AMERICAN UNIVERSITY OF BEIRUT

THE ZEROES OF PERIOD POLYNOMIALS

by
OMAR BASSAM SHAMI NASR

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OMAR BASSAM SHAMI NASR

Approved by:

Dr. Raji, Wissam, Associate Professor — Advisor
Mathematics

Dr. Abi – Khuzam, Faruk, Professor — Member of Committee
Mathematics

Dr. Khuri – Makdisi, Kamal, Professor — Member of Committee
Mathematics

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ACKNOWLEDGMENT

To the invaluable gift of existence

To the light that gathers us, and the nights that follow

To the beautiful mystery of numbers
The One and the Five and the Threes

To Mom,
For turning off her TV... and her dreams, to make me

To Dad,
For being closest no matter how far, for believing in happiness as a craft

To Ola,
for discovering infinities, and taking me with her

To my dog Snoopy,
For the thousand "love at first sight" he shows me. How his legs fail him but his love does not

To the tattoo I never did, and the many marks engraved

To you, wherever you are,
To the one who crossed my path, and the other living in a parallel universe, until we intersect

To partial failures,
To brave beginnings,
To the jumps not yet taken
And the stories yet to be lived

To the hope of missing wrong trains
With a company for a shared glory

To feelings, for staying untamed
And the "*Once Upon No Time*" story

To all of the above
And the ghosts in between,
I dedicate this paper

AN ABSTRACT OF THE THESIS OF

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A modular form is a function holomorphic in the upper half plane and at the cusps. It satisfies certain transformation conditions under the full modular group. The space of entire modular forms of integer weight on the full group is finite dimensional and the Fourier coefficients of these forms possess interesting arithmetical properties. Moreover, the zeroes of modular forms have been studied intensively and were the center of attention in the field. As an example, we show that the zeroes of the Eisenstein series of weight greater or equal to 4 lie on the portion of the unit circle. $\{z = e^{i\theta} : \frac{\pi}{2} \leq \theta \leq \frac{2\pi}{3}\}$.
The $(k + 1)$ fold integral of a modular form gives rise to what is known as the period polynomials. These polynomials satisfy certain consistency condition and have interesting connections to L-functions. We show that the zeroes of period polynomials lie on the unit circle.

Contents

Chapter 1

Basic Definitions

The importance of the study of modular forms arises from the following: The space of modular forms has a finite dimension , is computable, and can help solve several problems in many areas of mathematics.

1.1 The Full Modular Group Γ

Define $\mathbb{H}$ to be the *upper half plane* containing all complex numbers with positive imaginary part. Define the *full modular group* to be the set of all 2 x 2 matrices with integral entries and having one as a determinant. We denote this group by Γ. Γ acts on $\mathbb{H}$ by Mobius transformations in the following way:

Let $z \in H$

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

$$\gamma(z) = \frac{az+b}{cz+d}$$

Notice that $-\gamma z = \gamma z$. Thus, we identify each matrix with its negative. Under the group action, the image remains in $\mathbb{H}$ due to the fact that

$$Im(\gamma z) = \frac{Im(z)}{|cz+d|^2}$$

A modular function f is a complex-valued function from $\mathbb{H}$ to $\mathbb{C}$ satisfying:

$f(\gamma z) = f(z)$ for all $z \in \mathbb{H}$ and for all $\gamma \in \Gamma$.

A more interesting form would be modular forms, which we limit here to entire holomorphic functions from $\mathbb{H}$ to $\mathbb{C}$ satisfying the transformation law

$$f(\gamma z) = (cz+d)^k f(z)$$

for all $z \in \mathbb{H}$ and for all $\gamma \in \Gamma$ where k is the weight of the modular form.

An entire function is a complex-valued function holomorphic over $\mathbb{C}$.

Define $M_k(\Gamma)$ as the space of entire modular forms of weight k on Γ.

Consider

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma.$$

Then

$$f(z+1) = f(Tz) = (0.z+1)^k f(z) = f(z)$$

f is periodic of period 1,and can be written as

$$\sum_{n=0}^{\infty} a(n) e^{2\pi i n z}$$

Note that holomorphic modular forms of odd weights on the full modular group are the trivial ones.

$proof$:

$$f(Iz) = (1)^k f(z)$$

$$f(-Iz) = (-1)^k f(z)$$

$$f(Iz) = f(-Iz) = f(z)$$

Adding the first two equations we get $2f(z) = 0$.

Thus we consider modular forms of even weights in this paper.

1.2 The Full Modular Group and its Fundamental Region

The full modular group is generated by two matrices:

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$f(Tz) = f(z+1) = 1^k f(z) = f(z)$$

$$f(Sz) = f(-1/z) = (z^k)f(z)$$

We identify $\gamma \in \Gamma$ with $-\gamma$. forming the group $PSL_2(\mathbb{Z}) = \Gamma/\{\pm I\}$.

We conclude that a modular form of weight k as a periodic function of period 1 and satisfying $f(-1/z) = z^k f(z)$.

Definition 1.2.1. Two points z_1, z_2 in $\mathbb{H}$ are equivalent under the full modular group Γ if there exists $\gamma \in \Gamma$ such that $z_1 = \gamma z_2$.

Definition 1.2.2. A *fundamental domain*, denoted by F_Γ,is an open subset of the upper half plane such that no two distinct points of it are equivalent under Γ and every point in $\mathbb{H}$ is equivalent to some point in the closure of F_Γ.

Theorem 1.2.3. $F_1 = \{z \in H||z| > 1.|Re(z)| < 1/2\}$ *is a fundamental domain for the full modular group* Γ.

Proof. We first prove that any point in $\mathbb{H}$ is equivalent to some point in the closure of F_1 :

Let $z \in H$ and consider the lattice L= $\{gz + p|\ g, p \in \mathbb{Z}\}$. Since $z \in L$ for $g = 1$ and $p = 0$, L is not empty, and we can choose $z_1 \in$ L such that $z_1 \neq 0$ and having a minimal modulus. $z_1 = cz + d, c, d \in \mathbb{Z}$ with (c,d)=1. (If not, we choose $z_2 = \frac{cz}{(c,d)} + \frac{d}{(c,d)}$ with $|z_2| < |z_1|$)

Since (c,d)=1, then by *Bezout's Lemma* , there exists

$a, b \in \mathbb{Z}\ \ such\ \ that\ \ ac - bd = 1.$

Hence, we can consider

$$\gamma_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

We have

$$Im(\gamma_1 z) = \frac{Im(z)}{|cz+d|^2}$$

Since $|cz+d|$ is minimal, $Im(\gamma_1 z)$ is maximal in the set $\{Im(\gamma z), \gamma \in \Gamma\}$

Let $z_n = T^n \gamma_1 z = \gamma_1 z + n$ be such that $|Re(z_n)| \leq 1/2, n \in \mathbb{Z}$. Notice that $|z_n| \geq 1$ because if $|z_n| < 1, Im(Sz_n) = \frac{Im(z_n)}{|z_n|^2} > Im(z_n) = Im(\gamma_1 z)$ contradicting the fact that $Im(\gamma_z)$ is maximal. Also, $T^n\gamma_1 \in \Gamma$. Hence, z being an arbitrary point in $\mathbb{H}$ is equivalent to a point z_n belonging to the closure of F_1.

Next, we prove that no distinct points of F_1 are equivalent under Γ.

Suppose $z_1, z_2 \in F_1$ such that $z_2 = \gamma z_1$ and $\gamma \neq \pm I \in \Gamma$,

$$\gamma = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

For $z \in F_1$, $|z| > 1$ and thus $Im(z)^2 > 1 - Re(z)^2 = 3/4$ Now $Im(z) > 0$ and thus $Im(z) > \frac{\sqrt{3}}{2}$

Then, $\frac{\sqrt{3}}{2} < Im(z_2) = \frac{Im(z_1)}{|gz_1+h|^2} \leq \frac{Im(z_1)}{Im(gz_1+h)^2} = \frac{Im(z_1)}{Im(gz_1)^2} = \frac{1}{g^2 Im(z_1)} < \frac{2}{g^2\sqrt{3}}$ We get $3g^2 < 4$. Now g is an integer and thus is either a 0,1,or -1. If g is a zero,γ would be either $\pm I$ or T^n and the latter is not possible since z_2 would have its real part greater than 1/2. The only left case is $g = \pm 1$. Now assume that $Im(z_1) \leq Im(z_2)$, $Im(z_2) = \frac{Im(z_1)}{|\pm z_1+h|^2} \leq \frac{Im(z_1)}{|z_1|^2} < Im(z_1)$.

A contradiction arises. Thus no two distinct points of F_1 are equivalent under Γ.

This proves the claim. $\square$

1.3 Entire Modular Forms

A modular form of weight k is not a well-defined function on $\Gamma\backslash\mathbb{H}$

Proof Let z_1 , $z_2 \in H$. Suppose $z_1 \sim z_2$, then there exists

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

such that $z_1 = \gamma z_2$. $\gamma \neq I$

Now $f(z_1) = (cz_2 + d)^k f(z_2) \neq f(z_2)$ for $\gamma \neq T^b$.

Definition 1.3.1. Let $p \in \Gamma/H$,

$multi_p(f)$ is the local multiplicity.

$multi_\infty(f)$ is defined as the smallest integer n such that $a(n) \neq 0$ in the Fourier expansion $\sum_{n=1}^{\infty} a(n)e^{2\pi inz}$.Define the stabilizer of $z \in \mathbb{H}$ as the set of elements of the full modular group that fix z. n_p is the order of the stabilizer in the closure of the full modular group.

Note that the boundary points F_Γ are Γ-equivalent: If $z_1 = \pm\frac{1}{2} + y_1$ and $z_2 = \mp\frac{1}{2} + y_2$. $z_1 = Tz_2$. Now if $|z_3| = |z_4|$=1 and z_3 and z_4 belong to the left and right halves of the arc $|z_1|$=1, then $z_3 = Sz_4$. These are the only equivalences we have for the boundary points, and the points of F_Γ are not equivalent from the definition of a fundamental domain. That is why we define F_1 the semiclosure of F_Γ containing the boundary points with non-positive real parts. Thus, every point in the upper half plane has its unique Γ-equivalent point of F_1 .

Proposition 1.3.2. *Let f be a non-zero modular form of weight k on Γ .*

$$\sum_{\Gamma\backslash\mathbb{H}} \frac{1}{n_p} multi_p(f) + multi_\infty(f) = \frac{k}{12}$$

.

Proof. Since every point in H is equivalent to a unique point in F_1, and since the stabilizer of any point in F_Γ is trivial, and the stabilizers of $\omega = e^{2\pi i/3}$ and i are the cyclic groups generated by ST and S respectively, n_p is 1,2, or 3. The only cusp of Γ/H is ∞ and we add it to compactify the moduli space.

For $p \in H$, there exists a unique $z_1 \in \overline{F_\Gamma}$ and $\gamma_1 \in \overline{\Gamma}$ such that $p = \gamma_1 z_1$. Let γ be a stabilizer of $p.(\gamma p = p)$

Recall the *Argument Principle*: Let $f(z)$ be analytic inside and on a simple closed curve inside C except for a finite number of poles inside C. Then

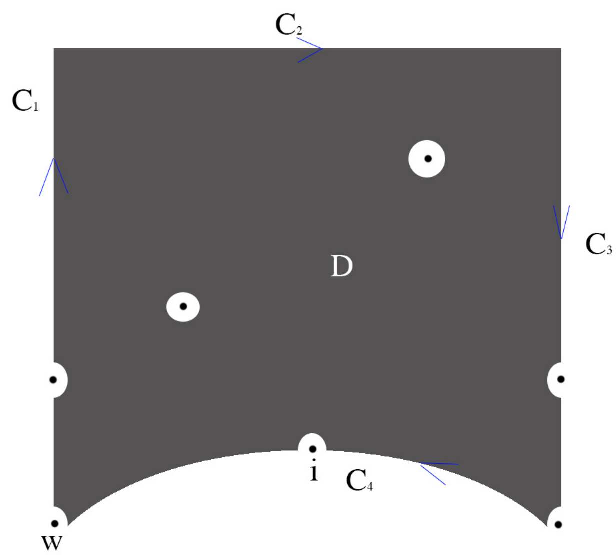
$$\int_C \frac{f'(z)}{f(z)} dz = 2\pi i(N - P) \tag{1.3.1}$$

where N is the number of zeroes of $f(z)$ inside C ,and P is the number of poles of $f(z)$ inside C. Delete ϵ-neighborhoods of all zeros of f and the neighborhood of infinity. Choose ϵ small enough so that the neighborhoods do not overlap. Let C be the closed set from F_1 with the deleted neighborhoods. Since f has no zeroes nor poles in the selected region, then by the argument principle:

$$\int_{\partial C} \frac{f'(z)}{f(z)} dz = 0$$

.

$$\int_{C_1} \frac{f'(z)}{f(z)} dz + \int_{C_2} \frac{f'(z)}{f(z)} dz + \int_{C_3} \frac{f'(z)}{f(z)} dz + \int_{C_4} \frac{f'(z)}{f(z)} dz + \int_{C_5} \frac{f'(z)}{f(z)} dz + \int_{C_6} \frac{f'(z)}{f(z)} dz = 0$$



(as in the figure)

Consider C_1 and C_3

Over C_1, $z_1 = -\frac{1}{2} + iY$, and over C_3, $z_2 = \frac{1}{2} + iY = z_1 + 1$, and recalling that f is periodic of period 1, we have:

$$\int_{C_1} \frac{f'(z_1)}{f(z_1)} dz_1 + \int_{C_3} \frac{f'(z_2)}{f(z_2)} dz = i \int_{\frac{\sqrt{3}}{2}}^{Y} \frac{f'(\frac{-1}{2} + iy)}{f(\frac{-1}{2} + iy)} dy + i \int_{Y}^{\frac{\sqrt{3}}{2}} \frac{f'(\frac{1}{2} + iy)}{f(\frac{1}{2} + iy)} dy = 0$$

Now,

$$\int_{C_2} \frac{f'(z)}{f(z)} dz = 2\pi i \; mult_{\infty}(f).$$

C_5 being the curve surrounding a zero inside the fundamental domain

Over C_5, if a is a zero of $f(z)$ of order p then where $F(z)$ is analytic inside C_5 and $F(a) \neq 0$ and

$$f'(z) = p(z-a)^{(p-1)}F(z) + (z-a)^p F'(z)$$

so the integral on the boundary is

$$\int_{C_5} \frac{f'(z)}{f(z)} dz = 2\pi i p.$$

In the case where $n_p = 1$:

$$\int_{C_5} \frac{f'(z)}{f(z)} dz = 2\pi i \; multi_p(f)$$

For $n_p = 2$ or 3:

$$\int_{C_5} \frac{f'(z)}{f(z)} dz = \frac{2\pi i \; multi_p(f)}{n_p}$$

What happens at ω and i is we integrate over third and half the circle respectively, so we divide by 2 or 3 respectively.

