AMERICAN UNIVERSITY OF BEIRUT

ON THE METHOD OF ANALYTIC DISCS

by
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An Abstract of the Thesis of

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Initiated by Riemann and Hilbert, the method of analytic discs is a powerful technique in Several Complex Variables. Such invariants are indeed particularly adapted to the study of CR extension, boundary extension of biholomorphisms or polynomial convexity. Based on a work of Alexander Tumanov, we propose to describe the techniques used to construct analytic discs with boundaries in real submanifolds of the n dimensional complex space and their application to the boundary extension of biholomorphisms.

CONTENTS

CHAPTER 1

INTRODUCTION

The Riemann mapping theorem states that if $\emptyset \neq D$ is a simply connected open proper subset of $\mathbb{C}$ then D is biholomorphically equivalent to the unit disc Δ. In a series of papers, O.D. Kellogg studied the boundary regularity of the Riemann mapping ([**?**], [**?**], [**?**]). In particular, he proved that if the domain D has a $\mathcal{C}^\infty$ boundary, then the Riemann mapping and its inverse, extend $\mathcal{C}^\infty$ smoothly up to the boundary. The equivalent smoothness result in higher dimension is due to C. Fefferman [**?**] who proved that if $F : D_1 \to D_2$ is a biholomorphism between two $\mathcal{C}^\infty$ bounded strictly pseudoconvex domains of $\mathbb{C}^n$ then F is of class $\mathcal{C}^\infty$ up to the boundary ∂D_1. C. Fefferman's original proof is rather technical and, nowadays, there exist simpler proofs of his theorem.

The aim of the present thesis is to study the extension of biholomorphism with a more geometrical approach. We propose to make use of some well-known invariants for real smooth hypersurfaces, namely the stationary holomorphic discs, that is, discs glued to an hypersurface and satisfying some differential condition at the boundary. These particular holomorphic discs were first introduced by L. Lempert [**?**] (see also [**?**]) in strongly convex domains of $\mathbb{C}^n$. Based on this approach and on the global properties of stationary discs, L. Lempert obtained a rather simple and elegant proof of Fefferman's theorem. Later on, A. Tumanov obtained an even simpler proof by using only local geometric properties of stationary discs. We will follow his approach to prove Fefferman's theorem.

In this thesis, we will focus our study on stationary discs and their importance in describing the geometry of the domains and their boundaries. In Section 1, we start by introducing the necessary concepts and tools. We first focus on real hypersurfaces and their local geometry. We also introduce totally real sets and their connection with real hypersurface. We finally introduce our main tools, namely the stationary discs. In Section 3, we explicitly describe all stationary discs attached to a Levi non-degenerate hyperquadric in $\mathbb{C}^2$ and deduce an important geometric property they satisfy. The Section 4 is the core of the present thesis. We construct and describe stationary discs attached to small deformations of an hyperquadric using the implicit function theorem in Banach spaces. Finally in Section 5, we present A. Tumanov's proof of Fefferman's theorem.

CHAPTER 2

DEFINITIONS

We start by fixing some notations. We denote by $\Delta = \{\zeta \in \mathbb{C} \mid \zeta\bar{\zeta} < 1\}$ the unit disc in $\mathbb{C}$. The coordinates in $\mathbb{C}^{n+1}$ are denoted by $z = (z_0, \ldots, z_n)$ and we write $z_\alpha = (z_1, \ldots, z_n)$ and $z_j = x_j + iy_j$ for $j = 0, \ldots, n$.

2.1 Real hypersurfaces

Definition 2.1.1. A *smooth real hypersurface* in $\mathbb{C}^N$ is a subset Γ of $\mathbb{C}^N$ such that for every point $p_0 \in \Gamma$, there is a neighborhood U and a smooth real valued function ρ such that:

$$\Gamma \cap U = \{z \in U \mid \rho(z, \bar{z}) = 0\}$$

with $d\rho$ non-vanishing in U. Such function ρ is called *local defining function* for Γ near p_0.

Example 2.1.2. *Consider for instance*

$$\Gamma = \{(z_0, z_1) \in \mathbb{C}^2 \mid |z_0|^2 + |z_1|^2 = 1\},$$

where $\rho = |z_0|^2 + |z_1|^2 - 1$. Note that in that case ρ is a global defining function.

Remark 2.1.3. *Let $\Gamma = \{\rho = 0\}$ be a smooth real hypersurface and let $F : \mathbb{C}^n \mapsto \mathbb{C}^n$ be* *a biholomorhism, that is a bijective holomorphic mapping. Then* $F(\Gamma) = \{\rho \circ F^{-1} = 0\}$*. Indeed, let* $F(z_0, z_\alpha) \in F(\Gamma)$ *then* $F(z_0, z_\alpha) \in \{\rho \circ F^{-1} = 0\}$ *since* $\rho \circ F^{-1}(F(z_0, z_\alpha)) = \rho(z_0, z_\alpha) = 0$*. Hence* $F(\Gamma) \subset \{\rho \circ F^{-1} = 0\}$*. On the other hand, let* $(z_0, z_\alpha) \in \{\rho \circ F^{-1} = 0\}$ *which is the same as saying* $\rho \circ F^{-1}(z_0, z_\alpha) = 0$*. Then* $F^{-1}(z_0, z_\alpha) \in \Gamma$*, hence* $(z_0, z_\alpha) \in F(\Gamma)$*. Therefore* $\{\rho \circ F^{-1} = 0\} \subset F(\Gamma)$*, hence we have equality.*

Let Γ be a real hypersurface. Denote by $T_p\Gamma$ its real tangent space and define its *complex tangent space at* $p \in \Gamma$ by

$$T_p^{\mathbb{C}}\Gamma = T_p\Gamma \cap iT_p\Gamma.$$

Definition 2.1.4. A smooth real hypersurface Γ is *Levi non-degenerate* at a point $p \in \Gamma$ if the restriction to $T_p^{\mathbb{C}}\Gamma$ of the Hermitian form

$$\sum_{0 \le i,j \le n} \frac{\partial^2 \rho}{\partial \bar{z}_j \partial z_i} \bar{Z}_j Z_i$$

is non-degenerate.

Example 2.1.5. *The hypersurfaces* $\{|z_0|^2 + |z_1|^2 = 1\}$ *and* $\{\Re e z_0 = |z_1|^2\}$ *are Levi non-degenerate while the hypersurfaces* $\{\Re e z_0 = 0\}$ *and* $\{\Re e z_0 = |z_1|^4\}$ *are Levi degenerate at* 0*.*

Let us understand the local geometry of a smooth real hypersurface Γ near $p \in \Gamma$. Up to a linear transformation,then using the implicit function theorem we can assume that $p = 0$ and the tangent space to Γ at 0 is $\Re e z_0 = 0$. Hence Γ is locally given by the equation

$$\Re e z_0 = (\text{real quadratic terms in} \quad \mathrm{y}_0, \mathrm{x}_1, \ldots, \mathrm{y}_\mathrm{n}) + O(|(y_0, z_\alpha)|^3),$$

more precisely

$$\Re e z_0 = \sum_{j=1}^{n} \Re e(a_j z_j^2) + {}^t\bar{z}_\alpha A z_\alpha + b_0 y_0^2 + \sum_{j=1}^{n} (b_j z_j + \bar{b}_j \bar{z}_j) y_0 + O(|(y_0, z_\alpha)|^3).$$

After the holomorphic change of coordinates $(z_0, z_\alpha) \mapsto (z_0 - \sum_{j=1}^n a_j z_j^2, z_\alpha)$, the hypersurface Γ is given in a neighborhood of $p = 0$ by the defining function:

$$\rho(z) = \Re e z_0 - {}^t\bar{z}_\alpha A z_\alpha + b_0 y_0^2 + \sum_{j=1}^n (b_j z_j + \bar{b}_j \bar{z}_j) y_0 + O(|(y_0, z_\alpha)|^3). \tag{2.1.1}$$

Note that this depends only on Γ and on the point $p \in \Gamma$. In case $\Gamma = \{\rho = 0\} \subset \mathbb{C}^2$,

$$\rho(z) = \Re e z_0 - a|z_1|^2 + b_0 y_0^2 + (b_1 z_1 + \bar{b}_1 \bar{z}_1) y_0 + O(|(y_0, z_1|^3). \tag{2.1.2}$$

It follows that in case ρ has the local expression (2.1.1), then $\Gamma = \{\rho = 0\}$ is Levi non-degenerate at $p = 0$ if and only if A is invertible; which reduces to $a \neq 0$ in complex dimension two.

We now define the Levi form associated to a smooth real valued function.

Definition 2.1.6. Consider a smooth real valued function ρ defined on an open set $U \subset \mathbb{C}^{n+1}$. Its *Levi form at $p \in U$ and $Z \in T_pU = \mathbb{C}^{n+1}$* is given by:

$$\mathcal{L}\rho(p, Z) = \sum_{i,j=0}^n \frac{\partial^2 \rho(p)}{\partial z_i \partial \bar{z}_j} Z_i \bar{Z}_j.$$

Example 2.1.7. *In case ρ has the local expression (2.1.2), then*

$$\mathcal{L}\rho(0, Z) = \sum_{i,j=0}^n \frac{\partial^2 \rho(0)}{\partial z_i \partial \bar{z}_j} Z_i \bar{Z}_j = -a|Z_1|^2.$$

Definition 2.1.8. An hypersurface $\Gamma = \{\rho = 0\}$ is a *strictly pseudoconvex at $p \in \Gamma$* if the levi form of ρ satisfies $\mathcal{L}\rho(p, Z) > 0$ for all $Z \in T_p^{\mathbb{C}}\Gamma \setminus \{0\}$.

Remark 2.1.9. *Notice that in case ρ is written as (2.1.1), then $\Gamma = \{\rho = 0\}$ is strictly pseudoconvex at 0 if and only if A is positive definite.*

Definition 2.1.10. *Consider a smooth hypersurface Γ, define by $\mathcal{L}_\rho(p, Z)$ its Levi form,we say that Γ is plurisubharmonic if its Levi form is positive definite, i.e $\mathcal{L}\rho(p, Z) > 0$, for all*

$Z \in \mathbb{C}^2 \setminus \{0\}$.

