y) = u_{xx}(L,y) + u_{yy}(L,y) & \forall y \in (0,L) \\ u_{xx}(x,0) + u_{yy}(x,0) = u_{xx}(x,L) + u_{yy}(x,L) & \forall x \in (0,L) \end{cases}$$

But recall from (i) and (ii) in the proof of Proposition 2.13, that

$$\begin{cases} u_{yy}(0,y) = u_{yy}(L,y) & \forall y \in (0,L) \\ u_{xx}(x,0) = u_{xx}(x,L) & \forall x \in (0,L) \end{cases}$$

so that

$$\begin{cases} u_{xx}(0,y) = u_{xx}(L,y) & \forall y \in (0,L) \\ u_{yy}(x,0) = u_{yy}(x,L) & \forall x \in (0,L) \end{cases}$$

and so both u_{xx} and u_{yy} satisfy PBC^0. Now recall from above that

$$\int_\Gamma v \nabla u \cdot \vec{\nu} \, ds = 0 \quad \forall v \in H_P^1(\Omega)$$

so by a reverse argument as in Proposition 2.19, we must have that u satisfies PBC^1. Therefore, $u \in H_P^3(\Omega)$. □

Proposition 3.7. *Let $m \geq 3$ be an integer. Then the operator $\Delta - I : H_P^m(\Omega) \longrightarrow H_P^{m-2}(\Omega)$ is invertible and*

$$\|u\|_m \leq C \|(\Delta - I)u\|_{m-2} \quad \forall u \in H_P^m(\Omega) \tag{3.8}$$

Proof. Equivalently, we show that the operator $I - \Delta : H_P^m(\Omega) \longrightarrow H_P^{m-2}(\Omega)$ is invertible for every integer $m \geq 3$. We prove this using induction on m, where the base case is established by Proposition 3.6. Assume now that the proposition holds for some $m \geq 3$. Then, clearly, if $u \in H_P^{m+1}(\Omega)$, then $(I - \Delta)u \in H_P^{m-1}(\Omega)$, where we have used Corollary 2.18. Now let $f \in H_P^{m-1}(\Omega)$, then $f \in H_P^{m-2}(\Omega)$ so that by the inductive assumption, there exists a unique $u \in H_P^m(\Omega)$ satisfying the weak formulation (3.6). With a usual elliptic regularity argument (see section 9.6 of [9], for example), it can be shown that $u \in H^{m+1}(\Omega)$ and $\|u\|_{m+1} \leq C \|f\|_{m-1}$. Now as in the proof of Proposition 3.6, $u - \Delta u = f$ a.e. on Ω, and so $\Delta u = u - f$

a.e. on Ω. But $u \in H_P^{m-1}(\Omega)$, so that $\Delta u \in H_P^{m-1}(\Omega)$, and so in particular Δu satisfies PBC^{m-2}, namely (in the Trace Operator sense)

$$\begin{cases} \partial_x^{m-2} u_{xx}(0,y) + \partial_x^{m-2} u_{yy}(0,y) = \partial_x^{m-2} u_{xx}(L,y) + \partial_x^{m-2} u_{yy}(L,y) & \forall y \in (0,L) \\ \partial_y^{m-2} u_{xx}(x,0) + \partial_y^{m-2} u_{yy}(x,0) = \partial_y^{m-2} u_{xx}(x,L) + \partial_y^{m-2} u_{yy}(x,L) & \forall x \in (0,L) \end{cases}$$

But by the spacial consequences (equation (2.7)) of the PBCs on $u \in H_P^m(\Omega)$, we have

$$\begin{aligned} \partial_x^{m-2} u_{yy}(0,y) &= \lim_{h \to 0} \frac{\partial_x^{m-3} u_{xy}(0, y+h) - \partial_x^{m-3} u_{xy}(0,y)}{h} \\ &= \lim_{h \to 0} \frac{\partial_x^{m-3} u_{xy}(L, y+h) - \partial_x^{m-3} u_{xy}(L,y)}{h} \\ &= \partial_x^{m-2} u_{yy}(L,y) \quad \forall y \in (0,L) \end{aligned}$$

where we have used the density of $C_P^m(\Omega)$ in $H_P^m(\Omega)$ (Theorem 2.11). Similarly,

$$\begin{aligned} \partial_y^{m-2} u_{xx}(x,0) &= \lim_{h \to 0} \frac{\partial_y^{m-3} u_{xy}(x+h, 0) - \partial_y^{m-3} u_{xy}(x,0)}{h} \\ &= \lim_{h \to 0} \frac{\partial_y^{m-3} u_{xy}(x+h, L) - \partial_y^{m-3} u_{xy}(y,L)}{h} \\ &= \partial_y^{m-2} u_{xx}(x,L) \quad \forall x \in (0,L) \end{aligned}$$

so that

$$\begin{cases} \partial_x^{m-2} u_{xx}(0,y) = \partial_x^{m-2} u_{xx}(L,y) & \forall y \in (0,L) \\ \partial_y^{m-2} u_{yy}(x,0) = \partial_y^{m-2} u_{yy}(x,L) & \forall x \in (0,L) \end{cases}$$

or equivalently, u satisfies PBC^m. But since $u \in H_P^m(\Omega)$, which implies that u satisfies PBC^k for $0 \le k \le m-1$. Thus, $u \in H_P^{m+1}(\Omega)$. □

Proposition 3.8. *The operator* $\lambda\Delta : H_P^3(\Omega) \longrightarrow H_P^1(\Omega)$ *is invertible and*

$$\|u\|_3 \le C \|\lambda \Delta u\|_1 \quad \forall u \in H_P^3(\Omega) \tag{3.9}$$

Proof. Similar to that of Proposition 3.6. Note that Poincaré inequality is essential for bilinear operator obtained from the weak formulation to be coercive. □

Proposition 3.9. *Let* $m \ge 3$ *be an integer. Then the operator* $\lambda\Delta : H_P^m(\Omega) \longrightarrow H_P^{m-2}(\Omega)$ *is invertible and*

$$\|u\|_m \le C \|\lambda \Delta u\|_{m-2} \quad \forall u \in H_P^m(\Omega) \tag{3.10}$$

Proof. Similar to that of Proposition 3.7. □

Theorem 3.10. *Let $m \geq 5$ be an integer. Then the operator $\lambda\Delta(\Delta - I) : H_P^m(\Omega) \longrightarrow H_P^{m-4}(\Omega)$ is invertible and*

$$\|u\|_m \leq C \|\lambda\Delta(\Delta - I)u\|_{m-4} \quad \forall u \in H_P^m(\Omega) \tag{3.11}$$

Proof. Compose $\lambda\Delta$ with $\Delta - I$ and use Propositions 3.7 and 3.9. □

Inspired from [8, p.210], we conjecture the following.

Conjecture 3.11. Let $A(x, D)$ be a strongly elliptic operator of order $2m$. For every η satisfying Re $\eta \geq \eta_0$ (as in Gårding's Inequality) and every $f \in H_P^1(\Omega)$, there exists a unique $u \in H_P^{2m+1}(\Omega)$ satisfying the equation

$$A(x, D)u + \eta u = f$$

i.e., $A(x, D) + \eta I : H_P^{2m+1}(\Omega) \longrightarrow H_P^1(\Omega)$ is invertible.

Theorem 3.12. *Conjecture 3.11 holds for A_λ for real η. Moreover*

$$\|u\|_5 \leq C \|(A_\lambda + \eta I)u\|_1 \quad \forall u \in H_P^5(\Omega) \tag{3.12}$$

Proof. Taking the L^2-innerproduct of the equation $A_\lambda u + \eta u = f$ with $v \in H_P^2(\Omega)$, we get the weak formulation

$$\langle A_\lambda u + \eta u, v\rangle = \langle f, v\rangle \quad \forall v \in H_P^2(\Omega) \tag{3.13}$$

Now $B(u, v) := \langle A_\lambda u + \eta u, v\rangle$ is a bicontinuous bilinear operator, which by Gårding's Inequality (Proposition 3.4) is coercive on $H_P^2(\Omega)$ because

$$\begin{aligned} B(u, u) &= \langle A_\lambda u + \eta u, u\rangle = \langle A_\lambda u, u\rangle + \eta \|u\|^2 \\ &= \text{Re } \langle A_\lambda u, u\rangle + \eta \|u\|^2 \\ &\geq c_0 \|u\|_2^2 + (\eta - \eta_0) \|u\|^2 \geq c_0 \|u\|_2^2 \end{aligned}$$

Hence by the Lax-Milgram theorem, there exists a unique $u \in H_P^2(\Omega)$ which satisfies the weak formulation. Now since $C_P^\infty(\Omega) \subset H_P^2(\Omega)$, then

$$\langle A_\lambda u + \eta u - f, v\rangle = 0 \quad \forall v \in C_P^\infty(\Omega)$$

so that by Theorem 2.12, we have $A_\lambda u + \eta u = f$ a.e. in Ω. Hence now $A_\lambda u = f - \eta u \in H_P^1(\Omega)$, so that by the invertibility of A_λ (Theorem 3.10), we get that $u \in H_P^5(\Omega)$ and

$$\|u\|_5 \leq C \|f - \eta u\|_1 \leq C \|f\|_1 + \eta \|u\|_1$$

But taking the H^1-innerproduct of $A_\lambda u + \eta u = f$ with u, it is easy to see that $\|u\|_1 \leq \frac{1}{\eta} \|f\|_1$. And so

$$\|u\|_5 \leq (C+1) \|f\|_1 = (C+1) \|A_\lambda u + \eta u\|_1$$

□

Corollary 3.13. *Let $\eta \geq \eta_0$ be real, and $k \geq 1$ be an integer. Then $A_\lambda + \eta I : H_P^{2m+k}(\Omega) \longrightarrow H_P^k(\Omega)$ is invertible and*

$$\|u\|_{2m+k} \leq C \|(A_\lambda + \eta I)u\|_k \quad \forall u \in H_P^{2m+k}(\Omega) \tag{3.14}$$

Proof. We prove this by induction on k. The base case $k = 1$ is exactly Theorem 3.12. Now assume the result is true for some $k \geq 1$, and let $f \in H_P^{k+1}(\Omega)$. Then using $f \in H_P^k(\Omega)$ with the induction hypothesis, there exists $u \in H_P^{2m+k}(\Omega)$ such that $A_\lambda u + \eta u = f$. Now $A_\lambda u = f - \eta u \in H_P^{k+1}(\Omega)$. So that by invertibility of A_λ, we get that $u \in H_P^{2m+k+1}(\Omega)$ and

$$\|u\|_{2m+k+1} \leq C \|f - \eta u\|_{k+1} \leq C \|f\|_{k+1} + \eta \|u\|_{k+1}$$

But taking the H^{k+1}-innerproduct of $A_\lambda u + \eta u = f$ with u, it is easy to see that $\|u\|_{k+1} \leq \frac{1}{\eta} \|f\|_{k+1}$. And so

$$\|u\|_{2m+k+1} \leq (C+1) \|f\|_{k+1} = (C+1) \|A_\lambda u + \eta u\|_{k+1}$$

□

3.3 As a Generator of an Analytic Semigroup

Consider the operator $A_{\eta_0} = A_\lambda + \eta_0 I$, where $\eta_0 > \lambda > 0$ is the constant from Gårding's Inequality (Proposition 3.4), with the domain $D(A_{\eta_0}) = D(A_\lambda) = H_P^5(\Omega)$. By Corollary 3.13, $A_{\eta_0} : H_P^5(\Omega) \longrightarrow H_P^1(\Omega)$ is invertible. Moreover, we have the following corollary.

Remark. Since in the proof of Gårding's Inequality, we picked any $\eta_0 > \lambda$, then all the subsequent results hold for any $\eta_0 > \lambda$.

Corollary 3.14. *There exists a constant $C > 0$ such that*

$$\|u\|_1 \leq C \|A_{\eta_0} u\|_1 \quad \forall u \in H_P^5(\Omega) \tag{3.15}$$

Proof. Let $u \in H^5_P(\Omega)$, and set $f = A_{\eta_0}u = A_\lambda u + \eta_0 u$. Then taking its H^1-innerproduct with u and using Cauchy-Schwatz Inequality, we get

$$\langle A_\lambda u, u\rangle_1 + \eta_0 \|u\|_1 = \langle A_{\eta_0} u, u\rangle_1 = \langle f, u\rangle_1 \leq \|f\|_1 \cdot \|u\|_1$$

where, by the proof of Proposition 3.4 (noting that $u \in H^5_P(\Omega)$ and $\eta_0 > \lambda$),

$$\begin{aligned}\langle A_\lambda u, u\rangle_1 &= \langle A_\lambda u, u\rangle + \langle A_\lambda u_x, u_x\rangle + \langle A_\lambda u_y, u_y\rangle \\ &= \lambda \|u\|_2^2 - \lambda \|u\|^2 + \lambda \|u_x\|_2^2 - \lambda \|u_x\|^2 + \lambda \|u_y\|_2^2 - \lambda \|u_y\|^2 \\ &= \lambda(\|u\|_2^2 + \|u_x\|_2^2 + \|u_y\|_2^2) - \lambda \|u\|_1^2\end{aligned} \tag{3.16}$$

and so

$$\lambda(\|u\|_2^2 + \|u_x\|_2^2 + \|u_y\|_2^2) + (\eta_0 - \lambda) \|u\|_1^2 \leq \|f\|_1 \cdot \|u\|_1$$

where $\eta_0 - \lambda > 0$, so that

$$\|u\|_1 \leq \frac{1}{\eta_0 - \lambda} \|f\|_1 = \frac{1}{\eta_0 - \lambda} \|A_{\eta_0} u\|_1$$

□

Now we aim to show that $-A_{\eta_0}$ generates an analytic semigroup of contractions over $H^1_P(\Omega)$, and conclude that $-A_\lambda$ generates a uniformly bounded analytic semigroup.

Proposition 3.15. *The operator $-A_{\eta_0}$ is the infinitesimal generator of a C_0-semigroup of contractions over $H^1_P(\Omega)$.*

Proof. The result follows immediately from the Hille-Yosida Theorem (Theorem B.11) after we prove its hypothesis as follows:

(i) Since $C^\infty_P(\Omega)$ is dense in $H^1_P(\Omega)$ and $C^\infty_P(\Omega) \subset D(-A_{\eta_0}) \subset H^1_P(\Omega)$, it follows that $\overline{D(-A_{\eta_0})} = H^1_P(\Omega)$, so that $-A_{\eta_0}$ is densely defined. Let $u, v \in D(-A_{\eta_0})$. Then using Green's formula and proposition 2.19, we have

$$\begin{aligned}\langle -A_{\eta_0} u, v\rangle = -\langle A_{\eta_0} u, v\rangle &= -\left\langle \lambda\Delta^2 u - \lambda\Delta u + \eta_0 u, v\right\rangle \\ &= -\lambda \left\langle \Delta^2 u, v\right\rangle - \lambda \langle \Delta u, v\rangle + \eta_0 \langle u, v\rangle \\ &= -\lambda \left\langle u, \Delta^2 v\right\rangle - \lambda \langle u, \Delta v\rangle + \eta_0 \langle u, v\rangle \\ &= -\left\langle u, \lambda\Delta^2 v - \lambda\Delta v + \eta_0 v\right\rangle \\ &= -\langle u, A_{\eta_0} v\rangle = \langle u, -A_{\eta_0} v\rangle\end{aligned}$$

so that $-A_{\eta_0}$ is self-adjoint on $D(-A_{\eta_0})$. Thus $-A_{\eta_0}$ is closed by Proposition 2.19 of [13].