Now for C_4, divide C_4 into C_4^1 and C_4^2 with C_4^1 being the curve joining $w+1$ to i and C_4^2 joining i to w. Notice that $S(w+1) = w$ and $S^{-1} = S$. Also $Si = i$.

$$\int_{C_4} = \int_{C_4^1} + \int_{C_4^2} = \int_{w+1}^{i} \frac{f'(z)}{f(z)} dz + \int_{i}^{w} \frac{f'(z)}{f(z)} dz.$$

But $f(Sz) = z^k f(z)$, so

$$f'(Sz)(Sz)' = kz^{k-1} f(z) + z^k f'(z)$$

and thus

$$\frac{f'(Sz)(Sz)'}{f(Sz)} = \frac{k}{z} + \frac{f'(z)}{f(z)}$$

We have

$$\log f(Sz) = k \log z + \log f(z)$$

. Thus,

$$\log f(z) = \log f(Sz) - k \log z$$

$$d \log f(z) = d \log f(Sz) - k \frac{dz}{z}$$

. As a result,

$$\begin{aligned}
&\int_{w+1}^{i} d \log f(z) + \int_{i}^{w} d \log f(z) \\
&= \int_{w+1}^{i} d \log f(Sz) - k\frac{dz}{z} + \int_{i}^{w} d \log f(z) \\
&= \int_{Sw+1}^{Si} d \log f(z) - k\frac{dz}{z} + \int_{i}^{w} d \log f(z) \\
&= \int_{w}^{i} d \log f(z) + \int_{i}^{w} d \log f(z) - \int_{i}^{w} k\frac{dz}{z} \\
&= 0 - k(log(w) - log(i)) \\
&= -k(i\frac{2\pi}{3} - i\frac{\pi}{2}) = -\frac{k\pi i}{6}
\end{aligned}$$

(the path of the third integral does not pass through zero and thus the function is the derivative of the analytic logarithmic function so the fundamental theorem of calculus applies)

Adding up all the above over the specified paths as in the figure we get:

$$\sum_{p \in \Gamma/H} 2\pi i \frac{1}{n_p} multi_p(f) + 2\pi i \; multi_\infty(f) - \frac{k\pi i}{6} = 0$$

Dividing the equation by $2\pi i$ we get

$$\sum_{p \in \Gamma/H} \frac{1}{n_p} multi_p(f) + multi_\infty(f) = \frac{k}{12}.$$

Notice that since $p \in \Gamma/H$ implies that there exists a $p_1 \in \overline{F_1}$ such that $p \sim p_1$, so we can choose $p_1 \in \overline{F_1}$ as a representative of the equivalence class $\square$

We denote the space of entire modular forms of weight k on Γ by $M_k(\Gamma)$ and the space of cusp forms of the same weight on Γ by $S_k(\Gamma)$.

1.4 Eisenstein Series and the Discriminant Function

Define

$$(f|_k\gamma)(z) = (cz+d)^{-k} f(\gamma z)$$

where $z \in \mathbb{C}, \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, and $a, b, c, d \in \mathbb{Z}$.

The map $f \to f|_k\gamma$ defines an operation of the group Γ on the vector space of holomorphic functions. Define Γ_∞ to be the stabilizer of the cusp at infinity

A matrix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ maps ∞ to $\frac{a}{c}$. For γ to map ∞ to ∞, c must be 0.

Thus $\gamma = \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, n \in \mathbb{Z}$. So $\gamma = \pm T^n$. In $PSL_2(\mathbb{Z})$, T and $-T$ are identified, and we deduce that $\Gamma_\infty =< T >$ is the infinite cyclic group generated by T.

Now, let

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$$

$$\gamma' = T^n\gamma = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+nc & b+nd \\ c & d \end{pmatrix}$$

has the same bottom row as γ.

Also, and knowing that $(T^n)^{-1} = T^{-n}$, if $\gamma' = \begin{pmatrix} a' & b' \\ c & d \end{pmatrix} \in \Gamma$. then

$$T^{-n}\gamma' = \begin{pmatrix} 1 & -n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a' & b' \\ c & d \end{pmatrix} = \begin{pmatrix} a'-nc & b'-nd \\ c & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \gamma.$$

and $ad - bc = 1$, which makes $(c, d) = 1$.

The Eisenstein Series of weight k is defined as

$$\sum_{\Gamma_\infty/\Gamma} 1|_k\gamma$$

Also, another way to express an Eisenstein Series is

$$E_k(z) = \frac{1}{2} \sum_{(c,d)=1, c,d\in\mathbb{Z}} (cz+d)^{-k}$$

$\frac{1}{2}$ comes from the fact that elements having(c,d) and (-c,-d) are identified

This sum is absolutely convergent for k>2, and for all $z \in H$, as well as uniformly

convergent on all compact subsets of $\mathbb{H}$.

Recall the Riemann-Zeta function

$$\zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k}$$

converges for Re(k)>1

Another way to define Eisenstein series would be

$$G_k(z) = \frac{1}{2} \sum_{m,n\in\mathbb{Z},(m,n)\neq(0,0)} \frac{1}{(mz+n)^k}$$

$G_k(z)$ is holomorphic in $\mathbb{H}$ since $z \in \mathbb{H}$ and thus $z \neq \frac{-n}{m}$.

Let $D = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. We have

$$\begin{aligned} G_k(Dz) &= \frac{1}{2}\sum_{(m,n)\in\mathbb{Z}^2}' (m\frac{az+b}{cz+d}+n)^{-k} \\ &= (cz+d)^k \frac{1}{2}\sum_{(m,n)\in\mathbb{Z}^2}' ((ma+nc)z+bm+nd)^{-k} \end{aligned}$$

$(a,c) = 1$ and $(b,d) = 1$. Now $ma + nc$ runs over all integers as n and m do.Same thing goes to $bm + nd$. To show it, let $m' = ma + nc$ and $n' = bm + nd$.Clearly, $m' \in \mathbb{Z}$ and $n' \in \mathbb{Z}$, Let $p \in \mathbb{Z}$. Since $(a,c) = 1$, then by Bezout's lemma, there exists $q \in \mathbb{Z}$ and $r \in \mathbb{Z}$ such that $qa + rc = 1$. Multiply both sides by p to get $pqa + prc = p$. Now, $pq \in \mathbb{Z}$ and $pr \in \mathbb{Z}$ and thus, $p = ma + nc$ where $m = pq$ and $n = pr$. Similar approach yields $bm + nd$ running over all integers.

Thus

$$G_k(Dz) = (cz+d)^k \frac{1}{2}\sum_{(m',n')\in\mathbb{Z}^2}' (m'z+n')^{-k} = (cz+d)^k G_k(z)$$

G_k is an entire modular form for $k \geq 4$. Both definitions relate in the sense that $G_k(z) = \zeta(k)E_k(z)$. This comes from the fact that any pair of integers can be expressed the product of their gcd by a pair of coprimes.

Proof: Let (m,n)=q(a,c) with q=gcd(m,n) and suppose a and c are not coprimes, then there is a divisor p for a and c, and thus q is not the greatest common divisor.)

$$G_k(z) = \frac{1}{2} \sum_{(m,n)\in\mathbb{Z}^2)}' (mz+n)^{-k} = \frac{1}{2} \sum_{(c,d)=1\in\mathbb{Z}^2}' (qcz+qd)^{-k} = \zeta(k)E_k(z)$$

1.5 Eisenstein Series and their Expansions

Define

$$\mathbb{G}_k(z) = \frac{(k-1)!}{(2\pi i)^k} G_k(z)$$

Proposition 1.5.1. *For $k > 2$,*

$$\mathbb{G}_k(z) = -\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

where B_k is the kth Bernoulli number and $\sigma_{k-1}(n)$ is the sum of the $k-1$ powers of the positive divisors of n and $q = e^{2\pi i z}$.

Proof. Euler identity states that for $z \in \mathbb{C}/\mathbb{Z}$

$$\frac{1}{z} + \sum_{m=1}^{\infty} (\frac{1}{z+m} + \frac{1}{z-m}) = \frac{\pi}{tan(\pi z)} \tag{1.5.1}$$

$\frac{\pi}{tan(\pi z)}$ is periodic of period 1 and its Fourier expansion is given by:

$\frac{\pi}{tan(\pi z)} = \pi \frac{cos(\pi z)}{sin(\pi z} = \pi i \frac{e^{\pi i z} + e^{-\pi i z}}{e^{\pi i z} - e^{-\pi i z}} = -\pi i \frac{1+q}{1-q} = -\pi i (\frac{1}{1-q} + q\frac{1}{1-q} = -2\pi i (\frac{1}{2} + \sum_{r=1}^{\infty} q^r)$

Differentiate (1.5.1) $k-1$ times to get:

$$\sum_{n\in\mathbb{Z}} \frac{1}{(z+n)^k} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{r=1}^{\infty} r^{k-1} q^r \quad (k \geq 2, z \in H) \tag{1.5.2}$$

$$
\begin{aligned}
G_k(z) &= \frac{1}{2} \sum_{(m,n)\in\mathbb{Z}^2}' \frac{1}{(mz+n)^k} \\
&= \frac{1}{2}[\sum_{n\neq 0} \frac{1}{n^k} + \sum' m \neq 0, (m,n) \in \mathbb{Z}^2 \frac{1}{(mz+n)^k}] \\
&= \sum_{n=1}^{\infty} \frac{1}{n^k} + \sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{(mz+n)^k} \\
&= \zeta(k) + \sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{(mz+n)^k}
\end{aligned}
$$

Using (1.5.2) and letting $z' = mz$

$$
G_k(z) = \zeta(k) + \sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{(z'+n)^k} = \zeta(k) + \sum_{m=1}^{\infty} \frac{(2\pi i)^k}{(k-1)!} \sum_{r=1}^{\infty} r^{k-1} e^{2\pi i r z'}
$$

$$
G_k(z) = \zeta(k) + \frac{(2\pi i)^k}{(k-1)!} \sum_{m=1}^{\infty} \sum_{r=1}^{\infty} r^{k-1} q^{mr}
$$

Let $l = mr$, and so m divides l, and since m and r vary from 1 to ∞, so does l and $\frac{l}{m} = r$.

$$
\begin{aligned}
G_k(z) &= \zeta(k) + \frac{(2\pi i)^k}{(k-1)!} \sum_{l=1}^{\infty} \sum_{m|l} ((\frac{l}{m})^{k-1} q^l \\
&= \zeta(k) + \frac{(2\pi i)^k}{(k-1)!} \sum_{l=1}^{\infty} \sigma_{k-1}(l) q^l \\
&= \frac{(2\pi i)^k}{(k_1)!} (-\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n) \\
\mathbb{G}_k(z) &= -\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n
\end{aligned}
$$

$\square$

1.5.1 Eisenstein Series of Weight 2

Since the Fourier expansion of $\mathbb{G}_k(z)$ defines a function which converges and is holomorphic even for k=2, we define $\mathbb{G}_2$, G_2, and E_2

$$\mathbb{G}_2(z) = \frac{-1}{24} + \sum_{n=1}^{\infty} \sigma_1(n) q^n$$

$$E_2(z) = \frac{6}{\pi^2} G_2(z)$$

$$G_2(z) = \frac{1}{2}\sum_{n\neq 0} \frac{1}{n^2} + \frac{1}{2}\sum_{m\neq 0}\sum_{n\in\mathbb{Z}} \frac{1}{(mz+n)^2}$$

Theorem 1.5.2.

The Eisenstein series is not a modular form.However, it is well defined on $\mathbb{H}$ and satisfies

$$G_2(z) = \frac{1}{2}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} \frac{1}{(mz+n)^2}$$

which converges conditionally.

Also,

$$G_2(z) = \frac{\pi i}{z} + \frac{1}{2}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \frac{1}{(mz+n)^2}$$

$$G_2(Sz) = z^2 G_2(z) - \pi i z$$

Proof. We recall the following: For $z \in \mathbb{C}/\mathcal{R}$ we have

$$\pi cot(\pi z) = \frac{1}{z} + \sum_{m=1}^{\infty}(\frac{1}{z+m} + \frac{1}{z-m})$$

For $d \in \mathcal{R}$

$$\lim_{d\to\infty}(\pi cot(\pi dz) - \pi cot(-\pi dz)) = \lim_{d\to\infty} 2\pi cot(\pi dz) = -2\pi i$$

Note that

$$\sigma_1(n) = \sum_{d|n} d < \sum_{d=1}^{n} d = n(\frac{n+1}{2}) \le (n+1)^2$$

$$|G_2(z)| \le \pi^2|B_2| + 4\pi^2 \sum_{n=1}^{\infty} |\sigma_1(n)||e^{2\pi inz}| < \infty$$

$$\lim_{n\to\infty} (n+1)^{\frac{2}{n}} e^{-2\pi Imz} < 1$$

Thus G_2 converges absolutely on $\mathbb{H}$.(By the root test)

We need to show the conditional convergence of $G_2(z)$, and the approach is to prove that changing the order of the summands of m and n yields different results.

Define the auxiliary function

$$Ga(z) = \frac{1}{2} \sum_{c=-\infty}^{\infty} \sum_{d=-\infty}^{\infty} \frac{1}{(cz+d)(cz+(d-1))} \quad ((c,d) \ne (0,0) \ and \ (0,1))$$

If $c \ne 0$,

$$\begin{aligned}
\sum_{d=-\infty}^{\infty} \frac{1}{(cz+d)(cz+(d-1))} &= \sum_{d=-\infty}^{\infty} \frac{1}{(cz+(d-1))} - \frac{1}{cz+d} \\
&= \sum_{d=0}^{\infty}(\frac{1}{(cz+(d-1))} - \frac{1}{cz+d}) + \sum_{d=-\infty}^{-1}(\frac{1}{(cz+(d-1))} - \frac{1}{cz+d}) \\
&= lim_{d\to\infty} \frac{1}{(cz+(d-1))} - \frac{1}{cz+d} + lim_{d\to-\infty} \frac{1}{(cz+(d-1))} - \frac{1}{cz+d} \\
&= 0
\end{aligned}$$

If $c = 0$ and $d \ne 0, d \ne 1$,

$$\sum_{d=-\infty}^{\infty} \frac{1}{d(d-1)} = 2\sum_{d=2}^{\infty} \frac{1}{d(d-1)} = 2$$

Thus $G_a(z) = 1$

Define

$$Gs(z) = \frac{1}{2}\sum_{d=-\infty}^{\infty}\sum_{c=-\infty}^{\infty}\frac{1}{(cz+d)(cz+(d-1)}\quad ((c,d)\neq(0,0),(0,1))$$

$$\begin{aligned} z^{-2}G_s(\frac{-1}{z}) &= \frac{1}{2}z^{-2}\sum_{d=-\infty}^{\infty}\sum_{c=-\infty}^{\infty}\frac{1}{(c(\frac{-1}{z})+d)(c(\frac{-1}{z})+d-1)} \\ &= \frac{1}{2}.\frac{1}{z}\sum_{d=-\infty}^{\infty}(\frac{1}{(d-1)z-c}-\frac{1}{dz-c}) \end{aligned}$$

We split the series according to the values of d.

