# CONCENTRATED FORCE IN
# QUARTER SPACE

By

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ABSTRACT

This thesis is an attempt at solving one of the outstanding problems of three-dimensional elasticity through a novel approach. The problem is that of a concentrated force in quarter space with a boundary free from stress. The approach is that of a double superposition based on the known solutions of the two problems of half space with the concentrated force perpendicular or parallel to the plane boundary as well as the related solutions of other nuclei of strain in half space.

Assume that the force is acting at P and that $\Sigma_{102}$ is the combination of nuclei required at $P_{102}$, mirror image of P with respect to one of the two plane boundaries, in order to eliminate stress on this plane. Assume further that $\Sigma_{102}^{202}$ is the union of combinations of nuclei required at $P_{202}$, mirror image of $P_{102}$ with respect to the second plane boundary, in order to eliminate stress on this plane due to $\Sigma_{102}$. Now, proceed in the opposite order and assume that $\Sigma_{201}$ and $\Sigma_{201}^{202}$ are the combination and union of combinations corresponding to $\Sigma_{102}$ and $\Sigma_{102}^{202}$ respectively. If $\Sigma_{201}^{202}$ is equivalent to $\Sigma_{102}^{202}$ then the union of the force, $\Sigma_{102}$, $\Sigma_{201}$, and $\Sigma_{102}^{202}$ will represent the solution to the problem.

However, the two unions of combinations did not turn out to be equivalent, and this means that this approach has not lead to the solution of this problem. It is hoped that this negative result will throw some light on future attempts at the solutions of this problem.

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CHAPTER I

# THE PROBLEM AND ITS SCOPE

## 1) Introduction

The theory of elasticity is concerned with the study of the response of elastic bodies to the action of forces. A body is elastic if it possesses the property of recovering its original shape when the forces causing the deformations are removed. This property is shared by all substances provided their deformations do not exceed certain limits; it is characterized mathematically by certain functional relationships connecting forces and deformations (displacements) (4)*.

Using Galerkin vector notation the fundamental equation of elasticity can be written as :

$$G(\Delta + \frac{1}{1-v} \text{ grad div}) \underline{u} = 0 \qquad [1]$$

where G is the modulus of elasticity in shear, v is the Poisson's ratio, $\Delta$ is the Laplace operator and $\underline{u} = \underline{i}\, u_x + \underline{j}\, u_y + \underline{k}\, u_z$ is the displacement vector. The body forces are neglected in [1] and in this thesis.

Equation [1] is an equilibrium equation stating the condition that the resultant force on any element of the elastic solid is zero. It makes use of the generalized Hooke's laws that express the relationships between the stresses and displacements in a perfectly elastic isotropic solid (2).

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* Numbers in parantheses refer to Bibliography at the end of the thesis.

Consequently, solving a problem in elasticity reduces to finding a vector function (or its scalar components) which satisfies [1] at every point in the solid and which produces the desired conditions on the boundary. The problem is a first boundary-value if the conditions are on the displacements, a second boundary-value if the conditions are on the stresses, and a mixed boundary-value if some conditions are on the displacements and some on the stresses.

2) <u>The Problem:</u>

This thesis is an attempt at solving the problem of quarter-space under the effect of a concentrated force. Referring to the cartesian coordinate system xyz, a quarter-space is chosen such that $x \geq 0$ and $z \geq 0$. The concentrated force P is assumed to act at (a,0,c), and the boundary conditions are such that the stresses $\sigma_{xx}$, $\sigma_{xy}$ and $\sigma_{xz}$ on $x = 0$ and $\sigma_{xz}$, $\sigma_{yz}$ and $\sigma_{zz}$ on $z = 0$ vanish; i.e., the boundaries $x = 0$ and $z = 0$ are free from stress. So, this is a second boundary-value problem.

This problem has resisted solution in spite of the different approaches that have been attempted. The approach chosen in this thesis, that of nuclei of strain, looks promising because of its straight-fordwardness. However, this approach, as such, proved to be fruitless, and the problem remains unsolved. Nevertheless, it is hoped that the negative result reported here will mark some progress towards solving the problem through providing a definitive analysis of one promising approach.

3) Symbols Used in This Thesis:

| | |
|---|---|
| $G$ | Modulus of elasticity in shear. |
| $v$ | Poisson's ratio. |
| $\underline{F}$ | Galerkin vector. |
| $F_x, F_y, F_z$ | Components of Galerkin vector. |
| $B_x, B_y, B_z, \beta$ | Papkovitch functions. |
| $\underline{u}$ | Displacement vector. |
| $u_x, u_y, u_z$ | Components of displacement vector. |
| $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}$ | Normal components of stress, |
| $\sigma_{xy}, \sigma_{xz}, \sigma_{yz}$ | Shear components of stress. |
| $\Delta$ | Laplace operator. |
| $\delta_x, \delta_y, \delta_z, \ldots$ etc., | Partial derivatives with respect to subscript variable. |
| $\int_x, \int_y, \int_z, \ldots$ etc., | Integration with respect to subscript variable. |
| $\frac{1}{2} \Delta F$ | Potential function for nuclei. |
| $(x,y,z)$ | Arbitrary point of elastic solid. |
| $R$ | Distance from origin to $(x,y,z)$. |
| $X, Y, Z, C$ | Single forces in x,y,z-directions and center of compression. |
| $\int_z \delta_x Z, \int_x C$ | Nuclei derived from single force and center of compression by indicated operations. |
| $(x',y',z')$ | Point at which nucleus is located. |
| $R_{ijk}$ | Distance from $(x',y',z')$ to $(x,y,z)$. |
| $Z_{ijk}, C_{ijk}, \ldots$ etc., | Nucleus located at $(x',y'z')$ where: |

$$
\begin{array}{llllll}
i = 0 & \text{if } x' = 0 & j = 0 & \text{if } y' = 0 & k = 0 & \text{if } z' = 0 \\
= 1 & x' = a & = 1 & y' = b & = 1 & z' = c \\
= -1 & x' = -a & = -1 & y' = -b & = -1 & z' = -c
\end{array}
$$

| | |
|---|---|
| $R_{.02}$ | $\sqrt{a^2 + y^2 + (z + c)^2}$. |
| $R_{20.}$ | $\sqrt{(x + a)^2 + y^2 + c^2}$. |
| $I^3(t)$ | $\frac{1}{R(R+t)}$ . |
| $II^3(t)$ | $\frac{1}{(R + t)}$ . The following relations hold |
| | $\delta_z I^n(z) = -z\, I^{n\ 2}(z) \qquad n = 3,5,7$ |
| | $\delta_z II^n(z) = - I^n(z) \qquad n = 3,5,7$ |
| $\Sigma_{102}$ | Combination of nuclei at (a,0,-c) such that $\Sigma_{102} \cup Z_{101}$ make free boundary at $z = 0$. |
| $\Sigma_{201}$ | Combination of nuclei at (-a,0,c) such that $\Sigma_{201} \cup Z_{101}$ make free boundary at $x = 0$. |
| $\Sigma^{202}_{102}$ | Combination of nuclei at (-a,0,-c) such that $\Sigma_{102} \cup \Sigma^{202}_{102}$ make free boundary at $x = 0$. |
| $\Sigma^{202}_{201}$ | Combination of nuclei at (-a,0,-c) such that $\Sigma_{201} \cup \Sigma^{202}_{102}$ make free boundary at $z = 0$. |
| $\perp$ - solution | Mindlin's solution for a force perpendicular to the boundary in half-space. |
| $\|\|$ - solution | Mindlin's solution for a force parallel to the boundary in half-space. |
| $\frac{P}{8\pi(1-v)}$ | force adjustment so that the solution represents a force of magnitude P. This adjustment factor is omitted from all the tables in the thesis. |

## CHAPTER II

## PREVIOUS WORK

1) Historical Note:

Lord Kelvin, in 1848, was the first to give the displacements and stresses at any point of a homogeneous isotropic solid of indefinite extent resulting from the application of a concentrated force at a point inside the solid. In 1936, Mindlin solved the problem of a concentrated force inside the semi-infinite solid making use of Kelvin's solution. Then, Mindlin in 1953, again, using two previously unrelated techniques: Papkovitch functions and Green's analysis, discovered a new approach suitable for the problem of a concentrated force in the semi-infinite solid. The Papkovitch functions approach has been used for concentrated force problems since that time. Shortly afterwards, in 1955, L. Rongved derived the solution for a force in the interior of a solid of semi-infinite extent with fixed plane boundary. Furthermore, the power of the approach initiated by Mindlin was shown in 1956 when W. Hijab solved mixed boundary problems and problems for bodies of composite boundaries (1,2).

2) Nuclei of Strain:

a) General Statement:-

By differentiation, integration or superposition other solutions can be derived from Kelvin's solution. This set of solutions obtained is called nuclei of strain; and it is found to be useful in solving many interesting problems of concentrated force in the interior of a solid. Mindlin's solution for semi-infinite solid, however, occupies the same position as that of Kelvin's. The nuclei derived from it by differentiation, integration or superposition satisfy the condition of vanishing stresses on a plane boundary. Hence, in solving a problem under the condition of zero stresses on at least one plane, a part of the conditions is satisfied (3).

Realizing the importance of such tools, Mindlin and Cheng calculated forty nuclei of the semi-infinite solid employing the methods of Love, and reported their results in terms of Galerkin vector stress functions (3).

