# AMERICAN UNIVERSITY OF BEIRUT

# PRICING AND INVENTORY DECISIONS OF AN ASSORTMENT UNDER EQUAL PROFIT MARGIN

by

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Submitted in partial fulfillment of the requirements
for the degree of Master of Engineering Management
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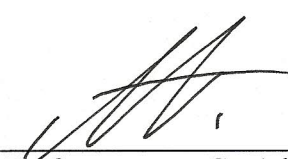at the American University of Beirut

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# AN ABSTRACT OF THE THESIS OF

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We consider the interdependent decisions on inventory and pricing of substitutable products in an assortment that is differentiated by some secondary attributes such as color, flavor, etc. We assume we have a newsvendor-type model with several products and one selling period under a logit consumer choice model. We also assume that all products in the assortment have equal profit margins. Demand assumes a *multiplicative-additive* structure where both variance and coefficient of variation depend on the pricing captured by the common profit margin. This is a realistic demand structure (Maddah et al., 2013) reflecting customers arriving according to a Poisson process and making purchases, at random, according to the logit model. Our problem is then to determine the common profit margin and the inventory levels of products in the assortment in a way that maximizes the expected profit. This problem has been studied in the literature for homogenous products, having equal unit costs or equal costs and average consumer valuations, but not for general assortments due to its complex nature. Under an adopted Taylor Series approximation, the profit function is proved unimodal in the common profit margin. Then, we compare the optimal profit margin to the "riskless" profit margin, where no inventory exists, in order to understand the effect of inventory considerations on pricing. We further perform a comparative static (sensitivity) analysis on demand and cost parameters to understand the environment impact on pricing. We continue to study the structure of the optimal assortment and establish dominance results.

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# INTRODUCTION

Quite often, when a consumer enters a store, she has to choose among a set of horizontally differentiated products (similar products that differ in color, flavor, etc.) available at the store. A major determinant that will affect her choice is the retailer's price. Price may not only affect her choice of whether to purchase her most favorable product or not, price may also shift her interest into a new product. Hence, pricing decisions does not only affect demand for every product but also has an impact on the overall demand and the allocation of demand among the set of products. For that reason, it is beneficial to decide on assortment, inventory and pricing jointly.

Our work is an attempt at exploring the benefit of joint decisions for a category of substitutable products. We first assume a Multinomial logit consumer choice model (MNL) which is a common approach in literature (e.g., Aydin and Porteus, 2008, van Ryzin and Mahajan, 1999) in a newsvendor inventory setting where the inventory is sold in a single time period. In order to simplify the pricing decision, we assume that all the products have the same profit margin. We verify that this assumption yields near optimal results via a vast numerical study. To avoid the independence-from-irrelevant-alternatives pitfall (IIA[*]) that the MNL model exhibits, we extend our results to the more general case of a nested logit model where consumers' purchase behavior is

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[*] IIA property: the property of a model where the ratio of the probabilities of choosing any two alternatives is independent of the attributes or the availability of a third alternative. For a detailed explanation, please refer to McFadden et al., 1977.

modeled through a two-step decision process; consumers first decide on the nest of preference then they choose to purchase an item within this nest. For instance, a customer entering a store wanting to buy a knit needs to first decide on whether to buy a cardigan, a V-neck sweater or a round neck sweater among other alernatives. Each of the three options can be seen as a distinct nest. Once she chooses to buy a cardigan; i.e. she chooses her nest, the consumer then decides on the color of cardigan she wants to purchase.

This paper is an extension of the work done by Maddah et al. (2014) where the authors analyze assortment, pricing, and inventory decisions for a category of substitutable products assuming (i) a "multiplicative-additive" demand model and logit choice, (ii) all products in the category have equal unit cost, and are accordingly priced at the same level, and (iii) a newsvendor-type inventory setting. This paper first relaxes Assumption (ii) of Maddah et al. (2014) by considering a category with products having heterogeneous costs. To keep the pricing decision trackable, we assume that all products have equal unit profit margins identified as the retail price minus the unit cost. We then relax Assumption (i) of Maddah et al. (2014) and analyze a nested-logit-based demand instead of the logit-based demand. The main contribution of this paper is showing that (i) many results in Maddah et al. (2014) continue to hold in our more general setting, and (ii) other results in that paper hold under reasonably weaker settings. Specifically, the unimodalarity of the profit function still holds under the more general case. We show that "popular sets" are "optimal sets" in the proposed upper and lower bounding functions, which conforms to the dominance result in Maddah et al. (2014). We compare risky and riskless prices as well, but we obtain different functions in terms of additional parameters that allows the comparison.

The remainder of this article will be organized as follows: In Section 2, we provide a review of common approaches to study the aforementioned joint decisions. In Section 3, we introduce our model. In Section 4, we establish some structural properties of the expected profit function to simplify the search for the optimal pricing policy. In Section 5, we compare riskless and "risky" profit margins. In Section 6, we provide a numerical analysis that validates the near optimality of assuming an equal profit margin. Finally in Section 7, we examine the structure of the optimal assortment. In Section 8, we extend some of our results on a nested-logit-based demand model.

# CHAPTER II

# LITERATURE REVIEW

As it affects the effectiveness of companies' decisions on product planning, pricing and control, product substitution has recently grasped substantial attention in the operations management literature (Shin et al., 2015). In this context, Shin et al. (2015) explain three substitution mechanisms: (1) assortment-based substitution, where the customer substitutes the initially preferred product by another newly introduced to the retailer's assortment, (2) inventory based substitution where the customer substitutes in the case of stock out of initially preferred product and (3) price based substitution, where the customer substitutes her initially preferred product due to change in relative prices of substitutable products. Considering these substitution mechanisms, the literature has focused on four major areas of decisions that can affect customers' substitution behavior: assortment planning, inventory decision, pricing decision, and capacity planning. In this paper, we examine the first three decisions in a two-facet approach. First, the assortment is considered fixed and optimal prices and inventory levels are determined. In the next step, prices are assumed fixed and the structure of optimal assortment and inventory levels are then determined. As shown in figure 1, our research follows suit of several works in literature.
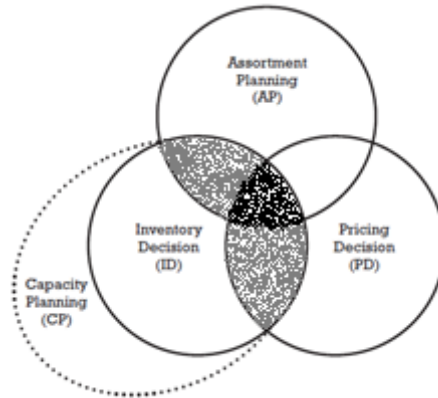
Figure 1: Area of decision and modeling environment under product substitution in the literature (adopted from Shin et al., 2015)

Examples of works that consider joint inventory and pricing decisions for a given assortment include Aydin and Porteus (2009), Maddah et al. (2014), Kocabiyikoglu and Popescu (2011), Li and Huh (2011), Roels (2012), Karakul and Chan (2010), and Akan et al. (2013).

Example of works on joint assortment and inventory decisions under exogeneous pricing include Van Ryzin and Mahajan (1999), Aggrawal and Smith (2000), Kok and fisher (2007), Maddah et al. (2014), and Honhon et al. (2010).

It is also worth mentioning that a stream of the literature considers all three namely assortment, pricing, and inventory decisions jointly (e.g. Maddah and Bish (2007), Ghoneim and Maddah (2015), and Tang and Yin (2010)). More details on these works are available in Ghoneim and Maddah (2015) and Shin et al. (2015).

Recent works on the joint decisions can be further classified based on their solution methodology; some utilize mathematical programming (MP), and others utilize stylized models. In the mathematical programming approach, most authors examine at most two of the three decisions (namely pricing, inventory, and assortment planning) by

optimizing on a data-driven model with the exception of Ghoneim and Maddah (2015) who study three decisions simultaneously. Example of works that utilize the MP approach include Dobson and Kalish (1993) and Subramanian and Sherali (2010). More details are available in Ghoneim and Maddah (2015).

The main difference between the MP approach and the stylized model approach is that in the first, authors seek an exact solution for large data-driven models whereas, authors utilizing stylized models are more concerned about gathering insight on the behavior of the profit function and the parameters affecting the profit. Furthermore, the MP approach takes on fewer assumptions but assumes that demand is deterministic, whereas in stylized models demand is more realistically taken as stochastic but several simplifying assumptions are often made. Examples of works that utilize stylized models include Aydin and Ryan (2000), Aydin and Porteus (2008), Besanko et al. (1998), Cattani et al. (2003), Hopp and Xu (2005), Van Ryzin and Mahajan (1999), Kök and Xu (2011), and Li and Huh (2011).  Our work uses the stylized approach and contributes to the literature by expanding the results of Maddah at al. as explained earlier.

Other authors assume exogenous demand models, where demand is predetermined for each product (e.g. Smith and Agrawal (2000) and Kok and Fisher (2007)). Locational demand has been widely used as well. Locational demand links the attractiveness of a product to its "distance" from an ideal product location (Gaur and Honhon, 2006.).

To avoid the IIA property that the MNL exhibits, many authors study consumer choice under the more general nested multinomial logit choice model

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(NMNL). Examples of such work includes Kalakesh (2006), Li and Huh (2011), and Kök and Xu (2011). Kalakesh (2006) concludes through a numerical study that "popular sets" are optimal sets and that the equal profit margin heuristic renders near optimal results under the nested logit choice model. Li and Huh (2011) study the pricing problem and show that revenue and profit are concave in the market share vector. Kök and Xu (2011) also assume a nested logit choice model and present their results under a brand primary and a demand primary choice model further differentiated under centralized and decentralized management. They derive closed form expressions for optimal profit margins and item revenues while taking on the assumption that inventory costs follow a function increasing and concave in the expected demand. In section 8 of this paper, we also consider the NMNL to study the validity of our findings. We use a model similar to that utilized by Kalakesh (2006) where customers can only choose not to purchase while choosing among nests. That is, if a nest is chosen, the costumer has to purchase an item within a nest and cannot leave empty handed.

# CHAPTER III

# MODEL AND ASSUMPTIONS

To study the problem, discussed in Section 1, we will assume a similar model to that utilized by Maddah et al. (2014). We define $\Omega = \{1, 2, \ldots, n\}$ as the set of products, differing in secondary characteristics, from which the retailer can form her assortment $S$. Variants in $\Omega$ have equal profit margins, denoted by $m$, defined as the difference between the retail price, $p_i$, and the unit cost, $c_i$, of a product $i \in \Omega$; $m = p_i - c_i$. A consumer has a mean reservation price for Product $i \in \Omega$ defined as $\alpha_i$. The customer chooses the product that maximizes her utility, defined by utilizing a logit choice model as $U_i = \alpha_i - p_i + \varepsilon_i$ for every product $i \in S$; and a utility of the no-purchase option given by $U_0 = \varepsilon_0$ where , without loss of generality, $\varepsilon_i$, $i \in S \cup \{0\}$, are independent and identically distributed (i.i.d.) Gumbel random variables with mean 0 and shape factor 1(e.g., Guadagni and Little (1983)). We assume that a consumer acts to maximize her utility. Hence the probability of purchasing $i \in \Omega$ is given by $q_i(m, S) = \Pr\left\{U_i = \max_{j \in S \cup \{0\}} U_j\right\}$, and the no-purchase probability is given by $q_0(m, S) = 1 - \sum_{i \in S} q_i(m, S)$, which simplifies to

$$q_i(m, S) = \frac{e^{\alpha_i - c_i - m}}{1 + \sum_{i \in S} e^{\alpha_i - c_i - m}} \qquad q_0(m, S) = \frac{1}{1 + \sum_{i \in S} e^{\alpha_i - c_i - m}}, \tag{1}$$

We assume that a consumer takes her decision of purchasing based only on price, variety and quality (assortment-based substitution). Assuming that the arrival

process to the store follows a Poisson distribution with rate $\lambda$, the demand for Product $i \in S$, $X_i$, is a normal random variable with mean $\lambda q_i(m, S)$ and standard deviation $\sqrt{\lambda q_i(m, S)}$. Since both the coefficient of variation and standard deviation (given by $1/\sqrt{\lambda q_i(m, S)}$ and $\sqrt{\lambda q_i(m, S)}$ respectively) of the demand are functions of the profit margin $m$, our demand model may be seen as "multiplicative–additive". This is because $X_i = \lambda q_i(m, S) + \sqrt{\lambda q_i(m, S)}Z_i$, where $Z_i$ are i.i.d. with a standard normal distribution.