For $d \neq 0$ or 1

$$\sum_{c=-\infty}^{\infty}\frac{1}{dz-c} = \pi cot(\pi dz)$$

Hence

$$\begin{aligned} z^{-2}G_s(\frac{-1}{z}) &= \frac{1}{2z}\sum_{d\in\mathbb{Z}\setminus\{0,1\}}(\pi cot(\pi(d-1)z)-\pi cot(\pi dz)) \\ &= \frac{1}{2z}(\pi cot(\pi z)+\pi cot(-\pi z))+\frac{1}{2z}\lim_{d\to\infty}\pi cot(-\pi dz)-\frac{1}{2z}\lim_{d\to\infty}\pi cot(\pi dz) \\ &= \frac{1}{z}\pi cot(\pi z)+\frac{\pi i}{z} \end{aligned}$$

For d=0

$$\frac{1}{z}\sum_{c\in\mathbb{Z}^*}(\frac{1}{-z-c}+\frac{1}{c}) = \frac{1}{z}(\pi cot(-\pi z)+\frac{1}{z}) = -\frac{\pi cot(\pi z)}{z}+\frac{1}{z^2}.$$

For d=1,

$$\frac{1}{z}\sum_{c\in\mathbb{Z}^*}(\frac{-1}{c}-\frac{1}{z-c}) = \frac{1}{z^2}-\pi\frac{cot(\pi z)}{z}.$$

Thus

$$z^{-2}Gs(-\frac{1}{z}) = \frac{1}{z^2} + \frac{\pi i}{z}$$

We get

$$Gs(-\frac{1}{z}) = 1 + \pi i z$$

$$Gs(z) = 1 - \frac{\pi i}{z} = Ga(z) - \frac{\pi i}{z}$$

$$Ga(z) - G_2(z) = \frac{1}{2}\sum_{c=-\infty}^{\infty}\sum_{d=-\infty}^{\infty\#}\frac{1}{(cz+d)(cz+(d-1)} - \frac{1}{2}\sum_{c=-\infty}^{\infty}\sum_{d=-\infty}^{\infty'}\frac{1}{(cz+d)^2}$$

Substituting the values when c=0 and d=1 by -1 we get

$$Ga(z) - G_2(z) = -\frac{1}{2} + \frac{1}{2}\sum_{c=-\infty}^{\infty}\sum_{d=-\infty}^{\infty\#}\frac{1}{(cz+d)^2(cz+d-1)} \tag{4}$$

Define

$$G_{se}(z) = \frac{1}{2}\sum_{d=-\infty}^{\infty}\sum_{c=-\infty}^{\infty}\frac{1}{(cz+d)^2(cz+d-1)}$$

Using same concept as in (4) but interchanging the order of summands, we get

$$Gs(z) - G_{se}(z) = -\frac{1}{2} + \frac{1}{2}\sum_{d=-\infty}^{\infty}\sum_{c=-\infty}^{\infty}\frac{1}{(cz+d)^2(cz+d-1)}$$

$$= Ga(z) - G_2(z)$$

(Here we can interchange the order since the series $\sum_{d=-\infty}^{\infty}\sum_{c=-\infty}^{\infty}\frac{1}{(cz+d)^2(cz+d-1)}$

coverges absolutely)

$G_2(z) - G_{se}(z) = \frac{\pi i}{z} \neq 0$

This proves the conditional convergence.

$$z^{-2}G_2(-\frac{1}{z}) = \frac{1}{2}\sum_{m=-\infty}^{\infty}\frac{1}{(-\frac{-m}{z}+n)^2} = \frac{1}{2}\sum_m\sum_n\frac{1}{(nz-m)^2}$$
$$= \frac{1}{2}\sum_m\sum_n\frac{1}{(mz+n)^2} = G_{se}(z) = G_2(z) - \frac{\pi i}{z}.$$

Thus,

$$G_2(Sz) = z^2G_2(z) - \pi iz$$

□

1.6 The Discriminant Function

Define

$$\Delta(z) = e^{2\pi iz}\prod_{n=1}^{\infty}(1-e^{2\pi inz})^{24}$$

for $z \in H$

The function converges $\forall z \in H$ and is holomorphic.

Theorem 1.6.1. *$\Delta(z)$ is a modular form of weight 12 on the full modular group.*

Proof. We first note that

$$\lim_{z\to i\infty}\Delta(z) = 0.$$

Also, $\Delta(z) \neq 0$ ($z \in H$ and $e^{2\pi nz} \neq 1$)

To prove $\Delta(Mz) = (cz+d)^{12}\Delta(z)$ where M=$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\in \Gamma$, it suffices to prove it for M=T or M=S as these are the only generators of Γ.

$$\Delta(Tz) = \Delta(z+1) = e^{2\pi i(z+1)}\prod_{n=1}^{\infty}(1-e^{2\pi in(z+1)})^{24} = e^{2\pi iz}\prod_{n=1}^{\infty}(1-e^{2\pi inz})^{24} = \Delta(z)$$

Since the function is non-zero, we can speak of logarithm, which would help turn the product into a sum and that should be easier to show that $\Delta(Sz) = z^{12}\Delta(z)$

$$\log(\Delta z) = 2\pi iz + 24\sum_{n=1}^{\infty} log(1 - e^{2\pi inz})$$

Let $x = e^{2\pi inz}$ and $|x| < 1$. We have:

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k - \log(1-x) = \sum_{k=1}^{\infty} \frac{x^k}{k}$$

$$\frac{\Delta'(z)}{\Delta(z)} = 2\pi i - 24(2\pi i)\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} ne^{2\pi inkz}$$

Let $l = nk$

$$\frac{\Delta'(z)}{\Delta(z)} = 2\pi i - 24(2\pi i)\sum_{l=1}^{\infty} \sigma_1(l)e^{2\pi ilz} = 2\pi i(-24\mathbb{G}_2(z)) = \frac{12i}{\pi}G_2(z)$$

As a result,

$$\frac{\Delta'(\frac{-1}{z})}{\Delta(\frac{-1}{z})} = \frac{12i}{\pi}G_2(\frac{-1}{z}) = \frac{12i}{\pi}(z^2G_2(z) - \pi iz) = \frac{\Delta'(z)}{\Delta(z)} + \frac{12}{z}$$

Integrating both sides we get:

$$\log(\Delta(Tz)) = \log(\Delta(z)) + 12\log(z) + c = \log(\Delta(z)z^{12}e^c)$$

We get,

$$\Delta(Tz) = \Delta(z)z^{12}e^c \quad \forall z \in H$$

We evaluate it at $z = i$ and get,

$$\Delta(i) = \Delta(i)i^{12}e^c.$$

We then have that $c = 0$ and hence

$$\Delta(Tz) = \Delta(z)z^{12}$$

$\square$

1.7 Dimensions of Space of Modular Forms

Definition 1.7.1. A **cusp form** is an entire modular form with a zero constant term in its Fourier expansion

We denote the space of cusp forms of weight k on the full modular group by $S_k(\Gamma)$.

Theorem 1.7.2. *Assume $S_k(\Gamma)$ is finite dimensional. For k even and $k \geq 4$, we have dim$M_k(\Gamma)$=1+dimS_k*

Proof. Let $V = \{v_1, v_2, ..., v_m\}$ be a basis for $S_k(\Gamma)$. We show that $V' = \{v_1, v_2, ..., v_m, G_k\}$ is a basis for $M_k(\Gamma)$

Let

$$a_1v_1 + a_2v_2 + a_3v_3 + ... + a_mv_m + \alpha G_k = 0$$

where $a_1, a_2, ...\alpha \in \mathbb{Z}$

$$a_1v_1 + a_2v_2 + a_3v_3 + ... + a_mv_m = -\alpha G_k$$

Letting $z \to i\infty$, we get

$$\alpha \frac{B_k}{2k} \frac{(2\pi i)^k}{(k-1)!} = 0$$

So $\alpha = 0$ and the fact that V is a basis makes all $a_i's$ zeroes ($v_i's$ are linearly independent),and hence we proved linear independence

We now prove M_k is spanned by V'

Let $f \in M_k(\Gamma)$, we have

$$f(z) = a(0) + \sum_{n=1}^{\infty} a(n)e^{2\pi inz}$$

$$f(z) - a(0)\frac{2k(k-1)!}{B_k(2\pi i)^k}G_k(z) \quad \in S_k(\Gamma)$$

$$f(z) - a(0)\frac{2k(k-1)!}{B_k(2\pi i)^k}G_k(z) = b_1v_1 + b_2v_2 + ... + b_mv_m$$

$$f(z) = a(0)\frac{2k(k-1)!}{B_k(2\pi i)^k}G_k(z)b_1v_1 + b_2v_2 + ... + b_mv_m$$

and thus V' spans $M_k(\Gamma)$. $\square$

Proposition 1.7.3. *$dim(S_k(\Gamma)) = 0$ for $k \in \{2,4,6,8,10\}$. $dimS_{12}(\Gamma) = 1$*

Proof. Consider $\frac{f(z)}{\Delta(z)}$.

$$\frac{f(Mz)}{\Delta(Mz)} = (cz+d)^{k-12}\frac{f(z)}{\Delta(z)}$$

and

$$\Delta(z) \neq 0 \quad \forall z \in H$$

Thus $\frac{f(z)}{\Delta(z)}$ is holomorphic in $\mathbb{H}$ and $\frac{f(z)}{\Delta(z)} \in M_{k-12}(\Gamma)$.

For $k = 2,4,6,8,10,\quad \frac{f(z)}{\Delta(z)}$ has a negative weight and thus $f(z) = 0$.

For $k = 12$, $\frac{f(z)}{\Delta(z)}$ is an entire modular form of weight 0 and thus is constant.

$$\frac{f(z)}{\Delta(z)} = c$$

So

$$f(z) = c\Delta(z)$$

and dim$S_{12}(\Gamma)$=1. $\square$

Definition 1.7.4. Let $F : H \to \mathbb{C}$ be a meromorphic function on $\mathbb{H}$ and $i\infty$. $F(z)$ is called an abelian integral if there exists $P \in \mathbb{C}$ such that

$$F(Mz) = F(z) + P_M$$

for $M \in \Gamma$

Definition 1.7.5. Let $f(z)$ be holomorphic in H, meromorphic at the cusp $i\infty$ and satisfying

$$f(Mz) = (cz+d)^k f(z) \quad \forall M \in \Gamma$$

then $f(z)$ is called weakly holomorphic.

Theorem 1.7.6. *Let f be weakly holomorphic of weight 2 and $f(z) = \sum_{n=-m}^{\infty} a(n)e^{2\pi i n z}$ $(m \in \mathbb{N})$. Then its antiderivative F is an abelian integral.*

Proof. To show F is an abelian integral, we show that $F(Mz) = F(z) + P_M$ for $M \in \Gamma$, through deriving $F(z)$ and making use of the fact that it is the antiderivative of $f(z)$.

$$\frac{d(F(z))}{dz} = f(z)$$

$$\begin{aligned}\frac{d(F(Mz))}{dz} &= \frac{d(F(Mz))}{dMz}\frac{dMz}{dz} \\ &= F'(Mz).(cz+d)^{-2} \\ &= f(Mz)(cz+d)^{-2} \\ &= f(z)\end{aligned}$$

$$\frac{d(F(Mz))}{dz} - \frac{d(F(z))}{dz} = 0$$

Thus

$$F(Mz) = F(z) + c, \quad c \in \mathbb{C}$$

$\square$

Proposition 1.7.7. *$dimM_2 = 0$*

Proof. Let $f \in M_2(\Gamma)$ and let F be its antiderivative. Then we have:

$$F(Sz) = F(z) + c_S = F(SSz) + c_S = F(Sz) + c_S$$

$$F(z) = F(SSz) = F(Sz) + c_S = F(z) + 2c_S$$

so $c_s = 0$

$$(TS)^3 = -I$$

and similarly we get

$$3c_{TS} = 0$$

so

$$c_{TS} = 0$$

$$F(Tz) = F(TSSz) = F(Sz) = F(z)$$

Thus F is periodic of period 1, and that yields $a(0)$ to be 0.Thus, f is a cusp form of weight 2, so $f = 0$ $\square$

Corollary 1.7.8.

$dim\ M_k(\Gamma) = 0$ for $k \leq 0$ or k odd.

$$dimM_k(\Gamma) = \begin{cases} [\frac{k}{12}] + 1 \quad if \quad k \neq 2\ mod 12 \\ \\ [\frac{k}{12}] \quad otherwise \end{cases}$$

Proof. Let $f \in M_k(\Gamma)$

- The case where k is odd: We have

$$f(Iz) = f(-Iz) = f(z)$$

Now by the transformation equations,

$$f(Iz) = 1^k f(z)$$

and

$$f(-Iz) = (-1)^k f(z)$$

Adding both we reach $2f(z) = 0$ so $f = 0$.

The case where k is negative: It will be proved in Section 1.8 that $|a(n)| \leq cy^{-\frac{k}{2}}$, and letting y tend to 0, we get $f = 0$.

For $0 \leq k \leq 14$, the result follows from Propositions 1.7.3 and 1.7.7.

For $k > 14$, we prove it by Induction.