The displacement vector $\underline{u}$ is given in terms of Galerkin vector $\underline{F}$ by:

$$2\,G\,\underline{u} = 2(1-\nu)\,\Delta\,\underline{F} - \text{grad div}\,\underline{F} \qquad [2]$$

where $\underline{F} = \underline{i}\,F_x + \underline{j}\,F_y + \underline{k}\,F_z$ is the Galerkin vector (3).

In order that the displacement vector derived from [2] should satisfy [1], the Galerkin vector should be biharmonic; i.e., $\Delta\Delta\,\underline{F} = 0$.

Lastly, the relationship between the Galerkin vector and Papkovitch functions is given by Mindlin as:

$$\underline{i}\,B_x + \underline{j}\,B_y + \underline{k}\,B_z = \frac{1-\nu}{G}\,\Delta\,\underline{F} \qquad [3]$$

$$\beta = \frac{1-\nu}{G}\left(2\,\text{div}\,\underline{F} - \underline{R}\cdot\Delta\,\underline{F}\right)$$

where $\underline{R} = \underline{i}\,x + \underline{j}y + \underline{k}\,z$ is the radius vector (2)

Also, the following relationships hold:

$$G\,\Delta\,B = -\underline{P}$$

$$G\,\Delta\beta = \underline{R}\cdot\underline{P}$$

where $\underline{B} = \underline{i}\, B_x + \underline{j}\, B_y + \underline{k}\, B_z$ is Papkovitch function and $\underline{P}$ is the body force vector (which is assumed to be zero in this thesis). The displacement vector is given in terms of Papkovitch functions by the following formula:

$$\underline{u} = \underline{B} - \frac{1}{4(1-v)} \quad \Delta\, (\underline{R} \cdot \underline{B} + \beta) \quad (1).$$

b) Notation for Nuclei of Strain:

The basis nuclei are the single forces and center of compression. Other nuclei are derived from these by differentiation and integration. Consequently, the following notation is followed:

X for single force in x-direction,

Y for single force in y-direction,

Z for single force in z-direction, and

C for center of compression (or dilatation).

As for the derived nuclei, the notation will be, by prefixing to the letter of the original nucleus, the operators encountered in derivation. Hence, $\delta$ will be used to denote partial differentiation with subscripts indicating: (1) the variables with respect to which the operation is performed and (2) the number of such differentiations.

Therefore, double forces will be denoted as:

$\delta_z Z$, for double force in z-direction, and

$\delta_x Z$, for double force in z-direction with moment about y-axis.

In contrast to the above, $\int$, the integral sign, will be used and similarly subscripted to indicate the variables and number of integrations. Here are two examples:

$-\int_z \delta_x Z$, a line, along the negative z-axis, of double forces in the z-direction with moment about the y-axis.

$- \int_x C$ , a line of centers of compression along the negative x-axis.

The minus sign appears so that the operation of differentiation of an integral will produce the integrand; where as the integrals resulting from the derivation of nuclei produce upon differentiation the negative of the integrand. (2)

Such abbreviated notation is acceptable because in handling these nuclei of strain, one deals, in fact, with a clearly defined set of functions; and the results of the differentiation or integration are unique.

c) Two Nuclei of Strain:

To clarify the fore-mentioned notions, the derivation of displacements and stresses derived from X and C is written. What follows is a synopsis of the work reported in Lesely's thesis.

i) Nuclei derived from X:

Displacements and stresses of X are derived from the Galerkin vector $\underline{F} = \underline{i}\, R$. Then, for any nucleus derived from X, its vector form is found by subjecting the above vector to the same operations of differentiation or integration as were performed on the displacements and stresses of X in the derivation. Therefore, it will be of the form $\underline{F} = \underline{i}\, F_x$.

The components of displacements $u_x$, $u_y$, $u_z$ are computed using [2] and the stresses $\sigma_{xx}$, $\sigma_{xy}$, $\sigma_{xz}$, $\sigma_{yy}$, $\sigma_{yz}$, $\sigma_{zz}$ follow from these by Hooke's Low.

They are:

$$2\,Gu_x = 2(1 - v)\ \Delta F_x - \delta_{xx} F_x$$

$$2\,Gu_y = \qquad\qquad - \delta_{yy} F_x$$

$$2\,Gu_z = \qquad\qquad - \delta_{zz} F_x$$

$$\sigma_{xx} = (2 - v)\delta_x \Delta F_x - \delta_{xxx} F_x$$

$$\sigma_{yy} = v\ \delta_x\ \Delta F_x - \delta_{yyx} F_x$$

$$\sigma_{zz} = v\ \delta_x\ \Delta F_x - \delta_{zzx} F_x$$

$$\sigma_{xy} = (1 - v)\ \delta_y \Delta F_x - \delta_{xyx} F_x$$

$$\sigma_{yz} = \qquad\qquad - \delta_{yzx} F_x$$

$$\sigma_{zz} = (1 - v)\,\delta_z\ \Delta F_x - \delta_{zxx} F_x$$

ii) Nuclei derived from C:

Let the vector for C be:

$$\underline{F} = - \frac{1}{2(1 - v)} \left(\underline{i}\,\frac{x}{R} + \underline{j}\,\frac{y}{R} + \underline{k}\,\frac{z}{R}\right)$$

Then $\underline{F}$ is the gradient of a scalar function:

$$\underline{i}\,\frac{x}{R} + \underline{j}\,\frac{y}{R} + \underline{k}\,\frac{z}{R} = \text{grad } R.$$

As a result, the vector for any nucleus derived from C is found by subjecting the vector for C to the same operations as performed on the displacements and stresses in derivation; this vector, however, will be the gradient of a function found by doing the same on R.

Denoting this latter function by F., then

$$\underline{F} = - \frac{1}{2(1-2v)} \text{ grad } F.$$

Therefore, by substituting back in [2] the displacement vector in terms of F is

$$2\, G\underline{u} = - \frac{1}{2(1-2v)} \left[\, 2(1-v)\, \Delta - \text{grad div} \right] \text{grad } F.$$

Since div. grad $= \Delta$, then the equation reduces to

$$2\, G\underline{u} = - \text{ grad } (\tfrac{1}{2}\, \Delta\, F).$$

where $(\frac{1}{2}\, \Delta\, F)$ is called the potential function.

As for displacements and stresses they will be given by:

$$2\, Gu_x = - \delta_x(\tfrac{1}{2}\, \Delta F)$$

$$2\, Gu_y = - \delta_y(\tfrac{1}{2}\Delta\, F)$$

$$2\, Gu_z = - \delta_z(\tfrac{1}{2}\, \Delta\, F)$$

$$\sigma_{xx} = - \delta_{xx}(\tfrac{1}{2}\, \Delta\, F)$$

$$\sigma_{yy} = - \delta_{yy}(\tfrac{1}{2}\, \Delta\, F)$$

$$\sigma_{zz} = - \delta_{zz}(\tfrac{1}{2}\, \Delta\, F)$$

$$\sigma_{xy} = - \delta_{xy}(\tfrac{1}{2}\, \Delta\, F)$$

$$\sigma_{yz} = - \delta_{yz}(\tfrac{1}{2}\, \Delta\, F)$$

$$\sigma_{zx} = - \delta_{zx}(\tfrac{1}{2}\, \Delta\, F).$$

d) <u>The Galerkin vectors of some nuclei:</u>

This section stands as a reference table to compare the nuclei of strain used in this thesis to their corresponding Galerkin vectors.

| Nucleus | Galerking Vector | |
|---|---|---|
| $X$ | $\underline{i}\ R$ | |
| $Y$ | $\underline{j}\ R$ | Single force |
| $Z$ | $\underline{k}\ R$ | |
| $\delta_x X$ | $\underline{i}\ \frac{x}{R}$ | |
| $\delta_y Y$ | $\underline{j}\ \frac{y}{R}$ | Double force |
| $\delta_z Z$ | $\underline{k}\ \frac{z}{R}$ | |
| $\delta_y X$ | $\underline{i}\ \frac{y}{R}$ | |
| $\delta_z X$ | $\underline{i}\ \frac{z}{R}$ | |
| $\delta_x Y$ | $\underline{j}\ \frac{x}{R}$ | Double force with moment |
| $\delta_z Y$ | $\underline{j}\ \frac{z}{R}$ | |
| $\delta_x Z$ | $\underline{k}\ \frac{x}{R}$ | |
| $\delta_y Z$ | $\underline{k}\ \frac{y}{R}$ | |
| $\delta_{zz} X$ | $\underline{i}\ (\frac{1}{R} - \frac{z^2}{R^3})$ | |
| $\delta_{xz} X$ | $\underline{i}\ - \frac{xz}{R^3}$ | |
| $\delta_{zz} Z$ | $\underline{k}\ (\frac{1}{R} - \frac{z^2}{R^3})$ | |
| $\delta_{zx} Z$ | $\underline{k}\ - \frac{xz}{R^3}$ | |

| | |
|---|---|
| $\int_z \delta_x X$ | $\underline{i}\ x \log(R + z)$ |
| $\int_x \delta_z X$ | $\underline{i}\ z \log(R + x)$ |
| $\int_x \delta_{zz} X$ | $\underline{i}\ (\log(R + x) + \frac{z^2}{R(R + x)})$ |
| $\int_{xx} \delta_{zz} Z$ | $\underline{k}\ \frac{-z}{R + x}$ |
| $C$ | $\underline{k} \log(R + z) + \underline{i} \log(R + x)$ |
| $\delta_z C$ | $\underline{k} \frac{1}{R} + \underline{i}\ \frac{z}{R(R + x)}$ |
| $\delta_x C$ | $\underline{i} \frac{1}{R} + \underline{k}\ \frac{x}{R(R + z)}$ |
| $\delta_{xx} C$ | $\underline{i}\ \frac{-x}{R^3}$ |
| $\delta_{zz} C$ | $\underline{k}\ \frac{-z}{R^3}$ |
| $\int_z C$ | $\underline{k}\ [\log(R + z) - R]$ |
| $\int_z \delta_x C$ | $\underline{i}\ \log(R + z)$ |
| $\int_{xx} \delta_z C$ | $\underline{k}\ [x \log(R + x) - R]$ |
| $\int_{xx} \delta_{zz} C$ | $\underline{k} - \frac{1}{R + x}$ |

CHAPTER III

METHOD OF APPROACH

The approach attempted is that of nuclei of strain. The nuclei of strain in the semi-infinite solid were calculated by Mindlin and Cheng in 1950. The half-space they considered is defined by $z \geq 0$. Point nuclei are at (0,0,c) and one end of each line nucleus is at the same point. The forty nuclei reported by them have the property of vanishing stresses at the plane boundary $z = 0$. (3) Illustrations 1 and 2 page 21 show the half-space and the quarter-space respectively.