Following established results on the newsvendor model under normal demand (e.g. Silver et al., 1998) the optimal inventory level, $y_i^*(m, S)$, for each product in $S$, and the total profit, $\Pi(m, S)$, from Assortment $S$ at optimal inventory levels can be written as

$$y_i^*(m, S) = \lambda q_i(m, S) + \phi^{-1}(1 - \frac{c_k}{m + c_k})\sqrt{\lambda q_i(m, S)}$$
(2)

$$\Pi(m, S) = \sum_{i=1}^{k} m\lambda q_i(m, S) - (m + c_i)\varphi(\phi^{-1}(1 - \frac{c_i}{m + c_i}))\sqrt{\lambda q_i(m, S)}$$
(3)

where $\varphi(.)$ and $\phi(.)$ define the probability density function and the cumulative distribution function of the standard normal distribution. In Equation (3) the first term is the profit with no inventory considerations, while the second term accounts for inventory cost. Through a numerical search, the profit margin and assortment size that maximize the expected profit function can be determined. Obtained optimal values can be then replaced in Equation (2) to determine optimal inventory levels. Finally, letting $k = |S|$ and rearranging (3), the expected profit can be written as

$$\Pi(m,k) = m\beta_k \lambda g(m,k) - \sqrt{\lambda g(m,k)} \sum_{i=1}^{k} (m+c_i)\varphi(\phi^{-1}(1-\frac{c_i}{m+c_i}))\sqrt{e^{\alpha_i-c_i}}$$

(4)

where $g(m,k) = \dfrac{e^{-m}}{1+e^{-m}\beta_k}$ , and $\beta_k = \sum_{i=1}^{k} e^{\alpha_i-c_i}$ .

# CHAPTER IV

# STRUCTURE OF THE PROFIT FUNCTION

In this section, we analyze the properties of the optimal profit margin one that

maximizes the expected total profit function; $m_k^* = \arg\max_{m>0} \Pi(m,k)$. Since the

expected profit is complex to analyze, we adopt a Taylor series-based approximation

developed by Maddah et al. (2014). Then, we prove that the resulting approximate

expected profit $\hat{\Pi}(m,k)$ is unimodal in the profit margin, $m$. To ensure that the retailer

is not better off by not selling anything, we first make the following assumption:

**Assumption 1:** The expected profit $\hat{\Pi}(m,k)$ is increasing in $m$ at $m=0$; that is

$$\left.\frac{\partial \hat{\Pi}(m,k)}{\partial m}\right|_{m=0} > 0$$
, rearranging, $\lambda > \dfrac{a^2(1+\beta_k)\mu_k}{\beta_k^{\,2}}$, where $\mu_k = \sum_{i=1}^{k} \sqrt{e^{\alpha_i - c_i}}$.

This assumption will hold in the remainder of the paper. Maddah et al. (2014)

approximation is as follows: $\varphi(\phi^{-1}(1-x)) \approx -ax(x-1)$, where $a = 1.66$. The profit

function becomes

$$\hat{\Pi}(m,k) = m\left[\lambda g(m,k)\beta_k - a\sqrt{\lambda g(m,k)}\theta(m,k)\right] \tag{5}$$

where $\theta(m,k) = \sum_{i=1}^{k} \dfrac{c_i}{m+c_i}\sqrt{e^{\alpha_i - c_i}}$.

Now, we will study the structure of the expected profit as function of the profit

margin. Since $\lim_{m\to\infty} \hat{\Pi}(m,k) \to 0^-$, it can be shown that $\hat{m}_k^*$ is an internal point

solution satisfying the first- and second-order optimality conditions under Assumption 1. The following lemma establishes our first set of structural properties on the expected profit as a function of $m$.

**Lemma 1.** *The expected profit* $\hat{\Pi}(m,k) > 0$ *if and only if $m \in (0, \bar{m}_k)$ where $\bar{m}_k > 0$ is the unique solution to the equation* $\lambda(\beta_k)^2 g(m,k) - a^2\theta(m,k)^2 = 0$ *. Furthermore,* $\hat{m}_k^* < \bar{m}_k$ *and* $\bar{m}_k$ *is decreasing in k.*

**Proof: See Appendix A**

Lemma 1 shows that the expected profit is positive on the interval extending from 0 to $\bar{m}_k$. This interval defines the logical set of profit margins consumers are willing to purchase at, and $\bar{m}_k$ may be seen as a logical upper bound on the optimal profit margin $\hat{m}_k^*$. Hence, a retailer offering a wide variety of products cannot assign high profit margins. This finding conforms to the behavior of the logit choice model, where we recognize demand thinning with high variety.

The following theorem describes the behavior of $\hat{\Pi}(m,k)$ on the interval defined in Lemma 1.

**Theorem 1.** *The expected profit* $\hat{\Pi}(m,k)$ *is unimodal in m on* $(0, \bar{m}_k)$ *.*

**Proof: See Appendix A**

This result is in-line with previous findings by Maddah et al. (2014), where the authors prove that the expected profit function is unimodal in price which is assumed to

be the same for all products having the same unit cost. Utilizing the same method, we prove that the expected profit is unimodal in *m* for homogeneous products having different unit costs. Theorem 1 shows that there exists a single optimal solution on $(0, \bar{m}_k)$ that maximizes the profit. This, along with Lemma 1, allows the determination of the optimal profit margin through any line search technique.

# CHAPTER V

# EFFECT OF INVENTORY CONSIDERATIONS ON PRICING

In this section, we study the properties of the optimal profit margin $m_k^*$, while establishing means by which we can compare it to the riskless profit margin. The riskless profit margin $m_k^0$, is one which maximizes the profit when no inventory exists; i.e. a make-to-order retailer case.

When no inventory costs are considered, the second part of Equation (4) is dropped, and the expected profit function can be written as

$$\Pi^0(m,k) = m\beta_k \lambda g(m,k) \tag{6}$$

In the following corollary, we adopt results established by Li and Huh (2011), who find a closed-form expression for the riskless optimal profit margin $m_k^0 = \arg\max \Pi^0(m,k)$ in terms of the Lambert $W$ (.) function. The $W$ $(x)$ function is defined as the unit solution of $we^w = x$; see Corless et al. (1998) for background on the Lambert function).

**Corollary 1.** The expected riskless profit $\Pi^0(m,k)$ is unimodal in m, with

$$m_k^0 = 1 + W(\sum_{j=1}^{k} e^{\alpha_j - c_j - 1}) \tag{7}$$

Moreover, $m_k^0$ is

a.    Increasing in the unit cost per item, $c$;

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b. Increasing in the mean reservation price of item $i \in S_k$, $\alpha_i$;

c. Insensitive to the demand volume, $\lambda$.

**Proof: See Appendix B**

In the related literature, the risky price, $p_k^*$, and the riskless price, $p_k^0$, compare as follows. In an additive demand, $p_k^* \leq p_k^0$, in multiplicative demand, $p_k^* \geq p_k^0$ (Petruzzi and Dadda, 1999) and in additive-multiplicative demand, $p_k^*$ may fall above or below $p_k^0$ (Maddah et al., 2014). In the following we present a result that closely relates to the result Maddah et al. provide.

After considering inventory costs, the riskless profit margin, $m_k^0$, shifts to a risky profit margin, $\hat{m}_k^*$. Thus, studying the inventory cost function will enable us to compare $m_k^*$ and $m_k^0$. From equation (5), the inventory cost is given by $\hat{C}_1(m,k) = am\sqrt{\lambda g(m,k)}\theta(m,k)$. We prove $\hat{C}_1(m,k)$ to be unimodal in $m$, and decreasing at $m = m_k^0$.

**Lemma2.** *The approximate inventory cost* $\hat{C}_1(m,k)$ *is unimodal in m. Moreover,*

$$\left.\frac{\partial \hat{C}_1(m,k)}{\partial m}\right|_{m=m_k^0} = a\sqrt{\lambda g(m_k^0,k)}\left[\frac{\theta(m_k^0,k)}{2} + \left.\frac{\partial\theta(m,k)}{\partial m}\right|_{m=m_k^0}\right]. \tag{8}$$

**Proof: See Appendix B**

Given Lemma 2, the comparison of the risky $\hat{m}_k^*$ and riskless profit margins $m_k^0$ becomes possible. The unimodilarity result indicates that $\hat{m}_k^*$ would fall above $m_k^0$ in a

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way as to obtain a lower inventory cost at $m_k^*$. The inventory cost, $\hat{C}_1(m,k)$, is decreasing

in $m_k^0$ when $\left|\dfrac{\partial\theta(m,k)}{\partial m}\right|_{m=m_k^0} > \dfrac{\theta(m_k^0,k)}{2}$ as $\left.\dfrac{\partial\theta(m,k)}{\partial m}\right|_{m=m_k^0}$ is negative. Thus, the risky

profit margin needs to fall above $m_k^0$ to obtain a lower inventory cost. When

$\left|\dfrac{\partial\theta(m,k)}{\partial m}\right|_{m=m_k^0} < \dfrac{\theta(m_k^0,k)}{2}$, $\hat{C}_1(m,k)$ is increasing in $m_k^0$ and thus $\hat{m}_k^*$ would need to fall

below $m_k^0$. Let $\overline{c}$ be the cost of the most expensive product in the assortment,

$\overline{c} = \max_{i\in S} c_i$ and $\underline{c}$ be the cheapest product, $\underline{c} = \min_{i\in S} c_i$. When $m_k^0 > 2 - \underline{c}$, we are

certain that $\left|\dfrac{\partial\theta(m,k)}{\partial m}\right|_{m=m_k^0} < \dfrac{\theta(m_k^0,k)}{2}$, hence, $\hat{m}_k^* < m_k^0$. Similarly, if $m_k^0 < 2 - \overline{c}$, then

$\hat{m}_k^* > m_k^0$.


**Lemma 3.** *If* $\beta_k < (1-\overline{c}).e^{2-\overline{c}}$, *then* $\hat{m}_k^* > m_k^0$. *Equivalently, i f* $\beta_k > (1-\underline{c}).e^{2-\underline{c}}$, *then*

$\hat{m}_k^* < m_k^0$.


**Proof: See appendix B**


The conditions on $\beta_k$ in Lemma 3 are equivalent to $m_k^0 > 2 - \underline{c}$ and $m_k^0 < 2 - \overline{c}$

but they allow the comparison of the riskless and risky profit margins without the need

to find the value of $m_k^0$.

