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SUPERSYMMETRIC GRAVITATIONAL SOLUTIONS IN N=2 SUPERGRAVITY THEORIES

by

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AN ABSTRACT OF THE THESIS OF

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Not only are black holes important objects in the field of gravitational physics, but also in studying other branches of physics such as quantum field theory [1], condensed matter physics and material science [2]. Within supergravity theories, black holes are considered as solutions of the low energy supergravity theories originating via compactification of the M-theory, and they can, in principle, be lifted to solutions in higher dimensions. In recent years, there has been a lot of research activities in physics and mathematics on the subject of finding and classifying black hole solutions and gravitational instantons admitting various fractions of supersymmetry (see for example [3], [4] and [5]). Supersymmetric solutions are those admitting Killing spinors, i.e., covariantly constant spinors with respect to the supercovariant connection. My thesis is based on learning spinorial geometry [6], a powerful method used in classifying and finding supersymmetric solutions in supergravity theories. We, specifically, discuss the ordinary and the fake $N = 2, D = 4$ supergravity theories coupled to vector multiplets. In the fake theory, the gauge fields have kinetic terms with a sign opposite to that present in the ordinary case. We solve the Killing spinor equations for the standard and the fake theories in a linked manner by introducing a parameter κ. The solutions found are fully determined in terms of algebraic conditions, the stabilisation equations, in which the symplectic sections are related to a set of functions. These functions are harmonic in the case of the ordinary supergravity theory and satisfy the wave-equation in flat (2+1)-space-time in the fake theory.

Contents

To whom sacrificed everything and are no longer with us.

Chapter 1

Introduction

Recently, investigation and classification of supersymmetric solutions in supergravity theories have been active areas of research. Piling up systematic classifications of candidate solutions of supergravity theories in different dimensions is a very beneficial thing to do. This is because it could give us an insight about possible background geometries for different strings from a low-energy perspective [7], for instance. In the early 80's, Tod [8] was capable of setting a first classification of the four-dimensional Einstein-Maxwell theory. Then, other significant findings and categorizations of solutions took place using various approaches. However, few years ago [9], the method of spinorial geometry was implemented as a powerful technique in the classification of solutions in supergravity theories. One of its traits is that it supplies us with a systematic way to solve the Killing spinor equations of supersymmetric theories.

In this thesis, we will use this method to discuss solutions in ordinary and fake $N = 2, D = 4$ supergravity theories coupled to vector multiplets. A special characteristic about the concerned fake supergravity theories is that their gauge fields' kinetic terms come with the non-conventional sign in the action. We shall refer to the solutions of these theories as phantom solutions. It is worth mentioning that fake de Sitter supergravity can be obtained by analytic continuation of anti-de

Sitter supergravity or by a non-linear Kaluza-Klein reduction of the * theories of Hull [10].

Phantom black hole solutions have been considered and analyzed by Gibbons and Rasheed [11]. Add to that, those solutions have been employed by different authors in the fields of astrophysics and dark matter [12]. Very recently, metrics with space-like Killing vectors admitting Killing spinors in four-dimensional Einstein gravity coupled to a phantom Maxwell field were found by Sabra [13]. In this work, we will generalize the results of [13] to four-dimensional $N = 2$ supergravity theory coupled to vector multiplets. Furthermore, we will present in a unified manner the ordinary scenario of the theory [3].

This thesis will next be divided into four main chapters. *Chapter 2* is a brief chapter that illustrates some mathematical concepts that are essential for later use. Since supergravity was born after introducing supersymmetry to general relativity, *chapter 3* was added to provide the reader with a summarized background about the latter triumph which stroke the world of theoretical physics. *Chapter 4* is more of a tool chapter that supplies us with some relations in special geometry which are necessary to carry out calculations in chapter 5. That *fifth chapter* is where we present detailed calculations of the solutions of Killing spinor equations for both, the standard and the fake, supergravity theories. Finally, *chapter 6* summarizes what we have done in few words.

Chapter 2

Mathematical Preliminaries

Since the break of their dawn, mathematics and physics have often complemented one another. Advances in both fields, no matter how specialized they grew independently, seemed to serve each other on many occasions. To physicists, a very gratifying branch of mathematics is the one that has to do with geometry and its concepts. The reason behind that is the sole intuition that one can extract from it. The intimate relation between the geometric perspective and the analytic one sets the stage for this chapter. The chapter will be concerned with few concepts in differential geometry which are essential tools to equip ourselves with in theoretical physics.

2.1 Manifolds

Manifolds, with their presentable mathematical aura, are one of the fundamental concepts in physics. For example, in Einstein's general theory of relativity, spacetime is taken to be a 4-dimensional differentiable manifold. In other situations, manifolds of arbitrarily higher dimensions play a good deal of parts in different theories. Putting this into context, a manifold corresponds to a space that may have complicated topology globally but locally can be seen as Euclidean space. In what follows, our treatment is close to that of [14].

Mathematically speaking, a D-dimensional manifold $\mathcal{M}$ is a topological space covered with a family of open sets (coordinate patches M_i) in a way that allows us to write: $\mathcal{M} = \cup_i M_i$. On each patch there is a one-to-one map ϕ_i, called a chart, from $M_i \to \mathbb{R}^D$. In other words, a point $p \in M_i \subset \mathcal{M}$ is mapped via ϕ_i to its local coordinates, i.e, $\phi_i(p) = (x^1, x^2, ..., x^D)$. However, in more exciting situations, p may belong to some overlap between two patches, $M_i \cap M_j$. When this is the case, then there exists another map $\phi_j(p) = (x'^1, x'^2, ..., x'^D)$ that leads us to a second set of coordinates for the point p. The set of functions $x'^\mu(x^\nu)$[1] then specify the compound map $\phi_j \circ \phi_i^{-1}$ from $\mathbb{R}^D \to \mathbb{R}^D$.

2.2 Scalars, Vectors and a Little More

Here, as in [14], we will refer to a less formal way in defining the properties of objects that will live on $\mathcal{M}$, and then we will elaborate on one definition. A manifold now should be understood as a topological space having points each of which can be linked to different coordinate systems such as the ones defined in the *Manifolds*' section. Now any two sets of these coordinates are related by smooth functions, C^∞, like those mentioned previously, i.e, $x'^\mu(x^\nu)$ with non-singular Jacobian $\frac{\partial x'^\mu}{\partial x^\nu}$. This certainly will ring a bell later on in chapter 3, for we will refer to such a change of coordinates as a general coordinate transformation.

From what preceded, we can give a flavor of the sort of ways objects are related on a manifold.

A *scalar field* is described by $f(x)$ in one set of coordinates and $f'(x')$ in the second set. The two functions must be pointwise equal

$$f'(x') = f(x). \tag{2.2.1}$$

[1]These functions and their inverses are required to be smooth.

A *contravariant vector field* is described by D functions $V^{\mu}(x)$ in one coordinate system and D functions $V'^{\mu}(x')$ in the second. Those are related by

$$V'^{\mu}(x') = \frac{\partial x'^{\mu}}{\partial x^{\nu}} V^{\nu}(x). \tag{2.2.2}$$

A *covariant vector field* $\omega_{\mu}(x)$ and a (mixed) *tensor* $T^{\mu}_{\nu}(x)$, by a similar fashion, can behave under general coordinate transformation respectively as

$$\omega'_{\mu}(x') = \frac{\partial x^{\nu}}{\partial x'^{\mu}} \omega_{\nu}(x), \tag{2.2.3}$$

$$T'^{\mu}_{\nu}(x') = \frac{\partial x'^{\mu}}{\partial x^{\sigma}} \frac{\partial x^{\rho}}{\partial x'^{\nu}} T^{\sigma}_{\rho}(x). \tag{2.2.4}$$

Using the notion of a contravariant vector field, one can consider this system of differential equations

$$\frac{dx^{\mu}}{d\lambda} = V^{\mu}(x). \tag{2.2.5}$$

It is the idea of an integral curve, a curve on the manifold $\mathcal{M}$, that we now aim to illustrate. This would be the solution $x^{\mu}(\lambda)$ of the equation (2.2.5) and which is a map from $\mathbb{R} \to \mathcal{M}$. Through every point of each of the $M_i's$ in which the vector field does not vanish, there is an integral curve. If the manifold is $\mathbb{R}^D$, then the tangent to the curve $x^{\mu}(\lambda)$ is well-known to be the vector

$$\frac{dx^{\mu}}{d\lambda}. \tag{2.2.6}$$

This interpretation is carried on for a general manifold.

It turns out that the vector fields evaluated at p determine the D-dimensional tangent space $T_p(\mathcal{M})$. Thus, as p varies over $\mathcal{M}$, a vector field $V^{\mu}(x)$ may then be thought of as a smooth assignment of a tangent vector in each $T_p(M)$.

2.3 Differential Forms and Hodge Duality

In this section, we discuss the algebra [14] of differential forms by listing some of the useful properties. This concept of differential forms is one that we will encounter a lot in chapters to come.

When considered together, the scalars (0-forms), the covariant vectors and the totally antisymmetric tensors (e.g., $\omega_{\mu\nu} = -\omega_{\nu\mu}$) make up a useful structure. A differential form of order p or a p-form for $p = 1, 2, ..., D$ is a totally antisymmetric tensor

$$\omega^{(p)} = \frac{1}{p!}\omega_{\mu_1\mu_2...\mu_p} dx^{\mu_1} \wedge dx^{\mu_2} \wedge ... \wedge dx^{\mu_p}. \tag{2.3.1}$$

For example,

$$\omega^{(1)} = \omega_\mu(x) dx^\mu, \tag{2.3.2}$$

$$\omega^{(2)} = \frac{1}{2}\omega_{\mu\nu}(x) dx^\mu \wedge dx^\nu, \tag{2.3.3}$$

where the wedge product, $\wedge$, is defined as antisymmetric. That is, upon flipping two neighboring dx^μ's, we pick up a minus sign. For instance,

$$dx^\mu \wedge dx^\nu = -dx^\nu \wedge dx^\mu, \qquad dx^\mu \wedge dx^\nu \wedge dx^\rho = -dx^\rho \wedge dx^\nu \wedge dx^\mu. \tag{2.3.4}$$

In addition to that, there is an exterior algebra of p-forms. A p-form $\omega^{(p)}$ and a q-form $\omega^{(q)}$ can be multiplied to give a $(p+q)$-form if $p+q \leqslant D$. This product has a property that says

$$\omega^{(p)} \wedge \omega^{(q)} = (-1)^{pq} \omega^{(q)} \wedge \omega^{(p)}. \tag{2.3.5}$$

Now in order to give a complete exterior calculus overview, we define the exterior and the interior derivative. The former maps p-forms into $(p+1)$-forms, whereas the latter maps p-forms into $(p-1)$-forms. We start with exterior derivatives which can be generally defined as

$$d\omega^{(p)} = \frac{1}{p!}\partial_\mu \omega_{\mu_1\mu_2...\mu_p} dx^\mu \wedge dx^{\mu_1} \wedge ... \wedge dx^{\mu_p}. \tag{2.3.6}$$

It is noteworthy that a p-form is called *closed* when it satisfies

$$d\omega^{(p)} = 0, \tag{2.3.7}$$

and *exact* when it can be written as

$$\omega^{(p)} = d\omega^{(p-1)}. \tag{2.3.8}$$

On the other hand, interior derivatives are expressed as

$$(i_V \omega^{(p)}) = \frac{1}{(p-1)!} V^\mu \omega_{\mu\mu_1...\mu_{p-1}} dx^{\mu_1} \wedge dx^{\mu_2} \wedge ... \wedge dx^{\mu_{p-1}}. \tag{2.3.9}$$

Finally, we list one more operation on differential forms. It is the Hodge duality. The Hodge star operator, $\star$, on an n-dimensional manifold takes p-forms to $(n-p)$-forms

$$\star(dx^{i_1} \wedge ... \wedge dx^{i_p}) = \frac{1}{(n-p)!} \epsilon^{i_1...i_p}{}_{j_1...j_{n-p}} dx^{j_1} \wedge ... \wedge dx^{j_{n-p}}. \tag{2.3.10}$$

This map is quite useful in the physics of Supergravity.

Chapter 3

General Relativity: A Review

General Relativity, dressed with an elegant mathematical structure, is one of the most revolutionary theories in physics. Questing for a relativistic theory of gravity, Einstein changed our perspective in viewing space and time. In 1915, General Relativity was brought into existence with its roots rested in the *Equivalence Principle*. Einstein related the fact that all bodies fall with the same acceleration in a gravitational field to a natural understanding of gravity in terms of the curvature of the four dimensional union of space and time [15]. Consequences of the Equivalence Principle were significant and gave General Relativity the spirit it needed. This chapter will start off from there shedding lights on few of the principle's results. These will be accompanied with the presentation of necessary technicalities in order to give a complete picture in a brief manner. The technicalities, in turn, will also serve in helping us find some of the solutions to Einstein's field equations. Finally, as this chapter comes to a close, we push things a little further so as to introduce the concept of vielbeins which will be of great use in chapters to come.

3.1 Gearing Up

We will begin this section by considering the effects of general coordinate transformations when performed in Minkowski space, the spacetime of Special Relativity. This strategy, used by [16], will pave the way for introducing essential concepts in our General Relativity journey. τ, the proper time, will be our starting point since as we know, it should not rely on what coordinates one uses. As such, we perform an arbitrary general coordinate transformation that will take us to new coordinates, say, $x^\mu(\xi^b)$ where ξ^a indicates a locally inertial coordinate system,

$$d\tau^2 = -\eta_{ab} d\xi^a d\xi^b = -\eta_{ab} \frac{\partial \xi^a}{\partial x^\mu} \frac{\partial \xi^b}{\partial x^\nu} dx^\mu dx^\nu. \tag{3.1.1}$$

Here, $\frac{\partial \xi^a}{\partial x^\mu}$ is the Jacobi matrix associated with the coordinate transformation $\xi^a = \xi^a(x^\mu)$, and $\eta_{ab} = \mathrm{diag}(-1, +1, +1, +1)$ is the Minkowski metric. What preceded calls up the first fact that in the new coordinates, proper time and distance are measured by the metric tensor $g_{\mu\nu}$ instead of the Minkowski metric η_{ab}, for now $d\tau^2$ can be written as

$$d\tau^2 = -g_{\mu\nu}(x) dx^\mu dx^\nu, \tag{3.1.2}$$

where $g_{\mu\nu}(x) = \eta_{ab} \frac{\partial \xi^a}{\partial x^\mu} \frac{\partial \xi^b}{\partial x^\nu}$.

In addition to that, it is well-known that the form of the equation of motion of a free particle in Minkowski space is

$$\frac{d^2}{d\tau^2} \xi^c(\tau) = 0. \tag{3.1.3}$$

So, upon using the conventional change of variable, we get

$$\frac{d}{d\tau} \xi^a = \frac{\partial \xi^a}{\partial x^\mu} \frac{dx^\mu}{d\tau}, \tag{3.1.4}$$

where $\frac{\partial \xi^a}{\partial x^\mu}$ is an invertible matrix at every point.

Differentiating (3.1.4), we obtain

$$\frac{d^2}{d\tau^2}\xi^a = \frac{\partial \xi^a}{\partial x^\mu}[\frac{d^2 x^\mu}{d\tau^2} + \frac{\partial x^\mu}{\partial \xi^b}\frac{\partial^2 \xi^b}{\partial x^\nu \partial x^\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau}]. \tag{3.1.5}$$

Setting (3.1.5) equal to zero as in (3.1.3), we arrive at

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} = 0, \tag{3.1.6}$$

where

$$\Gamma^\mu_{\nu\lambda} = \frac{\partial x^\mu}{\partial \xi^b}\frac{\partial^2 \xi^b}{\partial x^\nu \partial x^\lambda}, \tag{3.1.7}$$

represents what we will be referring to later as a Christoffel symbol.

Equation (3.1.6) brings up the second idea we intend to introduce here which is that (3.1.6) is known as the *geodesic* equation. This equation can be derived or deduced from a number of other ways such as the variational principle.

It is also noteworthy that the Christoffel symbol can be written in terms of the metric tensor as

$$\Gamma_{\mu\nu\lambda} = \frac{1}{2}(\partial_\lambda g_{\rho\nu} + \partial_\nu g_{\rho\lambda} - \partial_\rho g_{\nu\lambda}). \tag{3.1.8}$$

Now that we have introduced what we aimed to introduce, we will want to relate and extend ideas as we divert to the Equivalence Principle and its consequences [17]. Primarily, the Equivalence Principle tells us that physical equivalency exists between gravitational and inertial forces, therefore it is impossible to separate those by any physical experiment. In other words, gravitational forces (accelerations) must be described in the same manner as inertial forces (accelerations). Feeding on this last sentence, we take equation (3.1.3) in Minkowski space. This equation, after the usage of general coordinates, takes the form presented by equation (3.1.6). The second term of the latter equation is the inertial acceleration. Thus, it is obviously clear that inertial accelerations are described by Christoffel symbols, but then again, gravitational accelerations are the same thing as the inertial ones. This means that gravitational accelerations are also described by Christoffel symbols. At this point, it is insightful to see that (3.1.8) shows a reliance of these symbols on $g_{\mu\nu}$. This tells us that, in the General Relativity arena, $g_{\mu\nu}$ will play the role of the gravitational potential. That was one consequence. Another consequence could be deduced from turning again to Minkowski space and noticing that there $\Gamma^{\lambda}_{\mu\nu} = 0$. This implies that the sum of the inertial and the gravitational acceleration could be made equal to zero everywhere. Our experience with gravitational accelerations, though, tells us that when they exist, it is not possible to make them vanish everywhere. Thus, when a gravitational field is present, the space will be necessarily a curved space. That is to say, in general relativity, the gravitational field has gained a geometric interpretation. Therefore, the concept of curvatures, at least intuitively, is now within our grasp. Curvatures depend in a way or another on the metric which defines the geometry of our manifold. We would like to wrap this section up by giving the technical expression of curvatures that will be of use later on

$$R^{\lambda}{}_{\sigma\mu\nu} = \partial_{\mu}\Gamma^{\lambda}{}_{\sigma\nu} - \partial_{\nu}\Gamma^{\lambda}{}_{\sigma\mu} + \Gamma^{\lambda}{}_{\mu\rho}\Gamma^{\rho}{}_{\nu\sigma} - \Gamma^{\lambda}{}_{\nu\rho}\Gamma^{\rho}{}_{\mu\sigma}. \tag{3.1.9}$$

We can construct from it Ricci tensor and Ricci scalar respectively as

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}, \tag{3.1.10}$$

$$R = g^{\mu\nu}R_{\mu\nu}. \tag{3.1.11}$$

3.2 Vacuum Einstein Equations

We present here a sketch of the mathematical derivation that boils down to Einstein's vacuum equations:

We start with the famous Einstein-Hilbert (E-H) action

$$S_{EH} = \int \sqrt{-g} d^4 x R, \tag{3.2.1}$$

where R is the Ricci scalar.