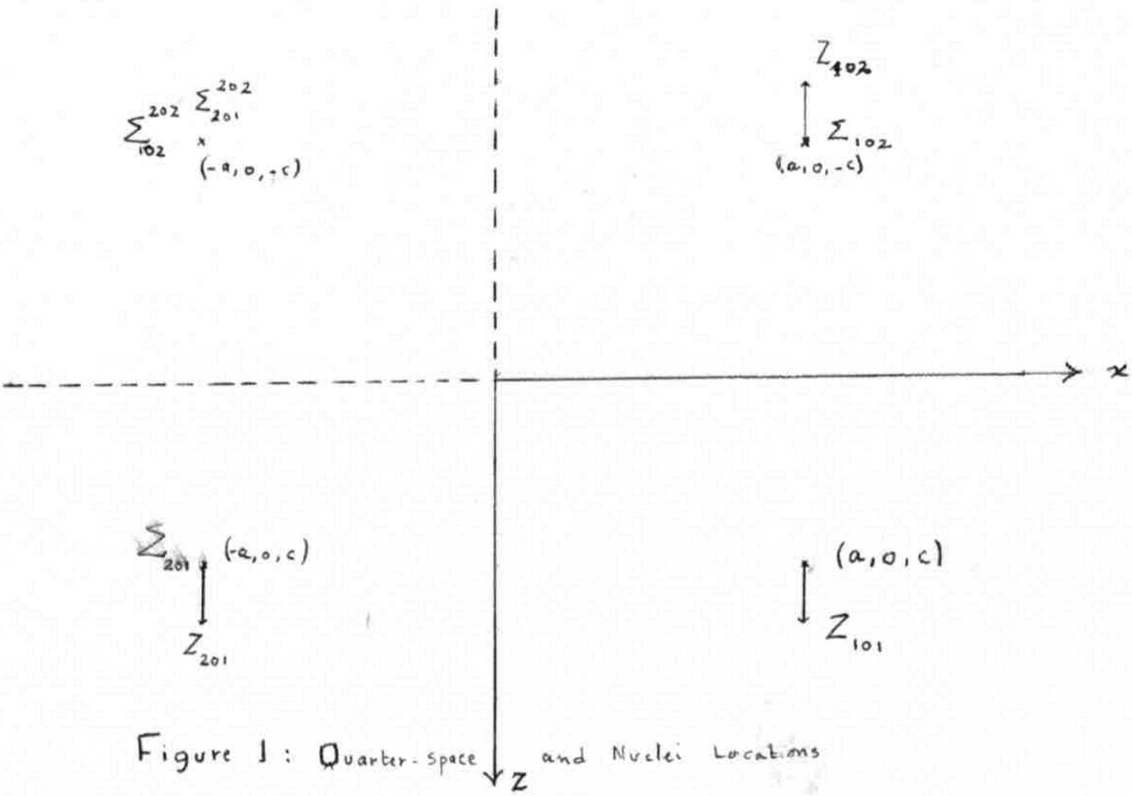


Figure 1: Quarter-space and Nuclei Locations

In order that the above mentioned nuclei can be used, the quarter-space should be looked upon as cases of half-space. Figure 1 shows the quarter-space defined by $x \geq 0$ and $z \geq 0$. $Z_{101}$ is a single force in z-direction located at (a,0,c); hence, it is perpendicular to the boundary $z = 0$ and parallel to the boundary $x = 0$. Considering, firstly, the half-space defined by $z \geq 0$, then $\perp$ -solution of Mindlin will give the combination of nuclei, denoted by $\Sigma_{102}$, located at (a,0,-c) so that $z = 0$ is free from stress due to the action of $Z_{101}$ and $\Sigma_{102}$. However, $Z_{101}$ and $\Sigma_{102}$ don't make free boundary at $x = 0$. To counteract such result, then, for each member of $\Sigma_{102}$ the corresponding combination of nuclei located at (-a,0,-c) is written such that the member of $\Sigma_{102}$ together with the corresponding combination make free boundary at $x = 0$. The combination of all such combinations is denoted by $\Sigma^{202}_{102}$.

Similarly, employing $||$ -solution of Mindlin, $\Sigma_{201}$ will denote the combination of nuclei located at (-a,0,c) so that $Z_{101}$ and $\Sigma_{201}$ make free boundary at $x = 0$ in the half-space defined by $x \geq 0$. Again, $Z_{101}$ and $\Sigma_{201}$ will introduce non-vanishing stresses at $z = 0$. So for each member of $\Sigma_{201}$, the corresponding combination of nuclei located at (-a,0,-c) is written in order that this member together with its corresponding combination make free boundary at $z = 0$. $\Sigma^{202}_{201}$ will denote the combination of all such combinations.

Having done this, $\Sigma^{202}_{102}$ and $\Sigma^{202}_{201}$ are compared. The solution will be complete if these two combinations come out to be the same. Since, some nuclei can be written as the sum of others, they are not. Hence, checking the boundary conditions is inevitable which is done in chapter V.

The following table serves as a summary of the approach and the plan of work. It gives the nuclei with their corresponding combinations and the conditions of stresses on plane boundaries.

TABLE 1

NUCLEI AND STRESSES ON THE BOUNDARIES

| Nucleus or Combinations of Nuclei | Stresses at z = 0 | Stresses at x = 0 |
|---|---|---|
| $Z_{101}$ | $\neq 0$ | $\neq 0$ |
| $Z_{101} \cup \Sigma_{102}$ (Mindlin $\perp$-solution) | $= 0$ | $= 0$ |
| $Z_{101} \cup \Sigma_{201}$ (Mindlin $\vert\vdash$solution) | $\neq 0$ | $= 0$ |
| $\Sigma_{102} \cup \Sigma^{202}_{102}$ | $\neq 0$ | $= 0$ |
| $\Sigma_{201} \cup \Sigma^{202}_{201}$ | $= 0$ | $\neq 0$ |
| $Z_{101} \cup Z_{102} \cup \Sigma_{201} \cup \Sigma^{202}_{201}$ | $= 0$ | $= ?$ |
| $Z_{101} \cup Z_{102} \cup \Sigma_{201} \cup \Sigma^{202}_{102}$ | $= ?$ | $= 0$ |

## CHAPTER IV

## REPORT OF THE WORK

This chapter includes the main body of the thesis. The problem stated again is the quarter space defined by $x \geq 0$ and $z \geq 0$ and the force $\underline{P}$ acts at $(a,0,c)$; the boundary conditions to be satisfied are: $\sigma_{xx} = \sigma_{xy} = \sigma_{xz} = 0$ ) on $x = 0$ and $(\sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0)$ on $z = 0$.

1) Nucleus $Z_{101}$ and Combinations $\Sigma_{102}$ and $\Sigma^{202}_{102}$

For a force in z-direction, Mindlin's $\perp$ -solution will give the following combination of nuclei located at $(a,0,c)$ and $(a,0,-c)$ so that $z = 0$ is free:

$$Z_{101} + (3-4v)Z_{102} - 2c\ \delta_z Z_{102} - 4c\ (1-2v)\ C_{102} + 4(1-v) \int_z C_{102} + 2c^2 \delta_z C_{102}.$$

However, this combination introduces non-vanishing stresses at $x = 0$. To account for this, for each term of the nuclei situated at $(a,0,-c)$ the corresponding combination located at $(-a,0,-c)$ is written. This corresponding combination to-gether with its initial term produce free boundary at $x = 0$. These combinations are taken from Mindlin (3) with suitable interpolation since the force, now, becomes parallel to the boundary plane $x = 0$.

| Nucleus at $(a,0,-c)$ | Corresponding Combination at $(-a,0,-c)$ so that $x = 0$ is free |
|---|---|
| $Z_{102}$ | $Z_{202} + 2a\ \delta_z X_{202} + 2(1-2v) \int_x \delta_z X_{202} - 2a^2\ \delta_z C_{202} + 4(1-v)(1-2v) \int_{xx} \delta_z C_{202}$. |

