AMERICAN UNIVERSITY OF BEIRUT

CONTINUOUS (s,S) INVENTORY POLICY WITH NON-STATIONARY STOCHASTIC DEMAND

by

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AN ABSTRACT OF THE THESIS OF

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We consider a single-item inventory model with non-stationary stochastic demand. Non-stationary stochastic demand is applicable to a large number of real world supply chain systems. Dynamically changing (s_t, S_t) policies are shown to be optimal in the existing literature, Song and Zipkin (1993).

In this thesis, we present relatively a new approach to model the non-stationary stochastic demand and inventory position processes. Our analytical model considers both a general phase-type (Ph_t) distribution and a special two-level mixture of Erlangs of common order (2-MECO) Ph_t distribution to serve as an approximation of the demand process. The approximate Ph_t distribution allows us to compute the expectation and variance of the demand, inventory position, net inventory and number of orders in function of time.

We then propose an optimization heuristic to compute the dynamic time dependent reorder and order up-to levels (s_t, S_t) that minimizes the total expected cost. Finally, we test our findings using numerical examples.

Contents

List of Figures

List of Tables

Chapter 1

INTRODUCTION

Most recent studies on inventory management focused on incorporating more realistic assumptions in their inventory models in order to reduce the gap between theory and practice. A realistic assumption is related to the demand and its non-stationary (time varying) vs. stationary (steady) nature. Non-stationary

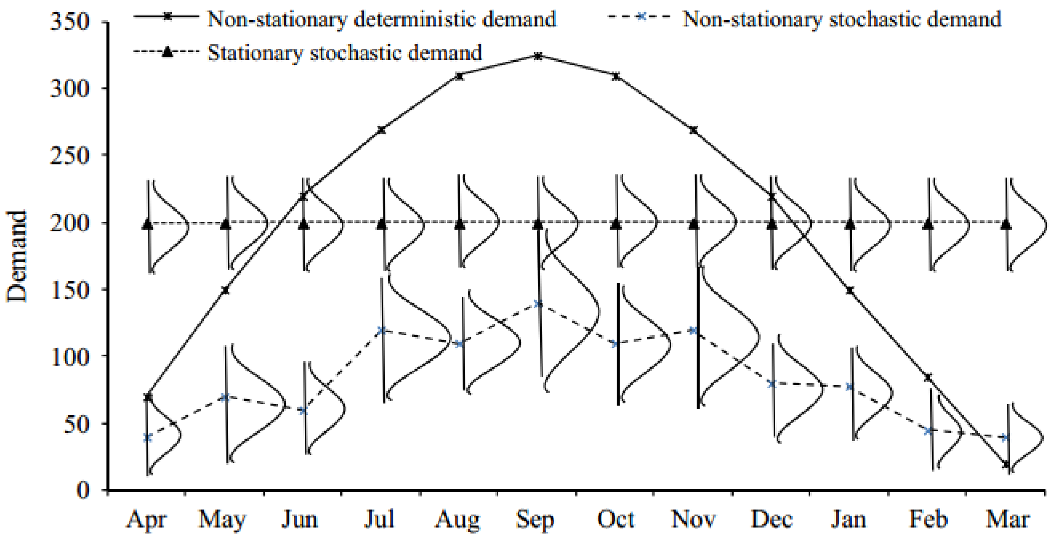

Figure 1.1: Various demand processes, Choudhary and Shankar (2015)

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stochastic demand is a pattern in which demand is random, uncertain and not constant for each time frame but varies due to seasonality, trend or other factors.

Demand mean and variance are not constant. Figure 1.1, clearly illustrates the difference between no-stationary stochastic demand and other demand processes. Inventory models that take into consideration the non-stationary nature of the demand is rather limited in the literature especially when compared to the body of work involving stationary demand. In practice, most real world applications of inventory management involve non-stationary demands, Silver (2008).

Non-stationarities in the demand may arise from factors like: short product life cycles, example: the life cycle for a Hewlett-Packard (HP) personal computer is often only 3 months! Neale and Willems (2009). On the other hand, the life cycle of an HP inkjet printer shows a lot of non-stationarity as the demand changes tremendously from the launch, ramp, peak and end of the product life-cycle , see Figure 1.2.

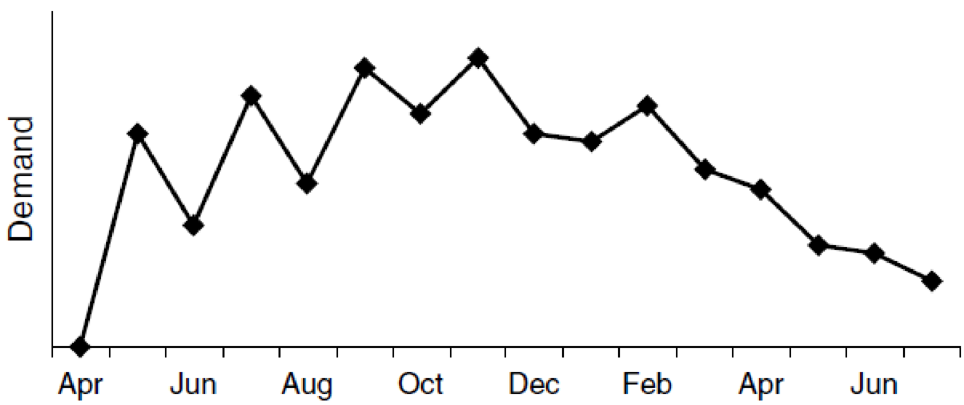


Figure 1.2: The figure shows the non-stationary demand of an HP inkjet printer. Neale and Willems (2009)

Another factor that causes non-stationary demand is the demand seasonality, example: Microsoft Xbox (Figure 1.3), Dell, Coca Cola, Kraft foods have reported seasonal demands that can be due to holidays, weather, back-to-school,. . . etc.

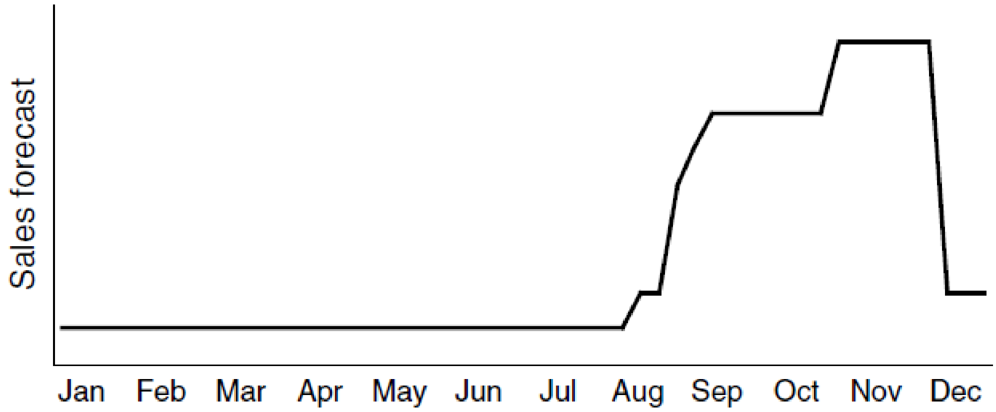


Figure 1.3: The figure shows Microsofts Xbox highly seasonal demand, where demand peaks in the weeks leading up to Christmas. Neale and Willems (2009)

Sales-force incentives and customer buying behavior are also factors that cause non-stationary demand, Figure 1.4.

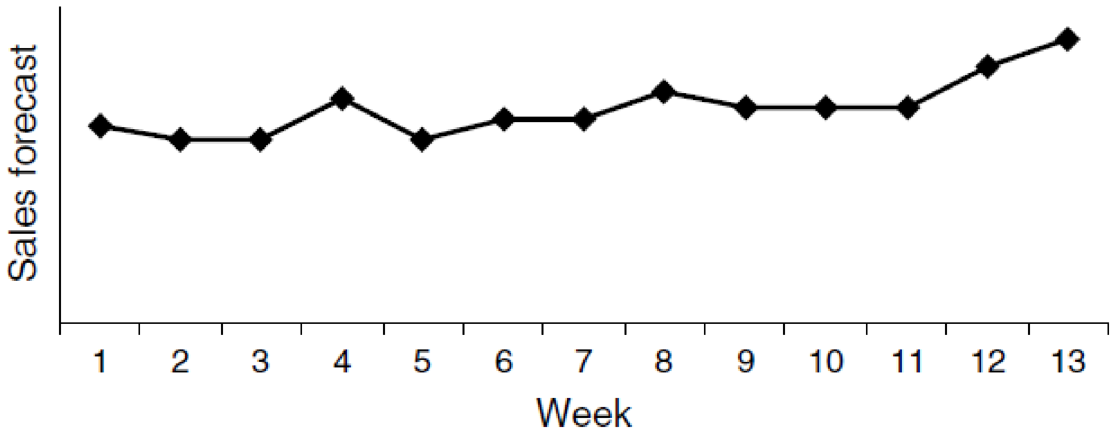


Figure 1.4: The figure shows Dell's projector end-of-month and end-of quarter demand peaks. The timeline covers one quarter with months ending in weeks 4, 8, and 13. Neale and Willems (2009)

Few authors have discussed the inventory problem with non-stationary demand from which we mention: Silver (1978), Bookbinder and Tan (1988), Bollapragada and Morton (1999) and Tarim and Kingsman (2006). Silver (1978) considered the stochastic time-varying demand by assuming normally distributed forecast errors. Bookbinder and Tan extended Silver's work by considering a rolling horizon with updated demand information. Bollapragada and Morton (1999) modeled the non-stationarity of the demand by describing the mean demand for a period as a product of a normal random variable and a seasonal factor or a geometric growth factor. They were able to compute the dynamic (s_n, S_n) by comparison with the stationary solution (s,S) of the problem. To the best of our knowledge, no one has studied the non-stationary demand by approximating it with a phase-type process. In this paper, our analytical model is similar to the one used by Nasr and Maddah (2015) but instead of using a stationary Markov Modulated Poisson Process (MMPP) correlated demand we first use a general time-dependent phase-type process and then a two-level mixture of Erlangs of common order (2-MECO) phase-type process to serve as an approximation of the non-stationary demand process. The 2-MECO phase-type distribution is adopted from Nasr and Taaffe (2012) which is utilized to fit the $Ph_t/M_t/s/c$ time-dependent departure process. We implement the fitting approach presented in Nasr and Taaffe (2012) to fit the time-dependent demand process to a 2-MECO. An efficient heuristic is then proposed to compute the optimal reorder and order up-to levels (s_t, S_t) at each period. Since its introduction by Richard Bellman especially after the release of his 1957 book, dynamic programming (DP) was ideal for inventory management problems. The value of the DP approach is in formulating models for inventory management. However, solving these models is most of the time difficult due to the dimensionality curse encountered when solving the recursive equations of the DP. For this, a heuristic optimization can be developed where the DP model comes to use in evaluating and verifying that this heuristic optimization is close to optimal. This is called policy iteration and it was first made popular by Howard (1960) in his paper about dynamic programming and Markov processes. Yet, a dynamic programming approach requires the demand data for the entire time horizon which is practically not available. Thus, the demand data will need to be obtained by forecasts that typically updates every period starting from the last period and moving backward, forcing the re-computation of all parameters every single period. Nonetheless, the computation for the dynamic programming will be prohibitively expensive especially when lead time is not zero, which complicates the dynamic process. For the above reasons we will not use dynamic programming to evaluate our heuristic and for that line-search heuristic will be used instead.

The rest of the thesis is structured as follows: Chapter 2 reviews the related literature. In Chapter 3, a background on phase-type processes is introduced. In Chapter 4, we present our analytical model. In Chapter 5, we propose a heuristic to compute the replenishment policy that minimizes expected cost. Chapter 6 contains some numerical examples. Finally, we conclude in Chapter 7 and propose some ideas for future work.

1.1 Statement of the Problem

Reducing the gap between theory and practice in inventory management problems is a challenge for researchers and practitioners. A good practical model should include all realistic assumptions essentially non-stationary stochastic demand. In this work demand is time-varying and stochastic, i.e., the distribution of the demand count occurred in a time interval depends on the length and position of the interval. Unfortunately, it is because of the analytical complexity of the problem that not much work has been done taking in to consideration the non-stationary condition of the demand. It is worth mentioning that all of the existing research does not focus their analytical study on the demand and inventory process itself but on the optimization or on the performance evaluation of the proposed policies. This does not provide a good analytical understanding of the problem on hand and thus does not open the door for more development in this field. A contribution of this work is in quantifying the impact of the time-dependent parameters and in illustrating the behavior and evolution of the inventory parameters and variables (inventory position, net inventory position, number of orders,...,etc.) all as a function of time. This will result in better decision making policies and improve our understanding of such problems, especially from a managerial perspective.

1.2 Statement of Purpose

The purpose of this research is threefold: to capture and investigate the non-stationary stochastic demand count process using the phase-type distribution approximation which is relatively a new approach to the literature, to find the system state probabilities of the inventory position, net inventory levels and number of orders in function of time relative to the (s_t, S_t) policy parameters and to propose a simple yet efficient heuristic to find the optimal (s_t, S_t) policy parameters.

Chapter 2

LITERATURE REVIEW

In this section, we review the literature related to our subject. We divide this section into three subsections. In section 2.1, we review the (s, S) policy literature; in section 2.2, we review the Markovian modulated demand literature; and finally in section 2.3 we review the literature about non-stationary demand.

2.1 (s,S) Policy Literature

Classical papers by Karlin (1958), Karlin and Scarf (1958), Bellman et al. (1955), Gaver Jr (1959, 1961) and Karlin (1960) have studied independent stochastic demands of a single product, periodic review with no fixed setup cost and successfully showed the optimality of a base-stock policy which is a special case of the (s,S) policy when s=S. Later work by Arrow et al. (1951), Dvoretzky et al. (1953), Karlin (1958), Scarf (1960), Veinott (1966) studied the optimality of the (s,S) policy of the same problem but with fixed ordering cost and backlogs. Scarf (1960) was able to prove the optimality of the (s,S) policy for a finite horizon stationary multi-period problem with certain conditions of the problem (assumed holding/backlog cost to be convex) by developing the K-convexity concept. Scarf's (1960) proof immediately extends to non-stationary costs and demand distributions. On the other hand, Veinott (1966) showed its optimality by showing that the negative of the expected cost is a unimodal function of the initial inventory level. In 1963, Iglehart proved the optimality of the (s,S)-type policy for a stationary infinite horizon model. Not until Veinott (1966), Shreve (1976), followed by Bensoussan et al. (1983) and Cheng and Sethi (1999), that lost sales have been considered. This is most probably because the lost sales model is much harder than that of the backlog model. Shreve (1976) and Bensoussan et al. (1983) used the concept of K-convexity to prove the optimality of the (s,S) policy. Veinott's (1966) approach was general for handling both lost sales and backlogs. It is worth to mention that all these results does not extend to the case at which the lead time is different from zero.

2.2 Markovian Modulated Demand Literature

More realistic inventory models that consider demand as a random variable that is dependent on environmental factors other than time was given by Karlin and Fabens (1960). The authors used a Markovian demand model that is unlike most classical inventory models depends on the state-of-the-world in each period. This takes into consideration the effects of the randomly changing environmental factors like the fluctuating economic and the uncertain market conditions on the demand which in turn affects the cost functions. The demand is modulated as a random variable having a distribution function that depends on the demand state in each period. Iglehart and Karlin (1962) used a discrete Markov processes to model dependent demand with no setup costs and managed to prove the optimality of a state-dependent base-stock policy. Song and Zipkin (1993) show that for a continuous review, state-dependent, Markov-modulated, Poisson distributed demand having linear costs, fixed order cost and backlogging in inventory, the optimality of a (s,S)-type policy. Zipkin (1989), Aviv and Federgruen (1997) and Kapuściński and Tayur (1998) investigated cyclical demand models. Beyer and Sethi (1997), Sethi and Cheng (1997) proved the optimality of (s,S)-type policy for a generalized Markovian demand distribution and a periodic review inventory model. Cheng and Sethi (1999) made an extension to their results by considering lost sales for unsatisfied demand. All proofs were given assuming a zero lead time distribution. Chen and Song (2001) studied the optimal policies for multistage inventory problems with non-stationary Markov-modulated Poisson demand process at which the demand process is governed by a discrete time Markov chain. An effective algorithm for the determination of the optimal base-stock is provided. Abhyankar and Graves (2001) studied a two-stage serial supply chain with a two-state Markov-modulated Poisson demand and were able to develop an optimization model to determine where it is best to hedge inventory. Nasr and Maddah (2015) utilized a Markov Modulated Poisson Process (MMPP) to model stochastic demand that is dependent on the state of the environment and has a fixed lead time. They were able to compute and provide an efficient optimization heuristic to derive the dynamic changing (s_n, S_n) policy for a single item, continuous and infinite horizon inventory problems. They studied and quantified the impact of autocorrelation of the MMPP demand-count process that is causing the variability in the demand. Their results show that when the demand is highly correlated the dynamically changing (s_n, S_n) policy significantly outperforms the common static heuristically computed replenishment (s, S) policies.

