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ON THE UNIQUENESS OF THE RADON TRANSFORM OVER LINES, PLANES AND SPHERES

by
NOUR GHASSAN AL HASSANIEH

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NOUR GHASSAN AL HASSANIEH

Approved by:

Faruk Abi-Khuzam

Dr. Faruk Abi-Khuzam, Professor — Advisor

Mathematics

Dr. Bassam Shayya, Professor — Member of Committee

Mathematics

Dr. Florian Bertrand, Assistant Professor — Member of Committee

Mathematics

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An Abstract of the Thesis of

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This thesis will discuss the uniqueness problem of the Radon transform over lines in $\mathbb{R}^2$, hyper planes in $\mathbb{R}^N$ for $N > 2$, and in more detail, the Radon transform over spheres whose centers are restricted subsets of $\mathbb{R}^N$ for $N \geq 2$. It will also examine the connection between probability theory, and the injectivity of the Radon transform over lines. The study of the uniqueness of the spherical Radon transform is important to the development of some medical imaging methods such as Thermoacoustic Tomography (TAT). It also has applications in approximation theory, integral geometry, inverse problems for PDE's, and other fields.

CONTENTS

INTRODUCTION

If we know the integrals of a function under certain conditions over all lines in the plane, hyper-planes in n-space or over all spheres for a given set of centers, can we obtain the function back uniquely?

Such integrals are expressed by what is called *the Radon transform* over lines, hyper-planes or circles. Ever since J. Radon investigated the uniqueness of the Radon transform in 1917, it has been studied extensively and formulated in different aspects (see [1, 2] for references). Its applications, as Lawrence Zalcman phrased, "range, truly, from heaven above to the earth beneath and the water under the earth" [2]. In this work, we discuss the uniqueness of the Radon transform in three forms: first taken over lines in the plane, second over hyper-planes in n-space and finally over circles in the plane.

We prove uniqueness theorems and construct counterexamples whenever certain conditions are removed. Moreover, we connect the problem of injectivity with probability theory to simplify matters of generalization. On the other hand, the circular Radon transform, whose methods of investigation differ from the other two forms, is discussed extensively. We perform algebraic characterizations of sets of non-injectivity and asymptotic analysis of certain algebraic curves in order to present and prove a major theorem by Agranovsky and Quinto [3].

Consequently, chapter one will introduce the Radon transform over lines in the plane. In it we prove the uniqueness of the Radon transform over lines, provided that the function is globally integrable, using uniqueness of Fourier coefficients. Further, we present in detail a construction given by Armitage and Goldstein [4], which disproves the theorem whenever global integrability is removed.

In chapter two, we discuss the connection of the injectivity of the Radon transform over lines with a major theorem in probability theory. This theorem, originally proved by Cramer and Wold [5], uses a well-known result that a distribution function is uniquely determined by its characteristic function. The connection is established using a method by Renyi [6], in which we convert integrals over lines to integrals over half-planes and vice versa.

Chapter three discusses the injectivity of the Radon transform over hyper-planes. We give an idea of the proof of the uniqueness theorem using methods of chapter two, and demonstrate another construction by Armitage and Goldstein [7] that uses approximation by harmonic functions.

Finally, and perhaps the most important of the work, chapter four discusses the uniqueness of the spherical then circular Radon transform. We prove uniqueness whenever the sets of centers are spheres. Most importantly, we present a necessary and sufficient condition on sets of centers for which the Radon transform is not injective.

CHAPTER 1

THE RADON TRANSFORM OVER LINES

In 1957, D.J Newman proposed the following problem: If a function f is continuous and integrable over the plane and is such that the line integral $\int_l f ds = 0$ for every line l that is infinite in both directions then $f \equiv 0$. After Newman himself showed this [8], many others formulated several proofs for it. Perhaps one of the most elegant of these is a proof by Laurence Zalcman which was discussed in this paper [1] where he uses the uniqueness of Fourier coefficients.

In this chapter, we will define the Radon transform over lines in the plane; present Zalcman's proof of Newman's problem and finally give a construction of a non-zero function that is not globally integrable and whose Radon transform vanishes over all lines in the plane. This construction is due to Armitage and Goldstein in their paper [4].

Let us then begin with a definition of the Radon transform

1.1 Newman's Problem and its Proof

Definition 1.1.1. Let f be a continuous and integrable function over lines in $\mathbb{C}$, the Radon transform of f with respect to plane Lebesgue measure is given by:

$$Rf = \int_l f ds$$

where l is a doubly infinite line and ds denotes the length measure.

Theorem 1.1.1. *If f is continuous and integrable over $\mathbb{C}$, with respect to plane Lebesgue measure, and if*

$$\int_l f ds = 0$$

for every line l (doubly infinite), where ds denotes length measure, then $f \equiv 0$.

Proof. First, notice that $\int_l f ds = 0$ over any line l. So let us consider the lines passing through the origin. Thus $\int_l f ds = 0 \iff \int_{-\infty}^{+\infty} f(x, mx)dx = 0$ for any $m \in \mathbb{R}$ and so we deduce that $\int_l f ds = 0$ over any line l is equivalent to $\int_{-\infty}^{+\infty} f(x, y)dx = 0$ for each fixed value of $y \in \mathbb{R}$.

Consider the Fourier transform of f given as

$$\hat{f}(\xi, \eta) = \iint f(x, y)e^{-i(x\xi + y\eta)}dxdy$$

With l being a line through the origin, choose orthogonal coordinates in a way that l becomes the imaginary axis. To be more precise, let us consider the following linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined as follows:
Since the line l has the form $l : y = mx$ with $x, m \in \mathbb{R}$ we would like to apply a rotation so that it becomes the imaginary axis. Since m is the slope let $\alpha = \tan^{-1} m$ so that α is the angle between the line l and the x-axis. Let A be a 2×2 matrix given as $A = \begin{pmatrix} \sin\alpha & -\cos\alpha \\ \cos\alpha & \sin\alpha \end{pmatrix}$ so that we can define $T\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix}$ for $x, y \in \mathbb{R}$.

Now $\int\int f(x,y)dxdy = \int\int f \circ T(u,v) \left|\frac{\delta(x,y)}{\delta(u,v)}\right| dudv$. It is easy to check that $\left|\frac{\delta(x,y)}{\delta(u,v)}\right| = 1$ and thus conclude that $\int\int f(x,y)dxdy = \int\int f \circ T(u,v)dudv$.

Therefore, since T is a rotation we may use the fact that $\widehat{f \circ T} = \hat{f} \circ T$ in order to obtain

$$\hat{f}(0,\eta) = \int\int e^{-iy\eta} f(x,y)dxdy$$

By Fubini's theorem we obtain

$$\hat{f}(0,\eta) = \int e^{-iy\eta} \left(\int f(x,y)dx \right) dy$$

Since $\int_l f ds = 0$, $\int_{-\infty}^{+\infty} f(x,y)dx$ vanishes for each fixed value of y. Therefore, $\hat{f}$ vanishes on l and hence on each line through 0. Thus $\hat{f} = 0$ and so by the uniqueness of Fourier coefficients $f = 0$ almost everywhere. Since f is assumed to be further continuous, we obtain that $f \equiv 0$.

$\square$

As this theorem was proved in several ways, it is important to discuss a method of proof that allows us to generalize to higher dimensions. In the next chapter, we will give a brief history behind the several formulations of theorem 1.1.1 and investigate a way of proof using probability theory.

But, the question now is, if we remove the integrability condition on our function f in theorem 1.1.1, would the result still follow?

1.2 Removing the Integrability Condition on f

Removing the integrability condition of the continuous function f would render the previous result untrue. That is, a continuous function whose integral over all lines is zero, but is not identically zero can be constructed (this function would not be globally integrable). Zalcman first constructed such a function using an approximation theorem by Arakelian for holomorphic functions [1]. However, Armitage and Goldstein did a similar construction using elementary complex analysis [4]. We will demonstrate their construction.

Let us assume that the constructed function has derivatives with vanishing integrals over all lines. Therefore, we need a function g with the following properties:

1.
$$\int_l |g^{(n+1)}|ds < \infty$$

2.
$$g^{(n)}(z) \to 0 \ as \ z \to \infty \quad (z \in l).$$

where we can define $f = g'$, then $f^{(n)}$ is integrable over every line l and $\int_l f^{(n)}(z)ds = 0, \ \forall n \in \mathbb{N}$.

We would like to clarify why $\int_l f^{(n)}(z)ds = 0, \ \forall n \in \mathbb{N}$. We know that $ds = |z'(t)|dt$, $(t \in \mathbb{R})$, so letting $z(t) = (1-t)z_1 + tz_2$, $(z_1, z_2 \in l)$, we obtain $dz = (z_2 - z_1)dt$.

With $z_2 - z_1 = |z_2 - z_1|e^{i\alpha}$, $\alpha \in \mathbb{R}$, and any continuous integrable function h in $\mathbb{C}$, $\int_l hds = e^{-i\alpha} \int_l h(z)dz = \lim_{z_1, z_2 \to \infty} e^{-i\alpha} \int_{z_1}^{z_2} h(t)dt = \lim_{z_1, z_2 \to \infty} e^{-i\alpha}(H(z_1) - H(z_2))$ with $H' = h$, by using the fundamental theorem of calculus.

So, by a similar argument, we have

$$\int_l g^{(n+1)}(z)ds = \lim_{z_1, z_2 \to \infty} \left(g^{(n)}(z_2) - g^{(n)}(z_1) \right) = 0$$

(by the second property $g^{(n)}(z) \to 0$ *as* $z \to \infty$).

And thus we have $\int_l f^{(n)}(z)ds = 0, \ \forall n \in \mathbb{N}$.

1.3 Construction of non-constant function f whose integral on every line is zero

To find f, we will construct the function g presented in the previous section. But, before we begin the construction of g, we will state and prove the following lemma:

Lemma 1.3.1. *Suppose that* $z_1, z_2 \in \mathbb{C}$ *and* $|z_1 - z_2| < 1$. *If* ϕ_1 *is holomorphic in* $\mathbb{C}\backslash z_1$ *and* $\varepsilon > 0$, *then there is* ϕ_2 *that is holomorphic in* $\mathbb{C}\backslash z_2$ *such that*

$$|(\phi_2 - \phi_1)(z)| < \varepsilon(1 + |z|)^{-2}, \quad (|z - z_2| > 1).$$

Proof. ϕ_1 has a Laurent expansion at z_2 given as

$$\phi_1(z) = \phi_0(z) + \sum_{j=1}^{\infty} a_j (z - z_2)^{-j} \quad (|z - z_2| > |z_1 - z_2|),$$

where ϕ_0 is an entire function.

Define

$$\phi_2(z) = \phi_0(z) + \sum_{j=1}^{m} a_j (z - z_2)^{-j} \quad (z \neq z_2)$$

where $m \in \mathbb{N}$

Now

$$\begin{aligned} |(\phi_2 - \phi_1)|(z) &= |\sum_{m+1}^{\infty} a_j (z - z_2)^{-j}| \\ &\leq \sum_{m+1}^{\infty} \frac{|a_j|}{|z - z_2|^j} \\ &= \frac{1}{|z - z_2|^{m+1}} \sum_{j=m+1}^{\infty} \frac{|a_j|}{|z - z_2|^{j-m-1}} \end{aligned}$$

If we assume that $|a_j| \leq M$ for some $M > 0$ and $|z - z_2| \geq c > 1$ for some constant c we obtain

$$\begin{aligned}
|(\phi_2 - \phi_1)|(z) &\leq \frac{1}{|z - z_2|^{m+1}} \sum_{j=m+1}^{\infty} \frac{M}{c^{j-m-1}} \\
&= \frac{Mc}{c-1} \cdot \frac{1}{|z - z_2|^{m+1}}
\end{aligned}$$

$|z - z_2| \geq |z_2| - |z| \geq (1 + |z_2|) - (1 + |z|)$ so that if we take in particular $1 + |z_2| > 2(1 + |z|)$ then $|z - z_2| \geq (1 + |z|)$ and therefore,

$$\begin{aligned}
|(\phi_2 - \phi_1)(z)| &\leq \frac{Mc}{c-1} \cdot \frac{2}{(1+|z|)^2} \frac{1}{|z - z_2|^{m-1}} \\
&< \frac{\varepsilon}{(1+|z|)^2}
\end{aligned}$$

for large m and with $\varepsilon > 0$.

In remaining annular region, we would have

$$\begin{aligned}
|(\phi_2 - \phi_1)(z)| &\leq \sum_{j=m+1}^{\infty} \frac{|a_j|}{|z - z_2|^j} \\
&\leq M \sum_{j=m+1}^{\infty} \frac{1}{c_j} \\
&= \frac{M}{c^m(c-1)} < \varepsilon \\
&= \frac{\varepsilon}{(1+|z|)^2} . (1+|z|)^2 \\
&\leq \frac{M\varepsilon}{(1+|z|)^2}
\end{aligned}$$

□

This Lemma would be our guide in the construction. The aim is to produce a sequence of functions that is locally uniformly convergent to an entire function having the properties mentioned above.

So, first, let (ξ_k) be a sequence of points on the parabolic arc $P = \{t + it^2 : t \geq 0\}$, such that $\xi_0 = 0$ and $|\xi_k - \xi_{k-1}| < 1,\ (k \geq 1)$ with $\xi_k \to \infty$ as $k \to \infty$.

Define $g_0(z) = z^{-2}$ which is holomorphic in $\mathbb{C}\backslash\xi_0$. Using Lemma 1.3.1, there exists a holomorphic function $g_1(z)$ in $\mathbb{C}\backslash\xi_1$ such that for $\varepsilon = \dfrac{1}{2}$ we have

$$|(g_1 - g_0)(z)| < \frac{1}{2}(1 + |z|)^{-2}, \quad (|z - \xi_1| > 1).$$

Doing this repeatedly, we obtain

$$|(g_k - g_{k-1})(z)| < 2^{-k}(1 + |z|)^{-2}, \quad (|z - \xi_k| > 1) \textit{ and } k \geq 1. \tag{1.1}$$

Because of the obtained equation 1.1, (g_k) is locally uniformly convergent to a limit function g which is entire. We explain this in detail in what follows.

Local uniform convergence of g_n

Let K be a compact subset of $\mathbb{C}$. Then, there is $R > 1$, such that $K \subseteq \{|z| \leq R\}$. Since $\xi_k \to \infty$ as $k \to \infty$, there is k_0 such that $|\xi_k| > 2R$ for every $k > k_0$, then $|z - \xi_k| > R$ and in particular $|z - \xi_k| > 1$ for every $z \in K$. Thus, (g_k) is well defined and holomorphic on K for every $k > k_0$.

Let $m > n > k_0$,

$$\begin{aligned} |(g_m - g_n)(z)| &\leq \sum_{k=n+1}^{m} |(g_k - g_{k-1})(z)| \\ &\leq \sum_{k=n+1}^{m} 2^{-k}(1 + |z|)^{-2} \\ &\leq \sum_{k=n+1}^{m} 2^{-k} \leq 2^{-n}. \end{aligned}$$

Thus, $\exists N$ such that $\forall n > N,\ 2^{-n} < \varepsilon$ for any given ε, and thus local uniform convergence is verified.