# CHAPTER VI

# NUMERICAL ANALYSIS ON THE EQUAL PROFIT MARGIN HEURISTIC

The objective of the following numerical study is to study the validity of the

equal profit margin heuristic under different scenarios. We assume a base case with

three products with equal unit costs ($c = 8$). Other base parameters were set to the

following: $\lambda = 100$, $\alpha_1 = 15$, $\alpha_2 = 14$, $\alpha_3 = 13$. In (a) we vary the arrival rate while

keeping all parameters constant. In (b), the consumer's utility is variable and in (c) we

study the heuristic under both variable costs and variable consumer utilities by

calculating the percentage optimality gap according to the following:

$$\frac{\Pi(\mathbf{p}^*, S) - \Pi(m^*)}{\Pi(\mathbf{p}^*, S)} \times 100 \quad {}^\dagger .$$

## A. Effect of arrival on heuristic.

In this section we study the effect of changing the arrival rate on the validity of

the equal profit margin. The arrival rate is increased from the minimum value where

$\lambda = a_k^2 [\beta_k + 1]$ following Assumption 1. At each arrival rate, optimal prices and the

optimal profit margin are determined. The percentage optimality gap is then computed.

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${}^\dagger$ The expected profit $\Pi(\mathbf{p}^*, S)$ is the optimal profit assuming that products in the assortment
have different prices given by $\mathbf{p} = (p_1, p_2, ..., p_k)$. $\Pi(\mathbf{p}^*, S) = \max \Pi(\mathbf{p}, S)$ where

$$\Pi(\mathbf{p}, S) = \sum_{i=1}^{K} (p_i - c_i) \lambda q_i(\mathbf{p}, S) - a \frac{c_i}{p_i} \sqrt{\lambda q_i(\mathbf{p}, S)} \text{ and } q_i(m, S) = \frac{e^{\alpha_i - p_i}}{1 + \sum_{j=1}^{K} e^{\alpha_j - p_j}} .$$

Figure 2 (a) shows that at reasonably high arrival rates the optimality gap tends to 0 validating the equal profit margin heuristic.

### B. Optimality with varying mean reservation prices $\alpha_i$.

Fixing the arrival rate and unit cost constant (at $\lambda = 100$, and $c = 8$), we vary the mean reservation price of each product and study the optimality gap. It is assumed that reservation prices do not change arbitrarily but vary in function of $k$ (assortment size), according to the following equation $\alpha_i = [\gamma + (k - i + 1)\delta]c$. A base case of $\gamma = 1.5$, $\delta = 0.125$ and k = 3 is assumed. The heuristic is then tested by calculating the percentage optimality gap according to [2] at different $k$, $\gamma$, and $\delta$ values. Figures (b) and (c) show a plot of the percentage optimality gaps in function of $\gamma$, and $\delta$. Ranges for $\gamma$, and $\delta$ were selected in conformity with Assumption 1.

Figures 2(b) and (c) show that the percentage optimality gap remains low as $\gamma$ and $\delta$ are varied. The percentage optimality gap at lower values may be explained by the presence of products having a cost exceeding reservation prices which is mostly not the case in reality. As for varying $\delta$, a maximum optimality gap of 2.1% was observed. This gap tends to 0 at higher $\delta$ values. With $\delta$ and $\gamma$ constant, the assumption of 3 products is relaxed as $k$ value varies. Figure 2(d) acquires the shape of a plateau when the assortment size, $k$, increases.

## C. Effect of simultaneous variation in product costs and mean reservation prices.

We utilize the same equation mentioned above and define the cost of product $i$ in function of variables $\gamma_c$, and $\delta_c$ with k = 3. Consumer's utility remains in function of the product cost but γ, δ and $k$ are set to their base values (γ = 1.5, δ = 0.125 and k = 3). Therefore, the cost of Product $i$ can be written as, $c_i = [\gamma_c + (4-i)\delta_c]8$. $\gamma_c$ and $\delta_c$ values were varied as to obtain positive costs for the three products with base values of $\gamma_c = 0$ and $\delta_c = 1$. Figure 2(e) (f) shows the variation of percentage optimality gap in function of the studied parameters. As $\gamma_c$ varies, the percentage optimality gap increases to a peak of less than 3.5% and then converges to 0. Setting $\gamma_c$ constant and varying $\delta_c$, a maximum error of less than 3% was obtained.
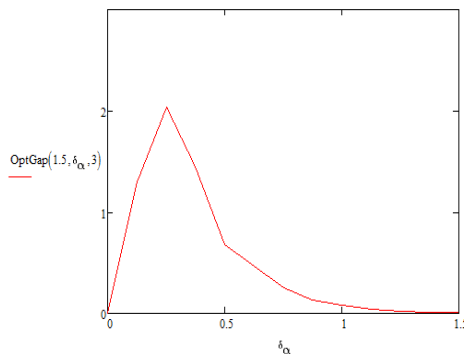
In conclusion, the obtained maximum optimality gaps further validate the equal profit margin heuristic. A retailer may be willing to accept such minimal decrease in profits to avoid computational complexity.
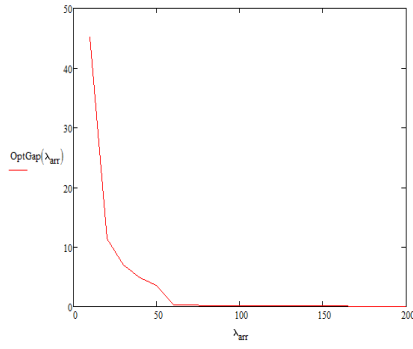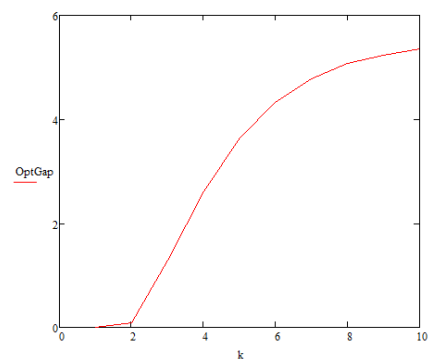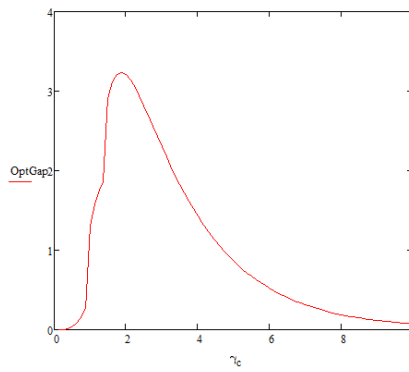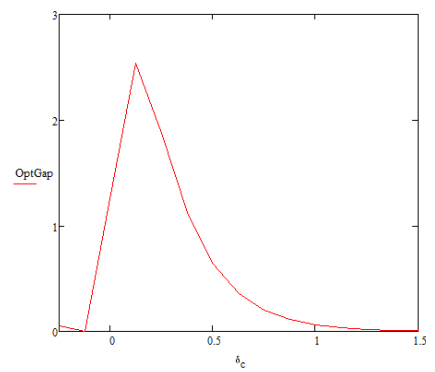
(a)
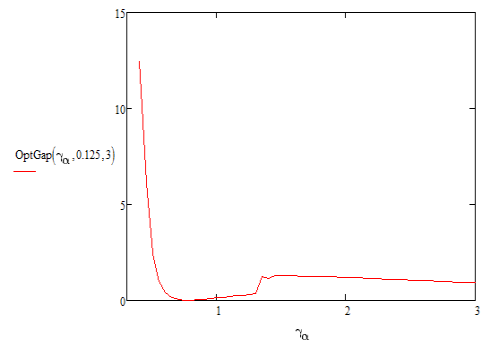
(b)

(c)

(d)

(e)

(f)

Figure 2. The variation of the Optimality Gap in % vs. studied parameters

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# CHAPTER VII

# STRUCTURE OF THE OPTIMAL ASSORTMENT

In this section, we study the problem of determining the products that constitute the optimal assortment under the MNL choice model.

To study the structure of the optimal assortment we introduce two new functions that serve as upper and lower bounds to the expected profit function.

Replacing $\dfrac{c_i}{m+c_i}$ by 1, define $\underline{\Pi}(m,S) = \sum_{i=1}^{k} m\lambda q_i(m,S) - a_k m\sqrt{\lambda q_i(m,S)}$ as a lower

bound and then, replacing $m$ by $\overline{m_k}$ in the same fraction yields an upper bound on

$\Pi(m,S)$ defined by $\overline{\Pi}(m,S) = \sum_{i=1}^{n} m\lambda q_i(m,S) - a_k m \dfrac{c_i}{\overline{m}+c_i}\sqrt{\lambda q_i(m,S)}$. We study the

nature of these bounds by varying the arrival rate, the assortment size, the mean utility prices and the unit costs in a similar process to that utilized in the preceding numerical study. Figure 3 ((e) to (f)) shows the obtained results. It can be inferred that the proposed functions provide a good upper and lower bound on the expected profit function.

Next, we show that an optimal assortment has a simple popular set structure under the tight bonds defined here, $\underline{\Pi}(m,S)$ and $\overline{\Pi}(m,S)$. That is, if one utilizes $\underline{\Pi}(m,S)$ and $\overline{\Pi}(m,S)$ as approximate profit functions, then an optimal assortment is a popular set.

**Lemma 4.** *Assume the profit margin is fixed at some profit margin m, and define* $\alpha_i'$ *as* $\alpha_i - c_i$ . *Then, the upper and lower bounds on the optimal profit functions,* $\underline{\Pi}(m, S, \alpha_i')$ *and* $\overline{\Pi}(m, S, \alpha_i')$ *are both pseudoconvex in* $\alpha_i'$ .

**Proof: See Appendix B**

As it is standard in literature (e.g. Van Ryzin and Mahajan (1992) and Maddah and Bish (2007)), the pseudoconvexity results in Lemma 4 imply a popular set structure of the optimal assortment under the lower and upper bound functions.

**Theorem 2.** Assume without loss of generality that $\alpha'_1 \geq \alpha'_2 \geq ....\alpha'_n$ . Under both the lower and upper bounds $\underline{\Pi}(m, S)$ and $\overline{\Pi}(m, S)$ , an optimal assortment is the popular set $S_k = \{1, 2,...n\}$ for some $k \leq n$ .