Assume this holds for all weights less than k. Define the following map

$$H : S_k(\Gamma) \to M_{k-12}(\Gamma)$$

$H(f(z)) = \frac{f(z)}{\Delta(z)} \in M_{k-12}(\Gamma)$:

Let

$$p(z) = \frac{f(z)}{\Delta(z)}$$

$$p(Mz) = \frac{f(Mz)}{\Delta(Mz)} = \frac{(cz+d)^k f(z)}{(cz+d)^{12}\Delta(z)} = (cz+d)^{k-12} p(z)$$

Also,

$$\Delta(z) \neq 0$$

Thus,

$$H(f(z)) \in M_{k-12}(\Gamma)$$

H is a vector space isomorphism (To prove surjectivity, take $g \in M_{k-12}(\Gamma)$, then $g\Delta \in S_k(\Gamma)$ is the right preimage). To prove injectivity, note that

$$\text{KerH} = \{p(z) \in S_k(\Gamma)) : H(p(z)) = 0\} = 0$$

Therefore,

$$\dim S_k = \dim M_{k-12}$$

$$\dim M_{k-12}(\Gamma) = [\frac{k}{12}] \quad \text{if} \quad k \neq 2 \ mod 12$$

$$\text{else} \qquad \dim M_{k-12}(\Gamma) = [\frac{k}{12}] - 1$$

$$\dim\ M_k(\Gamma) = 1 + \dim\ S_k(\Gamma)$$

□

Proposition 1.7.9. *G_4 and G_6 generate the space of entire modular forms $M_k(\Gamma)$*

Proof. We know that

$$\dim\ \ M_k(\Gamma) = 0 \quad \text{if} \quad k < 0 \quad \text{and} \quad k = 2$$

and

$$\dim\ M_k(\Gamma) = 1 \quad \text{if} \quad k = 0$$

To prove that the space of entire modular forms is spanned by G_4 and G_6, notice that since k is even (for $k \geq 4$), any weight can be obtained from linear combinations of 6 and 4. ($k = 6m + 4n = 2(3m + 2n) = 2\mathbb{Z}$ and $3m + 2n = \mathbb{Z}$, Let $a \in \mathbb{Z}$: If a is even then $a = 2n$ $n \in \mathbb{Z}$. If a is odd, then

$a = 2n + 1 = 2(t + 1) + 1 = 2t + 3$, where $m = 1$)

Assume that G_4 and G_6 generate entire modular forms of weight less than k, and let $F \in M_k(\Gamma)$ and let a_0 be the constant terms of F in the Fourier expansion at $i\infty$

Define

$$h := F - a_0 c G_4^m G_6^n \qquad \in S_k$$

(c is the normalizing constant of the product of powers of G_4 and G_6)

Let

$$g = \frac{h}{\Delta} \quad \in M_{k-12}(\Gamma)$$

By induction, g is a linear combination of powers of G_4 and G_6. Also,

$$\Delta = cG_4^3 - dG_6^2$$

Thus $h = \Delta g$ is also linear combination of powers of G_4 and G_6 and so is $F = h + a_0 c G_4^m G_6^n$. $\square$

1.8 Bounds on Fourier Coefficients of Entire Forms

Theorem 1.8.1. *Let $f(z)$ be a cusp form of weight k on Γ with Fourier expansion $\sum_{n=1}^{\infty} a_n q^n$.. Then*

$$|f(z)| \leq cy^{-\frac{k}{2}}$$

There exists a constant c such that

$$|a_n| \leq cn^{\frac{k}{2}} \quad \textit{for all } n$$

Proof. Consider

$$g(z) = y^{\frac{k}{2}}|f(z)|$$

with $g(z) \geq 0$, $z \in H$.

Let M be the usual typical matrix in the full modular group and consider

$$\begin{aligned} g(Mz) &= (Im(Mz))^{\frac{k}{2}}|f(Mz)| \\ &= (\frac{Im(z)}{|cz+d|^2})^{\frac{k}{2}}|(cz+d)^k f(z)| \\ &= y^{\frac{k}{2}}|f(z)| = g(z) \end{aligned}$$

So g is Γ-invariant

$$\lim_{z\to i\infty} g(z) = \lim_{z\to i\infty} y^{\frac{k}{2}}|\sum_{n=1}^{\infty} a_n e^{2\pi i n z}| \leq \lim_{y\to\infty} y^{\frac{k}{2}} \sum_{n=1}^{\infty} |a_n| e^{-2\pi n y}| \to 0$$

Also g is clearly continuous in $\mathbb{H}$ and at the cusp so g is bounded in $\mathbb{H}$. That is

$g(z) \leq c$. So $|f(z)| \leq cy^{\frac{-k}{2}}$

Consider the integral

$$\begin{aligned}
\int_z^{z+1} f(t)e^{-2\pi int}dt &= \int_z^{z+1} \sum_{m=1}^{m=\infty} a(m)e^{2\pi imt}e^{-2\pi int}dt \\
&= \int_z^{z+1} \sum_{m=1}^{m=\infty} a(m)e^{2\pi i(m-n)t}dt \\
&= a(n)
\end{aligned}$$

Hence

$$\begin{aligned}
|a(n)| &\leq |\int_z^{z+1} f(t)e^{-2\pi int}dt| \\
&\leq \int_z^{z+1} |f(t)|e^{2\pi ny}|dt| \\
&\leq cy^{\frac{-k}{2}}e^{2\pi ny} \quad \text{for all} \quad y > 0
\end{aligned}$$

Take $y = \frac{1}{n}$ and we get

$$|a_n| \leq cn^{\frac{k}{2}}$$

$\square$

Corollary 1.8.2. *There are no non-zero entire modular forms of negative weights*

Proof.

$$|a(n)| \leq cy^{\frac{-k}{2}}e^{2\pi ny}$$

Letting y tend to 0, we reach $a(n) = 0 \quad \forall n \in \mathbb{Z}$ $\square$

Chapter 2

The Hecke Algebra and Eigenforms and the L-series

Definition 2.0.1.

$T_n f$ would be

$$T_n f(z) = n^{k-1} \sum_{d|n} d^{-k} \sum_{b=0}^{d-1} f(\frac{nz + bd}{d^2})$$

Theorem 2.0.2. *If $f \in M_k$,*

$$f(z) = \sum_{m=0}^{\infty} a(m) e^{2\pi i m z}$$

Then

$$T_n f(z) = \sum_{m=0}^{\infty} \gamma_n(m) e^{2\pi i m z}$$

where

$$\gamma_n(m) = \sum_{d|(m,n)} d^{k-1}a(\frac{mn}{d^2})$$

Proof. The significance of this theorem is it linking the fourier coefficients of $T_nf(z)$ to that of $f(z)$.

We recall the definition of $T_nf(z)$ and then we substitute f inside by its Fourier expansion as below:

$$\begin{aligned} T_nf(z) &= n^{k-1}\sum_{d|n} d^{-k}\sum_{b=0}^{d-1}\sum_{m=0}^{\infty} a(m)e^{2\pi im(\frac{nz+bd}{d^2})} \\ &= \sum_{m=0}^{\infty}\sum_{d|n}(\frac{n}{d})^{k-1}a(m)e^{\frac{2\pi imnz}{d^2}}\frac{1}{d}\sum_{b=0}^{d-1} e^{\frac{2\pi imb}{d}} \end{aligned}$$

Note that

if $d|m$

$$\sum_{b=0}^{d-1} e^{\frac{2\pi imb}{d}} = \sum_{b=0}^{d-1} 1 = d$$

if $d\nmid m$

$$\sum_{b=0}^{d-1} e^{\frac{2\pi imb}{d}} = 0$$

$$T_nf(z) = \sum_{m=0}^{\infty}\sum_{d|n,d|m}(\frac{n}{d})^{k-1}a(m)e^{\frac{2\pi imnz}{d^2}}$$

As $d|m$ then we can write $m = qd$ for some $q \in \mathbb{N}$, and we get

$$T_nf(z) = \sum_{q=0}^{\infty}\sum_{d|n}(\frac{n}{d})^{k-1}a(qd)e^{\frac{2\pi iqnz}{d}}$$

Substitute d by $\frac{n}{d}$ and then reintroduce $m = qd$

$$T_nf(z) = \sum_{q=0}^{\infty}\sum_{d|n} d^{k-1}a(\frac{qn}{d})e^{2\pi iqdz} = \sum_{m=0}^{\infty}\sum_{d|n,d|m} d^{k-1}a(\frac{mn}{d^2})e^{2\pi imz}$$

$\square$

We now introduce another definition for $T_nf(z)$ involving an element from the full modular group. It will hep us show that $T_nf(z)$ maps entire modular forms to entire modular forms:

Since $d|n$, $n = ad$, $a \in \mathbb{Z}$.

Consider $A=\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ *where* $detA = ad = n$.

$$\begin{aligned} T_nf(z) &= n^{k-1} \sum_{ad=n,a\geq 1} d^{-k} \sum_{b=0}^{d-1} f(\frac{adz+bd}{d^2}) \\ &= n^{k-1} \sum_{ad=n,a\geq 1} d^{-k} \sum_{b=0}^{d-1} f(\frac{az+b}{d}) \\ &= n^{k-1} \sum_{a\geq 1,ad=n,0\leq b<d} d^{-k} f(Az) \\ &= \frac{1}{n} \sum_{a\geq 1,ad=n,0\leq b<d} (\frac{n}{d})^k f(Az) \\ &= \sum_{a\geq 1,ad=n,0\leq b<d} a^k f(Az) \end{aligned}$$

Theorem 2.0.3. *If $A_1 \in \Gamma(n)$ and $V_1 \in \Gamma$, there exists $A_2 \in \Gamma(n)$ and $V_2 \in \Gamma$ such that $A_1V_1 = V_2A_2$.*

Also, if $A_i = \begin{pmatrix} a_i & b_i \\ 0 & d_i \end{pmatrix}$ *and* $V_i = \begin{pmatrix} * & ** \\ \gamma_i & \sigma_i \end{pmatrix}$, $i = 1$ *or* 2

Then

$$a_1(\gamma_2 A_2 z + \sigma_2) = a_2(\gamma_1 z + \sigma_1)$$

Theorem 2.0.4. $V \in \Gamma, V = \begin{pmatrix} * & ** \\ \gamma & \sigma \end{pmatrix}$, $f \in M_k$ *then*

$$T_nf(Vz) = (\gamma z + \sigma)^k T_nf(z)$$

Proof.

$$\begin{aligned}
T_n f(Vz) &= \frac{1}{n} \sum_{a\geq 1, ad=n, 0\leq b<d} a^k f(AVz) \\
&= \frac{1}{n} \sum_{a\geq 1, ad=n, 0\leq b<d} a^k f(V_2 A_2 z) = \frac{1}{n} \sum_{a\geq 1, ad=n, 0\leq b<d} a^k (\gamma_2 A_2 z + \sigma_2)^k f(A_2 z) \\
&= \frac{1}{n} \sum_{a\geq 1, ad=n, 0\leq b<d} a_2^k (\gamma_1 z + \sigma_1)^k f(A_2 z) \\
&= \frac{1}{n} (\gamma_1 z + \sigma_1)^k \sum_{a\geq 1, ad=n, 0\leq b<d} a_2^k f(A_2 z) = (\gamma_1 z + \sigma_1)^k T_n f(z)
\end{aligned}$$

□

Theorem 2.0.5. *T_n maps $M_k(\Gamma)$ to $M_k(\Gamma)$ and S_k to S_k.*

Proof. $T_n f$ is holomorphic in $\mathbb{H}$ (from the definition of $T_n f$)

From its fourrier expansion in theorem 2.0.2, we deduce it is holomorphic at $i\infty$.

It satisfies the transformation law (by Theorem 2.0.4)

Hence, T_n maps $M_k(\Gamma)$ to $M_k(\Gamma)$.

To prove $T_n f$ maps S_k to S_k, notice that if $f \in S_k$,

$$T_n f(z) = \sum_{m=0}^{\infty} \sum_{d|n, d|m} d^{k-1} a(\frac{mn}{d^2}) e^{2\pi i m z} = \sigma_{k-1}(n) a(0) + a(n) q + ...$$

then if $a(0) = 0$, $\sigma_{k-1}(n)a(0) = 0$ and $\sigma_{k-1}(n)a(0)$ is the first coefficient in the expansion of $T_n f(z)$, so we deduce that $T_n f(z) \in S_k$. □

Definition 2.0.6. An entire modular form is said to be a simultaneous eigenform if $T_n f(z) = l(n) f$ for all n and th sequence of values $l(n)$ are knowns as the eigenvalues.

Theorem 2.0.7. *If f is a simultaneous eigenform with a weight ≥ 4, then $a(1) \neq 0$*

Proof. Since f is a simultaneous eigenform, we know that

$$\gamma_n(1) = a(n) = l(n)a(1) \text{ for all } n.$$

If $a(1) = 0$ then

$$a(n) = 0 \text{ for all } n \geq 1$$

$$f(z) = a(0)$$

. Thus $f = 0$. Hence

$$a(1) \neq 0$$

□

Theorem 2.0.8. *f is a normalized simultaneous eigenform with a weight ≥ 4 if and only if for all m and n*

$$\sum_{d|(m,n)} d^{k-1} a\left(\frac{mn}{d^2}\right) = a(m)a(n)$$

Proof. - Suppose f is a simultaneous eigenform, we have $a(n) = \gamma_n(1) = l(n)a(1)$.

Now $a(1) = 1$ so $a(n) = l(n)$.

Also $\gamma_n(m) = l(n)a(m) = a(n)a(m)$. Thus

$$\sum_{d|(m,n)} d^{k-1} a\left(\frac{mn}{d^2}\right) = a(n)a(m)$$

Note: If $(m, n) = 1$ we get $a(m)a(n) = a(mn)$.

- Now suppose that for all m and n

$$\sum_{d|(m,n)} d^{k-1} a\left(\frac{mn}{d^2}\right) = a(m)a(n)$$

$$T_nf(z) = \sum_{m=0}^{\infty} \gamma_n(m)e^{2\pi imz}$$

Replace $\gamma_n(m)$ by $a(m)a(n)$, we get

$$T_nf(z) = \sum_{m=0}^{\infty} a(n)a(m)e^{2\pi imz}$$

Thus

$$T_nf(z) = a(n)\sum_{m=0}^{\infty} a(m)e^{2\pi imz}$$

for all n.

Hence

$$T_nf(z) = a(n)f(z)$$

for all n and thus f is a simultaneous eigenform. □

2.1 The Multiplicative Property of Hecke Operators

Theorem 2.1.1. *$T_mT_n = T_{mn}$ if $(m,n) = 1$*

Proof. If $f \in M_k$,

$$T_n(z) = \frac{1}{n}\sum_{a\geq 1, ad=n, 0\leq b<d} a^k f(Az)$$

A= $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \Gamma$.

$$
\begin{aligned}
T_m(T_n(f))(z) &= \frac{1}{m} \sum_{\alpha\geq 1,\ \alpha\sigma=m,\ 0\leq\beta<\sigma} \alpha^k T_n f(Bz) \\
&= \frac{1}{m} \sum_{\alpha\geq 1,\ \alpha\sigma=m,\ 0\leq\beta<\sigma} \alpha^k \frac{1}{n} \sum_{a\geq 1, ad=n, 0\leq b<d} a^k f(ABz) \\
&= \frac{1}{mn} \sum_{\alpha\geq 1,\ \alpha\sigma=m,\ 0\leq\beta<\sigma} \sum_{a\geq 1, ad=n, 0\leq b<d} (\alpha a)^k f(Cz)
\end{aligned}
$$

where C= AB =

$$
\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ 0 & \sigma \end{pmatrix} = \begin{pmatrix} \alpha a & a\beta + b\sigma \\ 0 & d\sigma \end{pmatrix}
$$

d runs through the divisors of n, σ through that of m. $(m, n) = 1$ so σd runs through all divisors of mn. $\alpha b + \beta d$ runs through a complete residue system mod $d\sigma$ (by the Chinese Remainder Theorem). Thus C runs through a complete set of non-equivalent elements of $\Gamma(mn)$

Thus

$$
T_m T_n = T_{mn}.
$$

□

Theorem 2.1.2. *For all $m, n \in \mathbb{Z}$*

$$
T_m T_n = \sum_{d|(m,n)} d^{k-1} T(\frac{mn}{d^2})
$$

Proof. if $(m, n) = 1$, then $d = 1$ and $T_m T_n = T_{mn}$

Thus, it is enough to consider m and n as powers of the same prime p(if m and n had different primes we use the fact that the gcd of two different primes is one and thus $T_m T_n = T_{mn}$).