Proof that g is entire

Let $D = \{z; |z - z_0| \leq R\}$ for $R > 0$. D is clearly a compact subset of $\mathbb{C}$. Let $\gamma = \partial D$, $\exists k_0 > 0$ such that g_k is holomorphic on D, $\forall k > k_0$. Using Cauchy's integral formula we have

$$g_k(z) = \frac{1}{2\pi i} \int_\gamma \frac{g_k(\zeta)}{\zeta - z} d\zeta$$

$$g(z) = \frac{1}{2\pi i} \lim_{k \to \infty} \int_\gamma \frac{g_k(\zeta)}{\zeta - z} d\zeta$$

and so

$$g(z) = \frac{1}{2\pi i} \int_\gamma \frac{g(\zeta)}{\zeta - z} d\zeta \quad (by\ DCT).$$

So we deduce that g is differentiable and thus holomorphic over D. Since we chose an arbitrary region D, we conclude that g is entire.

The second step is to show that the limit function constructed is not identically zero and satisfies properties 1 and 2.

Proof that g is not identically zero

Define $P_a = \{z : \inf_{w \in P} |z - w| > a\}$.

For $z \in P_1$ we have

$$\begin{aligned}
|(g - g_0)(z)| &= |g_1 - g_0 + g_2 - g_1 + \dots| \\
&\leq |g_1 - g_0| + |g_2 - g_1| + \dots \\
&= \sum_{k=0}^{\infty} |g_k - g_{k-1}| \\
&< (1 + |z|)^{-2} < |g_0(z)|.
\end{aligned}$$

So, we deduce that $g \not\equiv 0$. Otherwise, using the previous equation, if we plug in 0 for $g(z)$, we obtain $|g_0(z)| < |g_0(z)|$ which is false. Also we see that

$$|g(z)| - |g_0(z)| < |(g - g_0)(z)| < |g_0(z)|$$
$$|g(z)| < (1 + |z|)^{-2} + |g_0(z)| < 2|z|^{-2}, \quad (z \in P_1).$$

Using Cauchy's estimates to finalize the construction

The writers used Cauchy's estimates to draw out a relation that includes the n^{th} derivative of g where it was obtained that

$$|g^{(n)}(z)| < 2n!(|z| - 1)^{-2} \quad n \in \mathbb{N} \quad (z \in P_2).$$

To verify this result, let C be a circle of radius ρ and z a point inside C, we know that

$$\begin{aligned} g^{(n)}(z) &= \frac{n!}{2\pi i}\int_C \frac{g(\zeta)}{(\zeta - z)^{n+1}} d\zeta \\ |g^{(n)}(z)| &\leq \frac{n!}{2\pi}\int_C \frac{2|\zeta|^{-2}|d\zeta|}{|\zeta - z|^{n+1}} \\ &= \frac{n!}{\pi\rho^{n+1}}\int_C \frac{|d\zeta|}{|\zeta|^2} \\ &\leq \frac{2n!}{\rho^n(\rho - |z|)^2}. \end{aligned}$$

Here we have used the fact that $|z - \zeta| \leq \rho$ which implies $|z| - |\zeta| \leq \rho$ and so $|\zeta| \geq |z| - \rho$ whereby we finally obtain $|\zeta|^{-2} \leq (|z| - \rho)^{-2}$.

Taking $\rho = 1$ we obtain the needed result

$$|g^{(n)}(z)| < 2n!(|z| - 1)^{-2}.$$

To finalize the construction notice that $l \backslash P_2$ is bounded (we will prove a general case of this fact in section 3.5 of chapter 3). We claim now that the properties 1 and 2 mentioned earlier are thus satisfied.

Therefore, it is to be proved that:

1.
$$\int_l |g^{(n+1)}| ds < \infty$$

2.
$$g^{(n)}(z) \to 0 \ as \ z \to \infty \quad (z \in l).$$

Proof that g satisfies the two properties

To prove property 1, divide the line l on which the integral is calculated into three portions: l_1, $l \backslash P_2$ and l_2 (l_1 and l_2 are the infinite portions left of l after removing P_2). Since $l \backslash P_2$ is bounded, $\int_{l \backslash P_2} |g^{(n+1)}(z)| ds < \infty$. Now,

$$\int_{l_1} |g^{(n+1)}(z)| ds < 2(n+1)! \int_{l_1} \frac{ds}{(|z|-1)^2}.$$

Noting that $|z| > |x|$ (for $z = x + iy$) and taking the following parametrization of $l_1(t) = (t, at+b)$, $-\infty < t < t_1$ ($t_1, a, b \in \mathbb{R}$ and $t_1 + iy_1$ ($y_1 \in \mathbb{R}$) is the point of intersection between l_1 and $l \backslash P_2$), we obtain the following

$$\begin{aligned} \int_{l_1} |g^{(n+1)}(z)| ds &< 2(n+1)! \int_{-\infty}^{t_1} \frac{\sqrt{1+a^2}}{(|t|-1)^2} dt \quad \text{assuming } t \neq 1 \\ &\leq 2(n+1)! \int_{-\infty}^{t_1} \frac{\sqrt{1+a^2}}{(t-1)^2} dt \\ &< 2(n+1)! \int_{-\infty}^{t_1} \frac{\sqrt{1+a^2}}{t^2+5} dt < \infty. \end{aligned}$$

Here $(t-1)^2 > t^2+5$ because $t_1 + iy_1 \in l \backslash P_2 \cap l_1$ and therefore $|t_1| \geq 2$ so $-2t_1 \leq -4$ and $-2t_1 \geq 4$. So, $(t-1)^2 = t^2 - 2t + 1 \geq t^2 - 2t_1 + 1 \geq t^2 + 5$ as needed.

We do the same for l_2.

To verify the second property 2, notice that we have already shown that

$$|g^{(n)}(z)| < 2n!\frac{1}{(|z|-1)^2} \quad z \in P_2,\ z \in l.$$

so as $z \to \infty$, $\lim_{z\to\infty} |g^{(n)}(z)| < 0$. $\therefore$ $\lim_{z\to\infty} g^{(n)}(z) = 0$ for $z \in P_2$.

If $z \in l \backslash P_2$, $|z| < M$ for some $M > 0$.

Thus as $z \to \infty$, $z \in P_2$ and so $\lim_{z\to\infty} g^{(n)}(z) = 0$.

Therefore, after obtaining our function g, letting $g' = f$ we obtain a function f that is not identically zero but whose integral over every line is zero.

The question now is whether or not we can generalize this construction and work in the space of $\mathbb{R}^n$. That is to say, can we find a non-zero harmonic function that has vanishing integrals over every hyper plane? In fact, we can, and this has been also shown by Armitage and Goldstein in their paper [7]. We will demonstrate the construction in detail in chapter 3.

CHAPTER 2

THE RADON TRANSFORM AND PROBABILITY THEORY

The proof we presented of theorem 1.1.1, does not help in the generalization to higher dimensions. Thus it is a must that we look at other methods that may be helpful.

In his paper [6], Renyi explains a method that relates integration over straight lines to integration over half planes. He then connects this to probability theory and paves the way for further results and generalizations in the area. Although we start with strict conditions on the function to be studied and its domain, the method holds new ideas that it was necessary to study it separately. It is also important to give credit to Cramer and Wold [5] who originally introduced and worked on several applications of the method.

Let us first begin with a brief history

Originally, in 1917 Radon proved the following theorem:
(*) If K is a bounded domain of the (x, y) plane, and the integral of the continuous function $f(x, y)$ vanishes along every chord of the domain K, then $f(x, y)$ is identically equal to zero.

Renyi then shows that theorem (*) is equivalent to the following theorem:

(**) A continuous and non-negative function $f(x, y)$ defined in the convex domain K, is uniquely determined if the value of its integral along every chord of K is given as a finite value.

Although the proof is straightforward, we will present it for completeness.

Lemma 2.0.1. *Theorem (*) $\iff$ theorem (**).*

Proof. ($\Rightarrow$) If $\int_l f ds = \int_l g ds$ for every chord l, then $\int_l f - g ds = 0$ (since the values of $\int_l f ds$ and $\int_l g ds$ are finite) and thus $f \equiv g$.
$\therefore$ (*) $\Rightarrow$ (**).

($\Leftarrow$) We are given that $\int_l f ds = 0$ for every line $l \subset K$. Let

$$f_1(x, y) = \begin{cases} f(x, y) & if\ f(x, y) \geq 0. \\ 0 & if\ f(x, y) < 0. \end{cases}$$

and $f_2(x, y) = f_1(x, y) - f(x, y)$ with $(x, y) \in \mathbb{R}^2$.
Clearly, $f_1,\ f_2$ are continuous and non-negative. Also $\int_l f_1 ds = \int_l f_2 ds\ \forall\ l \subset K$.
Hence (**) $\Rightarrow$ (*).

$\square$

The method we will represent in this chapter can easily prove theorems (*) and (**). But before we introduce the preliminaries, we will demonstrate a beautiful idea presented by Renyi in his paper [6], which allows us to convert, under certain conditions, the integral over all lines to an integral over all half-planes and vice versa.

2.1 From Integrals over lines to Integrals over Half-Planes

Let us state and prove the following lemma:

Lemma 2.1.1. *Let f be a continuous non-negative function over a bounded convex domain $K \subset \mathbb{R}^2$. Define $i(l) = \int_l f ds$. Then $i(l)$ is known as a finite value for every line $l \in K$ if and only if $I(H) = \int\int_H f(x,y)dxdy$ is known as a finite value for every half-plane H in the plane.*

Proof. ($\Leftarrow$) We claim that if $I(H)$ is known for every half-plane, then $\int\int_{S_\Delta} f(x,y)dxdy$ is known for every parallel strip S_Δ whose breadth is equal to Δ.
To see this, let l_1 be an arbitrary line in the plane. Take H_1 to be the half plane whose boundary line is parallel to l_1 and is at a distance Δ form l_1. Then, take H_2 to be the half-plane whose boundary line is l_1, in such a way that $H_1 \subset H_2$. Now, we know that $I(H_1)$ and $I(H_2)$ are both known and finite values, thus

$$\int\int_{H_2} f(x,y)dxdy - \int\int_{H_1} f(x,y)dxdy = \int\int_{S_\Delta} f(x,y)dxdy$$

is known for every parallel strip S_Δ.

Now, let l be the mid-line of the parallel strip S_Δ.
Consider,

$$\int\int_{S_\Delta} f(x,y)dxdy = \int\int_{S_{1/\Delta}} f(x,y)dxdy + \cdots + \int\int_{S_{1/\Delta}} f(x,y)dxdy$$

Δ times. That is

$$\int\int_{S_\Delta} f(x,y)dxdy = \Delta \int\int_{S_{1/\Delta}} f(x,y)dxdy$$

which implies that

$$\int\int_{S_{1/\Delta}} f(x,y)dxdy = \frac{1}{\Delta}\int\int_{S_\Delta} f(x,y)dxdy$$

so that finally we would have

$$\int_l f ds = \lim_{\Delta\to 0}\frac{1}{\Delta}\int\int_{S_\Delta} f(x,y)dxdy$$

which allows us to conclude that $i(l)$ is known for every chord l.

($\Rightarrow$) We know $i(l)$ over every line l in the plane. Let H be any half-plane, and take (d) to be the boundary line of H. Let (d') be the line perpendicular to (d) and passing through the origin. Define l_t to be the line parallel to (d) that cuts (d') at a point of abscissa t on the line (d'). Then, we claim that $I(H) = \int_{-\infty}^{+\infty} i(l_t)dt$.

To see this, and without loss in generality, we can consider a rotation T that takes the line (d') to the x-axis. Thus, any line l_t would be a vertical line passing through the point $(t,0)$. So then, $i(l_t) = \int_{l_t} f ds = \int_{-\infty}^{+\infty} f(t,y)dy$.

Since, we are working on a bounded domain K,

$$I(H) = \int\int_{K\cap H} f(x,y)dxdy = \int\int_{H} f(x,y)dxdy = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)dxdy$$

This is equivalent to saying $I(H) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(t,y)dtdy = \int_{-\infty}^{+\infty} i(l_t)dt$.

It is important to note that throughout this whole proof, the main worry was about the existence of the integrals we're working with. Having f defined over a bounded region K ensures this property. So then if we remove the condition of f being defined over the bounded region K, and replace it by the condition that f is integrable over the whole plane, the existence of the integrals would be also verified and the result of the lemma would follow accordingly.

□

In what follows we will give some preliminaries in probability theory that we will need to connect with theorems (*) and (**), and consequently prove theorem 1.1.1.

2.2 Probability Distribution and its Projection

Definition 2.2.1. A distribution function in $\mathbb{R}^n$ is a completely additive, non-negative set function $F(E)$ defined for all Borel sets E of $\mathbb{R}^n$ and is such that $F(\mathbb{R}^n) = 1$.

Definition 2.2.2. The characteristic function (or Fourier Stieltjes transform) of a probability distribution $F(E)$ is the function $f(t)$ defined for all $t \in \mathbb{R}^n$ by the Lebesgue Radon integral

$$f(t) = \int_{\mathbb{R}^n} e^{i(t_1 x_1 + \cdots + t_n x_n)} dF, \quad t \in \mathbb{R}^n$$

where $x \in \mathbb{R}^n = (x_1, x_2, \ldots, x_n)$.

(These definitions are obtained from [5]).

It well known that a distribution function is uniquely determined by its characteristic function. We will not prove this so as not to steer away from our subject. See [5] for references to the proof.

Let f be a non-negative, continuous function that is integrable over the plane. $\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x, y) dx dy < \infty$ and so we can take, without loss of generality, $\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x, y) dx dy = 1$. This way we can consider $f(x, y)$ to be a density function of a probability distribution.

Define the corresponding distribution function of $f(x, y)$ as follows

$$F(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(u, v) dv du$$

The characteristic function of $F(x, y)$ is given as

$$\psi(u, v) = \int_{-\infty}^{+\infty} e^{i(ux+vy)} dF(x, y)$$

Let l be an arbitrary line passing through the origin. Let u be the unit vector parallel to l. If the angle between the line l and the x-axis is θ, u can be chosen to be $u = (\cos\theta, \sin\theta)$. Define the *coordinate of the projection* at a random point $x = (x_1, x_2) \in \mathbb{R}^2$ on the line l as

$$p_{x,\theta} = u.x = x_1 \cos\theta + x_2 \sin\theta$$

The line l through the origin whose angle with the x-axis is θ, can be represented as follows $y = x \tan\theta$ which gives $y \cos\theta = x \sin\theta$ so that we finally obtain $l : y \cos\theta - x \sin\theta = 0$. Thus, any line perpendicular to l with coordinate p on l would have $(\cos\theta, \sin\theta)$ as a normal and hence be of the form $x \cos\theta + y \sin\theta = p$.

Now, let H_p denote the half-plane whose boundary line is perpendicular to l and intersects l at a point whose coordinate on l is equal to p. Define the distribution function of the projection on l of any point in the plane as

$$F_l(p) = \int\int_{H_p} f(x, y) dx dy.$$

Note that the equation representing H_p be would be $x \cos\theta + y \sin\theta \leq p$ which is the set of all lines perpendicular to l and whose coordinate on l is less than or equal to p.

We are ready to state and prove a theorem that will be very useful for our purposes.

Theorem 2.2.1. *Let $F(x,y)$ denote the distribution function of an arbitrary probability distribution on the plane, and suppose that the projection of this distribution is known on every straight line l through the origin i.e.*

$$F_{l_\varphi}(p) = \int\int_{x\cos\varphi + y\sin\varphi \leq p} dF(x,y)$$

is a known function of p for every value of φ $(0 \leq \varphi < \pi)$ where φ denotes the angle between the straight line l_φ and the x-axis. Then, $F(x,y)$ is uniquely determined for every value of x and y.