2.3 Non-Stationary Demand Literature

The literature on stochastic time-dependent demand in the context of inventory systems is scarce in comparison to stationary demand. Non-stationary demand can be due to seasonality or can be represented by a time varying function. Though the demand pattern may be stationary, the fact that the demand is stochastic makes the future demand unknown. However, most studies on the dynamic lot-sizing assume that future demand is known, Grewal et al. (2015). At the present time as the product life cycles are becoming shorter and a large variety of products are being introduced to the market and thus affecting customers order stability, the demands faced tend to be more non-stationary. The reason behind the small number of papers that deal with non-stationary demand relative to the large number of papers that deal with stationary demand is that non-stationary demand models are hard to compute because of their irregular structure. The literature on the non-stationary demand can be categorized in to two categories: papers that focus on the optimization and papers that focus on the performance evaluation of the proposed policies. Even though the (s,S) policy has been shown optimal for inventory problems with stationary and non-stationary demand, Karlin and Scarf (1958), Karlin (1960), Scarf (1960), not much work has been done for computing non-stationary (s,S) policies. The inventory problem with non-stationary demand was discussed by Silver (1978), Bookbinder and Tan (1988) and Tarim and Kingsman (2006) by using alternative policies such as the (R_n, S_n) policy. Silver (1978) considered the stochastic time-varying demand by assuming normally distributed forecast errors. A procedure involving sequential optimization is used then to find the periods in which to place orders, the number of periods of the horizon that has to be included in the next order and the size of the order, with the uncertainty of demands in these periods are given. This research is extended by Bookbinder and Tan (1988) by considering a rolling horizon with updated demand information. Tarim and Kingsman (2006) improved Bookbinder and Tan (1988) heuristic by making further improvements. Other heuristics used to compute near optimal (s,S) parameters, is by Bollapragada and Morton (1999). Bollapragada and Morton (1999) were able to compute optimal (s_n, S_n) levels for a single stage inventory problem facing a general non-stationary demand with proportional backorder and holding costs. They proposed a new myopic heuristic that is based on approximating part of the initial non-stationary problem with a stationary one and involves computing the static (s, S) policy replenishment values for different demand parameters from which the dynamic replenishment (s_t, S_t) policy of the non-stationary problem for period t is approximated by restricting the state space of the inventory position at the beginning of each time period to integer values. The main development behind this research is computing the upper and lower bounds of the optimal policy efficiently. Ettl et al. (2000) worked on minimizing the total expected inventory capital and on approximating the replenishment lead-times in a multistage inventory system. They modeled non-stationary demand by making the assumption that the horizon is made up of a set of stationary phases which form a rolling-horizon and then finding the optimal policy for each phase. Finally, it can be seen that the amount literature taking non-stationary demand into consideration is increasing. Tunc et al. (2011) studied the cost of using stationary inventory policies when demand is non-stationary. They took the (s,S) policy as a frame of reference, and they compared the optimal non-stationary (s,S) policy with the best possible stationary (s,S) policy in terms of cost performance. They showed that the cost of neglecting the non-stationarity of demand is significantly high for the majority of cases. There numerical study reveals that, the magnitude of the sub-optimality of stationary policies depends heavily on the variation of the demand pattern, i.e. the non-stationarity of demand, among other factors, such as, the stochasticity of demand, and cost parameters. Amaruchkul and Auwatanamongkol (2012) consider a periodic review inventory model with non-stationary stochastic demand under a dynamic (s, S) replenishment policy. They used genetic algorithm to compute the reorder and order-up-to levels which minimizes the expected total cost.

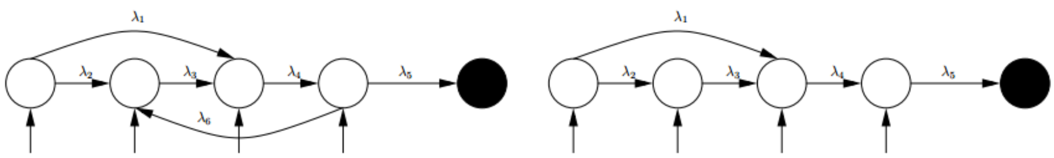
Chapter 3

PHASE-TYPE BACKGROUND

Phase-type (Ph_t) distributions were first introduced by Neuts (1975). Since then, they have been used in various stochastic modeling applications, from which we mention: Fackrell (2009) showed how and where Ph_t distributions have been used in the healthcare industry. Aalen (1995) studied the use of phase-type distribution in a series of survival analysis problems, one of which is the modification of an existing AIDS incubation model while preserving the properties and conditions of the original model. Faddy and McClean (1999) applied phase-type distribution to analyze some data on lengths of stay of hospital patients. Bladt (2005) showed the usage of Ph_t distribution in risk theory. Fazekas et al. (2002) involved phase-type distribution in their model of broadband cellular networks. Nasr and Taaffe (2012) utilized Ph_t distribution to fit time-dependent departure process in tandem queueing networks. Neuts (1975) defines a phase-type random variable as the time taken to progress through the states of a finite-state Markov chain until absorption. Phase-type distributions in stochastic models keep the Markovian structure in the models though they replace the well-known exponential distribution. Ph_t distributions are known for their great flexibility and their ability to replace exponential and non-exponential distributions while at the same time provide computational control to the model. The exponential distributions was first extended by Erlang (1917), with his method of stages, where he defines a non-negative random variable as the time needed to propagate through a fixed number of states, spending at each state an exponentially distributed amount of time. Distributions defined in this manner are now referred to as Erlang distribution. Erlang's extension opened the way for distributions that employ the well-known exponential distribution. The distribution where an exponentially distributed random variable is probabilistically chosen from a set of exponential distributions with different parameters is called the hyper-exponential distribution. The coxian distribution is similar to the Erlang distribution but in addition at each stage there is a possibility of jumping to the absorbing state directly.

For a better understanding of the Ph_t distribution parameters, we refer the reader to section 4.3 where we define clearly the parameters of the general Ph_t distribution.

Phase-type distribution classes: the most important distinction is into: Acyclic and General Ph_t distributions.



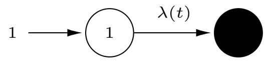
(a) General phase-type distribution containing cycles.

(b) Phase-type distribution without cycles (acyclic phase-type distribution).

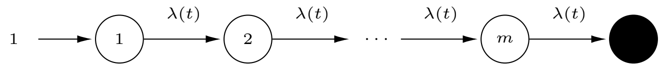
Figure 3.1: CTMC representations for cyclic and acyclic phase-type distributions

Most approaches in fitting and application of Ph_t distribution focus on the acyclic class, as this class offers better tractability than the general Ph_t class. Within acyclic Ph_t distribution, we distinguish the below sub-classes:

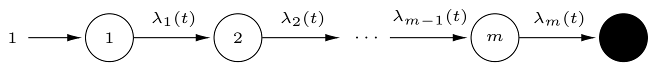
Exponential Distribution:



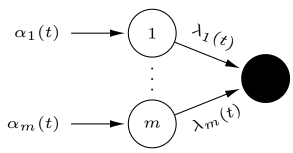
Erlang Distribution:



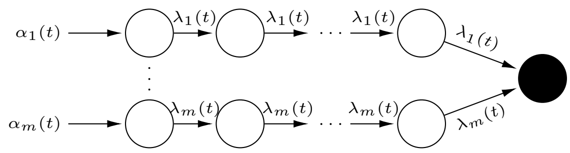
Hypo-exponential Distribution:



Hyper-exponential Distribution:



Hyper-Erlang Distribution:



The Erlang distribution is considered an acyclic phase-type distribution with low-variance. On the other hand, the hyper-exponential distribution is considered an acyclic phase-type distribution with high-variance since the Markov process can start from any phase. The structure of the hyper-Erlang distribution allows this distribution to have the flexibility to match high and low variability processes.

Fitting phase-type distribution to empirical data is extensively studied in literature. Approaches for fitting PH distributions to data include moment-matching, non-linear optimization and expectation maximization (EM) algorithms. A lot of work have been done towards fitting stationary Ph_t distributions. On the contrary the literature on fitting time-dependent Ph_t distributions is scarce due to its complexity. In this work, we use the fitting algorithm presented by Nasr and Taaffe (2012), that is based on moment-matching method, to fit time-dependent phase-type distributions.

Phase-type distributions are accredited for their great flexibility to capture moments or distribution shape and for their mathematical simplicity. The system to solve is lead back algorithmically to a Markovian system. Hence one only has to deal with systems of differential equations which is a field that has been extensively studied in the past and for which a lot of numerical solvers have already been made available.

Chapter 4

MODEL

The objective of this study is to model the non-stationary stochastic demand process and to determine the optimal reorder and order up-to levels (s_t, S_t). We present an approach to identify and capture the key characteristics of the demand process by fitting an approximate phase-type distribution to the demand process resulting in a demand forecast for the finite planning horizon. Inventory replenishment policies are investigated for the (s_t, S_t) continuous case for the finite horizon model. A Markovian representation of the system is presented along with the differential equations that model the behavior of the system over time. An algorithmic approach is presented to numerically solve the corresponding differential equations to calculate the performance measures of the system over time. Finally, We use line search optimization to compute the dynamic time-dependent reorder and order up-to levels (s_t, S_t).

To meet these objectives, the following steps will be followed:

1. Approximate the non-stationary demand process using:

   a) a general phase-type process.

   b) a two-level mixture of Erlangs of common order (2-MECO) phase-type process that will be used later on for the numerical example.

2. Write the set of differential equations (Kolmogorov Forward Equations, KFEs) that governs the approximated demand count process in a) and b).
3. Find the expected demand in function of time by numerically solving the KFEs.
4. Similarly, for a given dynamic reorder policy (s_t, S_t), write the inventory position KFEs for the general and 2-MECO phase-type processes.
5. Find the expected inventory position for a given dynamic reorder policy (s_t, S_t) by numerically solving the inventory position KFEs.
6. Find the expected net inventory position taking into consideration the lead-time.
7. Find the expected number of reorders.
8. Compute the expected inventory system costs.
9. Propose a heuristic policy that is based on line search to find the reorder and order up-to levels (s_t, S_t) that minimizes the expected costs.

4.1 Assumptions

We assume a single location, single-item and continuous review inventory control model. The demand faced is stochastic (i.e., uncertain) and non-stationary (i.e., time-dependent). Non-stationary demand is approximated with a phase-type process. The time that elapses from the time an order is placed until it arrives (lead time) is assumed fixed. Unmet demand is back-ordered. Our study is over a finite horizon. The inventory costs are the set-up cost each time an order is placed at \$$\omega$ per order, the unit order cost at \$c for each unit ordered, holding at \$h per unit held per unit time and penalty cost of \$b per unit of backordered demand.

4.2 Notation

4.2.1 Phase-Type Parameters

General $\mathbf{Ph_t}$ Parameters

- m: Total number of phases
- $\boldsymbol{\lambda}(\mathbf{t})$: $(m \times m)$ matrix containing the transition rates between the phases
- $\boldsymbol{\mu}(\mathbf{t})$: $(m \times 1)$ vector containing the absorption rates
- $\boldsymbol{\alpha(t)}$: A vector containing the initial probabilities of starting at the phases

2-MECO $\mathbf{Ph_t}$ Parameters

- m_1: Number of phases in level 1
- m_2: Number of phases in level 2
- $m = m_1 + m_2$: Total number of phases
- $\alpha(t)$: The probability of starting in phase 1
- $(1 - \alpha(t))$: The probability of starting in phase $(m_1 + 1)$
- $m_1\lambda_1(t)$: Represent the transition rate from phases 1,..., m_1
- $m_2\lambda_2(t)$: Represent the transition rate from phases $m_1 + 1, \ldots, m_1 + m_2$

4.2.2 Cost Parameters

- ω: Fixed ordering cost

- h: Holding cost
- b: Backorder cost

4.2.3 Inventory Characteristics

- $IP(t)$: Inventory position at time t
- $NI(t)$: Net inventory at time t
- $I(t)$: On-hand inventory at time t
- $B(t)$: Backorder amount at time t
- $O(t)$: Number of orders placed within the time interval $[0, t]$

4.2.4 Policy Parameters

- s_t: Reorder level at time t
- S_t: Order-up-to level at time t
- u: Upper limit on IP
- ℓ: lower limit on IP

4.3 Demand Process as Phase-Type:(Time Dependent)

In this section, we present an algorithm for computing the non-stationary demand distribution over the lead time by deriving the demand-count distribution conditional on the phase-type process being in a given state at the beginning of the lead time, and the related conditional moments at time t. This algorithm is based on numerically solving a set of differential equations (Kolmogorov Forward

Equations, KFEs). First, we define our demand model using the general phase-type process then we introduce the two-level mixture of Erlangs of common order (2-MECO) as defined by Nasr and Taaffe (2012) in their paper about fitting the $Ph_t/M_t/s/c$ time-dependent departure process for use in tandem queueing networks. The 2-MECO demand model will be used later for our numerical example.

The General $\mathbf{Ph_t}$ Demand-Count Process

In general, we define the phase-type process consisting of m transient states $S_T = \{1, 2, \ldots, m\}$ and a single absorbing state $S_A = \{m+1\}$. We write the infinitesimal generator matrix $\mathbf{Q(t)}$ at time $t \geq 0$ and the probability row vector $\boldsymbol{\alpha(t)}$ of the phase-type process as

$$\underset{(m+1,m+1)}{\mathbf{Q(t)}} = \begin{bmatrix} \underset{(m\times m)}{\boldsymbol{\lambda(t)}} & \underset{(m\times 1)}{\boldsymbol{\mu(t)}} \\ \underset{(1\times m)}{\mathbf{0}} & 0 \end{bmatrix} \text{ and } \boldsymbol{\alpha(t)} = [\alpha_1(t), \ldots, \alpha_m(t), \alpha_{m+1}(t)]$$

Where $\boldsymbol{\lambda(t)}$ is a $(m \times m)$ sub-matrix containing the transition rates at time $t \geq 0$ describing the transitions between the transient states.

$$\underset{(m\times m)}{\boldsymbol{\lambda(t)}} = \begin{bmatrix} \lambda_{1,1}(t) & \lambda_{1,2}(t) & \lambda_{1,3}(t) & \ldots & \lambda_{1,m}(t) \\ \lambda_{2,1}(t) & \lambda_{2,2}(t) & \lambda_{2,3}(t) & \ldots & \lambda_{2,m}(t) \\ \vdots & \ldots & \ddots & \ldots & \vdots \\ \vdots & \ldots & \ldots & \lambda_{m-1,m-1}(t) & \vdots \\ \lambda_{m,1}(t) & \ldots & \ldots & \ldots & \lambda_{m,m}(t) \end{bmatrix}$$

where $\lambda_{i,i}(t) = -\lambda_i(t)$ and $\lambda_i(t) = \sum_{\substack{j=1 \\ j\neq i}}^{m} \lambda_{i,j}(t) + \mu_i(t)$ for all $i = 1, \ldots, m$.

The $(m \times 1)$ vector $\boldsymbol{\mu(t)}$ contains the transition rates $\mu_i(t)$ for $i = 1, \ldots, m$ and

$t \geq 0$ from the transient states to the absorbing state. It is evident that the values of $\lambda_{i,i}(t)$, $\lambda_{i,j}(t)$ and $\mu_i(t)$ are dependent on the time $t \geq 0$ from which we intend to exploit the time-dependency in our phase-type model. The row vector $\mathbf{0}$ is a vector consisting entirely of zeros since no transitions are allowed from the absorbing state to the transient states. The last element of the matrix is 0 which represents the transition rate out of the absorbing state. The row vector $\boldsymbol{\alpha(t)}$ contains the initial probabilities that the embedded CTMC starts initially at a time t in transient state $i = 1, \ldots, m$ or directly starts in the absorbing state $m + 1$. We set $\alpha_{m+1}(t) = 0$ so the sum $\sum_{i=1}^{m} \alpha_i(t) = 1$ for all $t \geq 0$.

We first write the KFE of finding the general Ph_t process at state $\{A(t) = n\}$ at time $t \geq 0$ and $n = 1, \ldots, m$.

$$\mathrm{P}'(A(t) = n) = \sum_{w=1}^{m} \Big(\lambda_{w,n}(t) + \alpha_n(t)\, \mu_w(t)\Big) \mathrm{P}(A(t) = w) \tag{4.1}$$

To calculate the probability $\mathrm{P}(A(t) = n)$ of finding the general Ph_t process at state $\{A(t) = n\}$ for $n = 1, \ldots, m$ at time $t \geq 0$, we solve the differential equation in (4.1) with the following initial condition, $\mathrm{P}(A(0) = n) = \alpha_n(0)$.

Let $\{D_0(t) = k : 0 \leq t \leq L\}$ be the demand-count over the time interval $[0, L]$ and $\{D_{t-L}(\tau) = k : t - L > 0, \tau > 0\}$ be the demand-count over the time interval $[t - L, t - L + \tau]$ where $k = 0, 1, 2, 3, \ldots, \infty$ and L is the fixed lead time. We augment the demand-count process with the state of the Ph_t process, $\{A(t), t \geq 0\}$, and we define the state space of the demand-count and arrival-phase at time t for $0 \leq t \leq L$ and $t - L + \tau$ for $t - L > 0$ by $\{D_0(t) = k, A(t) = i\}$ and $\{D_{t-L}(\tau) = k, A(t - L + \tau) = i\}$ respectively. The KFEs of the augmented state space are presented below where the derivative is with respect to t in equation (eq.) (4.2) and τ in eq. (4.3).

For $0 \leq t < L$, $k = 0, 1, 2, \ldots, \infty$ and all $i = 1, 2, ..., m$,

$$\begin{aligned} \mathrm{P}'(D_0(t) = k, A(t) = i) = &\sum_{w=1}^{m} \lambda_{w,i}(t)\, \mathrm{P}(D_0(t) = k, A(t) = w) \\ &+ \sum_{w=1}^{m} \alpha_i(t)\, \mu_w(t)\, \mathrm{P}(D_0(t) = k-1, A(t) = w)\, \mathrm{I}_{(k>0)}. \end{aligned} \tag{4.2}$$

For $t \geq L$, $k = 0, 1, 2, \ldots, \infty$ and all $i = 1, 2, ..., m$,

let $t_s = t - L$,

$$\begin{aligned} \mathrm{P}'(D_{t_s}(\tau) = k, A(t_s + \tau) = i) = &\sum_{w=1}^{m} \lambda_{w,i}(t_s + \tau)\mathrm{P}(D_{t_s}(\tau) = k, A(t_s + \tau) = w) \\ &+ \sum_{w=1}^{m} \alpha_i(t_s + \tau)\, \mu_w(t_s + \tau)\, \mathrm{P}(D_{t_s}(\tau) = k-1, A(t_s + \tau) = w)\, \mathrm{I}_{(k>0)} \end{aligned} \tag{4.3}$$

where $\mathrm{I}_{(k>0)}$ is an indicator function such that, $\mathrm{I}_{(k>0)} = \begin{cases} 1 & \text{when } k > 0, \\ 0 & \text{otherwise.} \end{cases}$

The 2-MECO Demand-Count Process

The motivation behind adopting the 2-MECO phase-type process lies in its flexibility to match high and low variability processes over any time interval $[t, t+\tau]$, Nasr and Taaffe (2012). A count process $D_t(\tau)$ is said to have a high/low variability over an interval $[t, t+\tau]$ if $Var(D_t(\tau))(> / <)E[D_t(\tau)]$, where $Var(D_t(\tau))$ is the variance of $D_t(\tau)$. Now we present the 2-MECO process in Figure 4.1 and its properties as defined by Nasr and Taaffe (2012).