$\delta_z Z_{102}$ $\qquad$ $\delta_z Z_{202} + 2a\,\delta_{zz} X_{202} + 2(1-2v) \int_x \delta_{zz} X_{202} -$

$$2a^2 \delta_{zz} C_{202} + 4(1-v)(1-2v) \int_{xx} \delta_{zz} C_{202}$$

$C_{102}$ $\qquad$ $(1-4v) C_{202} + 2\,\delta_x X_{202} - 2a\,\delta_x C_{202}$

$\int_z C_{102}$ $\qquad$ $(1-4v) \int_z C_{202} - 2a \int_z \delta_x C_{202} + 2 \int_z \delta_x X_{202}$

$\delta_z C_{202}$ $\qquad$ $(1-4v)\,\delta_z C_{202} - 2a\,\delta_{xz} C_{202} + 2\,\delta_{xz} X_{202}.$

2) Nucleus $Z_{101}$ and Combinations $\Sigma_{201}$ and $\Sigma_{201}^{202}$

The force P, now, is parallel to the boundary $x = 0$. To make $x = 0$ free, then Mindlin's $||$ -solution will be applied. This gives nuclei at $(a,0,c)$ and at $(-a,0,c)$. It is:

$$Z_{101} + Z_{201} + 2a\,\delta_z X_{201} + 2(1-2v) \int_x \delta_z X_{201} - 2a^2 \delta_z C_{201}$$
$$+ 4(1-v)(1-2v) \int_{xx} \delta_z C_{201}.$$

Again, this combination introduces non-vanishing stresses at $z = 0$. So, each element of the combination located at $(-a,0,-c)$ is taken, and for it the corresponding combination of nuclei located at $(-a,0,-c)$ is written so that $z = 0$ is free.

Nucleus at $(-a,0,c)$ $\qquad$ Corresponding combination at $(-a,0,-c)$ so that $z = 0$ is free

$Z_{201}$ $\qquad$ $(3-4v) Z_{202} - 2c\,\delta_z Z_{202} - 4c(1-2v) C_{202}$

$$+ 4(1-v)(1-2v) \int_z C_{202} + 2c^2 \delta_z C_{202}$$

$\delta_z X_{201}$ $\qquad$ $-\delta_z X_{202} + 4c\,\delta_x C_{202} + 2c^2 \delta_{xz} C_{202}$

$$-4(1-v)(1-2v) \int_z \delta_x C_{202} - 2c\,\delta_{xz} Z_{202} - 4(1-v)\,\delta_x Z_{202}$$

$$\int_x \delta_z X_{201} \qquad - \int_x \delta_z X_{202} + 4c\, C_{202} + 2c^2\, \delta_z C_{202} - 4(1-v)(1-2v) \int_z C_{202}$$

$$- 2c\, \delta_z Z_{202} - 4(1-v)\, Z_{202}$$

$$\delta_z C_{201} \qquad (1+4v)\, \delta_z C_{202} + 2c\, \delta_{zz} C_{202} - 2\, \delta_{zz} Z_{202}$$

$$\int_{xx} \delta_z C_{201} \qquad (1+4v) \int_{xx} \delta_z C_{202} + 2c \int_{xx} \delta_{zz} C_{202} - 2 \int_{xx} \delta_{zz} Z_{202}$$

3) Comparison of Nuclei

Having written all the entries of the combinations $\Sigma_{102}^{202}$ and $\Sigma_{201}^{202}$ one needs to compare them. The following table provides such a comparison. The nucleus is written in the left column and the coefficients of that nucleus in $\Sigma_{102}^{202}$ and $\Sigma_{201}^{202}$ are written to the right of it.

It is clear from the table that all the nuclei except one ($\int_z \delta_x C_{202}$) have unequal coefficients. On the other hand, the coefficients of the nuclei: $\delta_{zz} C_{202}$, $\delta_{xz} C_{202}$, $\int_z C_{202}$, and $\int_{xx} \delta_{zz} C_{202}$ differ only in sign. Hence, it could be easily concluded that the solution of the problem seems not to be complete. The following chapter provides for the final checking whether the boundary conditions are satisfied.

TABLE 2

COMPARISON OF THE COEFFICIENTS OF NUCLEI IN $\Sigma_{102}^{202}$ AND $\Sigma_{201}^{202}$

| Nucleus | Coefficients of the Nucleus in $\Sigma_{102}^{202}$ | Coefficients in $\Sigma_{201}^{202}$ |
|---|---|---|
| $Z_{202}$ | $3-4v$ | $3-4v, -8(1-v)(1-2v)$ |
| $\delta_z Z_{202}$ | $-2c$ | $-2c, -4c(1-2v)$ |
| $\delta_x Z_{202}$ | | $-8a(1-v)$ |
| $\delta_{zz} Z_{202}$ | | $4a^2$ |
| $\delta_{zx} Z_{202}$ | | $-4ac$ |
| $\int_{xx} \delta_{zz} Z_{202}$ | | $-8(1-v)(1-2v)$ |
| $\delta_z X_{202}$ | $2a(3-4v)$ | $-2a$ |
| $\delta_{zz} X_{202}$ | $-4ac$ | |
| $\delta_x X_{202}$ | $-8c(1-2v)$ | |
| $\int_z \delta_x X_{202}$ | $8(1-v)(1-2v)$ | |
| $\int_x \delta_z X_{202}$ | $2(1-v)(3-4v)$ | $-2(1-2v)$ |
| $\int_x \delta_{zz} X_{202}$ | $-4c(1-2v)$ | |
| $\delta_{xz} X_{202}$ | $4c^2$ | |
| $C_{202}$ | $-4c(1-2v)(1-4v)$ | $-4c(1-2v), 8c(1-2v)$ |
| $\delta_z C_{202}$ | $-2a^2(3-4v), 2c^2(1-4v)$ | $2c^2, 4c^2(1-2v), -2a^2(1+4v)$ |
| $\delta_{zz} C_{202}$ | $4a^2c$ | $-4a^2c$ |
| $\delta_x C_{202}$ | $8ac\ (1-2v)$ | $8ac$ |

| | | |
|---|---|---|
| $\delta_{xz}\ C_{202}$ | $-4ac^2$ | $4ac^2$ |
| $\int_z C_{202}$ | $4(1-v)(1-2v)(1-4v)$ | $4(1-v)(1-2v)$, <br> $-8(1-v)(1-2v)(1-2v)$ |
| $\int_z \delta_x C_{202}$ | $-8a(1-v)(1-2v)$ | $-8a(1-v)(1-2v)$ |
| $\int_{xx} \delta_z C_{202}$ | $4(1-v)(1-2v)(3-4v)$ | $4(1-v)(1-2v)(1+4v)$ |
| $\int_{xx} \delta_{zz} C_{202}$ | $-8c(1-v)(1-2v)$ | $8c(1-v)(1-2v)$ |