Proof. Let $(\xi,\eta) \in \mathbb{R}^2$ and consider the distribution function $F(x,y)$ on the plane. Let $\psi(u,v)$ denote the characteristic function of the point (ξ,η) which is given as

$$\psi(u,v) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{i(ux+vy)} dF(x,y)$$

Now denote the coordinate of the projection of the point (ξ,η) on the line l_φ by ζ_φ and this is can be represented in the form

$$\zeta_\varphi = \xi\cos\varphi + \eta\sin\varphi$$

so that $F_{l_\varphi}(\zeta_\varphi)$ becomes the distribution function of ζ_φ. Since it is assumed that $F_{l_\varphi}(p)$ is a known function of p (p being the coordinate of projection of any point in the plane on the line l_φ), it's characteristic function

$$\psi_\varphi(t) = \int_{-\infty}^{+\infty} e^{it\zeta_\varphi} dF_{l_\varphi}(\zeta_\varphi) = \int_{-\infty}^{+\infty} e^{it(\xi\cos\varphi + \eta\sin\varphi)} dF_{l_\varphi}(\zeta_\varphi) = \psi(t\cos\varphi, t\sin\varphi)$$

This implies that for every $u, v \in \mathbb{R}$ the following holds and is known

$$\psi(u,v) = \psi_{\tan^{-1}\left(\frac{v}{u}\right)}(\sqrt{u^2+v^2})$$

Since $F(x,y)$ can be uniquely determined by $\psi(u,v)$, we conclude that $F(x,y)$ is known for every value of x and y which completes the proof.

□

2.3 Steps towards the generalization of Radon's Theorem:

We have started with a non-negative function f that is defined on a bounded domain K, and whose integral vanishes over every chord of K. The first step is to remove this restrictive domain and have f defined over the whole plane in such a way that $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)dxdy < \infty$ (i.e f is L^1 integrable).

The second step would be to remove the continuity condition so f then becomes non-negative and integrable. Here we can take f to be a density function of a probability distribution. The question then would be whether the values of $F(x,y)$ as defined in the previous section 2.2 would determine uniquely the value of our function f.

The final step of generalization would be to consider distributions that have no density functions. Theorem 2.2.1 tells us that any distribution function of any probability distribution on the plane can be uniquely determined by its projections on every straight line.

2.4 An alternative proof of theorem 1.1.1

If we weaken the condition on f stated in theorem 1.1.1 and require that f becomes non-negative, the above procedure can be applied. Again, let f be a non-negative, continuous and integrable function on the plane and assume with no loss of generality that $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)dxdy = 1$. (Note that with this assumption we are also assuming that $f \not\equiv 0$ so that we may argue by contradiction).
We know that $\int_l f ds = 0$ for every line l in the plane with ds denoting the length measure.
As we have shown earlier, if the integral of f is known over all lines in the plane, then it is also known over all half-planes. and thus since we are given that $\int_l f ds = 0$ over all lines, we obtain

$$\int\int_H f(x,y)dxdy = 0$$

for all half-planes H. Therefore, using the result of the previous theorem

$$F(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(x,y)dxdy \equiv 0$$

Since $F(x,y)$ is continuous $\lim_{x,y\to+\infty} F(x,y) = 0$ and thus $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)dxdy = 0$ which is a contradiction to our assumption that $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)dxdy = 1$ and hence $f \equiv 0$.

Likewise, theorem (*) which is equivalent to (**) are easily proven by this procedure.

2.5 More Results

One result that may generalize theorem 2.2.1 and which can be used to generalize theorem (*) is the following:

Theorem 2.5.1. *Let* (ξ, η) *be a point contained with probability 1 in a disc* D *of equation* $\xi^2 + \eta^2 \leq R^2$ *and the distribution function* $F_{l_\varphi}(\zeta_\varphi)$ *of the random variable* $\zeta_\varphi = \xi \cos\varphi + \eta \sin\varphi$ *is given for an infinity of mod* π *different values of* φ*, then the distribution function* $F(x, y) = Pr(\xi < x, \eta < y)$ *of the random point* (ξ, η) *is uniquely determined.*

Proof. Let $\varphi_1, \varphi_2, \ldots, \varphi_n, \ldots,$ be the values of φ for which F_{l_φ} is known.
Take φ_0 to be a limit point of the sequence φ_n .

So then $\psi(t\cos\varphi, t\sin\varphi)$ is known for each value of φ_i with $i = 1, 2, \ldots, n, \ldots.$ (see the proof of theorem 2.2.1 for details of the derivation)

Now we claim that ψ is an analytic function of φ for every fixed value of t. We can see, using dominated convergence, that

$$\frac{\partial \psi_\varphi(t)}{\partial \varphi} = it \int\int_{x^2+y^2 \leq R^2} (-x\sin\varphi + y\cos\varphi) e^{it(x\cos\varphi + y\sin\varphi)} dF(x, y)$$

exists for every value of φ. Since the terms $e^{it(x\cos\varphi+y\sin\varphi)}$, $\sin\varphi$ and $\cos\varphi$ are all analytic and the integral over the compact set $x^2 + y^2 \leq R^2$ is a finite number, then ψ is thus analytic.

Hence ψ will be known for any fixed value of t for values of $\varphi = \varphi_{n_k}$ with $\lim_{k\to\infty} \varphi_{n_k} = \varphi_0$.

Therefore, $\psi(t\cos\varphi, t\sin\varphi)$ is known for every value of t and φ. Hence, by the same way we proved theorem 2.2.1, the distribution function $F(x, y)$ can be determined uniquely.

□

Corollary 2.5.2. *Let* $f(x, y)$ *be a continuous function that is equal to* 0 *outside the* *disk* $x^2 + y^2 < R^2$ *for some* $R > 0$. *Suppose* $\int_l f ds = 0$ *for every line* l *parallel to some line belonging to an arbitrary infinite set of lines passing through the origin, then* $f \equiv 0$.

Proof. We follow the same procedure we used in section 2.4, but this time we use theorem 2.5.1 to show that $F(x, y) \equiv 0$ so that $f \equiv 0$.

□

In the next chapter, we will discuss briefly how the method described in section 2.2 can generalize the problem of injectivity of the Radon transform, from integrals taken over lines to integrals taken over hyper-planes. We see then that such integrals also determine uniquely the value of the function itself provided that this function is restricted to certain conditions.

CHAPTER 3

THE RADON TRANSFORM OVER HYPER-PLANES

After the research on the Radon transform over lines it is natural to generalize to $\mathbb{R}^N$. The steps of the construction presented in chapter 1 can be mimicked to obtain a similar result over hyper planes. However, it was a must that we include another construction by Armitage and Goldstein in their paper [7]. This paper is short, concise and holds truly beautiful ideas, particularly those relating to harmonic approximation that we found essential to include in our study of the Radon Transform.

3.1 Main Definitions, Theorems and Notation

Definition 3.1.1. Let f be real or complex valued on $\mathbb{R}^N$ where $N \geq 2$, and suppose f is integrable on each $(N-1)$-dimensional hyperplane P in $\mathbb{R}^N$, then the Radon transform $\hat{f}$ of f is defined on the set $\mathbb{P}^N$ of all such hyperplanes by:

$$\hat{f} = \int_P f d\lambda,$$

where λ denotes the $(N-1)$-dimensional Lebesgue measure on P.

To generalize theorem 1.1.1, we can state the following theorem

Theorem 3.1.1. *If f is continuous and integrable over $\mathbb{R}^N$, and $\hat{f} \equiv 0$ on $\mathbb{P}^N$, then $f \equiv 0$ on $\mathbb{R}^N$.*

The proof of this follows from the method represented in chapter 2. For example, if we take $N = 3$ we can see that theorem 2.2.1 can be generalized so that a probability distribution can be uniquely determined by its projection on every plane passing through a given line; or in fact, by its projection on every line passing through the origin, or through a combination of planes and lines that together cover the whole space (see [6]). So that then we can state the theorem

(***) A probability distribution in the n dimensional space is uniquely determined by its projections on such a set of subspaces of $1, 2, \ldots, (n-1)$ dimensions which together cover the whole space.

Keeping that in mind, the main objective of this chapter now is to prove the next theorem which is as follows:

Theorem 3.1.2. *There exists a non-constant harmonic function h on $\mathbb{R}^N$, $N \geq 2$ such that $\hat{h} \equiv 0$ on $\mathbb{P}^N$.*

Armitage and Goldstein presented a construction for the required function h [7] which we will discuss again in more detail. But let us first introduce some preliminary definitions and notation.

3.2 Preliminary Definitions and Notation for the Construction

Denote by S the unit sphere in $\mathbb{R}^N$, given as $\{y \in \mathbb{R}^N : ||y|| = 1\}$. Here we take the usual inner product on $\mathbb{R}^N$ as

$$\langle x, y \rangle = x_1 y_1 + \cdots + x_N y_N \;;$$

and the usual norm

$$||x|| = \sqrt{\langle x, x\rangle}\ ;$$

where $x, y \in \mathbb{R}^N$, $x = (x_1, \ldots, x_N)$ and $y = (y_1, \ldots, y_N)$.

For $y \in S$ and $t \in \mathbb{R}$, define

$$P(y,t) = \{x \in \mathbb{R}^N : \langle x, y\rangle = t\};$$

which is the $N-$dimensional hyperplane, and
for $-\infty \leq a < b \leq +\infty$, define

$$Q(y,a,b) = \bigcup_{a<t<b} P(y,t),$$

which is isometric to $\mathbb{R}^{N-1} \times (a,b)$.
Let $\mathcal{A}$ be the point at infinity of $\mathbb{R}^N$ and take the topology on $\mathbb{R}^N \cup \{\mathcal{A}\}$ to be the Aleksandroff one-point compactification topology.

What was rather remarkable in Armitage and Goldstein's proof [7] was the introduction of a certain special set so that they would make use of a theorem on harmonic approximation that was recent at their time. This theorem would give existence to the harmonic function we are seeking to prove theorem 3.1.2. So, let us first define this special set and then use it to build up other sets that are essential for our construction.

3.3 Special Set E and Construction of h

Let E be a non-empty subset of $\mathbb{R}^N$ with the following properties:

1. E is open in $\mathbb{R}^N$.

2. $E \cup \{\mathcal{A}\}$ is connected and locally connected in the topology of $\mathbb{R}^N \cup \{\mathcal{A}\}$.

3. If $y \in S$ and $0 < a < \infty$, then $E \cap Q(y, -a, a)$ is bounded.

4. If $y \in S$ then $\exists T > 0$ (depending on y) such that at least one of the sets $E \cap Q(y, -\infty, -T)$ and $E \cap Q(y, T, +\infty)$ is empty.

The authors presented an interesting example of this set which we will provide and discuss in further detail in the final section of this chapter.

Now, let $z \in E$ be a fixed point so that we define the following closed subsets of $\mathbb{R}^N$ as:

$$F_1 = \mathbb{R}^N \backslash E, \quad F_2 = \{z\}, \quad F = F_1 \cup F_2.$$

Let ω_1 and ω_2 be two open subsets of $\mathbb{R}^N$ such that $\omega_1 \supset F_1$, $\omega_2 \supset F_2$ and $\omega_1 \cap \omega_2 = \phi$. Define $u : \omega_1 \cup \omega_2 \to \mathbb{R}$ as:

$$u(x) = \begin{cases} 0 & if\ x \in \omega_1. \\ 1 & if\ x \in \omega_2. \end{cases}$$

First, notice that F is unbounded. Second, it is clear that u is harmonic on the open neighborhood $\omega_1 \cup \omega_2$ of F, $(\Delta u = 0)$.

Let us consider the set $\mathbb{R}^N \backslash F = \mathbb{R}^N \backslash F_1 \cup F_2 = E \backslash \{z\}$. It is clear that $E \backslash \{z\}$ is open in $\mathbb{R}^N$ and inherits the properties (1) through (4) of the set E.

To proceed with the construction we need the following theorem:

Theorem 3.3.1. *Let F be a non-empty closed subset of an open set Ω in $\mathbb{R}^N$ $(N \geq 2)$. If u is harmonic on an open set containing F except possibly for singularities, then for each $\varepsilon > 0$, each $\mu > 0$ and each $k \in \mathbb{N}$, there exists v that is harmonic on an* *open set containing* Ω, *except possibly for singularities, such that*

$$D^{\alpha}(v-u)(x) < \varepsilon(1+||x||)^{-\mu} \quad (x \in F, |\alpha| \leq k).$$

Moreover, if $\Omega^{*}\backslash F$ *is connected and locally connected and* u *is harmonic in an open set containing* F, *except possibly for removable singularities, then we may take* v *to be harmonic in an open set containing* Ω, *except possibly for removable singularities.*

Note: here $\Omega^{*} = \Omega \cup \{*\}$, is the Aleksandroff one-point compactification of Ω, where '$*$' denotes the ideal point of Ω.

Although we will not provide a proof of this theorem here, we urge the kind readers to check the following paper [9] which is also by Armitage and Goldstein and which presents and proves some essential results in harmonic approximation.

Let us take $\Omega = \omega_1 \cup \omega_2$, F to be the closed subset of $\mathbb{R}^N$ defined earlier, and $k = 0$. Thus, since the hypothesis of theorem 3.3.1 is satisfied, there is a function h that is harmonic on an open set containing Ω except possibly for removable singularities such that

$$|h(x) - u(x)| < (1+||x||)^{-N-1} \quad (x \in F). \tag{3.1}$$

It can be simply shown that $|h(x)-1| < 1$. So, if we assume that $h(x) = 0,\ \forall x \in \mathbb{R}^N$, we obtain $|0-1| < 1$ which is a contradiction and thus $h \not\equiv 0$.

Moreover, notice that $\lim_{x\to\mathcal{A}, x\in F} h(x) = 0$. This is because as $x \to \mathcal{A}$, $x \in \omega_1$ but $u(x) = 0,\ \forall x \in \omega_1$, so that equation 3.1 gives $|h(x)| < \dfrac{1}{(1+||x||)^{N+1}}$. Taking the limit as $x \to \mathcal{A}$ we see that $h(x) \to 0$. This step proves that h is non-constant.

3.4 Finalizing the Construction

Now that we obtained our harmonic function h, the objective from here is to prove that it is indeed the function needed to complete the proof of theorem 3.1.2. So we have to prove the following Lemma:

Lemma 3.4.1. *For a fixed $y \in S$, h is locally integrable on $P(y,t)$ and $\hat{h}(P(y,t)) = 0$ for all $t \in \mathbb{R}$.*

Before we proceed with the proof, define

$$B_N(r) = \{x \in \mathbb{R}^N : ||x|| < r\}.$$

Since E satisfies property (3), and for $0 < a < \infty$, $E \cap Q(y,-a,a)$ is bounded. So, there is $r > 0$ such that $E \cap Q(y,-a,a) \subseteq B_N(r)$. Given that $F_1 = \mathbb{R}^N \backslash E$, we have $E = \mathbb{R}^N \backslash F_1$ so that $E \cap Q(y,-a,a) = Q(y,-a,a)\backslash F_1$. However, $F = F_1 \cup F_2$ which implies $F_1 \subset F$ and so $Q(y,-a,a)\backslash F \subset Q(y,-a,a)\backslash F_1$. Thus, $Q(y,-a,a)\backslash F \subseteq B_N(r)$.