Given the good quality of the bounds $\underline{\Pi}(m, S)$ and $\overline{\Pi}(m, S)$ established in Figure 3, Theorem 2 suggests that restricting the search for the optimal assortment to popular sets, having the products with highest average margin $\alpha_i' = \alpha_i - c_i$ , will produce a near-optimal assortment.
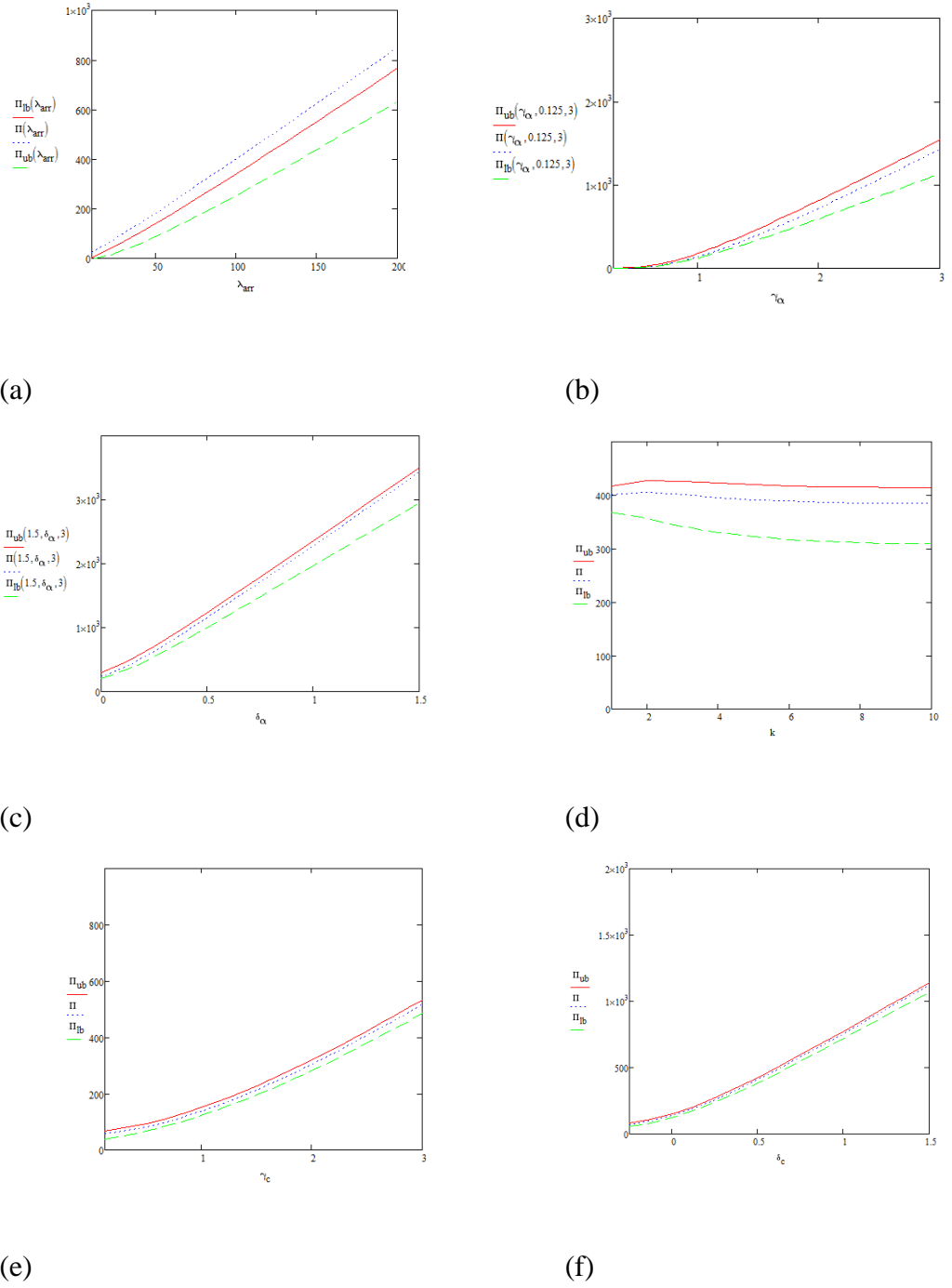
(a)

(b)

(c)

(d)

(e)

(f)

Figure 3. Assessing the quality of the upper ($\pi_{ub}$) and lower ($\pi_{lb}$) bounds as a function of $\lambda$, $\gamma$, $\delta$, k, $\gamma_c$ and $\delta_c$).

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# CHAPTER VIII

# THE NESTED LOGIT MODEL

To overcome the IIA limitation of the MNL, we utilize the nested multinomial choice model (NMNL) first introduced by Ben-akiva (1973). The NMNL assumes that the category $\Omega$ is partitioned into *n* nests (subsets); $\Omega = \bigcup_{i=1}^{n} N_i$.

The NMNL is characterized by two constants, $\mu_1 \geq 0$ that measures the degree of dissimilarity between nests, and $\mu_2 \geq 0$ that measures the dissimilarity among products in the same nest. Products across different nests exhibit more differences than products included in the same nest, which leads to $\mu_1 \geq \mu_2$. The NMNL assumes that customers first choose a certain subset or nest, and then choose an item within a nest. Following Kalakesh (2006), we assume that customers can only choose not to purchase in the first stage. Below is an illustration of the decision process adopted from Kalakesh (2006). In Figure 4, $q_i$ is defined as the probability of selecting nest *i*, $q_{li}$ as the probability of selecting product *l* from nest *i*, $k_i$ as the number of products offered in nest *i*, and *m* as the number of available nests. Kalakesh (2006) argues that a no purchase option within a nest is realistic since a high no purchase utility may make a nest highly attractive which is illogical.

In this section we aim at validating previous findings on the more general NMNL choice model. First we introduce our model; then, we show that the unimodality result still holds. We provide reviewed functions that allow the comparison of risky profit margins to those in a make-to-order setting. Finally, we perform a numerical

analysis to test the applicability of the equal profit margin heuristic under the NMNL choice model.
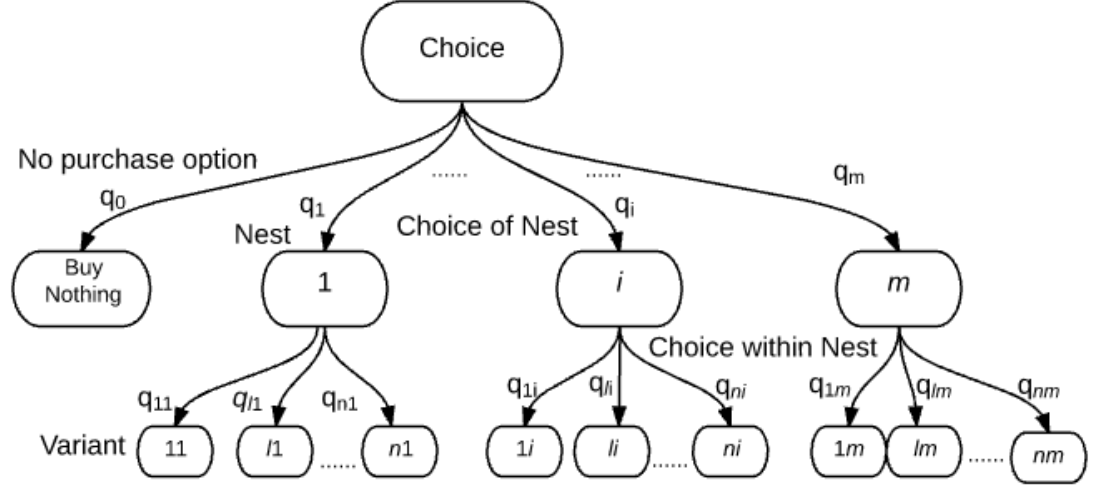


Figure 4. Two level decision process under NMNL (adopted from Kalakesh (2006))

## A. Model and Assumptions under NMNL choice model

The consumer's utility for product $i$ in nest $k$ is given by $U_{li} = \alpha_{li} - p_{li} + \varepsilon_{li}$;

and the utility of the no-purchase option is $U_0 = \varepsilon_0$, where $\varepsilon_i$ are independent and

identically distributed (i.i.d.) Gumbel random variables with mean zero and shape factor

$\mu_2$. The probability of purchasing a product $l$ given that a nest $N_i$ has been already

selected is $q_{l/i}(S_i, m) = \Pr\{U_{li} = \max_{l \in N_i} U_{li}\}$ which can be written as

$$q_{l/i}(S_i, m) = \frac{e^{\frac{\alpha_{li} - c_{li} - m}{\mu_2}}}{\sum_{l \in S_i} e^{\frac{\alpha_{li} - c_{li} - m}{\mu_2}}}$$

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where $S_i \subseteq N_i$ is the set of products offered in Nest $i$. The utility of each nest and the no purchase option is based on the attractiveness of nests, defined as

$$A_i = \mu_2 \ln \sum_{k \in S_i} e^{\frac{\alpha_{ki} - c_{ki} - m}{\mu_2}}$$

, for Nest $i$. Thus the utility of nest $i$ is given by $U_i = A_i + \varepsilon_i$,

and the utility of the no purchase option is $U_0 = \varepsilon_0$ where $\varepsilon_i$ are independent and identically distributed (i.i.d.) Gumbel random variables with mean zero and shape factor $\mu_1$. Hence, the probability of choosing $N_i$ is $q_i(m, N) = \Pr\left\{U_i = \max_{j \in \{1,2,..n\} \cup \{0\}} U_j\right\}$, which simplifies to

$$q_i(m, S) = \frac{\left(\sum_{l \in S_i} e^{\frac{\alpha_{li} - c_{li} - m}{\mu_2}}\right)^{\frac{\mu_2}{\mu_1}}}{\upsilon_0 + \sum_{j=1}^{n}\left(\sum_{l \in S_j} e^{\frac{\alpha_{lj} - c_{lj} - m}{\mu_2}}\right)^{\frac{\mu_2}{\mu_1}}}$$

Where $S = (S_1, S_2, ..., S_n)$ and $\upsilon_0 = e^{u_0 / \mu_1}$. Finally, the probability of purchasing a product $l$ in Nest $i$ can be written as $q_{li}(m, S) = q_{l/i}(m, N) \times q_i(m, N)$.

We assume that a consumer will take her decision of purchasing based only on price, variety and quality (assortment-based substitution). Then, the optimal inventory level, $y_{li}^*(m, S)$, for each product in the assortment, and the total profit, $\Pi(m, S)$, from the entire assortment at optimal inventory levels can be written as

$$y_{li}^*(m, S) = \lambda q_{li}(m, S) + \phi^{-1}(1 - \frac{c_{li}}{m + c_{li}})\sqrt{\lambda q_{li}(m, S)}$$

(9)

$$\Pi(m, S) = \sum_{i=1}^{n} \sum_{l \in S_i} m\lambda q_{li}(m, S) - (m + c_{li})\varphi(\phi^{-1}(1 - \frac{c_{li}}{m + c_{li}}))\sqrt{\lambda q_{li}(m, S)}$$

(10)

Similar to the MNL case, the profit margin and assortment size that maximize the expected profit function can be determined numerically. Obtained optimal values can be then replaced in Equation (10) to determine optimal inventory levels. Finally, letting $K_i = |S_i|$ and $K = (K_1, K_2, ..., K_n)$, and rearranging Equation (10) the expected profit can be written as

$$\Pi(m, K) = m\lambda p(m, K)\rho_K - \sqrt{\lambda p(m, K)} \sum_{i=1}^{n} \sum_{l=1}^{K_i} (m + c_{li})\varphi(\phi^{-1}(1 - \frac{c_{li}}{m + c_{li}}))\varsigma_{li}$$

(11)

Where $p(m, K) = \dfrac{e^{-m}}{1 + e^{-m}\rho_K}$ , $\rho_K = \sum_{i=1}^{n} \left( \sum_{l=1}^{K_i} e^{\frac{\alpha_{li} - c_{li}}{\mu_2}} \right)^{\frac{\mu_2}{\mu_1}}$ , and

$$\varsigma_{li} = \sqrt{e^{\frac{\alpha_{li} - c_{li}}{\mu_2}} \left( \sum_{l=1}^{K_i} e^{\frac{\alpha_{li} - c_{li}}{\mu_2}} \right)^{\frac{\mu_2}{\mu_1} - 1}}$$

Similar to the case in NML, we also take the following assumption to ensure that the retailer will not make higher profits when selling nothing.

**Assumption 1A.** The expected profit $\hat{\Pi}(m, K)$ is increasing in $m$ at $m = 0$; that is

$$\left. \frac{\partial \hat{\Pi}(m, K)}{\partial m} \right|_{m=0} > 0$$

rearranging,

$$\lambda \geq \frac{a(\upsilon_0 + \rho_K) \left( \sum_{i=1}^{n} \sum_{l=1}^{K_i} \sqrt{e^{\frac{\alpha_{li} - c_{li}}{\mu_2}} \left( \sum_{l=1}^{K_i} e^{\frac{\alpha_{li} - c_{li}}{\mu_2}} \right)^{\frac{\mu_2}{\mu_1} - 1}} \right)^2}{\rho_K}$$

.