If we unfold this, then we can write the Ricci scalar as $R = g^{\mu\nu} R_{\mu\nu}$, and thus (3.2) can be written as

$$S_{EH} = \int \sqrt{-g} d^4 x g^{\mu\nu} R_{\mu\nu}. \tag{3.2.2}$$

A classic approach to finding the Einstein's equations is to study the behavior of the E-H action under a variation of the metric, that is,

$$\delta S_{EH} = \delta \int \sqrt{-g} d^4 x g^{\mu\nu} R_{\mu\nu} \tag{3.2.3}$$

$$= \int d^4 x ((\delta\sqrt{-g}) g^{\mu\nu} R_{\mu\nu} + \sqrt{-g}(\delta g^{\mu\nu}) R_{\mu\nu} + \sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu}). \tag{3.2.4}$$

Using the identity

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu}, \tag{3.2.5}$$

we get

$$\delta S_{EH} = \int \sqrt{-g} d^4 x (R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R) \delta g^{\mu\nu} + \int \sqrt{-g} d^4 x g^{\mu\nu} \delta R_{\mu\nu}. \tag{3.2.6}$$

For $\frac{\delta S}{\delta \phi^i} = 0$, the first term would give the Einstein equation in vacuum

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0. \tag{3.2.7}$$

This is the case because the functional derivative of the action satisfies

$$\delta S = \int \sum_i (\frac{\delta S}{\delta \phi^i} \delta \phi^i) d^n x, \tag{3.2.8}$$

where ϕ^i is the field being varied. The second term in equation (3.2.6) gives no contribution[1].

In case we wanted to find the field equations in the presence of matter, then we can formally derive it using the same procedure but now with the necessary addition of the matter part to the action. We only state the result

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G_N T_{\mu\nu}, \tag{3.2.9}$$

where G_N is Newton's gravitational constant and $T_{\mu\nu}$ is the stress-energy-momentum tensor.

3.3 Spherical Symmetry

As a primer application of the field equations in vacuum, we shall concern ourselves with the gravitational field of a static and spherically symmetric body. Therefore, in this section we will illustrate the general form of a metric with spherical symmetry [18]. To require spherical symmetry, we write the metric of Minkowski space in polar coordinates (r, θ, ϕ)

$$ds^2_{Minkowski} = -dt^2 + dr^2 + r^2 d\Omega^2, \tag{3.3.1}$$

where $d\Omega^2$ is the metric on a unit two-sphere. Maintaining the form of $d\Omega^2$ is one requirement to preserve spherical symmetry, but we are otherwise free to multiply

[1]Showing that the term in the second integral is a total derivative gives rise to a boundary term in the variation of the action that sorts this out though we do not present this here.

all of the terms by separate coefficients as long as they are functions of the radial coordinate. In other words, we can implement the conditions above by writing the metric as

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + 2C(r)drdt + D(r)r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{3.3.2}$$

This can be simplified by introducing new time and radial coordinates [16]. First, defining a new time coordinate $T(t, r)$ by

$$T(t, r) = t + \psi(r), \tag{3.3.3}$$

this implies that

$$dT^2 = dt^2 + \psi'^2 dr^2 + 2\psi' drdt. \tag{3.3.4}$$

Thus, we can eliminate the $2C(r)drdt$-term in equation (3.3.2) by choosing ψ to satisfy

$$\frac{d\psi(r)}{dr} = -\frac{C(r)}{A(r)}. \tag{3.3.5}$$

Hence, (3.3.2) automatically boils down to

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + D(r)r^2(d\theta^2 + \sin^2\theta\phi^2). \tag{3.3.6}$$

At will, we could define a new radial coordinate so as to embed $D(r)$ in it. The line element then takes the form

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{3.3.7}$$

A useful representation of the line element is provided by setting $B(r) = D(r)$ upon a coordinate transformation

$$ds^2 = -A(r)dt^2 + B(r)(dr^2 + r^2 d\Omega^2). \tag{3.3.8}$$

This is known as the *isotropic* form.

3.4 Schwarzschild Solution

Schwarzschild geometry, after Karl Schwarzschild, is the simplest of curved spacetimes in General Relativity. The reasons behind the simplicity are that, first, the Schwarzschild solution is a solution to Einstein's equation for curved spacetime deprived of matter which practically makes things much easier than if this were not the case. Another reason will be the important feature that characterizes this solution and which is having a great deal of symmetry – spherical symmetry. Starting from Einstein equations in vacuum

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0, \tag{3.4.1}$$

we get

$$R_{\mu\nu} = 0. \tag{3.4.2}$$

Referring back to equation (3.3.7), we can write it using exponentials so that the signature of the metric doesn't change [18]. That is to say

$$ds^2 = -e^{2A(r)}dt^2 + e^{2B(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{3.4.3}$$

We directly deduce that

$$g_{\mu\nu} = \begin{pmatrix} -e^{2A} & 0 & 0 & 0 \\ 0 & e^{2B} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}, \tag{3.4.4}$$

where its inverse is

$$g^{\mu\nu} = \begin{pmatrix} -e^{-2A} & 0 & 0 & 0 \\ 0 & e^{-2B} & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{pmatrix}. \tag{3.4.5}$$

The non-vanishing Christoffel symbols are then

$$\left.\begin{aligned} \Gamma^1_{00} &= A'e^{2(A-B)}, \\ \Gamma^0_{10} &= \Gamma^0_{01} = A', \\ \Gamma^1_{11} &= B', \\ \Gamma^1_{22} &= -re^{-2B}, \\ \Gamma^1_{33} &= -r\sin^2\theta e^{-2B}, \\ \Gamma^2_{12} &= \Gamma^3_{13} = \frac{1}{r} = \Gamma^2_{21} = \Gamma^3_{31}, \\ \Gamma^2_{33} &= -\sin\theta\cos\theta, \\ \Gamma^3_{23} &= \Gamma^3_{32} = \cot\theta, \end{aligned}\right\} \tag{3.4.6}$$

where 0, 1, 2 and 3 represent the t, r, θ and ϕ coordinates respectively. The diagonal terms of $R_{\mu\nu}$ are found to be

$$R_{00} = e^{2A-2B}(A'' + A'^2 - A'B' + \frac{2A'}{r}), \tag{3.4.7}$$

$$R_{11} = -A'' + A'B' + \frac{2B'}{r} - A'^2, \tag{3.4.8}$$

$$R_{22} = (-1 - rA' + rB')e^{-2B} + 1, \tag{3.4.9}$$

$$R_{33} = \sin^2\theta R_{22}, \tag{3.4.10}$$

whereas the non-diagonal components of $R_{\mu\nu}$ vanish. Solving for (3.4.2) using what has just preceded, we promptly get

$$e^{2A} = 1 - \frac{2m}{r}, \tag{3.4.11}$$

where $m = G_N M$ is constant of integration; m is usually called the *gravitational radius* of the central body [17]. Therefore after the necessary substitution of (3.4.11), (3.4.3) can be written as

$$ds^2 = -(1 - \frac{2m}{r})dt^2 + (1 - \frac{2m}{r})^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{3.4.12}$$

This is the *Schwarzschild* solution.

3.5 Reissner-Nordström Solution

We turn sights to another physical application, this time describing the exterior geometry of a spherically symmetric *electrically* charged star or black hole. Here, as well, we solve the Einstein field equations for a static spherically symmetric spacetime. In this case, though, our $T_{\mu\nu}$ is present.

Since spherical symmetry is imposed here and since the object is considered static, the general form of the metric is given by (3.4.3)

$$ds^2 = -e^{2A(r)}dt^2 + e^{2B(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{3.5.1}$$

With $\phi(r)$ being the usual scalar potential, we take the vector potential A for the electrically charged scenario to have [16]:

$$A_t = A_t(r) \equiv -\phi(r), \qquad A_r = A_\theta = A_\phi = 0. \tag{3.5.2}$$

This implies that the only non-vanishing component of the field strength tensor is

$$F_{tr} = \partial_t A_r - \partial_r A_t = \phi'(r). \tag{3.5.3}$$

Now, for Maxwell theory, it can be easily derived starting from the theory's action that

$$T_{\mu\nu} = F_{\mu\gamma}F_\nu{}^\gamma - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}. \tag{3.5.4}$$

For example for the T_{tt} case, we get

$$T_{tt} = F_{tr}F_{tr}g^{rr} - \frac{1}{4}g_{tt}g^{tt}g^{rr}F_{tr}F_{tr}, \tag{3.5.5}$$

$$T_{tt} = \frac{1}{2}g^{rr}(F_{tr}F_{tr}). \tag{3.5.6}$$

We know what g^{rr} is from (3.4.5) and F_{tr} from the fact that [16]

$$\phi(r) = \frac{Q}{r}. \tag{3.5.7}$$

Hence, $E_r = Q/r^2$.

Summarizing the obtained results for $T_{\mu\nu}$, we write

$$T_{tt} = \frac{Q^2}{2r^4} e^{-2B}, \tag{3.5.8}$$

$$T_{rr} = -\frac{Q^2}{2r^4} e^{-2A}, \tag{3.5.9}$$

$$T_{\theta\theta} = \frac{Q^2}{2r^2} e^{-2(A+B)}. \tag{3.5.10}$$

That being mentioned, we only have to solve for Einstein's equation (3.2.9) using (3.4.7 - 3.4.10). Then, we follow the same procedure as in the Schwarzschild case in order to arrive at the Reissner-Nordstörm solution

$$ds^2 = -(1 - \frac{2m}{r} + \frac{q^2}{r^2}) dt^2 + (1 - \frac{2m}{r} + \frac{q^2}{r^2})^{-1} dr^2 + r^2 d\Omega^2, \tag{3.5.11}$$

where $q^2 = G_N Q^2$.

$f(r)$ being equal to

$$f(r) = (1 - \frac{2m}{r} + \frac{q^2}{r^2}), \tag{3.5.12}$$

has now two roots

$$r_{\pm} = m \pm \sqrt{m^2 - q^2}. \tag{3.5.13}$$

Having noticed that, we point out that there are three cases at hand now: $m^2 - q^2 < 0, > 0$ or $= 0$. The latter case will be the subject of the next section.

3.6 Extremal Reissner-Nordström Solution

The case where $m^2 = q^2$ characterizes the extremal Reissner-Nordström solution. Thus, equation (3.5.11) becomes

$$ds^2 = -(1 - \frac{m}{r})^2 dt^2 + (1 - \frac{m}{r})^{-2} dr^2 + r^2 d\Omega^2. \tag{3.6.1}$$

This equation can be written, upon setting $r' = r - m$, as

$$ds^2 = -(1 + \frac{m}{r'})^{-2} dt^2 + (1 + \frac{m}{r'})^2 (dr'^2 + r'^2 d\Omega^2). \tag{3.6.2}$$

This is the extremal Reissner-Nordström (ERN) metric in isotropic coordinates [16]. The (ERN) black hole has many special properties [19, 20]. One of those properties is that it has a multi-centered generalization. That is, as long as their horizons do not overlap and all their charges have equal sign, there exist static configurations of black holes. Those can be placed at arbitrary positions in space. Thus, the gravitational attraction and electrostatic repulsion cancel out without the slightest worry about the position. The corresponding metric belongs to the class of metrics which was discovered by Majumdar and Papapetrou

$$ds^2 = -H^{-2}(\vec{x}) dt^2 + H^2(\vec{x}) d\vec{x}^2, \tag{3.6.3}$$

where $H(\vec{x})$ is a harmonic function.

One possible choice of the harmonic funtion would be

$$H(\vec{x}) = 1 + \frac{M}{r}. \tag{3.6.4}$$

This single-center solution is the ERN black hole with mass M.

The most general choice of $H(\vec{x})$ would be

$$H(\vec{x}) = 1 + \sum_{i=1}^{N} \frac{M_i}{|\vec{x} - \vec{x}_i|}, \tag{3.6.5}$$

where M_i and x_i are the mass and position (of the event horizon) of the i-th black hole. Extremal black holes have other very special properties and it turns out that these can be translated in terms of a symmetry principle – the supersymmetry principle. After embedding gravity into extended supergravity, extremal black holes can be called supersymmetric solitons and a set of those black holes is the ERN one. It is good to note that besides the Majumdar-Papapetrou solutions, there are the IWP (Israel-Wilson-Perjes) solutions in the case of $N = 2$ supergravity. Those are rotating, stationary generalizations of the Majumdar-Papapetrou solutions.

3.7 The Frame Field

In this section, we discuss the formalism of few notions but this time in noncoordinate basis [14, 18]. We usually take advantage of the fact that for the tangent space at a point p, a natural basis is given by partial derivatives with respect to the coordinate at that point; that is to say, $\hat{e}_{(\mu)} = \partial_\mu$. But there is nothing that can keep us from choosing any basis we want. So in order to get started, we define a quantity called vierbein (or tetrad) when our physics is concerned with four dimensions and vielbein (or frame field) in general cases. The tetrad, e^a_μ, enters the game when expressing our old basis in terms of the new one via

$$\hat{e}_{(\mu)} = e_\mu{}^a \hat{e}_{(a)}. \tag{3.7.1}$$

These are chosen to form an orthonormal set of vectors at each point in the tangent space of $\mathcal{M}$. We note that upon the introduction of the notion of the vierbein, we can write $g_{\mu\nu}$, the known metric tensor, in terms of the Minkowski metric η_{ab} as follows

$$g_{\mu\nu} = e^a_\mu(x)\eta_{ab}e^b_\nu(x). \tag{3.7.2}$$

Equation (3.7.2) describes a general relation between the metric and the frame field. One thing that is worth mentioning here is that given the metric tensor $g_{\mu\nu}$, the frame field $e^a_\mu(x)$, obtained by diagonalization, is not the only solution out there. In fact, given any x-dependent matrix $\Lambda^a{}_b(x)$ which leaves η_{ab} invariant, we can get another solution of (3.7.2). More precisely,

$$e'^a_\mu(x) = \Lambda^{-1a}{}_b(x)e^b_\mu(x). \tag{3.7.3}$$

All choices of frame fields related by local Lorentz Transformations (LLT) are considered equivalent. Covariance under LLT is indeed the main principle behind this discussion. Translating what we know about tensors, for example, into noncoordinate basis is a matter of arranging vielbeins in the right place and order. The exception comes when we begin to differentiate things. Usually, in ordinary formalism, a covariant derivative differentiates tensors of type (p, q) to ones of type $(p, q+1)$. But for that to hold appropriately, we have to add affine connection $\Gamma^\rho_{\mu\nu}$. So, on vector fields, covariant derivative goes like

$$\nabla_\mu V^\rho = \partial_\mu V^\rho + \Gamma^\rho{}_{\mu\nu}V^\nu \quad , \quad \nabla_\mu V_\nu = \partial_\mu V_\nu - \Gamma^\rho{}_{\mu\nu}V_\rho \tag{3.7.4}$$

and it is quite straight forward to extend these to tensors. This will continue to be valid for the noncoordinate basis, with the exception that we have to replace the ordinary connection by the spin connection, denoted by $\omega_\mu{}^a{}_b$,

$$\nabla_\mu X^a{}_b = \partial_\mu X^a{}_b + \omega_\mu{}^a{}_c X^c{}_b - \omega_\mu{}^c{}_b X^a{}_c. \tag{3.7.5}$$

This was an essential point to mention and now that we are at it, spin connections are usually called that because of their importance in the description of spinors on manifolds. It is about time at this stage to take advantage of our freedom to suppress indices on differential forms, so we write

$$e^a = e_\mu{}^a dx^\mu. \tag{3.7.6}$$

Given the frame 1-forms e^a, we examine the 2-forms

$$de^a = \frac{1}{2}(\partial_\mu e^a_\nu - \partial_\nu e^a_\mu) dx^\mu \wedge dx^\nu. \tag{3.7.7}$$

The antisymmetric components transform as a (0, 2) tensor under coordinate transformations, but not as local Lorentz transformation, this can be seen through

$$de'^a = d(\Lambda^{-1a}{}_b e^b) = \Lambda^{-1a}{}_b de^b + d\Lambda^{-1a}{}_b \wedge e^b. \tag{3.7.8}$$

Clearly, the second term spoils the vector transformation property. To cancel it, we add the contribution from a 2-form involving the spin connection and consider

$$de^a + \omega^a{}_b \wedge e^b = T^a \tag{3.7.9}$$

If $\omega^a{}_b$ is defined to transform under LLT as

$$\omega'^a_b = \Lambda^{-1a}{}_c d\Lambda^c{}_b + \Lambda^{-1a}{}_c \omega^c{}_d \Lambda^d{}_b, \tag{3.7.10}$$

then T^a does indeed transform as a vector. Equation (3.7.9) is called the first *Cartan* structure equation. In most applications of differential geometry to gravity, the torsion term T vanishes, then

$$de^a + \omega^a{}_b \wedge e^b = 0. \tag{3.7.11}$$

Chapter 4

A Glimpse of Special Geometry

We will start this intermediate chapter from the definition of a complex manifold. Then, we will briefly take things from there to illustrate some properties of *Special Geometry* that we will be using in chapter 5. Special Geometry is the name given for a manifold determined by scalars of vector multiplets in $N = 2$ supergravity.

4.1 Complex Manifold

A complex manifold [14] is a manifold on which one can choose n complex coordinates, z^α, in a smooth fashion. Better stated, say the concerned manifold is covered by open sets U_I. On each of which there exists a one-to-one continuous map, $\psi_I(p) = (z^1, z^2, ..., z^n)$, such that $z^\alpha \in \mathbb{C}$.