Remark 2.1.11. *If $\Gamma = \{\rho = 0\}$ is globally defined by a strictly plurisubharmonic function then Γ is strictly pseudoconvex.*

Example 2.1.12. *The hypersurfaces $\{|z_0|^2 + |z_1|^2 = 1\}$ and $\{\Re e z_0 = |z_1|^2\}$ are strictly pseudoconvex while the hypersurfaces $\{\Re e z_0 = 0\}$, $\{\Re e z_0 = -|z_1|^2\}$ or $\{\Re e z_0 = |z_1|^4\}$ are not. Moreover $\{|z_0|^2 + |z_1|^2 = 1\}$ is globally defined by a strictly plurisubharmonic function.*

2.2 Totally real submanifolds

Definition 2.2.1. A submanifold M in $\mathbb{C}^{2n+2}$ of real dimension $2n+2$ is *totally real* if its complex tangent space at each point $p \in \Gamma$ is trivial, i.e

$$T_p^{\mathbb{C}}\Gamma = T_p\Gamma \cap iT_p\Gamma = \{0\}.$$

Example 2.2.2. *The real subspace*

$$M = \{x_0 = x_1 = x_2 = x_3 = 0\} = \mathbb{R} \times \{0\} \times \mathbb{R} \times \{0\} \times \mathbb{R} \times \{0\} \times \mathbb{R} \times \{0\} \subset \mathbb{C}^4$$

is totally real. Indeed let $X = (X_0, Y_0, X_1, Y_1, X_2, Y_2, X_3, Y_3) \in T_p^{\mathbb{C}}M = T_pM \cap iT_pM$. Since $X \in T_pM$ then $X_0 = X_1 = X_2 = X_3 = 0$. On the other hand, $X \in iT_pM$ implies that $iX = (-Y_0, X_0, -Y_1, X_1, -Y_2, X_2, -Y_3, X_3) \in T_pM$ and so $Y_0 = Y_1 = Y_2 = Y_3 = 0$. Hence $X = 0$.

Example 2.2.3. *The real subspace*

$$M = \{x_0 = y_0 = x_1 = y_1 = 0\} = \mathbb{C}^2 \times \{0\} \times \{0\} \subset \mathbb{C}^4$$

is not totally real. Indeed the vector $X = (0, 0, 0, 0, 0, 0, 1, 2)$ is a non zero vector in $T_p^{\mathbb{C}}M = T_pM \cap iT_pM$.

We now introduce a real submanifold of $\mathbb{C}^{2n+2}$ that will play an important role in our study:

Definition 2.2.4. Let $\Gamma = \{\rho = 0\}$ be a smooth real hypersurface of $\mathbb{C}^{n+1}$. For $p \in \Gamma$, we define the *conormal fiber at* p to be the real line generated by $\partial\rho(p) = \left(\frac{\partial\rho}{\partial z_0}(p), \ldots, \frac{\partial\rho}{\partial z_n}(p)\right)$, that is

$$N_p^*\Gamma = \mathrm{span}_{\mathbb{R}}\{\partial\rho(p)\} \subset T_p^*\mathbb{C}^{n+1} = \mathbb{C}^{n+1}.$$

The *conormal bundle* $N^*\Gamma$ of Γ is the bundle over Γ whose fiber at $p \in \Gamma$ is $N_p^*\Gamma$, namely

$$N^*\Gamma = \bigcup_{p\in\Gamma} N_p^*\Gamma.$$

Notice that $N^*\Gamma$ is a real submanifold of dimension $2n + 1 + 1 = 2n + 2$ of $T^*\mathbb{C}^{n+1} = \bigcup_{p\in\mathbb{C}^{n+1}} T_p^*\mathbb{C}^{n+1} = \mathbb{C}^{2n+2}$.

Example 2.2.5. *Consider the hyperquadric* $Q = \{\Re e z_0 - |z_1|^2 = 0\}$*. Let* $p = (z_0, z_1) \in Q$*. We have* $\partial r(p) = \left(\frac{1}{2}, -\bar{z}_1\right)$*. It follows that* $N_p^*Q = \mathrm{span}_{\mathbb{R}}\left\{\left(\frac{1}{2}, \bar{z}_1\right)\right\}$*. It is convenient to describe the conormal bundle using equations. Set* $r(z) = \Re e z_0 - |z_1|^2$*. Note that* $(z, w) = (z_0, z_1, w_0, w_1) \in N^*Q \subset \mathbb{C}^4$ *if and only if* $r(z) = 0$ *and* $w = c\partial r(z) = c\left(\frac{1}{2}, -\bar{z}_1\right)$*. Therefore* $(z, w) \in N^*Q$ *if and only if* $r(z) = 0$*,* $w_0 \in \mathbb{R}$ *and* $w_1 = -2w_0\bar{z}_1$*. Therefore*

$$N^*Q = \{r_1 = r_2 = r_3 = r_4 = 0\}$$

with

$$\begin{cases} r_1(z,w) = \Re e z_0 - |z_1|^2 & = 0 \\ \\ r_2(z,w) = \dfrac{1}{i}(w_0 - \bar{w_0}) & = 0 \\ \\ r_3(z,w) = w_1 + 2w_0\bar{z_1} + \overline{w_1 + 2w_0\bar{z_1}} & = 0 \\ \\ r_4(z,w) = i(w_1 + 2w_0\bar{z_1}) - i(\overline{w_1 + 2w_0\bar{z_1}}) & = 0 \end{cases}$$

or equivalently

$$\begin{cases} r_1 = x_0 - x_1^2 - y_1^2 & = 0 \\ \\ r_2 = y_2 & = 0 \\ \\ r_3 = x_3 + 2x_1x_2 + 2y_1y_2 & = 0 \\ \\ r_4 = -y_3 - 2x_1y_2 + 2x_2y_1 & = 0 \end{cases}$$

We will need the following essential result due to S. Webster [**?**]

Theorem 2.2.6 ([**?**])**.** *Let $\Gamma \subset \mathbb{C}^{n+1}$ be a smooth real hypersurface. Then Γ is Levi non-degenerate if and only if its conormal bundle $N^*\Gamma \subset \mathbb{C}^{2n+2}$ is totally real.*

Example 2.2.7. *Consider the non-degenerate hyperquadric $Q = \{\Re e z_0 - |z_1|^2 = 0\} \subset \mathbb{C}^2$. Let us check that N^*Q is indeed totally real, namely its complex tangent spaces are trivial*

$$T_p^{\mathbb{C}}Q = T_pQ \cap iT_pQ = \{0\}.$$

Let $(z,w) \in N^*Q$ and let $X = (X_0, Y_0, X_1, Y_1, X_2, Y_2, X_3, Y_3) \in T_{(z,w)}N^*Q \cap iT_{(z,w)}N^*Q$. The gradient of each function is

$$\nabla r_1 = (1, 0, -2x_1, -2y_1, 0, 0, 0, 0)$$

$$\nabla r_2 = (0,0,0,0,0,1,0,0)$$

$$\nabla r_3 = (0,0,2x_2,2y_2,2x_1,2y_1,1,0,)$$

$$\nabla r_4 = (0,0,-2y_2,2x_2,2y_1,-2x_1,0,-1).$$

Now that we have the gradients, we construct the system of equations as follows:

$$\begin{cases} X_0 - 2x_1X_1 - 2y_1Y_1 & = & 0 \\ Y2 & = & 0 \\ 2x_2X_1 + 2y_2Y_1 + 2x_1X_2 + 2y_1Y_2 + X_3 & = & 0 \\ -2y_2X_1 + 2x_2Y_1 + 2y_1X_2 - 2x_1Y_2 - Y_3 & = & 0 \\ -Y_0 + 2x_1Y_1 - 2y_1X_1 & = & 0 \\ X_2 & = & 0 \\ -2x_2Y_1 + 2y_2X_1 - 2x_1Y_2 + 2y_1X_2 - Y_3 & = & 0 \\ 2y_2Y_1 + 2x_2X_1 - 2y_1Y_2 - 2x_1X_2X_3 & = & 0 \end{cases}$$

Note that $r_2 = y_2 = 0$ *Hence the equations are reduced to the following:*

$$\begin{cases} X_0 - 2x_1X_1 - 2y_1Y_1 & = & 0 \\ Y2 & = & 0 \\ 2x_2X_1 + 2x_1X_2 + 2y_1Y_2 + X_3 & = & 0 \\ 2x_2Y_1 + 2y_1X_2 - 2x_1Y_2 - Y_3 & = & 0 \\ -Y_0 + 2x_1Y_1 - 2y_1X_1 & = & 0 \\ X_2 & = & 0 \\ -2x_2Y_1 - 2x_1Y_2 + 2y_1X_2 - Y_3 & = & 0 \\ 2x_2X_1 - 2y_1Y_2 - 2x_1X_2 - X_3 & = & 0 \end{cases}$$

Using the fact that $Y_2 = X_2 = 0$ *we get:*

$$\begin{cases} X_0 - 2x_1X_1 - 2y_1Y_1 &= 0 \\ 2x_2X_1 + X_3 &= 0 \\ 2x_2Y_1 - Y_3 &= 0 \\ -Y_0 + 2x_1Y_1 - 2y_1X_1 &= 0 \\ -2x_2Y_1 - Y_3 &= 0 \\ 2x_2X_1 - X_3 &= 0 \end{cases}$$

Notice that

$$\begin{cases} 2x_2Y_1 - Y_3 &= 0 \\ -2x_2Y_1 - Y_3 &= 0 \end{cases}$$

Which implies that $Y_3 = 0$*. Which gives us from the same equation that* $Y_1 = 0$*. Using these two equations combined together*

$$\begin{cases} -2x_2Y_1 - Y_3 &= 0 \\ 2x_2X_1 - X_3 &= 0 \end{cases}$$

We get that $X_3 = 0$ *Therefore* $X_1 = 0$*. Which will lead to* $X_0 = Y_0 = 0$*. Hence we are left with* $X_0 = Y_0 = X_1 = Y_1 = X_2 = Y_2 = 0$.