Proof of the Lemma

Fix $y \in S$ and let us consider the following

$$\int_{P(y,t)} |h| d\lambda = \int_{P(y,t)\backslash F} |h| d\lambda + \int_{P(y,t)\cap F} |h| d\lambda \tag{3.2}$$

For $-a < t < a$, we would have $P(y,t)\backslash F \subset B_N(r)$ since $Q(y,-a,a)\backslash F \subseteq B_N(r)$. Also, we have $|h(x)| < (1+||x||)^{-N-1} + |u(x)|$ for $x \in \Omega = \omega_1 \cup \omega_2$, so that for $x \in \omega_1$, $|h(x)| < (1+||x||)^{-N-1}$, and thus we obtain

$$\int_{P(y,t)} |h| d\lambda \leq \sup_{B_N(r)} |h| \int_{P(y,t)\backslash F} d\lambda + \int_{P(y,t)\cap F} (1+||x||)^{-N-1} d\lambda(x).$$

Now, $(1 + ||x||)^{-N-1}$ is at its maximum at the origin, and $P(y, t) \cap F \subset P(y, t)$, so we have

$$\int_{P(y,t)\cap F} (1 + ||x||)^{-N-1} d\lambda(x) \leq \int_{P(y,0)} (1 + ||x||)^{-N-1} d\lambda(x),$$

So that finally equation 3.2 becomes:

$$\int_{P(y,t)} |h| d\lambda \leq V(r) \sup_{B_N(r)} |h| + \int_{P(y,0)} (1 + ||x||)^{-N-1} d\lambda(x) \tag{3.3}$$

where $V(r)$ is $(N-1)-$dimensional volume of $B_{N-1}(r)$.

Define $f(t) = \int_{P(y,t)} |h| d\lambda$.

Let us prove that $f(t)$ is locally bounded. For $|t| < a$, and using equation 3.3, it is easy to see that the first term of the equation is bounded. It is left to show that $\int_{P(y,0)} (1 + ||x||)^{-N-1} d\lambda(x)$ is bounded.

Let us first take $N = 3$.

Let $y \in \mathbb{R}^3$ be such that y belongs to the unit sphere in $\mathbb{R}^3$ and is fixed. Let R be a rotation taking the vector y to the vector $(0, 0, 1)$, then $||Rx|| = ||x||$ and so $d\lambda(Rx) = d\lambda(x)$ for $x \in \mathbb{R}^3$.

Therefore, $\int_{P(y,0)} (1+||x||)^{-4} dxdy = \int_{x,yplane} (1+||x||)^{-4} dxdy$ where $x = (x, y)$ and thus $\int_{P(y,0)} (1 + ||x||)^{-4} dxdy = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left(\frac{1}{1 + \sqrt{x^2 + y^2}} \right)^4 dxdy = \int_0^{2\pi} \int_0^{\infty} \left(\frac{1}{1 + r} \right)^4 rdrd\theta$ which is a finite number.

If we go to higher dimensions, we work similarly to obtain

$$\int_0^{2\pi} \cdots \int_0^{2\pi} \int_0^{+\infty} \left(\frac{1}{1 + r} \right)^{N+1} r^{N-2} drd\theta_1 d\theta_2 \ldots d\theta_{N-2}$$

and we also get a finite number. Therefore, $f(t)$ defined above is bounded for $|t| < a$, so it is locally bounded.

In order to proceed, the authors presented the following result

If s is subharmonic on $\mathbb{R}^N$ and $t \mapsto \int_{P(y,t)} |s| d\lambda$ is locally bounded on $\mathbb{R}$, then the hyperplane mean $\hat{s}(P(y,t))$ is a convex function of t on $\mathbb{R}$.

Now take this s to be $s = h$ and $s = -h$. The conditions of the mentioned result are satisfied and so $\hat{h}(P(y,t))$ and $-(\hat{h})(P(y,t))$ are both convex functions of t, and therefore $\hat{h}(P(y,t))$ becomes a linear function of t.

By the 4th property of the set E, we see that $\exists T > 0$ such that $P(y,t) \subset F$, $\forall t > T$ *or* $\forall t < -T$. When $P(y,t) \subset F$, we would have

$$|\hat{h}(P(y,t))| < \int_{P(y,t)} (1 + ||x||)^{-N-1} d\lambda(x) = \int_{P(y,0)} (1 + \sqrt{||x||^2 + t^2})^{-N-1} d\lambda(x).$$

Let us explain the last step more. Take $N = 3$ so that we can visualize the result. For $x \in P(y,0)$ and $x' \in P(y,t)$, we can use Pythagoras formula to obtain $||x'||^2 = t^2 + ||x||^2$. Note that this method is independent on the dimension and thus holds for higher values of N.

Now,

$$\begin{aligned}\int_{P(y,0)} (1 + \sqrt{||x||^2 + t^2})^{-N-1} d\lambda(x) &= \int_{P(y,0)} \frac{1}{(1 + \sqrt{||x||^2 + t^2})^N} \frac{1}{(1 + \sqrt{||x||^2 + t^2})} d\lambda(x) \\ &< (1 + |t|)^{-1} \int_{P(y,0)} (1 + ||x||)^{-N} d\lambda(x)\end{aligned}$$

This is because, we can take $(1 + \sqrt{||x||^2 + t^2})^{-N} < (1 + ||x||)^{-N}$ and $(1 + \sqrt{||x||^2 + t^2})^{-1} < (1 + |t|)^{-1}$.

Since $\int_{P(y,0)} (1 + ||x||)^{-N} d\lambda(x)$ exists, letting $t \to +\infty$ or $t \to -\infty$, we get

$\hat{h}(P(y,t)) \to 0$. Since we proved that $\hat{h}$ is a linear function of t, it follows that

$$\hat{h}(P(y,t)) = 0 \ \forall t \in \mathbb{R}$$

and hence the proof of theorem 3.1.2 is complete.

So, we have obtained a non-constant harmonic function whose integral is zero over all hyper-planes which proved our theorem.

In the next subsection, we will demonstrate an example of the special set E defined earlier which was also presented by Armitage and Goldstein [7]. Notice that the proof of the properties of the constructed set is a generalization to the proof left out in chapter 1 (the one where we claimed that $l \backslash P_2$ is bounded in section 1.3).

3.5 Example of the special set E

The Goal now is to find a set E that satisfies the properties mentioned at the beginning of section 3.3. So let us begin the construction by the following:

let $I = [0, +\infty)$ and define $\psi : I \to \mathbb{R}^N$ by $\psi(\xi) = (\xi, \xi^2, \ldots, \xi^N)$.
Let $E = \{x \in \mathbb{R}^N : \inf_{\xi \in I} ||x - \psi(\xi)|| < 1\}$.
The first two properties of the special set defined in section 3.3 are clearly satisfied, so all that is left to prove are properties (3) and (4). Hence, we need to prove the following two claims:

Claim 1: for $y \in S$ and $0 < a < \infty$, $E \cap Q(y, -a, a)$ is bounded.

Claim 2: for $y \in S$, $\exists T > 0$ (depending on y), such that at least one of the sets $E \cap Q(y, -\infty, -T)$ and $E \cap Q(y, T, +\infty)$ is empty.

Proof of Claim 1

Let $y \in S$ be fixed and define $\eta : I \to \mathbb{R}$ as follows

$$\eta(\xi) = \sum_{j=1}^{N} y_j \xi^j$$

or in other words $\eta(\xi) = \langle y, \psi(\xi) \rangle$.

It is clear that as $\xi \to \infty$, $|\eta(\xi)| \to \infty$.

The authors here claim that $\eta(\xi)$ is bounded either below or above on I. That is, for every $\xi \in I$, $\exists M$ (or M') such that $\eta(\xi) \leq M$ (or $M' \leq \eta(\xi)$) for all $\xi \in I$.

$\eta(\xi)$ exists for any $0 \leq \xi \leq b$ with $0 < b < \infty$, and in fact, by the extreme value theorem, has a maximum or a minimum in that interval. Hence as $\xi \to \infty$, $\eta(\xi) \to +\infty$ or $-\infty$ depending on y, and so we see that η is either bounded above or below on I.

From here we need to utilize the fact that η is either bounded below or above to show that $E \cap Q(y, -a, a)$ is bounded, which is equivalent to showing that $\{x \in E : | \langle x, y \rangle | < a\}$ is bounded.

Now, notice that $\forall x \in E$, $\exists \xi_x \in I$ and $x' \in B_N(1)$ such that

$$x = \psi(\xi_x) + x' \tag{3.4}$$

If we take $N = 2$ or 3, we can see this geometrically. Any point in E belongs to a sphere of radius 1 centered at a point on the curve represented by ψ and this follows from the definition of E.

Take the inner product in equation 3.4 with respect to y ($y \in S$ fixed) to obtain

$$\langle x, y \rangle = \langle \psi(\xi_x), y \rangle + \langle x', y \rangle = \eta(\xi_x) + O(1) \quad (\text{as } x \to \mathcal{A},\ x \in E) \tag{3.5}$$

So for $x \in E \cap Q(y, -a, a)$, $\eta(\xi_x) = \langle \psi(\xi_x), y \rangle$ is bounded.

To prove that $E \cap Q(y, -a, a)$ is bounded, we may argue by contradiction. Let $x \in E \cap Q(y, -a, a)$ and assume that $||x|| \to \mathcal{A}$, then $\xi_x \to \infty$ and thus $|\eta(\xi_x)| \to \infty$. But we have just shown that $|\eta(\xi_x)|$ is bounded. Therefore, there must exist an integer $M > 0$ such that $||x|| < M$ for all $x \in E \cap Q(y, -a, a)$.

Proof of Claim 2

We need to show that there is $T > 0$ such that at least one of the sets $E \cap Q(y, -\infty, -T)$ and $E \cap Q(y, T, +\infty)$ is empty. That is to say, either $\{x \in E : \langle x, y \rangle < -T\}$ or $\{x \in E : \langle x, y \rangle > T\}$ is empty, which is also equivalent to saying that either $\langle x, y \rangle > -T$ or $\langle x, y \rangle < T$ for all $x \in E$. In fact, this follows immediately from equation 3.5, because we have shown that η is either bounded below or above on I, so then it follows that $\langle x, y \rangle$ is either bounded below or above on I. Hence, there is such a $T > 0$ with either $\langle x, y \rangle < T$ or $-T < \langle x, y \rangle$ which completes the proof.

We have now understood the behavior of the Radon transform over lines and planes. But what about its injectivity on other curves? This is a whole study on its own. One interesting paper may be Cormack's [10], where he studies the Radon transform over a family of curves in the plane. What is of more interest to us however is the circular Radon transform which we will discuss in the next chapter and which is truly fascinating, both theoretically and practically.

CHAPTER 4

THE RADON TRANSFORM OVER CIRCLES

We reach now a major part of the thesis where we will explore the circular Radon transform. The circular or spherical Radon transform (for $\mathbb{R}^n$, $n > 2$) takes a function and integrates it over all spheres centered at a given set of centers in $\mathbb{R}^n$. It has been studied vastly is many areas such as integral geometry, PDEs, sonar and radar imaging, approximation theory and many others (see [11] for references). Perhaps the field that required an extensive study of the circular Radon transform is a newly developed medical imaging technique called Thermoacoustic tomography (TAT).

To understand why such a transform is required, we will give a brief description of TAT which was presented initially in [11].

Given a biological object understudy, a short microwave or radiofrequency electromagnetic pulse is sent through it. The cells of the object absorb the energy from the wave so that given an internal location x, an energy $H(x)$ is absorbed. It is known that a cancerous cell absorbs much more energy than normal cells. Thus at tumorous locations, we can see a sudden increase in the values of $H(x)$.

Let us see now how we can measure the values of $H(x)$ so that we can detect the tumors. After the energy from the microwave or the radio frequency is absorbed, heat results causing thermo elastic expansion of cells which emit pressure waves. Such pressure waves are detected by transducers placed outside of the object understudy.

One can now measure effectively the integrals of $H(x)$ over all spheres centered at the locations of the transducers.

So then we ask the following questions: Is it possible to obtain the value of $H(x)$ from the data measured and would the reconstruction of $H(x)$ be unique?

In this chapter we will answer these questions by studying the injectivity of the circular Radon Transform. Let us then begin by introducing some notation and main definitions.

4.1 Notation and Main Definitions

The following notation will be used throughout the coming sections:

• $C(\mathbb{R}^n)$: Continuous real-valued functions endowed with the topology of the uniform convergence on compact sets.

• $C_c(\mathbb{R}^n) \subseteq C(\mathbb{R}^n)$: Subspace of compactly supported functions on $C(\mathbb{R}^n)$.

• $M(n)$: the groups of rigid motions of $\mathbb{R}^n$.

• For $N \in \mathbb{N}$ denote by Σ_N the Coxeter system of N lines $L_0, \ldots, L_{N-1}$ in the plane where $L_k = \{te^{ik\pi/N} | -\infty < t < +\infty\}$.

Definition 4.1.1. The Radon transform over spheres is given by:

$$Rf(x, r) = \int_{S(x,r)} f dA, \; f \in C_c(\mathbb{R}^n).$$

Here $x \in \mathbb{R}^n$, $r \in \mathbb{R}_+ = (0, \infty)$, $S(x, r)$ denotes the sphere centered at x and of radius r; dA is the normalized area measure on $S(x, r)$.

Definition 4.1.2. The Radon transform R is said to be injective on a set S if for any $f \in C_c(\mathbb{R}^n)$ the condition $Rf(x, r) = 0$, $\forall r \in \mathbb{R}_+$ and $\forall x \in S$ implies that $f \equiv 0$. S is thus called a set of injectivity of the Radon transform R.

The objective from here is to explore the injectivity sets of the spherical Radon transform and eventually be able to get a clear understanding or perhaps a solution of the following problem:

Problem: Describe all sets of injectivity for the Radon transform on $C_c(\mathbb{R}^n)$.

It has not been an easy task to characterize such sets of injectivity. Agranovsky and Quinto [3], two leading mathematicians in this area, based their results and proofs on the geometry of zero sets for harmonic polynomials and microlocal analysis of the circular Radon transform to finally obtain the following theorem:

Theorem 4.1.1. *The following condition is necessary and sufficient for S to be a set of injectivity for the Radon transform R over circles*
The set S is not contained in any set of the form $\omega(\Sigma_N) \cup F$, where $\omega \in M(2)$ and F is a finite set.

Notice that this theorem is only proved in $\mathbb{R}^2$. In fact, the work done by Agranovsky and Quinto [3] was considered to be a major breakthrough in the field. However, their methods were rather restrictive to the plane, so there was a need to find alternative ways, particularly those that use simple PDE techniques, that would allow for generalization to higher dimensions. Ambartsoumian and Kuchment discuss this matter in [11]; they do not prove theorem 4.1.1 but they open the roads to understand the problem in a different perspective so that we can get closer to solving the following conjecture:

Conjecture: *The following condition is necessary and sufficient for S to be a set of injectivity for the circular Radon transform on $C_c(\mathbb{R}^n)$:*
S is not contained in any set of the form $\omega(\Sigma) \cup F$, where $\omega \in M(\mathbb{R}^n)$, Σ is the zero set of a homogeneous harmonic polynomial, and F is an algebraic subset of $\mathbb{R}^n$ of co-dimension at least 2.

The subject is quite broad and we can tackle the problem from different directions. Nevertheless, we choose to present the study done by Agranovsky and Quinto [3] as it holds some marvelous ideas and connections. Because the topic is rather massive, we may state some results without proof, particularly those relating to microlocal Fourier analysis as they require a lot of background information.