(ii) Let η be a positive real number. Then $\eta I + A_{\eta_0} = (\eta+\eta_0)I + A_\lambda$ is invertible on $D(-A_{\eta_0})$ by Corollary 3.13. Finally,

$$\begin{aligned}\left\|(\eta I + A_{\eta_0})^{-1}\right\|_{\mathcal{L}(H^1_P(\Omega))} &= \sup_{f\in H^1_P(\Omega), \|f\|_1=1} \left\|(\eta I + A_{\eta_0})^{-1} f\right\|_1 \\ &= \sup_{f\in H^1_P(\Omega), \|f\|_1=1} \|u\|_1\end{aligned}$$

where $u \in D(-A_{\eta_0})$ is the unique solution to $(\eta I + A_{\eta_0})u = f$. Now taking the H^1-innerproduct of the last equation with u, we get

$$(\eta + \eta_0)\|u\|_1^2 + \langle A_\lambda u, u\rangle_1 = \langle f, u\rangle_1$$

where, by using equation (3.16), we get

$$\eta\|u\|_1^2 + (\eta_0 - \lambda)\|u\|_1^2 + \lambda(\|u\|_2^2 + \|u_x\|_2^2 + \|u_y\|_2^2) = \langle f, u\rangle_1$$

where $\eta_0 - \lambda > 0$, and so by Cauchy-Schwarz Inequality,

$$\eta\|u\|_1^2 < \|f\|_1 \|u\|_1 = \|u\|_1$$

and hence

$$\|u\|_1 < \frac{1}{\eta}$$

Therefore

$$\left\|(\eta I + A_{\eta_0})^{-1}\right\|_{\mathcal{L}(H^1_P(\Omega))} < \frac{1}{\eta}$$

□

Theorem 3.16. *The operator $-A_{\eta_0}$ is the infinitesimal generator of a uniformly bounded analytic semigroup over $H^1_P(\Omega)$.*

Proof. Observe that by Corollary 3.13 and Proposition 3.15, $-A_{\eta_0}$ is a uniformly bounded C_0 semigroup with $0 \in \rho(-A_{\eta_0})$, so that $-A_{\eta_0}$ satisfies the hypothesis of Theorem B.18, by which it suffices to show that $-A_{\eta_0}$ satisfies Theorem B.18(c), which states: There exists $0 < \delta < \pi/2$

and $M > 0$ such that

$$\rho(-A_{\eta_0}) \supset \Sigma = \left\{\eta : |\arg \eta| < \frac{\pi}{2} + \delta\right\} \cup \{0\} \tag{3.17}$$

and

$$\|R(\eta : -A_{\eta_0})\|_{\mathcal{L}(H^1_P(\Omega))} \leq \frac{M}{|\eta|} \quad \forall \eta \in \Sigma, \eta \neq 0 \tag{3.18}$$

Claim 1: The numerical range $\nu(-A_{\eta_0})$ of $-A_{\eta_0}$ is contained in the negative real axis.

Proof of Claim 1: Recall from Definition B.13 that

$$\nu(-A_{\eta_0}) = \left\{\langle -A_{\eta_0} u, u\rangle_1 \ : \ u \in D(-A_{\eta_0}), \|u\|_1 = 1\right\} \tag{3.19}$$

Now let $u \in D(-A_{\eta_0})$ be such that $\|u\|_1 = 1$. Then by equation (3.16)

$$\begin{aligned} \langle -A_{\eta_0} u, u\rangle_1 &= - \langle A_\lambda u, u\rangle_1 - \eta_0 \|u\|_1^2 \\ &= -\lambda(\|u\|_2^2 + \|u_x\|_2^2 + \|u_y\|_2^2) + \lambda \|u\|_1^2 - \eta_0 \|u\|_1^2 \end{aligned} \tag{3.20}$$

where $\eta_0 \geq \lambda$ as above, so that

$$\begin{aligned} \langle -A_{\eta_0} u, u\rangle_1 &\leq -\lambda(\|u\|_2^2 + \|u_x\|_2^2 + \|u_y\|_2^2) \\ &\leq -\lambda(\|u\|^2 + \|u_x\|^2 + \|u_y\|^2) \\ &= -\lambda \cancelto{1}{\|u\|_1} = -\lambda < 0 \end{aligned} \tag{3.21}$$

Let $\Lambda = \sup \overline{\nu(-A_{\eta_0})} \leq -\lambda < 0$, $\theta \in (\pi/2, \pi)$, and define $\Sigma_\theta = \{\eta \in \mathbb{C} \ : \ |\arg \eta| < \theta\}$.

Claim 2: There exists $C_\theta > 0$ such that $d(\eta : \overline{\nu(-A_{\eta_0})}) \geq C_\theta |\eta| \quad \forall \eta \in \Sigma_\theta$

Proof of Claim 2: Let $\eta \in \Sigma_\theta$, then:

(i) If Re $\eta \geq 0$, then $d(\eta : \overline{\nu(-A_{\eta_0})}) \geq d(\eta : 0) = |\eta| \geq |\text{Im } \eta|$

(ii) If Re $\eta \leq \Lambda$, then $d(\eta : \overline{\nu(-A_{\eta_0})}) \geq |\text{Im } \eta|$

(iii) If $\Lambda <$ Re $\eta < 0$, then $d(\eta : \overline{\nu(-A_{\eta_0})}) \geq d(\eta : \Lambda) \geq |\text{Im } \eta|$

We aim to show that $|\text{Im } \eta| \geq C_\theta |\eta|$. Since $\eta \in \Sigma_\theta$, then there exists $\widetilde{\eta} \in \partial\Sigma_\theta = \{x \pm ix \tan\theta \ : \ x \leq 0\}$ such that

$$\text{Re } \widetilde{\eta} = \text{Re } \eta \text{ and } |\text{Im } \widetilde{\eta}| \leq |\text{Im } \eta| \tag{3.22}$$

namely, $\widetilde{\eta} = \text{Re}\,\eta + i\text{Re}\,\eta\,|\tan\theta|$. Now we have

$$\begin{aligned} |\text{Im}\,\eta| \geq |\text{Im}\,\widetilde{\eta}| = |\text{Re}\,\eta| \cdot |\tan\theta| \iff & \frac{|\text{Im}\,\eta|^2}{\tan^2\theta} \geq |\text{Re}\,\eta|^2 \\ \iff & \frac{|\text{Im}\,\eta|^2}{\tan^2\theta} + |\text{Im}\,\eta|^2 \geq |\eta|^2 \\ \iff & |\text{Im}\,\eta| \geq \left(\frac{1}{\tan^2\theta} + 1\right)^{-1/2} |\eta| \end{aligned} \tag{3.23}$$

Hence setting $C_\theta = \left(\frac{1}{\tan^2\theta} + 1\right)^{-1/2} > 0$ completes the proof.

Now let Σ be the complement of $\overline{\nu(A)}$ in $\mathbb{C}$, then clearly $\Sigma_\theta \subset \Sigma$. Observe that, by Corollary 3.13, $\eta \in \rho(-A_{\eta_0}) \quad \forall \eta \geq \eta_0$, so that Σ_θ intersects $\rho(-A_{\eta_0})$. Thus by Theorem (B.14), $\Sigma_\theta \subset \rho(-A_{\eta_0})$ and along with Claim 2,

$$\|R(\eta : -A_{\eta_0})\|_{\mathcal{L}(H_P^1(\Omega))} \leq d(\eta : \overline{\nu(-A_{\eta_0})})^{-1} \leq \frac{1/C_\theta}{|\eta|} \quad \forall \eta \in \Sigma_\theta, \eta \neq 0 \tag{3.24}$$

Finally, setting $\delta = \theta - \pi/2 \in (0, \pi/2)$ and $M = 1/C_\theta > 0$, the proof is complete. □

Corollary 3.17. *For every $\lambda > 0$, the operator $-A_\lambda$ is the infinitesimal generator of an analytic semigroup over $H_P^1(\Omega)$.*

Proof. Since $\eta_0 I$ is a bounded linear operator, the result follows immediately from Proposition B.20. Or one can observe that if $-A_{\eta_0}$ is the infinitesimal generator of the analytic semigroup $T(t)$, then $-A_\lambda$ will be the infinitesimal generator of the analytic semigroup $S(t) = e^{t\eta_0}T(t)$. To see this, we compute

$$S(t) = e^{-tA_\lambda} = e^{t\eta_0 I - tA_{\eta_0}} = e^{t\eta_0 I}e^{-tA_{\eta_0}} = e^{t\eta_0}T(t)$$

where we have used the fact

$$e^{t\eta_0 I} = \sum_{n=0}^{\infty} \frac{(t\eta_0 I)^n}{n!} = \sum_{n=0}^{\infty} \frac{(t\eta_0)^n}{n!} I = e^{t\eta_0} I$$

Now clearly by definition, $S(t)$ is an analytic semigroup. □

Inspired from the result above, we conjecture the following.

Conjecture 3.18. If $A(x, D)$ is a strongly elliptic operator of order $2m$, then the operator $-A(x, D)$ with the domain $H_P^{2m+1}(\Omega)$ is the infinitesimal generator of an analytic semigroup of operators on $H_P^1(\Omega)$.

Chapter 4

The Perturbed Hasegawa-Mima Equation

In this chapter, we consider an equivalent version of the Hasegawa-Mima equation and perturb it into a semilinear abstract Cauchy problem. We follow the analytic semigroup methods found in [7] and [8] to establish the local existence of a solution. We also comment on the global existence.

4.1 As a Semilinear Abstract Cauchy Problem

Recall from Proposition 3.7 that linear operator $(\Delta - I)$ is invertible, then equation (1.1) is equivalent to

$$u_t = -(\Delta - I)^{-1}\{u, \Delta u\} - k(\Delta - I)^{-1}u_y \tag{4.1}$$

which we will refer to as the *equivalent Hasegawa-Mima equation*.

Now let $0 < \lambda < 1$, and consider the operator

$$B_\lambda = A_\lambda + 2\lambda I = \lambda\Delta(\Delta - I) + 2\lambda I$$

Recall from Corollary 3.13 and Proposition 3.16 that $B_\lambda : H_P^5(\Omega) \longrightarrow H_P^1(\Omega)$ is invertible with

$$\|u\|_5 \leq C\, \|B_\lambda u\|_1 \quad \forall u \in H_P^5(\Omega) \tag{4.2}$$

and generates a uniformly bounded analytic semigroup over $H_P^1(\Omega)$.

Now perturbing the equivalent Hasegawa-Mima equation with the linear operator B_λ, we get the semilinear equation

$$u_t + B_\lambda u = f(u) \tag{4.3}$$

where $f(u) = -(\Delta - I)^{-1}\{u, \Delta u\} - k(\Delta - I)^{-1}u_y$, a non-linearity.

Given $u_0 \in H^4_P(\Omega)$ and $0 < T < \infty$, consider the perturbed Hasegawa-Mima problem as the semilinear abstract Cauchy problem

$$\text{(pHM)} \begin{cases} u'(t) + B_\lambda u(t) = f(u(t)) & t \in (0, T] \\ u(0) = u_0 \end{cases} \tag{4.4}$$

Remark. Since the non-linearity f decreases spacial regularity by one, it cannot map $H^m_P(\Omega)$ into $H^m_P(\Omega)$ for any integer m. This means that we cannot apply nice (local) existence theorems for semilinear abstract Cauchy problems. And so we are obliged to use a version (Theorem B.30) of the only (local) existence theorem found in literature (for example, see [8]) that fits well with the non-linearity f; and this one deals with fractional powers of operators generating analytic semigroups.

4.2 Existence of a Local Solution

We begin with a lemma which gives new useful formulations of the Poisson bracket.

Lemma 4.1 (Lemma 2.1 of [7])**.** *We have*

$$\{u, \Delta u\} = Q(u_x^2 - u_y^2) - P(u_x u_y) \quad \forall u \in H^3(\Omega) \tag{4.5}$$

and

$$\{u, \Delta u\} = P[Q(u)u] - Q[P(u)u] \quad \forall u \in H^4(\Omega) \tag{4.6}$$

where P and Q are the differential operators $\partial_x^2 - \partial_y^2$ and ∂_{xy}, respectively.

Proof. We simply compute

$$\begin{aligned}
Q(u_x^2 - u_y^2) - P(u_x u_y) &= \partial_{xy}[u_x^2 - u_y^2] - (\partial_x^2 - \partial_y^2)[u_x u_y] \\
&= \partial_x[2u_x u_{xy} - 2u_y u_{yy}] - \partial_x[u_{xx}u_y + u_x u_{xy}] + \partial_y[u_{xy}u_y + u_x u_{yy}] \\
&= 2u_{xx}u_{xy} + 2u_x u_{xxy} - 2u_{xy}u_{yy} - 2u_y u_{xyy} - u_{xxx}u_y - u_{xx}u_{xy} \\
&\qquad - u_{xx}u_{xy} - u_x u_{xxy} + u_{xyy}u_y + u_{xy}u_{yy} + u_{xy}u_{yy} + u_x u_{yyy} \\
&= u_x u_{xxy} - u_y u_{xyy} - u_{xxx}u_y + u_x u_{yyy} \\
&= u_x \Delta u_y - u_y \Delta u_x \\
&= \{u, \Delta u\}
\end{aligned}$$

and

$$\begin{aligned}
P[Q(u)u] - Q[P(u)u] &= P[u_{xy}u] - Q[u_{xx}u - u_{yy}u] \\
&= \partial_x^2[u_{xy}u] - \partial_y^2[u_{xy}u] - \partial_{xy}[u_{xx}u - u_{yy}u] \\
&= \partial_x[u_{xxy}u + u_{xy}u_x] - \partial_y[u_{xyy}u + u_{xy}u_y] \\
&\qquad - \partial_x[u_{xxy}u + u_{xx}u_y - u_{yyy}u - u_{yy}u_y] \\
&= \partial_x[u_{xy}u_x - u_{xx}u_y + u_{yyy}u + u_{yy}u_y] - \partial_y[u_{xyy}u + u_{xy}u_y] \\
&= u_{xxy}u_x + u_{xy}u_{xx} - u_{xxx}u_y - u_{xx}u_{xy} + u_{xyyy}u + u_{yyy}u_x \\
&\qquad + u_{xyy}u_y + u_{yy}u_{xy} - u_{xyyy}u - u_{xyy}u_y - u_{xyy}u_y - u_{xy}u_{yy} \\
&= u_{xxy}u_x - u_{xxx}u_y + u_{yyy}u_x - u_{xyy}u_y \\
&= \{u, \Delta u\}
\end{aligned}$$

□

Now recall from Appendix B.2.2 that for every $\alpha \in (0,1)$, $D(B_\lambda^\alpha)$ endowed with the norm

$$|||\cdot|||_\alpha := \|B_\lambda^\alpha \cdot\|_1 \tag{4.7}$$

is a Banach space. For our purpose, we will use $\alpha = 3/4$.