To simplify the profit function, we also apply the approximation presented by Maddah et al. (2014) on equation (11) which reduces to the following:

$$\Pi(m, \mathbf{K}) = m\lambda p(m, \mathbf{K})\rho_{\mathbf{K}} - am\sqrt{\lambda p(m, \mathbf{K})}.\theta'(m, \mathbf{K}) \tag{12}$$

Where
$$\theta'(m, \mathbf{K}) = \sum_{i=1}^{n}\sum_{l=1}^{K_i} \frac{c_{li}}{m + c_{li}} \sqrt{e^{\frac{\alpha_{li}-c_{li}}{\mu_2}}\left(\sum_{l=1}^{K_i} e^{\frac{\alpha_{li}-c_{li}}{\mu_2}}\right)^{\frac{\mu_2}{\mu_1}-1}}.$$

## B. Structure of the Profit Function

In this subsection, we show that previous results on the structure of the profit function continue to hold under the NMNL choice model. We propose logical bounds for the optimal profit margin in Lemma 1A in the interval where the total profit is positive. Similarly, knowing that $\lim_{m\to\infty}\hat{\Pi}(m, \mathbf{K}) \to 0^-$, it can be shown that $m_{\mathbf{K}}^*$ is an internal point solution satisfying the first- and second-order optimality conditions under Assumption 1A. Thus we can write the following lemma.

**Lemma 1A.** *The expected profit* $\hat{\Pi}(m, \mathbf{K}) > 0$ *if and only if* $m \in (0, \bar{m}_{\mathbf{K}})$ *where* $\bar{m}_{\mathbf{K}} > 0$ *is the unique solution to the equation* $\lambda(\rho_{\mathbf{K}})^2 p(m, \mathbf{K}) - a^2\theta'(m, \mathbf{K})^2 = 0$. *Furthermore,* $\hat{m}_{\mathbf{K}}^* < \bar{m}_{\mathbf{K}}$ *and* $\bar{m}_{\mathbf{K}}$ *is decreasing in k.*

**Proof: See Appendix A**

The main result of this paper continues to be valid as we prove that the profit function under NMNL choice model is also unimodal in *m* established in Theorem 1A.

**Theorem 1A.** *The expected profit* $\hat{\Pi}(m, \boldsymbol{K})$ *is unimodal in m on* $(0, \bar{m}_K)$.

**Proof: See Appendix A**

### C. Effect of Inventory Considerations on Pricing under NMNL choice model.

In this section, we study the properties of the optimal profit margin $m_K^*$, and study how it compares to the riskless profit margin. When no inventory costs are considered, the expected profit function simplifies to

$$\Pi^0(m, \boldsymbol{K}) = m \rho_K \lambda p(m, \boldsymbol{K}).$$

(13)

To obtain a close form expression for the riskless profit margin, we apply in corollary 1A a result by Li and Huh (2011).

Corollary 1A. The expected riskless profit $\Pi^0(m, \boldsymbol{K})$ is unimodal in m, with

$$m_K^0 = 1 + W(\sum_{i=1}^{n} \sum_{l=1}^{K_i} e^{\alpha_{li} - c_{li} - 1}).$$

(14)

**Proof: See Appendix B**

As discussed earlier, accounting for inventory costs will adjust the optimal riskless profit margin $m_K^0$ to a risky profit margin, $m_K^*$. Thus, we compare $m_K^*$ and $m_K^0$ by studying the cost function as we did previously in section4 for the MNL case. From equation (12), the inventory cost is given by $\hat{C}_1(m, \boldsymbol{K}) = am\sqrt{\lambda p(m, \boldsymbol{K})} \theta'(m, \boldsymbol{K})$. We prove $\hat{C}_1(m, \boldsymbol{K})$ to be unimodal in m, and decreasing at $m = m_K^0$ under NMNL as well.

**Lemma2A.** *The approximate inventory cost* $\hat{C}_1(m, \boldsymbol{K})$ *is unimodal in m. Moreover,*

$$\left.\frac{\partial \hat{C}_1(m, \boldsymbol{K})}{\partial m}\right|_{m=m_K^0} = a\sqrt{\lambda p(m_K^0, \boldsymbol{K})}\left[\frac{\theta'(m_K^0, \boldsymbol{K})}{2} + \left.\frac{\partial \theta'(m_K^0, \boldsymbol{K})}{\partial m}\right|_{m=m_K^0}\right]. \tag{15}$$

**Proof: See Appendix B**

Given that the inventory cost function is unimodal in *m*, we can easily compare

the risky, $m_K^*$ and riskless profit margin , $m_K^0$ by adjusting $m_K^*$ above or below $m_K^0$ to

obtain a lower inventory cost. $\hat{C}_1(m, \boldsymbol{K})$ is decreasing in $m_K^0$ when

$$\left.\left|\frac{\partial \theta'(m, \boldsymbol{K})}{\partial m}\right|\right|_{m=m_K^0} > \frac{\theta'(m_K^0, \boldsymbol{K})}{2} \quad \text{as} \quad \left.\frac{\partial \theta'(m, \boldsymbol{K})}{\partial m}\right|_{m=m_K^0} \text{ is negative. Thus, the risky profit margin}$$

needs to fall above $m_K^0$. When $m_K^0 < 2 - \bar{c}$, we are certain that

$$\left.\left|\frac{\partial \theta'(m, \boldsymbol{K})}{\partial m}\right|\right|_{m=m_K^0} > \frac{\theta'(m_K^0, \boldsymbol{K})}{2} \text{ , hence, } m_K^* > m_K^0. \text{ Similarly, when}$$

$$\left.\left|\frac{\partial \theta'(m, \boldsymbol{K})}{\partial m}\right|\right|_{m=m_K^0} < \frac{\theta'(m_K^0, \boldsymbol{K})}{2} \text{ , i.e. when } m_K^0 > 2 - \underline{c}, \ \hat{C}_1(m, \boldsymbol{K}) \text{ is increasing in } m_K^0.$$

Thus, $m_K^*$ would need to fall below $m_K^0$. Using the closed form expression of the riskless

profit margin derived in (12), we can write Lemma 3A which enables us to perform the

comparison without optimizing on the riskless profit function.

**Lemma 3A.** *If* $\rho_K < (1-\bar{c}).e^{2-\bar{c}}$ *, then* $m_K^* > m_K^0$ *. Equivalently, if* $\rho_K > (1-\underline{c}).e^{2-\underline{c}}$ *, then*

$m_K^* < m_K^0$ *.*

**Proof: See appendix B**

### D. Numerical Study under the NMNL choice model

In order to study the magnitude of the loss in profit when assuming an equal profit margin, we will perform the following numerical study. First, we assume that a customer entering the store will have to choose among three nests or leave empty handed. Once the nest has been chosen, the customer will have to choose one of the two products in the nest. Recall that a customer who chose a nest in the preliminary step cannot leave empty handed. As a base case, assume that people arrive according to a Poisson process with rate $\lambda = 100$, and that the degree of dissimilarity among the nests is $\mu_1 = 2$ and within a nest is $\mu_2 = 1.2$. Other parameters are set to the following:

$\alpha_{11} = 12$, $\alpha_{21} = 11$, $\alpha_{12} = 13$, $\alpha_{22} = 14$, $\alpha_{13} = 13$, $\alpha_{23} = 16$ $c_{11} = 4$, $c_{21} = 5$, $c_{12} = 8$,

$c_{22} = 9$, $c_{13} = 8$, and $c_{23} = 7$. In the following subsections we study the validity of the equal profit margin heuristic by calculating the percentage optimality gap in a manner similar to that of the MNL. In each subsection, we keep all variables constant except for: (a) the arrival rate; (b) the consumer's utility; (c) item costs and consumer utilities; (d) degree of dissimilarity.

### 1. *Effect of arrival on heuristic.*

We will numerically test the equal profit margin heuristic as arrival varies. For this, we increase the arrival from the minimum case according to Assumption 1

$$\lambda \geq \frac{a(\upsilon_0 + \beta)\left(\sum_{i=1}^{I}\sum_{k=1}^{K}\sqrt{e^{\frac{\alpha_{ki}-c_{ki}}{\mu_2}}\left(\sum_{k \in S_i} e^{\frac{\alpha_{ki}-c_{ki}}{\mu_2}}\right)^{\frac{\mu_2}{\mu_1}-1}}\right)^2}{\beta}$$ while calculating the Optimality Gap at

each $\lambda$.

The optimality gap is low at reasonably large arrival rates, thus validating our heuristic.

## 2. *Optimality with varying mean reservation prices* $\alpha_{li}$.

In this section, we will study how the attractiveness the optimality gap varies as the mean utilities for products in Nest 1 vary. The arrival rate, degrees of dissimilarity and costs will remain constant. We assume that the mean reservation price of product $k$ in Nest 1 follows the following equation $\alpha_{k1} = [\gamma_{\alpha1} + (K-i+1)\delta_{\alpha1}]c_{k1}$, where K is the assortment size (K=2). At $\gamma_\alpha = 1$ and $\delta_\alpha = 0$, we obtain the base case. $\gamma_\alpha$ and $\delta_\alpha$ take values that conform with conditions of Assumption 1A. In graph 2 of figure 5(b), $\delta_\alpha$ is constant at its base value and $\gamma_\alpha$ varies. The optimality gap reaches a peak of 3% and declines to zero as $\gamma_\alpha$ increases. Figure 5 (c) shows the loss in profit when $\gamma_\alpha$ is set constant and $\delta_\alpha$ varies. The findings further confirm our heuristic as at the worst case conditions, the plot shows a loss of about 3.5% and the optimality tends to 0. In conclusion, varying mean reservation prices validates the equal margin approach.

**3.    *Effect of simultaneous variation in product costs and mean reservation prices.***

Now, we study the effect of changing the unit cost of items in Nest 1. The utilities of each items will vary accordingly similar to part (b) but with a constant $\gamma_\alpha = 1$ and $\delta_\alpha = 0$. The cost of product $i$ in Nest 1 will vary in function of variables $\gamma_c$, and $\delta_c$ as follows $c'_i = [\gamma_c + (3-i)\delta_c]c_{i1}$. When $\gamma_c = 1$ and $\delta_c = 0$, the parameters will be equal to the initial base case.

Figures 5 (d) and (e) show the variation of percentage optimality gap in function of the studied parameters. As $\gamma_c$ varies, the percentage optimality gap fluctuates with a maximum less than 3.5%. As $\delta_c$ varies, the optimality gap increases again to maximum less than 3.5 % and decreases thereafter. The obtained results prove that when the costs of items in a nest vary, assuming equal profit margins results in minimal loss in profits.

**4.    *Effect of degrees of dissimilarity on heuristic.***

In this section, we begin by varying the degree of dissimilarity among nests and then we set this parameter constant and change the degree of dissimilarity within each nest. In both cases, the percentage optimality gap fluctuates between maximums and minimums (Figure 5(f) and (g)). Examining the global maximum in each variation, we can conclude that the loss in profit is no more than 4% in case 1, and no more than 3% in the second case.