What preceded was worth introducing in order to settle the ground of what follows. The basic idea here is that the scalar fields of supersymmetric theories in four spacetime dimensions are a set of complex fields, z^α, which can be taken as coordinates of an essential type of complex manifold known as the Kähler manifold. This type of a manifold emerges naturally in supergravity. What we will be dealing with in the next chapter is a subclass of Kähler manifolds called special Kähler manifolds which are necessary for supersymmetric theories of the $N = 2$ vector multiplets.

4.2 Special Geometry Properties

If one is to obtain complex coordinates, one can start from a real coordinate set ϕ^i where i runs from 1 to $2n$ as was illustrated in [14]. Accordingly, one can write z^a as:

$$\begin{aligned} z^\alpha &= \phi^\alpha + i\phi^{\alpha+n}, \\ \bar{z}^{\bar{\alpha}} &= \phi^\alpha - i\phi^{\alpha+n}, \end{aligned} \tag{4.2.1}$$

where in the first set a runs through to define 'holomorphic' coordinates and through the second to define 'anti-holomorphic' coordinates.

The splitting of an index a into 'holomorphic' and 'antiholomorphic' components is not preserved by general transformation of complex coordinates, but it is preserved under the special class of holomorphic coordinate transformations.

That being said, the metric expressed in complex coordinates can be presented as

$$\begin{aligned} ds^2 &= g_{ab} dz^a dz^b \\ &= 2g_{\alpha\bar{\beta}} dz^\alpha d\bar{z}^{\bar{\beta}} + g_{\alpha\beta} dz^\alpha dz^\beta + g_{\bar{\alpha}\bar{\beta}} d\bar{z}^{\bar{\alpha}} d\bar{z}^{\bar{\beta}}. \end{aligned} \tag{4.2.2}$$

From there, we now move to define two conditions [14] on the metric g_{ab} which are preserved by holomorphic coordinate transformations. The metric is said to hermitian if $g_{\alpha\beta} = 0 = g_{\bar{\alpha}\bar{\beta}}$. Thus (4.2.2) takes the form

$$ds^2 = 2g_{\alpha\bar{\beta}} dz^\alpha d\bar{z}^{\bar{\beta}}. \tag{4.2.3}$$

In addition to that, given the hermitian metric, one can define a fundamental 2-form, [14]

$$\Omega = -2i g_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^{\bar{\beta}}. \tag{4.2.4}$$

However, for the manifold with a hermitian metric to be Kähler, its fundamental form must be *closed*. Thus upon that, we will have a necessary condition

$$\partial_\gamma g_{\alpha\bar{\beta}} - \partial_\alpha g_{\gamma\bar{\beta}} = 0. \tag{4.2.5}$$

This implies that the concerned metric is written as

$$\boxed{g_{\alpha\bar{\beta}} = \partial_\alpha \partial_{\bar{\beta}} K}, \tag{4.2.6}$$

where K is the Kähler potential.

An important idea [14] that characterizes special Kähler geometry as a subclass of Kähler manifolds is that they are Kähler manifolds in which the scalars 'sense' the *symplectic transformation* [14]. We will define what we mean by sympletic transformations next and then explain what we meant by the latter sentence as a whole.

In four dimensions, the duality transformations are transformations between field strengths. Those transformations preserve the field equations and Bianchi identities under the action of the symplectic matrix[1] S [21]

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \text{where} \quad S^T \Omega S = \Omega \quad \text{and} \quad \Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{4.2.7}$$

If you take a look at the action in (5.3.3), you would observe a coupling matrix, $\mathcal{N}_{IJ}$, that enters in the kinetic terms. This $\mathcal{N}_{IJ}$ depends on the scalars we introduced at the very beginning. The point of mentioning this is to say that $\mathcal{N}_{IJ}$ should transform under the action of the symplectic group above so that we get [21]

$$\tilde{\mathcal{N}} = (C + D\mathcal{N})(A + B\mathcal{N})^{-1}. \tag{4.2.8}$$

[1]Given the restriction that $\tilde{\mathcal{N}}_{IJ}$ should be symmetric [21].

This determines the action of symplectic transformations on the scalars and, in turn, gives an idea of how these 'special' manifolds are characterized.

A useful definition of a special Kähler manifold can be given by introducing a $(2n+2)$-dimensional symplectic bundle with covariantly holomorphic sections $\mathcal{V}$,

$$\mathcal{V} = \begin{pmatrix} L^I \\ M_I \end{pmatrix} = e^{K/2} \begin{pmatrix} X^I \\ F_I \end{pmatrix} \quad \text{where,} \quad I = 0, ..., n, \tag{4.2.9}$$

where [5]

$$\mathcal{D}_{\bar{A}}\mathcal{V} = (\partial_{\bar{A}} - \frac{1}{2}\partial_{\bar{A}}K)\mathcal{V} = 0, \tag{4.2.10}$$

and

$$U_A = \mathcal{D}_A\mathcal{V} = (\partial_A + \frac{1}{2}\partial_A K)\mathcal{V} = \begin{pmatrix} f_A^I \\ h_{AI} \end{pmatrix}. \tag{4.2.11}$$

These sections obey the symplectic constraint [3]

$$i(\bar{L}^I M_I - L^I \bar{M}_I) = 1. \tag{4.2.12}$$

The Kähler potential, that we previously introduced, can be obtained from the holomorphic sections by

$$e^{-K} = i(\bar{X}^I F_I - X^I \bar{F}_I). \tag{4.2.13}$$

Now, in general one can write [5],

$$F_I(z) = \mathcal{N}_{IJ}X^J(z), \quad \mathcal{D}_A F_I(z) = \bar{\mathcal{N}}_{IJ}\mathcal{D}_A X^I(z), \tag{4.2.14}$$

in addition to very handy equations

$$F_I\partial_\mu X^I - X^I\partial_\mu F_I = 0, \tag{4.2.15}$$

$$g_{A\bar{B}}\mathcal{D}_A L^M \mathcal{D}_{\bar{B}}\bar{L}^I = -\frac{1}{2}(Im\mathcal{N})^{MI} - \bar{L}^M L^I. \tag{4.2.16}$$

The $U(1)$ Kähler connection $\mathcal{A}$ is defined by

$$\mathcal{A} = -\frac{i}{2}(\partial_A K dz^A - \partial_{\bar{A}} K d\bar{z}^A), \tag{4.2.17}$$

where it was shown in [5] that this could be written as

$$\mathcal{A} = M_I d\bar{L}^I - L^I d\bar{M}_I. \tag{4.2.18}$$

Upon using (4.2.9), (4.2.10) and (4.2.11), we can derive another helpful equation as we will illustrate below:

$$\begin{aligned}
\partial_\mu L^I &= \partial_{\bar{A}} L^I \partial_\mu \bar{z}^A + \partial_A L^I \partial_\mu z^A \\
&= \frac{1}{2}\partial_{\bar{A}} K L^I \partial_\mu \bar{z}^A + \mathcal{D}_A L^I \partial_\mu z^A - \frac{1}{2}\partial_A K L^I \partial_\mu z^A.
\end{aligned} \tag{4.2.19}$$

Making use out of (4.2.17),

$$-i\mathcal{A}_\mu = \frac{1}{2}(-\partial_A K \partial_\mu z^A + \partial_{\bar{A}} K \partial_\mu \bar{z}^A), \tag{4.2.20}$$

we get the desired relation

$$\mathcal{D}_A L^I dz^A = (d + i\mathcal{A})L^I. \tag{4.2.21}$$

Furthermore, we plan to find another essential equation using equations (4.2.9), (4.2.10), (4.2.11) and (4.2.14):

$$\partial_\mu M_I = \partial_{\bar{A}} M_I \partial_\mu \bar{z}^A + \partial_A M_I \partial_\mu z^A$$
$$= \frac{1}{2}\partial_{\bar{A}} K M_I \partial_\mu \bar{z}^A + \mathcal{D}_A M_I \partial_\mu z^A - \frac{1}{2} M_I \partial_A K \partial_\mu z^A \quad = \bar{\mathcal{N}}_{IJ} \mathcal{D}_A L^J \partial_\mu z^A - i\mathcal{A}_\mu M_I. \tag{4.2.22}$$

Using equation (4.2.21), we get

$$dM_I = -i\mathcal{A}M_I + \bar{\mathcal{N}}_{IJ}(dL^J + i\mathcal{A}L^J). \tag{4.2.23}$$

Thus,

$$dM_I = i(Re\mathcal{N}_{IJ} - iIm\mathcal{N}_{IJ})\mathcal{A}L^J + \bar{\mathcal{N}}_{IJ} dL^J - i\mathcal{A}(Re\mathcal{N}_{IJ} + iIm\mathcal{N}_{IJ})L^J. \tag{4.2.24}$$

This guides us to the second desired relation

$$dM_I - \bar{\mathcal{N}}_{IJ} dL^J = 2\mathcal{A} Im\mathcal{N}_{IJ} L^J. \tag{4.2.25}$$

Chapter 5

Ordinary and Fake Supergravity

Supergravity is an extension of Einstein's general theory of relativity so as to carry supersymmetry. Supersymmetry, in simple terms, is a development relating the bosons (of integer spin) to fermions (of half odd integer spin). One of the main aspects of supergravity that caught a decent amount of attention is the investigation of supersymmetric solutions. These solutions are the result of solving a set of first-order Killing spinor differential equations. The concerned Killing spinor equations emerge from supersymmetric transformations that maintain the invariance of the theory.

In this chapter, we will be dealing with the ordinary and fake $N = 2$ supergravity theory in four dimensions coupled to vector multiplets. We will be going through detailed analysis of our theory's Killing spinor equations. Throughout this analysis, we are going to use the elegant spinorial geometry method which was first used in [9].

5.1 The Key Steps

We are going to start off this section by writing the spinors in terms of exterior forms which facilitates the way the Killing spinor equations act on the spinor representatives. This is applicable upon employing the isomorphism between Clifford algebra and exterior algebra [23]. After complexification, spinors can be written as complexified forms on $\mathbb{R}^2$. Now, if Δ were to denote the space of Dirac spinors, then $\Delta = \Lambda^\star(\mathbb{R}^2) \otimes \mathbb{C}$. Therefore, a generic spinor can be written as

$$\epsilon = \lambda 1 + \mu_i e^i + \sigma e^{12}, \tag{5.1.1}$$

where e^1, e^2 are 1-forms on $\mathbb{R}^2$, and $i = 1;2$; $e^{12} = e^1 \wedge e^2$. Here, λ, μ_i and σ are complex functions. That being said, the next step is to find representatives up to gauge transformations which can be used to simplify the Killing spinors of the theory we are working in. Those are referred to as the *canonical* forms of the spinor. After utilizing $Spin(3,1)$ gauge transformations, it was shown in [24] that one finds three canonical orbits:

$$\epsilon = 1 + \mu_2 e^2, \qquad \epsilon = 1 + \mu_1 e^1, \qquad \epsilon = e^2. \tag{5.1.2}$$

The first orbit, which we will be concerned with, represents the Killing spinor for the IWP metric which has a time-like Killing vector. In [13], phantom[1] solutions for the Killing spinor, $\epsilon = 1 + \mu e^2$, were found. And because our work is a generalization of the results of [13] to four-dimensional N = 2 supergravity theory coupled to vector multiplets, our attention will also be concentrated on

$$\epsilon = 1 + \mu e^2. \tag{5.1.3}$$

[1] We will refer to solutions of the fake theory, where the gauge fields come with the non-conventional sign of the kinetic terms, as phantom solutions.

Using the basis elements 1, e^1, e^2 and e^{12}, we can construct and define Clifford Gamma matrices in the following way [13]

$$\begin{aligned}
\Gamma_0 &= -e^2 \wedge + i_{e^2}, \\
\Gamma_1 &= e^1 \wedge + i_{e^1}, \\
\Gamma_2 &= e^2 \wedge + i_{e^2}, \\
\Gamma_3 &= i(e^1 \wedge - i_{e^1}),
\end{aligned} \tag{5.1.4}$$

where

$$\Gamma_5 = i\Gamma_{0123}. \tag{5.1.5}$$

It is assumed that it is known that the i operation represents the adjoint of the $\wedge$ one.

The third step, before moving on, is to write the Dirac matrices in (5.1.4) in terms of the oscillator basis. The usage of this basis simplifies the operations in a significant way. Thus,

$$\begin{aligned}
\Gamma_+ &= \frac{1}{\sqrt{2}}(\Gamma_2 + \Gamma_0) &&= \sqrt{2} i_{e^2}, \\
\Gamma_- &= \frac{1}{\sqrt{2}}(\Gamma_2 - \Gamma_0) &&= \sqrt{2} e^2 \wedge, \\
\Gamma_1 &= \frac{1}{\sqrt{2}}(\Gamma_1 + i\Gamma_3) &&= \sqrt{2} i_{e^1}, \\
\Gamma_{\bar{1}} &= \frac{1}{\sqrt{2}}(\Gamma_1 - i\Gamma_3) &&= \sqrt{2} e^1 \wedge,
\end{aligned} \tag{5.1.6}$$

where the metric components in the null basis are given by $g_{+-} = 1$ and $g_{1\bar{1}} = 1$.

Now that the Killing spinors are written in terms of forms and the Gamma matrices are written in terms of form-operators, this will facilitate the way we solve for the Killing spinor equations.

5.2 Essentials

It will be essential that we now present the action of the new gamma matrices on the desired forms – those that appeared in the previous section. This can be processed by a direct application of the Gammas defined in (5.1.6). We note that the anti-symmetrization convention, i.e, $\Gamma_{ab} = \frac{1}{2}(\Gamma_a \Gamma_b - \Gamma_b \Gamma_a)$ is used here. The results can be summarized as

$$
\begin{array}{llll}
\Gamma_{\bar{1}} 1 = \sqrt{2} e^1, & \Gamma_{\bar{1}} e^1 = 0, & \Gamma_{\bar{1}} e^2 = \sqrt{2} e^{12}, & \Gamma_{\bar{1}} e^{12} = 0, \\
\Gamma_1 1 = 0, & \Gamma_1 e^1 = \sqrt{2}(1), & \Gamma_1 e^2 = 0, & \Gamma_1 e^{12} = \sqrt{2} e^2, \\
\Gamma_+ 1 = 0, & \Gamma_+ e^1 = 0, & \Gamma_+ e^2 = \sqrt{2}(1), & \Gamma_+ e^{12} = -\sqrt{2} e^1, \\
\Gamma_- 1 = \sqrt{2} e^2, & \Gamma_- e^1 = -\sqrt{2} e^{12}, & \Gamma_- e^2 = 0, & \Gamma_- e^{12} = 0, \\
\Gamma_{1\bar{1}} 1 = 1, & \Gamma_{1\bar{1}} e^1 = -e^1, & \Gamma_{1\bar{1}} e^2 = e^2, & \Gamma_{1\bar{1}} e^{12} = -e^{12}, \\
\Gamma_{+\bar{1}} 1 = 0, & \Gamma_{+\bar{1}} e^1 = 0, & \Gamma_{+\bar{1}} e^2 = -2e^1, & \Gamma_{+\bar{1}} e^{12} = 0, \\
\Gamma_{+1} 1 = 0, & \Gamma_{+1} e^1 = 0, & \Gamma_{+1} e^2 = 0, & \Gamma_{+1} e^{12} = 2(1), \\
\Gamma_{-\bar{1}} 1 = -2e^{12}, & \Gamma_{-\bar{1}} e^1 = 0, & \Gamma_{-\bar{1}} e^2 = 0, & \Gamma_{-\bar{1}} e^{12} = 0, \\
\Gamma_{-1} 1 = 0, & \Gamma_{-1} e^1 = 2e^2, & \Gamma_{-1} e^2 = 0, & \Gamma_{-1} e^{12} = 0, \\
\Gamma_{+-} 1 = 1, & \Gamma_{+-} e^1 = e^1, & \Gamma_{+-} e^2 = -e^2, & \Gamma_{+-} e^{12} = -e^{12}.
\end{array}
\tag{5.2.1}
$$

Now, from what preceded, we can describe in a compact way

$$
\begin{aligned}
F^{ab}\Gamma_{ab}(1) &= 2F^{+-}\Gamma_{+-}(1) + 2F^{1\bar{1}}\Gamma_{1\bar{1}}(1) + 2F^{+\bar{1}}\Gamma_{+\bar{1}}(1) + 2F^{+1}\Gamma_{+1}(1) + 2F^{-\bar{1}}\Gamma_{-\bar{1}}(1) \\
&\quad + 2F^{-1}\Gamma_{-1}(1) \\
&= 2F^{+-}(1) + 2F^{1\bar{1}}(1) + 2F^{-\bar{1}}(-2e^{12}) \\
&= 2(F^{1\bar{1}} + F^{+-})1 - 4F^{-\bar{1}}e^{12},
\end{aligned}
\tag{5.2.2}
$$

$$
\begin{aligned}
F^{ab}\Gamma_{ab}(e^1) &= 2F^{+-}\Gamma_{+-}(e^1) + 2F^{1\bar{1}}\Gamma_{1\bar{1}}(e^1) + 2F^{+\bar{1}}\Gamma_{+\bar{1}}(e^1) + 2F^{+1}\Gamma_{+1}(e^1) + 2F^{-\bar{1}}\Gamma_{-\bar{1}}(e^1) \\
&\quad + 2F^{-1}\Gamma_{-1}(e^1) \\
&= 2F^{+-}(e^1) - 2F^{1\bar{1}}(e^1) + 2F^{-1}(2e^2) \\
&= 4F^{-1}e^2 - 2(F^{1\bar{1}} - F^{+-})e^1,
\end{aligned} \tag{5.2.3}
$$

$$
\begin{aligned}
F^{ab}\Gamma_{ab}(e^2) &= 2F^{+-}\Gamma_{+-}(e^2) + 2F^{1\bar{1}}\Gamma_{1\bar{1}}(e^2) + 2F^{+\bar{1}}\Gamma_{+\bar{1}}(e^2) + 2F^{+1}\Gamma_{+1}(e^2) + 2F^{-\bar{1}}\Gamma_{-\bar{1}}(e^2) \\
&\quad + 2F^{-1}\Gamma_{-1}(e^2) \\
&= -2F^{+-}(e^2) + 2F^{1\bar{1}}(e^2) - 4F^{+\bar{1}}(e^1) \\
&= -4F^{+\bar{1}}e^1 + 2(F^{1\bar{1}} - F^{+-})e^2,
\end{aligned} \tag{5.2.4}
$$

$$
\begin{aligned}
F^{ab}\Gamma_{ab}(e^{12}) &= 2F^{+-}\Gamma_{+-}(e^{12}) + 2F^{1\bar{1}}\Gamma_{1\bar{1}}(e^{12}) + 2F^{+\bar{1}}\Gamma_{+\bar{1}}(e^{12}) + 2F^{+1}\Gamma_{+1}(e^{12}) + 2F^{-\bar{1}}\Gamma_{-\bar{1}}(e^{12}) \\
&\quad + 2F^{-1}\Gamma_{-1}(e^{12}) \\
&= -2F^{+-}(e^{12}) - 2F^{1\bar{1}}(e^{12}) + 4F^{+1}(1) \\
&= 4F^{+1}1 - 2(F^{1\bar{1}} + F^{+-})e^{12}.
\end{aligned} \tag{5.2.5}
$$

In short, one obtains for a 2-form F, the following

$$
\begin{aligned}
F^{ab}\Gamma_{ab}1 &= 2(F^{1\bar{1}} + F^{+-})1 - 4F^{-\bar{1}}e^{12}, \\
F^{ab}\Gamma_{ab}e^1 &= 4F^{-1}e^2 - 2(F^{1\bar{1}} - F^{+-})e^1, \\
F^{ab}\Gamma_{ab}e^2 &= -4F^{+\bar{1}}e^1 + 2(F^{1\bar{1}} - F^{+-})e^2, \\
F^{ab}\Gamma_{ab}e^{12} &= 4F^{+1}1 - 2(F^{1\bar{1}} + F^{+-})e^{12}.
\end{aligned} \tag{5.2.6}
$$

5.3 The Killing Spinor Equations

As we said earlier this chapter, our theory incorporates supersymmetry. Since our background preserves supersymmetry, then it is invariant under supersymmetries of the supergravity theory [14]

$$\begin{aligned} \delta \text{ boson} &= \epsilon \text{ fermion}, \\ \delta \text{ fermion} &= \epsilon \text{ boson}. \end{aligned} \tag{5.3.1}$$

Our current aim is to find supersymmetric solutions. Because this is the case, we should seek an ϵ such that the supersymmetric variations vanish. Our solutions are bosonic, i.e., the fermions automatically vanish. That is, we need not worry about the first variation in (5.3.1). Thus, the relevant supersymmetric variation now is

$$\delta \text{ fermion} = \text{ boson } \epsilon = 0. \tag{5.3.2}$$

This, above, represents the Killing spinor equation. Once ϵ satisfies this equation, it is then called the Killing spinor.