2.3 Stationary discs

A *holomorphic disc* h is a holomorphic function $h : \Delta \to \mathbb{C}^{n+1}$ defined on the unit disc $\Delta \in \mathbb{C}$. We say that a holomorphic disc h is *attached to a submanifold* $M \subset \mathbb{C}^{n+1}$ if h is continuous up to the boundary $\partial\Delta$ and maps $\partial\Delta$ to M, that is $h(\partial\Delta) \subset M$. To a given real smooth hypersurface, the family of attached disc is invariant by biholomorphism. These discs were studied by many authors such as E. Bishop [?] or A. Tumanov [?]. However, it is important to note that there are usually too many discs attached to a real smooth hypersurface. In 1981, Lempert [?] defined the notion of stationary discs, that is discs attached to an hypersurface and satisfying some differential condition at the boundary.

Definition 2.3.1. A holomorphic disc h attached to a the real hypersurface Γ is *stationary for* Γ if there exists a holomorphic lift $\boldsymbol{h} = (h, g) : \Delta \to T^*\mathbb{C}^{n+1}$ of h to the cotangent bundle $T^*\mathbb{C}^{n+1}$, continuous up to the boundary and such that for all $\zeta \in \partial\Delta$

$$\boldsymbol{h}(\zeta) \in \mathscr{N}\Gamma(\zeta)$$

where

$$\mathscr{N}\Gamma(\zeta) = \{(z, \zeta w) \mid z \in \Gamma,\ w \in N_z^*\Gamma \setminus \{0\}\}.$$

The set

$$\mathscr{N}\Gamma = \bigcup_{\zeta \in \Delta} \mathscr{N}\Gamma(\zeta)$$

is called the *conormal fibration.* Moreover, the set of these lifted discs $\boldsymbol{h} = (h, g)$, with h non-constant, is denoted by $\mathcal{S}(\Gamma)$.

In case $\Gamma = \{\rho = 0\}$, Definition 2.3.1 is equivalent to the existence of a continuous function $c : \partial\Delta \to \mathbb{R} \setminus \{0\}$ such that $g(\zeta) = \zeta c(\zeta)\partial\rho(h(\zeta))$ on the boundary $\partial\Delta$ and extends holomorphically to the unit disk Δ.

Stationary discs attached to a real hypersurface $\Gamma \subset \mathbb{C}^{n+1}$ are of a big importance since they form a biholomorphic invariant family and are characterized by a finite number of parameters. More precisely if $F : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ a biholomorphism and if h is a stationary disc for $\Gamma = \{\rho = 0\}$, then $F \circ h$ is a stationary disc for $F(\Gamma) = \{\rho \circ F^{-1} = 0\}$. Indeed since h is a stationary disc attached to Γ, then there exists a continuous function $c : \partial\Delta \mapsto \mathbb{R}^*$ such that $\mathbf{h} = (h, g)$ where $g(\zeta) = \zeta c(\zeta) d\rho(h(\zeta))$. Hence $\zeta c(\zeta) d(\rho \circ F^{-1})(F \circ h)(\zeta) = \zeta c(\zeta) d\rho(h(\zeta))(dF(h(\zeta)))^{-1}$ extends holomorphically to the unit disc.

Due to Theorem 2.2.6 and to the regularity of holomorphic discs attached to a totally real submanifold [?] (see also the classical Schwarz reflection principle [?]), stationary discs inherit the smoothness of the hypersurface they are attached to. More precisely if Γ is $\mathcal{C}^4$ Levi non-degenerate, then $N^*\Gamma$ is a $\mathcal{C}^3$ totally real submanifold. According to [?], the lifts $\boldsymbol{h}$ of stationary discs are of class $\mathcal{C}^{2,\alpha}(\bar{\Delta}, T^*\mathbb{C}^{n+1})$ or, equivalently, $\boldsymbol{h}_{|\partial\Delta} \in \mathcal{C}^{2,\alpha}(\partial\Delta, T^*\mathbb{C}^{n+1})$ for any $0 < \alpha < 1$. The spaces $\mathcal{C}^{k,\alpha}(\partial\Delta)$, $0 < \alpha < 1$, $k \in \mathbb{N}$ are equipped with their usual norm:

$$\|\boldsymbol{h}\|_{\mathcal{C}^{k,\alpha}(\partial\Delta)} = \sum_{l=0}^{k} \|\boldsymbol{h}^{(l)}\|_\infty + \sup_{\zeta\neq\eta\in\partial\Delta} \frac{\|\boldsymbol{h}^{(k)}(\zeta) - \boldsymbol{h}^{(k)}(\eta)\|}{|\zeta - \eta|^\alpha},$$

where $\|\boldsymbol{h}^{(l)}\|_\infty := \max_{\partial\Delta} \|\boldsymbol{h}^{(l)}\|$.

CHAPTER 3

STATIONARY DISCS ATTACHED TO NON-DEGENERATE HYPERQUADRIC

Stationary discs attached to a non-degenerate hyperquadric Q are explicitly known. We first describe them in $\mathbb{C}^2$ and then deduce an important geometric property.

3.1 Description of stationary discs attached to a non-degenerate hyperquadric

Consider the hyperquadric $Q \in \mathbb{C}^2$ defined by:

$$r(z) = \Re e z_0 - |z_1|^2 = 0.$$

Proposition 3.1.1 (Proposition 2.1 [?]). *The stationary discs attached to Q are exactly of the form:*

$$h(\zeta) = \left(|v|^2 + 2\bar{v}w\frac{\zeta}{1-a\zeta} + \frac{|w|^2}{1-|a|^2}\frac{1+a\zeta}{1-a\zeta} + iy_0, v + w\frac{\zeta}{1-a\zeta} \right)$$

with $a \in \Delta, v \in \mathbb{C}, w \in \mathbb{C} \setminus \{0\}, y_0 \in \mathbb{R}$. Moreoever h^* is a regular lift of h if and only if *there exists* $b \in \mathbb{R} \setminus \{0\}$ *such that for all* $\zeta \in \Delta \setminus \{0\}$

$$h^*(\zeta) = \frac{b}{\zeta}\left(\frac{-\bar{a}}{1+|a|^2} + \zeta - \frac{a}{1+|a|^2}\zeta^2\right) \times \left(\frac{1}{2}, -\overline{h_1}\right)$$

Note that lift of stationary discs are parametrized by 8 real parameters a, v, w, y_0 and b.

Proof. Suppose h has a regular lift h^* then, by definition, there exists a continuous function $c : \partial\Delta \to \mathbb{R} \setminus \{0\}$ such that for $\zeta \in \partial\Delta$ we have:

$$h^*(\zeta) = c(\zeta)\frac{\partial r}{\partial z}(h(\zeta)).$$

Let $\mathbf{h}$ be a lift of h. The disc $\mathbf{h} = (h, g)$ is continuous up to the boundary $\partial\Delta$ and moreover $\mathbf{h}(\zeta) \in \mathscr{N}Q(\zeta)$ for all $\zeta \in \partial\Delta$. Therefore all $\zeta \in \partial\Delta$ we have $h(\zeta) \in Q$ and $\dfrac{g}{\zeta} \in N^*_{h(\zeta)}Q \setminus \{0\}$ with $g = \zeta h^*$ where h^* is the regular lift. Hence we have for all $\zeta \in \partial\Delta$:

$$h^*(\zeta) = c(\zeta)\frac{\partial r}{\partial z}(h(\zeta)) = c(\zeta)\left(\frac{1}{2}, -\bar{h_1}\right)$$

where $c : \partial\Delta \mapsto \mathbb{R} \setminus \{0\}$ is continuous and $\zeta\dfrac{c(\zeta)}{2}$ extends holomorphically to Δ. We now have

$$g_0 = \zeta h_0^*(\zeta) = \zeta\frac{c(\zeta)}{2}$$

and

$$g_1 = \zeta h_1^*\zeta = -\zeta c(\zeta)\bar{h_1}.$$

To find h_0, h_1, g_0 and g_1, we first find $c(\zeta)$. Since both g_0 and c are continuous on the boundary of Δ they admit a Fourier expansion on $\partial\Delta$. We have

$$g_0(\zeta) = a_0 + a_1\zeta + a_2\zeta^2 + \dots$$

since g_0 is holomorphic on Δ and

$$c(\zeta) = \ldots + \bar{c_2}\bar{\zeta}^2 + \bar{c_1}\bar{\zeta} + c_0 + c_1\zeta + c_2\zeta^2 + \ldots$$

since c is real. Therefore

$$\zeta c(\zeta) = \ldots + \bar{c_2}\bar{\zeta} + \bar{c_1} + c_0\zeta + c_1\zeta^2 + c_2\zeta^3 + \ldots$$

and since $\zeta c(\zeta)$ extends holomorphically on Δ it follows that $\zeta c(\zeta) = \bar{c_1} + c_0\zeta + c_1\zeta^2$. Hence

$$c(\zeta) = \bar{c_1}\bar{\zeta} + c_0 + c_1\zeta.$$

and

$$g_0(\zeta) = \frac{1}{2} \times \{\bar{c_1} + c_0\zeta + c_1\zeta^2\}.$$

We now turn on determining h_0, h_1 and g_1. We have

$$g_1(\zeta) = -\zeta c(\zeta)\bar{h_1} = -(\bar{c_1} + c_0\zeta + c_1\zeta^2)\bar{h_1}$$

and

$$\bar{h_1} = \bar{b_0} + \bar{b_1}\bar{\zeta} + \bar{b_2}\bar{\zeta}^2 + \ldots$$

since h_1 is continuous on ∂D and holomorphic on Δ. We distinguish two cases:

<u>First case: $c_1 = 0$</u>. Then $g_1(\zeta) = -c_0\zeta\bar{h_1}$ and so $\zeta\bar{h_1}$ holomorphic, i.e $\zeta(\bar{b_0} + \bar{b_1}\bar{\zeta} + \bar{b_2}\bar{\zeta}^2 + \ldots)$ is holomorphic. Hence

$$\bar{b_0}\zeta + \bar{b_1} + \bar{b_2}\bar{\zeta} + \bar{b_3}\bar{\zeta}^2 + \ldots$$

is holomorphic. Therefore $b_2 = b_3 = \ldots = 0$ and so

$$h_1(\zeta) = b_0 + b_1\zeta$$

is linear and

$$g_1(\zeta) = -c_0\bar{b_0}\zeta - c_0\bar{b_1}.$$

Finally

$$\begin{aligned}
\frac{1}{2}(h_0(\zeta) + \bar{h}_0(\zeta)) &= |h_1|^2 \\
&= h_1\bar{h_1} = (b_0 + b_1\zeta)(\overline{b_0 + b_1\zeta}) \\
&= |b_0|^2 + b_0\bar{b_1}\bar{\zeta} + \bar{b_0}b_1\zeta + |b_1|^2\zeta\bar{\zeta} \\
&= \bar{e}\bar{\zeta} + d + e\zeta
\end{aligned}$$

with $d \in \mathbb{R}$. Write

$$\sum_{n\geq 1} \frac{(\bar{h_0})_n}{2}\bar{\zeta}^n + \Re e(\hat{h}_n)_0 + \sum_{n\geq 1} \frac{(\hat{h}_0)_n}{2}\zeta^n = \bar{e}\bar{\zeta} + d + e\zeta$$

where $h(\zeta) = \sum_{n\geq 0}(\hat{h}_0)_n\zeta^n$. By comparison, we have that $(h_0)_n = 0$ for $n \geq 2$ and so h_0 is affine up to to a pure imaginary additive constant.

Second case: $c_1 \neq 0$. Denote by α_1, α_2 the roots of $c_1 + c_0\zeta + \bar{c_1}\zeta^2$ and assume $|\alpha_1| \leq |\alpha_2|$. Note that α_1 and α_2 are not of modulus 1 since $g(\zeta) = \zeta h^*(\zeta)$ and h^* does not vanish on $\partial\Delta$. Moreover $|\alpha_1\alpha_2| = |\frac{c_1}{\bar{c_1}}| = 1$ then $0 < |\alpha_1| < 1 < |\alpha_2|$. The map ζh^* extends holomorphically to Δ if and only if $(\bar{c_1} + c_0\zeta + c_1\zeta^2)\bar{h_1}(\zeta)$ extends holomorphically to Δ.

Using Fourier expansion again we have:

$$\begin{aligned}(\bar{c_1} + c_0\zeta + c_1\zeta^2)\bar{h_1}(\zeta) &= (\bar{c_1} + c_0\zeta + c_1\zeta^2)\sum_{k=0}^{\infty}\bar{b}_k\zeta^{-k} \\ &= \sum_{k=0}^{\infty}(\bar{c_1}\bar{b_k} + c_0\bar{b}_{k+1} + c_1\bar{b}_{k+2})\zeta^{-k} + (c_0\bar{b_0} + c_1\bar{b_1})\zeta + c_1\bar{b_0}\zeta^2.\end{aligned}$$

which extends holomorphically to Δ if and only if

$$\bar{c_1}\bar{b_k} + c_0\bar{b}_{k+1} + c_1\bar{b}_{k+2} = 0.$$

This is a linear recurrence of order 2 of which the caracteristic equation has two distinct roots which are exactly α_1 and α_2. There exists $v_1, w_1 \in \mathbb{C}$ independent of b_1 and b_2 such that for all $k \geq 1$

$$b_k = \bar{\alpha_1}^{k-1}v_1 + \bar{\alpha_2}^{k-1}w_1.$$

The function h_1 is holomorphic on Δ and so the series

$$\sum(v_1\bar{\alpha_1}^{k-1} + w_1\bar{\alpha_2}^{k-1})\zeta^k$$

converges in Δ, and hence has a radius of convergence greater than or equal to 1. Setting $d_k^1 = v_1\bar{\alpha_1}^{k-1} + w_1\bar{\alpha_2}^{k-1}$, we have

$$\frac{d_{k+2}^1}{d_{k+1}^1} = \frac{v_1\bar{\alpha_1}^{k+1} + w_1\bar{\alpha_2}^{k+1}}{v_1\bar{\alpha_1}^k + w_1\bar{\alpha_2}^k}$$

with $0 < |\alpha_1| < 1 < |\alpha_2|$. If $w_1 \neq 0$, then $\lim_{K\to\infty}\frac{d_{k+2}^1}{d_1^{k+1}}$ is $\approx \bar{\alpha}_2$. . Then the radius of convergence is equal to $\frac{1}{|\alpha_2|} < 1$ which is impossible. It follows that $w_1 = 0$. If $v_1 \neq 0$ then $\frac{d_{k+2}^1}{d_{k+1}^1} = \bar{\alpha_1}$ and the series has a radius of convergence equal to $\frac{1}{|\alpha_1|} > 1$; if $v_1 = 0$ the series has an infinite radius of convergence. Hence $b_0 = h_1(0), b_1 = h_1'(0)$ and

$$h_1(\zeta) = b_0 + b_1 \times \zeta \sum_{k=0}^{\infty} (\bar{\alpha_1}\zeta)^k.$$

We still have to express c_1 using α_1:

$$\alpha_1 = \frac{-1 + i\sqrt{4|c_1|^2 - 1}}{2c_1}$$

and

$$\alpha_2 = \frac{-1 - i\sqrt{4|c_1|^2 - 1}}{2c_1}.$$

and if $1 - 4|c_1|^2 = 0$ the equation has a double root. In both cases $|\alpha_1| = |\alpha_2|$ which is impossible. Therefore we have $1 - 4|c_1|^2 > 0$ and

$$\alpha_1 = \frac{-1 + \sqrt{1 - 4|c_1|^2}}{2|c_1|} e^{i\theta} = \frac{1 - \sqrt{1 - 4|c_1|^2}}{2|c_1|} e^{i(\theta+\pi)}.$$

Then $Arg(\bar{c_1}) = Arg(\alpha_1) - \pi$ and $|\alpha_1| = \frac{1-\sqrt{1-4|c_1|^2}}{2|c_1|}$ which gives $1 - 2|c_1||\alpha_1| = \sqrt{1 - 4|c_1|^2}$. This is equivalent to $|c_1| < \frac{1}{2|\alpha_1|}$ and $1 - 4|c_1||\alpha_1| + 4|c_1|^2|\alpha_1|^2 = 1 - 4|c_1|^2$. Since $c_1 \neq 0$ and $0 < |\alpha_1| < 1$ then $|c_1| = \frac{|\alpha_1|}{1+|\alpha_1|^2}$. Finally

$$c_1 = |c_1||e^{i\theta}| = \frac{|\alpha_1|}{1 + |\alpha_1|^2} \frac{-\alpha_1}{|\alpha_1|} = \frac{-\alpha_1}{1 + |\alpha_1|^2}.$$

Now that we have c_1, we can write

$$c(\zeta) = \frac{-\alpha_1}{1 + |\alpha_1|^2}\bar{\zeta} + c_0 - \frac{-\bar{\alpha}_1}{1 + |\alpha_1|^2}.$$

Therefore

$$h^*(\zeta) = \frac{c_0}{\zeta}\left(\frac{-\bar{a}}{1 + |a|^2} + \zeta - \frac{a}{1 + |a|^2}\zeta^2\right) \times \left(\frac{1}{2}, -\overline{h_1}\right).$$

We are left to find $h_0(\zeta)$:

$$
\begin{aligned}
\frac{1}{2}(h_0(\zeta)+\bar{h}_0(\zeta)) &= |h_1|^2 \\
&= h_1\bar{h_1} = (b_0 + b_1 \times \zeta \sum_{k=0}^{\infty}(\bar{\alpha_1}\zeta)^k) \times (\bar{b_0} + \bar{b_1} \times \bar{\zeta} \sum_{k=0}^{\infty} \overline{(\bar{\alpha_1}\zeta)^k}) \\
&= b_0\bar{b_0} + \frac{|b_1\zeta|^2}{(1-\alpha_1\bar{\zeta})(1-\bar{\alpha_1}\zeta)} + \frac{b_0\bar{b_1}\bar{\zeta}}{1-\alpha_1\bar{\zeta}} + \frac{\bar{b_0}b_1\zeta}{1-\bar{\alpha_1}\zeta}
\end{aligned}
$$

If we let $b_0 = v, b_1 = w, \bar{\alpha_1} = a$ we get that

$$
h_0(\zeta) = |v|^2 + 2\bar{v}w\frac{\zeta}{1-a\zeta} + \frac{|w|^2}{1-|a|^2}\frac{1+a\zeta}{1-a\zeta} + iy_0.
$$

□

We impose further restrictions on the set of lift of stationary discs and define $\mathcal{S}^*(Q)$:

$$
\mathcal{S}^*(Q) = \{\boldsymbol{h} = (h,g) \in \mathcal{S}(Q) \mid h(1) = (0,0), g(1) = (1,0)\}.
$$

Recall that $\mathcal{S}(Q)$ denotes the set of lifts of stationary discs for the hyperquadric Q. We have that $\boldsymbol{h} = (h,g) \in \mathcal{S}^*(Q)$ if and only if

$$
\begin{cases}
h(\zeta) = \left(2|v|^2\frac{(1-a)(1-\zeta)}{(1-a\zeta)(1-|a|^2)}, v\frac{(1-\zeta)}{(1-a\zeta)}\right) \\
\\
\zeta h^*(\zeta) = g(\zeta) = \frac{2}{|1-a|^2}\left(\frac{(\zeta-\bar{a})(1-a\zeta)}{2}, (1-\zeta)(1-a\zeta)\bar{v}\right)
\end{cases}
\tag{3.1.1}
$$

where $a \in \Delta$ and $v \in \mathbb{C}^2 \setminus \{0\}$. Indeed, since $h(1) = (0, 0)$ we have then:

$$w = -v(1-a)$$

and

$$iy_0 = |v|^2 \left(1 - \frac{(1-\bar{a})(1+a)}{1-|a|^2}\right).$$

Replacing these terms in 3.1.1, we get:

$$h(\zeta) = |v|^2 \left(2 - 2\frac{\zeta(1-a)}{1-a\zeta} + \frac{|1-a|^2(1+a\zeta)}{(1-|a|^2)(1-a\zeta)} - \frac{(1-a)(1+a)}{1-|a|^2}, v\frac{(1-\zeta)}{1-a\zeta}\right).$$

Hence

$$h(\zeta) = \left(2|v|^2 \frac{(1-a)(1-\zeta)}{(1-a\zeta)(1-|a|^2)}, v\frac{(1-\zeta)}{(1-a\zeta)}\right).$$

On the other hand, the condition $g(1) = (1, 0)$ implies that $c_0 = \frac{2}{|1-a|^2}$. Therefore, replacing c_0 in 3.1.1 we get that

$$\zeta h^*(\zeta) = g(\zeta) = \frac{2}{|1-a|^2}\left(\frac{(\zeta-\bar{a})(1-a\zeta)}{2}, (1-\zeta)(1-a\zeta)\bar{v}\right).$$

Note that the lifts in $\mathcal{S}^*(Q)$ are parametrized by 4 real parameters a and v, or equivalently a and w.

