A PROBLEM IN THREE-DIMENSIONAL ELASTICITY

By

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ABSTRACT

Papkovitch functions approach is a new method of solution of the three-dimensional problems of a concentrated force at an interior point of a homogeneous isotropic body. In this thesis the body under consideration is the quarter-space.

The boundary conditions of mixed conditions of zero normal displacement and zero shearing stresses on one plane and either zero displacements or zero stresses on the other, interpreted in terms of displacements and stresses, can be expressed in terms of Papkovitch functions.

The use of Green's Analysis makes possible evaluating the Papkovitch functions of the problem if a sufficient number of expressions, in Papkovitch functions, derived from the boundary conditions, can be found such that they are characterized by having vanishing values on the boundaries and known Laplacians throughout the region.

The determination of Papkovitch functions leads, after direct computations, to the determination of stresses and displacements and hence to the complete solution of the problem.

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NOTATION

$\vec{B}$:	Papkovitch vector function.
B_x, B_y, B_z:	Components of Papkovitch vector function.
C_1:	Limiting position of source point, Q_1, and point of application of Concentrated force.
C_2, C_3, C_4:	Limiting positions of image points, Q_2, Q_3, Q_4, respectively.
$\vec{F}$:	Body force vector.
F_x, F_y, F_z:	Components of body force vector.
G:	Green's function.
G_q:	Green's function for the quarter space.
P(x,y,z):	Arbitrary point of whole region.
$\vec{P}$:	Concentrated force vector.
P_x, P_y, P_z:	Components of Concentrated force vector.
Q:	Arbitrary point of region T.
Q_1:	Source point in region T.
Q_2, Q_3, Q_4:	Image points.
r:	Distance between P and Q.
r_1, r_2, r_3, r_4:	Distances between P and Q_1, Q_2, Q_3, Q_4 respectively.
R_1, R_2, R_3, R_4:	Distances between P and C_1, C_2, C_3, C_4 respectively.
$\vec{r}$:	Position vector.
T:	Region outside of which body forces vanish.
$\vec{U}$:	Displacement vector.
U_x, U_y, U_z:	Components of displacement vector.

x, y, z:	Cartesian Coordinates of P.
β	Fourth Papkovitch function.
μ	Modulus of rigidity.
ν	Poisson's ratio.
$\psi(P,Q)$:	Harmonic function in Coordinates of P.
$\sigma_x, \sigma_y, \sigma_z$:	Normal Components of stress.
$\tau_{xy}, \tau_{yz}, \tau_{zx}$:	Shearing Components of stress.
ξ, η, ζ	Cartesian Coordinates of Q.
$\vec{\nabla}$	$\vec{i}\dfrac{\partial}{\partial x} + \vec{j}\dfrac{\partial}{\partial y} + \vec{k}\dfrac{\partial}{\partial z}$.
Δ	$\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$.

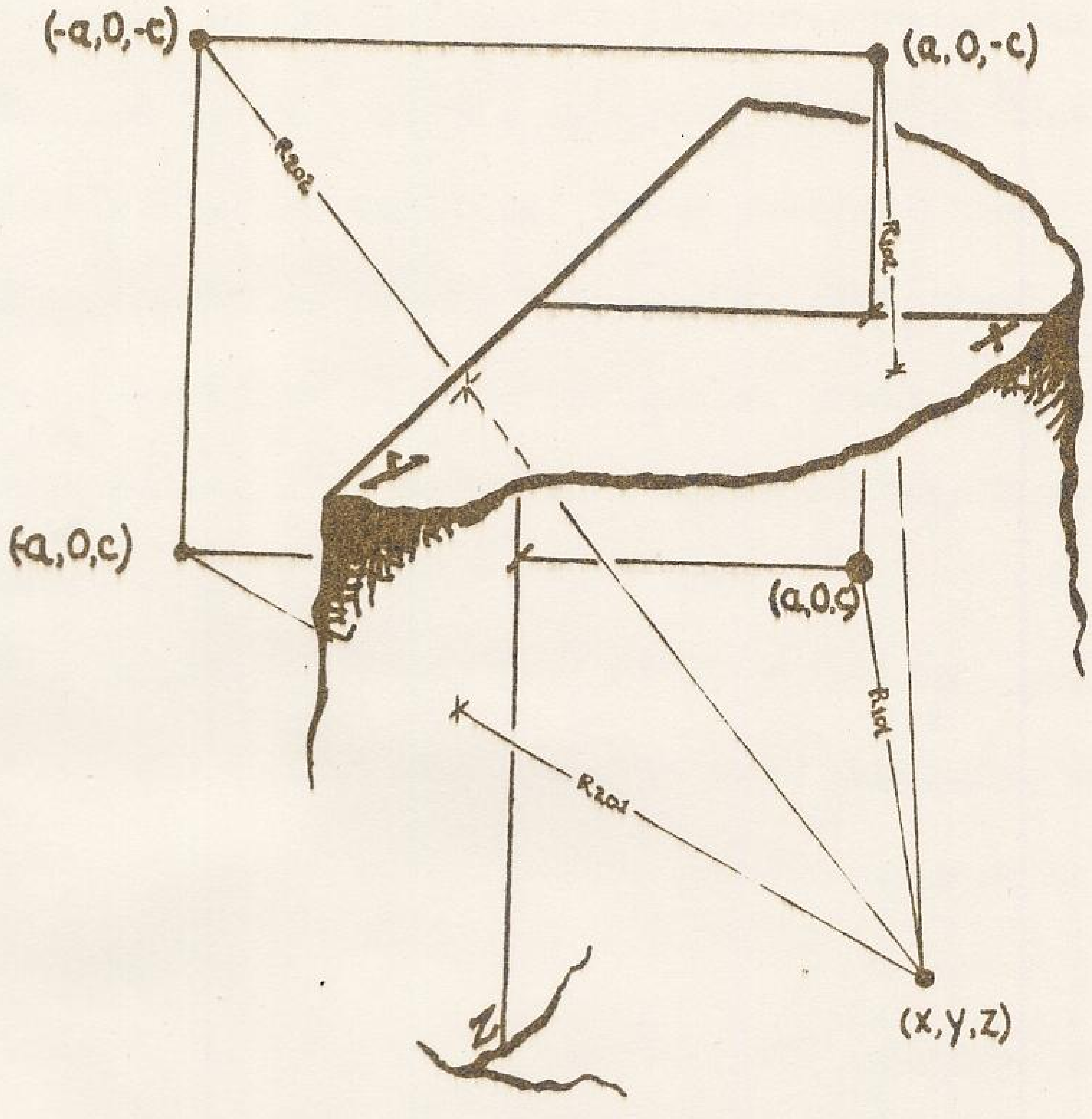

Fig. 1 : The Quarter-space

CHAPTER I

INTRODUCTION

1. Historical Sketch:

P.E. Papkovitch (4)[a] has developed a new solution of the three-dimensional elasticity equations for a homogeneous, isotropic solid in terms of harmonic functions. In 1953, R.D. Mindlin (9) showed how the solution of three-dimensional concentrated force problems may be obtained by an analysis applying Potential theory. The method of attack of concentrated force problems using Papkovitch functions approach has proved to be quite effective since Lord Kelvin's solution of the fundamental problem of a concentrated force acting at a point in a solid of indefinite extent in all directions.

The advantage of the Papkovitch functions approach is that it is a rigourous method that uses Green's analysis, where guessing as an essential procedure in solving problems, such as in the nuclei of strain approach, may be avoided.

In 1955, L. Rongved (5) derived the solution for a concentrated force in the interior of a semi-infinite solid with a fixed plane boundary, and in 1956, W. Hijab (2) solved the mixed boundary-value problem of the half-space using again Papkovitch function approach.

(a) Numbers in paranthesis refer to References at the end of the Thesis.

2. Problem of the Thesis:

This thesis aims to apply the method of Papkovitch functions approach to the solution of two problems of quarter-space:

1) Determination of displacements and stresses caused by a concentrated force acting at a point in the interior of a homogeneous isotropic quarter-space bounded by the planes $x = 0$, $z = 0$; where on $z = 0$ we have mixed boundary conditions of zero normal displacement and zero shearing stresses, and $x = 0$ is a fixed plane boundary.

2) The same problem except that the plane $x = 0$ is free from stress.

3. The Helmholtz Transformation:

It was shown by Helmholtz (6) that if $\vec{F}$ is a vector point-function which, along with its derivatives, is uniform, finite and continuous, and vanishes at infinity or outside a finite region, then it can be expressed as the sum of two other functions in the form:

$$\vec{F} = \text{grad}\ \phi + \text{curl}\ \vec{H}$$

ϕ, $\vec{H}$ are point functions, such that $\text{div}\ \vec{H} = 0$; ϕ is a scalar potential function; $\vec{H}$ is a vector potential function.

In particular the displacement vector $\vec{U}$ can be expressed in the form

$$\vec{U} = \text{grad}\ \phi + \text{curl}\ \vec{H} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

ϕ, $\vec{H}$ have the prescribed properties.

4. Equilibrium Equation:

The vector form of the equilibrium equation is:

$$\Delta \vec{U} + \frac{1}{1-2\nu} \text{ grad div } \vec{U} = - \frac{\vec{F}}{\mu} \quad \text{.................... (2)}$$

where Δ is Laplace's operator; ν is poisson's ratio; $\vec{F}$ is the vector body force per unit of volume; μ is the modulus of rigidity.