First consider the case when $m = p$ and $n = p^s, \quad s \geq 1$.

First we need to prove

$$T(p)T(p^s) = T(p^{s+1}) + p^{k-1}T(p^{s-1})$$

$$(T(p^s)f)(z) = \frac{1}{p^s} \sum_{d|p^s, ad=n, 0 \leq b_z \leq p^t - 1, 0 \leq t \leq s} p^{(s-t)k} f(\frac{p^{s-t}z + b_z}{p^t})$$

Also

$$\{T(p)g\}(z) = p^{k-1}g(pz) + \frac{1}{p}\sum_{b=0}^{p-1} g(\frac{z+b}{p})$$

$$\{T(p)T(p^s)f\}(z)$$

$$= p^{k-1}T(p^s)(pz) + \frac{1}{p}\sum_{b=0}^{p-1} T_{p^s}(\frac{z+b}{p})$$

$$= p^{k-1}\frac{1}{p^s} \sum_{0 \leq t \leq s, 0 \leq b_z \leq p^s - 1} p^{(s-t)k} f(\frac{p^{s+1-t}z + pb_z}{p^t}) + \frac{1}{p}\frac{1}{p^s}\sum_{b=0}^{p-1} \sum_{ad=p^s, 0 \leq b_z \leq p^t - 1} f(\frac{p^{s-t}(\frac{z+b}{p}) + b_z}{p^t})$$

$$= p^{k-1}\frac{1}{p^r} \sum_{0 \leq t \leq s, 0 \leq b_z \leq p^s - 1} p^{(s-t)k} f(\frac{p^{s+1-t}z + pb_z}{p^t}) + p^{-1-s} \sum_{0 \leq s, 0 \leq b_z \leq p^t} p^{(s-t)k} \sum_{b=0}^{p-1} f(\frac{p^{s-t}z + b_z + bp^t}{p^{t+1}})$$

$b_z + bp^t$ runs through a complete residue system mod p^{s+1}

$$T(p^{s+1})f(z) = \frac{1}{p^{s+1}} \sum_{0 \leq b_z \leq p^{s+1} - 1, d|ps+1, 0 \leq t \leq s+1} p^{(s+1-t)k} f(\frac{p^{s+1-t}z + b_z}{p^t})$$

Thus

$$\{T(p)T(p^s)f\}(z) = T(p^{s+1})f)(z) + p^{k-1-s} \sum_{0 \leq b_z \leq p^t, 1 \leq t \leq s} p^{(s-t)k} f(\frac{p^{s+1-t}z + pb_z}{p^t})$$

Let $T = t - 1$

$$= T((p^{s+1})f)(z) + p^{k-1-s} \sum_{0 \le b_z \le p^{T+1}, 0 \le T \le s-1} p^{(s-t)k} f(\frac{p^{s-T}z + pb_z}{p^{T+1}})$$

$$= T(p^{s+1})f(z) + p^{k-1}T(p^{s-1})$$

For the case when $m = p^s$ and $n = p^r$, we need to prove

$$T(p^r)T(p^s) = \sum_{t=o}^{r} p^{t(k-1)}T(p^{r+s-2t})$$

Suppose $r \le s$

For $r = 1$, the statement is true by what preceded.

Assume it is true for all $r \le s$, we have to prove it is true for $r + 1$. That is we need to prove

$$T(p^{r+1})T(p^s) = \sum_{t=o}^{r+1} p^{t(k-1)}T(p^{r+s-2t+1})$$

$$\begin{aligned} T(p^{r+1})T(p^s) &= [T(p)T(p^r) - p^{k-1}T(p^{r-1})]T(p^s) \\ &= T(p)T(p^r)T(p^s) - p^{k-1}T(p^{r-1})]T(p^s) \\ &= \sum_{t=0}^{r} p^{t(k-1)}T(p)T(p^{r+s-2t}) - p^{k-1}T(p^{r-1})T(p^s) \\ &= \sum_{t=0}^{r} p^{t(k-1)}[T(p^{r+s-2t+1}) + p^{k-1}T(p^{r+s-2t-1}] - p^{k-1}T(p^{r-1})T(p^s) \end{aligned}$$

We have

$$\sum_{t=0}^{r} p^{t(k-1)}[T(p^{r+s-2t+1}) + p^{k-1}\sum_{T=1}^{r+1} p^{T(k-1)}T(p^{r+s-2T+1}) - p^{k-1}\sum_{T=1}^{r} p^{T(k-1)}T(p^{r+s-1-2T})$$

$= \sum_{t=0}^{r+1} p^{t(k-1)}T(p^{r+s-2t+1})$ □

2.2 L-Series of Eigenforms

Definition 2.2.1. Let $f \in M_k(\Gamma)$ and $a(n)$ be its Fourrier coefficient. Define the Dirichlet series

$$L(f,s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$$

Recall that $a(n) = O(n^{\frac{k}{2}})$ for f a cusp form(Theorem 1.8.1). And $a(n) = O(n^{k-1})$ for entire modular forms (Theorem 2.2.5), we study the convergence of the L-Series based on the real part of s as below:

$$|L(f,s)| = |\sum_{n=1}^{\infty} \frac{a(n)}{n^s}| \leq \sum_{n=1}^{\infty} \frac{|a(n)|}{|n^s|}$$

For f a cusp form,

$$|L(f,s)| \leq c \sum_{n=1}^{\infty} \frac{n^{\frac{k}{2}}}{n^{Re(s)}}$$

$Re(s) - \frac{k}{2} > 1$ and so $Re(s) > 1 + \frac{k}{2}$

For f an entire form,

$$|L(f,s)| \leq c \sum_{n=1}^{\infty} \frac{n^{\frac{k}{2}}}{n^{Re(s)}}$$

We have $Re(s) - k + 1 > 1$ so $Re(s) > k$

Theorem 2.2.2. *If f is a normalized simultaneous eigenform then for p prime*

$$L(f,s) = \prod_{p=1}^{\infty} \frac{1}{1 - a(p)p^{-s} + p^{k-1-2s}}$$

Proof.

$$L(f,s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \sum_{n=1}^{\infty} \frac{a(p_1^{n_1} p_2^{n_2} ....p_m^{n_m})}{p_1^{n_1} p_2^{n_2} ....p_m^{n_m}}$$

Since the common divisors of p and p^n are 1 and p, we have

$$a(p)a(p^n) = a(p^{n+1}) + p^{k-1}a(p^{n-1})$$

Also,

$$(1-a(p)x+p^{k-1}x^2)(1+\sum_{n=1}^{\infty}a(p^n)x^n)=1 \qquad (3)$$

We claim that the euler product of $L(f,s)$ is the infinite product

$$\prod_{p=1}^{\infty}(1+\sum_{m=1}^{\infty}\frac{a(p^m)}{p^{ms}})$$

This infinite product converges absolutely since

$$\sum_{p=1}^{\infty}|\sum_{n=1}^{\infty}\frac{a(p^n)}{p^{ns}}|\leq|\sum_{n=1}^{\infty}\frac{a(n)}{n^s}|<\infty$$

We now show the product is equal to $L(f,s)$.

Consider

$$P_N(s)=\prod_{p\leq N}(1+\sum_{n=1}^{\infty}\frac{a(p^n)}{p^{ns}})$$

For $N\geq 2$, let $p_1,p_2,....,p_l$ denote all primes less than or equal to N. Noting that the terms can be writter as $\frac{a(p^n)}{p^{ns}}$ for $n=0$ ($a(1)=1$ for normalized eigenforms), we get

$$P_N(s)=\sum_{m_1=0}^{\infty}..........\sum_{m_1=0}^{\infty}\frac{a(p_1^{m_1})a(p_2^{m_2}).....a(p_l^{m_l}}{p_1^{m_1s}p_2^{m_2s}.........p_l^{m_ls}}$$

By the fundamental theorem of Arithmetic, every integer has a unique factorization as powers of primes.Also, the integers $p_1^{m_1},p_2^{m_2},.........p_l^{m_l}$ are elements in the set $S_N=\{n\in\mathbb{N}:p|n\ p\leq N\}$

Hence $P_n(s)=\sum_{n\in S_N}\frac{a(n)}{n^s}$

S_n contains all integers less than or equal to N. Now

$$|P_N(s)-L(f,s)|=|\sum_{n\notin S_N}\frac{a(n)}{n^s}|\leq\sum_{n=N+1}^{\infty}|\frac{|a(n)|}{|n^s|}$$

By the nth term test, since $\sum_{n=N+1}^{\infty} \frac{a(n)}{n^s}$ converges absolutely then

$\lim_{n\to\infty} \frac{|a(n)|}{|n|^s} = 0$ then $\lim_{n\to\infty} P_N(s) = L(f,s)$.

Using (3) and what was just proven, we get that

$$L(f,s) = \prod_{p=1}^{\infty}(1 + \sum_{m=1}^{\infty} \frac{a(p^m)}{p^{ms}}) = \prod_{p=1}^{\infty} \frac{1}{1 - a(p)p^{-s} + p^{k-1-2s}}$$

□

Theorem 2.2.3. *Let $f \in S_k(\Gamma)$.*

$$L_*(f,s) = \int_0^{\infty} f(iy)y^s \frac{dy}{y}$$

.

$L_*(f,s)$ is an entire function satisfying

$$L(f,s) = (2\pi)^s \frac{L_*(f,s)}{\Gamma(s)}$$

Proof. Recall

$$\Gamma(s) = \int_0^{\infty} e^{-t}t^{s-1}dt$$

$\mathrm{Re}(s) > 0$

We have

$$\begin{aligned} L_*(f,s) &= \int_0^{\infty} f(iy)y^s \frac{dy}{y} \\ &= \int_0^{\infty} \sum_{n=1}^{\infty} a(n)e^{-2\pi ny}y^{s-1}dy \end{aligned}$$

Let $\tau = 2\pi ny$, $d\tau = 2\pi n dy$. Substituting, we get

$$\begin{aligned} L_*(f,s) &= \int_0^\infty \sum_{n=1}^\infty a(n)e^{-\tau}\tau^{s-1}\frac{1}{(2\pi n)^{s-1}}\frac{dy}{2\pi n} \\ &= \frac{1}{(2\pi)^s}\int_0^\infty \sum_{n=1}^\infty \frac{a(n)}{n^s}e^{-\tau}\tau^{s-1}d\tau \\ &= \frac{1}{(2\pi)^s}L(f,s)\Gamma(s) \end{aligned}$$

As seen above, the integral of the sum is the sum of the integral and this can be shown using Dominated Convergence theorem as below:

$$|\sum_{n=1}^\infty \frac{a(n)}{n^s}| \leq \sum_{n=1}^\infty \frac{|a(n)|}{n^s} \leq \sum_{n=1}^\infty \frac{cn^{\frac{k}{2}}}{n^s}$$

The last series converges since $s - \frac{k}{2} > 1$. Thus

$$\int_0^\infty \sum_{n=1}^\infty \frac{a(n)}{n^s}d\tau = \sum_{n=1}^\infty \frac{a(n)}{n^s}\int_0^\infty d\tau$$

Hence

$$L(f,s) = (2\pi)^s)\frac{L_*(f,s)}{\Gamma(s)}$$

□

Theorem 2.2.4. *Suppose f is a cusp form, then $L(f,s)$ can be continued analytically beyond $s \geq 1 + \frac{k}{2}$ by the functional equation*

$$\frac{L(f,s)\Gamma(s)}{(2\pi)^s} = \frac{i^k L(f,k-s)\Gamma(k-s)}{(2\pi)^{k-s}}$$

where $\Gamma(s) = \int_0^\infty e^{-t}t^{s-1}dt$.

If $a(0) \neq 0$, and for $k \geq 4$, $L(f,s)$ is analytic for all s except a simple pole at $s = k$ with residue

$$\frac{(-1)^{\frac{k}{2}}a(0)(2\pi)^k}{\Gamma(k)}$$

Proof. Let $t = 2\pi n\tau$, $dt = 2\pi n d\tau$. We substitute and get:

$$\Gamma(s) = \int_0^\infty e^{-2\pi n\tau}(2\pi n\tau)^{s-1} 2\pi n d\tau$$

$$\frac{\Gamma(s)}{(2\pi n)^s} = \int_0^\infty e^{-2\pi n\tau}\tau^{s-1} d\tau$$

$$\frac{\Gamma(s)a(n)}{(2\pi)^s n^s} = \int_0^\infty e^{-2\pi n\tau} a(n)\tau^{s-1}$$

$$\frac{\Gamma(s)}{(2\pi)^s}\sum_{n=1}^\infty \frac{a(n)}{n^s} = \sum_{n=1}^\infty \int_0^\infty a(n)e^{-2\pi n\tau}\tau^{s-1} d\tau$$

By uniform covergence on compact sets

$$\begin{aligned}
\frac{\Gamma(s)}{(2\pi)^s}\sum_{n=1}^\infty \frac{a(n)}{n^s} &= \int_0^\infty \sum_{n=1}^\infty a(n)e^{-2\pi n\tau}\tau^{s-1} d\tau \\
&= \int_0^\infty (f(i\tau) - a(0))\tau^{s-1} d\tau \\
&= \int_0^1 (f(i\tau) - a(0))\tau^{s-1} d\tau + \int_1^\infty (f(i\tau) - a(0))\tau^{s-1} d\tau
\end{aligned}$$

Now

$$f(Si\tau) = (i\tau)^k f(i\tau)$$

$$\frac{\Gamma(s)}{(2\pi)^s}\sum_{n=1}^\infty \frac{a(n)}{n^s} = \int_0^1 [(f(\frac{i}{\tau})(i\tau)^{-k} - a(0))]\tau^{s-1} d\tau + \int_1^\infty (f(i\tau) - a(0))\tau^{s-1} d\tau$$

Let $w = \frac{1}{\tau}$ then

$$dw = \frac{-1}{\tau^s} d\tau = -w^2 d\tau$$

$$\begin{aligned}
\frac{\Gamma(s)}{(2\pi)^s}\sum_{n=1}^\infty \frac{a(n)}{n^s} &= \int_1^\infty [(f(iw)(\frac{i}{w})^{-k} - a(0))]w^{1-s}\frac{1}{w^2} dw + \int_1^\infty (f(i\tau) - a(0))\tau^{s-1} d\tau \\
&= \int_1^\infty f(iw)i^{-k}w^{k-s-1} - a(0))w^{-1-s}\frac{1}{w^2} dw + \int_1^\infty (f(i\tau) - a(0))\tau^{s-1} d\tau \\
&= f(iw)i^{-k}w^{k-s-1} dw + a(0)\frac{w^{-s}}{s}\Big|_1^\infty + \int_1^\infty (f(i\tau) - a(0))\tau^{s-1} d\tau \\
&= f(iw)i^{-k}w^{k-s-1} dw - \frac{a(0)}{s} + \int_1^\infty (f(i\tau) - a(0))\tau^{s-1} d\tau
\end{aligned}$$