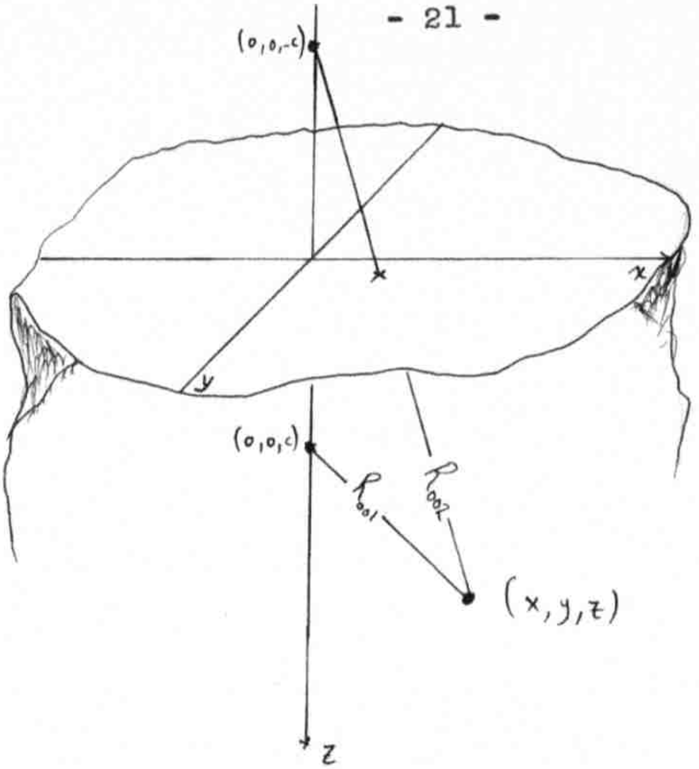


Fig. 1: The Half-space

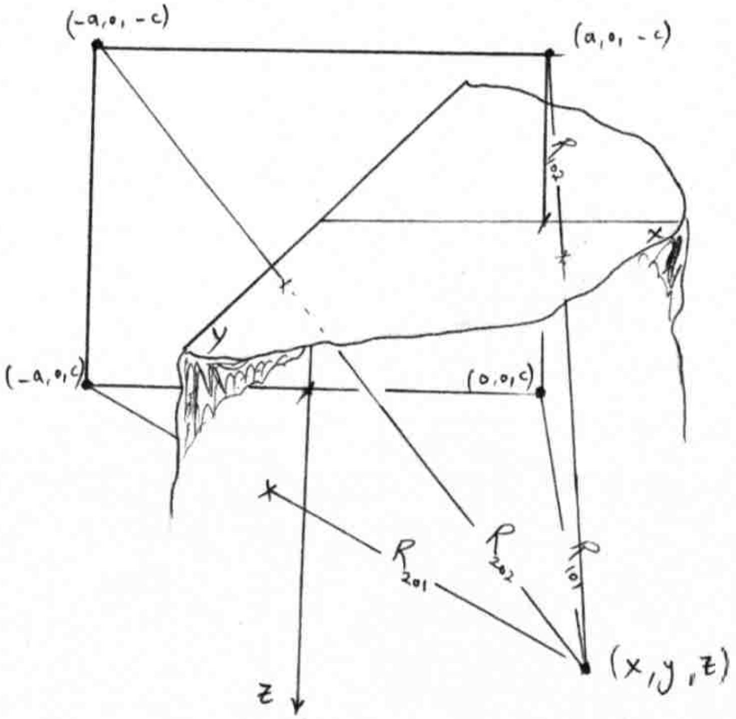


Fig. 3 : The Quarter-space

## CHAPTER V

## STRESSES ON THE BOUNDARIES

The coefficients of nuclei in $\Sigma^{202}_{102}$ and $\Sigma^{202}_{201}$ are not the same. As a last resort, the stresses on the boundaries $x = 0$ and $z = 0$ have to be checked. It is clear from the table at the end of Chapter III that at $z = 0$ the stresses due to $Z_{101} \cup \Sigma_{201} \cup \Sigma_{102} \cup \Sigma^{202}_{102}$ have to be computed. Since $Z_{101} \cup \Sigma_{102}$ make free boundary at $z = 0$ then the stresses due to $\Sigma_{201} \cup \Sigma^{202}_{102}$ , only, need to be written. At $x = 0$ the stresses due to $Z_{101} \cup \Sigma_{102} \cup \Sigma_{201} \cup \Sigma^{202}_{201}$ have to be computed. But $Z_{101} \cup \Sigma_{201}$ make free boundary at $x = 0$. Then, the stresses due to $\Sigma_{102} \cup \Sigma^{202}_{201}$ , only, need to be written.

The tables are arranged as follows: each component of the stress function is written to the left of the page and to its right the coefficients of that component. The stress functions for most of the nuclei are derived by the author. For reference, they were put in the Appendix.

1) Boundary $z = 0$

At this plane boundary there are three components of stress; one is normal $\sigma_{zz}$, and the other two are shear $\sigma_{xz}$ in x-direction and $\sigma_{yz}$ in y-direction along the plane $z = 0$.

a) $\sigma_{xz}$

| | |
|---|---|
| $\frac{a}{R^3_{20}}$. | $-2(1-2v)(3-4v)-8(1-v)(1-2v)-2(1-2v)$ |
| $\frac{(x+a)}{R^3_{20}}$. | $-(1-2v)(3-4v)+2(1-2v)(3-4v)+8(1-2v)(1+2v)(1-v)+4(1-v)(1-2v)(1-4v)$ $-(1-2v)$ |

| | |
|---|---|
| $\frac{a^2(x+a)}{R^5_{20}}.$ | $6(3-4v) + 6$ |
| $\frac{ac^2}{R^5_{20}}.$ | $6(1-2v)(3-4v)-36(1-2v)-24(1-2v) + 12 + 6(1-2v)$ |
| $\frac{c^2(x+a)}{R^5_{20}}.$ | $-3(3-4v)+ 6(1+ 2v)-6(1-2v)(3-4v) + 36(1-2v)-12(1-2v)-3$ |
| | $-6(1-2v).$ |
| | $+24(1-2v)(1+ 2v) + 12(1-2v)(1-4v)-6(1-4v)$ |
| $\frac{a(x+a)}{R^5_{20}}.$ | $- 6(3-4v)+ 24(1-v)(1-2v)$ |
| $\frac{(x+a)^3}{R^5_{20}}.$ | $-24(1-v)(1-2v)$ |
| $\frac{ac^4}{R^7_{20}}.$ | $60(1-2v)-60$ |
| $\frac{c^4(x+a)}{R^7_{20}}.$ | $-30-60(1-2v) + 60(1 + 2v) + 30(1-4v)$ |
| $\frac{a^2c^2(x+a)}{R^7_{20}}.$ | $-30(3-4v) + 180 - 30$ |
| $\frac{ac^2(x+a)^2}{R^7_{20}}.$ | $30(3-4v) - 180 + 120(1-2v) - 60 + 30$ |
| $\frac{c^2(x+a)^3}{R^7_{20}}.$ | $60 - 120(1-2v)$ |
| $\frac{ac^4(x+a)^2}{R^9_{20}}.$ | $420 + 420$ |
| $\frac{a^2c^4(x+a)^3}{R^9_{20}}.$ | $-420$ |
| $\frac{c^4(x+a)^3}{R^9_{20}}.$ | $-420$ |

| | |
|---|---|
| $I^3(x+a)$ | $-4(1-v)(1-2v)(3-4v)+4(1-v)(1-2v)(3-4v)+4(1-v)(1-2v)$ |
| | $-4(1-v)(1-2v) = 0$ |
| $c^2\, I^5(x+a)$ | $4(1-v)(1-2v)(3-4v)-24(1-v)(1-2v)-4(1-v)(1-2v)(3-4v)$ |
| | $+24(1-v)(1-2v)-4(1-v)(1-2v)+4(1-v)(1-2v) = 0$ |
| $c^4\, I^7(x+a)$ | $8(1-v)(1-2v)-8(1-v)(1-2v) = 0$ |

b) $\underline{\sigma_{zy}}$

| | |
|---|---|
| $\dfrac{y}{R^3_{20.}}$ | $-(1-2v)(3-4v)+2(1-2v)(3-4v)+8(1-v)(1-2v)+4(1-v)(1-2v)(1-4v)$ |
| | $-(1-2v)+2(1-2v)$ |
| $\dfrac{a^2 y}{R^5_{20.}}$ | $6(3-4v)+6$ |
| $\dfrac{c^2 y}{R^5_{20.}}$ | $-3(3-4v)+6(1+2v)-6(1-2v)(3-4v)+36(1-2v)-12$ |
| | $+12(1-2v)(1-4v)-3-6(1-2v)$ |
| | $-6(1-4v)+24(1-2v)$ |
| $\dfrac{ay(x+a)}{R^5_{20.}}$ | $-6(3-4v)+24(1-v)(1-2v)-6$ |
| $\dfrac{y(x+a)^2}{R^5_{20.}}$ | $-24(1-v)(1-2v)$ |
| $\dfrac{a^2c^2 y}{R^7_{20.}}$ | $30(3-4v)-180-30$ |
| $\dfrac{c^4 y}{R^7_{20.}}$ | $30-60(1-2v)+60+30(1-4v)$ |
| $\dfrac{ac^2y(x+a)}{R^7_{20.}}$ | $30(3-4v)-180+120(1-2v)-60+30$ |
| $\dfrac{c^2y(x+a)}{R^7_{20.}}$ | $60-120(1-2v)$ |

$\frac{ac^4y(x+a)}{R_{20}^9}$ $420 + 420$

$\frac{a^2c^4\,y}{R_{20}^9}$ $-420$

$\frac{c^4y(x+a)^2}{R_{20}^9}$ $-420$

$yII^5(x+a)$ $-4(1-v)(1-2v)(3-4v) - 4(1-v)(1-2v)$

$c^2y\,II^7(x+a)$ $4(1-v)(1-2v)(3-4v) - 24(1-v)(1-2v) + 4(1-v)(1-2v)$

$c^4\,II^9(x+a)$ $24(1-v)(1-2v)$

c) $\underline{\sigma_{zz}}$

$\frac{c}{R_{20}^3}$

$$-(1-2v)(3-4v) + 2(1-2v) + 2(1-2v)(3-4v)(3-2v) - 12(1-2v)(3-2v) + (1-2v)$$

$$-4(1-2v)(1-4v) + 4(1-v)(1-2v)(1-4v) + 16(1-v)(1-2v)$$

$$-8(1-2v)(1-2v) - 2(1-2v)(3-2v)$$

$\frac{c^3}{R_{20}^5}$

$$-3(3-4v) + 12(1+v) - 6(1-2v)(3-4v) + 24(1-2v)(3-v) - 12(3-2v) - 3$$

$$+(1-2v)(1-4v) - 18(1-4v) + 48(1-2v) + 6(1-2v)$$

$\frac{a^2c}{R_{20}^5}$ $18(3-4v) - 36 - 18$

$\frac{ac(x+a)}{R_{20}^5}$ $-6(3-2v)(3-4v) + 12(3-2v) - 24(1-2v) + 24(1-v)(1-2v) + 6(3-2v)$

$\frac{c(x+a)^2}{R_{20}^5}$ $-24(1-v)(1-2v) + 24(1-2v)(1-2v)$

$\frac{c^5}{R_{20}^7}$ $-30 - 60(1-2v) + 60 + 30(1-4v) = 0$

$\frac{a^2c^3}{R^7_{20}}$ $\quad -30(3-4v) + 360 + 30$

$\frac{ac^3(x+a)}{R^7_{20}}$ $\quad 30(3-4v) - 120(3-4v) + 120(1-2v) - 180 - 30$

$\frac{c^3(x+a)}{R^7_{20}}$ $\quad 60(3-2v) - 120(1-2v)$

$\frac{ac^5(x+a)}{R^9_{20}}$ $\quad 420 + 420$

$\frac{a^2c^5}{R^9_{20}}$ $\quad -420$

$\frac{c^5(x+a)^2}{R^9_{20}}$ $\quad -420$

$I^3(x+a)$ $\quad 8(1-v)(1-2v)(1+2v)$

$(x+a)^2\, I^5(x+a)$ $\quad -16v(1-v)(1-2v)$

$cII^5(x+a)$ $\quad -12(1-v)(1-2v)(3-4v) + 24(1-v)(1-2v) + 12(1-v)(1-2v)$

$c^3II^7(x+a)$ $\quad 4(1-v)(1-2v)(3-4v) - 48(1-v)(1-2v) - 4(1-v)(1-2v)$

$c^5II^9(x+a)$ $\quad 8(1-v)(1-2v)$

2) Boundary $x = 0$

The normal stress at this boundary is $\sigma_{xx}$, whereas $\sigma_{xy}$ and $\sigma_{xz}$ are shear components in y and z directions respectively.