In conclusion, the above numerical study further validates the equal profit margin heuristic as maximum percentage optimality gaps are acceptable. A retailer may

save on computational efforts which may exceed the loss in profit by assuming an equal profit margin over similar products.
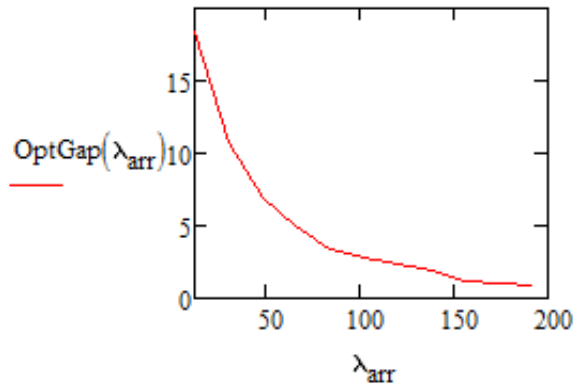


(a)
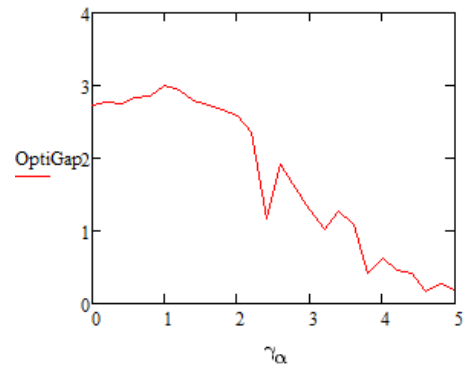
(b)

(c)

(d)

(e)                                              (f)



(g)

Figure 5. The variation of the Optimality Gap in % vs. studied parameters
(NMNL)

# CHAPTER IX

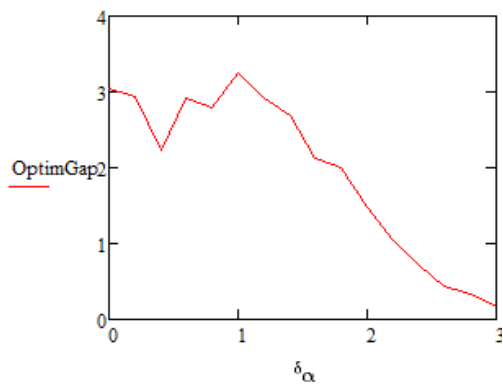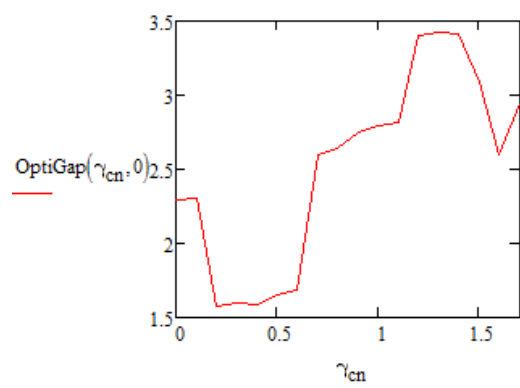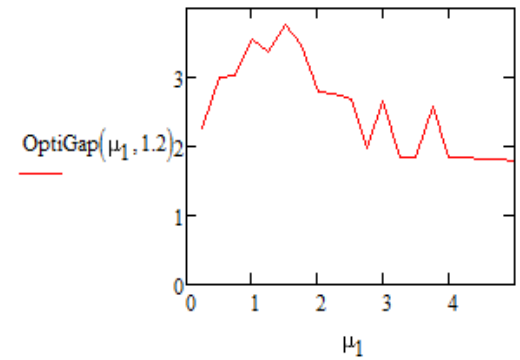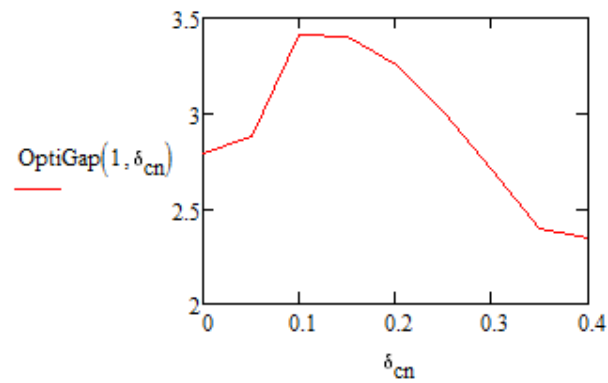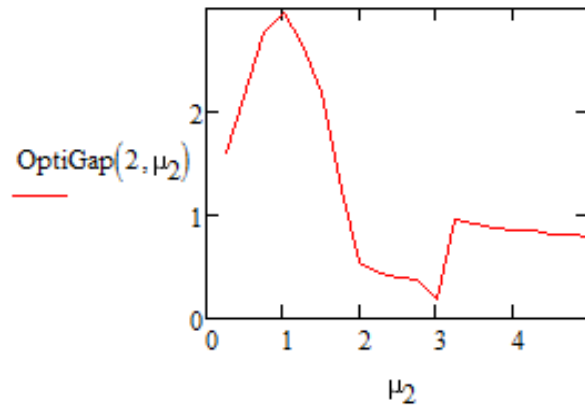# CONCLUSION AND FURTHER RESEARCH DIRECTIONS

This paper is an integration of marketing and operations management as it studies pricing and assortment planning while accounting for inventory costs. We consider the interdependent decisions on inventory, pricing and assortment planning of horizontally differentiated products (color, flavor, etc.) under a logit consumer choice model and the more general nested logit model within a newsvendor-type inventory setting. Demand assumes a *multiplicative-additive* function where both variance and coefficient of variation depend on the profit margin *m*. Under an accurate Taylor Series approximation, the profit function is proved unimodal in *m*. We further provide basis that allow the comparison between the optimal profit margin and the "riskless" profit margin, where no inventory exists. Finally, we study the optimal assortment and inventory decisions problem under exogenous prices. Under tight lower and upper bound approximations of the expected profit, we show that an optimal assortment is a popular set, having products with the highest $\alpha_i - c_i$, which suggests that restricting the search for the optimal assortment to popular sets will yield a "good" assortment with near-optimal results.

This research can be seen as an extension of the work by Maddah et al. (2014) through the validation of findings in a single variable problem, the profit margin, and on the nested logit consumer choice model.

Anderson and De Palma (1992), and Aydin and Ryan (2000) show that under the MNL demand model, equal profit margins are optimal. A limiting assumption both

research take is a "make to order" setting with no inventory considerations. The numerical analysis we presents in this paper shows that profits at optimal profit margins while considering inventory costs are near optimal. Thus, an important addition to this paper would be to derive an analytical upper bound on the loss of profit when assuming equal profit margins. The same applies to popular sets which this paper found to be also near-optimal. Deriving an analytical bound on the loss of profit from utilizing popular sets is also an important direction for future research.

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# APPENDIX

## A. Appendix A

### 1. Proof of Theorem 1

a. <u>MNL choice model</u>

Define $h(m) = \dfrac{\theta(m)}{\sqrt{g(m)}}$. We begin by proving that $h(m,k)$ is pseudonvex in $m$

for $m \geq 0$. We begin by rewriting $h(m)$ as follows: $h(m) = \dfrac{l(m)}{k(m)}$ where $l(m) = \sqrt{e^m + \beta_k}$

and $k(m) = \dfrac{1}{\theta(m)}$. For $m \in (0,\infty)$ Note that $\dfrac{\partial l(m)}{m} = \dfrac{e^m}{2\sqrt{e^m + \beta_k}} > 0$ increasing in m, and

$\dfrac{\partial^2 l(m)}{m^2} = \dfrac{e^m(e^m + 2\beta_k)}{2(e^m + \beta_k)} > 0$ which proves the convexity of $l(m)$. Now observe that

$\dfrac{\partial k(m)}{m} > 0$ and decreasing in m and

$\dfrac{\partial^2 k(m)}{m^2} = \dfrac{-2\sum\limits_{i=1}^{k}\dfrac{c_i}{(m+c_i)^3}\zeta_i\left(\sum\limits_{i=1}^{k}\dfrac{c_i}{m+c_i}\zeta_i\right) + 2\left(\sum\limits_{i=1}^{k}\dfrac{c_i}{(m+c_i)^2}\zeta_i\right)^2}{\left(\sum\limits_{i=1}^{k}\dfrac{c_i}{m+c_i}\zeta_i\right)^3} < 0$, proving that $k(m)$

is concave. To prove $\dfrac{\partial^2 k(m)}{m^2}$ negative, we start by assuming that

$\left(\sum\limits_{i=1}^{k}\dfrac{c_i}{(m+c_i)^3}\zeta_i\right)\left(\sum\limits_{i=1}^{k}\dfrac{c_i}{m+c_i}\zeta_i\right) \geq \left(\sum\limits_{i=1}^{k}\dfrac{c_i}{(m+c_i)^2}\zeta_i\right)^2$. Simplifying the above, we obtain

$\sum\limits_{j=1}^{k}\sum\limits_{i=1}^{k}\dfrac{c_ic_j\zeta_i\zeta_j}{(c_i+m)^3(c_j+m)} \geq \sum\limits_{j=1}^{k}\sum\limits_{i=1}^{k}\dfrac{c_ic_j\zeta_i\zeta_j}{(c_i+m)^2(c_j+m)^2}$. This implies that $(c_i - c_j)^2 \geq 0$,

which is true at all cost values. Since $h(m)$ can be written as a fraction of a positive

convex function over a positive concave function, $h(m)$ is then pseudoconvex in m. This follows from a result in Avriel (2003, p.156).

Setting $\Pi(m,k) = 0$ implies that $h(m) = \dfrac{\beta_k \sqrt{\lambda}}{a}$. Since $h(m,k)$ is pseudoconvex

in $m$ for $m \in (0, \infty)$, the function $\gamma_1(m,k) = \beta\sqrt{\lambda g(m,k)} - a\theta(m)$ has at two solutions in

$m \in (0, \infty)$. Observe that $\Pi(0,k) = 0$ and Assumption 1 imply that $\hat{\Pi}(0^+, k) > 0$. So

under Assumption 1, $\hat{\Pi}(m,k) = m\sqrt{\lambda g(m,k)}\gamma_1(m,k)$ equals zero at only one point $\overline{m_k}$

with $\overline{m_k} \in (0, \infty)$. This means that $\hat{\Pi}(m,k)$ is positive for $m \in (0, \overline{m_k})$ and is negative

otherwise. This shows that $\overline{m_k}$ constitutes an upper bound on $\hat{m}_k^*$.

b. <u>NMNL choice model</u>

Define $h'(m,k) = \dfrac{\theta'(m,k)}{\sqrt{p(m,k)}}$. We begin by proving that $h'(m,k)$ is pseudonvex in $m$

for $m \geq 0$ similar to the above proof. Note that $h'(m)$ can be written as follows:

$h'(m,k) = \dfrac{l'(m,k)}{k'(m,k)}$ where $l'(m) = \sqrt{e^m + \rho_k}$ and $k'(m) = \dfrac{1}{\theta'(m)}$. For $m \in (0, \infty)$ Note that

$\dfrac{\partial l'(m)}{m} = \dfrac{e^m}{2\sqrt{e^m + \rho_k}} > 0$ increasing in m, and $\dfrac{\partial^2 l'(m)}{m^2} = \dfrac{e^m(e^m + 2\rho_k)}{2(e^m + \rho_k)} > 0$ which proves the

convexity of $l'(m)$. Now observe that $\dfrac{\partial k'(m)}{m} > 0$ and decreasing in m and

$$\dfrac{\partial^2 k'(m)}{m^2} = \dfrac{-2\sum_{i=1}^{I}\sum_{k=1}^{K}\dfrac{c_{ki}}{(m+c_{ki})^3}\zeta_{ki}\sum_{i=1}^{I}\sum_{k=1}^{K}\dfrac{c_{ki}}{m+c_{ki}}\zeta_{ki} + 2\left(\sum_{i=1}^{I}\sum_{k=1}^{K}\dfrac{c_{ki}}{(m+c_{ki})^2}\zeta_{ki}\right)^2}{\left(\sum_{i=1}^{I}\sum_{k=1}^{K}\dfrac{c_{ki}}{(m+c_{ki})^3}\varsigma_{ki}\right)^3} < 0, \text{ proving}$$

that $k'(m)$ is concave. To prove $\dfrac{\partial^2 k'(m)}{m^2}$ negative, we start by assuming that

$$\sum_{i=1}^{I}\sum_{k=1}^{K}\frac{c_{ki}}{(m+c_{ki})^3}\zeta_{ki}\sum_{i=1}^{I}\sum_{k=1}^{K}\frac{c_{ki}}{m+c_{ki}}\zeta_{ki}\geq\left(\sum_{i=1}^{I}\sum_{k=1}^{K}\frac{c_{ki}}{(m+c_{ki})^2}\zeta_{ki}\right)^2$$ . Simplifying the above, we

obtain $(c_{ki}-c_{jl})^2\geq0$ , which is true at all cost values. Since $h(m)$ can be written as a fraction of

a positive convex function over a positive concave function, $h(m)$ is then pseudoconvex in m.