If in (1), and (2) we write

$$\alpha = \frac{2(1-\nu)}{1-2\nu} \text{ , we get:}$$

$$\Delta \left[\alpha \text{ grad } \emptyset + \text{curl } \vec{H}\right] = - \frac{\vec{F}}{\mu} \quad \text{.....................(3)}$$

5. The Galerkin Vector:

Since $\vec{H}$ is solenoidal, then it can be represented in the form:

$$\vec{H} = - \text{curl } \vec{W} \quad \text{.................... (4)}$$

The functions $\emptyset$, $\vec{H}$ are independent, therefore we can write

$$\emptyset = \frac{1}{\alpha} \text{ div } \vec{W} \quad \text{.........................(5)}$$

Substituting equations (4) and (5) in (3), we get:

$$\Delta \left[\text{grad div } \vec{W} - \text{cur curl } \vec{W} \right] = \frac{-\vec{F}}{\mu}$$

Using the identity:

curl curl = grad div − Δ , we obtain

$$\Delta \Delta \vec{W} = \frac{-\vec{F}}{\mu} \quad \text{..........................(6)}$$

Substituting equations (4) and (5) in (1) and employing the given identity, we get

$$\vec{U} = \Delta \vec{W} - \frac{1}{2(1-\nu)} \text{ grad div } \vec{W} \quad \text{..................(7)}$$

The vector $\vec{W}$ is identical with the Galeskin vector $\vec{F}$ in the form given by Papkovitch, where the relation between $\vec{W}, \vec{F}$ is given by:

$$\vec{W} = \frac{1-\nu}{\mu} \vec{F}$$

6. The Papkovitch Functions:

The quantity in Parentheses in equation (3) represents a vector function, say $\vec{B}$. Hence

$$\vec{B} = \alpha \operatorname{grad} \phi + \operatorname{curl} \vec{H} \qquad (8)$$

and therefore we have:

$$\operatorname{div} \vec{B} = \alpha \Delta \phi \qquad (9)$$

The complete solution of equation (9) may be written as:

$$\phi = \frac{1}{2\alpha} (\vec{r} . \vec{B} + \beta) \qquad (10)$$

where $\vec{r}$ is the position vector of a field point referred to the origin and

$$\Delta \vec{B} = \frac{-\vec{F}}{\mu} \qquad (11)$$

Hence β which is a scalar function should satisfy the equation:

$$\Delta \beta = - \vec{r} . \Delta \vec{B} \qquad (12)$$

by taking the Laplacian of both sides of equation (10), and observing equation (9).

Making use of equation (11), equation (12) is reduced to:

$$\Delta \beta = \frac{\vec{r} . \vec{F}}{\mu} \qquad (13)$$

Equations (1), (8), and (10) yield the expression for the displacement vector $\vec{U}$:

$$\vec{U} = \vec{B} - \frac{1}{4(1-\nu)} \text{ grad } (\vec{r} \cdot \vec{B} + \beta) \text{ ....(14)}$$

The function $\vec{B}$ and β that satisfy equations (11), (13) and produce the displacements according to equation (14), are the Papkovitch functions.

7. Displacements in terms of Papkovitch functions:

The vector equation (14) can be written in the non-vectorial form:

$$U_x = \frac{1}{4(1-\nu)}\left[(3-4\nu) B_x - x\frac{\partial B_x}{\partial x} - y\frac{\partial B_y}{\partial x} - z\frac{\partial B_y}{\partial x} - \frac{\partial \beta}{\partial x}\right] \quad (15)$$

$$U_y = \frac{1}{4(1-\nu)}\left[(3-4\nu) B_y - x\frac{\partial B_x}{\partial y} - y\frac{\partial B_y}{\partial y} - z\frac{\partial B_z}{\partial y} - \frac{\partial \beta}{\partial y}\right] \text{.....(16)}$$

$$U_z = \frac{1}{4(1-\nu)}\left[(3-4\nu) B_z - x\frac{\partial B_x}{\partial z} - y\frac{\partial B_y}{\partial z} - z\frac{\partial B_z}{\partial z} - \frac{\partial \beta}{\partial z}\right] \text{......(17)}$$

8. Stresses in Terms of Displacement Components:

In case of an isotropic body, the stress components are given in terms of the components of strain by means of the following equations:

$$\sigma_x = \frac{2\nu\mu}{1-2\nu} \text{ div } \vec{U} + 2\mu \frac{\partial U_x}{\partial x} \text{ ....(18)}$$

$$\sigma_y = \frac{2\nu\mu}{1-2\nu} \text{ div } \vec{U} + 2\mu \frac{\partial U_y}{\partial y} \text{ ....(19)}$$

$$\sigma_z = \frac{2\nu\mu}{1-2\nu} \text{ div } \vec{U} + 2\mu \frac{\partial U_z}{\partial z} \text{ ....(20)}$$

$$\tau_{xy} = \mu \gamma_{xy} \text{ ....(21)}$$

$$\tau_{yz} = \mu \gamma_{yz} \quad \text{.......................... (22)}$$

$$\tau_{zx} = \mu \gamma_{zx} \quad \text{.......................... (23)}$$

where $\frac{\partial U_x}{\partial x}$, $\frac{\partial U_y}{\partial y}$, $\frac{\partial U_z}{\partial z}$, γ_{xy}, γ_{yz}, γ_{zx} are the components of strain, with

$$\gamma_{xy} = \frac{\partial U_x}{\partial y} + \frac{\partial U_y}{\partial x} \quad \text{.......................... (24)}$$

$$\gamma_{xz} = \frac{\partial U_x}{\partial z} + \frac{\partial U_z}{\partial x} \quad \text{.......................... (25)}$$

$$\gamma_{yz} = \frac{\partial U_y}{\partial z} + \frac{\partial U_z}{\partial y} \quad \text{.......................... (26)}$$

Therefore, the stress components can be expressed in terms of the displacement components:

$$\sigma_x = 2\mu\left[\frac{\partial U_x}{\partial x} + \frac{\nu}{1 - 2\nu}\left(\frac{\partial U_x}{\partial x} + \frac{\partial U_y}{\partial y} + \frac{\partial U_z}{\partial z}\right)\right] \text{.... (27)}$$

$$\sigma_y = 2\mu\left[\frac{\partial U_y}{\partial y} + \frac{\nu}{1 - 2\nu}\left(\frac{\partial U_x}{\partial x} + \frac{\partial U_y}{\partial y} + \frac{\partial U_z}{\partial z}\right)\right] \text{.... (28)}$$

$$\sigma_z = 2\mu\left[\frac{\partial U_z}{\partial z} + \frac{\nu}{1 - 2\nu}\left(\frac{\partial U_x}{\partial x} + \frac{\partial U_y}{\partial y} + \frac{\partial U_z}{\partial z}\right)\right] \text{..... (29)}$$

$$\tau_{xy} = \mu\left(\frac{\partial U_x}{\partial y} + \frac{\partial U_y}{\partial x}\right) \quad \text{.......................... (30)}$$

$$\tau_{yz} = \mu\left(\frac{\partial U_y}{\partial z} + \frac{\partial U_z}{\partial y}\right) \quad \text{.......................... (31)}$$

$$\tau_{zx} = \mu\left(\frac{\partial U_z}{\partial x} + \frac{\partial U_x}{\partial z}\right) \quad \text{.......................... (32)}$$

9. Definition of Green's Function:

Let ψ be a harmonic point function within a certain region bounded by S, and having at a point of the boundary the value $\frac{1}{r}$, where r is the distance measured from a point $Q(\xi, \eta, \zeta)$ within the region. Then the function:

$$G = \frac{1}{r} - \varphi$$

vanishes at all points of the boundary, and satisfies Laplace's equation throughout the enclosed space, except at the point $Q(\xi, \eta, \zeta)$, where it becomes infinite. This function G is called Green's function for the given region, with pole at $Q(\xi, \eta, \zeta)$.

10. Green's function for the Quarter-space:

For the Quarter-space $x \geqslant 0$, $z \geqslant 0$, Green's function is

$$G_q = \frac{1}{r_1} - \frac{1}{r_2} - \frac{1}{r_3} + \frac{1}{r_4}$$

where

$$r_1 = PQ_1 = \left[(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2\right]^{\frac{1}{2}}$$

$$r_2 = PQ_2 = \left[(x-\xi)^2 + (y-\eta)^2 + (z+\zeta)^2\right]^{\frac{1}{2}}$$

$$r_3 = PQ_3 = \left[(x+\xi)^2 + (y-\eta)^2 + (z-\zeta)^2\right]^{\frac{1}{2}}$$

$$r_4 = PQ_4 = \left[(x+\xi)^2 + (y-\eta)^2 + (z+\zeta)^2\right]^{\frac{1}{2}}$$

where

$Q_1(\xi, \eta, \zeta)$ is the source point, $Q_2(\xi, \eta, -\zeta)$, $Q_3(-\xi, \eta, \zeta)$, $Q_4(-\xi, \eta, -\zeta)$ are the image points, $P(x,y,z)$ is a variable point in the region. Since

$$\varphi = \frac{1}{r_2} + \frac{1}{r_3} - \frac{1}{r_4}$$

We can easily verify that ψ is harmonic throughout the region in the coordinates of P and that G_q is zero when P os on the boundary since $r_1 = r_2$, $r_3 = r_4$ on $z = 0$, and $r_1 = r_3$, $r_2 = r_4$ on x 0.

11. Green's Formula:

If V is a function which is continuous along with its derivative in any direction, then its value at a point P in a region W, denoted by V_p, is given in terms of its values at the boundary of the region W, and the value of ΔV throughout the region, on condition we know the Green's function for the region. Hence V_p is determined by means of the formula

$$-4\pi V_p = \int_S V \frac{\partial G}{\partial n} \, d\sigma + \int_W G \, \Delta V \, d\tau \quad \ldots\ldots\ldots\ldots\ldots\ldots (33)$$

where $\frac{\partial G}{\partial n}$ is the derivative of G in the direction of the unit normal vector to the boundary.

This formula is known as Green's formula.

In particular if the boundary values of V are zero, then the first integral in (33) is zero, and we have:

$$V = -\frac{1}{4\pi} \int_W G \, \Delta V \, d\tau \quad \ldots\ldots\ldots\ldots\ldots\ldots (34)$$

12. Method of Solution:

The concentrated force $\vec{P}$ is considered to act at the point $C_1(a, 0, c)$. This force, by Kelvin's definition, is the limit of the integral $\int_T \vec{F} \, d\tau$ as $T \to 0$, where T is a closed region within the solid body surrounding the point $C_1(a,0,c)$, and $\vec{F}(F_x, F_y, F_z)$ is the vector of body forces distributed in T such that $\vec{F} = 0$

outside T and within the body. Hence we have:

$$\lim_{T \to 0} \int_T \vec{F} \, d\tau = \vec{P} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (35)$$

It was shown in section (7) that the displacement vector can be expressed in terms of the Papkovitch functions, and in section (8) stresses were determined in terms of the displacements, hence any boundary conditions whether of the stress or displacement type can be expressed in terms of Papkovitch functions.