Add and subtract $(-1)^{\frac{k}{2}} \int_1^\infty a(0)w^{k-s-1}dw$ to get:

$\frac{\Gamma(s)}{(2\pi)^s} \sum_{n=1}^\infty \frac{a(n)}{n^s} =$

$$(-1)^{\frac{k}{2}} \int_1^\infty (f(iw) - a(0))w^{k-s-1}dw + \int_1^\infty (f(i\tau) - a(0))\tau^{s-1}d\tau - \frac{a(0)}{s} + (-1)^{\frac{k}{2}} \int_1^\infty a(0)w^{k-s-1}dw$$

$$= \int_1^\infty (f(iw) - a(0))w^{k-s-1}i^{-k}dw + \int_1^\infty (f(i\tau) - a(0))\tau^{s-1}d\tau - a(0)(\frac{1}{s} + \frac{(-1)^{\frac{k}{2}}}{k-s})$$

Let $I = -a(0)(\frac{1}{s} + \frac{(-1)^{\frac{k}{2}}}{k-s})$ and let $w = t = \tau$

$$\frac{\Gamma(s)}{(2\pi)^s} \sum_{n=1}^\infty \frac{a(n)}{n^s} = \int_1^\infty (f(it) - a(0))t^{k-s-1}i^{-k}dt + \int_1^\infty [(it)^{-k}f(\frac{i}{t}) - a(0)]t^{s-1}dt + I$$

Let $T = \frac{1}{t}$. $dT = -T^2dt$

$$\frac{\Gamma(s)}{(2\pi)^s} \sum_{n=1}^\infty \frac{a(n)}{n^s} = \int_1^\infty (f(it) - a(0))t^{k-s-1}i^{-k}dt + \int_1^0 (i^{-k}T^kf(iT) - a(0))T^{1-s}.(-T^{-2})dT + I$$

$$= \int_1^\infty (f(it) - a(0))t^{k-s-1}i^{-k}dt + \int_0^1 (f(iT) - a(0))T^{k-s-1}i^{-k}dT + I$$

$$= \int_0^\infty (f(it) - a(0))t^{k-s-1}i^{-k}dt + I = \int_0^\infty (\sum_{n=1}^\infty e^{-2\pi nt})t^{k-s-1}i^{-k}dt + I$$

Let $B = 2\pi nt$ so $dB = 2\pi ndt$ and using Dominated Convergence Theorem as in Theorem 2.2.3

$$\frac{\Gamma(s)}{(2\pi)^s} \sum_{n=1}^\infty \frac{a(n)}{n^s} = \int_0^\infty (\sum_{n=1}^\infty e^{-B}B^{k-s-1}\frac{1}{(2\pi n)^{k-s}}i^{-k}dB\frac{1}{2\pi n} + I$$

$$= i^{-k}\sum_{n=1}^\infty \frac{a(n)}{n^{k-s}}\frac{1}{(2\pi)^{k-s}}\int_0^\infty e^{-B}B^{k-s-1}dB + I$$

$$= \Gamma(k-s)L(f, k-s)\frac{1}{(2\pi)^{k-s}}i^{-k} + I$$

Note that $i^k = i^{-k}$ since k is even and $i^{2k} = 1$. For $f \in S_k(\Gamma)$ $a(0) = 0$ and we get the equality needed. □

Theorem 2.2.5. *If f is an entire modular form and not a cusp form, then*

$a(n) = O(n^{k-1})$

Proof. If $f = G_k$,

$$a(n) = c\sigma_{k-1}(n)$$

We have

$$|a(n)| \leq |c|n^{k-1} \sum_{d=1}^{\infty} \frac{1}{d^{k-1}} = |c|n^{k-1}\zeta(k-1)$$

$$a(n) = O(n^{k-1})$$

Suppose f is an entire function of weight k Let

$$\alpha = \frac{f(i\infty)}{G_k(i\infty)}$$

Consider

$$f - \alpha G_k \quad \in S_k$$

Then

$$f = \alpha G_k + g$$

and so

$$a(n) = O(n^{k-1}) + O(n^{\frac{k}{2}}) = O(n^{k-1})$$

$\square$

Chapter 3

Zeroes of Eisenstein Series

For $k \geq 4$ and $z \in \mathbb{H}$, consider the Eisenstein series

$$E_k(z) = \frac{1}{2} \sideset{}{'}\sum_{(c,d)\in\mathbb{Z}^2,(c,d)=1} (cz+d)^{-k}$$

In the problem of locating the zeroes of the Eisenstein series in the fundamental domain on the full modular group, it was originally shown that the the zeroes lie on the unit circle when $k \leq 34$ and $k = 38$. Then the result was extended for all k in the paper of Rankin and Swinnerton-Dyer.

Theorem 3.0.1. *All zeros of $E_k(z)$ in the fundamental domain of the full modular group are located on the arc of the unit circle $\{z = e^{i\theta}, \frac{\pi}{2} \leq \theta \leq \frac{2\pi}{3}\}$*

Proof. From Proposition 1.3.2,

$$\sum_{p\in\Gamma/H} \frac{1}{n_p} multi_p(f) + multi_\infty(f) = \frac{k}{12}$$

. Developing it we get,

$$\sum_{p \in \Gamma/H, p \neq i \ or \ w} multi_p(f) + multi_\infty(f) + \frac{1}{2} multi_i(f) + \frac{1}{3} multi_w(f) = \frac{k}{12}$$

Write $k = 12n + s$ where $n \in \mathbb{Z}$ and s=4, 6, 8, 10, 0, or 14. Then

$$\sum_{p \in \Gamma/H, p \neq i \ or \ w} multi_p(f) + multi_\infty(f) + \frac{1}{2} multi_i(f) + \frac{1}{3} multi_w(f) = n + \frac{s}{12}$$

Note that the value of s determines the minimum number of zeros that $E_k(z)$ must have at the elliptic points i and w counting multiplicity (say for $s = 4$, f should have at least one zero at w. For s=0, f has no zeroes at w and i.For $s = 4$, f has at least one zero at w. For $s = 6$, f has at least one zero at i. For $s = 8$, f has at least two zeroes at w. For $s = 10$, f has at least one zero at w and one zero at i. And for $s = 14$, f has at least two zeroes at w and one zero at i)

Thus, it suffices to show that $E_k(z)$ has at least n zeroes on the arc of the unit circle $(\frac{\pi}{2}, \frac{2\pi}{3})$.

Write $F_k(\theta) = e^{i\frac{k\theta}{2}} E_k(e^{i\theta}) =$

$$\frac{1}{2} \sum_{(c,d) \in \mathbb{Z}^2, (c,d)=1}{}^{'} (ce^{\frac{i\theta}{2}} + de^{\frac{-i\theta}{2}})^{-k}$$

It is easy to see that $F_k(\theta) \in \mathbb{R}$ for real θ

We limit n to the positive integers yielding $k \geq 12$

Note that $c^2 + d^2 = 1$ for $(c, d) = \{(0, 1), (1, 0), (-1, 0), (0, -1)\}$ Let R_1 consists of the terms of the series for which $c^2 + d^2 > 1$, we have

$$F_k(\theta) = \frac{1}{2}\{e^{\frac{ik\theta}{2}} + e^{\frac{-ik\theta}{2}} + e^{\frac{-ik\theta}{2}} + e^{\frac{ik\theta}{2}}\} + R_1 = 2cos(\frac{k\theta}{2}) + R_1$$

We show that for $\frac{\pi}{2} < \theta < \frac{2\pi}{3}$, $|R_1| < 2$.

In the closed interval $[\frac{\pi}{2}, \frac{2\pi}{3}]$, cos $\theta > -\frac{1}{2}$,

$$|ce^{\frac{i\theta}{2}} + de^{\frac{-i\theta}{2}}|^2 = (ce^{\frac{i\theta}{2}} + de^{\frac{-i\theta}{2}})(ce^{-\frac{i\theta}{2}} + de^{\frac{i\theta}{2}}) = c^2 + 2cdcos\theta + d^2$$

$$\begin{aligned} c^2 + 2cdcos\theta + d^2 - \frac{1}{2}(c^2 + d^2) &= \frac{1}{2}(c^2 + d^2) + 2cdcos\theta \\ &> \frac{1}{2}(c^2 + d^2) - cd \\ &= \frac{1}{2}(c^2 + d^2 - 2cd) \\ &= \frac{1}{2}(c + d)^2 \\ &\geq 0 \end{aligned}$$

There are at most $2(2N^{\frac{1}{2}} + 1)$ couples such that the sum of their squares is N. Now for $N \geq 5$, $N^{\frac{1}{2}} > 2$ and thus

$$2(2N^{\frac{1}{2}} + 1) \leq 5N^{\frac{1}{2}}$$

For $c^2 + d^2 = 2$, we have 4 couples $\{(1, -1), (-1, 1), (1, 1), (-1, -1)\}$

For $c^2 + d^2 = 5$ we have 8

couples.$\{(1, 2), (-1, 2), (1, -2), (-1, -2), (2, 1), (-2, 1), (2, -1), (-2, -1)\}$

$$\begin{aligned} |R_1| &\leq \frac{1}{2} \sum_{c^2+d^2>1} |ce^{\frac{i\theta}{2}} + de^{\frac{-i\theta}{2}}|^{-k} \\ &\leq \frac{1}{2} \sum_{c^2+d^2>1} |\frac{1}{2}(c^2 + d^2)|^{-\frac{k}{2}} \\ &\leq 1 + 2^{\frac{-k}{2}} + 4(\frac{5}{2})^{-\frac{k}{2}} + \sum_{N=10}^{\infty} 5N^{\frac{1}{2}}(\frac{1}{2}N)^{-\frac{k}{2}} \\ &\leq 1 + 2^{\frac{-k}{2}} + 4(\frac{5}{2})^{-\frac{k}{2}} + \frac{20\sqrt{2}}{k-3}(\frac{9}{2})^{\frac{3-k}{2}} \end{aligned}$$

Clearly, the right hand side function is monotone decreasing and since $k \geq 12$, the max value is reached at $k = 12$ and is equal to 1.03562.Thus $|R_1| < 2$

Let $m \in \mathbb{Z}$, and consider $F_k(\frac{2m\pi}{k}) = 2cos(m\pi) + R_1$

For m even, $cos(m\pi) = 1$ and thus $F_k(\frac{2m\pi}{k}) = 2 + R_1 > 0$

For m odd, $cos(m\pi) = -1$ and thus $F_k(\frac{2m\pi}{k}) = -2 + R_1 < 0$

Since $\frac{\pi}{2} \leq \theta \leq \frac{2\pi}{3}$,

$$\frac{\pi}{2} \leq \frac{2m\pi}{k} \leq \frac{2\pi}{3}$$

and thus

$$\frac{1}{4}k \leq m \leq \frac{1}{3}k$$

Hence the number of zeroes of $F_k(\theta)$ in the interval $(\frac{\pi}{2}, \frac{2\pi}{3})$ is equal to the number of integers in the interval $[\frac{k}{4}, \frac{k}{3}]$ minus one, and this is exactly n.

$$\frac{1}{4}k \leq m \leq \frac{1}{3}k \ \text{ so } \ \frac{1}{4}(12n + s) \leq m \leq \frac{1}{3}(12n + s) \ \text{ so } \ 3n + \frac{s}{4} \leq m \leq 4n + \frac{s}{3}$$

Number of integers in this interval is $n + 1$

The number of zeroes of $F_k(\theta)$ is the same as that of $E_k(z)$ and the proof is complete $\square$

Chapter 4

Period Polynomials and their Roots

In this chapter, we study the zeroes of period polynomials. Those polynomials that arise from Eichler Shimura Integrals.

Definition 4.0.1. Define the n^{th} period of f as

$$r_n(f) = \int_0^\infty f(it)t^n dt$$

$(0 \leq n \leq k-2),\ f \in S_k)$

Theorem 4.0.2.

$$r_n(f) = n!(2\pi)^{-n-1}L(f, n+1)\Gamma(n+1)$$

Proof.

$$r_n(f) = \int_0^\infty f(it)t^n dt = \int_0^\infty t^n \sum_{l=1}^{\infty} a(l)e^{2\pi i(it)l}dt = \int_0^\infty t^n \sum_{l=1}^{\infty} a(l)e^{-2\pi lt}dt$$

Let $q = 2\pi lt$ so $dq = 2\pi l dt$

$$\begin{aligned} r_n(f) &= \int_0^\infty \frac{q^n}{(2\pi l)^n} \sum_{l=1}^\infty a(l) e^{-q} \frac{dq}{2\pi l} \\ &= \sum_{l=1}^\infty \frac{a(l)}{l^{n+1}} \frac{1}{(2\pi)^{n+1}} \int_0^\infty e^{-q} q^n dq \\ &= L(f, n+1) \frac{1}{(2\pi)^{n+1}} \Gamma(n+1) \\ &= L_*(f, n+1) \end{aligned}$$

□

Definition 4.0.3. The Eichler-Shimura Integral of an entire modular form is defined by:

$$E_f(z) = \int_z^{i\infty} (f(\tau) - a(0))(\tau - z)^{k-2} d\tau$$

Theorem 4.0.4. *We have*

$$E_f(z) = \frac{-(k-2)!}{(2\pi i)^{k-1}} \sum_{n=1}^\infty \frac{a(n)}{n^{k-1}} e^{2\pi i n z}$$

Proof.

$$E_f(z) = \int_z^{i\infty} \sum_{n=1}^\infty a(n) e^{2\pi i n \tau} (\tau - z)^{k-2} dt$$

Let $u = \tau - z$ (possible because f is analytic in $\mathbb{H}$)

$$E_f(z) = \int_o^{i\infty} \sum_{n=1}^\infty a(n) e^{2\pi i n u} e^{2\pi i n z} u^{k-2} du$$

Let $w = -2\pi i n u$

$$E_f(z) = \int_o^\infty \sum_{n=1}^\infty a(n) e^{-w} e^{2\pi i n z} (\frac{w}{-2\pi i n})^{k-2} \frac{dw}{-2\pi i n}$$
$$= -\frac{1}{(2\pi i)^{k-1}} \sum_{n=1}^\infty \frac{a(n)}{n^{k-1}} e^{2\pi i n z} \int_0^\infty e^{-w} w^{k-2} dw$$
$$= \frac{-(k-2)!}{(2\pi i)^{k-1}} \sum_{n=1}^\infty \frac{a(n)}{n^{k-1}} e^{2\pi i n z}$$

□

Theorem 4.0.5.

$$E_f(z) - z^{k-2} E_f(Sz) = r_f(z)$$

Proof.