3.2 A geometric property

We now state a geometric property satisfied by stationary discs that will be used to prove Theorem 5.0.7 (Fefferman's theorem) in Section 4.

Proposition 3.2.1. *The set of directions*

$$\left\{\frac{dh}{d\theta}|_{\theta=0} \mid \boldsymbol{h} = (h, g) \in \mathcal{S}^*(Q)\right\}$$

fills an open set in $T_{(0,0)}Q$.

The proof essentially relies on the explicit expression of stationary discs that we have obtained in Proposition 3.1.1.

Proof. The stationary discs attached to $Q = \{\Re e z_0 - |z_1|^2 = 0\}$ are exactly of the form:

$$h(\zeta) = \left(|v|^2 + 2\bar{v}w\frac{\zeta}{1-a\zeta} + \frac{|w|^2}{1-|a|^2}\frac{1+a\zeta}{1-a\zeta} + iy_0, v + w\frac{\zeta}{1-a\zeta}\right)$$

Then

$$h(e^{i\theta}) = \left(|v|^2 + 2\bar{v}w\frac{e^{i\theta}}{1-ae^{i\theta}} + \frac{|w|^2}{1-|a|^2}\frac{1+ae^{i\theta}}{1-ae^{i\theta}} + iy_0, v + w\frac{e^{i\theta}}{1-ae^{i\theta}}\right)$$

and

$$\frac{dh}{d\theta}|_{\theta=0} = \left(-2i\frac{|w|^2}{|1-a^2|(1-|a|^2)}, \frac{wi}{(1-a)^2}\right).$$

Moreover if $h(1) = 0$ then $v = \dfrac{-w}{1-a}$ and $\dfrac{2|w|^2}{(1-a)^2(1-|a|^2)} = y_0$. Note that the real tangent space at $(0,0)$ is

$$T_{(0,0)}Q = \{\Re e z_0 = 0\} = \{(X_0, Y_0, X_1, Y_1) \in \mathbb{R}^4 \mid X_0 = 0\}.$$

Let $(0, Y_0, X_1, Y_1) \in T_0Q$. Our aim is to find a, w such that:

$$\begin{cases} X_0 + iY_0 &= i\dfrac{w}{(1-a)^2} \\ \\ Y_1 &= \dfrac{-2|w|^2}{(1-a)^3(1+a)}. \end{cases}$$

Replacing w by $\alpha + i\beta$ we get:

$$\begin{cases} X_0 + iY_0 & = & \dfrac{i\alpha - \beta}{(1-a)^2} \\ \\ Y_1 & = & \dfrac{-2(\alpha^2+\beta^2)}{(1-a)^3(1+a)}. \end{cases}$$

Hence by comparison we obtain $X_0 = \frac{-\beta}{(1-a)^2}$ and $Y_0 = \frac{\alpha}{(1-a)^2}$. Then:

$$\begin{cases} \beta^2 & = & X_0^2(1-a)^2 \\ \\ \alpha^2 & = & Y_0^2(1-a)^2. \end{cases}$$

Hence

$$\frac{\alpha^2+\beta^2}{(1-a)^3(1+a)} = (X_0^2+Y_0^2)\frac{1-a}{1+a} = -\frac{Y_1}{2}.$$

Consider the change of variable

$$\frac{1-a}{1+a} = -\frac{Y_1}{2(X_0^2+Y_0^2)} = \gamma.$$

We get $a = \frac{1-\gamma}{1+\gamma}$. Now that we have a we can find w:

$$w = -i(X_0+iY_0)(1-a)^2 = -i(X_0+iY_0)\left(\frac{2\gamma}{1+\gamma}\right).$$

It follows that the set of directions

$$\left\{\frac{dh}{d\theta}|_{\theta=0} \mid h \in \mathcal{S}^*(Q)\right\}$$

fills an open set in $T_{(0,0)}Q$.

□

Remark 3.2.2. *Define the map* $\psi : \Delta \times \mathbb{C} \setminus \{0\} \to \{\Re e z_0 = 0\}$ *by*

$$\psi(a, w) = \frac{dh}{d\theta}|_{\theta=0} = \left(\frac{-2i|w|^2}{|1-a^2|(1-|a|^2)}, \frac{iw}{(1-a)^2} \right),$$

where h *is the unique stationary disc parametrized by* a *and* w*. Then a direct computation shows that the rank of its Jacobian* $d\psi$ *is* 3*.*

CHAPTER 4

DISCS ATTACHED TO A SMALL PERTURBATION OF A NON-DGENERATE HYPERQUADRIC

In this section we construct stationary discs attached to small deformations of the non-degenerate hyperquadric $Q = \{r = 0\} \subset \mathbb{C}^2$, where

$$r(z) = \Re e z_0 - |z_1|^2.$$

Our main tool is the usual implicit function theorem in Banach spaces. The existence of nearby discs, as well as the number of real number parametrizing the perturbed discs is completely determined by some integers, namely the partial indices and the Maslov index.

4.1 The implicit function theorem

We first recall the usual implicit function theorem in Banach spaces (see [?] for instance).

Theorem 4.1.1 ([?]). *Let X, Y, Z be banach spaces, let U an open subset of $X \times Y$, and let $\mathcal{F} : U \mapsto Z$ be a $\mathcal{C}^1$ map. Let $(a, b) \in U$ and suppose that*

$$\mathcal{F}(a, b) = 0.$$

Assume that the partial derivative in the second variable $d_Y \mathcal{F}(a, b) : Y \to Z$ is an *isomorphism from Y to Z. Then there exist an open neighborhood $V \subset U$ of (a,b), an open neighborhood W of a and a $\mathcal{C}^1$ map $g : W \to Y$ such that*

$$(x,y) \in V \quad \textit{and} \quad \mathcal{F}(x,y) = 0$$

if and only if

$$x \in W \quad \textit{and} \quad y = g(x).$$

We will use the following variation (see [**?**] for instance) which is a direct application of the standard implicit mapping theorem:

Theorem 4.1.2 ([**?**])**.** *Let X, Y, Z be banach spaces, let U an open neighbourhood of 0 in $X \times Y$, and let $\mathcal{F} : U \mapsto Z$ be a $\mathcal{C}^1$ map such that $\mathcal{F}(0) = 0$. Assume that $d_Y\mathcal{F}(0) : Y \to Z$ is onto and that $\mathrm{ker} d_Y\mathcal{F}(0)$ is complemented in Y, namely $Y = \mathrm{ker} d_Y\mathcal{F}(0) \oplus H$ where H is a closed subspace of Y. Identify $X \times Y$ with $X \times \mathrm{Ker} d_Y\mathcal{F}(0,0) \times H$ in the canonical way. Then there are neighborhoods V_1 of 0 in X, V_2 of 0 in $\mathrm{Ker} d_Y\mathcal{F}(0), V_3$ of 0 in H and a $\mathcal{C}^1$ map $g : V_1 \times V_2 \mapsto V_3$ such that*

$$(x_1, x_2, x_3) \in P \textit{ and } \mathcal{F}(x_1, x_2, x_3) = 0$$

if and only if

$$(x_1, x_2) \in P_1 \times P_2 \quad \textit{and} \quad x_3 = g(x_1, x_2).$$

In particular the set $\{x \in V_1 \times V_2 \mapsto V_3 \mid \mathcal{F}(x) = 0\}$ is a $\mathcal{C}^1$ submanifold of $V_1 \times V_2 \times V_3$ and for each $x_1 \in V_1$ the set $\{(x_2, x_3) \in V_2 \times V_3 \mid \mathcal{F}(x_1, x_2, x_3) = 0\}$ is a $\mathcal{C}^1$ submanifold of $V_2 \times V_3$.

Note that in particular if the kernel $\mathrm{ker} d_Y\mathcal{F}(0)$ has finite dimension N then $\mathrm{ker} d_Y\mathcal{F}(0)$ is complemented in Y and therefore $\{(x_2, x_3) \in V_2 \times V_3 \mid \mathcal{F}(x_1, x_2, x_3) = 0\}$ is a $\mathcal{C}^1$ submanifold of finite dimension N.