Suppose we could find in general differential expressions in terms of Papkovitch functions that vanish on the boundaries and have known Laplacian throughout the region, then these expressions can be known at every point of the region by applying Green's formula.

Thus if a sufficient number of such expression is determined and from which we can derive the Papkovitch functions that satisfy the prescribed boundary conditions, then the problem is considered to be solved.

At the point $C_1(a,0,c)$, point of application of the concentrated force, Green's function G_q becomes infinite, and Green's formula can't be applied at this point which is considered to be a singular point. Therefore, we exclude this point from the body under consideration. Hence at all points the Laplacians of the Papkovitch functions vanish, and the stresses can, therefore, have a simple form in terms of Papkovitch functions. From equation (14), and equations (27 - 32) we get by substitution:

$$\sigma_x = \frac{\mu}{2(1-\nu)} \left[2(1-\nu) \frac{\partial B_x}{\partial x} - x \frac{\partial^2 B_x}{\partial x^2} - y \frac{\partial^2 B_y}{\partial x^2} - z \frac{\partial^2 B_z}{\partial x^2} - \frac{\partial^2 \beta}{\partial x^2} + 2\nu \left(\frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} \right) \right] \quad \ldots\ldots\ldots\ldots (36)$$

$$\sigma_y = \frac{\mu}{2(1-\nu)}\left[2(1-\nu)\frac{\partial B_y}{\partial y} - x\frac{\partial^2 B_x}{\partial y^2} - y\frac{\partial^2 B_y}{\partial y^2} - z\frac{\partial^2 B_z}{\partial y^2} - \frac{\partial^2 \beta}{\partial y^2}\right.$$

$$\left. + 2\nu\left(\frac{\partial B_z}{\partial z} + \frac{\partial B_x}{\partial x}\right)\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (37)$$

$$\sigma_z = \frac{\mu}{2(1-\nu)}\left[2(1-\nu)\frac{\partial B_z}{\partial z} - x\frac{\partial^2 B_x}{\partial z^2} - y\frac{\partial^2 B_y}{\partial z^2} - z\frac{\partial^2 B_z}{\partial z^2}\right.$$

$$\left. - \frac{\partial^2 \beta}{\partial z^2} + 2\nu\left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y}\right)\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (38)$$

$$\tau_{xy} = \frac{\mu}{2(1-\nu)}\left[(1-2\nu)\left(\frac{\partial B_x}{\partial y} + \frac{\partial B_y}{\partial x}\right) - x\frac{\partial^2 B_x}{\partial x \partial y} - y\frac{\partial^2 B_y}{\partial x \partial y}\right.$$

$$\left. - z\frac{\partial^2 B_z}{\partial x \partial y} - \frac{\partial^2 \beta}{\partial x \partial y}\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (39)$$

$$\tau_{yz} = \frac{\mu}{2(1-\nu)}\left[(1-2\nu)\left(\frac{\partial B_y}{\partial z} + \frac{\partial B_z}{\partial y}\right) - x\frac{\partial^2 B_x}{\partial y \partial z} - y\frac{\partial^2 B_y}{\partial y \partial z}\right.$$

$$\left. - z\frac{\partial^2 B_z}{\partial y \partial z} - \frac{\partial^2 \beta}{\partial y \partial z}\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (40)$$

$$\tau_{zx} = \frac{\mu}{2(1-\nu)}\left[(1-2\nu)\left(\frac{\partial B_z}{\partial x} + \frac{\partial B_x}{\partial z}\right) - x\frac{\partial^2 B_x}{\partial z \partial x} - y\frac{\partial^2 B_y}{\partial z \partial x}\right.$$

$$\left. - z\frac{\partial^2 B_z}{\partial z \partial x} - \frac{\partial^2 \beta}{\partial z \partial x}\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (41)$$

Part I

One Plane Fixed And Zero Normal Displacement And Zero Shearing Stresse on The Other

CHAPTER II

FORCE NORMAL TO FIXED PLANE

1. Boundary Conditions in Terms of Papkovitch Functions:

The solid body under consideration is the quarter-space, $x \geqslant 0$, $z \geqslant 0$ with the exclusion of the point (a,0,c), point of application of the concentrated force. (Fig. 1).

The boundary conditions of the problem are:

Mixed Conditions on $z = 0$, i.e.

$$U_z = 0$$

$$\tau_{zx} = 0 \qquad \text{on } z = 0$$

$$\tau_{zy} = 0$$

The plane $x = 0$ is a fixed plane boundary, i.e.

$$U_x = 0$$

$$U_y = 0 \qquad \text{on } x = 0$$

$$U_z = 0$$

These boundary conditions can be stated in terms of Papkovitch functions:

$$(3 - 4\nu)\, B_z - \frac{\partial}{\partial z}\left(x\, B_x + y\, B_y + \beta\right) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

$$(1 - 2\nu)\left(\frac{\partial B_y}{\partial z} + \frac{\partial B_z}{\partial y}\right) - x\,\frac{\partial^2 B_x}{\partial y \partial z} - y\,\frac{\partial^2 B_y}{\partial y \partial z} - \frac{\partial^2 \beta}{\partial y \partial z} = 0 \quad \ldots\ldots \quad (2)$$

on z = 0

$$(1 - 2\nu)\left(\frac{\partial B_z}{\partial x} + \frac{\partial B_x}{\partial z}\right) - x\,\frac{\partial^2 B_x}{\partial z \partial x} - y\,\frac{\partial^2 B_y}{\partial z \partial x} - \frac{\partial^2 \beta}{\partial z \partial x} = 0 \quad \ldots\ldots \quad (3)$$

$$(3 - 4\nu)\, B_x - \frac{\partial}{\partial x}\left(y\, B_y + z\, B_z + \beta\right) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

$$B_y - \frac{1}{4(1-\nu)}\,\frac{\partial}{\partial y}\left(y\, B_y + z\, B_z + \beta\right) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (5)$$

on x = 0

$$B_z - \frac{1}{4(1-\nu)}\,\frac{\partial}{\partial z}\left(y\, B_y + z\, B_z + \beta\right) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (6)$$

2. Determination of Papkovitch Functions:

Since the concentrated force is taken to be parallel to the x-axis, then the distribution of body forces in the region is given by the following formulas:

$$\lim_{T \to 0} \int_T F_x \, d\tau = P_x \qquad \text{in } x \geqslant 0, \quad z \geqslant 0$$

$$F_y = 0 \qquad \text{in } x \geqslant 0, \quad z \geqslant 0$$

$$F_z = 0 \qquad \text{in } x \geqslant 0, \quad z \geqslant 0$$

Let

$$B_y = 0 \qquad \text{in } x \geqslant 0, \quad z \geqslant 0 \quad \ldots\ldots\ldots \quad (7)$$

$$B_z = 0 \qquad \text{in } x \geqslant 0, \quad z \geqslant 0 \quad \ldots\ldots\ldots \quad (8)$$

and see if it is possible to determine the other Papkovitch functions.

By substitution in equations (1 - 6), we get

$$- x \frac{\partial B_x}{\partial z} - \frac{\partial \beta}{\partial z} = 0 \qquad \text{.......... (9)}$$

$$- x \frac{\partial^2 B_x}{\partial y \partial z} - \frac{\partial^2 \beta}{\partial y \partial z} = 0 \qquad \text{.......... (10)} \quad \text{on } z = 0$$

$$(1 - 2\nu)\frac{\partial B_x}{\partial z} - x \frac{\partial^2 B_x}{\partial z \partial x} - \frac{\partial^2 \beta}{\partial z \partial x} = 0 \qquad \text{.......... (11)}$$

$$(3 - 4\nu) B_x - \frac{\partial \beta}{\partial x} = 0 \qquad \text{.......... (12)}$$

$$\frac{\partial \beta}{\partial y} = 0 \qquad \text{.......... (13)} \quad \text{on } x = 0$$

$$\frac{\partial \beta}{\partial z} = 0 \qquad \text{.......... (14)}$$

Equations (9 - 14) are satisfied if we let

$$\frac{\partial \beta}{\partial z} = 0 \qquad \text{on } x = 0, \quad z = 0 \qquad \text{.......... (15)}$$

$$(3 - 4\nu) \frac{\partial B_x}{\partial z} - \frac{\partial^2 \beta}{\partial x \partial z} = 0 \quad \text{on } x = 0, \quad z = 0 \qquad \text{.......... (16)}$$

Applying Green's formula to $\frac{\partial \beta}{\partial z}$, we get:

$$\frac{\partial \beta}{\partial z} = \lim_{T \to 0} - \frac{1}{4\pi\mu} \int_T G_q \frac{\partial}{\partial \xi} (\xi F_x)\, d\tau$$

$$\int_T G_q \frac{\partial}{\partial \xi} (\xi F_x) d\tau = \int_T \frac{\partial}{\partial \xi} (G_q \xi F_x)\, d\tau - \int_T \xi F_x \frac{\partial G_q}{\partial \xi}\, d\tau$$

$$= \iint G_q \xi F_x \, d\xi\, d\zeta + \int_T \xi F_x \frac{\partial}{\partial z}\left[\frac{1}{r_1} + \frac{1}{r_2} - \frac{1}{r_3} - \frac{1}{r_4}\right] d\tau$$

since

$$\frac{\partial}{\partial \xi}\left[\frac{1}{r_1} - \frac{1}{r_2} - \frac{1}{r_3} + \frac{1}{r_4}\right] = - \frac{\partial}{\partial z}\left[\frac{1}{r_1} + \frac{1}{r_2} - \frac{1}{r_3} - \frac{1}{r_4}\right]$$

Thus

$$\int_T G_q \frac{\partial}{\partial \xi}(\xi F_x)\, d\tau = \int_T \xi F_x \frac{\partial}{\partial z}\left[\frac{1}{r_1} + \frac{1}{r_2} - \frac{1}{r_3} - \frac{1}{r_4}\right] d\tau$$