This follows from a result in Avriel (2003, p.156).

Setting $\Pi(m,k)=0$ implies that $h(m)=\dfrac{\rho_k\sqrt{\lambda}}{a}$ . Since $h'(m,k)$ is pseudoconvex in $m$

for $m\in(0,\infty)$ , the function $\gamma'_1(m,k)=\rho_k\sqrt{\lambda p(m,k)}-a\theta'(m)$ has at two solutions in

$m\in(0,\infty)$ . Observe that $\Pi(0,k)=0$ and Assumption 1 imply that $\hat{\Pi}(0^+,k)>0$ . So under

Assumption 1, $\hat{\Pi}(m,k)=m\sqrt{\lambda p(m,k)}\gamma'_1(m,k)$ equals zero at only one point $\overline{m_k}$ with

$\overline{m_k}\in(0,\infty)$ . This means that $\hat{\Pi}(m,k)$ is positive for $m\in(0,\overline{m_k})$ and is negative otherwise.

This shows that $\overline{m_k}$ constitutes an upper bound on $\hat{m}_k^*$ .

## 2.     *Proof of Theorem 1.*

a.     <u>MNL choice Model</u>

Let $A=\lambda\beta_k$ , $B=a_k\sqrt{\lambda}$ . Rearranging Equation (6), we get

$$\hat{\Pi}(m,k)=\frac{A-B\sqrt{e^m+\beta_k}\theta(m,k)}{\dfrac{e^m}{m}+\dfrac{\beta_k}{m}}\qquad\text{(A2)}$$

Based on a result from Avriel (2003, p. 154, Theorem 6.9), we prove that this

function is pseudoconcave in *m*. Since the studied function is a single-variable function

with an internal point maximum, pseudo concavity implies unimodularity. (Assumption 1 and Lemma 2 imply that an internal-point maximum exists.) Let

$$f(m) \equiv -A + \frac{B}{\dfrac{1}{\theta(m)}}\sqrt{e^m + \beta_k},$$

$$y(m) \equiv (e^m + \beta_k)^{\frac{1}{4}}, \text{ and}$$

$$k(m) \equiv \frac{1}{\theta(m,k)}.$$

$k(m)$ is concave in $m$, as proven in Lemma 1, and

$$\frac{\partial y(m)}{\partial m} = \frac{1}{4}(e^m + \beta_k)^{-\frac{3}{4}} e^m = (\frac{1}{4})\frac{1}{(e^{-\frac{m}{3}} + \beta_k e^{-\frac{4}{3}m})^{\frac{3}{4}}} \text{ Since } \frac{\partial y(m)}{\partial m} \text{ is increasing in } m \text{ and}$$

$\dfrac{\partial^2 y(m)}{\partial m^2} > 0$, $y(m)$ is convex in $m$. Given that $f(y(m), z(m)) = B\dfrac{y^2(m)}{z(m)} - A$. The

Hessian of $f(y, z)$ is written as $\quad H = \begin{vmatrix} \dfrac{2B}{z} & -\dfrac{2By}{z^2} \\[2mm] -\dfrac{2By}{z^2} & 2B\dfrac{y^2}{z^3} \end{vmatrix}$ which is positive semi-definite.

This proves that $f(y, z)$ is convex. Following Theorem 6.9 in Avriel (2003), $f(m)$ is

convex in $m$. Define $r(m) \equiv \dfrac{e^m}{m} + \dfrac{\beta_k}{m}$. Then, $\dfrac{\partial r(m)}{\partial m} = \dfrac{e^m}{m} - \dfrac{e^m}{m^2} - \dfrac{\beta_k}{m^2}$ and

$\dfrac{\partial y^2(m)}{\partial m^2} = \dfrac{e^m}{m^3}\left[(m-1)^2 + 1\right] + \dfrac{2\beta_k}{m^3} > 0$. Note that $-\hat{\Pi}(m,k) = \dfrac{f(m,k)}{r(m,k)}$ where $f(m,k)$ is

convex and negative over $(0, \overline{m_k})$ and $r(m,k)$ is positive and convex. According to a

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theorem discussed by Avriel (2003, p. 156), $-\hat{\Pi}(m,k)$ is pseudo convex for $m \in (0, \overline{m_k})$

or, equivalently, $\hat{\Pi}(m,k)$ is pseudo concave for $m \in (0, \overline{m_k})$.

b.    NMNL choice model

We prove the profit function to be unimodal in m utilizing the exact above

method. That is, we rewrite the opposite of the profit function as a fraction of a convex

negative function over a positive convex one. Following the aforementioned result in

Avirel, we show that the opposite of the profit function is pseudoconvex in $m$. This

proves the profit function to be pseudoconcave in $m$. The constants and functions

utilized in the above proof take the following forms under NMNL: $A = \lambda \rho_k$, $B = a\sqrt{\lambda}$,

$$\hat{\Pi}(m,k) = \frac{A - B\sqrt{e^m + \rho_k}\,\theta'(m,k)}{\dfrac{e^m}{m} + \dfrac{\rho_k}{m}} \ , \ f(m) \equiv -A' + \frac{B'}{\dfrac{1}{\theta'(m)}}\sqrt{e^m + \rho_k} \ , \ y(m) \equiv (e^m + \rho_k)^{\frac{1}{4}},$$

$$k(m) \equiv \frac{1}{\theta'(m,k)} \text{ and } r(m) \equiv \frac{e^m}{m} + \frac{\rho_k}{m}.$$

**B.  Appendix B**

*1.    Proof of Corollary 1*

The profit with no inventory cost reduces to the following

$$\Pi^0(m,k) = \lambda m \beta_k \frac{e^{-m}}{1 + \beta_k e^{-m}} = \frac{\lambda m \beta_k}{e^m + \beta_k}. \text{ Setting } \frac{\partial \Pi^0(m,k)}{\partial m} = 0 \text{, we obtain:}$$

$$e^{m_k^0}(1 - m_k^0) = -\beta_k$$
$$\Leftrightarrow e^{m_k^0 - 1}(m_k^0 - 1) = \beta_k e^{-1}$$
$$\Leftrightarrow m_k^0 = 1 + W(\sum_{j=1}^{k} e^{\alpha_j - c_j - 1})$$

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We utilize Theorem 2 and Corollary 1 from Li and Huh (2011) to prove the unimodularity of $\Pi^0(m,k)$ and the closed-form expression in Equation (7). The monotonicity results follow from Equation (8) by noting that $W(x)$ is positive and increasing in $x > 0$ with $\dfrac{\partial w(x)}{\partial x} = \dfrac{w(x)}{x(1+W(x))}$ (e.g., Corless *et al.* (1996)). Using the same method, the established result can be easily proved under NMNL.

## 2. *Proof of Lemma 2*

a.  <u>MNL choice model</u>

Finding we rewrite $\hat{C}_1(m,k)$ as follows,

$$\hat{C}_1(m,k) = -a\sqrt{\lambda}\,\frac{(-m\theta(m,k))}{\sqrt{e^m + \beta_k}} = -a\sqrt{\lambda}\sigma(m,k).$$ Notice that the denominator is $y^2(m)$

and can be easily shown as a convex positive function. Note as well that ,

$$\frac{\partial(-m\theta(m,k))}{\partial m} = -\sum_{i=1}^{k}\frac{c_i}{m+c_i}\zeta_i + m\sum_{i=1}^{k}\frac{c_i}{(m+c_i)^2}\zeta_i$$ increasing in $m$ and where $\zeta_i = \sqrt{e^{\alpha_i - c_i}}$

and $\dfrac{\partial^2(-m\theta(m,k))}{\partial m^2} = -m\sum_{i=1}^{k}\dfrac{c_i}{(m+c_i)^3}\zeta_i + 2\sum_{i=1}^{k}\dfrac{c_i}{(m+c_i)^2}\zeta_i > 0$. The positive result extends

from comparing the positive and negative terms of the second derivative which simplifies to the condition of $m+2c_i > 0$. For obvious reasons, this condition is always satisfied, hence the second derivative is always positive and the above function is convex in m. $\sigma(m,k)$ can hence be written as a negative convex function over a positive convex function. Following a result by Avriel (2003, p.156), $\sigma(m,k)$ is pseudoconvex in m. Hence, $\hat{C}_1(m,k) = -a\sqrt{\lambda}\sigma(m,k)$ is pseudoconcave in m, completing the proof that the cost function is unimodal in m.

We write the first derivative of the cost function as follows,

$$\frac{\partial \hat{C}_1(m,k)}{\partial m} = a\sqrt{\lambda g(m,k)}\left[\theta(m,k) + m\frac{\partial \theta(m,k)}{\partial m} + m\frac{\theta(m,k)\frac{\partial g(m,k)}{\partial m}}{2g(m,k)}\right] \quad \text{(A3).}$$

Now, we study the behavior of $\hat{C}_1(m,k)$ at $m_k^0$. We define $m_k^0$ from the first-

order condition of $\Pi^0(m,k)$. Setting $\left.\frac{\partial \Pi^0(m,k)}{\partial m}\right|_{m=m_k^0} = 0$ implies that

$$g(m,k) = -m_k^0\left.\frac{\partial g(m,k)}{\partial m}\right|_{m=m_k^0} \quad \text{(A4). Replacing this expression into Equation (A3) gives}$$

$$\left.\frac{\partial \hat{C}_1(m,k)}{\partial m}\right|_{m=m_k^0} = a\sqrt{\lambda g(m_k^0,k)}\left[\frac{\theta(m_k^0,k)}{2} + \left.\frac{\partial \theta(m,k)}{\partial m}\right|_{m=m_k^0}\right].$$

b.  _NMNL choice model_

We rewrite $\hat{C}_1(m,k)$ as follows, $\hat{C}_1(m,k) = -a\sqrt{\lambda}\frac{(-m\theta'(m,k))}{\sqrt{e^m + \rho_k}} = -a\sqrt{\lambda}\sigma(m,k)$.