4.2 Birkhoff factorization and indices

Denote by $GL_N(\mathbb{C})$ the group of invertible $N \times N$ matrices with complex entries. Consider a map $G : \partial\Delta \to GL_N(\mathbb{C})$ and define for all $\zeta \in \partial\Delta$

$$B(\zeta) = -\overline{G(\zeta)}^{-1} G(\zeta).$$

One can find a *Birkhoff factorization* of B, namely two continuous functions $B^+ : \bar{\Delta} \to GL_N(\mathbb{C})$ and $B^- : (\mathbb{C} \cup \infty) \setminus \Delta \to GL_N(\mathbb{C})$ such that for all $\zeta \in \partial\Delta$

$$B(\zeta) = B^+(\zeta)\Lambda(\zeta)B^-(\zeta)$$

where

$$\Lambda(\zeta) = \begin{pmatrix} \zeta^{\kappa_1} & & (0) \\ & \ddots & \\ (0) & & \zeta^{\kappa_N} \end{pmatrix} \tag{4.2.1}$$

where B^+ and B^- are holomorphic on Δ and $\mathbb{C}\setminus\overline{\Delta}$ respectively. The integers $\kappa_1 \geq \ldots \geq \kappa_N$ do not depend on this factorization and are called the *partial indices* of B. The *Maslov index* of B is their sum $\sum_{j=1}^{N} \kappa_j$. Although computing the partial indices is rather challenging, the Maslov index is simply given by a winding number (see [**?**] for instance):

Lemma 4.2.1 ([**?**]). *Assume that the determinant* $\det B$ *is of class* $\mathcal{C}^1$ *on* $\partial\Delta$. *Then the Maslov index of* B *is given by*

$$\mathrm{Ind}_{\det B(\partial\Delta)}(0) = \frac{1}{2\pi i}\int_{\partial\Delta} \frac{(\det B)'(\zeta)}{\det B(\zeta)}\,\mathrm{d}\zeta.$$

Proof. Suppose ζ is on the unit disc, i.e suppose the partial indices are given by the following decomposition (4.2.1), for all θ

$$B(e^{i\theta}) = B^+(e^{i\theta}) \begin{pmatrix} e^{i\kappa_0\theta} & & (0) \\ & \ddots & \\ (0) & & e^{i\kappa_{2n}\theta} \end{pmatrix} B^-(e^{i\theta})$$

where B^+ can be holomorphically extended to Δ by an invertible matrix, and B^- can be anti-holomorphically extended to Δ by an invertible matrix, which is equivalent to the existence of $\tilde{B}^-$, that is holomorphic in Δ such that $B^-(\zeta) = \tilde{B}^-\left(\frac{1}{\zeta}\right)$ for all $\zeta \in \hat{\mathbb{C}} \setminus \Delta$. Consider $0 < r < 1$ and let

$$b_r^+(\theta) = det(B^+(re^{i\theta})),$$

$$b_r^-(\theta) = det(\tilde{B}^-(\overline{re^{i\theta}})) = det(\tilde{B}^-(re^{-i\theta}))$$

and

$$\beta_r(\theta) = b_r^+(\theta) r^\kappa e^{i\kappa\theta} b_r^-(\theta).$$

Since the curve $\gamma_r = \beta_r([0, 2\pi])$ does not pass by 0, because $\beta_r(\theta)$ is not 0 on $[0, 2\pi]$, we can then define the index:

$$2\pi i Ind_{\gamma_r}(0) = \int_{\gamma_r} \frac{d\zeta}{\zeta} = \int_0^{2\pi} \frac{b_r^{+'}(\theta)}{b_r^+(\theta)} d\theta + \int_0^{2\pi} i\kappa d\theta + \int_0^{2\pi} \frac{b_r^{-'}(\theta)}{b_r^-(\theta)} d\theta.$$

The integral

$$\int_0^{2\pi} \frac{b_r^{+'}(\theta)}{b_r^+(\theta)} d\theta = \int_{r\partial\Delta} \frac{det(B^+(\zeta))'}{det(B^+(\zeta))} d\zeta$$

is equal by Cauchy's argument principle to {number of zeros - number of poles} of the holomorphic function $det(B^+)$ in $r\bar{\Delta}$, which is equal to 0 since B^+ is invertible. Similarly

$$\int_0^{2\pi} \frac{b_r^{-'}(\theta)}{b_r^-(\theta)} d\theta = 0.$$

Hence we get for all $0 < r < 1$

$$Ind_{\gamma_r}(0) = \kappa.$$

Now since the compact set $\{\beta_r(\theta) \mid r \in [1/2, 1],\ \theta \in [0, 2\pi]\}$ does not contain 0, it is then contained in an open set Ω that does not contain 0. The closed curves $\gamma_{1/2}$ and γ_1 are of class $\mathcal{C}^1$ and homotopic in Ω by the application

$$(t, \theta) \mapsto \beta_{(1-t)/(2+t)}(\theta).$$

Since any two homotopic curves have the same index, we get:

$$Ind_{\gamma_{1/2}}(0) = \kappa = Ind_{\gamma_1}(0) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{\beta_1'(\theta)}{\beta_1(\theta)} d\theta.$$

and $\beta_1(\theta) = det(B(e^{i\theta}))$. $\square$

Note that a consequence of 4.2.1 is that the Maslov index is invariant under homotopy. This fact will play a major role in constructing stationary discs in Section 4.4.

4.3 Equations of the conormal fibration $\mathscr{N}\Gamma$

Let Q be the hyperquadric in $\mathbb{C}^2$ defined by

$$r(z) = \Re e z_0 - |z_1|^2.$$

Recall that for $\zeta \in \partial\Delta$

$$\mathscr{N}Q(\zeta) = \{(z, \zeta w) \mid z \in \Gamma,\ w \in N_z^* Q \setminus \{0\}\}.$$

In Example 2.2.5 we have described the equations of the conormal bundle N^*Q. In this section we find the defining equations for the conormal fibration $\mathscr{N}Q$. Although the computation is essentially the same as in Example 2.2.5 we include it for seek of completeness.

We have $\partial r(z) = \left(\frac{1}{2}, -\bar{z_1}\right)$, from which it follows that the conormal fiber is

$$N_z^* Q = \operatorname{span}_{\mathbb{R}} \left\{ \left(\frac{1}{2}, \bar{z_1}\right) \right\}.$$

Note that $(z, w) = (z_0, z_1, w_0, w_1) \in \mathscr{N}Q \subset \mathbb{C}^4$ if and only if $r(z) = 0$ and $w = c\zeta\partial r(z) = c\zeta\left(\frac{1}{2}, -\bar{z_1}\right)$. Therefore $(z, w) \in \mathscr{N}Q$ if and only if $r = 0$, $\frac{w_0}{\zeta} \in \mathbb{R}$ and $w_1 = -2w_0\bar{z_1}$. Therefore we obtain the following defining equations:

$$\begin{cases} \tilde{r}_0(\zeta)(z,w) & = & \dfrac{z_0 + \overline{z}_0}{2} - \bar{z_1} = 0, \\ \\ \tilde{r}_1(\zeta)(z,w) & = & i\dfrac{w_o}{\zeta} - i\zeta\overline{w}_0 = 0, \\ \\ \tilde{r}_2(\zeta)(z,w) & = & (w_1 + 2w_0\bar{z_1}) + (\overline{w_1 + 2w_0\bar{z_1}}) = 0, \\ \\ \tilde{r}_3(\zeta)(z,w) & = & i\left(w_1 + 2w_0\bar{z_1}\right) - i\left(\overline{w_1 + 2w_0\bar{z_1}}\right) = 0, \\ \end{cases} \tag{4.3.1}$$

and $(z, w) \in \mathscr{N}Q$ if and only if

$$\tilde{r}_0(\zeta)(z,w) = \tilde{r}_1(\zeta)(z,w) = \tilde{r}_2(\zeta)(z,w) = \tilde{r}_3(\zeta)(z,w) = 0.$$

Similarly if $\Gamma = \{\rho = 0\} \subset \mathbb{C}^2$ is a real smooth non-degenerate hypersurface the conormal fibration $\mathscr{N}\Gamma$ is also described by four real valued equations

$$\tilde{\rho}_0(\zeta)(z,w) = \tilde{\rho}_1(\zeta)(z,w) = \tilde{\rho}_2(\zeta)(z,w) = \tilde{\rho}_3(\zeta)(z,w) = 0.$$

We set

$$\tilde{\rho}(\zeta)(z,w) = \left(\tilde{\rho}_0(\zeta)(z,w), \tilde{\rho}_1(\zeta)(z,w), \tilde{\rho}_2(\zeta)(z,w), \tilde{\rho}_3(\zeta)(z,w)\right).$$

We can now investigate how these defining equations may help us in finding stationary discs. Recall that a holomorphic disc h attached to Γ is stationary if there exists a holomorphic lift $\boldsymbol{h} = (h,g) : \Delta \to T^*\mathbb{C}^{n+1}$ of h to the cotangent bundle $T^*\mathbb{C}^{n+1}$, continuous up to the boundary and such that for all $\zeta \in \partial\Delta$, $\boldsymbol{h}(\zeta) \in \mathscr{N}\Gamma(\zeta)$. Therefore h is stationary for Γ if and only if for all $\zeta \in \partial\Delta$

$$\tilde{\rho}(\zeta)(\boldsymbol{h}(\zeta)) = 0. \tag{4.3.2}$$

This boundary value problem (4.3.2) is generally called a non linear Riemann-Hilbert problem. Its solution is proposed in the next section.

4.4 Construction of stationary discs

Let Q be the hyperquadric in $\mathbb{C}^2$ defined by

$$r(z) = \Re e z_0 - |z_1|^2.$$

In Section 2, we have described all stationary for Q. Our goal is to construct stationary discs for small enough perturbations of Q using the implicit function theorem 4.1.2. We need to define the corresponding Banach spaces X, Y, Z and the mapping $\mathcal{F}$.