Lemma 4.2. *We have*

(i) $D(B_\lambda^{3/4}) = H_P^4(\Omega)$

(ii) $B_\lambda^{3/4} : H_P^4(\Omega) \longrightarrow H_P^1(\Omega)$ *is invertible*

(iii) $\|u\|_4 \leq C\, |||u|||_{3/4} \quad \forall u \in H_P^4(\Omega)$

Proof. (i) By Proposition B.22(a), for every $\alpha, \beta \geq 0$, $B_\lambda^{-(\alpha+\beta)} = B_\lambda^{-\alpha} \cdot B_\lambda^{-\beta}$. Hence

$$(B_\lambda^{-1/2})^2 = B_\lambda^{-1} : H_P^m(\Omega) \longrightarrow H_P^{m+4}(\Omega) \quad \forall m \geq 1$$

so that

$$B_\lambda^{-1/2} : H_P^m(\Omega) \longrightarrow H_P^{m+2}(\Omega) \quad \forall m \geq 1$$

Now

$$(B_\lambda^{-1/4})^2 = B_\lambda^{-1/2} : H_P^m(\Omega) \longrightarrow H_P^{m+2}(\Omega) \quad \forall m \geq 1$$

so that

$$B_\lambda^{-1/4} : H_P^m(\Omega) \longrightarrow H_P^{m+1}(\Omega) \quad \forall m \geq 1$$

Therefore

$$B_\lambda^{-3/4} = B_\lambda^{-1/2} B_\lambda^{-1/4} = H_P^m(\Omega) \longrightarrow H_P^{m+3}(\Omega) \quad \forall m \geq 1$$

and so

$$D(B_\lambda^{3/4}) = Range(B_\lambda^{-3/4}) = H_P^4(\Omega)$$

(ii) By definition and part (i).

(iii) It suffices to show that $B_\lambda^{3/4}$ is bounded on $H_P^4(\Omega)$. That is, there exists a constant $C > 0$ such that

$$\left\| B_\lambda^{3/4} u \right\|_1 \leq C \left\| u \right\|_4 \quad \forall u \in H_P^4(\Omega) \tag{4.8}$$

because then by the Bounded Inverse Theorem, $B_\lambda^{-3/4}$ will be bounded on $H_P^1(\Omega)$. Hence if $u \in H_P^4(\Omega)$, then letting $f = B_\lambda^{3/4} u \in H_P^1(\Omega)$, we have

$$\left\| u \right\|_4 = \left\| B_\lambda^{-3/4} f \right\|_4 \leq C \left\| f \right\|_1 = C \left\| B_\lambda^{3/4} u \right\|_1 = C \left\| |u| \right\|_{3/4} \tag{4.9}$$

for some constant $C > 0$.

We proceed by showing that $B_\lambda^{3/4}$ is bounded on $H_P^4(\Omega)$. We let $u \in H_P^4(\Omega)$ and by

(B.30) we have

$$\left\| B_\lambda^{3/4} u \right\|_1 = \frac{\sqrt{2}}{2\pi} \left\| \int_0^\infty t^{-1/4} B_\lambda (tI + B_\lambda)^{-1} u \, dt \right\|_1 \tag{4.10}$$

Let $w = (tI + B_\lambda)^{-1} u$, then $B_\lambda w = u - tw$ and so by equation (4.2) there exists a constant $C_1 > 0$ such that

$$\|w\|_1 \leq \|w\|_5 \leq C_1 \|u - tw\|_1 \leq C_1 \|u\|_1 + tC_1 \|w\|_1$$

so that

$$\|w\|_1 \leq \frac{C_1}{1 - tC_1} \|u\|_1$$

Now

$$\left\| B_\lambda (tI + B_\lambda)^{-1} u \right\|_1 = \|B_\lambda w\|_1 \leq \|u\|_1 + t \|w\|_1 \leq \left(1 + \frac{tC_1}{1 - tC_1} \right) \|u\|_1$$

so that $B_\lambda (tI + B_\lambda)^{-1}$ is bounded near $t = 0$. Thus there exist $\epsilon > 0$, such that $B_\lambda (tI + B_\lambda)^{-1}$ is bounded on $(0, \epsilon)$. Hence we have

$$\left\| B_\lambda^{3/4} u \right\|_1 \leq \frac{\sqrt{2}}{2\pi} \left[\int_0^\epsilon t^{-1/4} \left\| B_\lambda (tI + B_\lambda)^{-1} u \right\|_1 \, dt + \int_\epsilon^\infty t^{-1/4} \left\| B_\lambda (tI + B_\lambda)^{-1} u \right\|_1 \, dt \right] \tag{4.11}$$

By the argument above, there exists a constant $C_2 > 0$ such that the left term is bounded

$$\int_0^\epsilon t^{-1/4} \left\| B_\lambda (tI + B_\lambda)^{-1} u \right\|_1 \, dt \leq C_2 \int_0^\epsilon t^{-1/4} \, dt \, \|u\|_1 \leq \frac{4 C_2 \epsilon^{3/4}}{3} \|u\|_1$$

Now to bound the right term, recall from Corollary B.12 that for every $t > 0$,

$$(tI - B_\lambda)^{-1} = \int_0^\infty e^{-ts} S(s) \, ds$$

where $S(s)$ is the C_0 semigroup generated by B_λ. Now we have

$$\int_\epsilon^\infty t^{-1/4} \left\| B_\lambda (tI + B_\lambda)^{-1} u \right\|_1 \, dt = \int_\epsilon^\infty t^{-1/4} \left\| B_\lambda \int_0^\infty e^{-ts} S(s) \, ds \right\|_1 \, dt$$

But since B_λ is closed and the inner integral converges, we can put it inside the inner integral, and get

$$\int_{\epsilon}^{\infty} t^{-1/4} \left\|B_\lambda (tI + B_\lambda)^{-1} u\right\|_1 \, dt \leq \int_{\epsilon}^{\infty} t^{-1/4} \int_0^{\infty} e^{-ts} \left\|B_\lambda S(s)\right\|_1 \, ds \, dt$$

Now by Theorem B.18(d), there exists a constant $C_3 > 0$ such that

$$\|B_\lambda S(s)\|_{\mathcal{L}(H_P^1(\Omega))} \leq \frac{C_3}{t} \quad \forall t > 0$$

then

$$\begin{aligned}\int_{\epsilon}^{\infty} t^{-1/4} \left\|B_\lambda (tI + B_\lambda)^{-1} u\right\|_1 \, dt &\leq C_3 \int_{\epsilon}^{\infty} t^{-5/4} \int_0^{\infty} e^{-ts} \, ds \, dt \, \|u\|_1 \\ &= C_3 \int_{\epsilon}^{\infty} t^{-9/4} \, dt \, \|u\|_1 \leq C_4 \|u\|_1\end{aligned}$$

Therefore, there exists a constant $C_5 > 0$ such that

$$\left\|B_\lambda^{3/4} u\right\|_1 \leq C_5 \|u\|_1 \leq C_5 \|u\|_4$$

□

The following lemma shows that the nonlinearity f can be locally controlled by $|||\cdot|||_{3/4}$.

Lemma 4.3. *There exists a constant $C > 0$ depending only on Ω and k such that for every $u, v \in H_P^4(\Omega)$ we have*

$$\|f(u) - f(v)\|_1 \leq C \, |||u - v|||_{3/4} \, (1 + |||u|||_{3/4} + |||v|||_{3/4}) \tag{4.12}$$

Proof.

$$\begin{aligned}\|f(u) - f(v)\|_1 &= \left\|k(\Delta - I)^{-1}(v - u)_y + (\Delta - I)^{-1}[\{v, \Delta v\} - \{u, \Delta u\}]\right\|_1 \\ &\leq k \left\|(\Delta - I)^{-1}(v - u)_y\right\|_1 + \left\|(\Delta - I)^{-1} Q(v_x^2 - v_y^2 - u_x^2 + u_y^2)\right\|_1 \\ &\quad + \left\|(\Delta - I)^{-1} P(u_x u_y - v_x v_y)\right\|_1\end{aligned}$$

But for every $f \in H_P^1(\Omega)$, $(I - \Delta)u = f$ implies $\|u\|_1^2 \leq \|f\| \cdot \|u\| \leq \|f\|_1 \cdot \|u\|_1$, so that

$$\left\|(I - \Delta)^{-1} f\right\|_1 = \|u\|_1 \leq \|f\|_1$$

Now

$$\|f(u) - f(v)\|_1 \leq k \, \|(v - u)_y\|_1 + \left\|Q(v_x^2 - v_y^2 - u_x^2 + u_y^2)\right\|_1 + \|P(u_x u_y - v_x v_y)\|_1$$

But since P and Q are differential operators of order 2, then

$$\begin{aligned}\|f(u)-f(v)\|_1 &\le k\,\|u-v\|_4 + \left\|v_x^2 - v_y^2 - u_x^2 + u_y^2\right\|_3 + \|u_xu_y - v_xv_y\|_3 \\ &\le k\,\|u-v\|_4 + \left\|v_x^2 - u_x^2\right\|_3 + \left\|u_y^2 - v_y^2\right\|_3 + \|u_xu_y - u_xv_y\|_3 + \|u_xv_y - v_xv_y\|_3\end{aligned}$$

Now employing Theorem 2.9 which shows that $H_P^3(\Omega)$ is a Banach Algebra, we have

$$\begin{aligned}\|f(u)-f(v)\|_1 \le k\,\|u-v\|_4 &+ C_1\,\|(u-v)_x\|_3\cdot\|(u+v)_x\|_3 + C_1\,\|(u-v)_y\|_3\cdot\|(u+v)_y\|_3 \\ &+ C_1\,\|(u-v)_y\|_3\cdot\|u_x\|_3 + C_1\,\|(u-v)_x\|_3\cdot\|v_y\|_3\end{aligned}$$

where $C_1 > 0$ is a constant depending only on $m = 3$, $p = 2$ and the cone determining the cone condition for Ω. Now

$$\begin{aligned}\|f(u)-f(v)\|_1 &\le k\,\|u-v\|_4 + C_1\,\|u-v\|_4\cdot\|u+v\|_4 + C_1\,\|u-v\|_4\cdot\|u+v\|_4 \\ &\quad + C_1\,\|u-v\|_4\cdot\|u\|_4 + C_1\,\|u-v\|_4\cdot\|v\|_4 \\ &= \|u-v\|_4\,[k + 2C_1\,\|u+v\|_4 + C_1\,\|u\|_4 + C_1\,\|v\|_4] \\ &\le \|u-v\|_4\,[k + 3C_1\,\|u\|_4 + 3C_1\,\|v\|_4] \\ &\le C\,\|u-v\|_4\,(1 + \|u\|_4 + \|v\|_4)\end{aligned}$$

where C depends essentially only on Ω and k. Now by Lemma 4.2 the conclusion follows. □

At this point we can show that the nonlinearity f satisfies *Assumption (F)* found in the appendix B.2.3, where $\alpha = 3/4$, $X = H_P^1(\Omega)$, and $f(x,t) = f(x)$ are used.

Lemma 4.4. *Let U be an open subset of $(0,\infty)\times D(B_\lambda^{3/4})$. Then for every $(t,x)\in U$, there is a neighborhood $V\subset U$ and constants $L\ge 0, 0<\theta\le 1$ such that*

$$\|f(t_1,x_1) - f(t_2,x_2)\|_1 \le L(|t_1-t_2|^\theta + |||x_1-x_2|||_{3/4}) \quad \forall (t_i,x_i)\in V \tag{4.13}$$

Proof. Let $U = I\times W \subset (0,\infty)\times D(B_\lambda^{3/4})$ be open, and let $(t,x)\in U$. Then $t\in I$, so that by I being open, there exists $\delta>0$ such that $I' = (t-\delta, t+\delta)\subset I$. Also $x\in W$ which is open, we define

$$W' = \left\{ y\in W \;:\; |||x-y|||_{3/4} < |||x|||_{3/4} \right\}$$

so that $W' = W\cap B(x, |||x|||_{3/4})$ is an open neighborhood of x in W. Now set $V = I'\times W'$, then $V \subset U$ is an open neighborhood of (x, t) for which for every $(x_i, t_i) \in V$, with $i = 1, 2$,

$$
\begin{aligned}
\|f(t_1, x_1) - f(t_2, x_2)\|_1 &= \|f(x_1) - f(x_2)\|_1 \\
&\leq C\, |||x_1 - x_2|||_{3/4} \, (1 + |||x_1|||_{3/4} + |||x_2|||_{3/4}) \\
&\leq C\, |||x_1 - x_2|||_{3/4} \, (1 + 2\, |||x|||_{3/4})
\end{aligned}
$$

where we have used the independence of f on the temporal variable and Lemma 4.3. Setting $L = C(1 + 2\, |||x|||_{3/4})$ and $\theta = 1$, the conclusion follows. □

We finally prove that a unique local solution to the perturbed Hasegawa-Mima equation formulated as a semilinear abstract Cauchy problem exists. For this purpose, consider the Banach space $\mathcal{S}_{T^*}^{n,m}$ given by

$$
\mathcal{S}_{T^*}^{n,m} = C([0, T^*) : H_P^n(\Omega)) \cap C^1((0, T^*) : H_P^m(\Omega)) \tag{4.14}
$$

Theorem 4.5. *For every $\lambda > 0$ and initial value $u_0 \in H_P^4(\Omega)$, (pHM) has a unique local solution $u \in \mathcal{S}_{T^*}^{4,\infty}$, given by*

$$
u(t) = S(t)u_0 + \int_0^t S(t - s) f(s, u(s)) \, ds \quad 0 \leq t < T^* \tag{4.15}
$$

where $T^* > 0$ is a constant depending only on u_0, and $S(t)$ is the analytic semigroup generated by $-B_\lambda$.

Proof. Let $\lambda > 0$. By Corollary 3.13 and Proposition 3.16, $-B_\lambda$ is the infinitesimal generator of a uniformly bounded analytic semigroup over $H_P^1(\Omega)$ and $0 \in \rho(-B_\lambda)$. Also by Lemma 4.4, f satisfies the *Assumption (F)*. Therefore, by Theorem B.30 there exists a local solution $u \in \mathcal{S}_{T^*}^{4,5}$.