a) $\sigma_{xz}$

$\frac{a}{R^3_{20}}$ $\quad -(1-2v)(3-4v) + 8(1-v)(1-2v)(1-2v) + 2(1-2v) - 2(1-2v)$
$-4(1-v)(1-2v)(1-4v) - 8(1-v)(1-2v)\ \ 8(1-v)(1-2v) + (1-2v)(3-4v)$
$-4(1-v)(1-2v)$

| | |
|---|---|
| $\frac{ac^2}{R_{02}^5}$ | $-6(3-4v) - 12 + 6$ |
| $\frac{a^3}{R_{02}^5}$ | $-12(1+2v) + 6 + 6(1+4v) + 24(1-v)(1-2v) - 24(1-v)(1-2v) = 0$ |
| $\frac{ac(z+c)}{R_{02}^5}$ | $6(1+2v)(3-4v) - 12(1-2v) + 12(1+2v) - 24 - 6(1+2v) - 12(1-2v)$ |
| $\frac{a(z+c)^2}{R_{02}^5}$ | $-3(3-4v) + 24(1-v)(1-2v) - 6(1-2v) + 6(1-2v) + 24(1-v) + 3(3-4v)$ |
| $\frac{a^3c^2}{R_{02}^7}$ | $60$ |
| $\frac{ac(z+c)^3}{R_{02}^7}$ | $-30(3-4v) - 60 + 30$ |
| $\frac{a^3(z+c)^2}{R_{02}^7}$ | $120(2+v) - 30 - 30(1+4v) - 120(1-v)$ |
| $\frac{ac^2(z+c)^2}{R_{02}^7}$ | $30(3-4v) + 60 - 30$ |
| $\frac{a^3c(z+c)}{R_{02}^7}$ | $-180 - 60(1+2v) + 120$ |
| $\frac{a^3(z+c)^4}{R_{02}^9}$ | $-420$ |
| $\frac{a^3c^2(z+c)^2}{R_{02}^9}$ | $-420$ |
| $\frac{a^3c(z+c)^3}{R_{02}^9}$ | $420 + 420$ |
| $I^3(a)$ | $-4(1-v)(1-2v) - 4(1-v)(1-2v)(1+4v) + 8(1-v)(1-2v)(1+2v)$ |

| | |
|---|---|
| $(z+c)^2 I^5(a)$ | $4(1-v)(1-2v)+4(1-v)(1-2v)(1+4v)-16(1-v)(1-2v)(2+v)$ |
| $c(z+c)I^5(a)$ | $24(1-v)(1-2v)$ |
| $c(z+c)^3 I^7(a)$ | $-8(1-v)(1-2v)$ |
| $(z+c)^4 I^7(a)$ | $8(1-v)(1-2v)$ |

b) $\sigma_{xy}$

| | |
|---|---|
| $\frac{ay(z+c)}{R^5_{.02}}$ | $-3(3-4v)+2(1-v)(1-2v)-6(1-2v)+6(1-2v)+24(1-v)(1-2v)$ $+3(3-4v)$ |
| $\frac{acy}{R^5_{.02}}$ | $6(3-4v)-12(1-2v)+12-24-6-12$ |
| $\frac{acy(z+c)^2}{R^7_{.02}}$ | $-30(3-4v)-60+30$ |
| $\frac{a^3y(z+c)}{R^7_{.02}}$ | $180-30-30(1+4v)-120(1-v)=0$ |
| $\frac{ac^2y(z+c)}{R^7_{.02}}$ | $30(3-4v)+60-30$ |
| $\frac{a^3cy}{R^7_{.02}}$ | $-60-60+120=0$ |
| $\frac{a^3y(z+c)^3}{R^9_{.02}}$ | $-420$ |
| $\frac{a^2c^2y(z+c)^2}{R^9_{.02}}$ | $-420$ |
| $\frac{a^3cy(z+c)^2}{R^9_{.02}}$ | $420+420$ |

| | |
|---|---|
| $y(z+c)\ I^5(a)$ | $4(1-v)(1-2v)+4(1-v)(1-2v)(1+2v)-24(1-v)(1-2v)$ |
| $ay\ I^5(z+c)$ | $-4(1-v)(1-2v)(1-4v)-8(1-v)(1-2v)-4(1-v)(1-2v)$ |
| $cy\ I^5(a)$ | $8(1-v)(1-2v)$ |
| $cy(z+c)^2 I^7(a)$ | $-8(1-v)(1-2v)$ |
| $a^3y\ I^7(a)$ | $8(1-v)(1-2v)$ |
| $y(z+c)^3 I^7(z+c)$ | $8(1-v)(1-2v)$ |

c) $\underline{\sigma_{xx}}$

| | |
|---|---|
| $\frac{c}{R^3_{.02}}$ | $-2(1-2v)(3-4v)+4(1-2v)+8(1-v)(1-2v)-2(1-2v)-4(1-2v)$ |
| $\frac{z+c}{R^3_{.02}}$ | $(1-2v)(3-4v)-8(1-v)(1-2v)(1-2v)+2(1-2v)(3-4v)$ $+4(1-v)(1-2v)(1+4v)+(1-2v)(3-4v)$ |
| $\frac{a^2c}{R^5_{.02}}$ | $6(3-4v)-12(1-2v)+12+12(3-2v)-72+6+12(1-2v)$ |
| $\frac{a^2(z+c)}{R^5_{.02}}$ | $-3(3-4v)+24(1-v)(1-2v)-36(1-2v)-6(1-2v)+6(1-2v)+6(1+4v)$ $-3(3-4v)+24(1-v)(3-2v)$ |
| $\frac{c(z+c)^2}{R^5_{.02}}$ | $6(1-2v)(3-4v)-24(1-v)(1-2v)+6(1-2v)=0$ |
| $\frac{c^2(z+c)}{R^5_{.02}}$ | $-6(3-4v)-6$ |
| $\frac{ca^2(z+c)^2}{R^7_{.02}}$ | $-30(3-4v)-60-44(3-2v)-30$ |
| $\frac{a^2(z+c)^3}{R^7_{.02}}$ | $60(1-2v)$ |

| | |
|---|---|
| $\frac{a^4(z + c)}{R_{02}^7}$ | $180 - 30 - 30(1+4v) - 120(1-v)$ |
| $\frac{a^2c^2(z + c)}{R_{02}^7}$ | $30(3-4v) + 180 + 30$ |
| $\frac{a^4c}{R_{02}^7}$ | $-60 + 120$ |
| $\frac{a^4(z + c)^3}{R_{02}^9}$ | $-420$ |
| $\frac{a^2c^2(z + c)}{R_{02}^9}$ | $-420$ |
| $\frac{a^4c(z + c)^2}{R_{02}^9}$ | $420 + 420$ |
| $I^3(z + c)$ | $4(1-v)(1-2v)(1-4v) - 4(1-v)(1-2v)$ |
| $a^2\, I^5(z + c)$ | $-4(1-v)(1-2v)(1-4v) + 24(1-v)(1-2v) + 4(1-v)(1-2v)$ |
| $a^4\, I^7(z + c)$ | $8(1-v)(1-2v)$ |
| $(z + c)I^5(a)$ | $-24(1-v)(1-2v)$ |
| $(z + c)^3 I^7(a)$ | $8(1-v)(1-2v)$ |
| $(z + c)II^5(a)$ | $-48v(1-v)(1-2v)$ |
| $(z + c)II^7(a)$ | $-16v(1-2v)$ |

## CONCLUSION

The stresses on the plane boundaries do not vanish though most of the components vanish along y-axis; i.e. for $x = z = 0$. Again, the problem resisted soltuion. The nuclei of strain approach, then, is not adequate for solving this problem. The approach is so limited and is not deep enough, hence the result is little surprising. Such negative result indicates, plainly, that the problem is more deeper than it may be though of at the first sight. It calls for deeper analysis and a stronger approach. A researcher who is interested in contemplating this quarter-space problem should set himself, first, to its intrigning nature and, second, equip himself with more analytic tools. If he wishes to continue with the nuclei of strain approach, then it is necessary to use additional techniques. May be while developing such techniques and applying them he will find that the computing machines will relieve him from spending much time and effort on monotonous drudgery.

# APPENDIX

## TABLES OF STRESS FUNCTIONS

For reference, the tables of stress functions of the nuclei that are enountered are written. The nuclei are assumed to act at (-a,0,c) and the stresses are given for any point in the solid. A factor of $\frac{P}{8\pi(1-v)}$ is omitted all through. R, here, stands for $R_{202}$.