Since both G_q , F_x vanish on the boundaries. Hence; by taking into consideration that $\lim_{T \to 0} \int_T F_x \, d\tau = P_x$, we find that:

$$\lim_{T \to 0} \int_T G_q \frac{\partial}{\partial \xi}(\xi F_x)\, d\tau = a P_x \frac{\partial}{\partial z}\left[\frac{1}{R_1} + \frac{1}{R_2} - \frac{1}{R_3} - \frac{1}{R_4}\right]$$

Since as $T \longrightarrow 0$, then

$$\xi \longrightarrow a$$

$$\frac{\partial}{\partial z}\left[\frac{1}{r_1} + \frac{1}{r_2} - \frac{1}{r_3} - \frac{1}{r_4}\right] \rightarrow \frac{\partial}{\partial z}\left[\frac{1}{R_1} + \frac{1}{R_2} - \frac{1}{R_3} - \frac{1}{R_4}\right]$$

Therefore,

$$\frac{\partial \beta}{\partial z} = -\frac{aP_x}{4\pi\mu} \frac{\partial}{\partial z}\left[\frac{1}{R_1} + \frac{1}{R_2} - \frac{1}{R_3} - \frac{1}{R_4}\right]$$

Thus,

$$\beta = \frac{-aP_x}{4\pi\mu}\left[\frac{1}{R_1} + \frac{1}{R_2} - \frac{1}{R_3} - \frac{1}{R_4}\right] \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (17)$$

The arbitrary functions introduced by the above integration, which are functions of x and y, must vanish, since the Papkovitch functions vanish as $z \longrightarrow 0$.

Also, applying Green's formula to $(3 - 4\nu)\dfrac{\partial B_x}{\partial z} - \dfrac{\partial^2 \beta}{\partial z \partial x}$, we get

$$(3 - 4\nu)\frac{\partial B_x}{\partial z} - \frac{\partial^2 \beta}{\partial z \partial x} = \lim_{T \to 0} \frac{3 - 4\nu}{4\pi\mu} \int_T G_q \frac{\partial F_x}{\partial \xi}\, d\tau + \lim_{T \to 0} \frac{1}{4\pi\mu} \int_T G_q \frac{\partial^2(\xi F_x)}{\partial \xi \partial \xi}\, d\tau$$

But

$$\int_T G_q \frac{\partial^2(\zeta F_x)}{\partial\zeta\,\partial\xi}\,d\tau = \int_T \frac{\partial}{\partial\zeta}\left[G_q \frac{\partial(\zeta F_x)}{\partial\xi}\right] d\tau - \int_T \frac{\partial(\zeta F_x)}{\partial\xi}\,\frac{\partial G_q}{\partial\zeta}\,d\tau$$

$$= \iint G_q \frac{\partial(\zeta F_x)}{\partial\xi}\,d\eta\,d\xi + \int_T \frac{\partial(\zeta F_x)}{\partial\xi}\,\frac{\partial}{\partial x}\left[\frac{1}{r_1} - \frac{1}{r_2} + \frac{1}{r_3} - \frac{1}{r_4}\right] d\tau$$

$$\int_T G_q \frac{\partial^2(\zeta F_x)}{\partial\zeta\,\partial\xi}\,d\tau = \int_T \frac{\partial(\zeta F_x)}{\partial\xi}\,\frac{\partial}{\partial x}\left[\frac{1}{r_1} - \frac{1}{r_2} + \frac{1}{r_3} - \frac{1}{r_4}\right] d\tau$$

Since both F_x and G_q vanish on the boundaries, and

$$\frac{\partial}{\partial\zeta}\left[\frac{1}{r_1} - \frac{1}{r_2} - \frac{1}{r_3} + \frac{1}{r_4}\right] = -\frac{\partial}{\partial x}\left[\frac{1}{r_1} - \frac{1}{r_2} + \frac{1}{r_3} - \frac{1}{r_4}\right].$$

Also by integrating by parts, we get:

$$\int_T G_q \frac{\partial^2(\zeta F_x)}{\partial\zeta\,\partial\xi}\,d\tau = \int_T \frac{\partial}{\partial\xi}\left[\zeta F_x \frac{\partial}{\partial x}\left(\frac{1}{r_1} - \frac{1}{r_2} + \frac{1}{r_3} - \frac{1}{r_4}\right)\right] d\tau$$

$$- \int_T \zeta F_x \frac{\partial}{\partial\xi}\left[\frac{\partial}{\partial x}\left(\frac{1}{r_1} - \frac{1}{r_2} + \frac{1}{r_3} - \frac{1}{r_4}\right)\right] d\tau$$

$$= \iint \zeta F_x \frac{\partial}{\partial x}\left(\frac{1}{r_1} - \frac{1}{r_2} + \frac{1}{r_3} - \frac{1}{r_4}\right) d\zeta\,d\eta + \int_T \zeta F_x \frac{\partial^2}{\partial z\,\partial x}\left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4}\right) d\tau$$

$$= \int_T \zeta F_x \frac{\partial^2}{\partial x\,\partial z}\left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4}\right) d\tau$$

Hence,

$$\lim_{T\to 0} \int_T G_q \frac{\partial^2(\zeta F_x)}{\partial\zeta\,\partial\xi}\,d\tau = a\,P_x \frac{\partial^2}{\partial x\,\partial z}\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4}\right)$$

Similarly for $\int_T G_q \frac{\partial F_x}{\partial \xi}$, we have

$$\int_T G_q \frac{\partial F_x}{\partial \xi} d\tau = \int_T \frac{\partial}{\partial \xi}(G_q F_x) d\tau - \int_T F_x \frac{\partial G_q}{\partial \xi} d\tau$$

$$= \iint G_q F_x \, d\xi \, d\eta + \int_T F_x \frac{\partial}{\partial z}\left[\frac{1}{r_1} + \frac{1}{r_2} - \frac{1}{r_3} - \frac{1}{r_4}\right] d\tau$$

$$= \int_T F_x \frac{\partial}{\partial z}\left[\frac{1}{r_1} + \frac{1}{r_2} - \frac{1}{r_3} - \frac{1}{r_4}\right] d\tau$$

Thus,

$$\lim_{T \to 0} \int G_q \frac{\partial F_x}{\partial \xi} d\tau = P_x \frac{\partial}{\partial z}\left[\frac{1}{R_1} + \frac{1}{R_2} - \frac{1}{R_3} - \frac{1}{R_4}\right]$$

So finally we have

$$(3 - 4\nu)\frac{\partial B_x}{\partial z} - \frac{\partial^2 \beta}{\partial z \partial x} = \frac{P_x(3-4\nu)}{4\pi\mu} \frac{\partial}{\partial z}\left[\frac{1}{R_1} + \frac{1}{R_2} - \frac{1}{R_3} - \frac{1}{R_4}\right] +$$

$$\frac{a P_x}{4\pi\mu} \frac{\partial^2}{\partial z \partial x}\left[\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4}\right]$$

Integrating with respect to z, we get

$$(3 - 4\nu) B_x = \frac{\partial \beta}{\partial x} + \frac{P_x(3 - 4\nu)}{4\pi\mu}\left(\frac{1}{R_1} + \frac{1}{R_2} - \frac{1}{R_3} - \frac{1}{R_4}\right) + \frac{a P_x}{4\pi\mu} \frac{\partial}{\partial x}\left[\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4}\right]$$

From equation (17) we get by substitution:

$$(3 - 4\nu) B_x = \frac{\partial}{\partial x}\left[\frac{-aP_x}{4\pi\mu}\left(\frac{1}{R_1} + \frac{1}{R_2} - \frac{1}{R_3} - \frac{1}{R_4}\right)\right] + \frac{(3-4\nu)}{4\pi\mu} P_x\left(\frac{1}{R_1} + \frac{1}{R_2} - \frac{1}{R_3} - \frac{1}{R_4}\right)$$

$$+ \frac{aP_x}{4\pi\mu} \frac{\partial}{\partial x}\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4}\right)$$

The arbitrary functions introduced by the above integration, which are functions of x and y, must be equal to zero, since the

Papkovitch function for this problem must vanish as $z \to 0$, hence the appropriate Papkovitch functions for this problem are:

$$B_x = \frac{1}{3-4\nu}\left[\frac{-aP_x}{4\pi\mu}\frac{\partial}{\partial x}\left(\frac{1}{R_1}+\frac{1}{R_2}-\frac{1}{R_3}-\frac{1}{R_4}\right)+\frac{3-4\nu}{4\pi\mu}P_x\left(\frac{1}{R_1}+\frac{1}{R_2}\,\frac{1}{R_3}-\frac{1}{R_4}\right)\right.$$

$$\left.+\frac{a\,P_x}{4\pi\mu}\frac{\partial}{\partial x}\left(\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}+\frac{1}{R_4}\right)\right].$$

or

$$B_x = \frac{1}{3-4\nu}\left[\frac{aP_x}{2\pi\mu}\frac{\partial}{\partial x}\left(\frac{1}{R_3}+\frac{1}{R_4}\right)+\frac{3-4\nu}{4\pi\mu}P_x\left(\frac{1}{R_1}+\frac{1}{R_2}-\frac{1}{R_3}-\frac{1}{R_4}\right)\right] \quad \ldots (18)$$

$$B_y = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (19)$$

$$B_z = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (20)$$

$$\beta = \frac{-a\,P_x}{4\pi\mu}\left[\frac{1}{R_1}+\frac{1}{R_2}-\frac{1}{R_3}-\frac{1}{R_4}\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (21)$$

3. Verification:

We can easily verify that the Papkovitch functions (18 - 21) satisfy the boundaty conditions:

First we note that

on $x = 0$

$(3-4\nu)B_x - \dfrac{\partial\beta}{\partial x} = 0$ since

$$(3-4\nu)B_x - \frac{\partial\beta}{\partial x} = \frac{a\,P_x}{4\pi\mu}\frac{\partial}{\partial x}\left(\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}+\frac{1}{R_4}\right)+\frac{3-4\nu}{4\pi\mu}P_x\left(\frac{1}{R_1}+\frac{1}{R_2}-\frac{1}{R_3}-\frac{1}{R_4}\right)$$

$$= \frac{-aP_x}{4\pi\mu}\left(\frac{x-a}{R_1^3}+\frac{x-a}{R_2^3}+\frac{x+a}{R_3^3}+\frac{x+a}{R_4^3}\right)+$$

$$\frac{3-4\nu}{4\pi\mu}P_x\left(\frac{1}{R_1}+\frac{1}{R_2}-\frac{1}{R_3}-\frac{1}{R_4}\right)$$

the second side vanishes on $x = 0$, since on $x = 0$, we have

$$R_1 = R_3 \qquad R_2 = R_4$$

Also, for the same reason $\beta = 0$ on $x = 0$, which implies that $\frac{\partial \beta}{\partial y} = \frac{\partial \beta}{\partial z} = 0$ on $x = 0$

on $z = 0$

$\frac{\partial B_x}{\partial z} = 0$ since

$$\frac{\partial B_x}{\partial z} = \frac{1}{(3-4\nu)}\left[\frac{-3aP_x}{4\pi\mu}\left(\left(\frac{(x-a)(z-c)}{R_1^5} + \frac{(x-a)(z+c)}{R_2^5} - \frac{(x+a)(z-c)}{R_3^5} - \frac{(x+a)(z+c)}{R_4^5}\right)\right)\right.$$

$$- \frac{(3-4\nu)}{4\pi\mu} P_x\left(\frac{z-c}{R_1^3} + \frac{z+c}{R_2^3} - \frac{z-c}{R_3^3} - \frac{z+c}{R_4^3}\right) +$$

$$\left.\frac{3a\ P_x}{4\pi\mu}\left(\left(\frac{(x-a)(z-c)}{R_1^5} + \frac{(x-a)(z+c)}{R_2^5} + \frac{(x+a)(z-c)}{R_3^4} + \frac{(x+a)(z+c)}{R_4^5}\right)\right)\right]$$

The right hand side vanishes on $z = 0$ because on $z = 0$ $R_1 = R_2$, $R_3 = R_4$. This implies that $\frac{\partial^2 B_x}{\partial y \partial z} = \frac{\partial^2 B_x}{\partial x \partial z} = 0$ on $z = 0$.

Finally,

$$\frac{\partial \beta}{\partial z} = \frac{aP_x}{4\pi\mu}\left[\frac{z-c}{R_1^3} + \frac{z+c}{R_2^3} - \frac{z-c}{R_3^3} - \frac{z+c}{R_4^3}\right] = 0 \quad \text{on} \quad z = 0.$$

This implies that $\frac{\partial^2 \beta}{\partial y \partial z} = \frac{\partial^2 \beta}{\partial x \partial z} = 0$ on $z = 0$.

Now as the plane $x = 0$ recedes to infinity, then $\frac{1}{R_3}$, $\frac{1}{R_4}$ tend to zero, and equations (18 - 21) are reduced after replacing a by zero to:

$$B_x = \frac{P_x}{4\pi\mu}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)$$

$$B_y = 0$$

$$B_z = 0$$

$$\beta = 0$$

These functions coincide with Papkovitch functions for half-space given by W. Hijab (2) (force parallel to the plane). As the plane $z = 0$ recedes to infinity, then $\frac{1}{R_2}$, $\frac{1}{R_4}$ tend to zero, and we get from equations (18-21) after replacing C by zero:

$$B_x = \frac{P_x}{4\pi\mu}\left(\frac{1}{R_1}-\frac{1}{R_3}\right) + \frac{aP_x}{2\pi(3-4\nu)\mu}\frac{\partial}{\partial x}\left(\frac{1}{R_3}\right)$$

$$B_y = 0$$

$$B_z = 0$$

$$\beta = \frac{-aP_x}{4\pi\mu}\left(\frac{1}{R_1}-\frac{1}{R_3}\right)$$

These functions coincide with the Papkovitch functions given by L. Rongved (5) for half space (force normal to the plane boundary).

CHAPTER III

FORCE PARALLEL TO BOTH PLANES

1. Determination of Papkovitch Functions:

The boundary conditions for this case are identical with those given in chapter II. The concentrated force is taken again to pass through the point (a,0,c) but is parallel to the y-axis, hence the distribution of body forces in the region is given by the formulas:

$$F_x = 0 \qquad \text{in } x \geqslant 0,\ z \geqslant 0$$

$$\lim_{T \to 0} \int_T F_y \, d\tau = P_y \qquad " \quad "$$

$$F_z = 0 \qquad " \quad "$$

Let $\quad B_z = 0 \qquad$ in $x \geqslant 0,\ z \geqslant 0.$

From equations (1 - 6) we get by substitution:

$$\frac{\partial}{\partial z}\left(x B_x + y B_y + \beta\right) = 0 \qquad \ldots\ldots (22)$$

on $z = 0$

$$(1 - 2\nu)\frac{\partial B_y}{\partial z} - x\frac{\partial^2 B_x}{\partial y \partial z} - y\frac{\partial^2 B_y}{\partial y \partial z} - \frac{\partial^2 \beta}{\partial y \partial z} = 0 \qquad \ldots\ldots (23)$$

on $z = 0$

$$(1 - 2\nu)\frac{\partial B_x}{\partial z} - x\frac{\partial^2 B_x}{\partial z \partial x} - y\frac{\partial^2 B_y}{\partial z \partial x} - \frac{\partial^2 \beta}{\partial z \partial x} = 0 \qquad \ldots\ldots (24)$$

$$(3 - 4\nu)B_x - \frac{\partial}{\partial x}(y\, B_y + \beta) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (25)$$

$$B_y - \frac{1}{4(1-\nu)}\, \frac{\partial}{\partial y}(y\, B_y + \beta) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (26) \quad \text{on } x = 0$$

$$\frac{\partial}{\partial z}(y\, B_y + \beta) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (27)$$

These equations are satisfied if we set:

$$\frac{\partial B_y}{\partial z} = 0 \qquad \text{on } x = 0,\ z = 0 \quad \ldots\ldots \quad (28)$$

$$\frac{\partial \beta}{\partial z} = 0 \qquad \text{" \quad " \quad "} \quad \ldots\ldots \quad (29)$$

$$(3 - 4\nu)B_x - \frac{\partial}{\partial x}(y\, B_y + \beta) = 0 \quad \text{on } x = 0 \quad \ldots\ldots\ldots\ldots \quad (30)$$

Applying Green's formula to $\frac{\partial B_y}{\partial z}$, we get:

$$\frac{\partial B_y}{\partial z} = \frac{1}{4\pi\mu} \lim_{T \to 0} \int_T G_q \frac{\partial F_y}{\partial \xi}\, d\tau$$

$$= -\frac{1}{4\pi\mu} \lim_{T \to 0} \int_T F_y \frac{\partial G_q}{\partial \xi}\, d\tau$$

by integrating by parts and observing that G_q, F_y vanish on the boundaries, hence

$$\frac{\partial B_y}{\partial z} = \frac{1}{4\pi\mu} \lim_{T \to 0} \int_T F_y \frac{\partial}{\partial z}\left(\frac{1}{r_1} + \frac{1}{r_2} - \frac{1}{r_3} - \frac{1}{r_4}\right) d\tau$$

$$\frac{\partial B_y}{\partial z} = \frac{P_y}{4\pi\mu}\, \frac{\partial}{\partial z}\left(\frac{1}{R_1} + \frac{1}{R_2} - \frac{1}{R_3} - \frac{1}{R_4}\right)$$

$$B_y = \frac{P_y}{4\pi\mu}\left(\frac{1}{R_1} + \frac{1}{R_2} - \frac{1}{R_3} - \frac{1}{R_4}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (31)$$

Also, applying Green's formula to $\frac{\partial \beta}{\partial z}$, we get:

$$\frac{\partial \beta}{\partial z} = -\frac{1}{4\pi\mu} \lim_{T\to 0} \int_T G_q \frac{\partial(\eta F_y)}{\partial \xi} d\tau$$

$$\frac{\partial \beta}{\partial z} = \frac{1}{4\pi\mu} \lim_{T\to 0} \int_T \eta F_y \frac{\partial G_q}{\partial \xi} d\tau = 0$$

since $\eta \to 0$ as $T \to 0$, hence

$$\beta = 0.$$

Thus

$$B_x = \frac{y}{3-4\nu} \frac{\partial}{\partial x}\left[\frac{P_y}{4\pi\mu}\left(\frac{1}{R_1} + \frac{1}{R_2} - \frac{1}{R_3} - \frac{1}{R_4}\right)\right] \quad \text{on} \quad x = 0.$$

The last equation can be written in the form:

$$B_x = \frac{aP_y}{2\pi\mu(3-4\nu)} \left(\frac{y}{R_3{}^3} + \frac{y}{R_4{}^3}\right) = \frac{-aP_y}{2\pi\mu(3-4\nu)} \frac{\partial}{\partial y} \left(\frac{1}{R_3} + \frac{1}{R_3}\right) \quad \text{on } x = 0$$

if we let

$$B_x = \frac{-aP_y}{2\pi\mu(3-4\nu)} \frac{\partial}{\partial y} \left(\frac{1}{R_3} + \frac{1}{R_4}\right) \quad \text{in } x \geq 0,\ z \geq 0,$$

then B_x is harmonic and satisfies equations (25) and 11 (ch.I).