Now to show that $u(t) \in H_P^\infty(\Omega)$ for $0 < t < T^*$, we use a classical bootstrapping argument. For this purpose, we write the seminlinear equation as

$$
B_\lambda u(t) = -u_t(t) - (\Delta - I)^{-1} \{u(t), \Delta u(t)\} - k(\Delta - I)^{-1} u_y(t)
$$

where the RHS is in $H_P^4(\Omega)$ by invertibilty of $(\Delta - I)$ (see Proposition 3.7) and the Banach algebra (see Theorem 2.9) over $H_P^m(\Omega)$ for $m \geq 2$. Now writing

$$
u(t) = B_\lambda^{-1}(-u_t(t) - (\Delta - I)^{-1} \{u(t), \Delta u(t)\} - k(\Delta - I)^{-1} u_y(t))
$$

we see that $u(t)$ must be in $H_P^8(\Omega)$ for $0 < t < T^*$. Continuing in this fashion, we get that $u(t) \in H_P^\infty(\Omega)$ for $0 < t < T^*$. Hence $u \in \mathcal{S}_{T^*}^{4,\infty}$. □

4.3 Comments on the Existence of a Global Solution

One usually extends the local solution u on $(0, T^*)$ into a global one by considering the same problem but with a new initial condition $u(\tau_0)$ where $\tau_0 \in (0, T^*)$. Here, the uniqueness of the local solution plays an important role to make sure that we are really extending the local solution and not getting a different one. However, there are two big issues.

The first issue is that one needs to make sure that the newly picked initial condition is in the correct space required by the local existence theorem. In our case, we need to make sure that $u(t)$ stays in $D(B^{3/4})$, that is, $\left\| B_\lambda^{3/4} u(t) \right\|_1$ stays bounded on $0 \leq t \leq T^*$. Thanks to the proof of Lemma 4.2(iii) by which we have

$$\left\| B_\lambda^{3/4} u(t) \right\|_1 \leq C_1 \left\| u(t) \right\|_1 \quad \forall t$$

Also thanks to Corollary 5.10 (proven in the next section) that

$$\sup_{t \in [0, T^*)} \left\| u(t) \right\|_2 \leq C_2 \left\| u_0 \right\|_2$$

where C_2 is independent of λ and T^*. So that

$$\left\| B_\lambda^{3/4} u(t) \right\|_1 \leq C \left\| u_0 \right\|_2 \quad \forall t \in (0, T^*)$$

Hence the first issue is resolved.

Now if we pick $\tau_0 \in (0, T^*)$ and treat $u(\tau_0)$ as the new initial condition, we will get another local solution on (τ_0, τ_1) for some $\tau_1 > \tau_0$. And this τ_1 depends on $u(\tau_0)$, and not on the very initial condition u_0. So here comes the second issue that one needs to make sure that the new time steps accumulate and reach the desired time T or ∞. A way to resolve this, if possible, is to establish a lower bound on the length of the new time intervals. This could be a fixed positive real number or the terms of a divergent series with positive terms, for which the solution will extend to temporal infinity.

In our case, we couldn't find a lower bound on the length of the new time intervals yet, and so we leave this problem open for future investigation.

Chapter 5

The Hasegawa-Mima Equation

In this chapter, we begin with a series of lemmas for the purpose of proving some *a priori* estimates, which we will use to let $\lambda \to 0$ and obtain a local solution to the Hasegawa-Mima equation. We then establish the uniqueness of the local solution.

5.1 A Series of Lemmas

Lemma 5.1. *For every $u, v \in H_P^2(\Omega)$, we have*

$$\int_\Omega [(\Delta - I)u]v \, d\mu = \int_\Omega u[(\Delta - I)v] \, d\mu \tag{5.1}$$

Proof. Using Green's formula and Proposition 2.19, we have

$$\begin{aligned}
\int_\Omega [(\Delta - I)u]v \, d\mu &= \int_\Omega \Delta u v \, d\mu - \int_\Omega uv \, d\mu \\
&= \cancelto{0}{\int_\Gamma v \nabla u \cdot \vec{\nu} \, ds} - \int_\Omega \nabla u \cdot \nabla v \, d\mu - \int_\Omega uv \, d\mu \\
&= -\cancelto{0}{\int_\Gamma u \nabla v \cdot \vec{\nu} \, ds} + \int_\Omega u \Delta v \, d\mu - \int_\Omega uv \, d\mu \\
&= \int_\Omega u[(\Delta - I)v] \, d\mu
\end{aligned}$$

□

Lemma 5.2. *For every $u \in H_P^1(\Omega)$, we have*

$$\int_\Omega u_x u \, d\mu = \int_\Omega u_y u \, d\mu = 0 \tag{5.2}$$

Proof. Integrating by parts, we have

$$\begin{aligned}\int_\Omega u_x u \, d\mu &= \int_0^L \int_0^L u_x u \, dx dy \\ &= \cancelto{0}{\int_0^L [u^2]_0^L \, dy} - \int_\Omega u_x u \, d\mu\end{aligned}$$

so that

$$\int_\Omega u_x u \, d\mu = 0$$

Similarly, the other equality in u_y holds. □

Lemma 5.3. *For every $u \in H_P^2(\Omega)$, we have*

$$\int_\Omega u_x \Delta u \, d\mu = \int_\Omega u_y \Delta u \, d\mu = 0 \tag{5.3}$$

Proof. By Green's theorem and Proposition 2.19, we have

$$\begin{aligned}\int_\Omega u_x \Delta u \, d\mu &= \int_\Omega \nabla u_x \cdot \nabla u \, d\mu \\ &= \int_\Omega (u_x)_x (u_x) + (u_y)_x (u_y) \, d\mu\end{aligned}$$

which is equal to 0 by Lemma 5.2 because both u_x and u_y are in $H_P^1(\Omega)$.

Similarly, the other equality in $u_y \Delta y$ holds. □

Lemma 5.4. *For every $u \in H_P^4(\Omega)$, we have*

$$\int_\Omega \{u, \Delta u\} \, u \, d\mu = 0 \tag{5.4}$$

Proof.

$$\int_\Omega \{u, \Delta u\} \, u \, d\mu = \int_\Omega u_x \Delta u_y u \, d\mu - \int_\Omega u_y \Delta u_x u \, d\mu$$

where by integrating by parts,

$$\int_\Omega u_x \Delta u_y u \, d\mu = \int_0^L [\Delta u_y u^2]_0^L \, dy - \int_\Omega \Delta u_{yx} u^2 \, d\mu - \int_\Omega \Delta u_y u_x u \, d\mu$$

so that

$$\int_{\Omega} u_x \Delta u_y u \, d\mu = \frac{1}{2} \int_0^L [\Delta u_y u^2]_0^L \, dy - \frac{1}{2} \int_{\Omega} \Delta u_{yx} u^2 \, d\mu$$

But since u satisfies PBCs up to the third order, then

$$\frac{1}{2} \int_0^L [\Delta u_y u^2]_0^L \, dy = 0$$

so that

$$\int_{\Omega} u_x \Delta u_y u \, d\mu = -\frac{1}{2} \int_{\Omega} \Delta u_{yx} u^2 \, d\mu$$

Similarly, it can be shown that

$$\int_{\Omega} u_y \Delta u_x u \, d\mu = -\frac{1}{2} \int_{\Omega} \Delta u_{xy} u^2 \, d\mu$$

and so the result follows. □

Lemma 5.5. *For every $v, w \in H_P^4(\Omega)$, we have*

$$\int_{\Omega} \{w, \Delta v\} \, w \, d\mu = 0 \tag{5.5}$$

Proof. Same as the proof of (5.4). □

Lemma 5.6. *For every $v, w \in H_P^3(\Omega)$, we have*

$$\int_{\Omega} \{v, \Delta w\} \, w \, d\mu = \int_{\Omega} \{w, v\} \, \Delta w \, d\mu \tag{5.6}$$

Proof.

$$\int_{\Omega} \{v, \Delta w\} \, w \, d\mu = \int_{\Omega} v_x \Delta w_y w \, d\mu - \int_{\Omega} v_y \Delta w_x w \, d\mu$$

where with integration by parts,

$$\int_{\Omega} v_x \Delta w_y w \, d\mu = \cancelto{0}{\int_0^L [v_x w \Delta w]_0^L \, dx} - \int_{\Omega} v_{xy} w \Delta w \, d\mu - \int_{\Omega} v_x w_y \Delta w \, d\mu$$

and

$$\int_{\Omega} v_y \Delta w_x w \, d\mu = \cancelto{0}{\int_0^L [v_y w \Delta w]_0^L \, dy} - \int_{\Omega} v_{yx} w \Delta w \, d\mu - \int_{\Omega} v_y w_x \Delta w \, d\mu$$

where the boundary terms vanish because v, w satisfy PBCs up to the second order. Now

$$\int_\Omega \{v, \Delta w\}\, w \, d\mu = -\int_\Omega v_x w_y \Delta w \, d\mu + \int_\Omega v_y w_x \Delta w \, d\mu = \int_\Omega \{w, v\}\, \Delta w \, d\mu$$

$\square$

Lemma 5.7. *For every $u \in H_P^3(\Omega)$, we have*

$$\int_\Omega \{u, \Delta u\}\, \Delta u \, d\mu = 0 \tag{5.7}$$

Proof.

$$\int_\Omega \{u, \Delta u\}\, \Delta u \, d\mu = \int_\Omega u_x \Delta u_y \Delta u \, d\mu - \int_\Omega u_y \Delta u_x \Delta u \, d\mu$$

where by integrating by parts,

$$\int_\Omega u_x \Delta u_y \Delta u \, d\mu = \int_0^L [u_x (\Delta u)^2]_0^L \, dx - \int_\Omega u_{xy} (\Delta u)^2 \, d\mu - \int_\Omega u_x \Delta u_y \Delta u \, d\mu$$

so that

$$\int_\Omega u_x \Delta u_y \Delta u \, d\mu = \frac{1}{2} \int_0^L [u_x (\Delta u)^2]_0^L \, dx - \frac{1}{2} \int_\Omega u_{xy} (\Delta u)^2 \, d\mu$$

But since u satisfies PBCs up to the second order, then

$$\frac{1}{2} \int_0^L [u_x (\Delta u)^2]_0^L \, dx = 0$$

so that

$$\int_\Omega u_x \Delta u_y \Delta u \, d\mu = -\frac{1}{2} \int_\Omega u_{xy} (\Delta u)^2 \, d\mu$$

Similarly, it can be shown that

$$\int_\Omega u_y \Delta u_x \Delta u \, d\mu = -\frac{1}{2} \int_\Omega u_{yx} (\Delta u)^2 \, d\mu$$

and so the result follows. $\square$

Lemma 5.8. *Let α be a two-dimensional multi-index. Then for every $u \in H_P^{3+|\alpha|}(\Omega)$, we have*

$$\int_\Omega \{u, D^\alpha \Delta u\}\, \Delta D^\alpha u \, d\mu = 0 \tag{5.8}$$

Proof.

$$\int_\Omega \{u, D^\alpha \Delta u\} \, \Delta D^\alpha u \; d\mu = \int_\Omega u_x D^\alpha \Delta u_y \Delta D^\alpha u \; d\mu - \int_\Omega u_y D^\alpha \Delta u_x \Delta D^\alpha u \; d\mu$$

where integrating the first term by parts, we get

$$\int_\Omega u_x D^\alpha \Delta u_y \Delta D^\alpha u \; d\mu = \cancelto{0}{\int_0^L [u_x (D^\alpha \Delta u)^2]_0^L \; dx} - \int_\Omega u_{xy} (D^\alpha \Delta u)^2 \; d\mu - \int_\Omega u_x D^\alpha \Delta u_y \Delta D^\alpha u \; d\mu$$

so that

$$\int_\Omega u_x D^\alpha \Delta u_y \Delta D^\alpha u \; d\mu = -\frac{1}{2} \int_\Omega u_{xy} (D^\alpha \Delta u)^2 \; d\mu$$

Similarly,

$$\int_\Omega u_y D^\alpha \Delta u_x \Delta D^\alpha u \; d\mu = -\frac{1}{2} \int_\Omega u_{xy} (D^\alpha \Delta u)^2 \; d\mu$$

from which the result follows. □

5.2 *A Priori* Estimates

Lemma 5.9. *Suppose $u \in \mathcal{S}^{4,4}_{T^*}$ is a solution to the perturbed Hasegawa-Mima problem (pHM) with initial data $u_0 \in H^2_P(\Omega)$, then*

$$\|u(t)\|_2^2 + \lambda \int_0^t \|u(\tau)\|_4^2 \; d\tau \leq C \|u_0\|_2^2 \quad \forall t \in [0, T^*) \tag{5.9}$$

where C is independent of λ and T^*.