$$E_f(Sz) = \int_{Sz}^{i\infty} (f(\tau) - a(0))(\tau - Sz)^{k-2} d\tau$$

Since f is analytic over the region, we can do a change of variables. Let

$$\tau = Sw = \frac{-1}{w}$$

, so

$$d\tau = \frac{1}{w^2} dw$$

$$E_f(Sz) = \int_z^{S^{-1} i\infty} (f(Sw) - a(0))(\frac{-1}{w} + \frac{1}{z})^{k-2} \frac{1}{w^2} dw$$
$$= \int_z^0 (w^k)(f(w) - a(0))(\frac{-z+w}{zw})^{k-2} \frac{1}{w^2} dw$$
$$= \int_z^0 \frac{(f(w) - a(0))(w-z)^{k-2}}{z^{k-2}} dw$$

$$z^{k-2} E_f(Sz) = \int_z^0 (f(w) - a(0))(w-z)^{k-2} dw$$

$$E_f(z) - z^{k-2}E_f(Sz) = \int_z^{\infty}(f(w)-a(0))(w-z)^{k-2}dw + \int_0^z (f(w)-a(0))(w-z)^{k-2}dw$$
$$= \int_0^{i\infty}(f(w)-a(0))(w-z)^{k-2}dw$$

□

Thus we define the $r_f(z)$ as the period function given above

$$r_f(z) = E_f(z) - z^{k-2}E_f(Sz) = \int_0^{i\infty}(f(w)-a(0))(w-z)^{k-2}dw$$

Lemma 4.0.6. *The period function is a polynomial of degree less than or equal to*

$w = k-2$

Proof.

$$r_f(z) = \int_0^{i\infty}(f(w)-a(0))(w-z)^{k-2}dw$$

Using the binomial theorem,

$$r_f(z) = \int_0^{i\infty}\sum_{l=0}^{k-2} C_{k-2}^l w^l(-1)^{k-2-l}z^{k-2-l}(f(w)-a(0))dw$$
$$= \sum_{l=0}^{k-2}(-1)^{k-2-l}C_{k-2}^l z^{k-2-l}\int_0^{i\infty} w^l(f(w)-a(0))dw$$
$$= \sum_{l=0}^{k-2} a(l)z^{k-2-l}$$

where $a(l) = (-1)^{k-2-l}C_{k-2}^l \int_0^{i\infty} w^l(f(w)-a(0))dw$ □

Definition 4.0.7. The odd part of the period function is denoted by $r_f^-(z)$ and is equal to $\frac{r_f(z)-r_f(-z)}{2}$

The even part of the period function is denoted by $r_f^+(z)$ and is equal to $\frac{r_f(z)+r_f(-z)}{2}$

For $f \in S_k$ $(a(0)=0)$,we get

$$r_f(z) = \int_0^{i\infty} f(w)(w-z)^{k-2}dw$$

If z_0 is a zero of $r_f(z)$ then

$$E_f(z_0) = z_0^{k-2} E_f(\frac{-1}{z_0})$$

and we get

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^{k-1}} e^{2\pi i n z_0} = z_0^{k-2} \sum_{n=1}^{\infty} \frac{a(n)}{n^{k-1}} e^{-2\pi i \frac{n}{z_0}}$$

Definition 4.0.8. Let V_w be the space of complex polynomials with real-valued coefficients of degree less than or equal to w. Also V_w^- and V_w^+ be the subspace of odd and even polynomials respectively.

Let $\phi \in V_w$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.

Introduce the following group action of Γ on V_w

$$\phi \Big| \begin{pmatrix} a & b \\ c & d \end{pmatrix} (z) = (cz+d)^w \phi(\frac{az+b}{cz+d})$$

The action preserves $V_w^{\pm}$

Definition 4.0.9. Consider $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $U = TS = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$.

$S^2 = -I$ and $U^3 = -I$.

Define $Y_w = \{\phi \in V_w : \phi|(1+S) = \phi|(1+U+U^2) = 0\}$

$Y_w^{\pm} = Y_w \bigcap V_w^{\pm}$

Lemma 4.0.10. *Let $f \in S_k(\Gamma)$, $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, then*

$$r_f|M(z) = \int_{M^{-1}0}^{M^{-1}i\infty} f(w)(w-z)^{k-2} dw$$

Proof.

$$r_f(Mz) = \int_0^{i\infty} f(w)(w - Mz)^{k-2} dw$$

Since f is analytic over 0 and $i\infty$ we can safely do a change of variables.

Let

$$w = Mt = \frac{at+b}{ct+d}$$

For $w = 0, t = M^{-1}0$, and for $w = i\infty, t = M^{-1}i\infty$

$$dw = \frac{a(ct+d) - c(at+b}{(ct+d)^2}dt = \frac{ad-bc}{(ct+d)^2}dt = \frac{1}{(ct+d)^2}dt$$

$$\begin{aligned}
r_f(Mz) &= \int_{M^{-1}0}^{M^{-1}i\infty} f(Mt)(Mt - Mz)^{k-2}\frac{1}{(ct+d)^2}dt \\
&= \int_{M^{-1}0}^{M^{-1}i\infty} f(Mt)(\frac{at+b}{ct+d} - \frac{az+b}{cz+d})^{k-2}\frac{1}{(ct+d)^2}dt \\
&= \int_{M^{-1}0}^{M^{-1}i\infty} f(Mt)(\frac{t-z}{(ct+d)(cz+d))})^{k-2}\frac{1}{(ct+d)^2}dt \\
&= \int_{M^{-1}0}^{M^{-1}i\infty} f(Mt)(\frac{t-z}{(ct+d)(cz+d))})^{k-2}\frac{1}{(ct+d)^2}dt \\
&= \int_{M^{-1}0}^{M^{-1}i\infty} (ct+d)^k f(t)(\frac{t-z}{(ct+d)(cz+d))})^{k-2}\frac{1}{(ct+d)^2}dt
\end{aligned}$$

$$(cz+d)^{k-2}r_f(Mz) = \int_{M^{-1}0}^{M^{-1}i\infty} f(t)(t-z)^{k-2}dt$$

$$r_f|M(z) = \int_{M^{-1}0}^{M^{-1}i\infty} f(w)(w-z)^{k-2}dw$$

□

Lemma 4.0.11. $r_f \in Y_w$.

Proof. It was shown that $r_f \in V_w$ (lemma 3.0.5). So we show that

$r_f|(1+S) = r_f|(1+U+U^2) = 0$

$S^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$; $U^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$ and $U^{-2} = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$

$$r_f(z) = \int_0^{i\infty} f(w)(w-z)^{k-2}dw$$

$$r_f|S(z) = \int_{S^{-1}0}^{S^{-1}i\infty} f(w)(w-z)^{k-2}dw$$
$$r_f|S(z) = \int_{i\infty}^{0} f(w)(w-z)^{k-2}dw$$

Hence

$$r_f(z) + r_f(Sz) = \int_{0}^{i\infty} f(w)(w-z)^{k-2}dw + \int_{i\infty}^{0} f(w)(w-z)^{k-2}dw = 0$$

Now we show that

$$r_f(z) + r_f|U + r_f|U^2 = 0$$

$$r_f|U(z) = \int_{U^{-1}0}^{U^{-1}i\infty} f(w)(w-z)^{k-2}dw = \int_{1}^{0} f(w)(w-z)^{k-2}dw$$
$$r_f|U^2(z) = \int_{U^{-2}0}^{U^{-2}i\infty} f(w)(w-z)^{k-2}dw = \int_{i\infty}^{1} f(w)(w-z)^{k-2}dw$$

Hence

$$r_f(z) + r_f|U + r_f|U^2 = \int_{0}^{i\infty} f(w)(w-z)^{k-2}dw + \int_{1}^{0} f(w)(w-z)^{k-2}dw + \int_{i\infty}^{1} f(w)(w-z)^{k-2}dw$$
$$= 0$$

Thus

$$r_f(z) \in Y_w$$

□

According to the Eichler-Shimura theory, there is an isomorphism between the space of cusp forms and that of odd period functions. Thus a cusp form is uniquely determined by its odd period function. In this context,there would arise the relation between the zeroes of cusp forms and that of period functions(*also known as period polynomials*).According to Conrey, Farmer and Imamoglu, the odd period polynomial of a Hecke eigenform has simple zeroes at 0,±2 , double zeroes at ±1, and zeroes as complex numbers on the unit circle.

Here we include the proof that the zeroes of the full period polynomial of a Hecke cusp form are on the unit circle.

4.1 Inroduction to Period Polynomials

A (complex) polynomial $P(z) = \sum_{i=0}^{d} a_i z^i$ of degree d ($a_i \in \mathbb{C}$) is said to be self-inversive if $P(z) = cz^d\, \overline{P}(\frac{1}{z})$ for some constant c. If $\overline{P} = P$ and $c = 1$, $P(z)$ is self-reciprocal.

Theorem 4.1.1. *A necessary and sufficient condition for a polynomial to have all its zeroes on the unit circle is it being self-inversive and its derivative having its zeroes in the closed unit disc.*

Theorem 4.1.2. *Let $g(z)$ be a non-zero complex polynomial of degree n with all its zeroes in the closed unit disc. For $m \geq n$, and c with $|c| = 1$, the self-inversive polynomial $P^{(c)}(z) = z^{m-n}g(z) + cz^n\overline{g}(\frac{1}{z})$ has it zeroes on the unit circle.*

Proof. Let $g^*(z) = z^n\overline{g}(\frac{1}{z})$. Suppose all n zeroes of $g(z)$ are in the open unit disc.

$$g(z) = (z - a_1)(z - a_2)....(z - a_n)$$

with $|a_1|, |a_2|, ..., |a_n| < 1$. Then

$$\overline{g}(\frac{1}{z}) = (\frac{1}{\overline{z}} - \overline{a_1})(\frac{1}{\overline{z}} - \overline{a_2}).......(\frac{1}{\overline{z}} - \overline{a_n})$$

The zeroes of $\overline{g}(\frac{1}{z})$ are $\frac{1}{a_1}, \frac{1}{a_2}......\frac{1}{a_n}$ whose modulus is greater than 1.

$g^*(z)$ has all its zeroes in $|z| > 1$. Now $z^{m-n}g(z)$ has its m zeroes inside the open unit disk.

Let us recall *Rouche's Theorem*: If f and p are holomorphic functions inside and on a simple closed curve C with $|p(z)| < |f(z)|$ **on** C, then f and $f + p$ have the same number of zeroes **inside C**.

Although in the theorem stated, $|c| = 1$, we consider the two other cases(when $|c| < 1$ and $|c| > 1$) to deduce what happens at the value 1.

Case I: $|c| < 1$, set $f(z) = z^{m-n}g(z)$ and $p(z) = cg^*(z)$

On C,($\overline{z} = \frac{1}{z}$)

$|p(z)| = |cg^*(z)| < |g^*(z)| = |z^n\overline{g}(\frac{1}{z})| = |\overline{g}(\overline{z})| = |\overline{g(z)}| = |g(z)| = |z^{m-n}g(z)| = |f(z)|$

Now $f(z)$ has its m zeroes inside the open unit disk and thus $P^{(c)}(z)$ has its m zeroes inside as well.

Case II: $|c| > 1$, set $p(z) = z^{m-n}g(z)$ and $f(z) = cg^*(z)$

$|p(z)| = |z^{m-n}g(z)| = |g(z)| = |z^n g(\frac{1}{\overline{z}})| = |z^n\overline{g}(\frac{1}{z})| < |cg^*(z)| = |f(z)|$

Now $f(z)$ has its no zeroes inside the open unit disk and thus $P^{(c)}(z)$ has no zeroes inside as well. Thus, all its zeroes are on $|z| > 1$

The zeroes of $P^c(z)$ are continuous functions of c, and hence the lie on the unit circle for $|c| = 1$. Also this continuity leads to the same result if g is assumed to have its zeroes in the closed unit disk. (Further details can be found in the paper entitled "Unimodularity of zeros of self-inversive polynomials" by M.Murty and C.Smyth)

□

Given f a cusp form of weight $k = w+2$ $(f(z) = \sum_{n=1}^{\infty} a(n)e^{2\pi i n z})$, let $L(f,s)$ be its L-series.We have

$$
\begin{aligned}
r_f(X) &= -\sum_{n=0}^{w} \frac{w!}{n!} \frac{L(f, w-n+1)}{(2\pi i)^{w-n+1}} X^n \\
&= -\frac{w!}{(2\pi i)^{w+1}} \sum_{n=0}^{w} L(f, w-n+1) \frac{(2\pi i X)^n}{n!}
\end{aligned}
$$

Proof.

$$
\begin{aligned}
r_f(X) &= \int_0^{i\infty} f(t)(t-x)^w dx \\
&= \int_0^{i\infty} \sum_{l=1}^{\infty} a(l) e^{2\pi i l t} \sum_{n=0}^{w} C_n^w \ t^{w-n} (-1)^n x^n dt
\end{aligned}
$$

Let $b = -2\pi i l t$, then we have

$$
\begin{aligned}
r_f(X) &= \int_0^{\infty} \sum_{l=1}^{\infty} a(l) e^{-b} \sum_{n=0}^{w} C_n^w \ b^{w-n} (\frac{1}{-2\pi i l})^{w-n} (-1)^n x^n \frac{db}{-2\pi i l} \\
&= -(\frac{1}{2\pi i})^{w-n+1} \sum_{n=0}^{w} C_n^w \sum_{l=1}^{\infty} \frac{a(l)}{l^{w-n+1}} \int_0^{\infty} e^{-b} b^{w-n} db \ x^n \\
&= -(\frac{1}{2\pi i})^{w-n+1} \sum_{n=0}^{w} C_n^w \sum_{l=1}^{\infty} \frac{a(l)}{l^{w-n+1}} (w-n)! x^n \\
&= -\sum_{n=0}^{\infty} \frac{w!}{n!} \frac{L(f, w-n+1)}{(2\pi i)^{w-n+1}} X^n \\
&= -\frac{w!}{(2\pi i)^{w+1}} \sum_{n=0}^{w} L(f, w-n+1) \frac{(2\pi i X)^n}{n!}
\end{aligned}
$$

□

We normalize r_f to deal with period polynomials of real coefficients

$$P_f(X) = \sum_{n=0}^{w} L(f, w-n+1)\frac{(2\pi X)^n}{n!} = \frac{-(2\pi i)^{w+1}}{w!} r_f(\frac{X}{i})$$

Lemma 4.1.3. *$P_f(X)$ is self-inversive.*

Proof. We know that $r_f|1+S=0$. Recall that

$$P_f(X) = \frac{-(2\pi i)^{w+1}}{w!} r_f(\frac{X}{i})$$

Notice that

$$\begin{aligned}
r_f|S(\frac{X}{i}) &= (\frac{X}{i})^w r_f(\frac{-1}{\frac{X}{i}}) \\
&= (\frac{X}{i})^w r_f(\frac{1}{iX}) \\
&= (\frac{X}{i})^w P_f(\frac{1}{X})\frac{w!}{-(2\pi i)^{w+1}}
\end{aligned}$$