We are now going to solve the Killing spinor equations for the standard and fake supergravity theories in a linked manner. A necessity for the desired linkage is the introduction of a parameter, κ. The values that κ can take are: i and 1. For $\kappa = i$, we will be referring to the standard $N = 2, D = 4$ supergravity theory coupled to vector multiplet. For $\kappa = 1$, we will be referring to the fake theory where the gauge field terms in the action come with the opposite sign. We will reiterate on this throughout the thesis when necessary. As for now, we shall consider the action [25]

$$e^{-1}\mathcal{L} = \frac{1}{2}R - g_{A\bar{B}}\partial_\mu z^A \partial^\mu \bar{z}^B - \frac{\kappa^2}{4}(Im\mathcal{N}_{IJ}F^I \cdot F^J + Re\mathcal{N}_{IJ}F^I \cdot \tilde{F}^J), \tag{5.3.3}$$

where $F^I \cdot F^J \equiv F^I_{\mu\nu} F^{J\mu\nu}$ and $\tilde{F}^J$ is the dual of F^J, $I = 0, ..., n$. Here, $g_{A\bar{B}} = \partial_A \partial_{\bar{B}} K$ is the Kähler metric, and K is the Kähler potential. The n complex scalar fields, z^A, of the $N = 2$ vector multiplets are coordinates of the special Kähler manifold. $\mathcal{N}_{IJ}$, the coupling matrix, is characterized by being complex and symmetric.

The Killing spinor equations are given by [25]

$$(\nabla_\mu + \frac{i}{2}\mathcal{A}_\mu \Gamma_5 + \frac{\kappa}{4} Im\mathcal{N}_{IJ} \Gamma \cdot F^I (ImL^J - i\Gamma_5 ReL^J)\Gamma_\mu)\epsilon = 0, \tag{5.3.4}$$

and

$$\frac{\kappa}{2}(Im\mathcal{N})_{IJ}\Gamma \cdot F^J [Im(g^{A\bar{B}}\mathcal{D}_{\bar{B}}\bar{L}^I) - i\Gamma_5 Re(g^{A\bar{B}}\mathcal{D}_{\bar{B}}\bar{L}^I)]\epsilon + \Gamma^\mu \partial_\mu (Rez^A - i\Gamma_5 Imz^A)\epsilon = 0, \tag{5.3.5}$$

where $\nabla_\mu = \partial_\mu + \frac{1}{4}\omega_\mu{}^{ab}\Gamma_{ab}$. For $\kappa = i$, those stand for the vanishing of the supersymmetry variations, in a bosonic background, of the gravitini and gaugini in the standard $N = 2$, $D = 4$ supergravity theory coupled to vector multiplet. For $\kappa = 1$, those stand for the vanishing of the fake supersymmetry transformations for a theory where all the gauge field terms in the action come with the opposite sign.

5.4 Finding the Linear Systems

Our focus in this thesis will be on the canonical form, $\epsilon = 1 + \mu e^2$. As such we plug it in the above Killing spinor equations. Solving for the Killing spinor equations gives us geometric constraints as we will see later on, thus paving the path toward the complete solution.

Concerning equation (5.3.4), we will obtain four sets of equations after substituting μ by $+, -, 1$, and $\bar{1}$ respectively. Each set will have four equations which results in a total of sixteen equations. Making use out of (5.2) and (5.2.6), we illustrate the results below.

- For the + component, we get

$$\partial_+\mu(e^2)+\frac{1}{2}\omega_{+,-+}(1)+\frac{1}{2}\omega_{+,\bar{1}1}(1)-\omega_{+,+1}(e^{12})-\frac{\mu}{2}\omega_{+,-+}(e^2)+\frac{\mu}{2}\omega_{+,\bar{1}1}(e^2)-\mu\omega_{+,-1}(e^1)$$
$$+\frac{i}{2}\mathcal{A}_+(1)-\frac{i\mu}{2}\mathcal{A}_+(e^2)-\frac{i\mu\kappa\sqrt{2}}{4}L^J Im\mathcal{N}_{IJ}(2(F^{I1\bar{1}}+F^{I+-})(1)-4F^{I-\bar{1}}(e^{12}))=0. \tag{5.4.1}$$

Collecting terms results in the first set of four equations:

$$\left.\begin{aligned}\omega_{+,-1}&=0,\\ \partial_+\log\mu-\frac{1}{2}(\omega_{+,1\bar{1}}+\omega_{+,-+})-\frac{i}{2}\mathcal{A}_+&=0,\\ \frac{1}{2}(\omega_{+,-+}-\omega_{+,1\bar{1}}+i\mathcal{A}_+)-\frac{i\mu\kappa}{\sqrt{2}}Im\mathcal{N}_{IJ}(F^I_{-+}-F^I_{1\bar{1}})L^J&=0,\\ \omega_{+,+1}-i\mu\kappa Im\mathcal{N}_{IJ}F^I_{+1}L^J\sqrt{2}&=0.\end{aligned}\right\} \tag{5.4.2}$$

- For the − component, we get

$$\partial_-\mu(e^2)+\frac{1}{2}\omega_{-,-+}(1)-\frac{1}{2}\omega_{-,1\bar{1}}(1)-\omega_{-,+1}(e^{12})-\frac{\mu}{2}\omega_{-,-+}(e^2)-\frac{\mu}{2}\omega_{-,1\bar{1}}(e^2)-\mu\omega_{-,-1}(e^1)$$
$$+\frac{i}{2}\mathcal{A}_-(1)-\frac{i\mu}{2}\mathcal{A}_-(e^2)+\frac{i\kappa\sqrt{2}}{4}Im\mathcal{N}_{IJ}\bar{L}^J(-4F^{I+\bar{1}}(e^1)+2(F^{I1\bar{1}}-F^{I+-})(e^2))=0. \tag{5.4.3}$$

Collecting terms results in the second set of four equations:

$$\left.\begin{aligned}\omega_{-,+1}&=0,\\ \mu\omega_{-,-1}+i\kappa\sqrt{2}Im\mathcal{N}_{IJ}F^I_{-1}\bar{L}^J&=0,\\ \omega_{-,-+}-\omega_{-,1\bar{1}}+i\mathcal{A}_-&=0,\\ \partial_-\log\mu-\frac{1}{2}(\omega_{-,1\bar{1}}+\omega_{-,-+})-\frac{i}{2}\mathcal{A}_--\frac{i\kappa}{\mu\sqrt{2}}Im\mathcal{N}_{IJ}(F^I_{1\bar{1}}+F^I_{-+})\bar{L}^J&=0.\end{aligned}\right\} \tag{5.4.4}$$

- For the 1 component, we get

$$\partial_1\mu(e^2) + \frac{1}{2}\omega_{1,-+}(1) - \frac{1}{2}\omega_{1,1\bar{1}}(1) - \omega_{1,+1}(e^{12}) - \frac{\mu}{2}\omega_{1,-+}(e^2) - \frac{\mu}{2}\omega_{1,1\bar{1}}(e^2) - \mu\omega_{1,-1}(e^1)$$
$$+ \frac{i}{2}\mathcal{A}_1(1) - \frac{i\mu}{2}\mathcal{A}_1(e^2) = 0. \tag{5.4.5}$$

Collecting terms results in the third set of four equations:

$$\left.\begin{aligned} \omega_{1,-1} &= 0, \\ \omega_{1,+1} &= 0, \\ \partial_1 \log\mu - \frac{1}{2}(\omega_{1,1\bar{1}} + \omega_{1,-+}) - \frac{i}{2}\mathcal{A}_1 &= 0, \\ \omega_{1,-+} - \omega_{1,1\bar{1}} + i\mathcal{A}_1 &= 0. \end{aligned}\right\} \tag{5.4.6}$$

- For the $\bar{1}$ component, we get

$$\partial_{\bar{1}}\mu(e^2) + \frac{1}{2}\omega_{\bar{1},-+}(1) - \frac{1}{2}\omega_{\bar{1},1\bar{1}}(1) - \omega_{\bar{1},+1}(e^{12}) - \frac{\mu}{2}\omega_{\bar{1},-+}(e^2) - \frac{\mu}{2}\omega_{\bar{1},1\bar{1}}(e^2) - \mu\omega_{\bar{1},-1}(e^1)$$
$$+ \frac{i}{2}\mathcal{A}_{\bar{1}}(1) - \frac{i\mu}{2}\mathcal{A}_{\bar{1}}(e^2) + \frac{i\kappa\sqrt{2}}{4}\bar{L}^J Im\mathcal{N}_{IJ}(4F^{I-1}(e^2) - 2(F^{I1\bar{1}} - F^{I+-})(e^1))-$$
$$\frac{i\kappa\mu\sqrt{2}}{4}Im\mathcal{N}_{IJ}L^J(4F^{I+1} - 2(F^{I1\bar{1}} + F^{I+-})(e^{12})) = 0. \tag{5.4.7}$$

Collecting terms results in the fourth set of the last four equations:

$$\left.\begin{aligned} \mu\omega_{\bar{1},-1} - \frac{i\kappa}{\sqrt{2}}Im\mathcal{N}_{IJ}(F^I_{1\bar{1}} + F^I_{-+})\bar{L}^J &= 0, \\ \partial_{\bar{1}}\log\mu - \frac{1}{2}(\omega_{\bar{1},1\bar{1}} + \omega_{\bar{1},-+}) - \frac{i}{2}\mathcal{A}_{\bar{1}} + \frac{i\kappa}{\mu}Im\mathcal{N}_{IJ}F^I_{+\bar{1}}\bar{L}^J\sqrt{2} &= 0, \\ \frac{1}{2}(\omega_{\bar{1},-+} - \omega_{\bar{1},1\bar{1}} + i\mathcal{A}_{\bar{1}}) - i\kappa\mu Im\mathcal{N}_{IJ}F^I_{-\bar{1}}L^J\sqrt{2} &= 0, \\ \omega_{\bar{1},+1} + \frac{i\kappa\mu}{\sqrt{2}}Im\mathcal{N}_{IJ}(F^I_{1\bar{1}} - F^I_{-+})L^J &= 0. \end{aligned}\right\} \tag{5.4.8}$$

Thus, we combine now the linear systems, (5.4.2), (5.4.4), (5.4.6) and

(5.4.8), obtained from (5.3.4) into one system for ease of reference.

$$\omega_{+,-1} = 0, \quad (5.4.9)$$

$$\omega_{1,-1} = 0, \quad (5.4.10)$$

$$\omega_{-,+1} = 0, \quad (5.4.11)$$

$$\omega_{1,+1} = 0, \quad (5.4.12)$$

$$\mu\omega_{-,-1} + i\kappa\sqrt{2}Im\mathcal{N}_{IJ}F^I_{-1}\bar{L}^J = 0, \quad (5.4.13)$$

$$\mu\omega_{\bar{1},-1} - \frac{i\kappa}{\sqrt{2}}Im\mathcal{N}_{IJ}(F^I_{1\bar{1}} + F^I_{-+})\bar{L}^J = 0, \quad (5.4.14)$$

$$\partial_- \log\mu - \frac{1}{2}(\omega_{-,1\bar{1}} + \omega_{-,-+}) - \frac{i}{2}\mathcal{A}_- - \frac{i\kappa}{\mu\sqrt{2}}Im\mathcal{N}_{IJ}(F^I_{1\bar{1}} + F^I_{-+})\bar{L}^J = 0, \quad (5.4.15)$$

$$\partial_1 \log\mu - \frac{1}{2}(\omega_{1,1\bar{1}} + \omega_{1,-+}) - \frac{i}{2}\mathcal{A}_1 = 0, \quad (5.4.16)$$

$$\partial_+ \log\mu - \frac{1}{2}(\omega_{+,1\bar{1}} + \omega_{+,-+}) - \frac{i}{2}\mathcal{A}_+ = 0, \quad (5.4.17)$$

$$\omega_{1,-+} - \omega_{1,1\bar{1}} + i\mathcal{A}_1 = 0, \quad (5.4.18)$$

$$\omega_{-,-+} - \omega_{-,1\bar{1}} + i\mathcal{A}_- = 0, \quad (5.4.19)$$

$$\partial_{\bar{1}} \log\mu - \frac{1}{2}(\omega_{\bar{1},1\bar{1}} + \omega_{\bar{1},-+}) - \frac{i}{2}\mathcal{A}_{\bar{1}} + \frac{i\kappa}{\mu}Im\mathcal{N}_{IJ}F^I_{+\bar{1}}\bar{L}^J\sqrt{2} = 0, \quad (5.4.20)$$

$$\frac{1}{2}(\omega_{\bar{1},-+} - \omega_{\bar{1},1\bar{1}} + i\mathcal{A}_{\bar{1}}) - i\kappa\mu Im\mathcal{N}_{IJ}F^I_{-\bar{1}}L^J\sqrt{2} = 0, \quad (5.4.21)$$

$$\frac{1}{2}(\omega_{+,-+} - \omega_{+,1\bar{1}} + i\mathcal{A}_+) - \frac{i\mu\kappa}{\sqrt{2}}Im\mathcal{N}_{IJ}(F^I_{-+} - F^I_{1\bar{1}})L^J = 0, \quad (5.4.22)$$

$$\omega_{+,+1} - i\mu\kappa Im\mathcal{N}_{IJ}F^I_{+1}L^J\sqrt{2} = 0, \quad (5.4.23)$$

$$\omega_{\bar{1},+1} + \frac{i\kappa\mu}{\sqrt{2}}Im\mathcal{N}_{IJ}(F^I_{1\bar{1}} - F^I_{-+})L^J = 0. \quad (5.4.24)$$

Now, as we solve for the second equation (5.3.5), we obtain

$$\frac{\kappa}{2} Im\mathcal{N}_{IJ}[Im(g^{A\bar{B}}\mathcal{D}_{\bar{B}}\bar{L}^I) - iRe(g^{A\bar{B}}\mathcal{D}_{\bar{B}}\bar{L}^I)][2(F^{J1\bar{1}} + F^{J+-})(1) - 4F^{J-\bar{1}}(e^{12})]$$
$$+ \frac{\kappa\mu}{2} Im\mathcal{N}_{IJ}[Im(g^{A\bar{B}}\mathcal{D}_{\bar{B}}\bar{L}^I) + iRe(g^{A\bar{B}}\mathcal{D}_{\bar{B}}\bar{L}^I)][-4F^{J+\bar{1}}(e^1) + 2(F^{J1\bar{1}} - F^{J+-})(e^2)]+$$
$$\sqrt{2}\partial_+(Rez^A - iImz^A)(e^2) + \mu\sqrt{2}\partial_- Rez^A + iImz^A)(1) + \sqrt{2}\partial_{\bar{1}}(Rez^A - iImz^A)(e^1)$$
$$+ \mu\sqrt{2}\partial_1(Rez^A + iImz^A)(e^{12}) = 0. \tag{5.4.25}$$

Collecting terms boils this down to our second linear system, now resulting from (5.3.5). The set stands as

$$-i\kappa g^{A\bar{B}}\mathcal{D}_{\bar{B}}\bar{L}^I(Im\mathcal{N})_{IJ}(F^J_{-+} - F^J_{1\bar{1}}) + \partial_- z^A\mu\sqrt{2} = 0, \tag{5.4.26}$$
$$-i\bar{\kappa}\bar{\mu} g^{A\bar{B}}\mathcal{D}_{\bar{B}}\bar{L}^I(Im\mathcal{N})_{IJ}(F^J_{1\bar{1}} - F^J_{-+}) + \partial_+ z^A\sqrt{2} = 0, \tag{5.4.27}$$

$$2i\bar{\kappa}\bar{\mu} g^{A\bar{B}}\mathcal{D}_{\bar{B}}\bar{L}^I(Im\mathcal{N})_{IJ}F^J_{-\bar{1}} + \partial_{\bar{1}} z^A\sqrt{2} = 0, \tag{5.4.28}$$
$$2i\kappa g^{A\bar{B}}\mathcal{D}_{\bar{B}}\bar{L}^I(Im\mathcal{N})_{IJ}F^J_{+1} + \partial_1 z^A\mu\sqrt{2} = 0. \tag{5.4.29}$$

5.5 Studying the Linear Systems

5.5.1 Extracted Conditions

The analysis of the first linear system obtained from (5.3.4) produces geometric conditions which their derivation will be presented shortly. In addition to that and upon a similar analysis, conditions will also be imposed on the gauge field strengths. Furthermore, as this subsection comes to an end, we give two conditions that will come in handy later on.