Let us first begin by setting up an algebraic characterization of the sets of non-injectivity of the circular Radon transform.

4.2 Algebraic Characterization

Definition 4.2.1. Let $f \in C_c(\mathbb{R}^n)$ define

$$S[f] = \{x \in \mathbb{R}^n | Rf(x,r) = 0, \ \forall r \in \mathbb{R}_+\}$$

Let

$$Q_k = Q_k[f] = r^{2k} * f, \quad r^2 = x_1^2 + x_2^2 + \cdots + x_n^2. \tag{4.1}$$

which is the infinite family of polynomials of degree $\leq 2k$ associated with f. That is for each function f we have

$$Q_k(x) = Q_k[f](x) = \int_{\mathbb{R}^n} ||x - \xi||^{2k} f(\xi) d\xi.$$

with $||.||$ the usual norm on $\mathbb{R}^n$.

Finally, let the zero set of any polynomial Q with real coefficients be denoted by

$$V[Q] = \{x \in \mathbb{R}^n | Q(x) = 0\}.$$

We connect the three definitions above by the following lemma:

Lemma 4.2.1. $S[f] = \cap_{k=0}^{\infty} V[Q_k]$.

Proof. First we remember that any $y \in \mathbb{R}^n$ except for $y = 0$ can be represented uniquely as $y = ru$ where $r \in \mathbb{R}_+$ and $u \in S_{n-1}$ (Here S_{n-1} is the unit sphere in $\mathbb{R}^n$). So that, $\mathbb{R}^n - \{0\}$ can be expressed as the cartesian product $(0, \infty) \times S_{n-1}$. One can then prove the following formula

$$\int_{\mathbb{R}^n} f dm_n = \int_0^{\infty} r^{n-1} dr \int_{S_{n-1}} f(ru) d\sigma_{n-1}(u) \tag{4.2}$$

is valid for every non-negative Borel function f on $\mathbb{R}^n$.
Here, m_n is the Lebesgue measure on $\mathbb{R}^n$ and σ_{n-1} is a measure defined on S_{n-1} as follows: If $A \subset S_{n-1}$ and A is a Borel set, define

$$\sigma_{n-1}(A) = n.m_n(\tilde{A})$$

where $\tilde{A}$ is defined to be the set of all points ru, where $0 < r < 1$ and $u \in A$. (See exercise 6 chapter 8 in Rudin's Real and Complex Analysis).

Notice that the formula would work for $f \in C_c(\mathbb{R}^n)$.
Letting $\xi = ru + x$, where x is some point in $\mathbb{R}^n$ and due to invariance of Lebesgue measure under translation we obtain the following from equation 4.2

$$\int_{\mathbb{R}^n} f(y) dm_n(y) = \int_0^{\infty} r^{n-1} dr \int_{S_{n-1}} f(\xi) d\sigma_{n-1}(u)$$

Now, $\xi \in S(x, r)$ (which is the sphere of center x, radius r in $\mathbb{R}^{n-1}$), so we would have

$$d\sigma(\xi) = r^{n-1} d\sigma_{n-1}(u) \quad for\ u \in S_{n-1},\ \xi \in S(x, r).$$

Thus,

$$\int_{\mathbb{R}^n} f(y) dm_n(y) = \int_0^{\infty} \left(\int_{S(x,r)} f(\xi) d\sigma(\xi) \right) dr \tag{4.3}$$

Take $g(\xi) = ||x - \xi||^{2k} f(\xi)$ then $g \in C_c(\mathbb{R}^n)$. Substitute g in equation 4.3 to obtain

$$\begin{aligned}\int_{\mathbb{R}^n} ||x - \xi||^{2k} f(\xi) dm_n(\xi) &= \int_0^\infty \int_{S(x,r)} ||x - \xi||^{2k} f(\xi) d\sigma(\xi) dr \\ &= \int_0^\infty r^{2k} \int_{S(x,r)} f(\xi) d\sigma(\xi) dr \\ &= \int_0^\infty r^{2k} Rf(x,r) dr\end{aligned}$$

If $x \in S[f]$ then $Rf(x,r) = 0 \ \forall r > 0$ and thus $\int_{\mathbb{R}^n} ||x - \xi||^{2k} f(\xi) dm_n(\xi) = 0 \ \forall k$ so that $x \in \cap_{k=0}^{\infty} V(Q_k)$ and therefore $S[f] \subseteq \cap_{k=0}^{\infty} V(Q_k)$.

Let us suppose now that $x \in \cap_{k=0}^{\infty} V[Q_k]$ which implies that $\int_{\mathbb{R}^n} ||x - \xi||^{2k} f(\xi) dm_n(\xi) = 0$ for any k. We want to show $Rf(x,r) = 0$ for any $r > 0$. Since f is compactly supported, we see that for a fixed x, $Rf(x,r)$ is also compactly supported as a function of r. We also claim that $Rf(x,r)$ is a continuous function of r.

To see this, take $r, r' \in (0, \infty)$ with $r > r'$ and consider

$$\begin{aligned}|Rf(x,r) - Rf(x,r')| &= \left| \int_{S(x,r)} f ds - \int_{S(x,r')} f ds \right| \\ &= \left| \int_{S(x,r) - S(x,r')} f ds \right| \\ &\leq \int_{S(x,r) - S(x,r')} |f| ds \leq M \int_{S(x,r) - S(x,r')} ds \\ &= Mc(r^{n-1} - r'^{n-1}) \to 0 \ as \ r \to r'.\end{aligned}$$

M here is the supremum of f on $S(x,r) - S(x,r')$ and c is a constant.

$\therefore Rf(x,r) \in C_c((0,\infty))$ for a fixed $x \in \mathbb{R}^n$.

Now notice that if $\int_{\mathbb{R}^n} ||x - \xi||^{2k} f(\xi) d\xi = 0$ then equivalently for any polynomial α defined on $(0, \infty)$, the following holds

$$\int_{\mathbb{R}^n} \alpha(||x - \xi||^2) f(\xi) d\xi = 0$$

which, by equation 4.3, is equivalent to

$$\int_0^\infty \alpha(r^2)Rf(x,r)dr = 0$$

By Müntz -Szasz theorem, we know that all finite linear combinations of functions of the form $1, r^2, r^4, \ldots, r^{2k}, \ldots$ are dense in the space of continuous functions over compact support because $\sum_{k=1}^{\infty} \dfrac{1}{2k} = \infty$. Originally, the theorem is taken over the space of continuous complex functions on the closed unit interval, but since we are working on a compact set (due to the fact that Rf is of compact support), we can always do the following:
Take a linear transformation L that takes the unit interval to the compact support of Rf. Thus, by the Müntz -Szasz theorem, there is a sequence of polynomials spanned by $\{1, r^2, r^4, \ldots\}$ such that $\alpha_n \to (Rf \circ L)$ on the interval $[0,1]$, which implies that the sequence of polynomials $\alpha_n \circ L^{-1} \to Rf$ on the compact support of Rf. So then we can see that

$$\int_0^\infty \alpha_n(L^{-1}(r))Rf(x,r)dr = 0$$
$$\lim_{n\to\infty} \int_0^\infty \alpha_n(L^{-1}(r))Rf(x,r)dr = 0$$

so that by uniform convergence we have:

$$\int_0^\infty Rf^2(x,r)dr = 0$$

which implies that $Rf(x,r) = 0$ for any $r > 0$. And therefore the lemma follows.

□

Consider next the following proposition:

Proposition 4.2.2. *Let* $f \in C_c(\mathbb{R}^n)$*. If* $f \equiv 0$ *then* $Q_k[f] \equiv 0$*, for all* $k \in \mathbb{N}$*. If* $f \not\equiv 0$*, then* $P = Q_{k_{min}}$ *(which is the non-trivial polynomial of minimal degree in* *definition 4.2.1 equation 4.1) is harmonic. Denote this harmonic polynomial P as $P[f]$.*

Proof. Using the previous Lemma 4.2.1, $S[f] = \cap_{k=0}^{\infty} V[Q_k[f]]$, so if $f \equiv 0$, then $S[f] = \mathbb{R}^n$ which implies that $V[Q_k[f]] = \mathbb{R}^n$ for every $k \in \mathbb{N}$ and so $Q_k[f] \equiv 0$, $\forall k \in \mathbb{N}$.

Now assume $Q_k[f] \equiv 0,\ \forall k \in \mathbb{N}$, then $V[Q_k[f]] = \mathbb{R}^n\ \forall k$ so that $\cap_{k=0}^{\infty} V[Q_k[f]] = \mathbb{R}^n$ and thus $S[f] = \mathbb{R}^n$.

Hence, $Rf(x, r) = 0$ for all $x \in \mathbb{R}^n$, that is the integral of f vanishes over all spheres and so $f \equiv 0$.

To see the second part of the proposition, notice the following relation

$$\Delta Q_k = 2k(2k + n - 2)Q_{k-1}$$

To obtain this relation, start form $Q_k[f] = \int_{\mathbb{R}^n} ||x - \xi||^{2k} f(\xi)d\xi$ with $x = (x_1, x_2, \ldots, x_n)$. which is equivalent to $Q_k[f] = \int_{\mathbb{R}^n} [(x_1 - \xi_1)^2 + \cdots + (x_n - \xi_n)^2]^k f(\xi)d\xi$ where $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$.

Now evaluate $\dfrac{\partial Q_k}{\partial x_1}$ and $\dfrac{\partial^2 Q_k}{\partial {x_1}^2}$ to obtain the final result

$$\frac{\partial^2 Q_k}{\partial {x_1}^2} = 4k(k-1) \int_{\mathbb{R}^n} ||x - \xi||^{2k-4} (x_1 - \xi_1)^2 f(\xi)d\xi + 2k \int_{\mathbb{R}^n} ||x - \xi||^{2k-2} f(\xi)d\xi$$

which is the same as

$$\frac{\partial^2 Q_k}{\partial {x_1}^2} = 4k(k-1) \int_{\mathbb{R}^n} ||x - \xi||^{2k-4} (x_1 - \xi_1)^2 f(\xi)d\xi + 2kQ_{k-1}$$

Taking the 2nd partial derivative of Q_k with respect to every x_i for $i = 0, 1, \ldots, n$, we obtain similarly

$$\Delta Q_k = 2knQ_{k-1} + 4k(k-1) \int_{\mathbb{R}^n} ||x - \xi||^{2k-4} [(x_1 - \xi_1)^2 + \cdots + (x_n - \xi_n)^2] f(\xi)d\xi$$

which is equivalent to

$$2knQ_{k-1} + 4k(k-1)\int_{\mathbb{R}^n} ||x-\xi||^{2k-2} f(\xi)d\xi$$

so that finally we obtain $\Delta Q_k = 2knQ_{k-1} + 4k(k-1)Q_{k-1}$ and hence

$$\Delta Q_k = 2k(2k+n-1)Q_{k-1} \tag{4.4}$$

as required. Now since we are assuming that $f \not\equiv 0$ and if $P = Q_{k_{min}}$, we claim that $Q_{k_{min}-1} \equiv 0$ so that using the previous relation 4.4, $\Delta P = 0$ and thus P will be a harmonic polynomial.

To validate this, suppose that $Q_0(x) \neq 0$; this would imply that $\int_{\mathbb{R}^n} f(\xi)d\xi \neq 0$, also $P(x) = Q_0(x) = \int_{\mathbb{R}^n} f(\xi)d\xi$ which is harmonic. Now take $Q_0(x) \equiv 0$, and assume $Q_1(x) \neq 0$ then $P(x) = Q_1(x)$ and by equation 4.4 we obtain $\Delta Q_1 = 2(1+n)Q_0 = 0$ so then P is again harmonic. By choice, our polynomial $P(x) = Q_{k_{min}}$ is always such that $Q_{k_{min}-1} \equiv 0$.

□

Now that we have enough information, we can infer the following about injectivity sets of the spherical Radon transform:

(*)If the Radon transform R is not injective on a set S, then S is the zero set of a harmonic polynomial.

To confirm this, let S be a non-injectivity set of the Radon transform R. Take $f \in C_c(\mathbb{R}^n)$ with $f \neq 0$ and such that $Rf(x,r) = 0$ for every $x \in S$ and $r > 0$. Thus $S \subset S[f] = \cap_{k=0}^{\infty} V[Q_k] \subset V[P]$ where P is the harmonic polynomial presented in proposition 4.2.2. Therefore, S is the zero set of a harmonic polynomial.

From (*) follows the important corollary which solves the uniqueness problem of the spherical Radon transform for spherical locations of centers. Let us state and prove it.

Corollary 4.2.3. *If S is the uniqueness set of harmonic polynomials, then S is an injectivity set of the spherical Radon transform R.*

Let us define what is meant by *uniqueness sets of harmonic polynomials* before we proceed with the proof. Given any harmonic polynomial h, we say that S is an injectivity set on h, if the following holds:
If $h(x) = 0$ for every $x \in S$, then $h \equiv 0$. In other words, the harmonic polynomial is uniquely determined by its values on S.
For example, any sphere is a uniqueness set of a harmonic polynomial. This can be seen using the mean value property of harmonic functions. That is, since h is harmonic in $\mathbb{R}^n$, it is then harmonic over any sphere $S(a, r)$ in $\mathbb{R}^n$ (where $a \in \mathbb{R}^n$ is the center and r is some radius), and so the mean value property of harmonic functions tells us that:

$$h(a) = \frac{1}{n\omega_n r^{n-1}} \int_{S(a,r)} h d\sigma$$

(ω_n being the volume of the unit sphere in n dimensions and σ being the n-1 dimensional surface measure).

Let us now prove corollary 4.2.3.

Proof. Let S be a uniqueness set of harmonic polynomials and take $f \in C_c(\mathbb{R}^n)$. Assume that $f \not\equiv 0$ and that $Rf(x, r) = 0$ for any $x \in S$ and any $r \in \mathbb{R}_+$. Thus, we are assuming that R is not injective on S, so there is a harmonic polynomial h such that $S = V[h]$ by (*). Hence, for any $x \in S$, $h(x) = 0$. Since S is by hypothesis a uniqueness set of harmonic polynomials, we obtain that $h \equiv 0$ and so, $S = \mathbb{R}^n$, but also $S \subseteq S[f]$. This implies that $S[f] = \mathbb{R}^n$ and so $Q_k[f] \equiv 0$ for any $k \in \mathbb{N}$.

Therefore, $f \equiv 0$.

$\square$

Therefore, using the previous corollary, we see that spheres are injectivity sets of the Radon transform.

Before we proceed, we will introduce in a separate subsection an example which Agranovsky and Quinto included in their work [3] that will also show the importance of choosing functions in $C_c(\mathbb{R}^n)$. In fact, if these were replaced by bounded functions, even functions that vanish at infinity, corollary 4.2.3 will no longer hold.

4.2.1 An Example

Consider the following spherical function ϕ on $\mathbb{R}^n$ defined as

$$\phi(x) = J_k(||x||)||x||^{-k}$$

where $k = (n-2)/2$ and J_k is the Bessel function. The authors state that corollary 4.2.3 fails for ϕ.
They claim that if $E = \{x \in \mathbb{R}^n : ||x|| = \lambda \neq 0, J_k(\lambda) = 0\}$, then E is a uniqueness set of harmonic polynomials, yet it is not an injectivity set of the spherical Radon transform R.