We first introduce, for $0 < \alpha < 1$,

$$X = \mathcal{C}^{1,\alpha}(\partial\Delta, \mathcal{C}^3(\mathbb{C}^4, \mathbb{R}^4)).$$

Roughly speaking, X is the set of equations of possible conormal fibrations. The smoothness assumption is of purely technical nature. Define now

$$Y = \mathcal{A}^{1,\alpha}(\Delta, \mathbb{C}^4)$$

to be the set of holomorphic discs valued in $\mathbb{C}^4$ of class $\mathcal{C}^{1,\alpha}$ up to the boundary $\partial\Delta$. We set

$$Z = \mathcal{C}^{1,\alpha}(\partial\Delta, \mathbb{R}^4).$$

Recall that the conormal fibration of Q is given by $\tilde{r} = 0$ (see Equation (4.3.1)). Let U be a neighborhood of $\tilde{r}$ in X. Consider $\boldsymbol{h_0}$ a lift of a stationary disc in $\mathcal{S}^*(Q)$ and let V be a neighborhood of $\boldsymbol{h_0}$ in Y. Inspired by (4.3.2), we define the map

$$\mathcal{F} : U \times V \to Z$$

by

$$\mathcal{F}(\tilde{\rho}, \boldsymbol{h}) = \tilde{\rho}(.)(\boldsymbol{h}).$$

Note that, by the chain rule, the map is well defined. An important and technical result due to [?] states that $\mathcal{F}$ is $\mathcal{C}^1$. Moreoever if $\Gamma = \{\rho = 0\}$ is a non-degenerate hypersurface in $\mathbb{C}^2$ then $\mathcal{F}(\tilde{\rho}, \boldsymbol{h}) = 0$ if and only if $\boldsymbol{h}$ is a lift of a stationary disc for Γ. In particular we have $\mathcal{F}(\tilde{r}, \boldsymbol{h_0}) = 0$. In order to apply Theorem 4.1.2, we need to show that $d_Y\mathcal{F}(\tilde{r}, \boldsymbol{h_0})$ is onto, prove that its kernel has finite dimension and compute its dimension.

It derivative in the space Y is given by

$$d_Y\mathcal{F}(\tilde{r}, \boldsymbol{h_0})(\boldsymbol{h}) = 2\Re e \left(\overline{G}\boldsymbol{h}\right)$$

where $G : \partial\Delta \to GL_4(\mathbb{C})$ is the following 4×4 matrix map

$$G(\zeta) = \left(\frac{\partial \tilde{r}}{\partial \bar{z}}(\boldsymbol{h_0}(\zeta)), \frac{\partial \tilde{r}}{\partial \bar{w}}(\boldsymbol{h_0}(\zeta))\right) = \begin{pmatrix} 1/2 & -z_1(\zeta) & 0 & 0 \\ 0 & 0 & -i\zeta & 0 \\ 0 & 2w_0(\zeta) & 2z_1(\zeta) & 1 \\ 0 & 2iw_0(\zeta) & -2iz_1(\zeta) & -i \end{pmatrix}$$

where

$$z_1(\zeta) = 2|v|^2 \frac{(1-a)(1-\zeta)}{(1-a\zeta)(1-|a|^2)}$$

$$w_0(\zeta) = \frac{b\zeta|1-a\zeta|^2}{2}$$

since $\boldsymbol{h}_0$ is the lift of a stationary discs in $\mathcal{S}^*(Q)$ (see Equation (3.1.1)). It is important to point out that the fact that $G(\zeta)$ is invertible for all ζ follows from the fact the conormal fibration of the hyperquadric Q is totally real; which in turn follows from the non degeneracy of Q. In order to prove that $d_Y\mathcal{F}(\tilde{r}, \boldsymbol{h_0})$ is onto and to compute the dimension its kernel we use a result of Globevnik [**?**, **?**], namely that $d_Y\mathcal{F}(\tilde{r}, \boldsymbol{h_0})$ is onto if and only if the partial indices are greater than or equal to -1 and in that case the dimension of its kernel is equal to the Maslov index. Therefore, we need to compute the partial indices and the Maslov index of

$$B(\zeta) = -\overline{G(\zeta)}^{-1} G(\zeta).$$

To achieve this goal we perform operation on $G(\zeta)$. Right multiplication by the constant matrix $\begin{pmatrix} 2 & 0 \\ 0 & I_3 \end{pmatrix}$ does not change the partial indices and gives us the matrix

$$\begin{pmatrix} 1 & -2z_1(\zeta) & 0 & 0 \\ 0 & 0 & -i\zeta & 0 \\ 0 & 2w_0(\zeta) & 2z_1(\zeta) & 1 \\ 0 & 2iw_0(\zeta) & -2iz_1(\zeta) & -i \end{pmatrix}.$$

After permuting the rows, we get:

$$\begin{pmatrix} 1 & -2z_1(\zeta) & 0 & 0 \\ 0 & 2w_0(\zeta) & 2z_1(\zeta) & 1 \\ 0 & 2iw_0(\zeta) & -2iz_1(\zeta) & -i \\ 0 & 0 & -i\zeta & 0 \end{pmatrix}$$

Now we permute the columns and get a triangular by block matrix:

$$\begin{pmatrix} 1 & -2z_1(\zeta) & 0 & 0 \\ 0 & 2w_0(\zeta) & 1 & 2z_1(\zeta) \\ 0 & 2iw_0(\zeta) & -i & -2iz_1(\zeta) \\ 0 & 0 & 0 & -i\zeta \end{pmatrix}$$

with $(z, w) = \boldsymbol{h}(\zeta)$, $\zeta \in \partial\Delta$. By multiplying the second column by $\frac{1}{b(1-\bar{a}\bar{\zeta})}$ and the third column by $1 - \bar{a}\bar{\zeta}$, we do not change the partial indices and the resulting matrix is of the form:

$$G_1(\zeta) = \begin{pmatrix} 1 & & (*) & \\ & & P & \\ & (0) & & -i\zeta \end{pmatrix}$$

where

$$P = \begin{pmatrix} \zeta(1 - a\zeta) & 1 - \bar{a}\bar{\zeta} \\ i\zeta(1 - a\zeta) & -i(1 - \bar{a}\bar{\zeta}) \end{pmatrix}$$

and so

$$\overline{P^{-1}} = \begin{pmatrix} \frac{1}{2\bar{\zeta}(1-\bar{a}\bar{\zeta})} & \frac{i}{2\bar{\zeta}(1-\bar{a}\bar{\zeta})} \\ \frac{1}{2(1-a\zeta)} & \frac{-i}{2(1-a\zeta)} \end{pmatrix}.$$

It follows that we are reduced to compute the partial indices of the matrix

$$B_1(\zeta) = -\overline{G_1(\zeta)^{-1}}G_1(\zeta) = -\begin{pmatrix} 1 & & (*) \\ & \overline{P^{-1}}P & \\ & & \\ (0) & & -\zeta^2 \end{pmatrix} = -\begin{pmatrix} 1 & & (*) \\ & R & \\ & & \\ (0) & & -\zeta^2 \end{pmatrix}$$

where $R = \begin{pmatrix} 0 & \zeta \\ \zeta & 0 \end{pmatrix}$. In order to find the partial indices of $G_1(\zeta)$, we use the following lemma:

Lemma 4.4.1 ([?])**.** *Let $A : \partial\Delta \to GL_{2n+2}(\mathbb{C})$ of class $\mathcal{C}^\alpha$ $(0 < \alpha < 1)$, and denote by $\kappa_1 \geq \ldots \geq \kappa_{2n+2}$ the partial indices of the map $\zeta \mapsto A(\zeta)\overline{A(\zeta)^{-1}}$. Then there exists a map $\Theta : \bar{\Delta} \to GL_{2n+2}(\mathbb{C})$ of class $\mathcal{C}^\alpha$, holomorphic on Δ, such that for all $\zeta \in \partial\Delta$*

$$\Theta(\zeta)A(\zeta)\overline{A(\zeta)^{-1}} = \begin{pmatrix} \zeta^{\kappa_1} & & (0) \\ & \ddots & \\ (0) & & \zeta^{\kappa_{2n+2}} \end{pmatrix}\overline{\Theta(\zeta)}.$$

By applying this lemma to the matrix $A = i\overline{G_1(\zeta)^{-1}}$, we obtain a continuous map $\Theta : \bar{\Delta} \to GL_4(\mathbb{C})$, holomorphic on Δ such that for all $\zeta \in \partial\Delta$:

$$\Theta(\zeta)B_1(\zeta) = \begin{pmatrix} \zeta^{\kappa_1} & & (0) \\ & \ddots & \\ (0) & & \zeta^{\kappa_4} \end{pmatrix}\overline{\Theta(\zeta)}.$$

Denote by $l = (l_1, \ldots, l_4)$ the last row of the matrix Θ. It follows that for all $\zeta \in \partial\Delta$

$$l(\zeta)B_1(\zeta) = \zeta^{\kappa_4}\overline{l(\zeta)}. \tag{4.4.1}$$

- If $l_1 \not\equiv 0$ then (4.4.1) gives $-l_1(\zeta) = \zeta^{\kappa_4}\overline{l_1(\zeta)}$ and by holomorphy of Θ we get $\kappa_4 \geq 0$.

- If $l_1 \equiv 0$ then (4.4.1) gives two equations $-\zeta l_3(\zeta) = \zeta^{\kappa_4}\overline{l_2(\zeta)}$ and $-\zeta l_2(\zeta) = \zeta^{\kappa_4}\overline{l_3(\zeta)}$.
  - If $l_2 \not\equiv 0$ then $l_3 \not\equiv 0$ we obtain $\kappa_4 \geq 1$ by holomorphy.
  - If $l_2 \equiv 0$ then $l_3 \equiv 0$ then since the matrix $\Theta(\zeta)$ is invertible $l_4 \not\equiv 0$. In that case (4.4.1) gives $\zeta^2 l_4(\zeta) = \zeta^{\kappa_4}\overline{l_4(\zeta)}$ and by holomorphy of Θ we get $\kappa_4 \geq 2$.