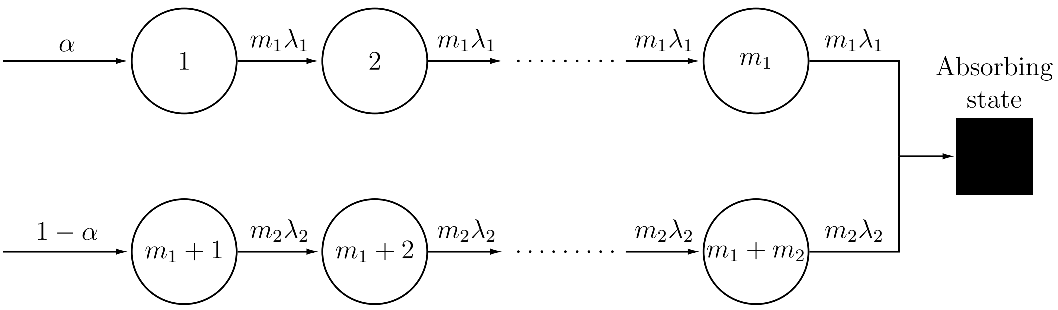


Figure 4.1: Balanced 2-MECO Phase-type Process

The 2-MECO phase-type process consists of two Erlang branches. The number of states in level 1 and 2 is m_1 and m_2, respectively. Thus, we have a total of $m = m_1 + m_2$ transient states $S_T = \{1, 2, \ldots, m_1 + m_2\}$ and a single absorbing state represented by the shaded box in Figure 4.1. α and $1-\alpha$ are the probabilities of starting at phase 1 and $m_1 + 1$, respectively. The transition rates between the states of the 1^{st} and 2^{nd} Erlang branch is $m_1\lambda_1$ and $m_2\lambda_2$, respectively. We write the infinitesimal generator matrix $\mathbf{Q(t)}$ at time $t \geq 0$ and the probability row vector $\boldsymbol{\alpha(t)}$ of the 2-MECO phase-type process as

$$\underset{(m_1+m_2+1, m_1+m_2+1)}{\mathbf{Q}(\mathbf{t})} = \begin{bmatrix} \underset{(m_1 \times m_1)}{\mathbf{Q_1}(\mathbf{t})} & 0 \\ 0 & \underset{(m_2 \times m_2)}{\mathbf{Q_2}(\mathbf{t})} \end{bmatrix}$$

$$\boldsymbol{\alpha(t)} = [\alpha(t), 0, \ldots, 0, (1 - \alpha(t)), 0, \ldots, 0]$$

Where matrix $\boldsymbol{Q_i(t)}$ for $i = 1, 2$ represents the infinitesimal matrix of the i^{th}

Erlang branch. We write the form of $\mathbf{Q_i(t)}$ for $i = 1, 2$ as

$$\underset{(m_i \times m_i)}{\boldsymbol{Q_i(t)}} = \begin{bmatrix} -m_i\lambda_i(t) & m_i\lambda_i(t) & 0 & \dots & 0 & 0 \\ 0 & -m_i\lambda_i(t) & m_i\lambda_i(t) & \dots & 0 & 0 \\ \vdots & \dots & \ddots & \dots & \vdots & \vdots \\ \vdots & \dots & \dots & -m_i\lambda_i(t) & m_i\lambda_i(t) & 0 \\ 0 & \dots & \dots & \dots & -m_i\lambda_i(t) & m_i\lambda_i(t) \\ 0 & \dots & \dots & \dots & 0 & 0 \end{bmatrix}$$

P_i being the steady state probability, i.e., $\mathrm{P}_i = \lim_{t\to\infty} \mathrm{P}_{ji}(t) \; \forall \; i, j \in \{1, \dots, m_1 + m_2\}$, the 2-MECO is said to be balanced if $\alpha\lambda_1^{-1} = (1-\alpha)\lambda_2^{-1}$ which results in

$$\sum_{i=1}^{m_1} \mathrm{P}_i = \sum_{i=m_1+1}^{m_1+m_2} \mathrm{P}_i = \frac{1}{2} \implies \mathrm{P}_i = \begin{cases} \frac{1}{2m_1} & \text{for } i = 1, \dots, m_1, \\ \frac{1}{2m_2} & \text{for } i = m_1+1, \dots, m_1+m_2. \end{cases}$$

Now, we write the KFEs of finding the 2-MECO Ph_t process at state $\{A(t) = n\}$ at time $t \geq 0$, for $n = 1, \dots, m_1$,

$$\begin{aligned} \mathrm{P}'(A(t) = n) &= -m_1\lambda_1(t)\,\mathrm{P}(A(t) = n) + m_1\lambda_1(t)\,\mathrm{P}(A(t) = n-1)(1 - \mathrm{I}_{(n=1)}) \\ &\quad + \alpha(t)\Big(m_1\lambda_1(t)\,\mathrm{P}(A(t) = m_1) + m_2\lambda_2(t)\mathrm{P}(A(t) = m_1 + m_2)\Big)\mathrm{I}_{(n=1)}, \end{aligned} \tag{4.4}$$

and for $n = m_1 + 1, \dots, m_1 + m_2$,

$$\begin{aligned} \mathrm{P}'(A(t) = n) &= -m_2\lambda_2(t)\,\mathrm{P}(A(t) = n) + m_2\lambda_2(t)\,\mathrm{P}(A(t) = n-1)(1 - \mathrm{I}_{(n=m+1)}) \\ &\quad + (1-\alpha(t))\Big(m_1\lambda_1(t)\,\mathrm{P}(A(t) = m_1) + m_2\lambda_2(t)\,\mathrm{P}(A(t) = m_1 + m_2)\Big)\,\mathrm{I}_{n=(m_1+1)} \end{aligned} \tag{4.5}$$

where $\mathrm{I}_{(i=j)}$ is an indicator function such that $\forall\, i,\, j,\, \mathrm{I}_{(i=j)} = \begin{cases} 1 & \text{when } i = j, \\ 0 & \text{otherwise.} \end{cases}$

To calculate the probability $\mathrm{P}(A(t) = n)$ of finding the 2-MECO Ph_t process at state $\{A(t) = n\}$ for $n = 1, \ldots, m$, where $m = m_1 + m_2$ at time $t \geq 0$, we solve the differential equation in (4.4) and (4.5) with the following initial condition, $\mathrm{P}(A(0) = n) = \alpha_n(0)$.

The 2-MECO demand-count KFEs of the augmented state space are presented below where the derivative is with respect to t in equations (4.6) and (4.7), and τ in (4.8) and (4.9). For $0 \leq t < L$, $k = 0, 1, 2, \ldots, \infty$, and $i = 1, \ldots, m_1$,

$$\begin{aligned} \mathrm{P}'(D_0(t) = k, A(t) = i) &= -m_1\lambda_1(t)\,\mathrm{P}(D_0(t) = k, A(t) = i) \\ &\quad + m_1\lambda_1(t)\,\mathrm{P}(D_0(t) = k, A(t) = i-1)\,(1 - \mathrm{I}_{(i=1)}) \\ &\quad + \alpha(t)\Big(\, m_1\lambda_1(t)\,\mathrm{P}(D_0(t) = k-1, A(t) = m_1) \\ &\quad + \; m_2\lambda_2(t)\,\mathrm{P}(D_0(t) = k-1, A(t) = m_1 + m_2)\Big)\,\mathrm{I}_{(k>0)}. \end{aligned} \tag{4.6}$$

For $0 \leq t < L$, $k = 0, 1, 2, \ldots, \infty$, and $i = m_1 + 1, \ldots, m_1 + m_2$,

$$\begin{aligned} \mathrm{P}'(D_0(t) = k, A(t) = i) &= -m_2\lambda_2(t)\,\mathrm{P}(D_0(t) = k, A(t) = i) \\ &\quad + m_2\lambda_2(t)\,\mathrm{P}(D_0(t) = k, A(t) = i-1)\,(1 - \mathrm{I}_{(i=m_1+1)}) \\ &\quad + (1-\alpha(t))\Big(\, m_1\lambda_1(t)\,\mathrm{P}(D_0(t) = k-1, A(t) = m_1) \\ &\quad + \; m_2\lambda_2(t)\,\mathrm{P}(D_0(t) = k-1, A(t) = m_1 + m_2)\Big)\,\mathrm{I}_{(k>0)}. \end{aligned} \tag{4.7}$$

For $t \geq L$, $k = 0, 1, 2, \ldots, \infty$, $t_s = t - L$, and $i = 1, ..., m_1$,

$$\begin{aligned}
\mathrm{P}'(D_{t_s}(\tau) = k, A(t_s + \tau) = i) &= -m_1\lambda_1(t_s + \tau)\,\mathrm{P}(D_{t_s}(\tau) = k, A(t_s + \tau) = i) \\
&+ m_1\lambda_1(t_s + \tau)\,\mathrm{P}(D_{t_s}(\tau) = k, A(t_s + \tau) = i - 1)\,(1 - \mathrm{I}_{(i=1)}) \\
&+ \alpha(t_s + \tau)\Big(m_1\lambda_1(t_s + \tau)\,\mathrm{P}(D_{t_s}(\tau) = k - 1, A(t_s + \tau) = m_1) \\
&+ m_2\lambda_2(t_s + \tau)\,\mathrm{P}(D_{t_s}(\tau) = k - 1, A(t_s + \tau) = m_1 + m_2)\Big)\,\mathrm{I}_{(k>0)}.
\end{aligned} \tag{4.8}$$

For $t \geq L$, $k = 0, 1, 2, \ldots, \infty$, $t_s = t - L$, and $i = m_1 + 1, ..., m_1 + m_2$,

$$\begin{aligned}
\mathrm{P}'(D_{t_s}(\tau) = k, A(t_s + \tau) = i) &= -m_2\lambda_2(t_s + \tau)\,\mathrm{P}(D_{t_s}(\tau) = k, A(t_s + \tau) = i) \\
&+ m_2\lambda_2(t_s + \tau)\,\mathrm{P}(D_{t_s}(\tau) = k, A(t_s + \tau) = i - 1)\,(1 - \mathrm{I}_{(i=m_1+1)}) \\
&+ (1 - \alpha(t_s + \tau))\Big(m_1\lambda_1(t_s + \tau)\,\mathrm{P}(D_{t_s}(\tau) = k - 1, A(t_s + \tau) = m_1) \\
&+ m_2\lambda_2(t_s + \tau)\,\mathrm{P}(D_{t_s}(\tau) = k - 1, A(t_s + \tau) = m_1 + m_2)\Big)\,\mathrm{I}_{(k>0)}.
\end{aligned} \tag{4.9}$$

The probability distribution of the Conditional Demand-Count (CDC) process can be calculated from the KFEs in (4.2) and (4.3) for the general Ph_t distribution and from the KFEs in (4.6),(4.7),(4.8) and (4.9) for the 2-MECO distribution by conditioning on the state of the Ph_t process at time $t \in [0, T]$. After which we can find the distribution of the demand-count process by applying the total probability theorem. Thus, in order to find the probability $P(D_0(t) = k)$ of having a demand count equal to $k = 0, 1, 2, \ldots, \infty$ over the time interval $0 \leq t \leq L$, we first find the probabilities $\mathrm{P}(D_0(t) = k | A(0) = n)$ of having a demand equal to $k = 0, 1, 2, \ldots, \infty$ over the interval $0 \leq t \leq L$, conditioned on being at all possible phases at time $t = 0$, i.e., for all $n = 1, \ldots, m$. Note that in the general

Ph_t distribution m in the number of transient states which is also the case in the 2-MECO distribution where $m = m_1 + m_2$. Hence, $\mathrm{P}(D_0(t) = k)$ will be equal to the sum $\sum_{n=1}^{m} \Big(\mathrm{P}(D_0(t) = k | A(0) = n) \times \mathrm{P}(A(0) = n) \Big)$.

Similarly, to find the probability $\mathrm{P}(D_{t_s}(L) = k)$, with $t_s = t - L$, of having a demand count equal to $k = 0, 1, 2, \ldots, \infty$ over the time interval $[t_s, t]$ such that $t_s \geq 0$, we first find the probabilities $\mathrm{P}(D_{t_s}(L) = k | A(t_s) = n)$ of having a demand equal to $k = 0, 1, 2, \ldots, \infty$ over the interval $[t_s, t]$ conditioned on being at all possible phases at the beginning of the lead time t_s, i.e., for all $n = 1, \ldots, m$. Hence, $\mathrm{P}(D_{t_s}(L) = k)$ will be equal to the sum $\sum_{n=1}^{m} \Big(\mathrm{P}(D_{t_s}(L) = k | A(t_s) = n) \times \mathrm{P}(A(t_s) = n) \Big)$.

To facilitate the computation later on in our model, specifically when computing the net inventory probabilities at time $t \geq 0$ at which we need to compute probabilities for all the possible conditional demand-count values over the lead time, we limit the demand-count to an upper limit d_{max}. The upper bound d_{max} is chosen such that it satisfies $\mathrm{P}(X > d_{max}) = \epsilon$ where X is a Poisson random variable having a mean $\lambda = \max(\lambda_{i,j}(t))$ for all $i = 1, \ldots, m$ and $t \in [0, T]$, where ϵ is a sufficiently small number. Note that for the 2-MECO, $\lambda_{i,j}(t)$ is either $m_1\lambda_1(t)$ or $m_2\lambda_2(t)$. It is worth to mention that this approach for selecting d_{max} preserves the independence of the CDC and Demand-count with the reorder policy.

Accordingly, we establish an algorithm to compute the probability distribution of the demand count process for a time interval $[0, T]$.

Demand-Count Distribution Algorithm

For $0 \leq t < L$ and $\forall\ k \in \{0, 1, \cdots, d_{max}\}$,

Step 1. Set $n = 1$ and $i = 1$.

Step 2. Numerically solve equations (4.2), (4.6) and (4.7) from 0 to t, $t < L$ with the following initial condition, $\mathrm{P}(D_0(0) = 0, A(0) = n) = \mathrm{P}(A(0) = n)$, to obtain $\mathrm{P}(D_0(t) = k, A(t) = i|A(0) = n)$.

Step 3. If $i < m$, set $i = i + 1$ and go to Step 2.

Step 4. Set $\mathrm{P}(D_0(t) = k|A(0) = n) = \sum_{i=1}^{m} \mathrm{P}(D_0(t) = k, A(t) = i|A(0) = n)$.

Step 5. If $n < m$, set $n = n + 1$ and $i = 1$ and go to Step 2.

Step 6. Set $\mathrm{P}(D_0(t) = k) = \sum_{n=1}^{m} \Big(\mathrm{P}(D_0(t) = k|A(0) = n) \times \mathrm{P}(A(0) = n)\Big)$.

For $t \geq L$ and $\forall\ k \in \{0, 1, \ldots, d_{max}\}$,

Step 1. Set $n = 1$ and $i = 1$.

Step 2. Numerically solve equations (4.3),(4.8) and (4.9) from $t_s = t - L$ to t, $t \geq L$ with the following initial condition, $\mathrm{P}(D_{t_s}(0) = 0, A(t_s) = n) = \mathrm{P}(A(t_s) = n)$, to obtain $\mathrm{P}(D_{t_s}(L) = k, A(t) = i|A(t_s) = n)$.

Step 3. If $i < m$, set $i = i + 1$ and go to Step 2.

Step 4. Set $\mathrm{P}(D_{t_s}(L) = k|A(t_s) = n) = \sum_{i=1}^{m} \mathrm{P}(D_{t_s}(L) = k, A(t) = i|A(t_s) = n)$.

Step 5. If $n < m$, set $n = n + 1$ and $i = 1$ and go to Step 2.

Step 6. Set $\mathrm{P}(D_{t_s}(L) = k) = \sum_{n=1}^{m} \Big(\mathrm{P}(D_{t_s}(L) = k|A(t_s) = n) \times \mathrm{P}(A(t_s) = n)\Big)$.

Next, we discuss finding the moments of the demand over the lead time. The expected demand-count over the interval $[0, t]$ where $t < L$ can be calculated by

$$\mathrm{E}[D_0(t)] = \sum_{k=1}^{d_{max}} k\, \mathrm{P}(D_0(t) = k) \tag{4.10}$$

and over an interval $[t_s, t]$ for $t_s = t - L$, $t_s \geq 0$ by

$$\mathrm{E}[D_{t_s}(L)] = \sum_{k=1}^{d_{max}} k\, \mathrm{P}(D_{t_s}(L) = k). \tag{4.11}$$

Similarly the r^{th} moment of the demand-count distribution over the interval $[0, t]$ where $t < L$ can be found by

$$\mathrm{E}[D_0^r(t)] = \sum_{k=1}^{d_{max}} k^r\, \mathrm{P}(D_0(t) = k) \tag{4.12}$$

and over an interval $[t_s, t]$ for $t_s = t - L$, $t_s \geq 0$ by

$$\mathrm{E}[D_{t_s}^r(L)] = \sum_{k=1}^{d_{max}} k^r\, \mathrm{P}(D_{t_s}(L) = k). \tag{4.13}$$

In the next section, we will present the analytical model of the inventory characteristics for a given $(\bar{s}, \bar{S})$ replenishment policy.