It is easy to see that the set E is a union of spheres which are uniqueness sets of harmonic polynomials by the previous discussion. However, proving that $R\phi(x, r) = 0$ for every $x \in E$ and any $r > 0$ requires some background in the theory of spherical functions. It basically uses the general integral equation for spherical functions. We refer the readers to proposition 2.4 in Chapter 4 of [12].

This next proposition will set us off to the next section of this chapter which deals with asymptotic analysis of the algebraic curve $V[P]$. We now restrict our discussion to the case where $f \in C_c(\mathbb{R}^2)$.

Proposition 4.2.4. *$f \in C_c(\mathbb{R}^2)$, $f \neq 0$, assume that $S[f]$ is infinite. There is a non-constant polynomial $\psi = \psi[f]$ and a finite set F such that:*

1. *$S[f] = V[\psi] \cup F$, F is a finite set.*
2. *$V[\psi] = S_1 \cup S_2 \cup \cdots \cup S_m$, where S_j is real analytic, topologically connected curve in $\mathbb{R}^2$.*
3. *ψ divides $P[f]$.*

Proof. Since $f \neq 0$, we have $P = Q_{k_{min}}$ is harmonic. So now let us decompose P into a product of irreducible polynomials $P = P_1 \dots P_l$. Choose k to be any arbitrary integer and consider Q_k. Bezout Theorem tells us that for real algebraic curves, the number of points of intersection

$$\#V[Q_k] \cap V[P_i] \leq degQ_k.degP_i$$

whenever Q_k and P_i have a common divisor (that is when P_i divides Q_k since P_i is irreducible).

Now pick $P_{i_1}, \dots, P_{i_m}$ to be all irreducible factors of P such that for any $\alpha = 1, \dots, m$, $V[P_{i_\alpha}] \cap V[Q_k]$ is infinite $\forall k \in \mathbb{N}$ (this is possible because $S[f]$ is infinite by hypothesis).

Using lemma 4.2.1, we know that $S[f] = \cap_{k=0}^{\infty} V[Q_k]$, which implies that

$$S[f] = V[P] \cap V[Q_{k_{min}+1}] \cap \dots$$

$$S[f] = (V[P_1] \cup \cdots \cup V[P_l]) \cap V[Q_k] \cap \dots$$

$$S[f] = (V[P_{i_1}] \cup \cdots \cup V[P_{i_m}]) \cup \underbrace{(V[P_i] \cap V[Q_k] \cap \dots)}_{which\ is\ finite}$$

Therefore, we finally obtain

$$S[f] = V[P_{i_1}] \cup \cdots \cup V[P_{i_m}] \cup F$$

where F is a finite set.

Define now $\psi = P_{i_1} \dots P_{i_m}$. This is just the greatest common divisor of the Q_ks. It is easy to see that ψ satisfies properties (1) and (3) by construction, so we need to only verify property (2) so that we have a complete proof.

$V[\psi]$ (the zero set of ψ) is a real algebraic curve. Take x_0 to be a singular point of $V[\psi]$. Thus, the gradient $\nabla\psi(x_0) = 0$. Without loss of generality, we may assume $x_0 = 0$ (otherwise we take a translation).

Let $\psi = \psi_k + (summands\ of\ higher\ degree)$ be the decomposition of ψ into homogeneous polynomials.

Since ψ is the divisor of a non-zero harmonic polynomial, we claim that $\psi_k = l_1 \dots l_k$, where l_j are linear functions defining k lines $l_j = 0$ where the angles between them are rational multiples of π.

This is due to the following theorem which we will state as a lemma but not prove. (see [13] for proof).

Lemma 4.2.5. *Given $p = p_n + p_{n-1} + \cdots + p_m$ where p_j $(m \leq j \leq n)$ is a homogeneous polynomial of degree j. A necessary condition for the existence of a harmonic polynomial u such that $p|u$ is that $p_n = \prod_{i=1}^{n} L_i$, $p_m = \prod_{i=1}^{m} K_i$ where L_is and K_is are real homogeneous linear factors; The angle between any two lines $L_i = 0$ $(1 \leq i \leq n)$ or $K_i = 0$ $(1 \leq i \leq m)$ is a rational multiple of π. If p is homogeneous, then the* *above condition is sufficient.*

We would like to write $V[\psi]$ as a union of real analytic curves. To do this we consider $\psi = 0$ is polar coordinates, divide by r^k and apply the implicit function theorem. Doing this provides us with k smooth curves that intersect transversely in a neighborhood of our singular point x_0. Also we see that the lines $l_i = 0$, where l_i for $i = 0, 1, \ldots, k$ are the linear factors constituting ψ_k, are each tangent to one of the smooth curves obtained in a neighborhood of $x_0 = 0$. (This will be written out rigorously in the next section 4.3).

So then, in a neighborhood of its singular points, we see that $V[\psi]$ is a union of k, non-singular smooth curves which intersect transversely at this singular point. Note that there are no self intersections, otherwise by the maximum modulus principle $P \equiv 0$. Hence, globally, $V[\psi]$ is the union of k smooth curves as is required of part (2) of the proposition.

$\square$

Notice throughout the proof of proposition 4.2.4, we begin to see how the Coxeter system of lines in the statement of theorem 4.1.1 starts to come into the picture. Knowing this, we will dedicate the next section to the asymptotic analysis of the algebraic curves $V[P]$ and $V[\psi]$.

4.3 Asymptotic Analysis Of $V[P]$ and $V[\psi]$

In this section, we have that $f \in C_c(\mathbb{R}^2)$ is a non-zero function. Assume that $S[f]$ is infinite. By proposition 4.2.2, we know that $P[f]$ is harmonic and during the proof of proposition 4.2.4, we introduced the function $\psi = gcd_k(Q_k)$ that is $\psi = P_{i_1} \ldots P_{i_m}$ where $P_{i_j} \cap V[Q_k]$ is infinite for every k.

From proposition 4.2.3, we know that $S[f]$ is contained in the zero set of the harmonic polynomial P. Thus it is important to study $V[P]$ so that we can obtain further information about the geometric properties of the set $S[f]$. Let us then proceed with the analysis.

4.3.1 Presentation of P as a product of linear factors

Let us first represent P as follows:

$$P(z) = Im(c_N z^N + c_{N-1} z^{N-1} + \cdots + c_0), \quad z = x + iy$$

We use a normalization to obtain $c_{N-1} = 0$ and $c_N > 0$ (this step is important for the upcoming discussions).
We may now let $P_k = Im(c_k z^k)$ for $k = 0, \ldots, N$ so that we have

$$P = P_N + P_{N-2} + \cdots + P_0$$

which is the decomposition of P into a sum of homogeneous polynomials. Because we chose $c_N > 0$, we see that P_N vanishes on the points of the form $z = re^{ik\pi/N}$ with $r \in \mathbb{R}_+$ for any $k = 0, \ldots, N-1$. This is due to the fact that

$$P_N = Im(c_N z^N) = c_N Im(e^{ik\pi}) = c_N \sin(k\pi) = 0.$$

Therefore, we can write P_N as follows:

$$P_N(x, y) = cst \prod_{k=0}^{N-1} (a_k x + b_k y)$$

where $a_k = \sin\dfrac{k\pi}{N}$ and $b_k = -\cos\dfrac{k\pi}{N}$. So P_n now is represented as a product of linear factors.

Let us explain this factorization in further detail. Since P_N vanishes on $z_k = re^{ik\pi/N}$ for $r \in \mathbb{R}$, the terms of the product must be of the form $Im\left(1 - \dfrac{z}{z_k}\right)$. For $z = x + iy$,

$$\begin{aligned}1 - \frac{z}{z_k} = 1 - \frac{z}{r}e^{-ik\pi/N} &= 1 - \frac{1}{r}(x+iy)\left(\cos\frac{k\pi}{N} - i\sin\frac{k\pi}{N}\right) \\ &= 1 - \frac{1}{r}\left[x\cos\frac{k\pi}{N} + y\sin\frac{k\pi}{N} - ix\sin\frac{k\pi}{N} + iy\cos\frac{k\pi}{N}\right]\end{aligned}$$

so then $Im\left(1 + \dfrac{z}{z_k}\right) = \dfrac{1}{r}\left(x\sin\dfrac{k\pi}{N} - y\cos\dfrac{k\pi}{N}\right)$
which verifies the product $P_N(x,y) = cst\prod_{k=0}^{N-1}\left(x\sin\dfrac{k\pi}{N} - y\cos\dfrac{k\pi}{N}\right)$.

4.3.2 Asymptotic analysis of $V[P]$

Before we proceed, remember that ψ is a divisor of the harmonic polynomial P and thus $V[P]$ has a similar structure as that of $V[\psi]$. Hence, due to proposition 4.2.4, we see that $V[P] = G_1 \cup G_2 \cup \cdots \cup G_N$, where each G_i is a real algebraic topologically connected curve in $\mathbb{R}^2$. We will call these the non-singular components of $V[P]$ throughout this section.

Notice that all the non-singular components of $V[P]$ are unbounded. To prove this, assume that some singular component G_i is bounded. Since algebraic curves are topologically closed sets, we obtain that G_i is in fact a closed curve in the plane. Thus $P = 0$ throughout the interior of G_i by the maximum modulus principle for harmonic functions and is thus identically zero throughout the whole plane. This is a contradiction to the fact that P is non-trivial.

Denote by L_k the line $L_k = \{(x,y)|a_kx + b_ky = 0\}$ and divide each L_k in half to obtain the following rays

$$L_k^{\pm} = \{te^{ik\pi/N}, t \in \mathbb{R}_{\pm}\}$$

Then, we can observe the following properties:

1. Each ray $L_k^{\pm}$ is an asymptote for some non-singular component of the algebraic curve $V[P]$.

   Proof. This can be verified by applying the implicit function theorem. We will present this in detail in the next subsection 4.3.3 during the asymptotic analysis of $V[\psi]$.

   $\square$

2. Each non-singular component has two asymptotes, each of which is one of the $2N$ rays $L_0^{\pm}, L_1^{\pm}, \ldots, L_{N-1}^{\pm}$.

   Proof. Let us take $P = 0$ in polar representation. So then we'll have for $z = re^{i\theta}$,

   $$Im(c_N r^N e^{iN\theta} + c_{N-2} r^{N-2} e^{i(N-2)\theta} + \cdots + c_0) = 0$$

   Divide by r^N to obtain the following

   $$Im(c_N e^{iN\theta} + \frac{c_{N-2}}{r^2} e^{i(N-1)\theta} + \cdots + \frac{c_0}{r^N}) = 0$$

   Now let r tend to infinity to obtain

   $$Im(c_N e^{iN\theta}) = 0 \overset{c_N \neq 0}{\Longleftrightarrow} \sin(N\theta) = 0 \iff \theta = \frac{k\pi}{N}$$

with $k = 0, 1, \ldots, N-1$. So then $L_k^{\pm}$ are asymptotes of $V[P] = G_1 \cup \cdots \cup G_N$. Therefore, each singular component G_i has two asymptotes of the $2N$ rays. Note that having the normalization $c_{N-1} = 0$ and $c_N > 0$ guarantees that the asymptotes coincide with two of the rays.

□

3. No ray $L_k^{\pm}$ can be the asymptote for two different non-singular components of $V[P]$.

   Proof. Noting that the zeros of the spherical harmonic $P_N(\cos\theta, \sin\theta)$ are simple, we see that each ray must be the asymptote of one curve G_i.

   □

4.3.3 Asymptotic analysis of $V[\psi]$

Lemma 4.3.1. *Let $f \not\equiv 0$ and assume that $S[f]$ is an infinite set, then there is a collection of rays*

$$L_{i_1}^{\pm}, \ldots, L_{i_M}^{\pm}$$

where $M = deg\psi$ such that

1. *Each curve S_j in proposition 4.2.4 has two asymptotes among the rays.*

2. *Each ray is an asymptote for some S_j.*

3. *No ray serves as an asymptote for different curves S_i, S_j.*

Proof. From the second property of $V[P]$ in the previous subsection 4.3.2, and knowing that the cuves S_i are unbounded and that $S_i \subset V[\psi] \subset V[P]$, part (1) of this lemma follows clearly.

In order to verify the second property, we need to select rays among $L_k^{\pm}$ (defined in the previous subsection) which are asymptotes to $V[\psi]$. First we divide ψ into homogeneous polynomials

$$\psi = \psi_M + \psi_{M-1} + \cdots + \psi_0$$

where $M = deg(\psi)$.

Since ψ divides P, ψ_M divides P_N and therefore ψ_M can be expressed as a product of linear factors as follows

$$\psi_M(x, y) = cst \prod_{\alpha}^{M} (a_{k_\alpha} x + b_{k_\alpha} y)$$

we remember that $a_{k_\alpha} = \sin \dfrac{k_\alpha \pi}{N}$ and $b_{k_\alpha} = -\cos \dfrac{k_\alpha \pi}{N}$.

Let us now set $\psi = 0$ and rewrite it in polar coordinates. To do this consider first

$$\begin{aligned}
\psi_M(r\cos\theta, r\sin\theta) &= cst \prod_{\alpha=1}^{M} (r a_{k_\alpha} \cos\theta + r b_{k_\alpha} \sin\theta) \\
&= r^M cst \prod_{\alpha=1}^{M} (a_{k_\alpha} \cos\theta + b_{k_\alpha} \sin\theta) \\
&= r^M \psi_M(\cos\theta, \sin\theta).
\end{aligned}$$

thus we obtain the following expression for $\psi = 0$ in polar

$$r^M \psi_M(\cos\theta, \sin\theta) + r^{M-1}\psi_{M-1}(\cos\theta, \sin\theta) + \cdots + \psi_0(\cos\theta + \sin\theta) = 0$$

now divide by r^M in order to obtain

$$\psi_M(\cos\theta, \sin\theta) + \frac{1}{r}\psi_{M-1}(\cos\theta, \sin\theta) + \cdots + \frac{1}{r^M}\psi_0(\cos\theta + \sin\theta) = 0$$

Set $\varepsilon = \dfrac{1}{r}$ and let $F(\varepsilon, \theta) = 0$ define $\psi = 0$ so that then we have

$$F(\varepsilon, \theta) = \psi_M(\cos\theta, \sin\theta) + \varepsilon\psi_{M-1}(\cos\theta, \sin\theta) + \cdots + \varepsilon^M \psi_0(\cos\theta + \sin\theta) = 0$$

Fix k_α and let $\theta_0 = \dfrac{k_\alpha \pi}{N}$ and notice that

$$\begin{aligned} F(0,\theta_0) &= \psi_M(\cos\theta_0, \sin\theta_0) \\ &= cst \prod_{\alpha=1}^{M} (a_{k_\alpha}\cos\theta_0 + b_{k_\alpha}\sin\theta_0) \\ &= cst \prod_{\alpha=1}^{M} (\sin\theta_0\cos\theta_0 - \cos\theta_0\sin\theta_0) \\ &= 0 \end{aligned}$$

Also notice that $\left.\dfrac{\partial F}{\partial \theta}\right|_{(0,\theta_0)} = \left.\dfrac{\partial \psi_M}{\partial \theta}\right|_{\theta_0}$.
It is easy to see that ψ_M can be written as

$$\begin{aligned} \psi_M &= cst \prod_{\alpha=1}^{M} \sin(\theta - \theta_\alpha) \\ &= cst \sin(\theta - \theta_{k_0})\sin(\theta - \theta_{k_1}) \ldots \sin(\theta - \theta_0) \ldots \sin(\theta - \theta_{k_M}) \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{\partial \psi_M}{\partial \theta} &= \cos(\theta - \theta_0)\left[\sin(\theta - \theta_{k_1}) \ldots \sin(\theta - \theta_{k_M})\right]\Big|_{\theta=\theta_0} \\ &\quad + \sin(\theta - \theta_0)\frac{d}{d\theta}\left[\sin(\theta - \theta_{k_1}) \ldots \sin(\theta - \theta_{k_M})\right]\Big|_{\theta=\theta_0} \end{aligned}$$

seeing that the second term is equal to zero, the expression becomes

$$\frac{\partial \psi_M}{\partial \theta} = \cos(\theta - \theta_0)\left[\sin(\theta - \theta_{k_1}) \ldots \sin(\theta - \theta_{k_M})\right]\Big|_{\theta=\theta_0} \neq 0$$

Thus, we may use the implicit function theorem to find $\theta = \theta(r)$ which satisfies $F(\varepsilon, \theta) = 0$ in a neighborhood of $\varepsilon = 0$, $\theta = \theta_0$ or equivalently $r = \infty$ and $\theta = \theta_0$.