Hence the appropriate Papkovitch functions are:

$$B_x = \frac{-aP_y}{2\pi\mu(3-4\nu)} \frac{\partial}{\partial y} \left(\frac{1}{R_3} + \frac{1}{R_4}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (32)$$

$$B_y = \frac{P_y}{4\pi\mu} \left(\frac{1}{R_1} + \frac{1}{R_2} - \frac{1}{R_3} - \frac{1}{R_4}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (33)$$

$$B_z = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (34)$$

$$\beta = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (35)$$

2. Verification:

To verify that equations (32 - 35) satisfy the boundary conditions, we notice that:

on $x = 0$

$$B_y = 0, \quad \text{hence} \quad \frac{\partial B_y}{\partial y} = \frac{\partial B_y}{\partial z} = 0$$

on $z = 0$

$$\frac{\partial B_x}{\partial z} = \frac{aP_y}{2\pi\mu(3 - 4\nu)} \frac{\partial}{\partial y} \left(\frac{z - c}{R_3^3} + \frac{z + c}{R_4^3}\right) = 0 \text{ , hence}$$

$$\frac{\partial^2 B_x}{\partial y \partial z} = \frac{\partial^2 B_x}{\partial x \partial z} = 0$$

$$\frac{\partial B_y}{\partial z} = \frac{-P_y}{4\pi\mu}\left[\frac{z - c}{R_1^3} + \frac{z + c}{R_2^3} - \frac{z - c}{R_3^4} - \frac{z + c}{R_4^3}\right] = 0 \text{ , hence}$$

$$\frac{\partial^2 B_y}{\partial y \partial z} = \frac{\partial^2 B_y}{\partial x \partial z} = 0.$$

Now as the plane $x = 0$ recedes to infinity (i.e. $\frac{1}{R_3}, \frac{1}{R_4} \rightarrow 0$, $a = 0$), we get:

$$B_x = 0$$

$$B_y = \frac{P_y}{4\pi\mu} \left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

$$B_z = 0$$

$$\beta = 0$$

when $z = 0$ recedes to infinity (i.e. $\frac{1}{R_2}, \frac{1}{R_4} \rightarrow 0$, $c = 0$), we get:

$$B_x = \frac{-aP_y}{2\pi\mu(3-4\nu)} \frac{\partial}{\partial y}\left(\frac{1}{R_3}\right)$$

$$B_y = \frac{P_y}{4\pi\mu}\left(\frac{1}{R_1} - \frac{1}{R_3}\right)$$

$$B_z = 0$$

$$\beta = 0$$

These coincide with the results for half-space.

CHAPTER IV

FORCE NORMAL TO PLANE WITH MIXED BOUNDARY CONDITIONS

1. Determination of Papkovitch functions:

This case is also identical with Chapters II, III, but the concentrated force is parallel this time to the z-axis. The distribution of body forces is:

$$F_x = 0 \qquad \text{in } x \geqslant 0, \quad z \geqslant 0$$

$$F_y = 0 \qquad \text{" \quad " \quad "}$$

$$\lim_{T \to 0} \int_T F_z \, d\tau = P_z \qquad \text{" \quad " \quad "}$$

Let $B_y = 0$ in $x \geqslant 0$, $z \geqslant 0$.

From equations (1 - 6), we get by substitution:

$$(3 - 4\nu)B_z - \frac{\partial}{\partial z}(x\,B_x + \beta) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (36)$$

$$(1 - 2\nu)\frac{\partial B_z}{\partial y} - x\frac{\partial^2 B_x}{\partial y\,\partial z} - \frac{\partial^2 \beta}{\partial y\,\partial z} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (37) \quad \text{on } z = 0$$

$$(1 - 2\nu)\left(\frac{\partial B_z}{\partial x} + \frac{\partial B_x}{\partial z}\right) - x\frac{\partial^2 B_x}{\partial z\,\partial x} - \frac{\partial^2 \beta}{\partial z\,\partial x} = 0 \quad \ldots\ldots\ldots\ldots \quad (38)$$

$$(3 - 4\nu)B_x - \frac{\partial}{\partial x}(z\,B_z + \beta) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (39$$

$$\frac{\partial}{\partial y}(z\,B_z + \beta) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (40) \quad \text{on } x = 0$$

$$B_z - \frac{1}{4(1-\nu)}\frac{\partial}{\partial z}(z\,B_z + \beta) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (41$$

Equations (36 - 41) are satisfied if we set:

$$B_z = 0 \qquad \text{on } x = 0, \quad z = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (42)$$

$$\frac{\partial\beta}{\partial z} = 0 \qquad \text{on } x = 0, \quad z = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (43)$$

$$(3 - 4\nu)B_x - \frac{\partial}{\partial x}(z\,B_z + \beta) = 0 \qquad \text{on } x = 0 \quad \ldots\ldots\ldots\ldots\ldots \quad (44)$$

By applying Green's formula to B_z, we get

$$B_z = \frac{P_z}{4\pi\mu}\left(\frac{1}{R_1} - \frac{1}{R_2} - \frac{1}{R_3} + \frac{1}{R_4}\right)$$

and to $\frac{\partial\beta}{\partial z}$, we get

$$\frac{\partial\beta}{\partial z} = \frac{-cP_z}{4\pi\mu}\frac{\partial}{\partial z}\left(\frac{1}{R_1} + \frac{1}{R_2} - \frac{1}{R_3} - \frac{1}{R_4}\right) \text{, hence}$$

$$\beta = \frac{-cP_z}{4\pi\mu}\left(\frac{1}{R_1} + \frac{1}{R_2} - \frac{1}{R_3} - \frac{1}{R_4}\right)\text{, Thus}$$

$$(3 - 4\nu)B_x = \frac{P_z}{4\pi\mu}\frac{\partial}{\partial x}\left(\frac{z-c}{R_1} - \frac{z+c}{R_2} - \frac{z-c}{R_3} + \frac{z+c}{R_4}\right) \quad \text{on } x = 0$$

which may be written:

$$(3 - 4\nu)B_x = \frac{P_z}{4\pi\mu}\left[\frac{2a(z-c)}{R_3^3} - \frac{2a(z+c)}{R_4^3}\right] \quad \text{on } x = 0$$

$$(3 - 4\nu)B_x = \frac{-aP_z}{2\pi\mu}\frac{\partial}{\partial z}\left(\frac{1}{R_3} - \frac{1}{R_4}\right) \quad \text{on } x = 0$$

If we let

$$(3 - 4\nu)B_x = \frac{-aP_z}{2\pi\mu}\frac{\partial}{\partial z}\left(\frac{1}{R_3} - \frac{1}{R_4}\right) \quad \text{in } x \geqslant 0, \quad z \geqslant 0$$

Then B_x is harmonic and satisfies equations (44) and (11, Ch. I).

Therefore, the Papkovitch functions corresponding to this problem are:

$$B_x = \frac{-aP_z}{2\pi\mu(3-4\nu)} \frac{\partial}{\partial z} \left(\frac{1}{R_3} - \frac{1}{R_4}\right) \quad \ldots\ldots\ldots\ldots \quad (45)$$

$$B_y = 0 \quad \ldots\ldots\ldots\ldots \quad (46)$$

$$B_z = \frac{P_z}{4\pi\mu} \left(\frac{1}{R_1} - \frac{1}{R_2} - \frac{1}{R_3} + \frac{1}{R_4}\right) \quad \ldots\ldots\ldots\ldots \quad (47)$$

$$\beta = \frac{-cP_z}{4\pi\mu} \left(\frac{1}{R_1} + \frac{1}{R_2} - \frac{1}{R_3} - \frac{1}{R_4}\right) ; \quad \ldots\ldots\ldots\ldots \quad (48)$$

2. Verification:

It is easy to see that

on $x = 0$

$B_z = 0$, Hence $\frac{\partial B_z}{\partial y} = \frac{\partial B_z}{\partial z} = 0$

$\beta = 0$, hence $\frac{\partial \beta}{\partial y} = \frac{\partial \beta}{\partial z} = 0$

and on $z = 0$

$B_z = 0$, hence $\frac{\partial B_z}{\partial x} = \frac{\partial B_z}{\partial y} = 0$

$$\frac{\partial \beta}{\partial z} = \frac{cP_z}{4\pi\mu}\left[\frac{z-c}{R_1^3} + \frac{z+c}{R_2^3} - \frac{z-c}{R_3^3} - \frac{z+c}{R_4^3}\right] = 0, \quad \text{hence}$$

$$\frac{\partial^2 \beta}{\partial y \partial z} = \frac{\partial^2 \beta}{\partial x \partial z} = 0, \quad \text{and}$$

$$(3-4\nu)\frac{\partial B_x}{\partial z} = \frac{-aP_z}{2\pi\mu}\frac{\partial^2}{\partial z^2}\left(\frac{1}{R_3}-\frac{1}{R_4}\right)$$

$$= \frac{-aP_z}{2\pi\mu}\left(-\frac{1}{R_3{}^3}+\frac{1}{R_4{}^3}+\frac{3(z-c)^2}{R_3{}^5}-\frac{3(z+c)^2}{R_4{}^5}\right) = 0,$$

hence

$$\frac{\partial^2 B_x}{\partial y \partial z} = \frac{\partial^2 B_x}{\partial y \partial z} = 0 .$$

As the plane $x = 0$ recedes to infinity (i.e. $a = 0$, $\frac{1}{R_3}$, $\frac{1}{R_4} \rightarrow 0$), we get:

$$B_x = 0$$

$$B_y = 0$$

$$B_z = \frac{P_z}{4\pi\mu}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$$

$$\beta = \frac{-cP_z}{4\pi\mu}\left(\frac{1}{R_1}+\frac{1}{R_2}\right).$$

These equations coincide with the equations found by W. Hijab (for a force normal to the plane.)

As now the plane $z = 0$ recedes to infinity, then $c = 0$, $\frac{1}{R_2}$, $\frac{1}{R_4} \rightarrow 0$, and we get:

$$B_x = \frac{-aP_z}{2\pi\mu(3-4\nu)}\frac{\partial}{\partial z}\left(\frac{1}{R_3}\right)$$

$$B_y = 0$$

$$B_z = \frac{P_z}{4\pi\mu}\left(\frac{1}{R_1}-\frac{1}{R_3}\right)$$

$$\beta = 0 .$$

These also coincide with the solution found by L. Rongved (for a force parallel to the plane boundary).