- We first multiply equation (5.4.20) by μ and then we take its conjugate.

Then, upon adding what we get in the previous step to equation (5.4.23) multiplied by κ and $-\bar{\kappa}/\mu$ respectively, we get:

$$\kappa\partial_1\bar{\mu} - \frac{\bar{\mu}\kappa}{2}(\omega_{1,-+} - \omega_{1,1\bar{1}}) + \frac{i\kappa\bar{\mu}\mathcal{A}_1}{2} - \frac{\bar{\kappa}}{\mu}\omega_{+,+1} = 0. \tag{5.5.1}$$

Addition of equation (5.5.1) and equation (5.4.18) after multiplying the latter by $(\bar{\mu}\kappa)/2$, gives:

$$\boxed{\omega_{+,+1} = \kappa^2\mu(\partial_1\bar{\mu} + i\bar{\mu}\mathcal{A}_1)}. \tag{5.5.2}$$

• Upon adding the conjugate of equation (5.4.21) to equation (5.4.13) multiplied by κ and $-\bar{\kappa}\bar{\mu}$ respectively, we get:

$$-\bar{\kappa}|\mu|^2\omega_{-,-1} + \frac{\kappa}{2}(\omega_{1,-+} + \omega_{1,1\bar{1}}) - \frac{i\kappa}{2}\mathcal{A}_1 = 0. \tag{5.5.3}$$

Addition of equation (5.5.3) and equation (5.4.16) after multiplying the latter by κ, gives:

$$\boxed{\omega_{-,-1} = \frac{\kappa^2}{|\mu|^2}(-i\mathcal{A}_1 + \partial_1\log\mu)}. \tag{5.5.4}$$

•Upon adding equation (5.4.14) to equation (5.4.15) after multiplying the latter by μ and former by -1, we get:

$$-\mu\omega_{\bar{1},-1} + \partial_-\mu - \frac{\mu}{2}(\omega_{-,1\bar{1}} + \omega_{-,-+}) - \frac{i\mu}{2}\mathcal{A}_- = 0. \tag{5.5.5}$$

Addition of equation (5.5.5) after dividing it by μ and the conjugate of equation (5.4.19) multiplied by $1/2$, gives:

$$\boxed{\omega_{\bar{1},-1} = \partial_-\log\mu - i\mathcal{A}_-}. \tag{5.5.6}$$

• Upon adding equation (5.4.22) to equation (5.4.24) after multiplying the latter by -1, we get:

$$-\omega_{\bar{1},+1} + \frac{1}{2}(\omega_{+,-+} - \omega_{+,1\bar{1}}) + \frac{i}{2}\mathcal{A}_+ = 0. \tag{5.5.7}$$

Addition of equation (5.5.7) and the conjugate of equation (5.4.17), gives:

$$\boxed{\omega_{\bar{1},+1} = \partial_+ \log \bar{\mu} + i\mathcal{A}_+}. \tag{5.5.8}$$

• Upon the addition of equation (5.4.16) to equation (5.4.18) after multiplying the latter by $1/2$, we get:

$$\boxed{\omega_{1,1\bar{1}} = \partial_1 \log \mu}. \tag{5.5.9}$$

• Substituting equation (5.5.9) in equation (5.4.18), gives:

$$\boxed{\omega_{1,-+} = -i\mathcal{A}_1 + \partial_1 \log \mu}. \tag{5.5.10}$$

• Upon the addition equation (5.4.22) to equation (5.4.24) after multiplying the latter by -1 and substituting $\omega_{\bar{1},+1}$ by its value in (5.5.8), we get:

$$\frac{1}{2}(\omega_{+,-+} - \omega_{+,1\bar{1}}) - \frac{i}{2}\mathcal{A}_+ - \partial_+ \log \bar{\mu} = 0. \tag{5.5.11}$$

Addition of equation (5.5.11) to equation (5.4.17), gives:

$$\boxed{\omega_{+,1\bar{1}} = -i\mathcal{A}_+ + \partial_+ \log \frac{\mu}{\bar{\mu}}}. \tag{5.5.12}$$

• Using equation (5.4.17) and after substituting $\omega_{+,1\bar{1}}$ by its value in equation (5.5.12), we get:

$$\boxed{\omega_{+,-+} = \partial_+ \log \mu\bar{\mu}}. \tag{5.5.13}$$

• Upon adding equation (5.4.14) to equation (5.4.15) multiplied by -1 and μ respectively, we get:

$$\partial_-\mu - \frac{\mu}{2}(\omega_{-,1\bar{1}} + \omega_{-,-+}) - \frac{i\mu}{2}\mathcal{A}_- - \mu\omega_{\bar{1},-1} = 0. \tag{5.5.14}$$

Addition of equation (5.5.14) after substituting its $\omega_{\bar{1},-1}$ by its value in equation (5.5.6) to equation (5.4.19) where the latter equation must first be multiplied by $\mu/2$, gives:

$$\boxed{\omega_{-,1\bar{1}} = i\mathcal{A}_-}. \tag{5.5.15}$$

Therefore, the existing geometric conditions can be summarized as:

$$\begin{aligned}
\omega_{-,1\bar{1}} &= i\mathcal{A}_-, & \omega_{+,-+} &= \partial_+ \log \mu\bar{\mu}, \\
\omega_{+,1\bar{1}} &= -i\mathcal{A}_+ + \partial_+ \log \frac{\mu}{\bar{\mu}}, & \omega_{1,-+} &= -i\mathcal{A}_1 + \partial_1 \log \mu, \\
\omega_{\bar{1},+1} &= \partial_+ \log \bar{\mu} + i\mathcal{A}_+, & \omega_{1,1\bar{1}} &= \partial_1 \log \mu, \\
\omega_{-,-1} &= \frac{\kappa^2}{|\mu|^2}(-i\mathcal{A}_1 + \partial_1 \log \mu), & \omega_{+,+1} &= \kappa^2\mu(\partial_1\bar{\mu} + i\bar{\mu}\mathcal{A}_1), \\
\omega_{\bar{1},-1} &= \partial_- \log \mu - i\mathcal{A}_-. & &
\end{aligned} \tag{5.5.16}$$

We now move to the next necessary step and which is finding the conditions involving the gauge field strengths.

• Using equation (5.4.20) multiplied by μ, we replace $\omega_{\bar{1},1\bar{1}}$ and $\omega_{\bar{1},-+}$ by their values in equations (5.5.9) and (5.5.10) respectively. It is noteworthy that the values could be subjected to certain manipulations from conjugation and flipping indices where necessary. Accordingly, we get:

$$\partial_{\bar{1}}\mu - i\mu\mathcal{A}_{\bar{1}} + i\kappa Im\mathcal{N}_{IJ}F^I_{+\bar{1}}\bar{L}^J\sqrt{2} = 0. \tag{5.5.17}$$

Taking the conjugate of equation (5.5.17), gives:

$$\boxed{Im\mathcal{N}_{IJ}F^{I}_{+1}L^{J} = -\frac{i\kappa}{\sqrt{2}}(\partial_{1} + i\mathcal{A}_{1})\bar{\mu}}. \tag{5.5.18}$$

• In equation (5.4.13), after replacing $\omega_{-,-1}$ by its value in equation (5.5.4), we take the conjugate of the result to get:

$$\boxed{Im\mathcal{N}_{IJ}F^{I}_{-\bar{1}}L^{J} = -\frac{i\bar{\kappa}}{\sqrt{2}|\mu|^{2}}(\partial_{\bar{1}} + i\mathcal{A}_{\bar{1}})\bar{\mu}}. \tag{5.5.19}$$

• In equation (5.4.14), we substitute $\omega_{\bar{1},-1}$ by its value in equation (5.5.6). After that, we take the conjugate of the result to get:

$$\boxed{Im\mathcal{N}_{IJ}(F^{I}_{-+} - F^{I}_{1\bar{1}})L^{J} = i\kappa\sqrt{2}(\partial_{-} + i\mathcal{A}_{-})\bar{\mu}}. \tag{5.5.20}$$

It is worth mentioning that equation $Im\mathcal{N}_{IJ}(F^{I}_{-+} - F^{I}_{1\bar{1}})$ could be obtained also from equation (5.4.24) to get:

$$Im\mathcal{N}_{IJ}(F^{I}_{-+} - F^{I}_{1\bar{1}})L^{J} = \frac{-i\sqrt{2}\bar{\kappa}}{|\mu|^{2}}(\partial_{+} + i\mathcal{A}_{+})\bar{\mu}. \tag{5.5.21}$$

What preceded can be summed up as:

$$\begin{aligned} Im\mathcal{N}_{IJ}F^{I}_{-\bar{1}}L^{J} &= -\frac{i\bar{\kappa}}{\sqrt{2}|\mu|^{2}}(\partial_{\bar{1}} + i\mathcal{A}_{\bar{1}})\bar{\mu}, \\ Im\mathcal{N}_{IJ}F^{I}_{+1}L^{J} &= -\frac{i\kappa}{\sqrt{2}}(\partial_{1} + i\mathcal{A}_{1})\bar{\mu}, \\ Im\mathcal{N}_{IJ}(F^{I}_{-+} - F^{I}_{1\bar{1}})L^{J} &= i\kappa\sqrt{2}(\partial_{-} + i\mathcal{A}_{-})\bar{\mu}. \end{aligned} \tag{5.5.22}$$

There are yet other conditions that will certainly come in handy. One of which relates $\partial_{-}, \partial_{+}, \mathcal{A}_{-}$ and $\mathcal{A}_{+}$. It comes from setting equations (5.5.20) and (5.5.21) equal:

$$\boxed{\mu\partial_{-}\bar{\mu} + \kappa^{2}\partial_{+}\log\bar{\mu} = -i(\mathcal{A}_{-}|\mu|^{2} + \kappa^{2}\mathcal{A}_{+})}. \tag{5.5.23}$$

Another one comes from adding equation (5.5.23) to its conjugate:

$$\boxed{\partial_-(\mu\bar{\mu}) + \kappa^2\partial_+ \log\mu\bar{\mu} = 0}. \tag{5.5.24}$$

5.5.2 Torsion Equation

It is a good idea at this stage to collect the ω-terms derived in the previous section in order to make it easier for us to plug things in the first *Cartan* structure equation with a vanishing torsion term. Using equation (5.5.1), we can write the following relations for the spin connection:

$$\begin{aligned}
\omega_{1\bar{1}} &= \left(\partial_+ \log\frac{\mu}{\bar{\mu}} - i\mathcal{A}_+\right)\mathbf{e}^+ + i\mathcal{A}_-\mathbf{e}^- + \partial_1 \log\mu\mathbf{e}^1 - \partial_{\bar{1}}\log\bar{\mu}\mathbf{e}^{\bar{1}},\\
\omega_{-1} &= \frac{\kappa^2}{\mu\bar{\mu}}\left(\partial_1\log\mu - i\mathcal{A}_1\right)\mathbf{e}^- + \left(\partial_-\log\mu - i\mathcal{A}_-\right)\mathbf{e}^{\bar{1}},\\
\omega_{-+} &= \left(\partial_1\log\mu - i\mathcal{A}_1\right)\mathbf{e}^1 + \left(\partial_{\bar{1}}\log\bar{\mu} + i\mathcal{A}_{\bar{1}}\right)\mathbf{e}^{\bar{1}} + \partial_+\log\bar{\mu}\mu\mathbf{e}^+,\\
\omega_{+1} &= \left(\partial_+\log\bar{\mu} + i\mathcal{A}_+\right)\mathbf{e}^{\bar{1}} + \kappa^2\mu\left(\partial_1\bar{\mu} + i\mathcal{A}_1\bar{\mu}\right)\mathbf{e}^+.
\end{aligned} \tag{5.5.25}$$

Now, using equation (3.7.11),

$$d\mathbf{e}^a + \omega^a_{\ b}\wedge\mathbf{e}^b = 0, \tag{5.5.26}$$

that we saw in *The Frame Field* section, we now derive expressions for $d\mathbf{e}^1$, $d\mathbf{e}^+$ and $d\mathbf{e}^-$.

- Expression for $d\mathbf{e}^1$:

Expanding equation (5.5.26), we get:

$$d\mathbf{e}^1 + \omega_{\bar{1}+}\wedge\mathbf{e}^+ + \omega_{\bar{1}-}\wedge\mathbf{e}^- + \omega_{\bar{1}1}\wedge\mathbf{e}^1 = 0. \tag{5.5.27}$$

We now substitute the desired terms in (5.5.2), where some of which could be subjected to manipulations from conjugation and flipping lower indices, in equation

(5.5.27):

$$d\mathbf{e}^1 + (i\mathcal{A}_+ - \partial_+ \log\mu)\mathbf{e}^1\wedge\mathbf{e}^+ - (\partial_- \log\bar{\mu} + i\mathcal{A}_-)\mathbf{e}^1\wedge\mathbf{e}^- + (i\mathcal{A}_+ - \partial_+ \log\frac{\mu}{\bar{\mu}})\mathbf{e}^+\wedge\mathbf{e}^1$$
$$- (i\mathcal{A}_-)\mathbf{e}^-\wedge\mathbf{e}^1 + \partial_{\bar{1}}\log\bar{\mu}\mathbf{e}^{\bar{1}}\wedge\mathbf{e}^1 = 0 \qquad (5.5.28)$$

$$d\mathbf{e}^1 - \partial_-\log\bar{\mu}\mathbf{e}^1\wedge\mathbf{e}^- + \partial_{\bar{1}}\log\bar{\mu}\mathbf{e}^{\bar{1}}\wedge\mathbf{e}^1 + \partial_+\log\bar{\mu}\mathbf{e}^+\wedge\mathbf{e}^1 = 0 \qquad (5.5.29)$$
$$d\mathbf{e}^1 + (d\log\bar{\mu})\wedge\mathbf{e}^1 = 0. \qquad (5.5.30)$$

- Expression for $d\mathbf{e}^+$:

We perform the same technique used for $d\mathbf{e}^1$. We start by expanding equation (5.5.26):

$$d\mathbf{e}^+ + \omega_{-+}\wedge\mathbf{e}^+ + \omega_{-1}\wedge\mathbf{e}^1 + \omega_{-\bar{1}}\wedge\mathbf{e}^{\bar{1}} = 0. \qquad (5.5.31)$$

Using concerned terms in (5.5.2), we obtain:

$$d\mathbf{e}^+ + (\partial_1\log\mu - i\mathcal{A}_1)\mathbf{e}^1\wedge\mathbf{e}^+ + (\partial_{\bar{1}}\log\bar{\mu} + i\mathcal{A}_{\bar{1}})\mathbf{e}^{\bar{1}}\wedge\mathbf{e}^+ + \frac{\kappa^2}{\mu\bar{\mu}}(\partial_1\log\mu - i\mathcal{A}_1)\mathbf{e}^-\wedge\mathbf{e}^1$$
$$+ (\partial_-\log\mu - i\mathcal{A}_-)\mathbf{e}^{\bar{1}}\wedge\mathbf{e}^1 + \frac{\kappa^2}{\mu\bar{\mu}}(\partial_{\bar{1}}\log\bar{\mu} + i\mathcal{A}_{\bar{1}})\mathbf{e}^-\wedge\mathbf{e}^{\bar{1}} + (\partial_-\log\bar{\mu} + i\mathcal{A}_-)\mathbf{e}^1\wedge\mathbf{e}^{\bar{1}}$$
$$= 0. \qquad (5.5.32)$$

After a straight forward simplification, we get:

$$d\mathbf{e}^+ = -(\partial_-\log\frac{\mu}{\bar{\mu}} - 2i\mathcal{A}_-)\mathbf{e}^{\bar{1}}\wedge\mathbf{e}^1 - (\frac{\kappa^2}{\mu\bar{\mu}}\mathbf{e}^- - \mathbf{e}^+)\wedge[(\partial_{\bar{1}}\log\bar{\mu} + i\mathcal{A}_{\bar{1}})\mathbf{e}^{\bar{1}}$$
$$+ (\partial_1\log\mu - i\mathcal{A}_1)\mathbf{e}^1]. \qquad (5.5.33)$$

- Expression for $d\mathbf{e}^-$:

In a similar fashion,

$$d\mathbf{e}^{-}+\omega_{+-}\wedge\mathbf{e}^{-}+\omega_{+1}\wedge\mathbf{e}^{1}+\omega_{+\bar{1}}\wedge\mathbf{e}^{\bar{1}}=0. \tag{5.5.34}$$

That is,

$$\begin{aligned}&d\mathbf{e}^{-}+(-\partial_1\log\mu+i\mathcal{A}_1)\mathbf{e}^{1}\wedge\mathbf{e}^{-}+(-\partial_{\bar{1}}\log\bar{\mu}-i\mathcal{A}_{\bar{1}})\mathbf{e}^{\bar{1}}\wedge\mathbf{e}^{-}+(-\partial_{+}\log\mu\bar{\mu})\mathbf{e}^{+}\wedge\mathbf{e}^{-}+\\&(\partial_{+}\log\bar{\mu}+i\mathcal{A}_{+})\mathbf{e}^{\bar{1}}\wedge\mathbf{e}^{1}+\kappa^2\mu(\partial_1\bar{\mu}+i\mathcal{A}_1\bar{\mu})\mathbf{e}^{+}\wedge\mathbf{e}^{1}+(\partial_{+}\log\mu-i\mathcal{A}_{+})\mathbf{e}^{1}\wedge\mathbf{e}^{\bar{1}}+\\&\kappa^2\bar{\mu}(\partial_{\bar{1}}\mu-i\mathcal{A}_{\bar{1}}\mu)\mathbf{e}^{+}\wedge\mathbf{e}^{\bar{1}}=0.\end{aligned} \tag{5.5.35}$$