4.4 System Prob. - Inventory Characteristics

In order to obtain a better control over the inventory, it is typically classified into different types. From the operational perspective, inventories may be classified based on their availability. As a matter of fact, at time t inventory can be either ordered, but not yet delivered (*inventory on order, IO(t)*) or it is physically on the shelf and immediately available to satisfy demand (*on-hand inventory, I(t)*). Silver et al. (1998) point out that sometimes even the physical on the shelf inventory might not be available since it could be already committed, for example, due to unmet demands from previous periods, so called *backorders, B(t).* The remaining quantity that is not backordered is referred to as *net inventory, NI(t)* or *inventory level, IL(t)*, and it holds that

$$\underbrace{\text{net inventory}}_{NI(t)} = \underbrace{\text{on-hand inventory - backorders}}_{I(t)-B(t)}$$

Since at time t, we can have backorders only whenever the on-hand inventory is zero, it does not make sense in single-customer setting to hold both, on-hand inventory and backorders. Thus, the positive net inventory is the on-hand inventory. When the net inventory is negative, the demand is backordered until inventory on-order is delivered. Thus, the amount by which the net inventory is negative is the backorder. That is,

$$\underbrace{\text{on-hand inventory}}_{I(t)} = \underbrace{\text{amount of positive net inventory}}_{NI(t)^+} \tag{4.14}$$

$$\underbrace{\text{backorders}}_{B(t)} = \underbrace{\text{amount of negative net inventory}}_{NI(t)^-}. \tag{4.15}$$

Let $IP(t)$ denotes the inventory position, which is defined by Silver et al. (1998) as

$$\begin{aligned} &\text{inventory position} \\ &= \text{on-hand inventory + inventory on-order - backorders} \\ &= \text{net inventory + inventory on-order.} \end{aligned}$$

Moreover, having $D_t(t+L)$ refer to the amount of demand that takes place in the interval $[t, t+L]$ and all the inventory that was on order at time t will be delivered at $t+L$, we can write,

$$\begin{aligned} NI(t+L) &= NI(t) + IO(t) - D_t(t+L) \\ &= IP(t) - D_t(t+L). \end{aligned} \tag{4.16}$$

4.4.1 Inventory Position

The replenishment policy for a planning horizon of length T having N periods is $(\bar{s}, \bar{S})$ where, $\bar{s} = \{s_1, s_2, \ldots, s_N\}$ and $\bar{S} = \{S_1, S_2, \ldots, S_N\}$, Figure 4.2. The dynamic ordering policy at any time $t \in [0, T]$ is defined by the reorder level s_n and the order up-to level S_n where $n = \text{ceil}(\frac{tN}{T}) \equiv \lceil \frac{tN}{T} \rceil$.

For a given dynamic reorder policy, $(\bar{s},\ \bar{S})$, the state of the dynamic inventory position at time $t \in [0, T]$ is represented by the value of the inventory position and the state of the phase-type process at time $t \in [0, T]$.

The inventory position at time $t \in [0, T]$ is $\text{IP}(t) \in \{\ell, \ldots, u\}$, where ℓ is the lowest achievable inventory position level such that $\ell = \min_{\text{t}\in[0,\text{T}]} (s_t) + 1$ and u is the inventory position upper limit such that $u = \max_{\text{t}\in[0,\text{T}]} (S_t)$.

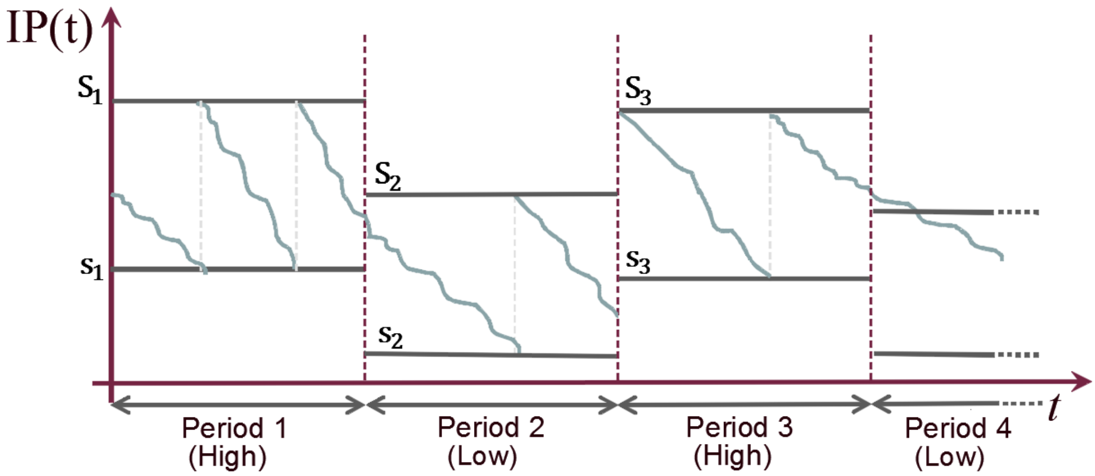


Figure 4.2: Inventory Position under dynamic (s, S) policy.

We augment the states of the Ph_t demand count process $A(t) \in \{1, 2, \cdots, m\}$ with the inventory position level $\text{IP}(t)$ at time $t \geq 0$.

Let $\text{P}_{i,n}(t) = \text{P}(\text{IP}(t) = i, A(t) = n)$ for $i = \ell, \ldots, u$ and $n = 1, \ldots, m$, be the resulting Markovian process state probabilities. Recall that m is the total number of transient states in the general Ph_t distribution and the 2-MECO distribution where $m = m_1 + m_2$. Similar to the demand-count process the inventory position level can be defined by a general and a 2-MECO Ph_t processes. The resulting inventory position Kolmogorov Forward Equations (IP-KFEs) for the general Ph_t and 2-MECO processes are presented in equations (4.17) to (4.24).

The General $\mathbf{Ph_t}$ IP Process KFEs

For $t \geq 0$, $n = 1, \ldots, m$ and $i > s(t)$,

$$\begin{aligned} \mathrm{P}'_{i,n}(t) &= \sum_{w=1}^{m} \lambda_{w,n}(t)\, \mathrm{P}_{i,w}(t) \\ &\quad + \alpha_n(t) \Big(\sum_{w=1}^{m} \mu_w(t)\, \mathrm{P}_{i+1,w}(t) \Big) \left(1 - \mathrm{I}_{(i=u)}\right) \\ &\quad + \alpha_n(t) \Big(\sum_{w=1}^{m} \sum_{q=\ell}^{s(t)+1} \mu_n(t)\, \mathrm{P}_{q,n}(t) \Big)\, \mathrm{I}_{(i=S(t))}. \end{aligned} \tag{4.17}$$

For $t \geq 0$, $n = 1, \ldots, m$ and $i \leq s(t)$,

$$\mathrm{P}'_{i,n}(t) = \sum_{w=1}^{m} \lambda_{w,n}(t)\, \mathrm{P}_{i,w}(t). \tag{4.18}$$

The 2-MECO $\mathbf{Ph_t}$ IP Process KFEs

For $t \geq 0$, $n = 2, \ldots, m_1$ and $\ell \leq i \leq u$,

$$\mathrm{P}'_{i,n}(t) = -m_1\, \lambda_1(t)\, \mathrm{P}_{i,n}(t) + m_1\, \lambda_1(t)\, \mathrm{P}_{i,n-1}(t). \tag{4.19}$$

For $t \geq 0$, $n = m_1 + 2, \ldots, m_1 + m_2$ and $\ell \leq i \leq u$,

$$\mathrm{P}'_{i,n}(t) = -m_2\, \lambda_2(t)\, \mathrm{P}_{i,n}(t) + m_2\, \lambda_2(t)\, \mathrm{P}_{i,n-1}(t). \tag{4.20}$$

For $t \geq 0$, $n = 1$ and $i > s(t)$,

$$\begin{aligned}\mathrm{P}'_{i,1}(t) &= -m_1\,\lambda_1(t)\,\mathrm{P}_{i,1}(t) \\ &\quad + \alpha(t)\Big(\, m_1\,\lambda_1(t)\,\mathrm{P}_{i+1,m_1}(t) +\, m_2\,\lambda_2(t)\,\mathrm{P}_{i+1,m_1+m_2}(t)\Big)\,(1 - \mathrm{I}_{(i=u)}) \\ &\quad + \alpha(t)\Big(\sum_{w=\ell}^{s(t)+1}(\, m_1\,\lambda_1(t)\,\mathrm{P}_{w,m_1}(t) +\, m_2\,\lambda_2(t)\,\mathrm{P}_{w,m_1+m_2}(t))\Big)\,\mathrm{I}_{(i=S(t))}.\end{aligned} \tag{4.21}$$

For $t \geq 0$, $n = m_1 + 1$ and $i > s(t)$,

$$\begin{aligned}\mathrm{P}'_{i,m_1+1}(t) &= -m_2\,\lambda_2(t)\,\mathrm{P}_{i,m_1+1}(t) \\ &\quad + (1 - \alpha(t))\Big(m_1\,\lambda_1(t)\,\mathrm{P}_{i+1,m_1}(t) + m_2\,\lambda_2(t)\,\mathrm{P}_{i+1,m_1+m_2}(t)\Big)\,(1 - \mathrm{I}_{(i=u)}) \\ &\quad + (1 - \alpha(t))\Big(\sum_{w=\ell}^{s(t)+1}(m_1\,\lambda_1(t)\,\mathrm{P}_{w,m_1}(t) + m_2\,\lambda_2(t)\,\mathrm{P}_{w,m_1+m_2}(t))\Big)\,\mathrm{I}_{(i=S(t))}.\end{aligned} \tag{4.22}$$

For $t \geq 0$, $n = 1$ and $i \leq s(t)$,

$$\mathrm{P}'_{i,1}(t) \;=\; -m_1\,\lambda_1(t)\,\mathrm{P}_{i,1}(t). \tag{4.23}$$

For $t \geq 0$, $n = m_1 + 1$ and $i \leq s(t)$,

$$\mathrm{P}'_{i,m_1+1}(t) \;=\; -m_2\,\lambda_2(t)\,\mathrm{P}_{i,m_1+1}(t). \tag{4.24}$$

where, $I_{i=n} = \begin{cases} 1 & \text{when } i = n, \\ 0 & \text{otherwise.} \end{cases}$

The KFEs in (4.17) and (4.18) for the general Ph_t distribution and the KFEs in (4.19) to (4.24) for the 2-MECO Ph_t distribution allow us to compute the joint probability distribution of the inventory position and the Ph_t state at time $t \geq 0$. After which the distribution of the inventory position at time $t \geq 0$ can be found by the sum of the joint probability distribution for all phases. Thus, in order to find the probability $\mathrm{P}(IP(t) = i)$ of having an inventory position equal to $i = \ell, \ldots, u$ at time $t \geq 0$, we first find the probabilities $\mathrm{P}(IP(t) = i, A(t) = n)$ of having an inventory position equal to $i = \ell, \ldots, u$ and being in phase $n = 1, \ldots, m$ at time $t \geq 0$. Then, the $\mathrm{P}(IP(t) = i)$ will be equal to the sum $\sum_{n=1}^{m} \mathrm{P}(IP(t) = i, A(t) = n)$.

IP Distribution Algorithm

For $t \geq 0$ and $\forall\, i \in \{\ell, \ldots, u\}$,

Step 1. Set $n = 1$.

Step 2. Numerically solve the differential equations in (4.17) to (4.24) over the time interval $[0, T]$ with the following initial condition, $\mathrm{P}(IP(0) = i_0, A(0) = n) = 1$, where i_0 is the initial inventory at time $t = 0$, to obtain $\mathrm{P}(IP(t) = i, A(t) = n)$.

Step 3. If $n < m$, set $n = n + 1$ and go to Step 2.

Step 4. Set $\mathrm{P}(IP(t) = i) = \sum_{n=1}^{m} \mathrm{P}(IP(t) = i, A(t) = n)$.

Finally, the expected inventory position at time $t \geq 0$ can be calculated by

$$E[IP(t)] = \sum_{i=\ell}^{u} i\, \mathrm{P}(IP(t) = i) \tag{4.25}$$

Similarly the r^{th} moment of the inventory position at time $t \geq 0$ can be found by

$$E[IP^r(t)] = \sum_{i=\ell}^{u} i^r\, \mathrm{P}(IP(t) = i) \tag{4.26}$$

Next, we will find the net inventory position probabilities and its moments.

4.4.2 Net Inventory Position

Now, we present the equations which allow us to find the distribution of the net inventory position at time $t \geq 0$. As we have demonstrated in section 4.4 equation (4.16) and as thoroughly discussed in Maddah et al. (2004), it can be seen that

$$NI(t) = IP(0) - D_0(t) \qquad \text{for } 0 \leq t \leq L, \tag{4.27}$$

$$NI(t) = IP(t-L) - D_{t-L}(L) \qquad \text{for } t > L. \tag{4.28}$$

Which allows us to write,

for $0 \leq t \leq L$ and $\ell - d_{max} \leq i \leq u$,

$$P(NI(t) = i) = \sum_{y=\ell}^{u} P(IP(0) = y, D_0(t) = y - i), \tag{4.29}$$

for $t > L$ and $\ell - d_{max} \leq i \leq u$,

$$P(NI(t) = i) = \sum_{y=\ell}^{u} P(IP(t-L) = y, D_{t-L}(L) = y - i). \tag{4.30}$$

By conditioning equations (4.29) and (4.30) on the Ph_t system state $n = \{1, \ldots, m\}$ we obtain: for $0 \leq t \leq L$ and $\ell - d_{max} \leq i \leq u$,

$$\begin{aligned} P(NI(t) = i | A(0) = n) = \sum_{y=\ell}^{u} \Big(& P(IP(0) = y | A(0) = n) \\ & \times P(D_0(t) = y - i | A(0) = n) \Big), \end{aligned} \tag{4.31}$$

for $t > L$ and $\ell - d_{max} \leq i \leq u$,

$$P(NI(t) = i|A(t-L) = n) = \sum_{y=\ell}^{u} \Big(P(IP(t-L) = y|A(t-L) = n) \times P(D_{t-L}(L) = y - i|A(t-L) = n) \Big). \tag{4.32}$$

Which can be written as,

for $0 \leq t \leq L$ and $\ell - d_{max} \leq i \leq u$,

$$P(NI(t) = i, A(0) = n) = \sum_{y=\ell}^{u} \Big(P(IP(0) = y, A(0) = n) \times P(D_0(t) = y - i|A(0) = n) \Big), \tag{4.33}$$

for $t > L$ and $\ell - d_{max} \leq i \leq u$,

$$P(NI(t) = i, A(t-L) = n) = \sum_{y=\ell}^{u} \Big(P(IP(t-L) = y, A(t-L) = n) \times P(D_{t-L}(L) = y - i|A(t-L) = n) \Big). \tag{4.34}$$

Accordingly, we establish the following equations of the net inventory probability distribution,

for $0 \leq t \leq L$ and $\ell - d_{max} \leq i \leq u$,

$$P(NI(t) = i) = \sum_{n=1}^{m} P(NI(t) = i, A(0) = n) \tag{4.35}$$

for $t > L$ and $\ell - d_{max} \leq i \leq u$,

$$P(NI(t) = i) = \sum_{n=1}^{m} P(NI(t) = i, A(t-L) = n) \tag{4.36}$$

Finally, the expected net inventory position at time $t \geq 0$ can be calculated by

$$E[NI(t)] = \sum_{i=\ell-d_{max}}^{u} i\, P(NI(t) = i) \tag{4.37}$$

Similarly the r^{th} moment of the net inventory position at time $t \geq 0$ can be found by

$$E[NI^r(t)] = \sum_{i=\ell-d_{max}}^{u} i^r\, P(NI(t) = i) \tag{4.38}$$

4.4.3 On-hand Inventory

As discussed earlier in section 4.4 the on-hand inventory $I(t)$ is the inventory on the shelf that is available for sale or use at a particular time t. Now, we present the equations which allow us to find the distribution of the on-hand inventory at time $t \geq 0$. Based on equations (4.14), (4.27) and (4.28), we can write

$$I(t) = NI(t)^+ = \Big(IP(0) - D_0(t)\Big)^+ \qquad \text{for } 0 \leq t \leq L, \tag{4.39}$$

$$I(t) = NI(t)^+ = \Big(IP(t-L) - D_{t-L}(L)\Big)^+ \qquad \text{for } t > L. \tag{4.40}$$

Hence, for $t \geq 0$,

$$\mathrm{P}(\mathrm{I}(t) = i) = \begin{cases} \mathrm{P}(\mathrm{NI}(t) = i) & \text{for } i = 1, \ldots, u, \\ \displaystyle\sum_{j=\ell-d_{max}}^{0} \mathrm{P}(\mathrm{NI}(t) = j) & \text{for } i = 0. \end{cases} \tag{4.41}$$

Therefore, the expected on-hand inventory at time $t \geq 0$ can be calculated by

$$E[I(t)] = \sum_{i=0}^{u} i\, P(I(t) = i) \tag{4.42}$$

Similarly the r^{th} moment of the on-hand inventory at time $t \geq 0$ can be found by

$$E[I^r(t)] = \sum_{i=0}^{u} i^r\, P(I(t) = i) \tag{4.43}$$

4.4.4 Backorders

As we've seen earlier in section 4.4 the backorders also called backlogs, denoted $B(t)$, is the inventory on the shelf that is already committed due to unmet demands from previous periods. In this section, we present the equations which allow us to find the distribution of the backorders at time $t \geq 0$. Based on equations (4.15), (4.27) and (4.28), we can write

$$B(t) = NI(t)^- = (IP(0) - D_0(t))^- = (D_0(t) - IP(0))^+ \quad \text{for } 0 \leq t \leq L, \tag{4.44}$$

$$B(t) = NI(t)^- = (IP(t-L) - D_{t-L}(L))^- = (D_{t-L}(L) - IP(t-L))^+ \quad \text{for } t > L. \tag{4.45}$$

Hence, for $t \geq 0$,

$$\mathrm{P(B}(t) = i) = \begin{cases} \mathrm{P(NI}(t) = -i) & \text{for } i = 1, \ldots, (d_{max} - \ell), \\ \sum_{j=0}^{u} \mathrm{P(NI}(t) = j) & \text{for } i = 0. \end{cases} \tag{4.46}$$

Therefore, the expected backorders at time $t \geq 0$ can be calculated by

$$E[B(t)] = \sum_{i=0}^{d_{max}-\ell} i\, P(B(t) = i) \tag{4.47}$$

Similarly the r^{th} moment of the backorders at time $t \geq 0$ can be found by

$$E[B^r(t)] = \sum_{i=0}^{d_{max}-\ell} (i)^r\, P(B(t) = i) \tag{4.48}$$

4.4.5 Number of Orders

In this section, we find the expected number of orders placed in an interval of length $[0, t]$. Let $R(t)$ be the number of orders placed up-to time $t \geq 0$. Let $\Gamma_{r,i,n}(t) = \mathrm{P}(R(t) = r,\, \mathrm{IP}(t) = i,\, A(t) = n)$ be the probability of having $R(t) = r$, $\mathrm{IP}(t) = i$ and $A(t) = n$ at time $t \geq 0$ for $r \geq 0$, $i = \ell, \ldots, u$ and $n = 1, \ldots, m$. We present the below KFEs where the derivative is with respect to t.