Proof. Let $t \in [0, T]$. Taking the L^2-innerproduct of the perturbed Hasegawa-Mima equation (4.3) with the function $-2(\Delta - I)u$ and using equations (5.1), (5.2) and (5.4), we get

$$\begin{aligned} &- \int_\Omega 2u_t (\Delta u - u) \; d\mu - 2\lambda \int_\Omega \Delta(\Delta - I)u (\Delta - I)u \; d\mu - 4\lambda \int_\Omega u (\Delta - I)u \; d\mu \\ &\qquad = 2 \cancelto{0}{\int_\Omega \{u, \Delta u\} \, u \; d\mu} + 2k \cancelto{0}{\int_\Omega u_y u \; d\mu} \end{aligned}$$

which, by Green's formula and Proposition 2.19, is equivalent to

$$\int_\Omega 2uu_t \; d\mu + \int_\Omega 2\nabla u \cdot \nabla u_t \; d\mu + 2\lambda \int_\Omega |\nabla(\Delta - I)u|^2 \; d\mu + 4\lambda \int_\Omega |\nabla u|^2 \; d\mu + 4\lambda \int_\Omega u^2 \; d\mu = 0$$

which, in turn, is equivalent to

$$\frac{d}{dt}\left\|u(t)\right\|_1^2 + 2\lambda\left\|\nabla(\Delta - I)u(t)\right\|^2 + 4\lambda\left\|u(t)\right\|_1 = 0$$

Now integrating over the temporal interval $[0, t]$, we get

$$\left\|u(t)\right\|_1^2 + 2\lambda\int_0^t \left\|\nabla(\Delta - I)u(\tau)\right\|^2 \, d\tau + 4\lambda\int_0^t \left\|u(\tau)\right\|_1 \, d\tau = \left\|u_0\right\|_1^2 \tag{5.10}$$

On the other hand, taking the L^2-innerproduct of (4.3) with the function $2\Delta(\Delta - I)u$ and using equations (5.1), (5.3) and (5.7), we get

$$\begin{aligned}\int_\Omega 2u_t\Delta(\Delta u - u)\, d\mu + 2\lambda\left\|\Delta(\Delta - I)u(t)\right\|^2 + 4\lambda\int_\Omega u\Delta(\Delta - I)u\, d\mu \\ = -2\cancelto{0}{\int_\Omega \{u, \Delta u\}\Delta u\, d\mu} - 2k\cancelto{0}{\int_\Omega u_y\Delta u\, d\mu}\end{aligned}$$

which, by Green's formula and Proposition 2.19, is equivalent to

$$\int_\Omega 2\Delta u\Delta u_t\, d\mu + \int_\Omega 2\nabla u\cdot\nabla u_t\, d\mu + 2\lambda\left\|\Delta(\Delta - I)u(t)\right\|^2 + 4\lambda(\left\|\Delta u\right\|^2 + \left\|\nabla u\right\|^2) = 0$$

which, in turn, is equivalent to

$$\frac{d}{dt}(\left\|\Delta u\right\|^2 + \left\|\nabla u\right\|^2) + 2\lambda\left\|\Delta(\Delta - I)u(t)\right\|^2 + 4\lambda(\left\|\Delta u\right\|^2 + \left\|\nabla u\right\|^2) = 0$$

Now integrating over the temporal interval $[0, t]$, we get

$$\begin{aligned}\left\|\Delta u(t)\right\|^2 + \left\|\nabla u(t)\right\|^2 + 2\lambda\int_0^t \left\|\Delta(\Delta - I)u(\tau)\right\|^2 \, d\tau + 4\lambda\int_0^t (\left\|\Delta u(\tau)\right\|^2 + \left\|\nabla u(\tau)\right\|^2)\, d\tau \\ = \left\|\Delta u_0\right\|^2 + \left\|\nabla u_0\right\|^2\end{aligned} \tag{5.11}$$

Now adding equations (5.10) and (5.11), we get

$$\begin{aligned}\left\|u(t)\right\|^2 + 2\left\|\nabla u(t)\right\|^2 + \left\|\Delta u(t)\right\|^2 + 2\lambda\int_0^t \left(\left\|\nabla(\Delta - I)u(\tau)\right\|^2 + \left\|\Delta(\Delta - I)u(\tau)\right\|^2\right)\, d\tau \\ + 4\lambda\int_0^t (\left\|u(\tau)\right\|^2 + 2\left\|\nabla u(\tau)\right\|^2 + \left\|\Delta u(\tau)\right\|^2)\, d\tau \\ = \left\|u_0\right\|^2 + 2\left\|\nabla u_0\right\|^2 + \left\|\Delta u_0\right\|^2\end{aligned}$$

which implies

$$\|u(t)\|_2^2 + \lambda \int_0^t \left(\|\nabla(\Delta - I)u(\tau)\|^2 + \|\Delta(\Delta - I)u(\tau)\|^2 + \|u(\tau)\|^2 \right) \, d\tau \leq 2\, \|u_0\|_2^2 \quad (5.12)$$

where the integrand is much larger than $\|u(\tau)\|_4^2$. And therefore, the result follows. □

Corollary 5.10. *Suppose $u \in \mathcal{S}^{4,4}_{T^*}$ is a solution to the perturbed Hasegawa-Mima problem (pHM) with initial data $u_0 \in H^2_P(\Omega)$, then*

$$\sup_{t \in [0,T^*)} \|u(t)\|_2 \leq C \, \|u_0\|_2 \quad (5.13)$$

where C is independent of λ and T^*.

For higher regularities of the solution, we need the following estimates.

Lemma 5.11. *Let $m \geq 2$ be an integer. Suppose $u \in \mathcal{S}^{4,m+2}_{T^*}$ is a solution to the perturbed Hasegawa-Mima problem (pHM) with initial data $u_0 \in H^m_P(\Omega)$, then*

$$\|u(t)\|_m^2 + 2\lambda \int_0^t \|u(\tau)\|_{m+2}^2 \, d\tau \leq C \, \|u_0\|_m \quad \forall t \in [0, T^*) \quad (5.14)$$

where C is independent of λ and T^*.

Proof. We prove this by strong induction on m. Lemma 5.9 establishes the base case $m = 2$. Now assume that (5.14) holds for every $k = 2, \ldots m$ for some $m \geq 2$. We consider a two-dimensional multi-index $|\alpha| = m - 1$ and apply the operator D^α to equation (4.3) and get

$$\begin{aligned} D^\alpha u_t + \lambda\Delta(\Delta - I)D^\alpha u + 2\lambda D^\alpha u &= -(\Delta - I)^{-1} D^\alpha \{u, \Delta u\} - k(\Delta - I)^{-1} D^\alpha u_y \\ &= -(\Delta - I)^{-1} \sum_{\beta_1+\beta_2=\alpha} \left\{ D^{\beta_1} u, D^{\beta_2} \Delta u \right\} - k(\Delta - I)^{-1} D^\alpha u_y \end{aligned}$$

Now taking its L^2-innerproduct with the function $2\Delta(\Delta - I)D^\alpha u$ and employing equation (5.1), we get

$$\begin{aligned} 2 \int_\Omega D^\alpha u_t \Delta(\Delta - I) D^\alpha u \, d\mu + 2\lambda \, \|\Delta(\Delta - I) D^\alpha u\|^2 + 4\lambda \int_\Omega D^\alpha u \Delta(\Delta - I) D^\alpha u \, d\mu \\ = -2 \sum_{\beta_1+\beta_2=\alpha} \int_\Omega \left\{ D^{\beta_1} u, D^{\beta_2} \Delta u \right\} \Delta D^\alpha u \, d\mu - 2k \int_\Omega D^\alpha u_y \Delta D^\alpha u \, d\mu \end{aligned} \quad (5.15)$$

where by using Proposition 2.19, we have

$$\begin{aligned} 2\int_\Omega D^\alpha u_t \Delta(\Delta - I) D^\alpha u \, d\mu &= 2\int_\Omega \Delta D^\alpha u_t (\Delta - I) D^\alpha u \, d\mu \\ &= 2\int_\Omega \Delta D^\alpha u_t \Delta D^\alpha u \, d\mu - 2\int_\Omega \Delta D^\alpha u_t D^\alpha u \, d\mu \\ &= \frac{d}{dt}\left(\|\Delta D^\alpha u\|^2 + \|\nabla D^\alpha u\|^2 \right) \end{aligned}$$

$$\begin{aligned} \int_\Omega D^\alpha u \Delta(\Delta - I) D^\alpha u \, d\mu &= \int_\Omega \Delta D^\alpha u (\Delta - I) D^\alpha u \, d\mu \\ &= \|\Delta D^\alpha u\|^2 + \|\nabla D^\alpha u\|^2 \end{aligned}$$

$$\begin{aligned} \int_\Omega D^\alpha u_y \Delta D^\alpha u \, d\mu &= \int_\Omega \nabla D^\alpha u_y \cdot \nabla D^\alpha u \, d\mu \\ &= \int_\Omega (D^\alpha u_x)_y (D^\alpha u_x) \, d\mu + \int_\Omega (D^\alpha u_y)_y (D^\alpha u_y) \, d\mu \\ &= 0 \quad \text{by Lemma 5.2} \end{aligned}$$

$$\begin{aligned} &\sum_{\beta_1+\beta_2=\alpha} \int_\Omega \left\{ D^{\beta_1} u, D^{\beta_2} \Delta u \right\} \Delta D^\alpha u \, d\mu \\ &\quad = \sum_{\substack{\beta_1+\beta_2=\alpha \\ 1 \le |\beta_1| \le m-2}} \int_\Omega \left\{ D^{\beta_1} u, D^{\beta_2} \Delta u \right\} \Delta D^\alpha u \, d\mu \\ &\qquad + \underbrace{\int_\Omega \{u, D^\alpha \Delta u\} \Delta D^\alpha u \, d\mu}_{\to 0} + \int_\Omega \{D^\alpha u, \Delta u\} \Delta D^\alpha u \, d\mu \end{aligned}$$

where we have used equation (5.8), and

$$\int_\Omega \{D^\alpha u, \Delta u\} \Delta D^\alpha u \, d\mu = \int_\Omega D^\alpha u_x \Delta u_y \Delta D^\alpha u \, d\mu - \int_\Omega D^\alpha u_y \Delta u_x \Delta D^\alpha u \, d\mu$$

Thus, putting all together, we get

$$
\begin{aligned}
&\frac{d}{dt}\left(\|\Delta D^\alpha u\|^2 + \|\nabla D^\alpha u\|^2\right) + 2\lambda\,\|\Delta(\Delta - I)D^\alpha u\|^2 + 4\lambda(\|\Delta D^\alpha u\|^2 + \|\nabla D^\alpha u\|^2)\\
&\qquad = -2 \sum_{\substack{\beta_1+\beta_2=\alpha\\ 1\le|\beta_1|\le m-2}} \left(\int_\Omega D^{\beta_1}u_x D^{\beta_2}\Delta u_y \Delta D^\alpha u\, d\mu - \int_\Omega D^{\beta_1}u_y D^{\beta_2}\Delta u_x \Delta D^\alpha u\, d\mu\right)\\
&\qquad\qquad - 2\int_\Omega D^\alpha u_x \Delta u_y \Delta D^\alpha u\, d\mu + 2\int_\Omega D^\alpha u_y \Delta u_x \Delta D^\alpha u\, d\mu\\
&\qquad \le 2 \sum_{\substack{\beta_1+\beta_2=\alpha\\ 1\le|\beta_1|\le m-2}} \left(\left\|D^{\beta_1}u_x\right\|_\infty \left\|D^{\beta_2}\Delta u_y\right\| \|\Delta D^\alpha u\| + \left\|D^{\beta_1}u_y\right\|_\infty \left\|D^{\beta_2}\Delta u_x\right\| \|\Delta D^\alpha u\|\right)\\
&\qquad\qquad + 2\,\|D^\alpha u_x\|_{L^4}\,\|\Delta u_y\|_{L^4}\,\|\Delta D^\alpha u\| + 2\,\|D^\alpha u_y\|_{L^4}\,\|\Delta u_x\|_{L^4}\,\|\Delta D^\alpha u\|
\end{aligned}
\tag{5.16}
$$

But when $u \in H_P^{|\beta_1|+3}(\Omega)$, the continuous embedding of $H^2(\Omega)$ into $L^\infty(\Omega)$ (see Corollary A.3) implies

$$\left\|D^{\beta_1}u_x\right\|_\infty \le C_1 \left\|D^{\beta_1}u_x\right\|_2 \text{ and } \left\|D^{\beta_1}u_y\right\|_\infty \le C_2 \left\|D^{\beta_1}u_y\right\|_2$$

and so

$$\left\|D^{\beta_1}u_x\right\|_\infty \le C_1 \left\|D^{\beta_1}u\right\|_3 \le C_1 \|u\|_{|\beta_1|+3} \text{ and } \left\|D^{\beta_1}u_y\right\|_\infty \le C_2 \left\|D^{\beta_1}u\right\|_3 \le C_2 \|u\|_{|\beta_1|+3}$$

Also when $u \in H_P^{|\alpha|+2}$, a particular case of Gagliardo-Nirenberg's inequality (see Theorem A.5) implies

$$\|D^\alpha u_x\|_{L^4} \le C_3 \|D^\alpha u_x\|^{1/2} \|D^\alpha u_x\|_1^{1/2} \text{ and } \|D^\alpha u_y\|_{L^4} \le C_4 \|D^\alpha u_y\|^{1/2} \|D^\alpha u_y\|_1^{1/2}$$

and so

$$\|D^\alpha u_x\|_{L^4} \le C_3 \|D^\alpha u\|_2 \le C_3 \|u\|_{|\alpha|+2} \text{ and } \|D^\alpha u_y\|_{L^4} \le C_4 \|D^\alpha u\|_2 \le C_4 \|u\|_{|\alpha|+2}$$

and when $u \in H_P^4$,

$$\|\Delta u_x\|_{L^4} \le C_5 \|\Delta u_x\|^{1/2} \|\Delta u_x\|_1^{1/2} \text{ and } \|\Delta u_y\|_{L^4} \le C_6 \|\Delta u_y\|^{1/2} \|\Delta u_y\|_1^{1/2}$$

and so

$$\|\Delta u_x\|_{L^4} \le C_5 \|\Delta u\|_2 \le C_5 \|u\|_4 \text{ and } \|\Delta u_y\|_{L^4} \le C_6 \|\Delta u\|_2 \le C_6 \|u\|_4$$

Now inequality (5.16) becomes

$$\frac{d}{dt}\left(\|\Delta D^\alpha u\|^2 + \|\nabla D^\alpha u\|^2\right) + 2\lambda \|\Delta(\Delta - I)D^\alpha u\|^2 + 4\lambda(\|\Delta D^\alpha u\|^2 + \|\nabla D^\alpha u\|^2)$$
$$\leq 4C_7 \sum_{\substack{\beta_1+\beta_2=\alpha \\ 1\leq|\beta_1|\leq m-2}} \|u\|_{|\beta_1|+3} \|u\|_{|\beta_2|+3} \|u\|_{|\alpha|+2} + 4C_8 \|u\|^2_{|\alpha|+2} \|u\|_4 \tag{5.17}$$
$$\leq C_9 \|u\|^3_{m+1}$$

Now integrating this over the temporal interval $[0, t]$, we get

$$\|\Delta D^\alpha u(t)\|^2 + \|\nabla D^\alpha u(t)\|^2 + 2\lambda \int_0^t \|\Delta(\Delta - I)D^\alpha u(\tau)\|^2 \, d\tau + 4\lambda \int_0^t \|\Delta D^\alpha u(\tau)\|^2 \, d\tau$$
$$+ 4\lambda \int_0^t \|\nabla D^\alpha u(\tau)\|^2 \, d\tau \leq C_9 \int_0^t \|u(\tau)\|^3_{m+1} \, d\tau + \|\Delta D^\alpha u_0\|^2 + \|\nabla D^\alpha u_0\|^2 \tag{5.18}$$

Recall that the strong induction hypothesis implies equation (5.14) for $k = 2, \ldots, m$. And so

$$\int_0^t \|u(\tau)\|^3_{m+1} \, d\tau \leq \int_0^t \|u(\tau)\|^3_{m+2} \, d\tau \leq C_{10} \|u_0\|^2_m \tag{5.19}$$

Now adding equations (5.14) for $k = 2, \ldots, m$ to (5.18) and using (5.19), the result follows for $m + 1$. □

Corollary 5.12. *Let $m \geq 2$ be an integer. Suppose $u \in \mathcal{S}^{4,m+2}_{T^*}$ is a solution to the perturbed Hasegawa-Mima problem (pHM) with initial data $u_0 \in H^m_P(\Omega)$*

$$\sup_{t\in[0,T^*)} \|u(t)\|_m \leq C \|u_0\|_m \tag{5.20}$$

where C is independent of λ and T^*.