After collecting terms, we get

$$\begin{aligned}d\mathbf{e}^{-}=&-(\partial_{+}\log\frac{\bar{\mu}}{\mu}+2i\mathcal{A}_{+})\mathbf{e}^{\bar{1}}\wedge\mathbf{e}^{1}+\partial_{+}\log\bar{\mu}\mu\mathbf{e}^{+}\wedge\mathbf{e}^{-}-\kappa^2\mathbf{e}^{+}\wedge[(\mu\partial_1\bar{\mu}+i\mathcal{A}_1\mu\bar{\mu})\mathbf{e}^{1}+\\&(\bar{\mu}\partial_{\bar{1}}\mu-i\mathcal{A}_{\bar{1}}\bar{\mu}\mu)\mathbf{e}^{\bar{1}}]-\frac{1}{\mu\bar{\mu}}\mathbf{e}^{-}\wedge[(\bar{\mu}\partial_1\mu-i\mu\bar{\mu}\mathcal{A}_1)\mathbf{e}^{1}+(\mu\partial_{\bar{1}}\bar{\mu}+i\mu\bar{\mu}\mathcal{A}_{\bar{1}})\mathbf{e}^{\bar{1}}].\end{aligned} \tag{5.5.36}$$

5.5.3 Total Differential and the Killing Vector

At this point, it would be rather illuminating to notice that using the results that just preceded, we can write:

$$d(\mu\bar{\mu}\mathbf{e}^{+}-\kappa^2\mathbf{e}^{-})=0. \tag{5.5.37}$$

In order to prove that equation (5.5.37) holds, we start by expanding it in the following manner:

$$\begin{aligned}d(\mu\bar{\mu}-\kappa^2\mathbf{e}^{-})=&d(\mu\bar{\mu})\mathbf{e}^{+}-\mu\bar{\mu}\partial_1\log\mu\mathbf{e}^{1}\wedge\mathbf{e}^{+}+i\mu\bar{\mu}\mathcal{A}_1\mathbf{e}^{1}\wedge\mathbf{e}^{+}-\mu\bar{\mu}\partial_{\bar{1}}\log\bar{\mu}\mathbf{e}^{\bar{1}}\wedge\mathbf{e}^{+}\\&-i\mu\bar{\mu}\mathcal{A}_{\bar{1}}\mathbf{e}^{\bar{1}}\wedge\mathbf{e}^{+}-\kappa^2\partial_1\log\mu\mathbf{e}^{-}\wedge\mathbf{e}^{1}+i\mathcal{A}_1\kappa^2\mathbf{e}^{-}\wedge\mathbf{e}^{1}\end{aligned} \tag{5.5.38}$$

$$-\kappa^2\partial_{\bar{1}}\log\bar{\mu}\mathbf{e}^-\wedge\mathbf{e}^{\bar{1}} - i\kappa^2\mathcal{A}_{\bar{1}}\mathbf{e}^-\wedge\mathbf{e}^{\bar{1}} - \mu\bar{\mu}(\partial_-\log\frac{\mu}{\bar{\mu}} - 2i\mathcal{A}_-)\mathbf{e}^{\bar{1}}\wedge\mathbf{e}^1$$

$$-\kappa^2\partial_1\log\mu\mathbf{e}^1\wedge\mathbf{e}^- + i\kappa^2\mathcal{A}_1\mathbf{e}^1\wedge\mathbf{e}^- - \kappa^2\partial_{\bar{1}}\log\bar{\mu}\mathbf{e}^{\bar{1}}\wedge\mathbf{e}^- - i\kappa^2\mathcal{A}_{\bar{1}}\mathbf{e}^{\bar{1}}\wedge\mathbf{e}^-$$

$$-\kappa^2\partial_+\log\mu\bar{\mu}\mathbf{e}^+\wedge\mathbf{e}^- + \mu\kappa^4\partial_1\bar{\mu}\mathbf{e}^+\wedge\mathbf{e}^1 + i\mu\bar{\mu}\kappa^4\mathcal{A}_1\mathbf{e}^+\wedge\mathbf{e}^1$$

$$+\kappa^4\bar{\mu}\partial_{\bar{1}}\mu\mathbf{e}^+\wedge\mathbf{e}^{\bar{1}} - i\mu\bar{\mu}\kappa^4\mathcal{A}_{\bar{1}}\mathbf{e}^+\wedge\mathbf{e}^{\bar{1}} + \kappa^2(\partial_+\log\frac{\bar{\mu}}{\mu} + 2iA_+)\mathbf{e}^{\bar{1}}\wedge\mathbf{e}^1.$$

Making use of (5.5.23) and its conjugate in addition to equation (5.5.24), we indeed prove that equation (5.5.37) holds. Therefore, this tells us that

$$(\mu\bar{\mu}\mathbf{e}^+ - \kappa^2\mathbf{e}^-), \tag{5.5.39}$$

is a total differential. From here, we now want to check if

$$V = |\mu|^2\partial_- + \kappa^2\partial_+, \tag{5.5.40}$$

is a Killing vector. For a Killing vector to exist, it should obey the equation:

$$\partial_A V_B + \partial_B V_A = \omega_{A,CB}V^C + \omega_{B,CA}V^C. \tag{5.5.41}$$

And it turns out, after summing over all possible valid indices, that when $A = +$ and $B = -$, that equation (5.5.40) is valid, whereas for other cases, things vanish. We note that in (5.5.40), if $\kappa^2 = 1$, then the Killing vector is space-like. Meanwhile, if $\kappa^2 = -1$, then it is time-like.

But, we are also aware that the *dual* is defined by the map

$$\frac{\partial}{\partial x^\mu} \to g_{\mu\nu}dx^\mu, \tag{5.5.42}$$

so,

$$\partial_- \to g_{-+}\mathbf{e}^+ \to \mathbf{e}^+,$$

$$\partial_+ \to g_{+-}\mathbf{e}^- \to \mathbf{e}^-.$$

As a result, the one-form could be written as

$$V = \frac{1}{\sqrt{2}}(|\mu|^2\mathbf{e}^+ + \kappa^2\mathbf{e}^-) = \kappa^2|\mu|^2(dt+\sigma), \tag{5.5.43}$$

where $\sigma = \sigma_x dx + \sigma_y dy + \sigma_z dz$ is a 1-form independent of the coordinate t.

5.5.4 Prerequisites

The conditions noted at the end of section (5.5.3) permit us to introduce the (t, x, y, z) coordinates. In order to proceed with the analysis, we will start by extracting expressions for $\mathbf{e}^+$, $\mathbf{e}^-$, $\mathbf{e}^1$, $\mathbf{e}^{\bar{1}}$.

Upon the addition of equations (5.5.39) and (5.5.43), we simply get:

$$\boxed{\mathbf{e}^+ = \frac{1}{\sqrt{2}|\mu|^2}(dz+V) \equiv \frac{1}{\sqrt{2}|\mu|^2}\left(dz + \kappa^2|\mu|^2(dt+\sigma)\right)}. \tag{5.5.44}$$

Whereas, upon subtraction, we get:

$$\boxed{\mathbf{e}^- = -\frac{\kappa^2}{\sqrt{2}}(dz-V) \equiv -\frac{\kappa^2}{\sqrt{2}}\left(dz - \kappa^2|\mu|^2(dt+\sigma)\right)}. \tag{5.5.45}$$

We, then, extracted $\mathbf{e}^1$ by the following procedure:

Knowing that $\mathbf{e}^1$ is a one-form, we make use out of equation (5.5.30), to get

$$\boxed{\mathbf{e}^1 = \frac{1}{\bar{\mu}\sqrt{2}}(dx+idy)}. \tag{5.5.46}$$

We can take conjugate of (5.5.46) to arrive at the expression for $\mathbf{e}^{\bar{1}}$ which is

$$\boxed{\mathbf{e}^{\bar{1}} = \frac{1}{\mu\sqrt{2}}(dx-idy)}. \tag{5.5.47}$$

We summarize the results,

$$\begin{aligned}
\mathbf{e}^{+} &= \frac{1}{\sqrt{2}|\mu|^2}(dz+V) \equiv \frac{1}{\sqrt{2}|\mu|^2}\big(dz+\kappa^2|\mu|^2(dt+\sigma)\big),\\
\mathbf{e}^{-} &= -\frac{\kappa^2}{\sqrt{2}}(dz-V) \equiv -\frac{\kappa^2}{\sqrt{2}}\big(dz-\kappa^2|\mu|^2(dt+\sigma)\big),\\
\mathbf{e}^{1} &= \frac{1}{\bar{\mu}\sqrt{2}}(dx+idy),\\
\mathbf{e}^{\bar{1}} &= \frac{1}{\mu\sqrt{2}}(dx-idy).
\end{aligned} \tag{5.5.48}$$

Now, the metric is independent of the coordinate t and can be written as

$$ds^2 = 2\mathbf{e}^1\mathbf{e}^{\bar{1}} + 2\mathbf{e}^{+}\mathbf{e}^{-} = \kappa^2|\mu|^2(dt+\sigma)^2 + \frac{1}{|\mu|^2}(-\kappa^2dz^2+dx^2+dy^2). \tag{5.5.49}$$

It is important to mention here that using equations (5.4.26) and (5.4.27) we deduce,

$$(\mu\bar{\mu}\partial_- + \kappa^2\partial_+)z^A = \partial_t z^A = 0 \tag{5.5.50}$$

Which means that the scalar fields are independent of the coordinate t.

We also note that another condition could be deduced from equation (5.5.50) along with equation (4.2.17),

$$\begin{aligned}
\mathcal{A}_+ &= -\frac{i}{2}(\partial_A K\partial_+ z^A - \partial_{\bar{A}}K\partial_+\bar{z}^{\bar{A}}),\\
&\text{and}\\
\mathcal{A}_- &= -\frac{i}{2}(\partial_A K\partial_- z^A - \partial_{\bar{A}}K\partial_-\bar{z}^{\bar{A}}),
\end{aligned} \tag{5.5.51}$$

and which is

$$\kappa^2\mathcal{A}_+ + \mu\bar{\mu}\mathcal{A}_- = 0. \tag{5.5.52}$$

This implies and upon making use of equation (5.5.23) that

$$\partial_t\mu = 0. \tag{5.5.53}$$

We will now find what ∂_+, ∂_-, ∂_1 and $\partial_{\bar{1}}$ are in terms of ∂_x, ∂_y and ∂_z. We know that

$$dX = \partial_+ X\mathbf{e}^+ + \partial_- X\mathbf{e}^- + \partial_1 X\mathbf{e}^1 + \partial_{\bar{1}} X\mathbf{e}^{\bar{1}}, \tag{5.5.54}$$

that is

$$\begin{aligned} dX &= \partial_+ X[\frac{1}{\sqrt{2}|\mu|^2}(dz+V)] + \partial_- X[-\frac{\kappa^2}{\sqrt{2}}(dz-V)] + \partial_1 X[\frac{1}{\sqrt{2}\bar{\mu}}(dx+idy)] + \\ &\quad \partial_{\bar{1}} X[\frac{1}{\sqrt{2}\mu}(dx-idy)], \\ &= (\frac{1}{\sqrt{2}\bar{\mu}}\partial_1 X + \frac{1}{\sqrt{2}\mu}\partial_{\bar{1}} X)dx + (\frac{i}{\sqrt{2}\bar{\mu}}\partial_1 X - \frac{i}{\sqrt{2}}\partial_{\bar{1}} X)dy + (\frac{1}{\sqrt{2}|\mu|^2}\partial_+ X - \frac{\kappa^2}{\sqrt{2}}\partial_- X)dz \\ &\quad + (\frac{1}{\sqrt{2}|\mu|^2}\partial_+ X + \frac{\kappa^2}{\sqrt{2}}\partial_- X)V. \end{aligned} \tag{5.5.55}$$

But we also can write dX as

$$dX = \partial_x X dx + \partial_y X dy + \partial_z X dz. \tag{5.5.56}$$

So by comparing equation (5.5.55) to the second and neater equation (5.5.56) followed by elementary simplification, we get[2]

$$\begin{aligned} \partial_+ &= \frac{|\mu|^2}{\sqrt{2}}\partial_z, \\ \partial_- &= -\frac{\kappa^2}{\sqrt{2}}\partial_z, \\ \partial_1 &= \frac{\bar{\mu}}{\sqrt{2}}(\partial_x - i\partial_y), \\ \partial_{\bar{1}} &= \frac{\mu}{\sqrt{2}}(\partial_x + i\partial_y). \end{aligned} \tag{5.5.57}$$

Our next step at this point is to find an expression for $d\sigma$. This can be done using,

[2] 5.5.57 is true provided we are acting on time-independent functions.

for example, equation (5.5.44). For ease of reference, the equation says:

$$\mathbf{e}^{+} = \frac{1}{\sqrt{2}|\mu|^2}\big(dz + \kappa^2|\mu|^2(dt+\sigma)\big). \tag{5.5.58}$$

We start by taking the differential of the above equation to get:

$$\sqrt{2}d\mathbf{e}^{+} - d\left(\frac{1}{|\mu|^2}\right)\wedge dz = \kappa^2 d\sigma. \tag{5.5.59}$$

We start one thing at a time, that is, we start by the expression of $d\mathbf{e}^{+}$ which is already derived in (5.5.33). Unfolding the concerned equation, we get:

$$\begin{aligned} d\mathbf{e}^{+} =& -\partial_1 \log\mu\mathbf{e}^1\wedge\mathbf{e}^{+} + i\mathcal{A}_1\mathbf{e}^1\wedge\mathbf{e}^{+} - \partial_{\bar{1}}\log\bar{\mu}\mathbf{e}^{\bar{1}}\wedge\mathbf{e}^{+} - i\mathcal{A}_{\bar{1}}\mathbf{e}^{\bar{1}}\wedge\mathbf{e}^{+} \\ & - \frac{\kappa^2}{\mu\bar{\mu}}\partial_1\log\mu\mathbf{e}^{-}\wedge\mathbf{e}^1 + i\frac{\kappa^2}{\mu\bar{\mu}}\mathcal{A}_1\mathbf{e}^{-}\wedge\mathbf{e}^1 - \frac{\kappa^2}{\mu\bar{\mu}}\partial_{\bar{1}}\log\bar{\mu}\mathbf{e}^{-}\wedge\mathbf{e}^{\bar{1}} \\ & - i\frac{\kappa^2}{\mu\bar{\mu}}\mathcal{A}_{\bar{1}}\mathbf{e}^{-}\wedge\mathbf{e}^{\bar{1}} - \partial_{-}\log\frac{\mu}{\bar{\mu}}\mathbf{e}^{\bar{1}}\wedge\mathbf{e}^1 + 2i\mathcal{A}_{-}\mathbf{e}^{\bar{1}}\wedge\mathbf{e}^1. \end{aligned} \tag{5.5.60}$$

We now make use of equations (5.5.48) and (5.5.57) by substituting them in the equation that just preceded. We arrive at

$$\begin{aligned} d\mathbf{e}^{+} =& -\frac{\bar{\mu}}{\sqrt{2}}\left(\frac{1}{\sqrt{2}\bar{\mu}}\right)\left(\frac{1}{\sqrt{2}\mu\bar{\mu}}\right)(\partial_x\log\mu - i\partial_y\log\mu)(dx+idy)\wedge(dz+V) \\ & + i\frac{\bar{\mu}}{\sqrt{2}}\left(\frac{1}{\sqrt{2}\bar{\mu}}\right)\left(\frac{1}{\sqrt{2}\mu\bar{\mu}}\right)(\mathcal{A}_x - i\mathcal{A}_y)(dx+idy)\wedge(dz+V) \\ & - \frac{\mu}{\sqrt{2}}\left(\frac{1}{\sqrt{2}\mu}\right)\left(\frac{1}{\sqrt{2}\mu\bar{\mu}}\right)(\partial_x\log\bar{\mu} + i\partial_y\log\bar{\mu})(dx-idy)\wedge(dz+V) \\ & - i\frac{\mu}{\sqrt{2}}\left(\frac{1}{\sqrt{2}\mu}\right)\left(\frac{1}{\sqrt{2}\mu\bar{\mu}}\right)(\mathcal{A}_x + i\mathcal{A}_y)(dx-idy)\wedge(dz+V) \\ & + \frac{\kappa^2}{\mu\bar{\mu}}\frac{\bar{\mu}}{\sqrt{2}}\left(\frac{\kappa^2}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}\bar{\mu}}\right)(\partial_x\log\mu - i\partial_y\log\mu)(dz-V)\wedge(dx+idy) \\ & - i\frac{\kappa^2}{\mu\bar{\mu}}\frac{\bar{\mu}}{\sqrt{2}}\left(\frac{\kappa^2}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}\bar{\mu}}\right)(\mathcal{A}_x - i\mathcal{A}_y)(dz-V)\wedge(dx+idy) \end{aligned}$$

$$
\begin{aligned}
&+\frac{\kappa^2}{\mu\bar{\mu}}\frac{\mu}{\sqrt{2}}\left(\frac{\kappa^2}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}\mu}\right)(\partial_x \log\bar{\mu} + i\partial_y \log\bar{\mu})(dz - V)\wedge(dx - idy)\\
&+ i\frac{\kappa^2}{\mu\bar{\mu}}\frac{\mu}{\sqrt{2}}\left(\frac{\kappa^2}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}\mu}\right)(\mathcal{A}_x + i\mathcal{A}_y)(dz - V)\wedge(dx - idy)\\
&+\frac{\kappa^2}{\sqrt{2}}\left(\frac{1}{\sqrt{2}\mu}\right)\left(\frac{1}{\sqrt{2}\bar{\mu}}\right)\partial_z \log\frac{\mu}{\bar{\mu}}(dx - idy)\wedge(dx + idy)\\
&- 2i\frac{\kappa^2}{\sqrt{2}}\left(\frac{1}{\sqrt{2}\mu}\right)\left(\frac{1}{\sqrt{2}\bar{\mu}}\right)\mathcal{A}_z(dx - idy)\wedge(dx + idy).
\end{aligned}
\tag{5.5.61}
$$