$Z_{202}$

$$\sigma_{zx} = \frac{-(1-2v)(x+a)}{R^3} - \frac{3(x+a)(z+c)^2}{R^5}$$

$$\sigma_{xy} = \frac{-3y(x+a)(z+c)}{R^5}$$

$$\sigma_{yz} = \frac{-(1-2v)y}{R^3} - \frac{3y(z+c)^2}{R^5}$$

$$\sigma_{xx} = \frac{(1-2v)(z+c)}{R^3} - \frac{3(x+a)^2(z+c)}{R^5}$$

$$\sigma_{zz} = \frac{-(1-2v)(z+c)}{R^3} - \frac{3(z+c)^2}{R^5}$$

$\delta_z Z_{202}$

$$\sigma_{zx} = \frac{-3(1+2v)(x+a)(z+c)}{R^5} \quad \frac{15(x+a)(z+c)^3}{R^7}$$

$$\sigma_{yz} = \frac{-3(1+2v)y(z+c)}{R^5} + \frac{15y(z+c)^3}{R^7}$$

$$\sigma_{zz} = \frac{-(1-2v)}{R^3} - \frac{6(1+v)(z+c)^2}{R^5} + \frac{15(z+c)^4}{R^7}$$

$$\sigma_{yx} = \frac{-3y(x+a)}{R^5} + \frac{15y(x+a)(z+c)^2}{R^7}$$

$$\sigma_{xx} = \frac{(1-2v)}{R^3} - \frac{3(1-2v)(z+c)^2}{R^5} - \frac{3(x+a)^2}{R^5} - \frac{15(x+a)^2(z+c)^2}{R^7}$$

$\delta_{zz}\, Z_{202}$

$$\sigma_{xz} = \frac{-3(1+2v)(x+a)}{R^5} + \frac{30(2+v)(x+a)(z+c)^2}{R^7} - \frac{105(x+a)(z+c)^4}{R^9}$$

$$\sigma_{yz} = \frac{-(1+2v)y}{R^5} + \frac{30(2+v)y(z+c)^2}{R^7} - \frac{105\, y(z+c)^4}{R^9}$$

$$\sigma_{zz} = \frac{-9(1+2v)(z+c)}{R^5} + \frac{30(2+v)(z+c)^3}{R^7} - \frac{105(z+c)^4}{R^9}$$

$$\sigma_{xy} = \frac{45y(x+a)(z+c)}{R^7} - \frac{105y(x+a)(z+c)^3}{R^9}$$

$$\sigma_{xx} = \frac{-9(1-2v)(z+c)}{R^5} + \frac{15(1-2v)(z+c)^3}{R^7} + \frac{45(x+a)^2(z+c)}{R^7} - \frac{105(x+a)^2(z+c)^3}{R^9}$$

$\delta_{x}\, Z_{202}$

$$\sigma_{zx} = \frac{-(1-2v)}{R^3} + \frac{3(1-2v)(x+a)^2}{R^5} - \frac{3(z+c)^2}{R^5} + \frac{15(x+a)^2(z+c)^2}{R^7}$$

$$\sigma_{zy} = \frac{3(1-2v)y(x+a)}{R^5} + \frac{15y(x+a)(z+c)^2}{R^7}$$

$$\sigma_{zz} = \frac{3(1-2v)(x+a)(z+c)}{R^5} + \frac{15(x+a)(z+c)^3}{R^7}$$

$$\sigma_{yx} = \frac{-3y(z+c)}{R^5} + \frac{15y(x+a)^2(z+c)}{R^7}$$

$$\sigma_{xx} = \frac{-3(3-2v)(x+a)(z+c)}{R^5} + \frac{15(x+a)^3(z+c)}{R^7}$$

$\delta_{xz}\, Z_{202}$

$$xz = \frac{-3(1+2v)(z+c)}{R^5} + \frac{15(1+2v)(x+a)^2(z+c)}{R^7} + \frac{15(z+c)^3}{R^7} - \frac{105(x+a)^2(z+c)^3}{R^9}$$

$$\sigma_{yx} = \frac{15(1+2v)y(x+a)(z+c)}{R^7} - \frac{105y(x+a)(z+c)^3}{R^9}$$

$$\sigma_{zz} = \frac{3(1-2v)(x+a)}{R^5} + \frac{30(1+v)(x+a)(z+c)^2}{R^7}$$

$$- \frac{105(x+a)(z+c)^4}{R^9}$$

$$\sigma_{xy} = \frac{-3y}{R^5} + \frac{15y(x+a)^2}{R^7} + \frac{15y(z+c)^2}{R^7} - \frac{105y(x+a)^2(z+c)^2}{R^9}$$

$$\sigma_{xx} = \frac{-3(3-2v)(x+a)}{R^5} + \frac{11(3-2v)(x+a)(z+c)^2}{R^7}$$

$$- \frac{105(x+a)^3(z+c)^2}{R^9}$$

$\int_{xx} \delta_{zz} Z_{202}$

$$\sigma_{xz} = -(1+2v)I^3(x+a) + 2(2+v)(z+c)^2 I^5(x+a)$$
$$-(z+c)^4 I^7(x+a)$$

$$\sigma_{zy} = -(1+2v)y\, II^5(x+a) + 2(2+v)y(z+c)^2\, II^7(x+a)$$
$$-y(z+c)^4\, II^9(x+a)$$

$$\sigma_{zz} = -3(1+2v)(z+c)II^5(x+a) + 2(3+v)(z+c)^3\, II^7(x+a)$$
$$-(z+c)^5\, II^9(x+a)$$

$$\sigma_{xy} = 3y(z+c)I^5(x+a) - y(z+c)^3 I^7(x+a)$$

$$\sigma_{xx} = 3(z+c)I^5(x+a) - (z+c)^3 I^7(x+a) + 6v(z+c)II^5(x+a)$$
$$2v(z+c)II^7(x+a).$$

$\delta_z X_{202}$

$$\sigma_{zx} = \frac{-(1-2v)}{R^3} + \frac{3(1-2v)(z+c)^2}{R^5} - \frac{3(x+a)^2}{R^5}$$

$$+ \frac{15(x+a)^2(z+c)^2}{R^7}$$

$$\sigma_{zy} = \frac{-3y(x+a)}{R^5} + \frac{15y(x+a)(z+c)^2}{R^7}$$

$$\sigma_{zz} = \frac{-3(3-2v)(x+a)(z+c)}{R^5} + \frac{15(x+a)(z+c)^3}{R^7}$$

$$\sigma_{xy} = \frac{3(1-2v)y(z+c)}{R^5} + \frac{15y(x+a)^2(z+c)}{R^7}$$

$$\sigma_{xx} = \frac{3(1-2v)(x+a)(z+c)}{R^5} + \frac{15(x+a)^3(z+c)}{R^7}$$

$\delta_{zz}\ X_{202}$

$$\sigma_{zx} = \frac{9(1-2v)(z+c)}{R^5} - \frac{15(1-2v)(z+c)^3}{R^7} + \frac{45(x+a)^2(z+c)}{R^7} - \frac{105(x+a)^2(z+c)^3}{R^9}$$

$$\sigma_{zy} = \frac{45y(x+a)(z+c)}{R^7} - \frac{105y(x+a)(z+c)^3}{R^9}$$

$$\sigma_{zz} = \frac{-3(3-2v)(x+a)}{R^5} + \frac{30(3-v)(x+a)(z+c)^2}{R^7} - \frac{105(x+a)(z+c)^4}{R^9}$$

$$\sigma_{xy} = \frac{3(1-2v)y}{R^5} - \frac{15(1-2v)y(z+c)^2}{R^7} + \frac{15y(x+a)^2}{R^7} - \frac{105y(x+a)^2(z+c)^2}{R^9}$$

$$\sigma_{xx} = \frac{3(1-2v)(x+a)}{R^5} - \frac{15(1-2v)(x+a)(z+c)^2}{R^7} + \frac{15(x+a)^3}{R^7} - \frac{105(x+a)^3(z+c)^2}{R^9}$$

$\delta_x X_{202}$

$$\sigma_{xz} = \frac{-3(1+2v)(x+a)(z+c)}{R^5} + \frac{15(x+a)^3(z+c)}{R^7}$$

$$\sigma_{yz} = \frac{-3y(z+c)}{R^5} + \frac{15y(x+a)^2(z+c)}{R^7}$$

$$\sigma_{zz} = \frac{(1-2v)}{R^3} - \frac{3(1-2v)(x+a)^2}{R^5} - \frac{6(z+c)^2}{R^5} \quad \frac{15(x+a)^2}{R^7}$$

$$\sigma_{xy} = \frac{-3(1+2v)y(x+a)}{R^5} + \frac{15y(x+a)^3}{R^7}$$

$$\sigma_{xx} = \frac{-(1-2v)}{R^3} - \frac{6(1+v)(x+a)^2}{R^5} + \frac{15(x+a)^4}{R^7}$$