Let us obtain a description of the asymptotic behavior of $\theta = \theta(\varepsilon)$ or equivalently $\theta = \theta(r)$. We know that $\theta(\varepsilon)$ is the solution of $F(\varepsilon, \theta)$. By the implicit function theorem we can obtain that as $\varepsilon \to 0$

$$\begin{aligned}
\frac{d\theta}{d\varepsilon} &= \frac{\partial F/\partial \varepsilon}{\partial F/\partial \theta} \\
&= \frac{\psi_{M-1}(\cos\theta_0, \sin\theta_0)}{\sin(\theta_0 - \theta_{k_1}) \ldots \sin(\theta_0 - \theta_{k_m})} + o(\varepsilon) \\
&= cst\psi_{M-1}(\cos\theta_0, \sin\theta_0) + o(\varepsilon)
\end{aligned}$$

so that then we obtain

$$\theta(\varepsilon) = cst.\psi_{M-1}(\cos\theta_0, \sin\theta_0)\varepsilon + o(\varepsilon) + \theta_0$$

which can be equivalently expressed as

$$\theta\left(\frac{1}{r}\right) = C\left(\frac{1}{r}\right) + o\left(\frac{1}{r}\right) + \theta_0$$

with $C = cst.\psi_{M-1}(\cos\theta_0, \sin\theta_0)$.

Thus, we see that the ray $L^+_{k_\alpha}$ is parallel to an asymptote of the curve $\theta = \theta(r)$, $r > r_0$. But, since this curve is a subset of $V[\psi]$ and $L^+_{k_\alpha}$ is the only half line parallel and in the same direction as an asymptote of $V[\psi]$, we conclude that $L^+_{k_\alpha}$ is an asymptote for the curve $\theta = \theta(r)$. The same discussion can be repeated for the case of $L^-_{k_\alpha}$. So then part (2) of this lemma 4.3.1 follows.

The third property is also inherited from property (3) of $V[P]$. Seeing that for each pair of m asymptotes corresponds a pair of M rays, we conclude that $m = M = \deg\psi$. $\square$

The next proposition is very important for the final proof of theorem 4.1.1. Although we will not provide a proof, we urge the readers to read its proof in [3].

Proposition 4.3.2. *Let $f \neq 0$ and assume that $S[f]$ is infinite. Only the two following cases are possible:*

1. *There exists $t \in \mathbb{R}^2$ such that the shifted polynomial $\psi^t(x) = \psi(x+t)$ is homogeneous;*

2. *At least two non-singular components S_i, S_j of $V[\psi]$ are disjoint.*

4.4 Proof of theorem 4.1.1

Before we proceed with the proof of theorem 4.1.1, a very important and fascinating result is needed. Perhaps the most significant of the work done on the subject by Agranovsky and Quinto in their paper [3], is the use of Micro-local Fourier Analysis to prove the support theorem. Although the proof branches into different concepts that are very interesting to revise and study regarding the circular Radon transform, we will not present it in this work. [3] contains the full proof.

Theorem 4.4.1. *(Support Theorem) Let S be a regular real-analytic curve (possibly disconnected). Assume that S contains two points, a and b, $a \neq b$, such that the segment $\overline{ab}$ is perpendicular to the tangent lines L_a and L_b at the point a and b respectively. Then, the Radon transform R is injective on S.*

Now that we are equipped with all the knowledge needed, we are ready to discuss the proof of theorem 4.1.1 by Agranovsky and Quinto [3].
We will state the theorem again

Theorem: *The following condition is necessary and sufficient for S to be a set of injectivity for the Radon transform R over circles:*
(*) The set S is not contained in any set of the form $\omega(\Sigma_N) \cup F$, where $\omega \in M(2)$ and F is a finite set.

We will divide the proof into two parts: *Sufficiency* and *Necessity* of the condition (*).

4.4.1 Sufficiency

Let us take a function $f \in C_c(\mathbb{R}^2)$ and consider the set $S[f]$. We wish to show that either $S[f] = \mathbb{R}^2$ or $S[f] \subset \omega(\Sigma_N) \cup F$.

Notice that if $S[f] = \mathbb{R}^2$ then $f \equiv 0$.

So assume that $f \not\equiv 0$. By corollary 4.2.3, we know that uniqueness sets of harmonic polynomials are injectivity sets of the Radon transform, which indicates that $S[f] \neq \mathbb{R}^2$ ($\mathbb{R}^2$ is clearly a uniqueness set of harmonic polynomials).

If $S[f]$ is finite, then we are done. So assume that $S[f]$ is infinite. Therefore, by propositions 4.2.2 and 4.2.4, we have the polynomials $P = P[f]$ which is harmonic and $\psi = \psi[f]$ as defined earlier.

From proposition 4.3.2, we see that two cases are possible:

1. For some $t \in \mathbb{R}^2$, the shifted polynomial $\psi^t = \psi(x + t)$ is homogeneous. Thus, by proposition 4.2.4, we have that ψ^t divides the leading homogeneous part P_N^t of the shifted polynomial P^t. This implies that $V[\psi^t] \subset V[P_N^t]$ and we recall that $V[P_N]$ is in fact Σ_N.
Notice that $V[\psi] = V[\psi^t] + t$ and remember that in section 4.3, subsection 4.3.2, we have used a rotation and translation to normalize the polynomial P. Therefore $V[\psi] \subset \omega(\Sigma_N)$ for some $\omega \in M(2)$ so that proposition 4.2.4, which says $S[f] = V[\psi] \cup F$, yields the following

$$S[f] \subset \omega(\Sigma_N) \cup F.$$

2. At least two non-singular components, say S_1 and S_2 of $V[\psi]$ are disjoint. First of all, notice that the distance between S_1 and S_2 can never be infinite. This is because, by lemma 4.3.1, we know the curves S_1 and S_2 have two different and non-parallel asymptotes. Thus, we can find two points $a \in S_1$ and $b \in S_2$ such that

$$d = dist(S_1, S_2) = dist(a, b) > 0$$

We will now use the support theorem 4.4.1 on the regular real analytic curve $S_1 \cup S_2$. Let us first check if $S_1 \cup S_2$ satisfies the conditions of theorem 4.4.1.

We know that d is the minimal distance between the point $a \in S_1$ and any other point in S_2. This implies that the circle centered at a of radius d is tangent to S_2 at the point b. Note that this is emphasized by the fact that S_2 is a regular curve and that b is not an end point of S_2, so then the circle cannot meet S_2 transversally. Therefore, the segment $\overline{ab}$ is perpendicular to the tangent to S_2 at b, and similarly $\overline{ab}$ is perpendicular to the tangent to S_1 at a.

Thus, $S_1 \cup S_2$ satisfies the conditions of the support theorem and so the Radon transform R is injective on $S_1 \cup S_2$. Again, by proposition 4.2.4, we know that $S[f] = V[\psi] \cup F$ where F is a finite set, and thus $Rf(x, r) = 0$ for any $x \in S_1 \cup S_2$ and all $r > 0$. Therefore, because $S_1 \cup S_2$ is an injectivity set of the transform R, f is then identically zero, which contradicts our assumption.

This completes the proof of the fact that condition (*) is sufficient for S to be a set of non-injectivity for the Radon transform R over circles.

4.4.2 Necessity

To prove the necessity of the condition (*), we need to construct a non-zero function $f \in C_c(\mathbb{R}^2)$ such that $Rf(a,r) = 0$ for any $r > 0$ and every $a \in \Sigma_N \cup F$ where F is a finite set. Notice that taking a rigid motion $\omega \in M(2)$ here is not important. Consider now the following lemma:

Lemma 4.4.2. *For any function $f \in C_c(\mathbb{R}^2)$ of the form*

$$f(x) = \sum_{j=1}^{l} f_j(r)\sin(jN\theta), \qquad x = re^{i\theta}, \tag{4.5}$$

with l being some integer, the Radon transform $Rf(a,.) \equiv 0$ for all $a \in \Sigma_N$.

Proof. First, for $k = 0, 1, \ldots, N-1$, we claim that f is odd with respect to the reflection w_k about the line $L_k = \{te^{ik\pi/N} | t \in \mathbb{R}\} \subset \Sigma_N$, that is $f(w_k(x)) = -f(x)$ for all $x \in \mathbb{R}^2$.

To see this, pick $x = re^{i\theta}$, $r > 0$, and consider $f(w_k(x))$. w_k is a reflection about the line L_k which has an angle $\theta_k = \dfrac{k\pi}{N}$ with the x axis. So then it is easy to see that under the action of w_k, x will be rotated by an angle of $2(\theta_k - \theta)$. Thus

$$w_k(x) = re^{i(\theta + 2(\theta_k - \theta))} = re^{i(2\theta_k - \theta)}.$$

Now,

$$
\begin{aligned}
f(w_k(x)) &= f(re^{i(2\theta_k-\theta)}) \\
&= \sum_{j=1}^{l} f_j(r)\sin(jN(2\theta_k-\theta)) \\
&= \sum_{j=1}^{l} f_j(r)\sin(jN(\frac{2k\pi}{N}-\theta)) \\
&= \sum_{j=1}^{l} f_j(r)\sin(2k\pi j - jN\theta)) \\
&= \sum_{j=1}^{l} f_j(r)\left[\sin(2k\pi j)\cos(jN\theta)-\cos(2k\pi j)\sin(jN\theta)\right] \\
&= -\sum_{j=1}^{l} f_j(r)\sin(jN\theta) = -f(x)
\end{aligned}
$$

Thus for $C(a,r)$ being a circle of center a and radius r, we see that if $a \in L_k$, we have

$$\int_{C(a,r)} f dA = \int_{C(a,r)} (f \circ w_k) dA = -\int_{C(a,r)} f dA$$

so that $2\int_{C(a,r)} f dA = 0$ and therefore $Rf(a,r) = 0$ for any $r > 0$, and all $a \in L_k$.

$\square$

We still have to find a non-negative function that satisfies the remaining finite number of cases for which the Radon transform R is zero.
Let $F = \{a_1, a_2, \ldots, a_q\}$ with $a_s = r_s e^{i\theta_s}$, $s = 1, 2, \ldots, q$, and assume further that $a_s \notin \Sigma_N$. We need to find a function $f \in C_c(\mathbb{R}^2)$ such that $Rf(a_s, r) = 0$ for any radius $r > 0$ and any $s = 1, \ldots, q$.

We claim that if we solve the following condition

$$\int_0^{2\pi} f(a_s + ze^{i\theta})d\theta = 0 \qquad \text{for } \; s = 1, \ldots, q \; \text{ and } \; z = x + iy \tag{4.6}$$

then our function f would be the desired function.

To validate this, take for some center $a_s \in F$ and a positive radius r, $Rf(a_s, r) = 0$. This implies that $\int_{C(a_s,r)} f dA = 0$. If we let $z(\theta) = a_s + re^{i\theta}$ for a fixed $r > 0$ and since $dA = |z'(\theta)|d\theta$, we have $\int_0^{2\pi} f(a_s + re^{i\theta}) r d\theta = 0$ which is equivalent to $\int_0^{2\pi} f(a_s + re^{i\theta}) d\theta = 0$ since $r \neq 0$.

Notice now that this is true for any $r > 0$ because by hypothesis $Rf(a_s, r) = 0$.

So then if we define $F : \mathbb{C} \to \mathbb{R}$ as

$$F(z) = \int_0^{2\pi} f(a_s + ze^{i\theta}) d\theta$$

we see that $F(z) \equiv 0$.

The goal from here is to find this function f that satisfies the condition 4.6.

Before venturing into that, notice that the Fourier transform of F is well defined and $\hat{F}(\xi) = 0$ for all $\xi \in \mathbb{R}^2$. Also, we know that any rotation commutes with the Fourier transform. That is, if T is a rotation then $\widehat{f \circ T} = \hat{f} \circ T$.

Consider then

$$\begin{aligned}
\hat{F}(\xi) &= \int e^{-2\pi i \langle \xi, z \rangle} F(z) dz \\
&= \int e^{-2\pi i \langle \xi, z \rangle} \int_0^{2\pi} f(a_s + ze^{i\theta}) d\theta dz \\
&\overset{Fubini}{=} \int_0^{2\pi} \int e^{-2\pi i \left\langle \xi e^{i\theta}, z \right\rangle} f(a_s + z) dz d\theta \\
&= \int_0^{2\pi} e^{2\pi i \left\langle \xi e^{i\theta}, a_s \right\rangle} \hat{f}(\xi e^{i\theta}) d\theta
\end{aligned}$$

Here, we take $\left\langle \xi e^{i\theta}, a_s \right\rangle$ to be the real inner product in $\mathbb{R}^2$. Letting $\xi e^{i\theta} = \lambda e^{i\varphi}$, we obtain the following system of integral equations

$$\int_0^{2\pi} \hat{f}(\lambda e^{i\varphi}) e^{i\langle a_s, \lambda e^{i\varphi}\rangle} d\varphi = 0 \tag{4.7}$$

for all $\lambda \in \mathbb{C}$ and $s = 1, \dots, q$.

The plan from here is to find a solution for equations 4.7, that has the form of the functions introduced in lemma 4.4.2 and defined in equation 4.5. We do this in the following steps:

Step 1: Find the Fourier transform of functions of the form 4.5 in polar coordinates (ρ, φ). These then can be written as

$$\begin{aligned}
\hat{f}(\rho e^{i\varphi}) &= \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\infty} \sum_{k=1}^{l} f_k(r) \sin kN\theta e^{-i\langle \rho e^{i\varphi}, re^{i\theta}\rangle} r dr d\theta \\
&= \sum_{k=1}^{l} \frac{1}{2\pi} \int_0^{\infty} f_k(r) \left(\int_0^{2\pi} e^{-i\rho r \cos(\theta - \varphi)} \sin kN\theta d\theta \right) r dr
\end{aligned}$$

We can now do the following

$$\begin{aligned}
\sin(kN\theta) &= \sin(kN\theta + kN\varphi - kN\varphi) \\
&= \sin(kN\varphi)\cos(kN\theta - kN\varphi) + \cos(kN\varphi)\sin(kN\theta - kN\varphi)
\end{aligned}$$

and we note that

$$\int_0^{2\pi} \sin(kN(\theta - \varphi)) e^{-i\rho r \cos(\theta - \varphi)} d\theta = 0$$

so that we obtain the following

$$\hat{f}(\rho,\varphi) = \frac{1}{2\pi}\sum_{k=1}^{l}\int_0^\infty f_k(r)\sin kN\varphi\left(\int_0^{2\pi}\cos(kN(\theta-\varphi))e^{-i\rho r\cos(\theta-\varphi)}d\theta\right)rdr$$

We know that

$$J_n(z) = \frac{1}{2\pi i^n}\int_0^{2\pi} e^{iz\cos\theta}\cos n\theta d\theta$$

and by simple calculation we can obtain $J_n(-z) = i^{2n}J_n(z)$. so then

$$\begin{aligned}\hat{f}(\rho,\varphi) &= \frac{1}{2\pi}\sum_{k=1}^{l}\sin kN\varphi\int_0^\infty f_k(r)J_{kN}(-\rho r)(2\pi i^{kN})rdr\\ &= \sum_{k=1}^{l}\sin kN\varphi\int_0^\infty f_k(r)J_{kN}(\rho r)(i^{-kN})rdr\end{aligned}$$

(this is true because the integrands we are dealing with are periodic, so applying a change of variable doesn't affect the value of the integral after changing the bounds). Finally, we obtain

$$\hat{f}(\rho,\varphi) = \sum_{k=1}^{l} i^{-kN}\hat{f}_k(\rho)\sin(kN\varphi) \tag{4.8}$$

where $\hat{f}_k(\rho) = \int_0^\infty f_k(r)J_{kN}(r\rho)rdr$ is the Fourier Bessel transform.