Notice that the denominator is $y^2(m)$ and can be easily shown as a convex positive

function. Note as well that  , $\frac{\partial(-m\theta'(m,k))}{\partial m} = -\sum_{i=1}^{I}\sum_{k=1}^{k}\frac{c_{ki}}{m+c_{ki}}\zeta_{ki} + m\sum_{i=1}^{I}\sum_{k=1}^{k}\frac{c_{ki}}{(m+c_{ki})^2}\zeta_{ki}$

increasing in $m$ and  where $\zeta_i = \sqrt{e^{\alpha_i - c_i}}$ and

$\frac{\partial^2(-m\theta'(m,k))}{\partial m^2} = -m\sum_{i=1}^{I}\sum_{k=1}^{k}\frac{c_{ki}}{(m+c_{ki})^3}\zeta_{ki} + 2\sum_{i=1}^{I}\sum_{k=1}^{k}\frac{c_{ki}}{(m+c_{ki})^2}\zeta_{ki} > 0$. The positive result

extends from comparing the positive and negative terms of the second derivative which

simplifies to the condition of $m + 2c_{ki} > 0$. For obvious reasons, this condition is always

satisfied, hence the second derivative is always positive and the above function is

convex in m. $\sigma(m,k)$ can hence be written as a negative convex function over a positive

convex function. Following a result by Avriel (2003, p.156), $\sigma(m,k)$ is pseudoconvex in

m. Hence, $\hat{C}_1(m,k) = -a\sqrt{\lambda}\sigma(m,k)$ is pseudoconcave in m, completing the proof that

the cost function is unimodal in m.

We write the first derivative of the cost function as follows,

$$\frac{\partial \hat{C}_1(m,k)}{\partial m} = a\sqrt{\lambda p(m,k)}\left[\theta'(m,k) + m\frac{\partial \theta'(m,k)}{\partial m} + m\frac{\theta'(m,k)\dfrac{\partial p(m,k)}{\partial m}}{2p(m,k)}\right] \text{(A3').}$$

Now, we study the behavior of $\hat{C}_1(m,k)$ at $m_k^0$. We define $m_k^0$ from the first-

order condition of $\Pi^0(m,k)$. Setting $\left.\dfrac{\partial \Pi^0(m,k)}{\partial m}\right|_{m=m_k^0} = 0$ implies that

$g(m,k) = -m_k^0 \left.\dfrac{\partial p(m,k)}{\partial m}\right|_{m=m_k^0}$ (A4'). Replacing this expression into Equation (A3') gives

$$\left.\frac{\partial \hat{C}_1(m,k)}{\partial m}\right|_{m=m_k^0} = a\sqrt{\lambda p(m_k^0,k)}\left[\frac{\theta'(m_k^0,k)}{2} + \left.\frac{\partial \theta'(m,k)}{\partial m}\right|_{m=m_k^0}\right].$$

### 3. *Proof of Lemma 3*

a. <u>NML choice model</u>

Since $\hat{\Pi}(m,k) = \Pi^0(m,k) - \hat{C}_1(m,k)$, $\hat{\Pi}(m_k^0,k) \leq \hat{\Pi}(m_k^*,k)$, and

$\hat{\Pi}^0(m_k^0,k) \geq \hat{\Pi}^0(m_k^*,k)$, it can be concluded that $\hat{C}_1(m_k^*,k) \leq \hat{C}_1(m_k^0,k)$. Utilizing Lemma

2 we can establish the following finding. When $m_k^0 < 2 - c_m$, we are certain that $\hat{C}_1(m,k)$

is decreasing at $m_k^0$. This, along with the fact that $\hat{C}_1(m,k)$ is unimodal, shows that

$\hat{m}_k^* > m_k^0$. Note that $c_m$ is the highest cost in the assortment and $c_l$ is the least cost. The

opposite happens if $m_k^0 > 2 - c_l$. Finally, we show that $m_k^0 < 2 - c_m$ is equivalent to

$\beta_k < (1 - c_m)e^{2 - c_m}$. This follows from the closed form of $m_k^0$ in Equation (7) and the

monotonicity of the Lambert $W(.)$ function. Specifically, based on Equation (7),

$m_k^0 > 2 - c_l$ is equivalent to $W(\beta_k e^{-1}) > 2 - c_l - 1$, which, in turn, is equivalent to

$\beta_k e^{-1} > (1 - c_l)e^{1 - c_l}$, which, upon simplification, completes the proof.

b.    <u>NMNL choice model</u>

Since $\hat{\Pi}(m,k) = \Pi^0(m,k) - \hat{C}_1(m,k)$, $\hat{\Pi}(m_k^0,k) \le \hat{\Pi}(m_k^*,k)$, and

$\hat{\Pi}^0(m_k^0,k) \ge \hat{\Pi}^0(m_k^*,k)$, it can be concluded that $\hat{C}_1(m_k^*,k) \le \hat{C}_1(m_k^0,k)$. Utilizing Lemma

2 we can establish the following finding. Define $c_l$ as the cost of the least expensive

item and $c_m$ as the cost of the most expensive item. When $m_k^0 > 2 - c_l$, we are certain that

$\hat{C}_1(m,k)$ is increasing at $m_k^0$. This, along with the fact that $\hat{C}_1(m,k)$ is unimodal, shows

that $m_k^* < m_k^0$. The opposite happens if $m_k^0 < 2 - c_m$. Finally, we show that $m_k^0 < 2 - c_m$ is

equivalent to $\rho_k < (1 - c_m)e^{2 - c_m}$. This follows from the closed form of $m_k^0$ in Equation

(7A) and the monotonicity of the Lambert $W(.)$ function. Specifically, based on

Equation (7A), $m_k^0 > 2 - c_l$ is equivalent to $W(\rho_k e^{-1}) > 1 - c_l$, which, in turn, is equivalent

to $\rho_k e^{-1} > (1 - c_l)e^{1 - c_l}$, which, upon simplification, completes the proof.

## 4.    *Proof of Lemma 4*

We extend the proof by Maddah and Bish (2007) on our equal profit margin

case. We set $v_i = e^{\alpha_i - c_i - m} = e^{\alpha_i' - m}$, hence the lower bound function can be written as

$$\underline{\Pi}(m,S,v_i) = \sum_{i=1}^{n} m\lambda \frac{v_i}{1+\sum_j v_j} - a_k m \sqrt{\lambda \frac{v_i}{1+\sum_j v_j}}$$ and the upper bound function can be

written as $$\overline{\Pi}(m,S,v_i) = \sum_{i=1}^{n} m\lambda \frac{v_i}{1+\sum_j v_j} - a_k m \frac{c_i}{m_1+c_i} \sqrt{\lambda \frac{v_i}{1+\sum_j v_j}}$$ . In order to prove

Lemma 5, we first prove $\underline{\pi}(S,m,v_i)$ to be pseudoconvex in $v_i$. For this purpose, we write

$\underline{\Pi}(m,S,v_i) = \dfrac{\gamma(v_i)}{\delta(v_i)}$ . If $\gamma(v_i)$ is convex and $\delta(v_i)$ is linear, then $\underline{\Pi}(S,m,v_i)$ is

pseudoconvex in $v_i$. Let $\underline{\Pi}(m,S,v_i) = (K_1 + K_2 v_i) - (K_3 + K_4 v_i^{\frac{1}{2}})(K_5 + v_i)^{\frac{1}{2}}$ where

$K_1 = \displaystyle\sum_{\substack{i \neq j \\ j \in s}} m\lambda v_j$ , $K_2 = m\lambda$ , $K_3 = \displaystyle\sum_{\substack{i \neq j \\ j \in s}} a_k \sqrt{\lambda v_j}$ , $K_4 = \sqrt{\lambda}$ , and $K_5 = \displaystyle\sum_{\substack{i \neq j \\ j \in s}} v_j + 1$ . The second

derivative of $\gamma(v_i)$ with respect to $v_i$ simplifies to

$\dfrac{\partial^2 \gamma(v_i)}{\partial v_i^2} = \dfrac{1}{4}(K_5 + v_i)^{-\frac{3}{2}}(K_3 + K_4 K_5^2 v_i^{-\frac{3}{2}}) > 0$ . The positive result proves the convexity of

$\gamma(v_i)$ . We define $\delta(v_i) = \displaystyle\sum_{\substack{j \neq i \\ j \in S}} 1 + v_i + v_j$ which is linear in $v_i$ . This proves $\underline{\Pi}(m,S,v_i)$ to be

pseudoconvex in $v_i$ . Taking $K_3 = \displaystyle\sum_{\substack{i \neq j \\ j \in s}} a_k \frac{c_j}{m+c_j} \sqrt{\lambda v_j}$ and $K_4 = \frac{c_i}{m+c_i} \sqrt{\lambda}$ , $\overline{\Pi}(m,S,v_i)$ can

be similarly proved to be pseudoconvex in $v_i$ . From the definition of $v_i = e^{\alpha_i' - m}$ , $v_i$ is

strictly increasing in $\alpha_i'$ . Hence, $\underline{\Pi}(m,S,\alpha_i')$ and $\overline{\Pi}(m,S,\alpha_i')$ are pseudoconvex in $\alpha_i'$ .

A similar proof is used to show that $\overline{\pi}_i(S,m,\alpha_i')$ and are pseudoconvex in $\alpha_i'$ where

$K_1 = K_3 = 0$, $K_5 = 1$ and $\delta(v_i) = 1 + v_i$ .

## 5.     *Proof of Theorem 2*

We prove theorem 2 through the Lemma 5 and 6.

**Lemma 5.** Suppose $S^* = \{1, ..., k\}$ is the optimal assortment, then $\left.\dfrac{\partial \Pi}{\partial \alpha}\right|_{\alpha = \alpha_1'} \geq 0$.

For this Lemma to be valid, $\underline{\Pi}(m, S, \alpha_i')$ and $\overline{\Pi}(m, S, \alpha_i')$ need to be increasing in $\alpha_i'$ at $\alpha_i' = \alpha_1'$. Lemma 5 implies that $\underline{\Pi}(m, S, \alpha_i')$ and $\overline{\Pi}(m, S, \alpha_i')$ are pseudoconvex in $\alpha_i'$. By contradiction, assume that $\underline{\Pi}(m, S, \alpha_i')$ and $\overline{\Pi}(m, S, \alpha_i')$ are nonincreasing in $\alpha_i'$ at $\alpha_i' = \alpha_1'$. Since $\underline{\Pi}(m, S, \alpha_i')$ and $\overline{\Pi}(m, S, \alpha_i')$ are pseudoconvex in $\alpha_i'$, then $\underline{\Pi}(m, S, \alpha_i')$ and $\overline{\Pi}(m, S, \alpha_i')$ are strictly decreasing in $\alpha_i'$ for $\alpha_i' < \alpha_1'$. Setting $\alpha_1' = -\infty$ (which is equivalent to removing item *1* from $S^*$) implies that $\Pi(m^*, S^*, -\infty) > \Pi(m^*, S^*, \alpha_i')$. Thus, $\Pi(m^*_{-1}, S^* / \{i\}) > \Pi(m^*_i, S^*)$ which contradicts with the definition of $S^*$ as the optimal assortment.

**Lemma 6.** Suppose $\alpha_2' \geq \alpha_1'$; if $\{2\} \in S^*$, then $\{1\} \in S^*$.

We prove this Lemma by contradiction. Assume $\{2\} \in K^*$ but $\{1\} \notin K^*$. Let $K_1^* = K^* / \{2\} \bigcup \{1\}$. Then, $\underline{\Pi}(K_1^*) \geq \underline{\Pi}(K^*)$. This result contradicts with Lemma 6 where $\underline{\Pi}(K^*) \geq \underline{\Pi}(K_1^*)$. Thus, item $\{1\} \in K^*$.