5.3 Existence of a Local Solution

We finally prove the main results of this thesis.

Theorem 5.13. *Given $u_0 \in H^m_P(\Omega)$ with integer $m \geq 4$, the two-dimensional Periodic Hasegawa-*

Mima problem

$$\text{(HM)} \begin{cases} (\Delta - I)u_t + \{u, \Delta u\} + ku_y = 0 & \text{on } \Omega \times (0, T] \\ u(x, y, 0) = u_0(x, y) & \text{on } \Omega \end{cases} \tag{5.21}$$

has a local solution $u \in \mathcal{S}^{m,m}_{T^*}$*, where* $T^* > 0$ *depends only on the initial condition* u_0*.*

Proof. By Theorem 4.5, for each $n \in \mathbb{N}$, there exists a local solution $u_n \in \mathcal{S}^{4,\infty}_{T^*}$ of the perturbed Hasegawa-Mima equation (4.4) with $\lambda = 1/n$. But since $u_n(t) \in H^m_P(\Omega)$ for every $0 \le t < T^*$, then $u_n \in \mathcal{S}^{m,\infty}_{T^*}$. Now observe that by Corollary 5.12 with $u_0 \in H^m_P(\Omega)$, the sequence $\{u_n\}$ is uniformly bounded in $\mathcal{S}^{m,m}_{T^*}$. Hence it must have a convergent subsequence $\{u_{n_k}\}$, say with limit $u \in \mathcal{S}^{m,m}_{T^*}$. We now proceed to show that u is a local solution to (HM). On the temporal interval $(0, T^*)$, we have

$$\begin{aligned} &\left\| u_{n,t} + (\Delta - I)^{-1} \{u_n, \Delta u_n\} + k(\Delta - I)^{-1} u_{n,y} \right\|_1 \\ &= \left\| B_{1/n} u_n \right\|_1 \\ &= \frac{1}{n} \left\| \Delta(\Delta - I)u_n + 2u_n \right\|_1 \\ &\le \frac{C_1}{n} \left\| u_n \right\|_4 \quad \text{for some constant } C_1 > 0 \\ &\le \frac{C_1 C_2}{n} \left\| u_0 \right\|_4 \quad \text{for some constant } C_2 > 0 \text{ which exists by Lemma 5.11} \end{aligned} \tag{5.22}$$

so that as $n \to \infty$, $u_{n,t} + (\Delta - I)^{-1} \{u_n, \Delta u_n\} + k(\Delta - I)^{-1} u_{n,y} \to 0$ in $\|\cdot\|_1$. Thus

$$u_t + (\Delta - I)^{-1} \{u, \Delta u\} + k(\Delta - I)^{-1} u_y = 0 \quad \text{a.e. on } \Omega,\ \forall t \in (0, T^*)$$

or equivalently,

$$(\Delta - I)u_t + \{u, \Delta u\} + ku_y = 0 \quad \text{a.e. on } \Omega,\ \forall t \in (0, T^*)$$

Therefore $u \in \mathcal{S}^{m,m}_{T^*}$ is a local solution to (HM). □

Corollary 5.14. *Given* $u_0 \in H^m_P(\Omega)$ *with integer* $m \ge 4$*, the two-dimensional Periodic Hasegawa-Mima problem*

$$\text{(HM)} \begin{cases} (\Delta - I)u_t + \{u, \Delta u\} + ku_y = 0 & \text{on } \Omega \times (0, T] \\ u(x, y, 0) = u_0(x, y) & \text{on } \Omega \end{cases} \tag{5.23}$$

has a local classical solution $u \in C([0, T^*), C^{m-2}_P(\Omega)) \cap C^1((0, T^*), C^{m-2}_P(\Omega))$*, where* $T^* > 0$ *depends only on the initial condition* u_0*. In fact,* $u \in C([0, T^*), C^{m-2}_P(\Omega)) \cap C^\infty((0, T^*), C^{m-2}_P(\Omega))$*.*

Proof. By Theorem 5.13, it suffices to check that $u(t) \in H_P^m(\Omega)$ is in $C_P^{m-2}(\Omega)$ for $t \in (0, T^*)$. So let $u(t) \in H_P^m(\Omega)$, then

$$D^\alpha u(t) \in H_P^1(\Omega) \quad \forall\, |\alpha| \leq m - 1$$

where $H^1(\Omega)$ is continuously embedded in $L^4(\Omega)$ by a particular case of Gagliardo-Nirenberg's inequality (see Theorem A.5). Hence

$$D^\alpha u(t) \in L^4(\Omega) \quad \forall\, |\alpha| \leq m - 1$$

so that $u \in W^{m-1,4}(\Omega)$, which by Sobolev Embedding Theorems (see Theorem A.1)), is continuously embedded in $C^{m-2}(\Omega)$. Thus $u \in C^{m-2}(\Omega)$, and so $u \in C_P^{m-2}(\Omega)$. Now with a usual bootstrapping argument with respect to the temporal variable, we easily get that $u \in C([0, T^*), C_P^{m-2}(\Omega)) \cap C^\infty((0, T^*), C_P^{m-2}(\Omega))$. □

5.4 Uniqueness of the Local Solution

We first recall Gronwall's Inequality from [14, p.146].

Theorem 5.15 (Gronwall's Inequality)**.** *Let α, ϕ and ψ be nonnegative continuous real-valued functions defined on the interval $[a, b]$. Moreover, suppose that α is differentiable on (a, b) with nonnegative continuous derivative. If for every $t \in [a, b]$,*

$$\phi(t) \leq \alpha(t) + \int_a^t \psi(s)\phi(s)\, ds \tag{5.24}$$

then

$$\phi(t) \leq \alpha(t) e^{\int_a^t \psi(s)\, ds} \quad \forall t \in [a, b] \tag{5.25}$$

Now we establish uniqueness.

Theorem 5.16. *The local (classical) solution of the two-dimensional Periodic Hasegawa-Mima problem obtained in Theorem 5.13 and Corollary 5.14 is unique.*

Proof. Assume that $u, v \in \mathcal{S}_{T^*}^{4,4}$ are solutions to (HM). Then

$$u_t + (\Delta - I)^{-1} \{u, \Delta u\} + k(\Delta - I)^{-1} u_y = 0 \tag{5.26}$$

and

$$v_t + (\Delta - I)^{-1}\{v, \Delta v\} + k(\Delta - I)^{-1} v_y = 0 \tag{5.27}$$

Write $w = u - v$, then subtracting equations (5.26) and (5.27), we get

$$w_t + (\Delta - I)^{-1}[\{u, \Delta u\} - \{v, \Delta v\}] + k(\Delta - I)^{-1} w_y = 0 \tag{5.28}$$

Now taking the L^2-innerproduct of equation (5.28) with $(\Delta - I)w$, we obtain

$$\begin{aligned}\int_\Omega w_t(\Delta - I)w \, d\mu + \int_\Omega (\Delta - I)^{-1}[\{u, \Delta u\} - \{v, \Delta v\}](\Delta - I)w \, d\mu \\ + k\int_\Omega (\Delta - I)^{-1} w_y (\Delta - I)w \, d\mu = 0\end{aligned}$$

which by Lemma 5.1, is equivalent to

$$\int_\Omega w_t(\Delta - I)w \, d\mu + \int_\Omega [\{u, \Delta u\} - \{v, \Delta v\}]w \, d\mu + k\int_\Omega w_y w \, d\mu = 0$$

where the last term vanishes by Lemma 5.2, leaving us with

$$\int_\Omega w_t(\Delta - I)w \, d\mu + \int_\Omega [\{u, \Delta u\} - \{v, \Delta v\}]w \, d\mu = 0 \tag{5.29}$$

where by Green's formula and Proposition 2.19,

$$\begin{aligned}\int_\Omega w_t(\Delta - I)w \, d\mu &= \int_\Omega w_t \Delta w \, d\mu - \int_\Omega w_t w \, d\mu \\ &= \cancelto{0}{\int_\Gamma w_t \nabla w \cdot \vec{\nu} \, ds} - \int_\Omega \nabla w_t \cdot \nabla w \, d\mu - \int_\Omega w_t w \, d\mu \\ &= -\frac{1}{2}\frac{d}{dt}\left[\int_\Omega |\nabla w|^2 \, d\mu - \int_\Omega w^2 \, d\mu\right] \\ &= -\frac{1}{2}\frac{d}{dt}\|w\|_1^2\end{aligned}$$

and

$$\begin{aligned}\int_\Omega [\{u, \Delta u\} - \{v, \Delta v\}]w \, d\mu &= \int_\Omega [\{w + v, \Delta w + \Delta v\} - \{v, \Delta v\}]w \, d\mu \\ &= \int_\Omega [\{w, \Delta w\} + \{w, \Delta v\} + \{v, \Delta w\}]w \, d\mu \\ &= \cancelto{0}{\int_\Omega \{w, \Delta w\} \, w \, d\mu} + \cancelto{0}{\int_\Omega \{w, \Delta v\} \, w \, d\mu} + \int_\Omega \{v, \Delta w\} \, w \, d\mu\end{aligned}$$

where vanishing is due to Lemmas 5.4 and 5.5. Now by Lemma 5.6, we have

$$\begin{aligned}
\int_\Omega [\{u, \Delta u\} - \{v, \Delta v\}]w \, d\mu &= \int_\Omega \{w, v\} \, \Delta w \, d\mu \\
&= \int_\Omega v_x \left[-\frac{1}{2}(w_x^2 - w_y^2)_y + (w_x w_y)_x \right] \, d\mu \\
&\quad - \int_\Omega v_y \left[\frac{1}{2}(w_x^2 - w_y^2)_x + (w_x w_y)_y \right] \, d\mu \\
&= -\frac{1}{2} \int_\Omega v_x (w_x^2 - w_y^2)_y \, d\mu + \int_\Omega v_x (w_x w_y)_x \, d\mu \\
&\quad - \frac{1}{2} \int_\Omega v_y (w_x^2 - w_y^2)_x \, d\mu - \int_\Omega v_y (w_x w_y)_y \, d\mu
\end{aligned}$$

which integrating by parts, and using the fact that v and w satisfy PBCs upto the second order, we get

$$\begin{aligned}
\int_\Omega [\{u, \Delta u\} - \{v, \Delta v\}]w \, d\mu &= \frac{1}{2} \int_\Omega v_{xy}(w_x^2 - w_y^2) \, d\mu - \int_\Omega v_{xx} w_x w_y \, d\mu \\
&\quad + \frac{1}{2} \int_\Omega v_{yx}(w_x^2 - w_y^2) \, d\mu + \int_\Omega v_{yy} w_x w_y \, d\mu \\
&= \int_\Omega v_{xy}(w_x^2 - w_y^2) \, d\mu - \int_\Omega (v_{xx} - v_{yy}) w_x w_y
\end{aligned}$$

Now equation (5.29) becomes

$$\frac{1}{2} \frac{d}{dt} \|w\|_1^2 = \int_\Omega v_{xy}(w_x^2 - w_y^2) \, d\mu - \int_\Omega (v_{xx} - v_{yy}) w_x w_y$$

which by Cauchy-Schwarz Inequality becomes

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|w\|_1^2 &\leq \|v_{xy}\|_\infty \cdot \|w_x\|^2 + \|v_{xy}\|_\infty \cdot \|w_y\|^2 + \|v_{xx} - v_{yy}\|_\infty \cdot \|w_x\| \cdot \|w_y\| \\
&\leq (2\|v_{xy}\|_\infty + \|v_{xx} - v_{yy}\|_\infty) \cdot \|w\|_1^2
\end{aligned}$$

But since for every $t \in (0, T^*), \; v(t) \in H_P^4(\Omega)$ so that $v_{xx}(t), v_{xy}(t), v_{yy}(t) \in H_P^2(\Omega) \subset L^\infty(\Omega)$ by Corollary A.3, then $2\|v_{xy}\|_\infty + \|v_{xx} - v_{yy}\|_\infty < \infty$. Hence, there exists a constant $C > 0$ such that

$$\frac{d}{dt} \|w\|_1^2 \leq C \|w\|_1^2 \tag{5.30}$$

and so

$$\|w(t)\|_1^2 \leq C \int_0^t \|w(s)\|_1^2 \, ds + C\cancelto{0}{\|w(0)\|_1^2} \quad \forall t \in (0, T^*) \tag{5.31}$$

Thus by Gronwall's Inequality,

$$\|w(t)\|_1^2 \leq 0 \qquad \forall t \in (0, T^*) \tag{5.32}$$

so that

$$\|w(t)\|_1 = 0 \qquad \forall t \in (0, T^*) \tag{5.33}$$

and therefore $u(t) = v(t)$ (a.e. on Ω) for every $t \in (0, T^*)$. $\square$

Appendix A

Preliminary Results from Functional Analysis

A.1 Special Sobolev Embeddings

Theorem A.1. *[9, Cor 9.13 & 9.15] Let $U \subseteq \mathbb{R}^N$ be open, $m \geq 1$ be an integer, and $1 \leq p < \infty$. Then we have*

$$\begin{array}{llll} W^{m,p}(U) \subset L^q(\Omega), & \text{where} \quad \frac{1}{q} = \frac{1}{p} - \frac{m}{N} & \text{if} & \frac{1}{p} - \frac{m}{N} > 0 \\ W^{m,p}(U) \subset L^q(\Omega) & \forall q \in [p, \infty) & \text{if} & \frac{1}{p} - \frac{m}{N} = 0 \\ W^{m,p}(U) \subset L^\infty(\Omega) & & \text{if} & \frac{1}{p} - \frac{m}{N} < 0 \end{array}$$

where all these injections are continuous. Moreover, if $m - N/p > 0$ is not an integer, let k be the integral part of $m - N/p$, then

$$W^{m,p}(\Omega) \subset C^k(\overline{\Omega})$$

where $C^k(\overline{\Omega}) = \left\{u \in C^k(\Omega) \ : \ D^\alpha u \text{ has a continuous extension on } \overline{\Omega} \text{ for all } |\alpha| \leq k\right\}$.

Theorem A.2. *Let $U = \mathbb{R}^N$ or be an open set of class C^1 with bounded boundary or else that $U = \mathbb{R}^N_+$. Also let $p \in [1, \infty)$, and $m > N/p$ be an integer. Then for every integer r such that $N/p < r \leq m$, $W^{m,p}(U) \subset W^{m-r,\infty}(U)$ with continuous injection, i.e. there exists a constant $C > 0$ depending only on N and p such that $\|\cdot\|_{m-r,\infty} \leq C \|\cdot\|_{m,p}$.*

Proof. $m > N/p \iff N - pm < 0 \iff (N - pm)/pN < 0 \iff 1/p - m/N < 0$ so that by Theorem A.1, $W^{m,p}(U) \subset L^\infty(U)$ with $\|\cdot\|_\infty \leq C_1 \|\cdot\|_{m,p}$, where $C_1 > 0$ depends only on N and p. Similarly since $r > N/p$, $W^{r,p}(U) \subset L^\infty(U)$ with $\|\cdot\|_\infty \leq C_2 \|\cdot\|_{r,p}$, where $C_2 > 0$ depends only on N and p.