Step 2: Substitute the equation 4.8 inside equation 4.7 in order to obtain the following

$$\begin{aligned}&\int_0^{2\pi}\sum_{k=1}^{l} i^{-kN}\hat{f}_k(\rho)\sin(kN\varphi)e^{i\langle a_s,\rho e^{i\varphi}\rangle}d\varphi = 0\\ \iff &\sum_{k=1}^{l} i^{-kN}\int_0^{2\pi}\hat{f}_k(\rho)sin(kN\varphi)e^{i\langle a_s,\rho e^{i\varphi}\rangle}d\varphi = 0\end{aligned}$$

Now since we took $a_s = r_s e^{\theta_s}$, we have

$$\begin{aligned}e^{i\langle a_s,\rho e^{i\varphi}\rangle} &= e^{i(r_s\cos\theta_s, r_s\sin\theta_s).(\rho\cos\varphi,\rho\sin\varphi)}\\ &= e^{ir_s\rho\cos(\varphi-\theta_s)}\end{aligned}$$

Notice again that

$$\begin{aligned}\sin(kN\varphi) &= \sin(kN\varphi + kN\theta_s - kN\theta_s)\\ &= \sin(kN\theta_s)\cos(kN\varphi - kN\theta_s) + \cos(kN\theta_s)\sin(kN\varphi - kN\theta_s)\end{aligned}$$

So then

$$\begin{aligned}&\sum_{k=1}^{l} i^{-kN}\hat{f}_k(\rho)\int_0^{2\pi}\sin(kN\varphi)e^{ir_s\rho\cos(\varphi-\theta_s)}d\varphi = 0\\ \iff &\sum_{k=1}^{l} i^{-kN}\hat{f}_k(\rho)\sin(kN\theta_s)\int_0^{2\pi}\cos(kN(\varphi-\theta_s))e^{ir_s\rho\cos(\varphi-\theta_s)}d\varphi\\ &+\sum_{k=1}^{l} i^{-kN}\hat{f}_k(\rho)\cos(kN\theta_s)\int_0^{2\pi}\sin(kN(\varphi-\theta_s))e^{ir_s\rho\cos(\varphi-\theta_s)}d\varphi = 0\end{aligned}$$

But, it can be seen clearly by using a substitution that

$$\sum_{k=1}^{l} i^{-kN}\hat{f}_k(\rho)\cos(kN\theta_s)\int_0^{2\pi}\sin(kN(\varphi-\theta_s))e^{ir_s\rho\cos(\varphi-\theta_s)}d\varphi = 0$$

So then we are left with

$$\sum_{k=1}^{l} i^{-kN}\hat{f}_k(\rho)\sin(kN\theta_s)\int_0^{2\pi}\cos(kN(\varphi-\theta_s))e^{ir_s\rho\cos(\varphi-\theta_s)}d\varphi = 0 \tag{4.9}$$

We can now apply the following formula again

$$\int_0^{2\pi} e^{i\beta\cos x}cosnxdx = 2i^n\pi J_n(\beta)$$

so that equation 4.9 becomes

$$\sum_{k=1}^{l}\hat{f}_k(\rho)\sin(kN\theta_s)(2\pi J_{kN}(r_s\rho)) = 0$$

We have thus obtained q linear equations for l functions. Let us take $l = q + 1$ to get

$$\sum_{k=1}^{q+1} M_{s,k}(\rho)\hat{f}_k(\rho) = 0 \qquad \text{for } s = 1, \ldots, q. \tag{4.10}$$

where $M_{s,k}(\rho) = \sin(kN\theta_s)J_{kN}(\rho r_s)$.

Denote by $M(\rho) = [M_{s,k}(\rho)]_{s,\ k=1}^{q,\ q+1}$ which is the matrix of the system presented in equation 4.10.

Step 3: Now the objective from here is to find a suitable solution for the system of linear equations obtained in equation 4.10. To do this, let us take

$$\bar{q} = \max \{\text{rank } (M(\rho))|\rho \in \mathbb{R}_+\}$$

and let ρ_0 be such that $M(\rho_0) = \bar{q}$. Since we chose $a_s \notin \Sigma_N$, we see that $M(\rho) \not\equiv 0$. Therefore, $\bar{q} > 0$ and there is a neighborhood W of ρ_0 such that $M(\rho) = \bar{q}$ for every $\rho \in W$. Also, some $\bar{q} \times \bar{q}$ minor of the matrix doesn't vanish on W.

Without loss of generality, take this $\bar{q} \times \bar{q}$ minor to be a principal minor and denote it by $\Delta(\rho)$.

Take the first $\bar{q}$ equations in the system presented in 4.10 and set $\hat{f}_{\bar{q}+1} = \cdots = \hat{f}_q = 0$. Define now a new truncated system of 4.10 as follows

$$\tilde{M}(\rho)\hat{F}(\rho) = -\hat{F}_{q+1}(\rho) \tag{4.11}$$

where $\tilde{M}(\rho) = [M_{s,k}(\rho)]_{s,k=1}^{\bar{q},\bar{q}}$, $\hat{F} = (\hat{f}_1, \ldots, \hat{f}_{\bar{q}})^T$ and $\hat{F}_{q+1} = (M_{1,q+1}\hat{f}_{q+1}, \ldots, M_{\bar{q},q+1}\hat{f}_{q+1})^T$.

Let us now solve 4.11. Let $\hat{f}_{q+1}(\rho) = \Delta(\rho)\hat{u}(\rho)$, where u is an arbitrary, fixed, smooth, non-zero radial function of compact support that satisfies $\Delta(\rho)\hat{u}(\rho) \not\equiv 0$ on W.

By Cramer's Rule, we can see that

$$\hat{f}_k(\rho) = -\Delta_k(\rho)$$

where $\Delta_k(\rho)$ is the determinant obtained by replacing the kth column in $\Delta(\rho)$ by the column

$$(M_{1,q+1}(\rho)\hat{u}(\rho), \dots, M_{\bar{q},q+1}(\rho)\hat{u}(\rho))^T$$

$\hat{f}_1, \dots, \hat{f}_{q+1}$ gives a solution to 4.11. For $\rho \in W$ this becomes a solution for the whole system as well. This is because the last $q - \bar{q}$ equations are linear combinations of the first $\bar{q}$ equations.

Notice that the functions we are dealing with that are in terms of ρ are real analytic so the solution becomes valid for all ρ.

To see this more clearly, and since $\hat{f}_k(\rho) = -\Delta_k(\rho)$, we can use Laplace expansion, to obtain

$$\Delta_k(\rho) = \hat{u}(\rho) \sum_{s=1}^{\bar{q}} \sin(kN\theta_s) J_{kN}(\rho r_s) A_{s,k}(\rho)$$

where $A_{s,k}(\rho)$ is the co-factor of the matrix $\tilde{M}_k(\rho)$, this being the matrix $\tilde{M}(\rho)$ with the kth column replaced by $(M_{1,q+1}(\rho)\hat{u}(\rho), \dots, M_{\bar{q},q+1}(\rho)\hat{u}(\rho))^T$ (so it is made up of $\hat{u}(\rho)$ and $J_{kN}(\rho r_s)$). We chose u to be smooth and of compact support, thus $\hat{u}$ is in $C^\infty(\mathbb{R}_+)$. The Bessel function is also a real analytic function of ρ.

From here we can define $\hat{f}$ as in equation 4.8. We claim that from the construction $\hat{f} \in L^2(\mathbb{R}^2)$.

To prove this, It is enough to show that $\sqrt{\rho}\hat{f}_k(\rho)$ is in $L^2(\mathbb{R}_+)$. Now, we know that

$\hat{f}_k$ are made up of functions of the form $\hat{u}(\rho)J_{kn}(\rho r_s)$. First notice that

$$\begin{aligned}
|J_{kN}(\rho r_s)|^2 &= \left|\frac{1}{\pi i^{kN}}\int_0^\pi e^{i\rho r_s \cos x}\cos(kNx)dx\right|^2 \\
&\leq \left(\frac{1}{\pi}\int_0^\pi |\cos(kNx)|dx\right)^2 \\
&= \left(\frac{2\sin(kN\pi/2)}{kN\pi}\right)^2 \qquad (k, N \neq 0)
\end{aligned}$$

which is a constant and can be taken out of the integral. So we only need to study the L^2 integrability of $\sqrt{\rho}\hat{u}(\rho)$, that is whether $\int_0^\infty |\hat{u}(\rho)|^2\rho d\rho$ is finite or not. Since we chose $u \in C_c^\infty(\mathbb{R}_+)$, we can say that for any $N > 0$ there is a constant C_N such that $|\hat{u}(\rho)| \leq \dfrac{C_N}{(1+\rho)^N}$ for all $\rho \in \mathbb{R}_+$, which makes $\hat{u} \in L^2(\mathbb{R}_+)$. Thus, consider $\int_0^\infty \dfrac{C_N^2\rho}{(1+\rho)^{2N}}d\rho$. We can see that $\int_0^1 \dfrac{C_N^2\rho}{(1+\rho)^{2N}}d\rho < \infty$, so then let us look at

$$\begin{aligned}
\int_1^\infty \frac{C_N^2\rho}{(1+\rho)^{2N}}d\rho &\leq \int_1^\infty \frac{C_N^2\rho}{1+\rho^{2N}}d\rho \\
&\leq \int_1^\infty \frac{C_N^2}{\rho^{2N-1}}d\rho < \infty.
\end{aligned}$$

We then obtain that $\sqrt{\rho}\hat{f}_k(\rho) \in L^2((0,\infty))$, and therefore, $\hat{f} \in L^2(\mathbb{R}^2)$.

Now, we claim that $\hat{f}$ has analytic extension to $\mathbb{C}^2$, as a function of exponential growth.

Analytic continuation of $\hat{f}$: First, we know that $\hat{f}(\rho,\varphi) = \sum_{k=1}^{q+1} i^{-kN}\hat{f}_k(\rho)\sin(kN\varphi)$ where $\rho \in \mathbb{R}_+$ and $\varphi \in \mathbb{R}$. It is easy to see that $\sin(kN\varphi)$ can be defined for $\varphi \in \mathbb{C}$. So let us then study the analytic continuation of $\hat{f}_k$. To do this, we need to consider the analytic the continuation of $\hat{u}(\rho)$ and $J_{kN}(\rho r_s)$. Since u was chosen to be of compact support, say supp$(u)\subset \{x \in \mathbb{C} : |x| < A\}$, then we see that $\hat{u}(z) = \int_{|x|<A} e^{-2\pi i z.x}u(x)dx$ converges absolutely for all $z \in \mathbb{C}$. This is due to the fact that $|e^{-2\pi i z.x}| = |e^{-2\pi i x Re(z)}e^{2\pi x Im(z)}| \leq e^{AIm(z)}$. As for the functions $J_{kN}(\rho r_s)$,

analytic continuation is possible due to the following relation

$$J_\nu(xe^{m\pi i}) = e^{m\nu\pi i}J_\nu(x) \qquad m \in \mathbb{Z}, \quad \nu \in \mathbb{C} \quad x \in \mathbb{R}.$$

Proof that $\hat{f}$ is of exponential type: We will show now that for non-negative integers N, there is $C_N > 0$ such that

$$|\hat{f}(z)| \leq C_N(1+|z|)^{-N}e^{A|Im(z)|} \qquad \forall z \in \mathbb{C}^2$$

Again we look at $\hat{f}_k(\rho) = \hat{u}(\rho)\sum_{s=1}^{\bar{q}}\sin(kN\theta_s)J_{kN}(\rho r_s)A_{s,k}(\rho)$ but after extension we have $\rho \in \mathbb{C}$. We know that that there is $C_N > 0$ for every non-negative integer N such that $|\hat{u}(\rho)| \leq \dfrac{C_N}{(1+|\rho|)^N}$ for all $\rho \in \mathbb{C}$. So let us consider now $J_{kN}(\rho r_s)$. A well known Bessel integral equation we can use is the following

$$J_n(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in\theta+iz\sin\theta}d\theta$$

so then $|J_n(z)| \leq \dfrac{1}{2\pi}\int_{-\pi}^{\pi} e^{|Im(z)|d\theta} = e^{|Im(z)|}$.
Therefore, $|J_{kN}(\rho r_s)| \leq e^{r_s|Im(\rho)|}$.
Functions of the form $|\hat{u}(\rho)J_{kN}(\rho)| \leq \dfrac{C_N}{(1+|\rho|)^N}e^{r_s|Im(\rho)|}$ for all $\rho \in \mathbb{C}$ are then of exponential type. $\hat{f}_k$ is a finite sum of powers (at most 2) of such functions. This wouldn't change the type particularly because the sum is independent of ρ and so $\hat{f}_k$ is of exponential type as well. We conclude then that $\hat{f}(\rho,\varphi) = \sum_{k=1}^{q+1} i^{-kN}\hat{f}_k(\rho)\sin(kN\varphi)$ is also of exponential type.

To proceed, we will need Paley Wiener's theorem which we will state as follows:

Theorem (Paley Wiener): *Let $U(z)$ be analytic on $\mathbb{C}^n$ and such that for non-*

negative integers N, there is $C_N > 0$ such that

$$|U(z)| \leq C_N(1+|z|)^{-N} e^{A|Im(z)|} \qquad \forall z \in \mathbb{C}^n,$$

then there exists $f \in C_c^\infty(\mathbb{R}^n)$ with supp$f \subset B_A(0)$ (ball of radius A center 0) such that $U(\xi) = \hat{f}(\xi) \quad \forall \xi \in \mathbb{R}^n$.

So then using Paley Wiener's theorem, we obtain that $f \in C_c^\infty(\mathbb{R}^2)$. We have, by construction that $Rf(a_s, r) = 0$ for any $r > 0$ and $s = 1, \ldots, q$. Further, $f \not\equiv 0$ since $\hat{f}_{q+1} \not\equiv 0$ in the neighborhood W which completes our proof of necessity.

As Charles Colton, an English critic and writer, once said "The study of mathematics, like the Nile, begins in minuteness but ends in magnificence." Throughout the years, the study and applications of the Radon transform proved such magnificence. The topics presented in this thesis are three out of many forms of the Radon transform and the field of integral geometry is still vast and open for further results and explorations.

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