The General $\mathbf{Ph_t}$ Number of Orders KFEs

For $t \geq 0$, $n = 1, \ldots, m$, $s(t) < i \leq u$, and $r > 0$,

$$\begin{aligned}\Gamma'_{r,i,n}(t) &= \textstyle\sum_{w=1}^{m} \lambda_{w,n}(t)\Gamma_{r,i,w}(t) \\ &\quad +\alpha_n(t)\Big(\textstyle\sum_{w=1}^{m} \mu_w(t)\,\Gamma_{r,i+1,w}(t)\Big)(1 - I_{i=u}) \\ &\quad +\alpha_n(t)\Big(\textstyle\sum_{w=1}^{m} \sum_{q=\ell}^{s(t)+1} \mu_n(t)\,\Gamma_{r-1,q,n}(t)\Big) I_{i=S(t)} I_{(r>0)}\end{aligned} \tag{4.49}$$

and for $n = 1, \ldots, m$, $\ell \leq i \leq s(t)$ and $r \geq 0$,

$$\Gamma'_{r,i,n}(t) = \textstyle\sum_{w=1}^{m} \lambda_{w,n}(t)\Gamma_{r,i,w}(t). \tag{4.50}$$

The 2-MECO $\mathbf{Ph_t}$ Number of Orders KFEs

For $t \geq 0$, $n = 2, \ldots, m_1$, $\ell \leq i \leq u$ and $r \geq 0$,

$$\Gamma'_{r,i,n}(t) = -m_1\,\lambda_1(t)\,\Gamma_{r,i,n}(t) + m_1\,\lambda_1(t)\,\Gamma_{r,i,n-1}(t). \tag{4.51}$$

For $t \geq 0$, $n = m_1 + 2, \ldots, m_1 + m_2$, $\ell \leq i \leq u$ and $r \geq 0$,

$$\Gamma'_{r,i,n}(t) = -m_2\,\lambda_2(t)\,\Gamma_{r,i,n}(t) + m_2\,\lambda_2(t)\,\Gamma_{r,i,n-1}(t). \tag{4.52}$$

For $t \geq 0$, $n = 1$, $i > s(t)$ and $r \geq 0$,

$$\begin{aligned}\Gamma'_{r,i,1}(t) &= -m_1\,\lambda_1(t)\,\Gamma_{r,i,1}(t) \\ &\quad +\alpha(t)\Big(\,m_1\,\lambda_1(t)\,\Gamma_{r,i+1,m_1}(t) \\ &\quad +m_2\,\lambda_2(t)\,\Gamma_{r,i+1,m_1+m_2}(t)\Big)(1-I_{i=u}) \\ &\quad +\alpha(t)\Big(\,\textstyle\sum_{w=\ell}^{s(t)+1}(\,m_1\,\lambda_1(t)\,\Gamma_{r-1,w,m_1}(t) \\ &\quad +m_2\,\lambda_2(t)\,\Gamma_{r-1,w,m_1+m_2}(t))\Big)I_{i=S(t)}\,I_{r>0}.\end{aligned} \tag{4.53}$$

For $t \geq 0$, $n = m_1 + 1$, $i > s(t)$ and $r \geq 0$,

$$\begin{aligned}\Gamma'_{r,i,m_1+1}(t) &= -m_2\,\lambda_2(t)\,\Gamma_{r,i,m_1+1}(t) \\ &\quad +(1-\alpha(t))\Big(\,m_1\,\lambda_1(t)\,\Gamma_{r,i+1,m_1}(t) \\ &\quad +m_2\,\lambda_2(t)\,\Gamma_{r,i+1,m_1+m_2}(t)\Big)(1-I_{i=u}) \\ &\quad +(1-\alpha(t))\Big(\,\textstyle\sum_{w=\ell}^{s(t)+1}(\,m_1\,\lambda_1(t)\,\Gamma_{r-1,w,m_1}(t) \\ &\quad +m_2\,\lambda_2(t)\,\Gamma_{r-1,w,m_1+m_2}(t))\Big)I_{i=S(t)}\,I_{r>0}.\end{aligned} \tag{4.54}$$

For $t \geq 0$, $n = 1$, $i \leq s(t)$ and $r \geq 0$,

$$\Gamma'_{r,i,1}(t) = -m_1\,\lambda_1(t)\,\Gamma_{r,i,1}(t). \tag{4.55}$$

For $t \geq 0$, $n = m_1 + 1$ and $i \leq s(t)$ and $r \geq 0$,

$$\Gamma'_{r,i,m_1+1}(t) = -m_2\,\lambda_2(t)\,\Gamma_{r,i,m_1+1}(t). \tag{4.56}$$

To compute the probabilities $\Gamma_{r,i,n}$ we can numerically solve the differential equations in (4.49) and (4.50) for the general Ph_t distribution and equations (4.51), (4.52), (4.53), (4.54), (4.55) and (4.56) for the 2-MECO distribution. But since $r = 0, 1, \ldots, \infty$ this approach requires setting an upper bound (r_{max}) on the number of orders such that $P(R(T) > r_{max})$ is sufficiently small. An alternate method for the computation of the expected number of orders is to derive a finite set of moment differential equations to calculate the moments. Next, we write the general and 2-MECO Ph_t distributions differential equations for the first two moments of the reorder count process.

The General $\mathbf{Ph_t}$ $\mathbf{k^{th}}$ Moment of the Number of Orders

For $n = 1, \ldots, m$, $s(t) < i \leq u$ and $r > 0$,

$$\begin{aligned}
\mathrm{E}'[R^k(t); i, n] &= \sum_{r=0}^{\infty} r^k \, \Gamma'_{r,i,n}(t) \\
&= \sum_{v=1}^{m} \lambda_{v,n}(t) \, \mathrm{E}[R^k(t); i, v] \\
&+ \alpha_n(t) \Big(\sum_{v=1}^{m} \mu_v(t) \, \mathrm{E}[R^k(t); i+1, v] \Big) \, \mathrm{I}_{(i \neq u)} \\
&+ \alpha_n(t) \Big(\sum_{v=1}^{m} \sum_{j=\ell}^{s(t)+1} \mu_v(t) \Big(\sum_{z=1}^{k} \binom{k}{z} \mathrm{E}[R^z(t); j, v] + P_{j,v}(t) \Big) \Big) \, \mathrm{I}_{(i=S(t))}.
\end{aligned} \tag{4.57}$$

For $n = 1, \ldots, m$, $\ell \leq i \leq s(t)$ and $r \geq 0$,

$$\mathrm{E}'[R^k(t); i, n] = \sum_{v=1}^{m} \lambda_{v,n}(t) \, \mathrm{E}[R^k(t); i, v]. \tag{4.58}$$

Setting $k = 1, 2$ in Equation 4.57 and 4.58, the first and second moment differential equations for $n = 1, \ldots, m$,

$$\mathrm{E}'[R(t);i,n] = \sum_{v=1}^{m} \lambda_{v,n}(t)\,\mathrm{E}[R(t);i,v] + \alpha_n(t)\Big(\sum_{v=1}^{m} \mu_v(t)\,\mathrm{E}[R(t);i+1,v]\Big)\,\mathrm{I}_{(i\neq u)} + \alpha_n(t)\Big(\sum_{v=1}^{m}\sum_{j=\ell}^{s(t)+1} \mu_v(t)\Big(\mathrm{E}[R(t);j,v] + P_{j,v}(t)\Big)\Big)\,\mathrm{I}_{(i=S(t))}. \tag{4.59}$$

$$\mathrm{E}'[R^2(t);i,n] = \sum_{v=1}^{m} \lambda_{v,n}(t)\,\mathrm{E}[R^2(t);i,v] + \alpha_n(t)\Big(\sum_{v=1}^{m} \mu_v(t)\,\mathrm{E}[R^2(t);i+1,v]\Big)\,\mathrm{I}_{(i\neq u)} + \alpha_n(t)\Big(\sum_{v=1}^{m}\sum_{j=\ell}^{s(t)+1} \mu_v(t)\Big(\mathrm{E}[R^2(t);j,v] + 2\,\mathrm{E}[R(t);j,v] + P_{j,v}(t)\Big)\Big)\,\mathrm{I}_{(i=S(t))}. \tag{4.60}$$

For $n = 1, \ldots, m$, and $\ell \le i \le s_t$,

$$\mathrm{E}'[R(t);i,n] = \sum_{v=1}^{m} \lambda_{v,n}(t)\,\mathrm{E}[R(t);i,v]. \tag{4.61}$$

$$\mathrm{E}'[R^2(t);i,n] = \sum_{v=1}^{m} \lambda_{v,n}(t)\,\mathrm{E}[R^2(t);i,v]. \tag{4.62}$$

The 2-MECO $\mathbf{Ph_t}$ $\mathbf{k^{th}}$ Moment of the Number of Orders

For $t \ge 0$, $n = 2, \ldots, m_1$, $\ell \le i \le u$ and $r \ge 0$,

$$\begin{aligned} \mathrm{E}'[R^k(t);i,n] &= \sum_{r=0}^{\infty} r^k\,\Gamma'_{r,i,n}(t) \\ &= -m_1\,\lambda_1(t)\,\mathrm{E}[R^k(t);i,n] + m_1\,\lambda_1(t)\,\mathrm{E}[R^k(t);i,n-1]. \end{aligned} \tag{4.63}$$

For $t \ge 0$, $n = m_1+2, \ldots, m_1+m_2$, $\ell \le i \le u$ and $r \ge 0$,

$$\mathrm{E}'[R^k(t);i,n] = -m_2\,\lambda_2(t)\,\mathrm{E}[R^k(t);i,n] + m_2\,\lambda_2(t)\,\mathrm{E}[R^k(t);i,n-1]. \tag{4.64}$$

For $t \geq 0$, $n = 1$, $i > s(t)$ and $r > 0$,

$$\begin{aligned}
\mathrm{E}'[R^k(t); i, 1] &= -m_1\,\lambda_1(t)\,\mathrm{E}[R^k(t); i, 1] \\
&\quad + \alpha(t)\Big(\,m_1\,\lambda_1(t)\,\mathrm{E}[R^k(t); i+1, m_1] \\
&\quad + m_2\,\lambda_2(t)\,\mathrm{E}[R^k(t); i+1, m_1+m_2]\Big)(1 - I_{i=u}) \\
&\quad + \alpha(t)\Big(\,\textstyle\sum_{w=\ell}^{s(t)+1}(\,m_1\,\lambda_1(t)\,(\sum_{z=1}^{k}\binom{k}{z}\mathrm{E}[R^k(t); w, m_1] + \mathrm{P}_{w,m_1}) \\
&\quad + m_2\,\lambda_2(t)\,(\textstyle\sum_{z=1}^{k}\binom{k}{z}\mathrm{E}[R^k(t); w, m_1+m_2] + \mathrm{P}_{w,m_1+m_2})\Big)I_{i=S(t)}.
\end{aligned} \tag{4.65}$$

For $t \geq 0$, $n = m_1 + 1$, $i > s(t)$ and $r > 0$,

$$\begin{aligned}
\mathrm{E}'[R^k(t); i, m_1+1] &= -m_2\,\lambda_2(t)\,\mathrm{E}[R^k(t); i, m_1+1] \\
&\quad + (1-\alpha(t))\Big(\,m_1\,\lambda_1(t)\,\mathrm{E}[R^k(t); i+1, m_1] \\
&\quad + m_2\,\lambda_2(t)\,\mathrm{E}[R^k(t); i+1, m_1+m_2]\Big)(1 - I_{i=u}) \\
&\quad + (1-\alpha(t))\Big(\,\textstyle\sum_{w=\ell}^{s(t)+1}(\,m_1\,\lambda_1(t)\,(\sum_{z=1}^{k}\binom{k}{z}\mathrm{E}[R^k(t); w, m_1] + \mathrm{P}_{w,m_1}) \\
&\quad + m_2\,\lambda_2(t)\,(\textstyle\sum_{z=1}^{k}\binom{k}{z}\mathrm{E}[R^k(t); w, m_1+m_2] + \mathrm{P}_{w,m_1+m_2})\Big)I_{i=S(t)}.
\end{aligned} \tag{4.66}$$

For $t \geq 0$, $n = 1$, $i \leq s(t)$ and $r \geq 0$,

$$\mathrm{E}'[R^k(t); i, 1] = -m_1\,\lambda_1(t)\,\mathrm{E}[R^k(t); i, 1]. \tag{4.67}$$

For $t \geq 0$, $n = m_1 + 1$ and $i \leq s(t)$ and $r \geq 0$,

$$\mathrm{E}'[R^k(t); i, m_1+1] = -m_2\,\lambda_2(t)\,\mathrm{E}[R^k(t); i, m_1+1]. \tag{4.68}$$

Setting $k = 1, 2$ in equations 4.63 and 4.68, the first and second moment differential equations for $t \geq 0$, $n = 2, \ldots, m_1$, $\ell \leq i \leq u$ and $r \geq 0$,

$$\begin{aligned} E'[R(t); i, n] &= \sum_{r=0}^{\infty} r\, \Gamma'_{r,i,n}(t) \\ &= -m_1\, \lambda_1(t)\, E[R(t); i, n] + m_1\, \lambda_1(t)\, E[R(t); i, n-1]. \end{aligned} \tag{4.69}$$

$$\begin{aligned} E'[R^2(t); i, n] &= \sum_{r=0}^{\infty} r^2\, \Gamma'_{r,i,n}(t) \\ &= -m_1\, \lambda_1(t)\, E[R^2(t); i, n] + m_1\, \lambda_1(t)\, E[R^2(t); i, n-1]. \end{aligned} \tag{4.70}$$

For $t \geq 0$, $n = m_1 + 2, \ldots, m_1 + m_2$, $\ell \leq i \leq u$ and $r \geq 0$,

$$E'[R(t); i, n] = -m_2\, \lambda_2(t)\, E[R(t); i, n] + m_2\, \lambda_2(t)\, E[R(t); i, n-1]. \tag{4.71}$$

$$E'[R^2(t); i, n] = -m_2\, \lambda_2(t)\, E[R^2(t); i, n] + m_2\, \lambda_2(t)\, E[R^2(t); i, n-1]. \tag{4.72}$$

For $t \geq 0$, $n = 1$, $i > s(t)$ and $r > 0$,

$$\begin{aligned} E'[R(t); i, 1] &= -m_1\, \lambda_1(t)\, E[R(t); i, 1] \\ &\quad + \alpha(t) \Big(m_1\, \lambda_1(t)\, E[R(t); i+1, m_1] \\ &\quad + m_2\, \lambda_2(t)\, E[R(t); i+1, m_1 + m_2] \Big) (1 - I_{i=u}) \\ &\quad + \alpha(t) \Big(\textstyle\sum_{w=\ell}^{s(t)+1} (\, m_1\, \lambda_1(t)\, (E[R(t); w, m_1] + P_{w,m_1}) \\ &\quad + m_2\, \lambda_2(t)\, (E[R(t); w, m_1 + m_2] + P_{w,m_1+m_2}) \Big) I_{i=S(t)}. \end{aligned} \tag{4.73}$$

$$\begin{aligned} E'[R^2(t); i, 1] &= -m_1\, \lambda_1(t)\, E[R^2(t); i, 1] \\ &\quad + \alpha(t) \Big(m_1\, \lambda_1(t)\, E[R^2(t); i+1, m_1] \\ &\quad + m_2\, \lambda_2(t)\, E[R^2(t); i+1, m_1 + m_2] \Big) (1 - I_{i=u}) \\ &\quad + \alpha(t) \Big(\textstyle\sum_{w=\ell}^{s(t)+1} (\, m_1\, \lambda_1(t)\, (E[R^2(t); w, m_1] + 2E[R(t); w, m_1] + P_{w,m_1}) \\ &\quad + m_2\, \lambda_2(t)\, (E[R^2(t); w, m_1 + m_2] + 2E[R(t); w, m_1 + m_2] + P_{w,m_1+m_2}) \Big) I_{i=S(t)}. \end{aligned} \tag{4.74}$$

For $t \geq 0$, $n = m_1 + 1$, $i > s(t)$ and $r > 0$,

$$\begin{aligned} \mathrm{E}'[R(t); i, m_1+1] &= -m_2\,\lambda_2(t)\,\mathrm{E}[R(t); i, m_1+1] \\ &\quad +(1-\alpha(t))\Big(\,m_1\,\lambda_1(t)\,\mathrm{E}[R(t); i+1, m_1] \\ &\quad +m_2\,\lambda_2(t)\,\mathrm{E}[R(t); i+1, m_1+m_2]\Big)(1-I_{i=u}) \\ &\quad +(1-\alpha(t))\Big(\,\textstyle\sum_{w=\ell}^{s(t)+1}(\,m_1\,\lambda_1(t)\,(\mathrm{E}[R(t); w, m_1] + \mathrm{P}_{w,m_1}) \\ &\quad +m_2\,\lambda_2(t)\,(\mathrm{E}[R(t); w, m_1+m_2] + \mathrm{P}_{w,m_1+m_2})\Big)I_{i=S(t)}. \end{aligned} \tag{4.75}$$

$$\begin{aligned} \mathrm{E}'[R^2(t); i, m_1+1] &= -m_2\,\lambda_2(t)\,\mathrm{E}[R^2(t); i, m_1+1] \\ &\quad +(1-\alpha(t))\Big(\,m_1\,\lambda_1(t)\,\mathrm{E}[R^2(t); i+1, m_1] \\ &\quad +m_2\,\lambda_2(t)\,\mathrm{E}[R^2(t); i+1, m_1+m_2]\Big)(1-I_{i=u}) \\ &\quad +(1-\alpha(t))\Big(\,\textstyle\sum_{w=\ell}^{s(t)+1}(\,m_1\,\lambda_1(t)\,(\mathrm{E}[R^2(t); w, m_1] + 2\mathrm{E}[R(t); w, m_1] + \mathrm{P}_{w,m_1}) \\ &\quad +m_2\,\lambda_2(t)\,(\mathrm{E}[R^2(t); w, m_1+m_2] + 2\mathrm{E}[R(t); w, m_1+m_2] + \mathrm{P}_{w,m_1+m_2})\Big)I_{i=S(t)}. \end{aligned} \tag{4.76}$$