$\int_z \delta_x X_{202}$

$$\sigma_{xz} = \frac{(1+2v)(x+a)}{R^3} - \frac{3(x+a)^3}{R^5}$$

$$\sigma_{yz} = \frac{y}{R^3} - \frac{3y(x+a)^2}{R^5}$$

$$\sigma_{zz} = (1+2v)I^3(z+c) - 2v(x+a)^2\, I^5(z+c) \quad \frac{2(z+c)}{R^3}$$

$$- \frac{3(z+c)(x+a)^2}{R^5}$$

$$\sigma_{xy} = (1+2v)I^5(z+c) - (x+a)^3 y\, I^7(z+c)$$

$$\sigma_{xx} = (1-2v)I^3(z+c) + 2(1+v)(x+a)^2\, I^5(z+c) - (x+a)^4 I^7(z+c)$$

$\int_x \delta_z X_{202}$

$$\sigma_{zx} = 2(1-v)I^3(x+a) - 2(1-v)(z+c)^2\, I^5(x+a) \quad + \frac{(x+a)}{R^3}$$

$$- \frac{3(x+a)(z+c)^2}{R^5}$$

$$\sigma_{zy} = \frac{y}{R^3} - \frac{3y(z+c)^2}{R^5}$$

$$\sigma_{zz} = \frac{(3-2v)(z+c)}{R^3} - \frac{3(z+c)^3}{R^5}$$

$$\sigma_{xy} = -\frac{3y(x+a)(z+c)}{R^5} - 2(1-v)y(z+c)\ I^5(x+a)$$

$$\sigma_{xx} = \frac{-(3-2v)(z+c)}{R^3} - \frac{3(x+a)^2(z+c)}{R^5}$$

$\delta_{xz}\ X_{202}$

$$\sigma_{xz} = \frac{-3(1-2v)(x+a)}{R^5} + \frac{15(1+2v)(x+a)(z+c)^2}{R^7}$$

$$+ \frac{15(x+a)^3}{R^7} - \frac{105(z+c)^2(x+a)^3}{R^9}$$

$$\sigma_{zy} = \frac{-3y}{R^5} + \frac{15y(x+a)^2}{R^7} + \frac{15y(z+c)^2}{R^7} - \frac{105(x+a)^2(z+c)^2}{R^9}$$

$$\sigma_{zy} = \frac{-3y}{R^5} + \frac{15y(x+a)^2}{R^7} + \frac{15y(z+c)^2}{R^7} - \frac{105y(x+a)^2(z+c)^2}{R^9}$$

$$\sigma_{zz} = \frac{-3(3-2v)(z+c)}{R^5} + \frac{15(3-2v)(x+a)^2(z+c)}{R^7}$$

$$+ \frac{15(z+c)^3}{R^7} - \frac{105(x+a)^2(z+c)^3}{R^9}$$

$$\sigma_{xy} = \frac{15(1+2v)y(x+a)(z+c)}{R^7} - \frac{105y(x+a)^3(z+c)}{R^9}$$

$$\sigma_{xx} = \frac{3(1-2v)(z+c)}{R^5} \quad \frac{30(1+v)(x+a)^2(z+c)}{R^7}$$

$$- \frac{105(x+a)^4(z+c)}{R^9} .$$

$\int_x \delta_{zz}\ Z_{202}$

$$\sigma_{xz} = -6(1-v)(z+c)I^5(x+a) + 2(1-v)(z+c)^3 I^7(x+a)$$

$$- \frac{9(z+c)(x+a)}{R^5} + \frac{15(x+a)(z+c)^3}{R^7}$$

$$\sigma_{yz} = \frac{-9y(z+c)}{R^5} + \frac{15y(z+c)^3}{R^7}$$

$$\sigma_{zz} = \frac{(3-2v)}{R^3} - \frac{6(3-v)(z+c)^2}{R^5} + \frac{15(z+c)^4}{R^7}$$

$$\sigma_{xy} = \frac{-3y(x+a)}{R^5} + \frac{15(x+a)y(z+c)^2}{R^7} - 2(1-v)yI^5(x+a)$$

$$+ 2(1-v)y(z+c)^2 I^7(x+a)$$

$$\sigma_{xx} = \frac{-(3-2v)}{R^3} + \frac{3(3-2v)(z+c)^2}{R^5} - \frac{3(x+a)^2}{R^5}$$

$$+ \frac{15(x+a)^2(z+c)^2}{R^7}$$

$C_{202}$

$$\sigma_{xz} = \frac{-3(x+a)(z+c)}{R^5}$$

$$\sigma_{zy} = \frac{-3y(z+c)}{R^5}$$

$$\sigma_{zz} = \frac{1}{R^3} - \frac{3(z+c)^2}{R^5}$$

$$\sigma_{xy} = \frac{-3y(x+a)}{R^5}$$

$$\sigma_{xx} = \frac{1}{R^3} - \frac{3(x+a)^2}{R^5}$$

$\delta_z\, C_{202}$

$$\sigma_{zx} = \frac{-3(x+a)}{R^5} + \frac{15(x+a)(z+c)^2}{R^7}$$

$$\sigma_{yz} = \frac{-3y}{R^5} + \frac{15y(z+c)^2}{R^7}$$

$$\sigma_{zz} = \frac{-9(z+c)}{R^5} + \frac{15(z+c)^3}{R^7}$$

$$\sigma_{xy} = \frac{15(x+a)y(z+c)}{R^7}$$

$$\sigma_{xx} = \frac{-3(z+c)}{R^5} + \frac{15(x+a)^2(z+c)}{R^7}$$

$\delta_{zz}\, c_{202}$

$$\sigma_{xz} = \frac{45(x+a)(z+c)}{R^7} - \frac{105(x+a)(z+c)^3}{R^9}$$

$$\sigma_{zy} = \frac{45y(z+c)}{R^7} - \frac{105y(z+c)^3}{R^9}$$

$$\sigma_{zz} = \frac{-9}{R^5} + \frac{90(z+c)^2}{R^7} - \frac{105c^4}{R^9}$$

$$\sigma_{xy} = \frac{15(x+a)y}{R^7} - \frac{105(x+a)y(z+c)^2}{R^9}$$

$$\sigma_{xx} = \frac{-3}{R^5} + \frac{15(z+c)^2}{R^7} + \frac{15(x+a)^2}{R^7} - \frac{105(x+a)^2(z+c)^2}{R^9}$$

$\delta_{x}\, c_{202}$

$$\sigma_{xz} = \frac{-3(z+c)}{R^5} + \frac{15(x+a)^2(z+c)}{R^7}$$

$$\sigma_{yz} = \frac{15(x+a)y(z+c)}{R^7}$$

$$\sigma_{zz} = \frac{-3(x+a)}{R^5} + \frac{15(x+a)(z+c)^2}{R^7}$$

$$\sigma_{xx} = \frac{-9(x+a)}{R^5} + \frac{15(x+a)^3}{R^7}$$

$$\sigma_{xy} = \frac{-3y}{R^5} + \frac{15(x+a)^2y}{R^7}$$

$\delta_{xz}\, c_{202}$

$$\sigma_{zx} = \frac{-3}{R^5} + \frac{15(x+a)^2}{R^7} + \frac{15(z+c)^2}{R^7} - \frac{105(x+a)^2(z+c)^2}{R^9}$$

$$\sigma_{yz} = \frac{15(x+a)y}{R^7} - \frac{105(x+a)y(z+c)^2}{R^9}$$

$$\sigma_{zz} = \frac{45(x+a)(z+c)}{R^7} - \frac{105(x+a)(z+c)^3}{R^9}$$

$$\sigma_{xy} = \frac{15y(z+c)}{R^7} - \frac{105(x+a)^2 y(z+c)}{R^9}$$

$$\sigma_{xx} = \frac{45(x+a)(z+c)}{R^7} - \frac{105(x+a)^3(z+c)}{R^9}$$

$\int_z c_{202}$

$$\sigma_{zx} = \frac{(x+a)}{R^3}$$

$$\sigma_{yz} = \frac{y}{R^3}$$

$$\sigma_{zz} = \frac{(z+c)}{R^3}$$

$$\sigma_{xy} = (x+a)y\ I^5(z+c)$$

$$\sigma_{xx} = -I^3(z+c) + (x+a)^2\ I^5(z+c)$$

$\int_z \delta_x\ c_{202}$

$$\sigma_{zx} = \frac{1}{R^3} - \frac{3(x+a)^2}{R^5}$$

$$\sigma_{yz} = \frac{-3(x+a)y}{R^5}$$

$$\sigma_{zz} = \frac{-3(x+a)(z+c)}{R^5}$$

$$\sigma_{xy} = y\ I^5(z+c) - (x+a)^2\ y\ I^7(z+c)$$

$$\sigma_{xx} = 3(x+a)\ I^5(z+c) - (x+a)^3\ I^7(z+c)$$

$\int_{xx} \delta_z\ c_{202}$

$$\sigma_{zx} = -I^3(x+a) + (z+c)^2\ I^5(x+a)$$

$$\sigma_{yz} = -y\ II^5(x+a) + y(z+c)^2\ II^7(x+a)$$

$$\sigma_{zz} = -3(z+c)II^5(x+a) + (z+c)^3\ II^7(x+a)$$

$$\sigma_{xy} = y(z+c)\ I^5(x+a)$$

$$\sigma_{xx} = \frac{z+c}{R^3}$$

$\int_{xx} \delta_{zz}\, C_{202}$

$$\sigma_{zx} = 3(z+c)\ I^5(x+a) - (z+c)^3\ I^7\ (x+a)$$

$$\sigma_{yz} = 3y(z+c)\ I^7\ (x+a) - y(z+c)^3\ II^9(x+a)$$

$$\sigma_{zz} = -3II^5(x+a) + 6(z+c)^2\ II^7(x+a) - (z+c)^4 II^9(x+a)$$

$$\sigma_{xy} = y\ I^5(x+a) - y(z+c)^2\ I^7(x+a)$$

$$\sigma_{xx} = \frac{1}{R^3} - \frac{3(z+c)^2}{R^5}$$

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