PART II

One Plane Free And Zero Normal Displacement And Zero Shearing Stresses On The Other

CHAPTER V

FORCE NORMAL TO FREE PLANE

1. Boundary Conditions in Terms of Papkovitch Functions:

The boundary conditions of this problem are:

Mixed Conditions on $z = 0$, i.e.

$$U_z = 0$$

$$\tau_{zx} = 0 \qquad \text{on} \quad z = 0$$

$$\tau_{zy} = 0$$

The plane $x = 0$ is free of stress, i.e.

$$\sigma_x = 0$$

$$\tau_{xy} = 0 \qquad \text{on} \quad x = 0$$

$$\tau_{xz} = 0$$

These boundary conditions can be expressed in terms of Papkovitch functions:

$$(3 - 4\nu)B_z - \frac{\partial}{\partial z}(x\, B_x + y\, B_y + \beta) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

$$(1 - 2\nu)\left(\frac{\partial B_y}{\partial z} + \frac{\partial B_z}{\partial y}\right) - x\,\frac{\partial^2 B_x}{\partial y \partial z} - y\,\frac{\partial^2 B_y}{\partial y \partial z} - \frac{\partial^2 \beta}{\partial y \partial z} = 0 \quad \ldots\ldots \quad (2) \quad \text{on } z = 0$$

$$(1 - 2\nu)\left(\frac{\partial B_z}{\partial x} + \frac{\partial B_x}{\partial z}\right) - x\,\frac{\partial^2 B_x}{\partial z \partial x} - y\,\frac{\partial^2 B_y}{\partial z \partial x} - \frac{\partial^2 \beta}{\partial z \partial x} = 0 \quad \ldots\ldots \quad (3)$$

$$2(1 - \nu)\frac{\partial B_x}{\partial x} - y\,\frac{\partial^2 B_y}{\partial x^2} - z\,\frac{\partial^2 B_z}{\partial x^2} - \frac{\partial^2 \beta}{\partial x^2} + 2\nu\left(\frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}\right) = 0 \quad \ldots\ldots \quad (4)$$

$$(1 - 2\nu)\left(\frac{\partial B_x}{\partial y} + \frac{\partial B_y}{\partial x}\right) - y\,\frac{\partial^2 B_y}{\partial x \partial y} - z\,\frac{\partial^2 B_z}{\partial x \partial y} - \frac{\partial^2 \beta}{\partial x \partial y} = 0 \quad \ldots\ldots \quad (5) \quad \text{on } x = 0$$

$$(1 - 2\nu)\left(\frac{\partial B_z}{\partial x} + \frac{\partial B_x}{\partial z}\right) - y\,\frac{\partial^2 B_y}{\partial z \partial x} - z\,\frac{\partial^2 B_z}{\partial z \partial x} - \frac{\partial^2 \beta}{\partial z \partial x} = 0 \quad \ldots\ldots \quad (6)$$

2. Determination of Papkovitch Functions:

The distribution of body forces in the region is given by:

$$\lim_{T \to 0} \int F_x \, d\tau = P_x \qquad \text{in } x \geqslant 0,\ z \geqslant 0$$

$$F_y = 0 \qquad \text{"} \qquad \text{"}$$

$$F_z = 0 \qquad \text{"} \qquad \text{"}$$

Let

$$B_y = B_z = 0 \qquad \text{in } x \geqslant 0,\ z \geqslant 0 \quad \ldots\ldots \quad (7)$$

By substituting in equations (1 - 6), we get

$$-x\,\frac{\partial B_x}{\partial z} - \frac{\partial \beta}{\partial z} = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

$$-x\,\frac{\partial^2 B_x}{\partial y \partial z} - \frac{\partial^2 \beta}{\partial y \partial z} = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (9) \quad \text{on } z=0$$

$$(1 - 2\nu)\,\frac{\partial B_x}{\partial z} - x\,\frac{\partial^2 B_x}{\partial z \partial x} - \frac{\partial^2 \beta}{\partial z \partial x} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (10)$$

$$2(1-\nu)\frac{\partial B_x}{\partial x}-\frac{\partial^2\beta}{\partial x^2}=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (11)$$

$$(1-2\nu)\frac{\partial B_x}{\partial y}-\frac{\partial^2\beta}{\partial x\partial y}=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (12) \qquad \text{on } x=0$$

$$(1-2\nu)\frac{\partial B_x}{\partial z}-\frac{\partial^2\beta}{\partial z\partial x}=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (13)$$

Equations (8 - 13) are satisfied if we get:

$$(1-2\nu)\frac{\partial B_x}{\partial z}-\frac{\partial^2\beta}{\partial z\partial x}=0 \qquad \text{on } x=0,\ z=0 \ \ldots\ldots \ (14)$$

$$2(1-\nu)\frac{\partial^2 B_x}{\partial x\partial z}-\frac{\partial^3\beta}{\partial x^2\partial z}=0 \qquad \text{on } x=0,\ z=0 \ \ldots\ldots \ (15)$$

Applying Green's formula to $(1-2\nu)\frac{\partial B_x}{\partial z}-\frac{\partial^2\beta}{\partial z\partial x}$, we get:

$$(1-2\nu)\frac{\partial B_x}{\partial z}-\frac{\partial^2\beta}{\partial z\partial x}=\frac{1-2\nu}{4\pi\mu}\lim_{T\to 0}\int_T G_q\frac{\partial F_x}{\partial\xi}\,d\tau+\frac{1}{4\pi\mu}\lim_{T\to 0}\int_T G_q\frac{\partial^2(\xi F_x)}{\partial\xi\,\partial\xi}\,d\tau$$

but

$$\int_T G_q\frac{\partial F_x}{\partial\xi}\,d\tau=-\int_T F_x\frac{\partial G_q}{\partial\xi}\,d\tau=\int_T F_x\frac{\partial}{\partial z}\left(\frac{1}{r_1}+\frac{1}{r_2}-\frac{1}{r_3}-\frac{1}{r_4}\right)d\tau$$

and

$$\int_T G_q\frac{\partial^2(\xi F_x)}{\partial\xi\,\partial\xi}\,d\tau=\int_T\frac{\partial}{\partial\xi}\left[G_q\frac{\partial(\xi F_x)}{\partial\xi}\right]d\tau-\int_T\frac{\partial(\xi F_x)}{\partial\xi}\,\frac{\partial G_q}{\partial\xi}\,d\tau$$

$$=\int_T\frac{\partial(\xi F_x)}{\partial\xi}\,\frac{\partial}{\partial x}\left(\frac{1}{r_1}-\frac{1}{r_2}+\frac{1}{r_3}-\frac{1}{r_4}\right)d\tau$$

$$=\int_T\frac{\partial}{\partial\xi}\left[\xi F_x\frac{\partial}{\partial x}\left(\frac{1}{r_1}-\frac{1}{r_2}+\frac{1}{r_3}-\frac{1}{r_4}\right)\right]d\tau-$$

$$\int_T\xi F_x\frac{\partial}{\partial x}\left[\frac{\partial}{\partial\xi}\left(\frac{1}{r_1}-\frac{1}{r_2}+\frac{1}{r_3}-\frac{1}{r_4}\right)\right]d\tau$$

$$=\int_T\xi F_x\frac{\partial^2}{\partial x\partial z}\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4}\right)d\tau.$$ Hence

$$(1 - 2\nu)\frac{\partial B_x}{\partial z} - \frac{\partial^2\beta}{\partial z\partial x} = \frac{(1-2\nu)P_x}{4\pi\mu}\frac{\partial}{\partial z}\left(\frac{1}{R_1} + \frac{1}{R_2} - \frac{1}{R_3} - \frac{1}{R_4}\right)$$

$$+ \frac{aP_x}{4\pi\mu}\frac{\partial^2}{\partial x\partial z}\left[\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4}\right] \ldots. \quad (16)$$

Also, applying Green's formula to

$$2(1-\nu)\frac{\partial^2 B_x}{\partial x\partial z} - \frac{\partial^3\beta}{\partial x^2\partial z}\text{, we get}$$

$$2(1-\nu)\frac{\partial^2 B_x}{\partial x\partial z} - \frac{\partial^3\beta}{\partial x^2\partial z} = \frac{2(1-\nu)}{4\pi\mu}\lim_{T\to 0}\int_T G_q \frac{\partial^2 F_x}{\partial\xi\partial\zeta}\,d\tau$$

$$+ \frac{1}{4\pi\mu}\lim_{T\to 0}\int_T G_q \frac{\partial^3(\xi F_x)}{\partial\xi^2\partial\zeta}\,d\tau$$

But

$$\int_T G_q \frac{\partial^2 F_x}{\partial\xi\partial\zeta}\,d\tau = \int_T \frac{\partial F_x}{\partial\zeta}\frac{\partial}{\partial x}\left(\frac{1}{r_1} - \frac{1}{r_2} + \frac{1}{r_3} - \frac{1}{r_4}\right)d\tau$$

$$= \int_T F_x \frac{\partial^2}{\partial x\partial z}\left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4}\right)d\tau$$

and

$$\int_T G_q \frac{\partial^3(\xi F_x)}{\partial\xi^2\partial\zeta}\,d\tau = \int_T \frac{\partial}{\partial\xi}\left[G_q \frac{\partial^2(\xi F_x)}{\partial\xi\partial\zeta}\right]d\tau - \int_T \frac{\partial^2(\xi F_x)}{\partial\xi\partial\zeta}\frac{\partial G_q}{\partial\xi}\,d\tau$$

$$= \int_T \frac{\partial^2(\xi F_x)}{\partial\xi\partial\zeta}\frac{\partial}{\partial x}\left(\frac{1}{r_1} - \frac{1}{r_2} + \frac{1}{r_3} - \frac{1}{r_4}\right)d\tau$$

$$= \int_T \frac{\partial}{\partial\xi}\left[\frac{\partial(\xi F_x)}{\partial\zeta}\frac{\partial}{\partial x}\left(\frac{1}{r_1} - \frac{1}{r_2} + \frac{1}{r_3} - \frac{1}{r_4}\right)\right]d\tau$$

$$\int_T \frac{\partial(\xi F_x)}{\partial\zeta}\frac{\partial}{\partial x}\left[\frac{\partial}{\partial\xi}\left(\frac{1}{r_1} - \frac{1}{r_2} + \frac{1}{r_3} - \frac{1}{r_4}\right)\right]d\tau$$

$$= \int_T \frac{\partial(\xi F_x)}{\partial\zeta}\frac{\partial^2}{\partial x^2}\left(\frac{1}{r_1} - \frac{1}{r_2} - \frac{1}{r_3} + \frac{1}{r_4}\right)d\tau$$

since $\dfrac{\partial(\xi F_x)}{\partial \zeta}$ vanishes on the boundaries (F_x is identically zero on and close to $z = 0$, $x = 0$).