Now, upon careful simplification of equation (5.5.61), we get:

$$
\begin{aligned}
d\mathbf{e}^+ = &-\frac{1}{\sqrt{2}\mu\bar{\mu}}\partial_x \log\mu(dx\wedge dz) - i\frac{1}{\sqrt{2}\mu\bar{\mu}}\partial_x \log\mu(dy\wedge dz) + i\frac{1}{\sqrt{2}\mu\bar{\mu}}\partial_y \log\mu(dx\wedge dz)\\
&-\frac{1}{\sqrt{2}\mu\bar{\mu}}\partial_y \log\mu(dy\wedge dz) - \frac{1}{\sqrt{2}\mu\bar{\mu}}\partial_x \log\bar{\mu}(dx\wedge dz) + i\frac{1}{\sqrt{2}\mu\bar{\mu}}\partial_x \log\bar{\mu}(dy\wedge dz)\\
&- i\frac{1}{\sqrt{2}\mu\bar{\mu}}\partial_y \log\bar{\mu}(dx\wedge dz) - \frac{1}{\sqrt{2}\mu\bar{\mu}}\partial_y \log\bar{\mu}(dy\wedge dz) + i\frac{\kappa^2}{\sqrt{2}\mu\bar{\mu}}\partial_z \log\frac{\mu}{\bar{\mu}}(dx\wedge dy)\\
&-\frac{\sqrt{2}}{\mu\bar{\mu}}\mathcal{A}_x(dy\wedge dz) + \frac{\sqrt{2}}{\mu\bar{\mu}}\mathcal{A}_y(dx\wedge dz) + \frac{\sqrt{2}\kappa^2}{\mu\bar{\mu}}\mathcal{A}_z(dx\wedge dy).
\end{aligned}
\tag{5.5.62}
$$

Plugging related terms in equation (5.5.59) with further simplification, this gives:

$$
\begin{aligned}
&\frac{1}{\mu\bar{\mu}}\left(-i\partial_y \log\frac{\mu}{\bar{\mu}}(dz\wedge dx) + i\partial_x \log\frac{\mu}{\bar{\mu}}(dz\wedge dy) + i\kappa^2\partial_z \log\frac{\mu}{\bar{\mu}}(dx\wedge dy)\right)\\
&+\frac{1}{\mu\bar{\mu}}(-2\mathcal{A}_x(dy\wedge dz) + 2\mathcal{A}_y(dx\wedge dz) + 2\kappa^2\mathcal{A}_z(dx\wedge dy)) = \kappa^2 d\sigma.
\end{aligned}
\tag{5.5.63}
$$

This then implies that

$$
\boxed{d\sigma = -\frac{\kappa^2}{|\mu|^2}\star_3\left(id\log\frac{\mu}{\bar{\mu}} + 2\mathcal{A}\right)},
\tag{5.5.64}
$$

where $\star_3$ is the Hodge dual with the metric $(-\kappa^2 dz^2 + dx^2 + dy^2)$.

5.5.5 Equations of Motion

Given the action

$$S = \int d^4x\sqrt{G}[R - 2g_{A\bar{B}}\partial z^A \partial \bar{z}^B - \frac{\kappa^2}{4}(Im\mathcal{N}_{IJ}F^I \cdot F^J + Re\mathcal{N}_{IJ}F^I \cdot \tilde{F}^J)], \quad (5.5.65)$$

we now want to derive the equations of motion using

$$G_K^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma}\frac{\delta S}{\delta F^{\rho\sigma K}}. \quad (5.5.66)$$

Starting with the term in the action containing $Im\mathcal{N}_{IJ}$, we calculate for $\frac{\delta S}{\delta F^{\rho\sigma K}}$ as follows:

$$= \frac{\kappa^2}{4}Im\mathcal{N}_{IJ}g_{\mu\mu'}g_{\nu\nu'}(\frac{\delta F^{\mu'\nu' I}}{\delta F^{\rho\sigma K}}F^{\mu\nu J} + \frac{\delta F^{\mu\nu J}}{\delta F^{\rho\sigma K}}F^{\mu'\nu' I})$$

$$= \frac{\kappa^2}{4}Im\mathcal{N}_{IJ}g_{\mu\mu'}g_{\nu\nu'}(\delta^{\mu'}_{\rho}\delta^{\nu'}_{\sigma}\delta^I_K F^{J\mu\nu} + \delta^J_K\delta^{\mu}_{\rho}\delta^{\nu}_{\sigma}F^{\mu'\nu' I})$$

$$= \frac{\kappa^2}{4}(Im\mathcal{N}_{IJ}F^J_{\rho\sigma} + ImN_{IJ}F^J_{\rho\sigma})$$

$$= \frac{\kappa^2}{2}Im\mathcal{N}_{IJ}F^J_{\rho\sigma}. \quad (5.5.67)$$

Multiplying the previous result by $\epsilon^{\mu\nu\rho\sigma}$, we arrive at

$$\kappa^2 Im\mathcal{N}_{IJ}\tilde{F}^{J\mu\nu}. \quad (5.5.68)$$

Following the same procedure for the part of the Lagrangian with $Re\mathcal{N}_{IJ}$, we get

$$-\kappa^2 Re\mathcal{N}_{IJ}F^{J\mu\nu}. \quad (5.5.69)$$

Combining both, we obtain

$$G_K^{\mu\nu} = \kappa^2(-ReN_{IJ}F^{J\mu\nu} + ImN_{IJ}\tilde{F}^{J\mu\nu}). \tag{5.5.70}$$

Thus, the Bianchi identities and Maxwell equations are

$$dF^I = 0, \qquad \text{and } d(Re\mathcal{N}_{IJ}F^J - Im\mathcal{N}_{IJ}\tilde{F}^J) = 0. \tag{5.5.71}$$

In order to find the equations of motion, we first start by finding an expression for F^I. In order to do that, we modify by simple means equations (5.4.26-5.4.29). By this, we mean, multiplying each by $\mathcal{D}_A L^M$ and using the relations (4.2.16) and (4.2.21),

$$\begin{gathered} g^{A\bar{B}}\mathcal{D}_A L^M \mathcal{D}_{\bar{B}}\bar{L}^I = -\frac{1}{2}(Im\mathcal{N})^{MI} - \bar{L}^M L^I, \\ \text{and} \\ \mathcal{D}_A L^I dz^A = (d + i\mathcal{A})L^I. \end{gathered} \tag{5.5.72}$$

As a result, we arrive at

$$\begin{aligned} \frac{i\kappa}{2}\left(F^M_{-+} - F^M_{1\bar{1}}\right) + i\kappa(\,\mathrm{Im}\mathcal{N})_{IJ}\left(F^J_{-+} - F^J_{1\bar{1}}\right)\bar{L}^M L^I + \left(\partial_- + i\mathcal{A}_-\right)L^M\mu\sqrt{2} = 0, \\ -2i\bar{\kappa}\bar{\mu}(\mathrm{Im}\mathcal{N})_{IJ}F^J_{-\bar{1}}\bar{L}^M L^I - i\bar{\kappa}F^M_{-\bar{1}}\bar{\mu} + \left(\partial_{\bar{1}} + i\mathcal{A}_{\bar{1}}\right)L^M\sqrt{2} = 0, \\ -2i\kappa(\mathrm{Im}\mathcal{N})_{IJ}F^J_{+1}\bar{L}^M L^I - i\kappa F^M_{+1} + \left(\partial_1 + i\mathcal{A}_1\right)L^M\mu\sqrt{2} = 0. \end{aligned}$$

The set of equations in (5.5.73) can be altered using (5.5.22) and (5.5.57), in order to obtain

$$\begin{aligned} F^I_{1\bar{1}} &= i\left[-\kappa L^I\left(\partial_z - i\mathcal{A}_z\right)\mu + \kappa\mu\left(\partial_z + i\mathcal{A}_z\right)L^I + \bar{\kappa}\bar{\mu}\left(\partial_z - i\mathcal{A}_z\right)\bar{L}^I - \bar{\kappa}\bar{L}^I\left(\partial_z + i\mathcal{A}_z\right)\bar{\mu}\right], \\ F^I_{-+} &= -i\kappa^2\partial_z\left(\bar{\kappa}\mu L^I - \kappa\bar{L}^I\bar{\mu}\right), \ , \\ F^I_{-\bar{1}} &= i\frac{1}{\bar{\mu}}\left[\bar{\kappa}\bar{L}^I\left(\left(\partial_x + i\partial_y\right) + i\left(\mathcal{A}_x + i\mathcal{A}_y\right)\right)\bar{\mu} - \kappa\mu\left(\left(\partial_x + i\partial_y\right) + i\left(\mathcal{A}_x + i\mathcal{A}_y\right)\right)L^I\right], \\ F^I_{+1} &= i\bar{\mu}\left[\kappa\bar{L}^I\left(\left(\partial_x - i\partial_y\right) + i\left(\mathcal{A}_x - i\mathcal{A}_y\right)\right)\bar{\mu} - \bar{\kappa}\mu\left(\left(\partial_x - i\partial_y\right) + i\left(\mathcal{A}_x - i\mathcal{A}_y\right)\right)L^I\right]. \end{aligned}$$

We now shall find expression for the gauge field strength,

$$F^I = F^I_{-+}\mathbf{e}^- \wedge \mathbf{e}^+ + F^I_{1\bar{1}}\mathbf{e}^1 \wedge \mathbf{e}^{\bar{1}} + F^I_{-1}\mathbf{e}^- \wedge \mathbf{e}^1 + F^I_{+1}\mathbf{e}^+ \wedge \mathbf{e}^1 + F^I_{\bar{1}+}\mathbf{e}^{\bar{1}} \wedge \mathbf{e}^+ + F^I_{\bar{1}-}\mathbf{e}^{\bar{1}} \wedge \mathbf{e}^-$$

Upon substitution

$$\begin{aligned} F^I = &- i\kappa^2\partial_z(\bar{\kappa}\mu L^I - \kappa\bar{L}^I\bar{\mu})\left(-\frac{\kappa^2}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}\mu\bar{\mu}}\right)(dz - V)\wedge(dz+V) \\ &+ i[-\kappa L^I(\partial_z - i\mathcal{A}_z)\mu + \kappa\mu(\partial_z + i\mathcal{A}_z)L^I + \bar{\kappa}\bar{\mu}(\partial_z - i\mathcal{A}_z)\bar{L}^I \\ &- \bar{\kappa}\bar{L}^I(\partial_z + i\mathcal{A}_z)\bar{\mu}]\left(\frac{1}{\sqrt{2}\bar{\mu}}\right)\left(\frac{1}{\sqrt{2}\mu}\right)(dx + idy)\wedge(dx - idy) \\ &- i\frac{1}{\mu}[\kappa L^I((\partial_x - i\partial_y) - i(\mathcal{A}_x - i\mathcal{A}_y))\mu - \bar{\kappa}\bar{\mu}((\partial_x - i\partial_y) - i(\mathcal{A}_x \\ &- i\mathcal{A}_y))\bar{L}^I]\left(-\frac{\kappa^2}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}\bar{\mu}}\right)(dz - V)\wedge(dx + idy) \\ &+ i\bar{\mu}[\kappa\bar{L}^I((\partial_x - i\partial_y) + i(\mathcal{A}_x - i\mathcal{A}_y))\bar{\mu} - \bar{\kappa}\mu((\partial_x - i\partial_y) + i(\mathcal{A}_x \\ &- i\mathcal{A}_y))L^I]\left(\frac{1}{\sqrt{2}\mu\bar{\mu}}\right)\left(\frac{1}{\sqrt{2}\bar{\mu}}\right)(dz + V)\wedge(dx + idy) \\ &+ i\mu[\bar{\kappa}L^I((\partial_x + i\partial_y) - i(\mathcal{A}_x + i\mathcal{A}_y))\mu - \kappa\bar{\mu}((\partial_x + i\partial_y) - i(\mathcal{A}_x \\ &+ i\mathcal{A}_y))\bar{L}^I]\left(\frac{1}{\sqrt{2}\mu}\right)\left(\frac{1}{\sqrt{2}\mu\bar{\mu}}\right)(dx - idy)\wedge(dz + V) \\ &- i\frac{1}{\bar{\mu}}[\bar{\kappa}\bar{L}^I((\partial_x + i\partial_y) + i(\mathcal{A}_x + i\mathcal{A}_y))\bar{\mu} - \kappa\mu((\partial_x + i\partial_y) + i(\mathcal{A}_x \\ &+ i\mathcal{A}_y))L^I]\left(\frac{1}{\sqrt{2}\mu}\right)\left(-\frac{\kappa^2}{\sqrt{2}}\right)(dx - idy)\wedge(dz - V). \end{aligned} \tag{5.5.73}$$

Upon serious simplification of what preceded, we get:

$$\begin{aligned} F^I = &- \frac{2i}{\mu\bar{\mu}}(\kappa\bar{\mu}\mathcal{A}_z\bar{L}^I - \bar{\kappa}\mu\mathcal{A}_z L^I)\kappa^2(dx\wedge dy) \\ &- \frac{2i}{\mu\bar{\mu}}(\bar{\kappa}\mu L^I\mathcal{A}_x - \kappa\bar{\mu}\mathcal{A}_x\bar{L}^I)(dy\wedge dz) \\ &- \frac{2i}{\mu\bar{\mu}}(\bar{\kappa}\mu L^I\mathcal{A}_y - \bar{\mu}\mathcal{A}_y\bar{L}^I\kappa)(dz\wedge dx) \\ &- \partial_z(i\bar{\kappa}\bar{L}^I\bar{\mu} - i\kappa\mu L^I)dz\wedge(dt+\sigma) \\ &+ \partial_x(-i\bar{\kappa}\bar{L}^I\bar{\mu} + i\kappa L^I\mu)dx\wedge(dt+\sigma) \end{aligned} \tag{5.5.74}$$

$$+\partial_y(-i\bar\kappa\bar L^I\bar\mu + i\kappa L^I\mu)dy\wedge(dt+\sigma)$$
$$+\frac{1}{\mu\bar\mu}(-\bar\kappa L^I\partial_z\mu + \bar\kappa\mu\partial_z L^I + \kappa\bar\mu\partial_z\bar L^I - \kappa\bar L^I\partial_z\bar\mu)\kappa^2(dx\wedge dy)$$
$$-\frac{1}{\mu\bar\mu}(-\kappa\bar L^I\partial_x\bar\mu + \mu\bar\kappa\partial_x L^I - \bar\kappa L^I\partial_x\mu + \kappa\bar\mu\partial_x\bar L^I)(dy\wedge dz)$$
$$-\frac{1}{\mu\bar\mu}(-\bar\kappa L^I\partial_y\mu + \kappa\bar\mu\partial_y\bar L^I - \kappa\bar L^I\partial_y\bar\mu + \bar\kappa\mu\partial_y L^I)(dz\wedge dx).$$

So,

$$F^I = d\left(i\kappa\mu L^I - i\bar\kappa\bar L^I\bar\mu\right)\wedge(dt+\sigma) - \frac{1}{|\mu|^2}\star_3\left[\kappa\bar\mu d\bar L^I - \kappa\bar L^I d\bar\mu + \bar\kappa\mu dL^I - \bar\kappa L^I d\mu\right]$$
$$-\frac{2i}{|\mu|^2}\star_3\left(\bar\kappa\mu L^I - \kappa\bar L^I\bar\mu\right)\mathcal{A}. \qquad (5.5.75)$$

With the help of equation (5.5.64), (5.5.75) can be written as

$$F^I = d\left[\left(i\kappa\mu L^I - i\bar\kappa\bar L^I\bar\mu\right)(dt+\sigma)\right] - *d\left[\kappa\left(\frac{\bar L^I}{\mu}\right) + \bar\kappa\left(\frac{L^I}{\bar\mu}\right)\right]. \qquad (5.5.76)$$

Next, we want to find an expression for $\tilde F^I$.