Now let $f \in W^{m,p}(U) \subset L^\infty(U)$, then for every multi-index $|\alpha| \leq m - r$, $D^\alpha f \in W^{r,p}(U) \subset L^\infty(U)$. Thus, $f \in W^{m-r,\infty}(U)$. To see that $W^{m,p}(U) \subset W^{m-r,\infty}(U)$ is a continuous injection, we consider $f \in W^{m,p}(U)$, then

$$\begin{aligned}
\|f\|_{m-r,\infty}^2 &= \|f\|_\infty^2 + \sum_{|\alpha| \leq m-r} \|D^\alpha f\|_\infty^2 \\
&\leq C_1^2 \|f\|_{m,p}^2 + C_2^2 \sum_{|\alpha| \leq m-r} \|D^\alpha f\|_{r,p}^2 \\
&\leq C_1^2 \|f\|_{m,p}^2 + C_2^2 \sum_{|\alpha| \leq m-r} \|f\|_{m,p}^2 \\
&= C \|f\|_{m,p}^2
\end{aligned}$$

Hence, $\|f\|_{m-r,\infty} \leq C \|f\|_{m,p}$, where $C > 0$ is a constant depending only on N and p. □

Corollary A.3. *In the context of this thesis, $U = \Omega \subset \mathbb{R}^2$ is open, bounded and of class (at least) C^1. Also, $N = p = 2$. Therefore, for every integer $m \geq 2$, $H^m(\Omega) \subset W^{m-2,\infty}$ with continuous injection.*

Corollary A.4. *For every integer $m \geq 2$, the norms $\|\cdot\|_m$ and $\|\cdot\|_{m,\infty}$ are equivalent.*

Proof. Let $u \in H_P^m(\Omega)$. Then by Corollary A.3, there exists $C_1 > 0$ such that

$$\|u\|_{m,\infty} \leq C_1 \|u\|_m$$

Observe that $\|u\|_{m,\infty} < \infty$, so that for every multi-index $|\alpha| \leq m$,

$$\|D^\alpha u\|_\infty < \infty$$

Hence

$$\|u\|_m^2 = \sum_{|\alpha| \leq m} \|D^\alpha u\|^2 = \sum_{|\alpha| \leq m} \int_\Omega |D^\alpha u|^2 \; d\mu \leq \mu(\Omega) \sum_{|\alpha| \leq m} \|D^\alpha u\|_\infty^2 \leq \mu(\Omega) \|u\|_{m,\infty}^2$$

Setting $C_2 = \mu(\Omega)^{-\frac{1}{2}} = 1/L$, we have

$$C_2 \|u\|_m \leq \|u\|_{m,\infty} \leq C_1 \|u\|_m$$

□

We now give a particular case of the Gagliardo-Nirenberg's inequality.

Theorem A.5. *[9, p.314] Let $U \subset \mathbb{R}^2$ be a regular bounded open set, then*

$$\|u\|_{L^4(U)} \leq C \|u\|_{L^2(U)}^{1/2} \|u\|_{H^1(U)}^{1/2} \quad \forall u \in H^1(U) \tag{A.1}$$

and so $H^1(U)$ is continuously embedded in $L^4(U)$.

A.2 $W^{m,p}(U)$ as a Banach Algebra

Let $U \subset \mathbb{R}^N$ be an open domain, and u, v in $W^{m,p}(U)$. In general, one cannot expect that the pointwise product uv belongs to $W^{m,p}(U)$. According to the following propositions, this is the case most of the time.

In the case when $m = 0$, we require one of the functions to be essentially bounded on U.

Proposition A.6. *Let $1 \leq p \leq \infty$. If $u \in L^p(U)$ and $v \in L^\infty(U)$, then $uv \in L^p(U)$ and*

$$\|uv\|_{L^p} \leq \|u\|_{L^p} \|v\|_\infty \tag{A.2}$$

Proof. If $p = \infty$, then $|u| \leq \|u\|_\infty < \infty$ and $|v| \leq \|v\|_\infty < \infty$ both a.e. on U. Hence

$$|uv| = |u|\,|v| \leq \|u\|_\infty \|v\|_\infty < \infty \text{ a.e. on } U$$

so that $\|uv\|_\infty \leq \|u\|_\infty \|v\|_\infty$, and so $uv \in L^\infty(U)$. If $p < \infty$, then

$$\|uv\|_{L^p}^p = \int_U |uv|^p \; d\mu \leq \|v\|_\infty^p \int_U |u|^p \; d\mu = \|v\|_\infty^p \|u\|_{L^p}^p \; < \infty$$

so that $\|uv\|_{L^p} \leq \|u\|_{L^p} \|v\|_\infty$, and so $uv \in L^p(U)$. □

In the case when $m = 1$, we require both functions to be essentially bounded on U.

Proposition A.7 (differentiation of a product, Prop 9.4 of [9])**.** *Let $1 \leq p \leq \infty$. If $u, v \in W^{1,p}(U) \cap L^\infty(U)$, then $uv \in W^{1,p}(U) \cap L^\infty(U)$ and*

$$\frac{\partial}{\partial x_i}(uv) = \frac{\partial u}{\partial x_i} v + u \frac{\partial v}{\partial x_i} \quad \forall i = 1, 2, \cdots N \tag{A.3}$$

In the case when $mp > N$ or $p = 1$ with $m \geq N$, we have the following due to the Sobolev embedding theorem.

Theorem A.8 (Thm 4.39 of [15])**.** *Let $U \subset \mathbb{R}^N$ be an open domain satisfying the cone condition below, and $mp > N$ or $p = 1$ with $m \geq N$. If $u, v \in W^{m,p}(U)$, then $uv \in W^{m,p}(U)$*

and

$$\|uv\|_{m,p} \leq C\,\|u\|_{m,p}\,\|v\|_{m,p} \tag{A.4}$$

where $C > 0$ is a constant depending on N, m, p, and the cone determining the cone condition for U.

The Cone Condition. $U \subset \mathbb{R}^N$ satisfies the *cone condition* if there exists a finite cone $\mathcal{C}$ such that each point $p \in U$ is the vertex of some finite cone $\mathcal{C}_p$ contained in U that is congruent to $\mathcal{C}$.

Appendix B

A Brief Overview of Semigroups and Their Applications

In this chapter, we give a brief overview of strongly continuous and analytic semigroups from a classical book on the topic [8] by A. Pazy, titled “Semigroups of Linear Operators and Applications to Partial Differential Equations”.

Throughout this chapter, X will be a Banach space.

B.1 Strongly Continuous or C_0 Semigroups

B.1.1 Definitions

Definition B.1. [8, Def 1.2.1] A one parameter family $S(t)$, with $0 \leq t < \infty$, of bounded linear operators from X to X is called *strongly continuous or C_0 semigroup* if

(i) $S(0) = I$, where I is the identity operator on X.

(ii) $S(t+s) = S(t)S(s) \quad$ for every $t, s \geq 0$.

(iii) $\lim_{t\to 0} S(t)x = x \quad$ for every $x \in X$.

Axiom (ii) is referred as the semigroup property.

Definition B.2. [8, Def 1.1.1] Let $S(t)$ be a C_0 semigroup (of bounded linear operators), then the linear operator A defined on

$$D(A) = \left\{ x \in X : \lim_{t\to 0} \frac{S(t)x - x}{t} \text{ exists} \right\} \tag{B.1}$$

by

$$Ax = \lim_{t \to 0} \frac{S(t)x - x}{t} \tag{B.2}$$

is called the *infinitesimal generator* of $S(t)$, and $D(A)$ is called the *domain* of A.

It can be shown (see Theorem 1.1.2 of [8]) that if the infinitesimal generator of a C_0 semigroup $S(t)$ is a bound linear operator, then

$$\lim_{t \to 0} \|S(t) - I\|_{\mathcal{L}(X)} = 0 \tag{B.3}$$

and consequently we have the following result. As a cultural note, a semigroup of bounded linear operators satisfying (B.3) is called *uniformly continuous*.

Theorem B.3. *If the infinitesimal generator A of a C_0 semigroup $S(t)$ is a bounded linear operator, then*

$$S(t) = e^{tA} := \sum_{n=0}^{\infty} \frac{(tA)^n}{n!} \tag{B.4}$$

Proof. Theorem 1.1.2, Theorem 1.1.3, and Corollary 1.1.4 of [8]. □

An important question is whether the infinitesimal generator of a C_0 semigroup generates a unique semigroup. It turns out that it can, if it is bounded.

Theorem B.4. *Let $T(t)$ and $S(t)$ be C_0 semigroups generated by the bounded linear operator. Then $T(t) = S(t)$ for all $t \geq 0$.*

Proof. Observe that $T(t)$ and $S(t)$ are uniformly continuous semigroups generated by the same bounded linear operator. Hence by Theorem 1.1.3 of [8], the result follows. □

B.1.2 Properties and Terminology

An essencial property of C_0 semigroups is that it is exponentially bounded.

Theorem B.5. *[8, Thm 1.2.2] Let $S(t)$ be a C_0 semigroup. Then there exist constants $\omega \geq 0$ and $M \geq 1$ such that*

$$\|S(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t} \quad \forall t \geq 0 \tag{B.5}$$

From this we can deduce the following properties.

Corollary B.6. *[8, Cor 1.2.3] If $S(t)$ is a C_0 semigroup, then for every $x \in X$, the map $t \mapsto S(t)x$ is continuous from $[0, \infty) \longrightarrow X$.*

Theorem B.7. *[8, Thm 1.2.4] Let $S(t)$ be a C_0 semigroup, and let A be its infinitesimal generator. Then*

a) For $x \in X$

$$\lim_{h \to 0} \frac{1}{h} \int_t^{t+h} S(\tau)x \, d\tau = S(t)x \tag{B.6}$$

b) For $x \in X$, $\int_0^t S(\tau)x \, d\tau \in D(A)$ and

$$A\left(\int_0^t S(\tau)x \, d\tau\right) = S(t)x - x \tag{B.7}$$

c) For $x \in D(A)$, $S(t)x \in D(A)$ and

$$\frac{d}{dt} S(t)x = AS(t)x = S(t)Ax \tag{B.8}$$

d) For $x \in D(A)$,

$$S(t)x - S(s)x = \int_s^t S(\tau)Ax \, d\tau = \int_s^t AS(\tau)x \, d\tau \tag{B.9}$$

Definition B.8. [8, p.8] In the context of Theorem B.5, if $\omega = 0$, we say that $S(t)$ is *uniformly bounded*. Moreover if $M = 1$, we say it is a C_0 *semigroup of contractions*.

Definition B.9. [8, p.8] For any linear operator A on a Banach space X, we define the *resolvent set* $\rho(A)$ of A as the set of all complex numbers η for which $\eta I - A : D(A) \longrightarrow X$ is invertible. The family $R(\eta : A) := (\eta I - A)^{-1}$, with $\eta \in \rho(A)$, of bounded linear operators is called the *resolvent* of A.

B.1.3 Hill-Yosida Characterizations of C_0 Semigroups

We now give a complete characterizations of C_0 semigroups and C_0 semigroups of contractions.

Theorem B.10 (Hille-Yosida for a C_0 semigroups)**.** *[8, Thm 1.5.3] Let X be a Banach space. A (unbounded) linear operator $A : D(A) \subseteq X \longrightarrow X$ is the infinitesimal generator of a C_0 semigroup $S(t)$ satisfying $\|S(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t}$ if and only if*

(i) A is closed and $D(A)$ is dense in X.

(ii) The resolvent set $\rho(A)$ of A contains the ray (ω, ∞), and

$$\|R(\eta : A)^n\|_{\mathcal{L}(X)} \leq \frac{M}{(\eta - \omega)^n} \quad \forall \eta > \omega \text{ and } \forall n = 1, 2, \cdots \tag{B.10}$$

Theorem B.11 (Hille-Yosida for a C_0 semigroup of contractions)**.** *[8, Thm 1.3.1] Let X be a Banach space. A (unbounded) linear operator $A : D(A) \subseteq X \longrightarrow X$ is the infinitesimal generator of a C_0 semigroup of contractions if and only if*

(i) A is closed and $D(A)$ is dense in X.

(ii) The resolvent set $\rho(A)$ of A contains the ray $(0, \infty)$, and

$$\|R(\eta : A)\|_{\mathcal{L}(X)} \leq \frac{1}{\eta} \tag{B.11}$$

From the proof of the Hille-Yosida theorems, we have the following corollary.

Corollary B.12. *If A is the infinitesimal generator of a C_0 semigroup $S(t)$, then*

$$R(\eta : A) = (\eta I - A)^{-1} = \int_0^\infty e^{-\eta t} S(t)\, dt \quad \forall \eta > 0 \tag{B.12}$$

Finally, we give a useful result for bounding the resolvent of a densely defined closed operator on a Hilbert space.

Definition B.13. [16, p.345] For an arbitrary linear operator A on a Hilbert space H (possibly over $\mathbb{C}$), we define the *numerical range* $\nu(A)$ of A by

$$\nu(A) = \{\langle Ax, x\rangle_H \;:\; x \in D(A), \|x\|_H = 1\} \subset \mathbb{C} \tag{B.13}$$

Theorem B.14. *[8, Thm 1.3.9] Let A be a closed linear operator with dense domain $D(A)$ in a Hilbert space H. Also let Σ be the complement of $\overline{\nu(A)}$ in $\mathbb{C}$. If $\eta \in \Sigma$, then $\eta I - A$ is one-to-one and has a closed range. Moreover, if Σ_0 is a component of Σ that intersects $\rho(A)$, then $\Sigma_0 \subset \rho(A)$, and*

$$\|R(\eta : A)\|_{\mathcal{L}(H)} \leq d(\eta : \overline{\nu(A)})^{-1} \tag{B.14}$$

where $d(\eta : \overline{\nu(A)})$ is the distance from η to $\overline{\nu(A)}$.

B.1.4 Applications to Some Abstract Cauchy Problems

The following are some basic results about initial value problems known as abstract Cauchy problems.

Theorem B.15. *[8, Thm 4.1.3] Let X be a Banach space, and $A : D(A) \subset X \longrightarrow X$ be a densly defined linear operator with a nonempty resolvent set $\rho(A)$. Then for every initial value $x \in D(A)$, the homogeneous abstract Cauchy Problem*

$$\begin{cases} \frac{du(t)}{dt} = Au(t) \quad t > 0 \\ u(0) = x \end{cases} \tag{B.15}$$

has a unique solution $u(t)$, which is continuously differentiable on $[0, \infty)$, if and only if A is the inifinitesimal generator of a C_0 semigroup.