Since $\kappa_1 \geq \ldots \geq \kappa_4$, we have proved that the partial indices of $B(\zeta) = -\overline{G(\zeta)}^{-1}G(\zeta)$ are nonnegative. Therefore by a result of Globevnik [**?**, **?**], the linear map $d_Y\mathcal{F}(\tilde{r}, \boldsymbol{h_0})$ is onto. Furthermore, by Lemma 4.2.1 its Maslov index is equal

$$\kappa = \frac{1}{2\pi i}\int_{\partial\Delta}\frac{(\mathrm{det}B)'(\zeta)}{\mathrm{det}B(\zeta)}\,\mathrm{d}\zeta = \frac{1}{2\pi i}\int_{\partial\Delta}\frac{(\mathrm{det}B_1)'(\zeta)}{\mathrm{det}B_1(\zeta)}\,\mathrm{d}\zeta = 4$$

since $\det(B_1(\zeta)) = \zeta^4$. It follows from [**?**, **?**] that the kernel of $d_Y\mathcal{F}(\tilde{r}, \boldsymbol{h_0})$ has dimension $\kappa + dim_{\mathbb{C}}(\mathbb{C}^4) = 4 + 4 = 8$. By applying Theorem 4.1.2, we finally obtain

Theorem 4.4.2. *There are neighborhoods V_1 of $\tilde{r}$ in X and V_2 of $\boldsymbol{h_0}$ in Y such that for each $\tilde{\rho} \in V_1$, the set*

$$\{\boldsymbol{h} \in V_2 \mid \mathcal{F}(\tilde{\rho}, \boldsymbol{h}) = 0\}$$

is a $\mathcal{C}^1$ submanifold of X of real dimension 8*.*

In other words

Theorem 4.4.3. *Let $Q = \{r = 0\}$ where $r(z) = \Re e z_0 - |z_1|^2$. Fix $\boldsymbol{h_0} \in \mathcal{S}^*(Q)$ and $0 < \alpha < 1$. There exist $\delta_1, \delta_2 > 0$, both depending on Q and $\boldsymbol{h_0}$, such that if $\|\rho - r\|_{\mathcal{C}^4} < \delta_1$ (with ρ in normal form (2.1.2)), the set*

$$\mathcal{S}^{\delta_2}(\Gamma) = \{\boldsymbol{h} = (h, g) \in \mathcal{S}(\Gamma) \mid \|\boldsymbol{h} - \boldsymbol{h_0}\|_{\mathcal{C}^{1,\alpha}(\partial\Delta)} < \delta_2\}$$

forms a 8 real parameter family, where $\Gamma = \{\rho = 0\}$.

Moreover since this family is obtained by a deformation argument, the geometric property stated in Proposition 3.2.1 remains true:

Proposition 4.4.4. *Let $Q = \{r = 0\}$ where $r(z) = \Re e z_0 - |z_1|^2$. Fix $\boldsymbol{h_0} \in \mathcal{S}^*(Q)$ and $0 < \alpha < 1$. There exists $\delta_1, \delta_2 > 0$ such that if $\|\rho - r\|_{\mathcal{C}^4} < \delta_1$ then the set of directions*

$$\left\{ \frac{dh}{d\theta}|_{\theta=0} \mid \boldsymbol{h} = (h, g) \in \mathcal{S}^{\delta_2}(\Gamma) \cap \mathcal{S}^*(\Gamma) \right\}$$

fills an open set in $T_{(0,0)}\Gamma$.

Proof. The implicit Theorem 4.4.2 gives a neighborhood V_1 of $\tilde{r}$ and a neighborhood U of the origin in $\mathbb{R}^8$ and a parametrization map of stationary discs

$$\mathcal{H} : V_1 \times U \to \mathcal{A}^{1,\alpha}(\Delta, \mathbb{C}^4)$$

of class $\mathcal{C}^1$ with

i. $\mathcal{H}(r, 0) = \boldsymbol{h_0}$,

ii. for all $\rho \in V_2$, the map $\mathcal{H}(\rho, \cdot) : U \to \mathcal{S}^{\delta_2}(\Gamma)$ is bijective. Here $\Gamma = \{\rho = 0\}$.

Now consider the $\mathcal{C}^1$ map $\psi : V_1 \times U \to \{0\} \times \mathbb{R}^3$ defined by

$$\psi(\rho, t) = \frac{d(\pi \circ \mathcal{H}(\rho, t))}{d\theta}|_{\theta=0}$$

where π is the canonical projection onto the first 2 components. According to Remark 3.2.2, $\frac{\partial}{\partial t}\psi(r, 0)$ has rank 3 and more precisely the submatrix of $\frac{\partial}{\partial t}\psi(r, 0)$ formed by the derivatives in the direction of a and w is of rank 3. Hence for ρ sufficiently close to r, we still get that the submatrix of $\frac{\partial}{\partial t}\psi(r, 0)$ formed by the derivatives in the direction of a and w is of rank 3 and the result follows. $\square$

CHAPTER 5

EXTENSION OF BIHOLOMORPHISM

First we recall the classical Riemann mapping theorem (see [?] for instance):

Theorem 5.0.5 ([?]). *Let U be a non empty open subset of $\mathbb{C}$ not equal to $\mathbb{C}$ with U simply connected, then there exists a biholomorphic mapping f from U onto the open unit disk Δ.*

O.D. Kellogg studied the boundary regularity of the Riemann mapping ([?], [?], [?]). However, Kellogg's original theorem and its proof do not provide sharp regularity. Nowadays, there are several methods to obtain the exact regularity for Riemann mappings for a fixed domain; for instance, see the monography of G.M. Goluzin [?] p. 426.

Theorem 5.0.6 (Kellogg's theorem [?] p. 426). *Let U be a non empty open subset of $\mathbb{C}$ not equal to $\mathbb{C}$ with U simply connected. Suppose U has a $\mathcal{C}^{k,\alpha}$ boundary. Then the Riemann mapping and its inverse extend as $\mathcal{C}^{k,\alpha}$ up to the boundary.*

In particular if the domain $U \in C$ has a $\mathcal{C}^\infty$ boundary, then the Riemann mapping and its inverse, extend $\mathcal{C}^\infty$ smoothly to the boundary. The equivalent smoothness result in higher dimension is due to C. Fefferman [?]:

Theorem 5.0.7 (Fefferman [?]). *Let $D_1, D_2 \subset \mathbb{C}^2$ be two $\mathcal{C}^\infty$ smoothly bounded strictly pseudoconvex domains and let $F : D_1 \to D_2$ be a biholomorphic mapping. Then F is of class $\mathcal{C}^\infty$ up to the boundary ∂D_1.*

Note that the original proof by Fefferman, based on estimates on the Bergman metric, is rather technical. Simpler proofs were later provided by Bell and Ligocka [?], Lempert [?, ?], Pinchuk and Khasanov [?] or Forstnerič [?]. We propose to follow the approach of A. Tumanov [?] based on the local theory of stationary discs. His method is very similar to the one of L. Lempert, but instead of using the difficult global theory of [?], A. Tumanov's approach uses local results. In particular we will use a separate smoothness result due to A. Tumanov [?]

Proposition 5.0.8 (Proposition 3.1 [?]). *Let k be a nonnegative integer and let $0 < \alpha < 1$. Let F_j, $1 \leq j \leq n$, be $\mathcal{C}^{k+1,\alpha}$ smooth foliations of a domain $\Omega \subset \mathbb{R}^n$ such that for every point $p \in \Omega$ the tangent vectors to the curves $\gamma_j \in F_j$ passing through p are linearly independent. Let f be a function on Ω such that the restrictions $f|\gamma$, $\gamma \in F_j$, $1 \leq j \leq n$ are of class $\mathcal{C}^{k,\alpha}$ and are uniformly bounded in the $\mathcal{C}^{k,\alpha}$ norm. Then f is $\mathcal{C}^{k,\alpha}$ smooth.*

By a $\mathcal{C}^{k,\alpha}$ smooth foliation, we mean a family of curves that cover all of a domain $\Omega \in \mathbb{R}^n$, and that, after a change of coordinates, they become parallel lines.

Remark 5.0.9. *The uniform smoothness assumption in 5.0.8 is essential as illustrated by the following example:*

$$f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2} & if \quad (x,y) \neq (0,0) \\ 0 & if \quad (x,y) = (0,0) \end{cases}$$

Indeed, the restriction of f to each line parallel to either the x-axis or the y-axis is $\mathcal{C}^\infty$. However, the function f is discontinuous at the origin since on the line $y = x$ we have $f(x,x) = 1$. Note that on the lines $L_n : y = \frac{1}{n}$ the norms

$$\sup_{(x,y)\in L_n} \left\| \frac{\partial f}{\partial x}(x,y) \right\| = \sup \left\| \frac{2n - 2n^3x^2}{(1+n^2x^2)^2} \right\|$$

blow up as $n \to \infty$.

We are now able to prove Theorem 5.0.7

Proof. We first point out that F extends as a $\mathcal{C}^{\frac{1}{2}}$ homeomorphism from ∂D_1 to ∂D_2 (see [**?**] for instance); this step is rather classical and since it does not use the theory of stationary disc we will omit its proof.

Let $p \in \partial D_1$. Assume that $p = 0 \in \partial D_1$ and that ∂D_1 has the local definition function (2.1.2):

$$\rho(z) = \Re e z_0 + |z_1|^2 + b_0 y_0^2 + (b_1 z_1 + \bar{b}_1 \bar{z}_1) y_0 + O(|(y_0, z_1|^3).$$

Since ∂D_1 is strictly pseudoconvex. Consider a small enough neighborhood U of the origin $p = 0$ such that ρ is close enough to $r^+ = \Re e z_0 + |z_1|^2$ in the $\mathcal{C}^4$ topology in order to obtain, by Proposition 4.4.4, that the set

$$\left\{ \frac{dh}{d\theta}|_{\theta=0} \mid \boldsymbol{h} = (h, g) \in \mathcal{S}^{\delta_2}(\Gamma) \cap \mathcal{S}^*(\Gamma) \right\}$$

for some $\delta_2 > 0$, fills an open set in $T_p \partial D_1$. Therefore the directions of boundaries of stationary discs span all directions in $T\partial D_1$. Please note that Proposition 4.4.4 was obtained for small perturbations of $\Re e z_0 - |z_1|^2$ but remains true when considering small perturbations of $\Re e z_0 + |z_1|^2$.

Since $F : \partial D_1 \to \partial D_2$ is of class $\mathcal{C}^{\frac{1}{2}}$, F maps stationary discs to stationary discs. Recall also that stationary discs inherit the smoothness of the hypersurface they are attached to (see Section 2.3). Therefore F maps smooth boundaries of stationary discs to smooth boundaries of stationary discs. Moreover since the lifts of stationary discs are attached to a totally real submanifold, their $\mathcal{C}^k$ norms, for any k, is controlled by their $\mathcal{C}^\alpha$ norm which in turn are uniformly bounded. Therefore, the separate smoothness principle allows to conclude that F is smooth up to ∂D_1.

□