For $t \geq 0$, $n = 1$, $i \leq s(t)$ and $r \geq 0$,

$$\mathrm{E}'[R(t); i, 1] = -m_1\,\lambda_1(t)\,\mathrm{E}[R(t); i, 1]. \tag{4.77}$$

$$\mathrm{E}'[R^2(t); i, 1] = -m_1\,\lambda_1(t)\,\mathrm{E}[R^2(t); i, 1]. \tag{4.78}$$

For $t \geq 0$, $n = m_1 + 1$ and $i \leq s(t)$ and $r \geq 0$,

$$\mathrm{E}'[R(t); i, m_1+1] = -m_2\,\lambda_2(t)\,\mathrm{E}[R(t); i, m_1+1]. \tag{4.79}$$

$$\mathrm{E}'[R^2(t); i, m_1+1] = -m_2\,\lambda_2(t)\,\mathrm{E}[R^2(t); i, m_1+1]. \tag{4.80}$$

Therefore, for $t \geq 0$, $n = 1, \ldots, m$, $\ell \leq i \leq u$ and $r \geq 0$ the expected number of orders is

$$\mathrm{E}[R(t)] = \sum_{i=\ell}^{u} \sum_{n=1}^{m} \mathrm{E}[R(t); i, n] \tag{4.81}$$

and the second moment

$$\mathrm{E}[R^2(t)] = \sum_{i=\ell}^{u} \sum_{n=1}^{m} \mathrm{E}[R^2(t); i, n]. \tag{4.82}$$

4.4.6 Inventory Total Cost

In this section, we find the expected total cost for a given reorder policy $(\bar{s}, \bar{S})$. The expected total cost includes the expected holding, back-ordering and number of orders costs. Total expected costs up-to time $t \geq 0$,

$$\Phi_{(\bar{s},\bar{S})}(t) = H \int_0^t E[I(t)]\, dt + P \int_0^t E[B(t)]\, dt + K\, E[R(t)] \tag{4.83}$$

Total expected cost for a given replenishment $(\bar{s}, \bar{S})$ $t \geq 0$ policy at the end of the planning horizon,

$$\Phi_{(\bar{s},\bar{S})}(T) = h \int_0^T E[I(t)]\, dt + b \int_0^T E[B(t)]\, dt + \omega E[R(T)] \tag{4.84}$$

The below algorithm summarizes the computation of the expected cost for a given reorder policy $(\bar{s}, \bar{S})$.

Total Expected Cost Algorithm

For $t \geq 0$; $t_s = t - L$; $d = 0, 1, \ldots, d_{max}$; $i = \ell, \ldots, u$ and $n = 1, \ldots, m$

Step 1. Follow the demand-count distribution algorithm in §4.3 in order to find $\mathrm{P}(D_0(t) = d | A(0) = n)$ and $\mathrm{P}(D_{t_s}(L) = d | A(t_s) = n)$

Step 2. Follow the IP distribution algorithm in §4.4.1 in order to find $\mathrm{P}(IP(t) = i, A(t) = n)$.

Step 3. Use equations (4.33) and (4.34) to find $P(NI(t) = i, A(0) = n)$ and $P(NI(t) = i, A(t_s) = n)$ and then equations (4.35) and (4.36) to compute $P(NI(t) = i)$.

Step 4. Use equation (4.41) to compute $P(I(t) = i)$.
Set $E[I(t)] = \sum_{i=0}^{u} i\, P(I(t) = i)$, eq. (4.42).

Step 5. Use equation (4.46) to compute $P(B(t) = i)$.
Set $E[B(t)] = \sum_{i=0}^{d_{max}-\ell} i\, P(B(t) = i)$, eq. (4.47)

Step 6. Solve equations (4.59), (4.61), (4.69), (4.71), (4.73), (4.75), (4.77) and (4.79) to find $\mathrm{E}[R(t); i, n]$
Set $\mathrm{E}[R(t)] = \sum_{i=\ell}^{u} \sum_{n=1}^{m} \mathrm{E}[R(t); i, n]$, eq. (4.81)

Step 7. Set $\Phi_{(\bar{s},\bar{S})}(T) = h \int_0^T E[I(t)]\, dt + b \int_0^T E[B(t)]\, dt + \omega E[R(T)]$, eq. (4.84).

Chapter 5

HEURISTIC

In this chapter, we propose a heuristic optimization that utilizes line-search algorithm to minimize the cost function that was demonstrated in §4.4.6. The proposed heuristic initiates the line-search from the solution of the static ordering policy which is presented in some operations management textbooks; see e.g., Silver et al. (1998), Simchi-Levi et al. (1999) and Nahmias and Cheng (2009). We first introduce the static ordering policy in section 5.1 after which the line-search heuristic is presented in section 5.2.

5.1 Static Ordering Policy

The static ordering policy is a simple heuristic that could be easily implemented in practice to approximate the (s, S) values that minimizes costs. This approximation is based on the assumption that the demand during lead time $D(L)$ is normally distributed. In period n, the reorder and order-up-to levels, rounded to the nearest integer values, are computed using equation (5.1).

$$\begin{aligned} s_n &= E[D(L)] + z\,\text{Stdv}(D(L)) \\ S_n &= s_n + Q \end{aligned} \tag{5.1}$$

where z is the solution to the equation,

$$G_u(z) = \frac{Q}{\text{Stdv}(D(L))}\Big(\frac{h}{b+h}\Big) = \int_z^{\infty} (u-z)\frac{1}{\sqrt{2\pi}}e^{-u^2/2}du$$

$G_u(z)$ is the expected value of backordered units per order cycle.

$E[D(L)]$ and $\text{Stdv}(D(L))$ are the mean and standard deviation of the demand-count over lead time. Q is the order quantity based on the Economic Order Quantity (EOQ) model,

$$Q = \sqrt{\frac{2\,\omega\,D_e}{h}}$$

where D_e is the effective demand rate, $D_e = E[D(\tau)]/\tau$.

Thus, the reorder point, s_n, is the sum of the average demand and the safety stock, which is the safety factor z times the standard deviation of demand. The order-up-to level, S_t, is the sum of s_t and the economic order quantity Q.

5.2 Line Search Heuristic

To find the replenishment $(\bar{s}, \bar{S})$ policy of this heuristic a one-way line search algorithm is utilized. The (s_n, S_n) values obtained from the static ordering policy in eq. (5.1) for each period n, is used as a starting point for the line search. In each period n, take $E[D(L)]$ ($\text{Stdv}(D(L))$ as the average value of all $E[D_t(L)]$ ($\text{Stdv}(D_t(L))$) $\forall\, t \in$ Period n and $D_e = E[D(L)]/L$. A line search is done on the s_n values to find the best $s_n \in \{1, \ldots, S_n - 1\}$ value that minimize cost in period n while keeping the S_n values fixed. After finding the best s_n values in each period, another line search is done on the S_n values to find the best $S_n \in \{s_n + 1, \ldots, S_{max}\}$ for each period while keeping the s_n values previously found fixed. S_{max} represents the upper bound of the order-up-to level and should be large enough, like the double of the total demand quantity, $S_{max} = 2 \times d_{max}$. The algorithm is repeated again with the resulting $(\bar{s}, \bar{S})$ ordering policy as a starting point until no further improvement is possible.

The below algorithm summarizes the computation reorder policy $(\bar{s^p}, \bar{S^p})$ that minimizes the total expected costs.

Heuristics: Optimization by Line Search - Algorithm

Algorithm 5.1: Proposed Heuristic Policy

Input : Cost function $\Phi_{(\bar{s},\bar{S})}(T)$
Output: Optimal ordering policy $(\bar{s}, \bar{S})$
Initialization: solve eq.(5.1) to find the static ordering policy $(\bar{s^i}, \bar{S^i})$ values;
Set $(\bar{s^p}, \bar{S^p}) = (\bar{s^i}, \bar{S^i})$;
repeat
    for $n = 1$ *to* N **do**
        Set $s_n^t = \underset{s}{\operatorname{argmin}}\, \Phi_{(\bar{s^p},\bar{S^p})}(T)$;
        Set $s_n^p = s_n^t$;
    end
    for $n = 1$ *to* N **do**
        Set $S_n^t = \underset{S}{\operatorname{argmin}}\, \Phi_{(\bar{s^p},\bar{S^p})}(T)$;
        Set $S_n^p = S_n^t$;
    end
    if $(\bar{s^p}, \bar{S^p}) \neq (\bar{s^i}, \bar{S^i})$ **then**
        Set $(\bar{s^i}, \bar{S^i}) = (\bar{s^p}, \bar{S^p})$;
    end
until $(\bar{s^p}, \bar{S^p}) = (\bar{s^i}, \bar{S^i})$;
return $(\bar{s^p}, \bar{S^p})$;

Chapter 6

ANALYSIS

6.1 Numerical Example 1

Consider a single-location inventory control problem of a single product where demand is stochastic and time-dependent. The inventory cost parameters are the fixed setup cost $\omega = \$200$/order incurred every time we make an order, the holding cost $h = \$1$/unit/unit time and the back-ordering cost $b = \$10$/unit/unit time. The lead time until orders are received is $L = 4$ time units. Let $T = [0, 40]$ be the planning horizon. We divide the planning horizon T into 4 periods such that each period has a length of 10 time units. Each period is then divided into 100 sub-intervals of length equals to 0.1 time units. The period intervals are used to approximate the demand-count 2^{nd} moment and the sub-intervals in each period are used to approximate the demand-count 1^{st} moment. The 1^{st} and 2^{nd} moments of the demand-count are shown in Table 6.1.

Period:	1	2	3	4
1^{st} moment:	20	40	10	50
2^{nd} moment:	490	1707	140	2600

Table 6.1: First and second moment demand count

For simplicity, the demand-count 1^{st} moment is held constant over the sub-intervals of each period. This is reflected by the instantaneous demand rate $r(t)$ that is constant through out each period as it can be seen in Figure 6.1.

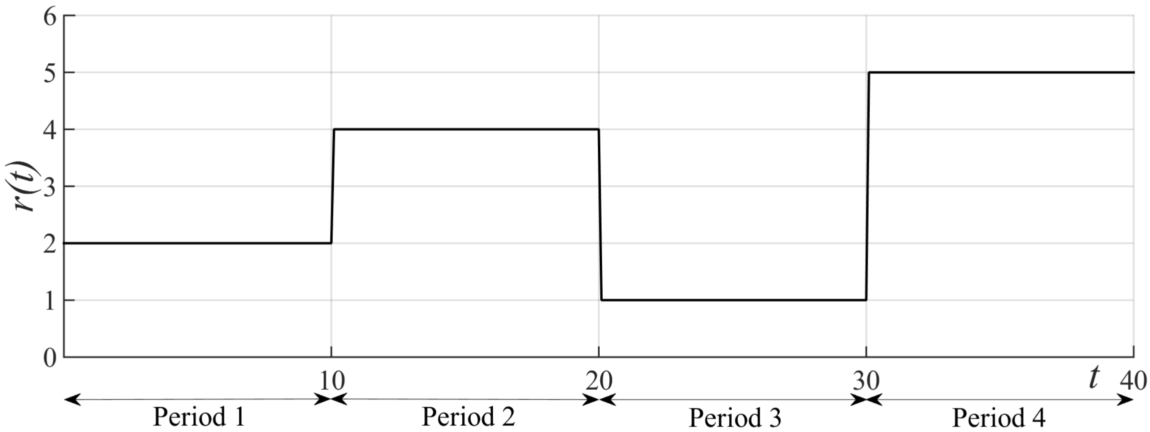


Figure 6.1: Instantaneous Demand Rate.

We fit a two-level mixture of Erlangs of common order (2-MECO) phase-type distribution with $m_1 = 2$ and $m_2 = 3$ to the demand-count moments.

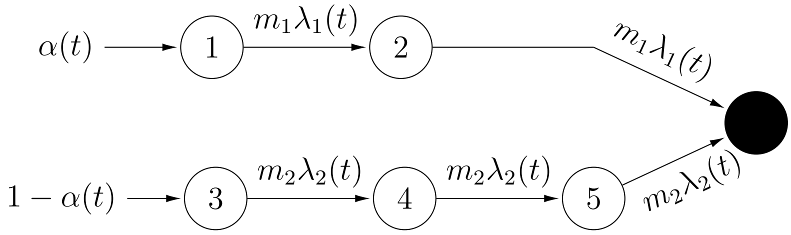


Figure 6.2: 2-MECO Structure

The fitting is done by utilizing the time-dependent fitting algorithm presented by Nasr and Taaffe (2012). Figure 6.3 shows the output of Nasr-Taaffe fitting algorithm which contains the time-dependent 2-MECO parameters $\lambda_1(t)$, $\lambda_2(t)$ and $\alpha(t)$. These parameters are then used to find the demand count distribution over $[0, T]$ using the demand-count distribution algorithms presented in §4.3. From the distribution of the demand-count during lead time, we find the mean

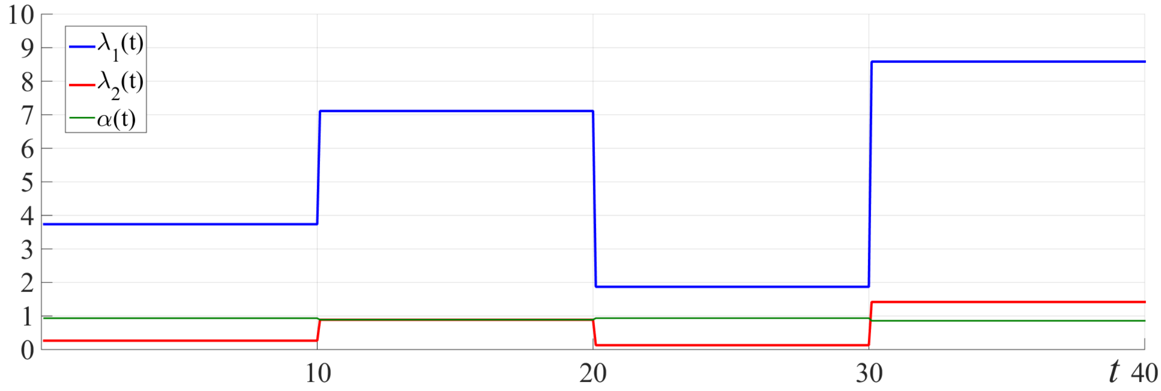


Figure 6.3: 2-MECO Time-dependent Parameters.

and standard deviation of the demand over lead time, Figure 6.4.

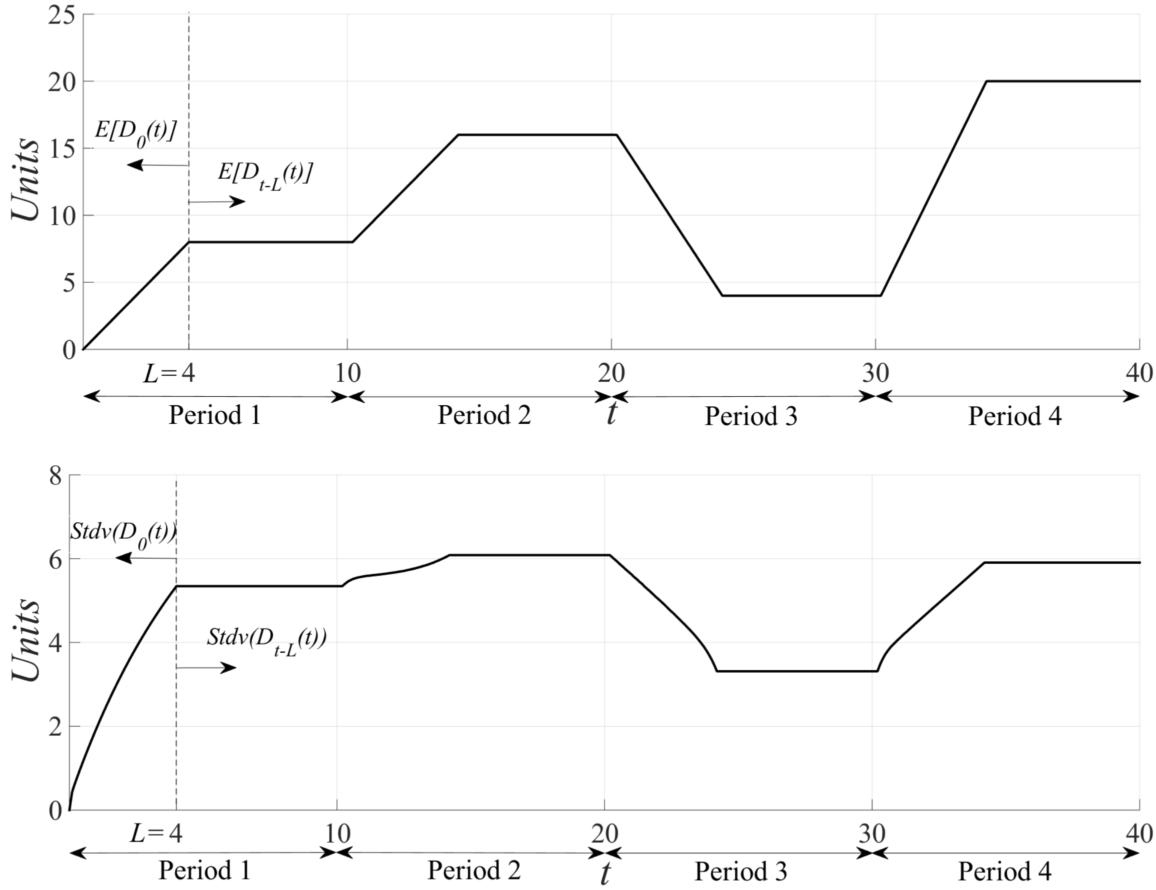


Figure 6.4: Mean and Standard Deviation of the Demand over Lead Time.