Theorem B.16. *[8, Cor 4.2.11] Let X be a reflexive Banach space, and $A : D(A) \subset X \longrightarrow X$ be the inifinitesimal generator of a C_0 semigroup $S(t)$ on X. If $f : [0,T] \longrightarrow X$ is Lipschitz continuous, then for every initial value $x \in D(A)$, the inhomogeneous abstract Cauchy Problem*

$$\begin{cases} \frac{du(t)}{dt} = Au(t) + f(t) \quad t > 0 \\ u(0) = x \end{cases} \tag{B.16}$$

has a unique solution $u(t)$, which is continuously differentiable on $[0,T]$, given by

$$u(t) = S(t)x + \int_0^t S(t-s)f(s)\, ds \tag{B.17}$$

B.2 Analytic Semigroups

C_0 and other types of semigroups discussed above had the non-negative real-axis as their domains. In this section, we define a special type of semigroups called analytic semigroups, whose domains are sectors in the complex plane which contain the non-negative real axis.

B.2.1 Definition and Characterizations

Definition B.17. [8, Def 2.5.1] Let $\mathcal{R} = \{z \in \mathbb{C} : a < \arg z < b, \text{ with } a < 0 < b\}$ be sector in $\mathbb{C}$. A family $S(z)$, with $z \in \mathcal{R}$, of bounded linear operators from a Banach space X to X is called an *analytic semigroup* if

(i) $S(z_1 + z_2) = S(z_1)S(z_2)$ for every $z_1, z_2 \in \mathcal{R}$.

(ii) $S(0) = I$, and $\lim\limits_{\substack{z \to 0 \\ z \in \mathcal{R}}} S(z)x = x$ for every $x \in X$.

(iii) $z \mapsto T(z)$ is analytic on $\mathcal{R}$

A semigroup $S(t)$ will be called *analytic* if it is analytic in some sector $\mathcal{R}$ containing the non-negative real axis.

Clearly, the restriction of an analytic group to the real axis is a C_0 semigroup. Conversely, the following two theorems give essential ways of extending or characterizing C_0 semigroups into analytic semigroups.

Theorem B.18. *[8, Thm 2.5.2] Let $S(t)$ be a uniformly bounded C_0 semigroup, and A be its infinitesimal generator with $0 \in \rho(A)$. Then the following are equivalent:*

(a) $S(t)$ can be extended to an analytic semigroup in a sector $\mathcal{R}_\delta = \{z \in \mathbb{C} : |\arg z| < \delta\}$ and $\|S(z)\|_{\mathcal{L}(X)}$ is uniformly bounded in every closed subsector $\mathcal{R}'_\delta$, with $\delta' < \delta$, of $\mathcal{R}_\delta$.

(b) There exists a constant $C > 0$ such that for every $\sigma > 0$ and $\tau \neq 0$,

$$\|R(\sigma + i\tau : A)\|_{\mathcal{L}(X)} \leq \frac{C}{|\tau|} \tag{B.18}$$

(c) There exists $0 < \delta < \pi/2$ and $M > 0$ such that

$$\rho(A) \supset \Sigma = \left\{\eta : |\arg \eta| < \frac{\pi}{2} + \delta\right\} \cup \{0\} \tag{B.19}$$

and

$$\|R(\eta : A)\|_{\mathcal{L}(X)} \leq \frac{M}{|\eta|} \quad \forall \eta \in \Sigma, \eta \neq 0 \tag{B.20}$$

(d) $S(t)$ is differentiable for $t > 0$ and there exists a constant $C > 0$ such that

$$\|AS(t)\|_{\mathcal{L}(X)} \leq \frac{C}{t} \quad \forall t > 0 \tag{B.21}$$

Theorem B.19. *[8, Thm 2.5.5] Let A be the infinitesimal generator of a C_0 semigroup $S(t)$ satisfying $\|S(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t}$. Then $S(t)$ is analytic if and only if there exist constants $C > 0$ and $\Lambda \geq 0$ such that*

$$\left\|AR(\eta : A)^{n+1}\right\|_{\mathcal{L}(X)} \leq \frac{C}{n\eta^n} \quad \forall \eta > n\Lambda, \quad n = 1, 2, \cdots \tag{B.22}$$

Sometimes it is easier to show that a perturbation of an operator is the infinitesimal generator of an analytic semigroup, than working directly. In this case, the following proposition is very useful.

Proposition B.20. *[8, Cor 3.2.2] Let A be the infinitesimal generator of an analytic semigroup. If B is a bounded linear operator, then $A + B$ is the infinitesimal generator of an analytic semigroup.*

B.2.2 Fractional Powers of Infinitesimal Generators of Analytic Semigroups

Throughout this subsection, suppose that $-A$ is the infinitesimal generator of the analytic semigroup $S(t)$.

Definition B.21. [8, p.70] For every real $\alpha > 0$, we define

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} S(t)\, dt \tag{B.23}$$

and

$$A^{-0} = I \tag{B.24}$$

Proposition B.22. *[8, p.70-72]*

(a) For every $\alpha, \beta \geq 0$,

$$A^{-(\alpha+\beta)} = A^{-\alpha} \cdot A^{-\beta} \tag{B.25}$$

(b) There exists a constant $C > 0$ such that for every $0 \leq \alpha \leq 1$

$$\left\| A^{-\alpha} \right\|_{\mathcal{L}(X)} \leq C \tag{B.26}$$

(c) For every $x \in X$, we have

$$\lim_{\alpha \to 0} A^{-\alpha} x = x \tag{B.27}$$

Corollary B.23. *For every $0 \leq \alpha \leq 1$, $A^{-\alpha}$ is a uniformly bounded C_0 semigroup.*

Proposition B.24. *[8, Lem 2.6.6] $A^{-\alpha}$ is one-to-one (note that $A^{-\alpha}$ is densely defined on X).*

This leads us to define A^{α} as follows.

Definition B.25. [8, Def 2.6.7] For every real $\alpha > 0$, we define

$$A^{\alpha} = (A^{-\alpha})^{-1} \tag{B.28}$$

on the domain $D(A^\alpha) = \text{Range of } A^{-\alpha}$. We also set $A^0 = I$.

Proposition B.26. *[8, Thm 2.6.8]*

(a) A^α *is closed.*

(b) $\alpha \geq \beta > 0$ *implies* $D(A^\alpha) \subset D(A^\beta)$.

(c) $\overline{D(A^\alpha)} = X$ *for every* $\alpha \geq 0$.

(d) If α, β *are real, then*

$$A^{\alpha+\beta}x = A^\alpha \cdot A^\beta x \quad \forall x \in D(A^\gamma) \tag{B.29}$$

where $\gamma = \max\{\alpha, \beta, \alpha+\beta\}$.

An explicit formula for A^α is given by the following theorem.

Theorem B.27. *[8, Thm 2.6.9] Let* $0 < \alpha < 1$, $\gamma > \alpha$ *and* $x \in D(A^\gamma)$, *then*

$$A^\alpha x = \frac{\sin \pi\alpha}{\pi} \int_0^\infty t^{\alpha-1} A(tI + A)^{-1} x \, dt \tag{B.30}$$

Now let $\alpha > 0$, then the closedness of A^α implies that $D(A^\alpha)$ endowed with the graph norm of A^α, namely $\|\cdot\|_X + \|A^\alpha \cdot\|_X$, is a Banach space. But since A^α is invertible, then the graph norm is equivalent to the norm $\|A^\alpha \cdot\|_X$, so that we have the following definition [8, p.135].

Definition B.28. Let $\alpha > 0$. Then $D(A^\alpha)$ endowed with the norm

$$|||\cdot|||_\alpha := \|A^\alpha \cdot\|_X \tag{B.31}$$

is a Banach space, which we will denote as X_α .

We finally relate A^α to the analytic semigroup $S(t)$ generated by $-A$.

Theorem B.29. *[8, Thm 2.6.13] If* $0 \in \rho(A)$, *then*

(a) $S(t) : X \longrightarrow D(A^\alpha)$ *for every* $t > 0$ *and* $\alpha \geq 0$.

(b) For every $x \in D(A^\alpha)$, *we have* $S(t)A^\alpha x = A^\alpha S(t)x$.

(c) For every $t > 0$, *the operator* $A^\alpha S(t)$ *is bounded and*

$$\|A^\alpha S(t)\|_{\mathcal{L}(X)} \leq M_\alpha t^{-\alpha} e^{-\delta t} \tag{B.32}$$

for some constants $M_\alpha > 0$ *(depending on* α*) and* $\delta > 0$.

(d) If $0 < \alpha \leq 1$ and $x \in D(A^\alpha)$, then

$$\|S(t)x - x\|_{\mathcal{L}(X)} \leq C_\alpha t^\alpha \, |||x|||_\alpha \tag{B.33}$$

for some constant $C_\alpha > 0$ (depending on α).

B.2.3 Applications to a Semilinear Abstract Cauchy Problem

Analytic semigroups have vast applications in the theory of evolutionary PDEs, however in this subsection we only recall a result from [8, p.196] we will rely on.

Consider the semilinear Abstract Cauchy Problem

$$\begin{cases} \frac{du(t)}{dt} + Au(t) = f(t, u(t)) & t > t_0 \\ u(t_0) = x_0 \end{cases} \tag{B.34}$$

where $-A$ is the infinitesimal generator of a uniformly bounded analytic semigroup of bounded linear operators on a Banach space X. For the purpose of solving (B.34), we first introduce some assumptions on the function f as follows.

Assumption (F). Let U be an open subset of $(0, \infty) \times X_\alpha$, where $0 < \alpha < 1$. The function $f : U \longrightarrow X$ satisfies the *assumption (F)* if for every $(t, x) \in U$, there is a neighborhood $V \subset U$ and constants $L \geq 0, 0 < \theta \leq 1$ such that

$$\|f(t_1, x_1) - f(t_2, x_2)\|_X \leq L(|t_1 - t_2|^\theta + |||x_1 - x_2|||_\alpha) \quad \forall (t_i, x_i) \in V \tag{B.35}$$

Theorem B.30. *[8, Thm 6.3.1] Let $-A$ be the infinitesimal generator of an analytic semigroup $S(t)$ satisfying $\|S(t)\|_{\mathcal{L}(X)} \leq M$, and assume further that $0 \in \rho(-A)$. If f satisfies the assumption (F), then for every initial data $(t_0, x_0) \in U$ the initial value problem (B.34) has a unique local solution $u \in C([t_0, t_1) : X_\alpha) \cap C^1((t_0, t_1) : D(A))$, given by*

$$u(t) = S(t - t_0)x_0 + \int_{t_0}^{t} S(t - s)f(s, u(s)) \, ds \quad t_0 \leq t < t_1 \tag{B.36}$$

where $t_1 > t_0$ is a constant depending only on (t_0, x_0).

Appendix C

Abbreviations and Notation

C_0	Strongly Continuous
$C^m(\Omega)$	Space of up to m-times continuously differentiable functions on Ω
$C^\infty(\Omega)$	Space of infinitely continuously differentiable functions on Ω
$C_0^\infty(\Omega)$	Space of infinitely continuously differentiable functions on Ω with compact support
D^α	(Weak) Differential operator with respect to spacial multi-index α
$\mathcal{F}(A;B)$	Space of functions from space A to space B
$W^{m,p}(\Omega)$	Sobolev space
$H^m(\Omega)$	Sobolev space $W^{m,2}(\Omega)$
$H^m_\infty(\Omega)$	Sobolev space $W^{m,\infty}(\Omega)$
$H^m_P(\Omega)$	Periodic Sobolev space
$C^m_P(\Omega)$	$C^m(\Omega) \cap H^m_P(\Omega)$
$\mathcal{S}^{n,m}_{T^*}$	Banach space $C([0,T^*) : H^n_P(\Omega)) \cap C^1((0,T^*) : H^m_P(\Omega))$
$\|\cdot\|$	$L^2(\Omega)$ norm
$\|\cdot\|_m$	$H^m(\Omega)$ norm
$\|\cdot\|_{m,p}$	$W^{m,p}(\Omega)$ norm
$\langle\cdot,\cdot\rangle$	$L^2(\Omega)$ innerproduct
$\langle\cdot,\cdot\rangle_m$	$H^m(\Omega)$ innerproduct
$\|A\|_{\mathcal{L}(X)}$	Operator norm of the linear operator $A : X \to X$
$D(A)$	Domain of the linear operator A
A^α	Linear operator A raised to the fractional power α
$\|\|\|\cdot\|\|\|_\alpha$	Fractional norm $\|A^\alpha \cdot\|_X$
$\rho(A)$	Resolvent set of the linear operator A
$R(\eta : A)$	Resolvent of A, equals to $(\eta I - A)^{-1}$
$S(t)$	One-parameter semigroup of bounded linear operators

Bibliography

[1] ITER Physics Expert Groups on Confinement and Transport and Confinement Modeling and Database, "Plasma confinement and transport," *Nuclear Fusion*, vol. 39, no. 12, pp. 2175–2249, 1999.

[2] A. Hasegawa and K. Mima, "Pseudo-three-dimensional turbulence in magnetized nonuniform plasma," *Physics of Fluids*, vol. 21, pp. 87–92, Jan 1978.

[3] A. Hasegawa and M. Wakatani, "Plasma edge turbulence," *Phys. Rev. Lett.*, vol. 50, pp. 682–686, Feb 1983.

[4] B. Shivamoggi, "Charney-hasegawa-mima equation: A general class of exact solutions," *Physics Letters A*, vol. 138, pp. 37–42, Jun 1989.

[5] F. A. Hariri, "Simulating bi-dimensional turbulence in fusion plasma with linear geometry," Master's thesis, American University of Beirut, 2010.

[6] B. Guo and Y. Han, "Existence and uniqueness of global solution of the hasegawa-mima equation," *Journal of Mathematical Physics*, Apr 2004.

[7] L. Paumond, "Some remarks on a hasegawa-mima-charney-obukhov equation," *Physica D: Nonlinear Phenomena*, vol. 195, p. 379390, Aug 2004.

[8] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Springer, 1983.

[9] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. Springer, 2011.

[10] L. C. Evans, *Partial Differential Equations*. American Mathematical Society, 2010.

[11] J. C. Robinson, *Infinite-Dimensional Dynamical Systems: An Introduction to Dissipative Parabolic PDEs and the Theory of Global Attractors*. Cambridge University Press, 2001.

[12] G. C. Hsiao and W. L. Wendland, *Boundary Integral Equations*. Springer, 2008.

[13] W. O. Amrein, *Hilbert Space Methods in Quantum Mechanics*. EPFL Press, 2009.

[14] C. Chicone, *Ordinary Differential Equations with Applications*. Springer, second ed., 2006.

[15] R. A. Adams and J. J. Fournier, eds., *Sobolev Spaces*. Academic Press, Inc, second ed., 2003.

[16] G. Grubb, *Distributions and Operators*. Springer, 2009.