Thus we get

$$\begin{aligned}
P_f(X) &= \frac{-(2\pi i)^{w+1}}{w!}(\frac{X}{i})^w P_f(\frac{1}{X})\frac{w!}{(2\pi i)^{w+1}} \\
&= -(\frac{X}{i})^w P_f(\frac{1}{X}) \\
&= i^k X^w P_f(\frac{1}{X})
\end{aligned}$$

$\square$

Let

$$q_f(X) = \sum_{n=0}^{\frac{w}{2}-1} L(f, w-n+1)\frac{(2\pi X)^n}{n!} + \frac{1}{2}L(f, \frac{k}{2})\frac{(2\pi X)^{\frac{w}{2}}}{\frac{w}{2}}$$

then

$$i^k P_f(X) = q_f(X) + i^k X^w q_f(\frac{1}{X})$$

Proof. We prove that

$$i^k P_f(X) - i^k X^w q_f(\frac{1}{X}) = q_f(X)$$

$$i^k(\sum_{n=0}^{w} L(f, w-n+1)\frac{(2\pi X)^n}{n!} - \sum_{n=0}^{\frac{w}{2}-1} L(f, w-n+1)\frac{(2\pi)^n}{n!}X^{w-n} - \frac{1}{2}L(f, \frac{k}{2})\frac{(2\pi X)^{\frac{w}{2}}}{(\frac{w}{2})!})$$
$$= i^k(\sum_{n=0}^{\frac{w}{2}-1} L(f, w-n+1)\frac{(2\pi X)^n}{n!} + \frac{1}{2}L(f, \frac{k}{2})\frac{(2\pi X)^{\frac{w}{2}}}{(\frac{w}{2})!} +$$
$$\sum_{n=\frac{w}{2}+1}^{w} L(f, w-n+1)\frac{(2\pi X)^n}{n!} - \sum_{n=0}^{\frac{w}{2}-1} L(f, w-n+1)\frac{(2\pi)^n}{n!})X^{w-n})$$

Consider

$$\sum_{n=\frac{w}{2}+1}^{w} L(f, w-n+1)\frac{(2\pi X)^n}{n!}$$

For $n > \frac{w}{2}$, we have

$$w - n + 1 < w - \frac{w}{2} + 1$$

and thus

$$w - n + 1 < \frac{k}{2}$$

Hence we use the functional equation to get

$$\sum_{n=\frac{w}{2}+1}^{w} i^k (2\pi)^{w-n+1} L(f, n+1)\Gamma(n+1)\frac{1}{\Gamma(w-n+1)}\frac{1}{(2\pi)^{n+1}}\frac{(2\pi X)^n}{n!} = \sum_{n=\frac{w}{2}+1}^{w} i^k \frac{(2\pi)^{w-n}}{(w-n)!} L(f, n+1)X^n$$

Let $N = w - n$, so we have

$$\sum_{N=0}^{N=\frac{w}{2}-1} i^k \frac{(2\pi)^N}{(N)!} L(f, w-N+1)X^{w-N}$$

$$
\begin{aligned}
i^k P_f(X) - i^k X^w q_f(\frac{1}{X}) &= i^k(\sum_{n=0}^{\frac{w}{2}-1} L(f, w-n+1)\frac{(2\pi X)^n}{n!} \\
&+ \frac{1}{2}L(f, \frac{k}{2})\frac{(2\pi X)^{\frac{w}{2}}}{(\frac{w}{2})!} + \sum_{n=0}^{n=\frac{w}{2}-1} L(f, w-n+1)\frac{(2\pi)^n}{n!}X^{w-n} \\
&- \sum_{n=0}^{\frac{w}{2}-1} L(f, w-n+1)\frac{(2\pi)^n}{n!})X^{w-n}) \\
&= q_f(X)
\end{aligned}
$$

□

Notice that $r_f(z_0) = 0$ if and only if

$$
P_f(iz_0) = -\frac{(2\pi i)^{w+1}}{w!} r_f(\frac{iz_0}{i})
$$

and $|iz_0| = |z_0|$.

Hence $r_f(X)$ has its zeroes on the unit circle if and only if $P_f(X)$(or $i^k P_f(X)$) has its zeroes there as well.

By theorem 3.1.2, If $q_f(X)$ has all its zeroes inside the closed unit disk, $i^k P_f(X)$ would have all its zeroes on the unit circle, and so would $r_f(X)$.

4.2 Zeroes of Period Polynomials

We now show that the zeroes of the period polynomials lie on the unit circle.

Let

$$
T_m(z) = \sum_{n=0}^{m} \frac{(2\pi)^n}{n!} z^n
$$

and

$$
H_m(z) = z^m T_m(\frac{1}{z}) = \sum_{n=0}^{m} \frac{(2\pi)^n}{n!} z^{m-n}
$$

$$P_m^{(c)}(z) = z^m H_m(z) + cT_m(z)$$

with $|c| = 1$

Theorem 4.2.1. *For $m \geq 20$, the zeroes of $H_m(z)$ lie in the open unit disk, and thus $P_m^{(c)}$ has its zeroes on the unit circle.*

Proof. Recall that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$f(z) = e^{2\pi z} = \sum_{n=0}^{\infty} \frac{(2\pi)^n}{n!} z^n = T_m(z) + R_m(z)$$

$f^{(m+1)}(z) = (2\pi)^{m+1} e^{2\pi z}$ Notice

$|e^{2\pi z}| = |\sum_{n=0}^{\infty} \frac{(2\pi)^n}{n!} z^n| < \sum_{n=0}^{\infty} \frac{(|2\pi z|)^n}{n!} = e^{|2\pi z|}$

$|f^{(m+1)}(z)| \leq (2\pi)^{m+1} e^{|2\pi z|} = (2\pi)^{m+1} e^{2\pi}$ for $|z| = 1$

Hence $|R_m(z)| \leq \frac{(2\pi)^{m+1} e^{2\pi}}{(m+1)!} |z|^{m+1} = \frac{(2\pi)^{m+1} e^{2\pi}}{(m+1)!}$

$|R_m(z)| \leq 0.0007513$ for $m \geq 25$

For $z = e^{i\theta}$ $|e^{2\pi z}| = |e^{2\pi(cos\theta + isin\theta)}| = e^{2\pi cos\theta} \geq e^{-2\pi} \geq 0.001867$

Thus $|T_m(z)| = |e^{2\pi z} - R_m(z)| \geq 0.001867 - 0.0007513 = 0.001157$

Thus we found a lower bound for $T_m(z)$ on the circle, and this would be the lower bound for $H_m(z)$

For $m \geq 25$, $H_m(z)$ can be written as

$H_m(z) = \sum_{n=0}^{m} \frac{(2\pi)^n}{n!} z^{m-n} = \sum_{n=0}^{25} \frac{(2\pi)^n}{n!} z^{m-n} + \sum_{n=26}^{m} \frac{(2\pi)^n}{n!} z^{m-n}$

$\sum_{n=0}^{25} \frac{(2\pi)^n}{n!} z^{m-25} z^{25-n} + \sum_{n=26}^{m} \frac{(2\pi)^n}{n!} z^{m-n}$

Let $g_m(z) = \sum_{n=26}^{m} \frac{(2\pi)^n}{n!} z^{m-n}$

For $|z| = 1$,

we have $|g_m(z)| \leq \sum_{n=26}^{m} \frac{(2\pi)^n}{n!} \leq \sum_{n=26}^{\infty} \frac{(2\pi)^n}{n!} = e^{2\pi} - H_{25}(1) \leq 0.000001823 <$

$|H_{25}(z)| = |z^{m-25} H_{25}(z)|$

Now $g_m(z)$ and $z^{m-25} H_{25}(z)$ are holomorphic inside and on C and

$|g_m(z)| < |z^{m-25} H_{25}(z)|$ so $z^{m-25} H_{25}(z)$ and $g_m(z) + z^{m-25} H_{25}(z) = H_m(z)$ have the same number of zeroes inside C. PARI (as given in [1]) shows that $H_{25}(z)$ has all its 25 roots inside the unit circle.Hence, $z^{m-25} H_{25}(z)$ has all its m zeroes inside C, and so does $H_m(z)$ by Rouche's theorem. Using PARI (as given in [1]) for $m \geq 20$ and $m \leq 24$, the same result is reached. This proves (by Theorem 3.1.2) that $P_m^c(z)$ has its zeroes on the unit circle. □

Lemma 4.2.2. *Let $f \in S_k$ be a normalized Hecke eigenform and $L(f,s)$ be its L-function. Thus for $Re(s) \geq \frac{3k}{4}$, $|L(f,s) - 1| \leq 5 * 2^{-\frac{k}{4}}$ and for $Re(s) \geq \frac{k}{2}$,*

$L(f,s) \leq 1 + 2\sqrt{k} log(2k)$

Theorem 4.2.3. *If $f \in S_k$ is a Hecke eigenform, then $r_f(x)$ has all its zeroes on the unit circle*

Proof. Note that $r_{\lambda f}(X) = \lambda r_f(X)$ for $\lambda \in \mathbb{C}$ (Proof:

$r_{\lambda f}(X) = -\frac{w!}{(2\pi i)^{w+1}} \sum_{n=0}^{w} L(\lambda f, w-n+1) \frac{(2\pi i X)^n}{n!}$

$L(\lambda f, w-n+1) = (2\pi)^s \frac{L^*(\lambda f, s)}{\Gamma(s)} = \frac{(2\pi)^s}{\Gamma(s)} \int_0^\infty (\lambda f)(iy) \frac{y^s}{y} dy$

$= \lambda \frac{(2\pi)^s}{\Gamma(s)} \int_0^\infty f(iy) \frac{y^s}{y} dy = \lambda \frac{(2\pi)^s}{\Gamma(s)} L^*(f,s)$ So

$r_{\lambda f}(X) = \lambda \frac{-w!}{(2\pi i)^{w+1}} \sum_{n=0}^{w} L(\lambda f, w-n+1) \frac{(2\pi i X)^n}{n!}$)

So we can suppose that f is normalized without loss in generality(if not, let

$\lambda = \frac{1}{a(1)}$ and $r_{\lambda f}(X) = \lambda r_f(X)$)

As noted earlier, we prove the roots of $q_f(x)$ are inside C.

Let $m = \frac{k}{2} - 1$ and $|X| = 1$ ($m = \frac{w+2}{2} - 1 = \frac{w}{2}$))

Using Lemma 3.2.2, and on C

$$\begin{aligned}
|H_m(X) - q_f(X)| &\leq \sum_{n=0}^{m-1} |L(f, k-n-1)| \frac{(2\pi)^n}{n!} + \frac{1}{2}|L(f, \frac{k}{2})| \frac{(2\pi)^m}{m!} + \sum_{n=0}^{m} \frac{(2\pi)^n}{n!} \\
&= \sum_{n=0}^{m-1} |L(f, k-n-1)| \frac{(2\pi)^n}{n!} + \frac{1}{2}|L(f, \frac{k}{2})| \frac{(2\pi)^m}{m!} + \sum_{n=0}^{m-1} \frac{(2\pi)^n}{n!} + \frac{(2\pi)^m}{m!} \\
&\leq \sum_{n=0}^{m-1} |L(f, k-n-1) - 1| \frac{(2\pi)^n}{n!} + |1 - \frac{L(f, \frac{k}{2})}{2}| \frac{(2\pi)^m}{m!}
\end{aligned}$$

To make use of the inequalities mentioned in Lemma 4.2.2, we split the series on

the value of $\lfloor \frac{k}{4} \rfloor$

For $n \leq \lfloor \frac{k}{4} \rfloor - 1$

So $k - n - 1 \geq k - \lfloor \frac{k}{4} \rfloor + 1 - 1 \geq \frac{3k}{4}$

Thus

$$\begin{aligned}
|H_m(X) - q_f(X)| &\leq \sum_{n=0}^{\lfloor \frac{k}{4} \rfloor - 1} 5 * 2^{-\frac{k}{4}} \frac{(2\pi)^n}{n!} + \sum_{n=\lfloor \frac{k}{4} \rfloor}^{m-1} |L(f, k-n-1) - 1| \frac{(2\pi)^n}{n!} + |1 - \frac{L(f, \frac{k}{2})}{2}| \frac{(2\pi)^m}{m!} \\
&= \sum_{n=0}^{\lfloor \frac{k}{4} \rfloor - 1} 5 * 2^{-\frac{k}{4}} \frac{(2\pi)^n}{n!} + \sum_{n=\lfloor \frac{k}{4} \rfloor}^{m-1} |L(f, k-n-1) - 1| \frac{(2\pi)^n}{n!} + \\
&\quad |L(f, \frac{k}{2}) - 1| \frac{(2\pi)^m}{m!} + |L(f, \frac{k}{2}) - 1| \frac{(2\pi)^m}{m!} + |1 - \frac{L(f, \frac{k}{2})}{2}| \frac{(2\pi)^m}{m!}
\end{aligned}$$

$$|H_m(X) - q_f(X)| \leq \sum_{n=0}^{\lfloor \frac{k}{4} \rfloor - 1} 5 * 2^{-\frac{k}{4}} \frac{(2\pi)^n}{n!} + \sum_{n=\lfloor \frac{k}{4} \rfloor}^{m} (1 + |L(f, k-n-1)|) \frac{(2\pi)^n}{n!}$$

for $\lfloor \frac{k}{4} \rfloor \leq n \leq m$

for $-m \leq -n \leq -\lfloor \frac{k}{4} \rfloor$

for $-m + k - 1 \leq k - n - 1 \leq -\lfloor \frac{k}{4} \rfloor + k - 1$

For $|X| = 1$ and $k \geq 124$

$$|H_m(X) - q_f(X)| \leq \sum_{n=0}^{\infty} 5 * 2^{-\frac{k}{4}} \frac{(2\pi)^n}{n!} + (1 + 1 + 2\sqrt{k} log 2k) \sum_{n=\lfloor \frac{k}{4} \rfloor}^{\infty} \frac{(2\pi)^n}{n!}$$

$$= 5 * 2^{\frac{-k}{4}}.e^{2\pi} + (2 + 2\sqrt{k} log(2k)) R_{\lfloor \frac{k}{4} \rfloor}(1)$$

$$R_{\lfloor \frac{k}{4} \rfloor}(z) \leq \frac{(2\pi)^{\lfloor \frac{k}{4} \rfloor}}{\lfloor \frac{k}{4} \rfloor !} e^{2\pi}$$

Also, for $k \geq 124$

$(2 + 2\sqrt{k} log 2k) e^{2\pi} \frac{(2\pi)^{\lfloor \frac{k}{4} \rfloor}}{\lfloor \frac{k}{4} \rfloor !} \leq 0.000045$

$5e^{2\pi} 2^{-\frac{k}{4}} \leq 0.0000025$

$|H_m(X) - q_f(X)| < |H_m(X)|$ (using theorem 4.2.1) ,both are holomorphic, $H_m(z)$ has all its m zeroes inside the circle so $q_f(X)$ has its m zeroes inside the circle .

For $12 \leq k \leq 122$, results can be verified using PARI(as given in [1]). The result this is achieved for all k. $\square$

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