To do that, we first find

$$\tilde F^I_{+-} = iF^I_{1\bar1},$$
$$\tilde F^I_{1\bar1} = iF^I_{+-},$$
$$\tilde F^I_{-1} = iF^I_{1-},$$
$$\tilde F^I_{+1} = iF^I_{+1}. \qquad (5.5.77)$$

We then make use of what preceded

$$\tilde F^I = \tilde F^I_{-+}\mathbf{e}^-\wedge\mathbf{e}^+ + \tilde F^I_{1\bar1}\mathbf{e}^1\wedge\mathbf{e}^{\bar1} + \tilde F^I_{-1}\mathbf{e}^-\wedge\mathbf{e}^1 + \tilde F^I_{+1}\mathbf{e}^+\wedge\mathbf{e}^1 + \tilde F^I_{\bar1+}\mathbf{e}^{\bar1}\wedge\mathbf{e}^+$$
$$+\tilde F^I_{\bar1-}\mathbf{e}^{\bar1}\wedge\mathbf{e}^-$$

$$
= -iF^I_{1\bar{1}}\mathbf{e}^- \wedge \mathbf{e}^+ + iF^I_{+-}\mathbf{e}^1 \wedge \mathbf{e}^{\bar{1}} + iF^I_{1-}\mathbf{e}^- \wedge \mathbf{e}^1 + iF^I_{+1}\mathbf{e}^+ \wedge \mathbf{e}^1 + iF^I_{+\bar{1}}\mathbf{e}^{\bar{1}} \wedge \mathbf{e}^+ \\
+ iF^I_{\bar{1}-}\mathbf{e}^{\bar{1}} \wedge \mathbf{e}^-
$$

$$
\begin{aligned}
\tilde{F}^I = & -\kappa L^I(\partial_z - i\mathcal{A}_z)\mu + \kappa\mu(\partial_z + i\mathcal{A}_z)L^I + \bar{\kappa}\bar{\mu}(\partial_z - i\mathcal{A}_z)\bar{L}^I \\
& - \bar{\kappa}\bar{L}^I(\partial_z + i\mathcal{A}_z)\bar{\mu}]\left(-\frac{\kappa^2}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}\mu\bar{\mu}}\right)(dz - V)\wedge(dz + V) \\
& - \kappa^2\partial_z(\bar{\kappa}\mu L^I - \kappa\bar{L}^I\bar{\mu})\left(\frac{1}{\sqrt{2}\bar{\mu}}\right)\left(\frac{1}{\sqrt{2}\mu}\right)(dx + idy)\wedge(dx - idy) \\
& - \frac{1}{\mu}[\kappa L^I((\partial_x - i\partial_y) - i(\mathcal{A}_x - i\mathcal{A}_y))\mu - \bar{\kappa}\bar{\mu}((\partial_x - i\partial_y) - i(\mathcal{A}_x \\
& - i\mathcal{A}_y))\bar{L}^I]\left(-\frac{\kappa^2}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}\bar{\mu}}\right)(dz - V)\wedge(dx + idy) \\
& - \bar{\mu}[\kappa\bar{L}^I((\partial_x - i\partial_y) + i(\mathcal{A}_x - i\mathcal{A}_y))\bar{\mu} - \bar{\kappa}\mu((\partial_x - i\partial_y) + i(\mathcal{A}_x \\
& - i\mathcal{A}_y))L^I]\left(\frac{1}{\sqrt{2}\mu\bar{\mu}}\right)\left(\frac{1}{\sqrt{2}\bar{\mu}}\right)(dz + V)\wedge(dx + idy) \\
& + \mu[\bar{\kappa}L^I((\partial_x + i\partial_y) - i(\mathcal{A}_x + i\mathcal{A}_y))\mu - \kappa\bar{\mu}((\partial_x + i\partial_y) - i(\mathcal{A}_x \\
& + i\mathcal{A}_y))\bar{L}^I]\left(\frac{1}{\sqrt{2}\mu}\right)\left(\frac{1}{\sqrt{2}\mu\bar{\mu}}\right)(dx - idy)\wedge(dz + V) \\
& + \frac{1}{\bar{\mu}}[\bar{\kappa}\bar{L}^I((\partial_x + i\partial_y) + i(\mathcal{A}_x + i\mathcal{A}_y))\bar{\mu} - \kappa\mu((\partial_x + i\partial_y) + i(\mathcal{A}_x \\
& + i\mathcal{A}_y))L^I]\left(\frac{1}{\sqrt{2}\mu}\right)\left(-\frac{\kappa^2}{\sqrt{2}}\right)(dx - idy)\wedge(dz - V).
\end{aligned} \tag{5.5.78}
$$

Upon serious simplification, we get:

$$
\begin{aligned}
\tilde{F}^I = & -2i\mathcal{A}_z(\mu\kappa L^I - \bar{\mu}\bar{\kappa}\bar{L}^I)dz \wedge (dt+\sigma) \\
& -2i\mathcal{A}_x(\mu\kappa L^I - \bar{\kappa}\bar{\mu}\bar{L}^I)dx \wedge (dt+\sigma) \\
& -2i\mathcal{A}_y(-\bar{\kappa}\bar{\mu}\bar{L}^I + \kappa L^I\mu)dy \wedge (dt+\sigma) \\
& + \frac{i}{\mu\bar{\mu}}\partial_y(\kappa\bar{L}^I\bar{\mu} - \bar{\kappa}\mu L^I)(dz\wedge dx) \\
& + \frac{i}{\mu\bar{\mu}}\partial_x(-\bar{\kappa}L^I\mu + \kappa\bar{\mu}\bar{L}^I)(dy\wedge dz) \\
& - \frac{i\kappa^2}{\mu\bar{\mu}}\partial_z(-\bar{\kappa}\mu L^I + \kappa\bar{L}^I\bar{\mu})(dx\wedge dy) \\
& + (\kappa L^I\partial_z\mu - \kappa\mu\partial_z L^I - \bar{\kappa}\bar{\mu}\partial_z\bar{L}^I + \bar{\kappa}\bar{L}^I\partial_z\bar{\mu})dz\wedge(dt+\sigma) \\
& + (\kappa L^I\partial_x\mu - \bar{\kappa}\bar{\mu}\partial_x\bar{L}^I + \bar{\kappa}\bar{L}^I\partial_x\bar{\mu} - \kappa\mu\partial_x L^I)dx\wedge(dt+\sigma) \\
& + (\bar{\kappa}\bar{L}^I\partial_y\bar{\mu} - \kappa\mu\partial_y L^I + \kappa L^I\partial_y\mu - \bar{\kappa}\bar{\mu}\partial_y\bar{L}^I)dy\wedge(dt+\sigma),
\end{aligned}
\tag{5.5.79}
$$

this boils down to

$$
\begin{aligned}
\tilde{F} \;=\; & \frac{i}{|\mu|^2}\star_3 d\left[\kappa\bar{L}^I\bar{\mu} - \bar{\kappa}\mu L^I\right] + \left(-2i\mathcal{A}\left(\kappa\mu L^I - \bar{\kappa}\bar{\mu}\bar{L}^I\right) + \left(\bar{\kappa}\bar{L}^I d\bar{\mu} - \kappa\mu dL^I\right)\right. \\
& \left. + \left(\kappa L^I d\mu - \bar{\kappa}\bar{\mu}d\bar{L}^I\right)\right)\wedge(dt+\sigma)\,.
\end{aligned}
$$

We plug what preceded in $Re\mathcal{N}_{IJ}F^J - Im\mathcal{N}_{IJ}\tilde{F}^J$ to get

$$
\begin{aligned}
& Re\mathcal{N}_{IJ}F^J - Im\mathcal{N}_{IJ}\tilde{F}^J = \\
& Re\mathcal{N}_{IJ}(d[(i\kappa\mu L^J - i\bar{\kappa}\bar{L}^J\bar{\mu})(dt+\sigma)] - \star_3 d[\kappa(\frac{\bar{L}^J}{\mu}) + \bar{\kappa}(\frac{L^J}{\bar{\mu}})]) \\
& - Im\mathcal{N}_{IJ}(\frac{i}{|\mu|^2}\star_3 d(\kappa\bar{L}^J\bar{\mu} - \bar{\kappa}L^J\mu) + [(\bar{\kappa}\bar{L}^J d\bar{\mu} - \kappa\mu dL^J) + \\
& (\kappa L^J d\mu - \bar{\kappa}\bar{\mu}d\bar{L}^J) - 2i\mathcal{A}(\kappa\mu L^J - \bar{\kappa}\bar{\mu}\bar{L}^J)]\wedge(dt+\sigma)) =
\end{aligned}
\tag{5.5.80}
$$

$$\begin{aligned}
&\big(i\kappa d\mu Re\mathcal{N}_{IJ}L^J + i\kappa\mu Re\mathcal{N}_{IJ}L^J - i\bar{\kappa}\bar{\mu}Re\mathcal{N}_{IJ}d\bar{L}^J - i\bar{\kappa}d\bar{\mu}Re\mathcal{N}_{IJ}\bar{L}^J \\
&- \bar{\kappa}d\bar{\mu}Im\mathcal{N}_{IJ}\bar{L}^J + \kappa\mu Im\mathcal{N}_{IJ}dL^J - \kappa d\mu Im\mathcal{N}_{IJ}L^J + \bar{\kappa}\bar{\mu}Im\mathcal{N}_{IJ}d\bar{L}^J \\
&+2iA\kappa\mu Im\mathcal{N}_{IJ}L^J - 2iA\bar{\kappa}\bar{\mu}Im\mathcal{N}_{IJ}\bar{L}^J\big) \wedge (dt+\sigma) \\
&+ (i\kappa\mu Re\mathcal{N}_{IJ}L^J - i\bar{\kappa}\bar{\mu}Re\mathcal{N}_{IJ}\bar{L}^J)d\sigma - Re\mathcal{N}_{IJ} \star_3 d(\frac{k\bar{L}^J}{\mu} + \frac{\bar{\kappa}L^J}{\bar{\mu}}) - \\
&Im\mathcal{N}_{IJ}\frac{i}{|\mu|^2} \star_3 d(\kappa\bar{L}^J\bar{\mu} - \bar{\kappa}L^J\mu) =
\end{aligned} \tag{5.5.81}$$

Using equations of dM_I and M_I and their conjugates

$$\begin{aligned}
&(i\kappa\mu dM_I - i\bar{\kappa}\bar{\mu}d\bar{M}_I + i\kappa M_I d\mu - i\bar{\kappa}\bar{M}_I d\bar{\mu}) \wedge (dt+\sigma) \\
&+ (i\kappa\mu Re\mathcal{N}_{IJ}L^J - i\bar{\kappa}\bar{\mu}Re\mathcal{N}_{IJ}\bar{L}^J)d\sigma - Re\mathcal{N}_{IJ} \star_3 d(\frac{k\bar{L}^J}{\mu} + \frac{\bar{\kappa}L^J}{\bar{\mu}}) \\
&- Im\mathcal{N}_{IJ}\frac{i}{|\mu|^2} \star_3 d(\kappa\bar{L}^J\bar{\mu} - \bar{\kappa}L^J\mu) =
\end{aligned} \tag{5.5.82}$$

$$\begin{aligned}
&d[(i\kappa\mu M_I - i\bar{\kappa}\bar{\mu}\bar{M}_I)(dt+\sigma)] - (i\kappa\mu M_I - i\bar{\kappa}\bar{\mu}\bar{M}_I - i\kappa\mu Re\mathcal{N}_{IJ}L^J + \\
&i\bar{\kappa}\bar{\mu}Re\mathcal{N}_{IJ}\bar{L}^J)d\sigma - Re\mathcal{N}_{IJ} \star_3 d(\frac{k\bar{L}^J}{\mu} + \frac{\bar{\kappa}L^J}{\bar{\mu}}) \\
&- Im\mathcal{N}_{IJ}\frac{i}{|\mu|^2} \star_3 d(\kappa\bar{L}^J\bar{\mu} - \bar{\kappa}L^J\mu) =
\end{aligned} \tag{5.5.83}$$

$$\begin{aligned}
&d[(i\kappa\mu M_I - i\bar{\kappa}\bar{\mu}\bar{M}_I)(dt+\sigma)] + (\kappa\mu Im\mathcal{N}_{IJ}L^J + \bar{\kappa}\bar{\mu}Im\mathcal{N}_{IJ}\bar{L}^J)d\sigma \\
&- Re\mathcal{N}_{IJ} \star_3 d(\frac{k\bar{L}^J}{\mu} + \frac{\bar{\kappa}L^J}{\bar{\mu}}) - Im\mathcal{N}_{IJ}\frac{i}{|\mu|^2} \star_3 d(\kappa\bar{L}^J\bar{\mu} - \bar{\kappa}L^J\mu) =
\end{aligned} \tag{5.5.84}$$

Substituting $d\sigma$ by its value followed by simplifying and collecting terms, we obtain

$$\begin{aligned}
&d[(i\kappa\mu M_I - i\bar{\kappa}\bar{\mu}\bar{M}_I)(dt+\sigma)] - \star_3(\frac{\kappa}{\mu}Re\mathcal{N}_{IJ}d\bar{L}^J - \frac{d\mu\kappa}{\mu\mu}Re\mathcal{N}_{IJ}\bar{L}^J \\
&+ \frac{\bar{\kappa}}{\bar{\mu}}Re\mathcal{N}_{IJ}dL^J - \frac{d\bar{\mu}\bar{\kappa}}{\bar{\mu}\bar{\mu}}Re\mathcal{N}_{IJ}L^J + i\frac{\kappa}{\mu}Im\mathcal{N}_{IJ}d\bar{L}^J + i\frac{d\bar{\mu}\kappa}{\mu\bar{\mu}}Im\mathcal{N}_{IJ}\bar{L}^J \\
&- i\frac{d\mu\bar{\kappa}}{\mu\bar{\mu}}Im\mathcal{N}_{IJ}L^J - i\frac{\bar{\kappa}}{\bar{\mu}}Im\mathcal{N}_{IJ}dL^J + i\frac{d\bar{\mu}\bar{\kappa}}{\mu\bar{\mu}}Im\mathcal{N}_{IJ}L^J - i\frac{d\bar{\mu}\bar{\kappa}}{\bar{\mu}\bar{\mu}}Im\mathcal{N}_{IJ}L^J \\
&+ 2A\frac{\bar{\kappa}}{\bar{\mu}}Im\mathcal{N}_{IJ}L^J + i\frac{d\mu\kappa}{\mu\mu}Im\mathcal{N}_{IJ}\bar{L}^J - i\frac{d\bar{\mu}\kappa}{\mu\bar{\mu}}Im\mathcal{N}_{IJ}\bar{L}^J + 2A\frac{\kappa}{\mu}Im\mathcal{N}_{IJ}\bar{L}^J).
\end{aligned} \tag{5.5.85}$$

Using dM_I and M_I along with their conjugates, we get:

$$Re\mathcal{N}_{IJ}F^J - Im\mathcal{N}_{IJ}\tilde{F}^J = d(i\kappa\mu M_I - i\bar{\kappa}\bar{\mu}\bar{M}_I)(dt+\sigma)] - \star_3 d[\kappa(\frac{\bar{M}_I}{\mu}) + \bar{\kappa}(\frac{M_I}{\bar{\mu}})]. \tag{5.5.86}$$

This implies, after applying (5.5.71), the conditions

$$\left(\frac{\kappa\bar{L}^I}{\mu} + \frac{\bar{\kappa}L^I}{\bar{\mu}}\right) = \psi^I, \qquad \left(\frac{\kappa\bar{M}_I}{\mu} + \frac{\bar{\kappa}M_I}{\bar{\mu}}\right) = \psi_I, \tag{5.5.87}$$

where

$$\nabla^2\psi^I = \nabla^2\psi_I = 0, \text{ with } \nabla^2 = \partial_x^2 + \partial_y^2 - \kappa^2\partial_z^2. \tag{5.5.88}$$

Using (5.5.87) along with the three special geometry-relations that we rewrite here for ease of reference:

$$\begin{aligned}
&i(\bar{L}^I M_I - L^I\bar{M}_I) = 1, \\
&F_I\partial_\mu X^I - X^I\partial_\mu F_I = 0, \\
&\mathcal{A} = M_I d\bar{L}^I - L^I d\bar{M}_I,
\end{aligned} \tag{5.5.89}$$

we get

$$\psi_I d\psi^I = \left[\frac{\kappa\bar{M}_I}{\mu} + \frac{\bar{\kappa}M_I}{\bar{\mu}}\right]\left[\frac{\kappa d\bar{L}^I}{\mu} + \kappa\bar{L}^I d\left(\frac{1}{\mu}\right) + \frac{\bar{\kappa}dL^I}{\bar{\mu}} + \bar{\kappa}L^I d\left(\frac{1}{\bar{\mu}}\right)\right],$$

$$\psi^I d\psi_I = \left[\frac{\kappa\bar{L}^I}{\mu} + \frac{\bar{\kappa}L^I}{\bar{\mu}}\right]\left[\frac{\kappa d\bar{M}_I}{\mu} + \kappa\bar{M}_I d\left(\frac{1}{\mu}\right) + \frac{\bar{\kappa}dM_I}{\bar{\mu}} + \bar{\kappa}M_I d\left(\frac{1}{\bar{\mu}}\right)\right],$$

$$\psi_I d\psi^I - \psi^I d\psi_I = \frac{1}{|\mu|^2}(\bar{M}_I dL^I - \bar{L}^I dM_I) + \frac{1}{|\mu|^2}(M_I d\bar{L}^I - L^I d\bar{M}_I) + i\frac{1}{|\mu|^2}d\log\frac{\mu}{\bar{\mu}}, \tag{5.5.90}$$

thus,

$$\mathcal{A} = \frac{|\mu|^2}{2}(\psi_I d\psi^I - \psi^I d\psi_I) - \frac{i}{2}d\log\frac{\mu}{\bar{\mu}}. \tag{5.5.91}$$

Substituting equation (5.5.91) in the expression of $d\sigma$, we get:

$$d\sigma = -\kappa^2 \star_3 (\psi_I d\psi^I - \psi^I d\psi_I). \tag{5.5.92}$$

For $\kappa = i$, we get the known solutions of [3, 26] which are generalizations of the solutions first obtained in [27]. The new derivation here, tells us that these are the unique solutions with time-like Killing vector as has also been illustrated in [28]. For $\kappa = 1$, we obtain new phantom solutions for the theories with the non-conventional signs for the gauge kinetic terms. In this case, the functions ψ^I and ψ_I satisfy the wave equation

$$\left(\partial_x^2 + \partial_y^2\right)\psi^I = \partial_z^2\psi^I, \qquad \left(\partial_x^2 + \partial_y^2\right)\psi_I = \partial_z^2\psi_I. \tag{5.5.93}$$

These solutions are the unique solutions with space-like Killing vectors admitting Killing spinors.

Chapter 6

Summary

We started this thesis by giving a short presentation of necessary mathematical concepts needed for later chapters. Then, in a brief manner, we illustrated an overview of few ideas in Einstein's general theory of relativity. This was essential, since we were heading toward supergravity which was founded after incorporating supersymmetry into General Relativity. Afterwards, we intended to collect few of the Special Geometry relations in a separate chapter. Those relations were used to carry out the calculations in chapter 5. This is because the scalars in vector multiplets of $N = 2$ supergravity theories in four dimensions exhibit this sort of geometry – the special Kähler geometry. Chapter 5, which is the core of my thesis work, contains detailed calculations for solving the theory's Killing spinor equations using the method of spinorial geometry.

The method presented is very elegant and systematic. The first key step of spinorial geometry is to describe the spinors in terms of exterior forms. Next, one should seek the canonical forms of the spinor which are practically representatives up to gauge transformations. This prescription reduces the Killing spinor equations to sets of linear systems. From these systems, one derives geometric constraints involving field strengths and spin connections. In their turn, those constraints occupy their roles in the interplay that leads to the solutions.

Not only did we use the spinorial geometry to discuss the ordinary $N = 2, D = 4$ supergravity theory coupled to vector multiplets, but we also used it to tackle the fake scenario of that theory. Phantom solutions with Killing spinors were first discussed by Sabra in July 2015. Our work here was a generalization of that paper. That is, we obtained new phantom solutions admitting Killing spinors in fake N = 2, D = 4 supergravity where the Abelian $U(1)$ gauge fields have kinetic terms with the non-conventional sign. The solutions found are characterized in terms of algebraic constraints where the symplectic sections are related to a set of functions satisfying the wave-equation in flat (2+1)-space-time in the fake theory. However, in the ordinary one, those functions are harmonic.

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