Now, for a given dynamic reorder policy, $(\bar{s},\ \bar{S})$ where $\bar{s} = \{7, 14, 3, 17\}$ and $\bar{S} = \{36, 54, 23, 62\}$, the inventory position distribution is found following the algorithm proposed in §4.4.1. The resulting mean and standard deviation of the inventory position level at any time t, $IP(t)$, are shown in Figure 6.5.

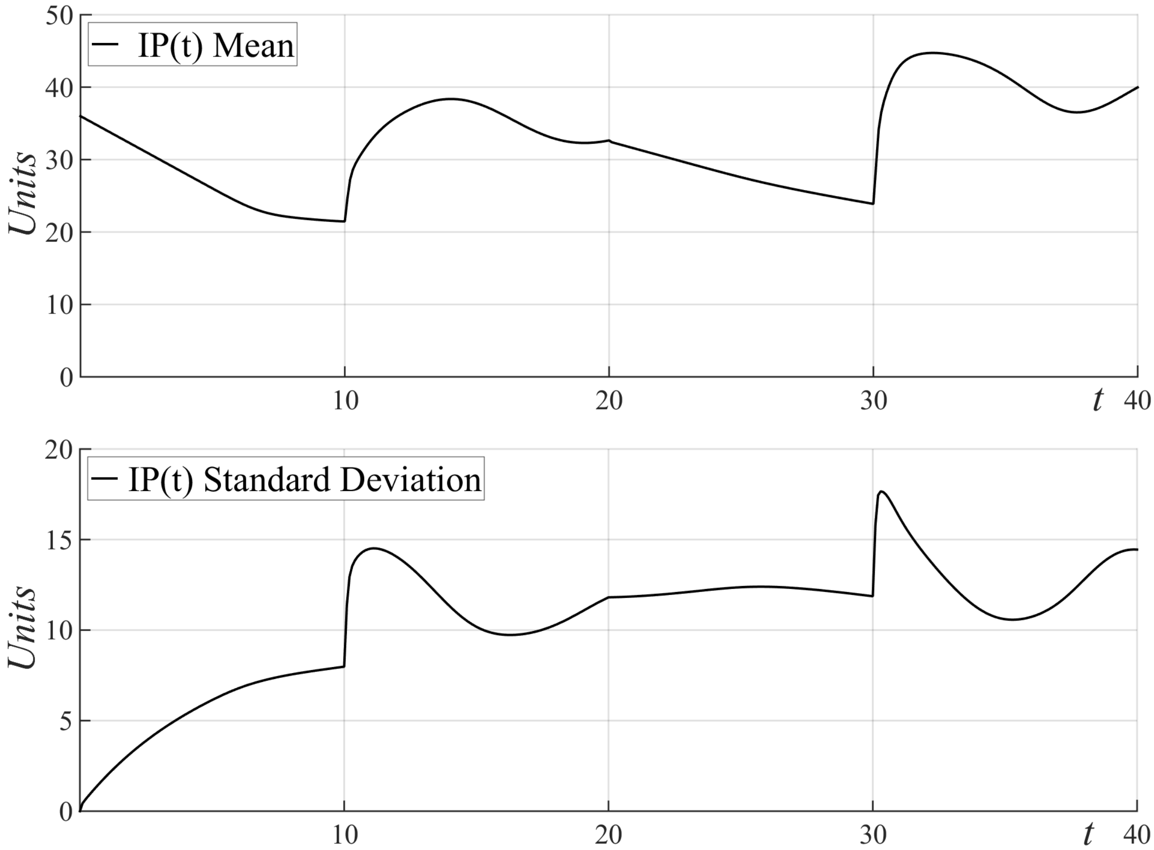


Figure 6.5: IP(t) Mean and Standard Deviation.

The $NI(t)$ distribution can be found by solving equations (4.33), (4.34), (4.35) and (4.36). The resulting mean and standard deviation of the net inventory level at any time t, $NI(t)$, are shown in Figure 6.6.

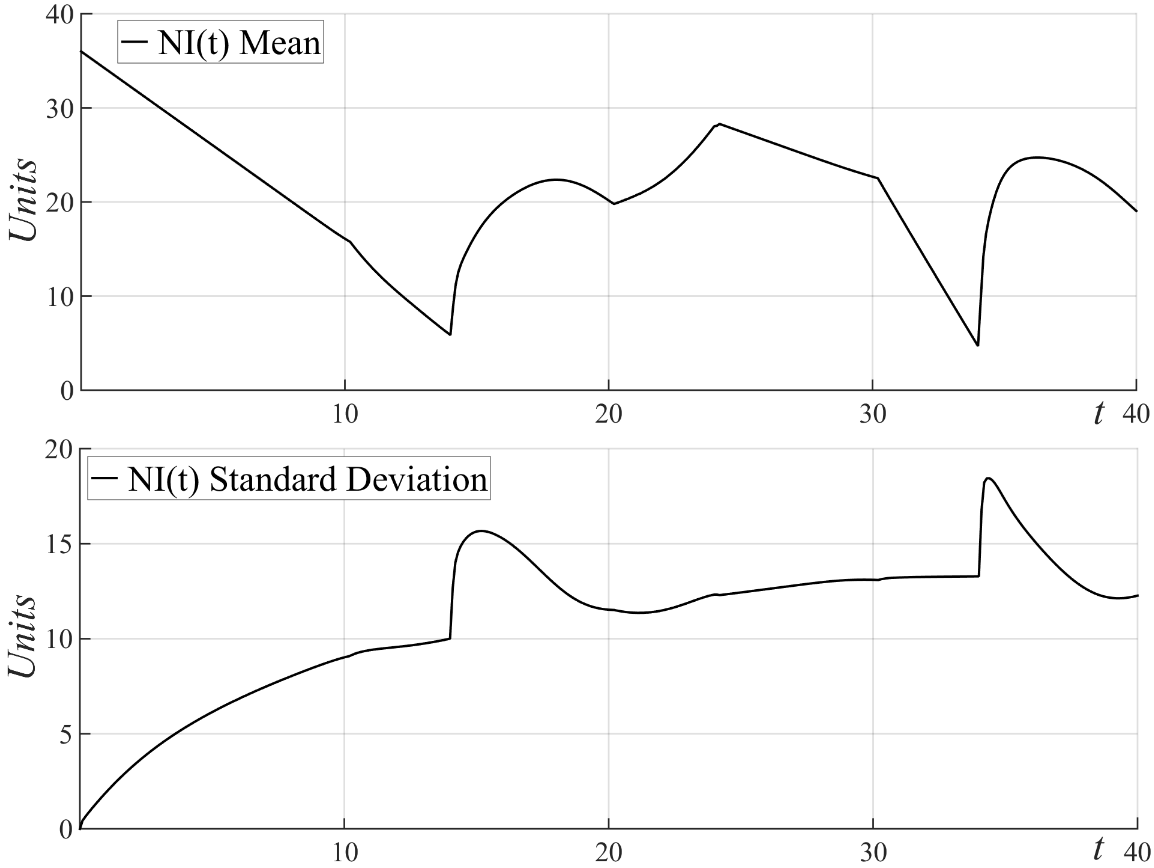


Figure 6.6: NI(t) Mean and Standard Deviation.

Using equation (4.41) we find the on-hand inventory distribution. The mean and standard deviation of the on-hand inventory levels $I(t)$ are shown below in Figure 6.7.

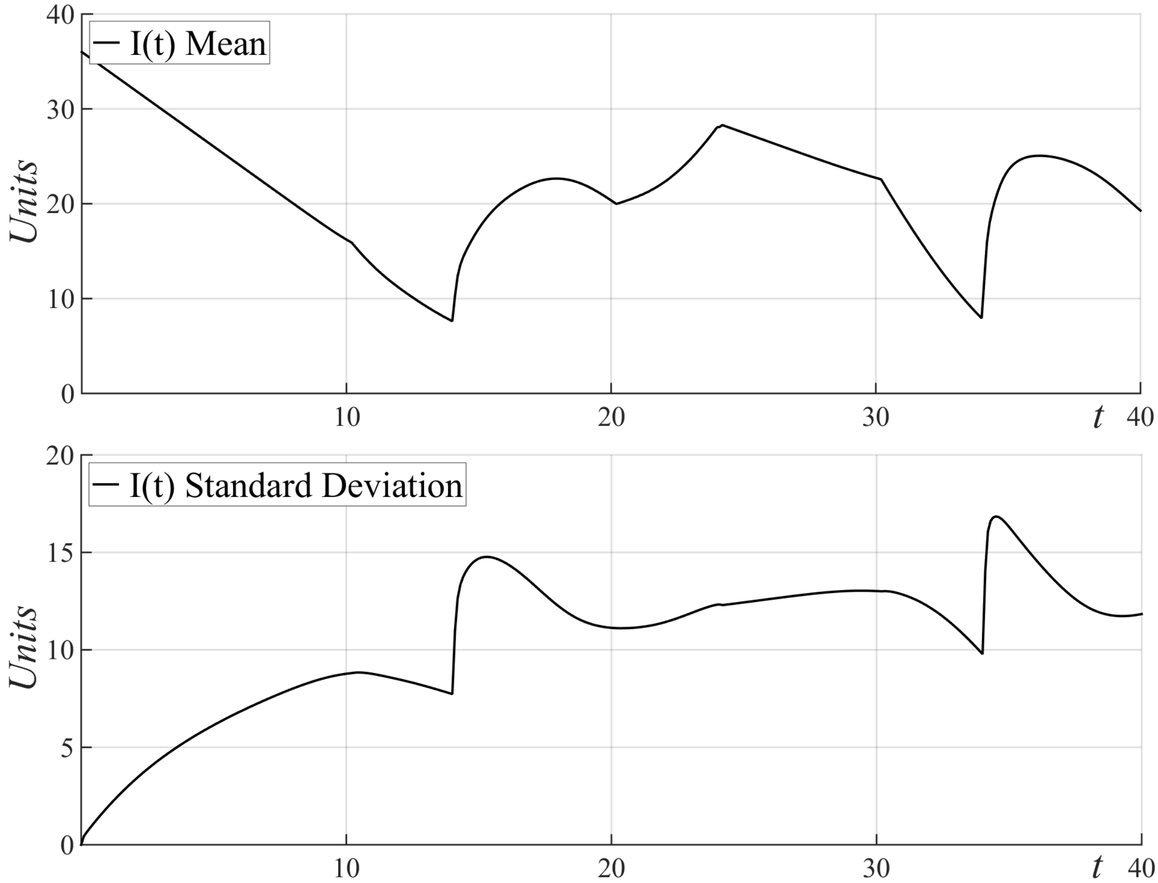


Figure 6.7: I(t) Mean and Standard Deviation.

Similarly, using equation (4.46) we find the backordered inventory distribution. The mean and standard deviation of the backorders, $B(t)$, are shown below in Figure 6.8.

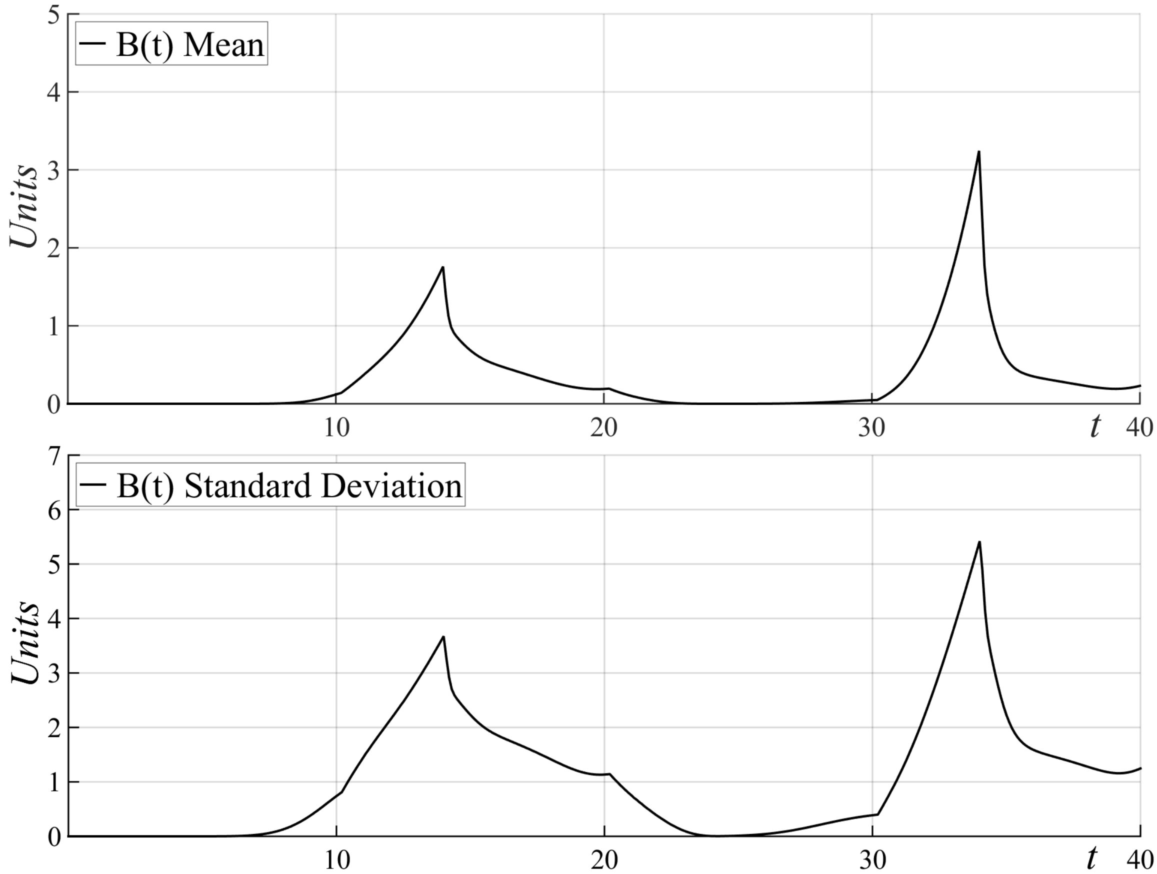


Figure 6.8: B(t) Mean and Standard Deviation.

Figure 6.9 shows the evolution of all inventory measures over time.

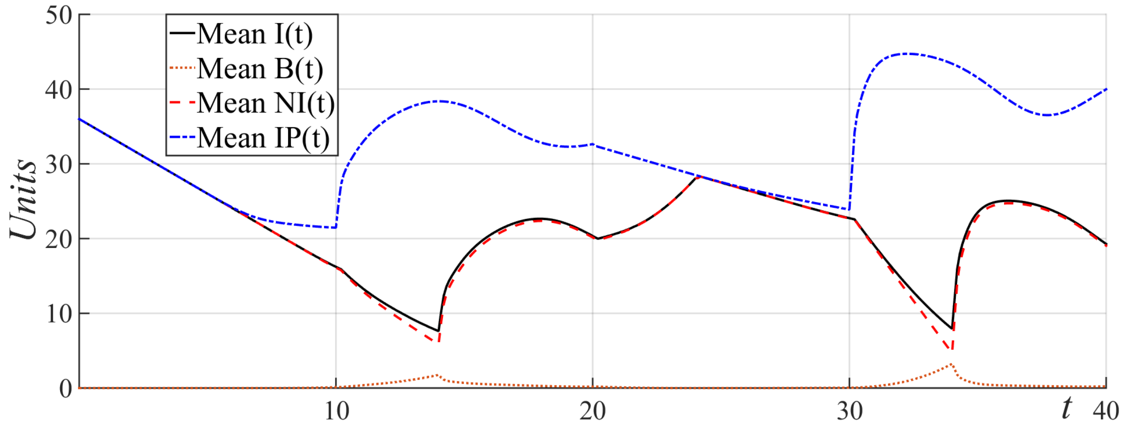


Figure 6.9: Evolution of Inventory Measures Over Time.

The mean and standard deviation of the number of orders is found by solving equations (4.69) to (4.82).

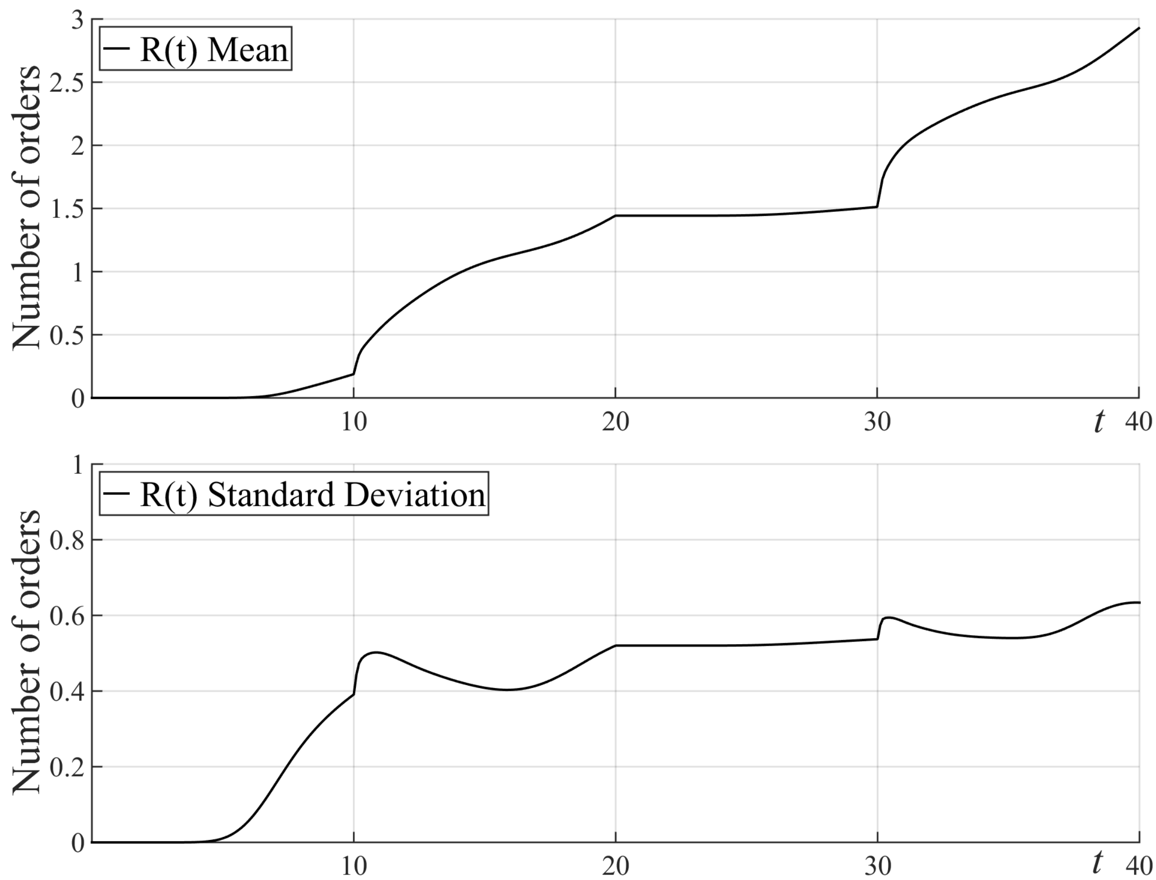


Figure 6.10: R(t) Mean and Standard Deviation.

Using equation (4.84) we find the inventory expected costs at any time t for our replenishment policy $\bar{s} = \{7, 14, 3, 17\}$ and $\bar{S} = \{36, 54, 23, 62\}$.

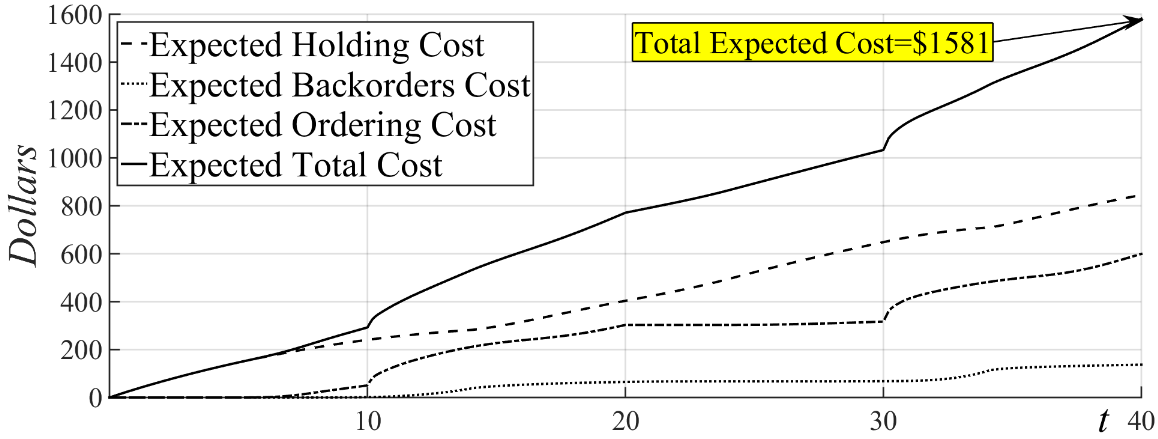


Figure 6.11: Inventory Expected Costs.

To find the $(\bar{s}, \bar{S})$ policy the minimizes the expected cost we first compute the static ordering policy:

$$\left.\begin{aligned} s_n &= E[D(L)] + z\,\text{Stdv}(D(L)) \\ S_n &= s_n + Q \end{aligned}\right\} \rightarrow \left\{\begin{aligned} \bar{s^i} &= \{7\ 14\ 3\ 17\} \\ \bar{S^i} &= \{36\ 54\ 23\ 62\} \end{aligned}\right.$$

Then following Algorithm 5.1 the optimal heuristic policy is obtained:

$$\boxed{\begin{aligned} \bar{s^*} &= \{8\ 10\ 5\ 12\} \\ \bar{S^*} &= \{34\ 49\ 51\ 66\} \end{aligned}}$$

Figure 6.12 shows the expected inventory costs at any time t for the optimal heuristic policy.

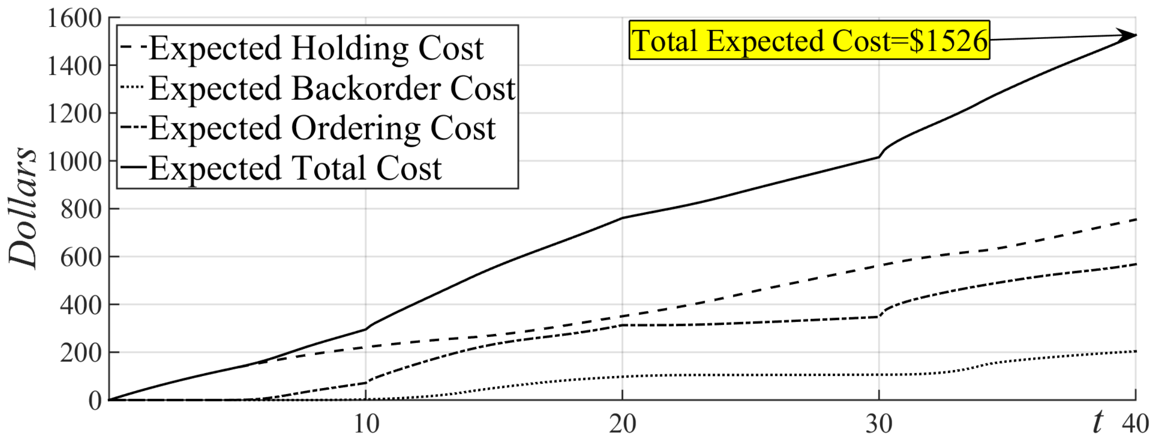


Figure 6.12: Inventory Expected Costs.

Now, we study the option of incorporating a service level constraint (SL=0.9) and accordingly recompute the optimal heuristic policy. The results are shown below in Figure 6.13.

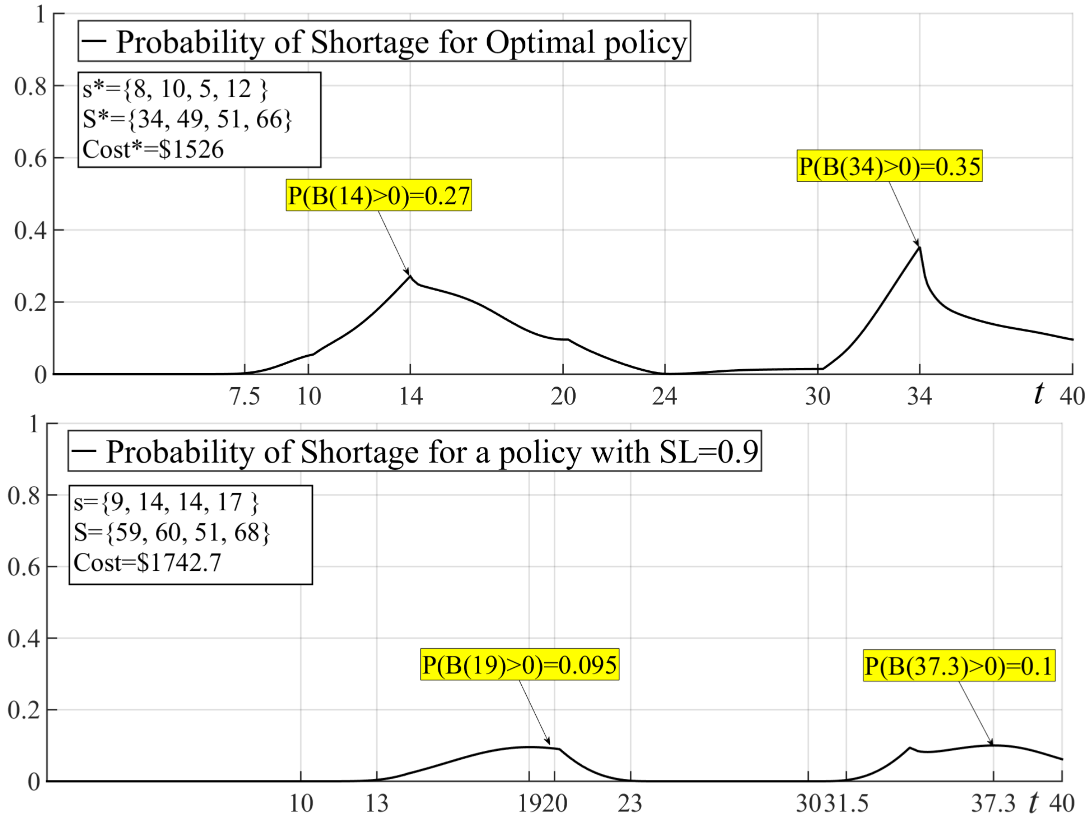


Figure 6.13: Probability of shortage

6.2 Numerical Example 2

Consider Numerical Example 1 but in this example we illustrate the case where the demand rate is continuous and follows the following function:

$$r(t) = 1 + \frac{t}{10} + 0.75 \times \sin(0.2\,\pi\,t)$$

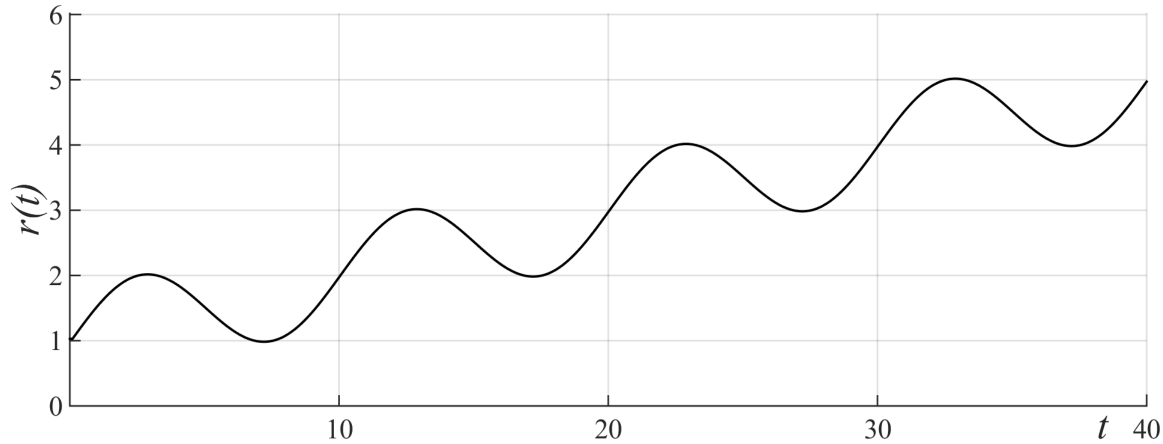


Figure 6.14: Instantaneous Demand Rate.

The output of Nasr-Taaffe fitting algorithm, Figure 6.15, contains the time-dependent 2-MECO parameters $\lambda_1(t)$, $\lambda_2(t)$ and $\alpha(t)$.

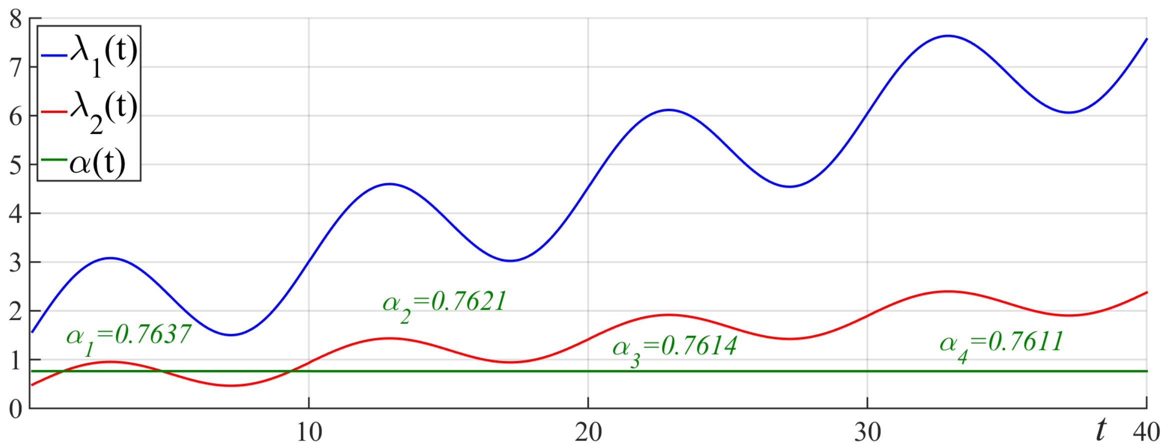


Figure 6.15: 2-MECO Time-dependent Parameters.

These parameters are then used to find the demand count distribution over the planning horizon $[0, T]$.

The resulting mean and standard deviation of the demand-count over lead time is shown in Figure 6.16.

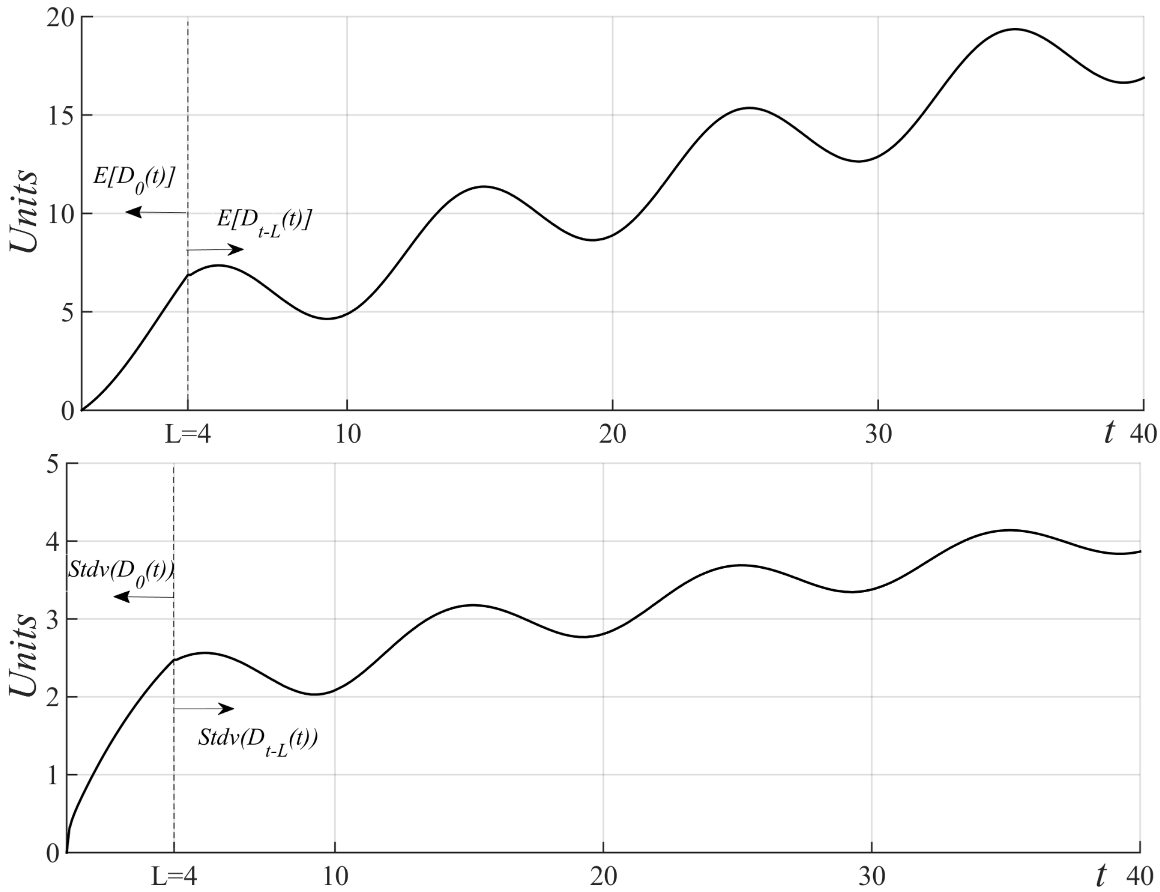


Figure 6.16: Mean and Standard Deviation of the Demand over Lead Time.

Chapter 7

CONCLUSION AND FUTURE WORK

Due to the complicated nature of time-dependent demand models not much work have been done dealing with non-stationary stochastic demand.

In this work we quantify the impact of time-dependent parameters and illustrate the behavior and evolution of the inventory measures over time. This results in a better decision making policies and improve our understanding of such problems, especially from the managerial perspective.

To the best of our knowledge, we are the first to utilize time-dependent phase-type distribution to model non-stationary stochastic demand and inventory characteristics.

Our results show the feasibility of approximating non-stationary demand-count distribution with time-dependent Ph_t distributions in inventory management problems. Moreover, we present computationally efficient numerical algorithms to compute all expected inventory measures values over time. The resulting inventory measures at any point in time allow for sensitivity analysis study which provides great insights for what-if analysis.

In addition to stochastic time-dependent demand future work could include models that incorporates correlated demand. In this case, demand will be uncertain, time-varying and depends on the environment factors. Thus, incorporating the utter most case that could be encountered in practice. This thesis considers fixed lead time. Future work could extend this assumption to the case of a stochastic lead time that could be modeled also using phase-type distribution, see Song and Zipkin (1993). It would be also nice to study the effect of sudden changes (such as a sudden increase or decrease) that could happen in the demand pattern on the inventory costs and number of backorders for a given replenishment policy. Future work could utilize alternate ordering policies such as the batch ordering policy (R, nQ). In addition, the inventory costs in this research could be extended to include discounted costs. Finally, future work could include models that deal with the end of horizon effect because of the fact that cost accounting continues beyond the last decision period due to lead time. Thus, for instance to deal with this problem, excess inventory at the end of horizon could have a salvage value which could be subtracted from the total costs.

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