$$\int_T G_q \frac{\partial^3(\xi F_x)}{\partial \xi^2 \partial \zeta} d\tau = \int_T \frac{\partial}{\partial \zeta}\left[\xi F_x \frac{\partial^2}{\partial x^2}\left(\frac{1}{r_1} - \frac{1}{r_2} - \frac{1}{r_3} + \frac{1}{r_4}\right)\right] d\tau -$$

$$\int_T \xi F_x \frac{\partial^2}{\partial x^2}\left[\frac{\partial}{\partial \zeta}\left(\frac{1}{r_1} - \frac{1}{r_2} - \frac{1}{r_3} + \frac{1}{r_4}\right)\right] d\tau$$

$$= \int_T \xi F_x \frac{\partial^3}{\partial x^2 \partial z}\left(\frac{1}{r_1} + \frac{1}{r_2} - \frac{1}{r_3} - \frac{1}{r_4}\right) d\tau .$$

Hence:

$$2(1-\nu)\frac{\partial^2 B_x}{\partial x \partial z} - \frac{\partial^3 \beta}{\partial x^2 \partial z} = \frac{2(1-\nu)P_x}{4\pi\mu}\frac{\partial^2}{\partial x \partial z}\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4}\right) +$$

$$\frac{aP_x}{4\pi\mu}\frac{\partial^3}{\partial x^2 \partial z}\left(\frac{1}{R_1} + \frac{1}{R_2} - \frac{1}{R_3} - \frac{1}{R_4}\right)$$

Integrating with respect to x, we get:

$$2(1-\nu)\frac{\partial B_x}{\partial z} - \frac{\partial^2 \beta}{\partial x \partial z} = \frac{2(1-\nu)P_x}{4\pi\mu}\frac{\partial}{\partial z}\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4}\right) +$$

$$\frac{aP_x}{4\pi\mu}\frac{\partial^2}{\partial x \partial z}\left(\frac{1}{R_1} + \frac{1}{R_2} - \frac{1}{R_3} - \frac{1}{R_4}\right) \quad \ldots\ldots (17)$$

Solving Equations (16), (17) for $\dfrac{\partial B_x}{\partial z}$, $\dfrac{\partial^2 \beta}{\partial x \partial z}$, we get:

$$\frac{\partial B_x}{\partial z} = \frac{P_x}{4\pi\mu}\frac{\partial}{\partial z}\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{3-4\nu}{R_3} + \frac{3-4\nu}{R_4}\right) - \frac{2aP_x}{4\pi\mu}\frac{\partial^2}{\partial x \partial z}\left(\frac{1}{R_3} + \frac{1}{R_4}\right)$$

$$\frac{\partial^2 \beta}{\partial x \partial z} = \frac{4(1-\nu)(1-2\nu)}{4\pi\mu} P_x \frac{\partial}{\partial z}\left(\frac{1}{R_3} + \frac{1}{R_4}\right) + \frac{aP_x}{4\pi\mu}\frac{\partial^2}{\partial x \partial z}\left(-\frac{1}{R_1} - \frac{1}{R_2} + \frac{4\nu-3}{R_3} + \frac{4\nu-3}{R_4}\right)$$

Integrating these two equations with respect to z, we get:

$$B_x = \frac{P_x}{4\pi\mu}\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{3-4\nu}{R_3} + \frac{3-4\nu}{R_4}\right) - \frac{2aP_x}{4\pi\mu}\frac{\partial}{\partial x}\left(\frac{1}{R_3} + \frac{1}{R_4}\right) \quad \ldots\ldots (18)$$

$$\frac{\partial \beta}{\partial x} = \frac{(1-\nu)(1-2\nu)P_x}{\pi\mu}\left(\frac{1}{R_3}+\frac{1}{R_4}\right) - \frac{aP_x}{4\pi\mu}\frac{\partial}{\partial x}\left(\frac{1}{R_1}+\frac{1}{R_2}+\frac{3-4\nu}{R_3}+\frac{3-4\nu}{R_4}\right) \qquad \ldots\ldots\ldots\ldots(19)$$

Integrating Equation (19) with respect to x, we get

$$\beta = \frac{(1-\nu)(1-2\nu)P_x}{\pi\mu}\int\left(\frac{1}{R_3}+\frac{1}{R_4}\right)dx - \frac{aP_x}{4\pi\mu}\left(\frac{1}{R_1}+\frac{1}{R_2}+\frac{3-4\nu}{R_3}+\frac{3-4\nu}{R_4}\right)\ldots\ (20)$$

Therefore, the required Papkovitch functions are:

$$B_x = \frac{P_x}{4\pi\mu}\left(\frac{1}{R_1}+\frac{1}{R_2}+\frac{3-4\nu}{R_3}+\frac{3-4\nu}{R_4}\right) - \frac{aP_x}{2\pi\mu}\frac{\partial}{\partial x}\left(\frac{1}{R_3}+\frac{1}{R_4}\right) \ldots\ldots\ldots\ (21)$$

$$B_y = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ (22)$$

$$B_z = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ (23)$$

$$\beta = \frac{(1-\nu)(1-2\nu)P_x}{\pi\mu}\int\left(\frac{1}{R_3}+\frac{1}{R_4}\right)dx - \frac{aP_x}{4\pi\mu}\left(\frac{1}{R_1}+\frac{1}{R_2}+\frac{3-4\nu}{R_3}+\frac{3-4\nu}{R_4}\right)\ldots\ (24)$$

3. Verification:

We note that on $z = 0$

$$\frac{\partial B_x}{\partial z} = 0 \qquad \text{since}$$

$$\frac{\partial B_x}{\partial z} = \frac{-P_x}{4\pi\mu}\left[\frac{z-c}{R_1^3}+\frac{z+c}{R_2^3}+\frac{(3-4\nu)(z-c)}{R_3^3}+\frac{(3-4\nu)(z+c)}{R_4^3}\right]$$

$$+ \frac{aP_x}{2\pi\mu}\frac{\partial}{\partial x}\left(\frac{z-c}{R_3^3}+\frac{z+c}{R_4^3}\right)$$

also,

$$\frac{\partial \beta}{\partial z} = -\frac{(1-\nu)(1-2\nu)P_x}{\pi\mu}\int\left(\frac{z-c}{R_3^3}+\frac{z+c}{R_4^3}\right)dx + \frac{aP_x}{4\pi\mu}\left(\frac{z-c}{R_1^3}+\frac{z+c}{R_2^3}+\frac{(3-4\nu)(z-c)}{R_3^3}\right.$$

$$\left. + \frac{(3-4\nu)((z+c)}{R_4^3}\right) = 0$$

It follows that

$$\frac{\partial^2 B_x}{\partial x \partial z} = \frac{\partial^2 B_x}{\partial y \partial z} = 0$$

and

$$\frac{\partial^2 \beta}{\partial x \partial z} = \frac{\partial^2 \beta}{\partial y \partial z} = 0$$

on $x = 0$

$$(1 - 2\nu)B_x - \frac{\partial \beta}{\partial x} = \frac{(1-2\nu)P_x}{4\pi\mu}\left(\frac{1}{R_1} + \frac{1}{R_2} - \frac{1}{R_3} - \frac{1}{R_4}\right) + \frac{aP_x}{4\pi\mu}\frac{\partial}{\partial x}\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4}\right)$$

$$= \frac{(1-2\nu)}{4\pi\mu} P_x \left(\frac{1}{R_1} + \frac{1}{R_2} - \frac{1}{R_3} - \frac{1}{R_4}\right) - \frac{aP_x}{4\pi\mu}\left(\frac{x-a}{R_1{}^3} + \frac{x-a}{R_2{}^3}\right.$$

$$\left. + \frac{x+a}{R_3{}^3} + \frac{x+a}{R_4{}^3}\right) = 0$$

Hence

$$(1 - 2\nu)\frac{\partial B_x}{\partial y} - \frac{\partial^2 \beta}{\partial x \partial y} = 0$$

$$(1 - 2\nu)\frac{\partial B_x}{\partial z} - \frac{\partial^2 \beta}{\partial x \partial z} = 0$$

Also

$$2(1 - \nu)\frac{\partial B_x}{\partial x} - \frac{\partial^2 \beta}{\partial x^2} = \frac{2(1-\nu)P_x}{4\pi\mu}\frac{\partial}{\partial x}\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4}\right)$$

$$+ \frac{aP_x}{4\pi\mu}\frac{\partial^2}{\partial x^2}\left(\frac{1}{R_1} + \frac{1}{R_2} - \frac{1}{R_3} - \frac{1}{R_4}\right) = \frac{-2(1-\nu)P_x}{4\pi\mu}\left(\frac{x-a}{R_1{}^3} + \frac{x-a}{R_2{}^3} + \frac{x+a}{R_3{}^3} + \frac{x+a}{R_4{}^3}\right) +$$

$$\frac{aP_x}{4\pi\mu}\left[-\frac{1}{R_1{}^3} - \frac{1}{R_2{}^3} + \frac{1}{R_3{}^3} + \frac{1}{R_4{}^3} + \frac{3(x-a)^2}{R_1{}^5} + \frac{3(x-a)^2}{R_2{}^5}\right.$$

$$\left. - \frac{3(x+a)^2}{R_3{}^5} - \frac{3(x+a)^2}{R_4{}^5}\right] = 0$$

As the plane $x = 0$ recedes to infinity ($\frac{1}{R_3}$, $\frac{1}{R_4} \rightarrow 0$, $a = 0$),

we get:

$$B_x = \frac{P_x}{4\pi\mu}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

$$B_y = 0$$

$$B_z = 0$$

$$\beta = 0$$

These results coincide with the results given by Hijab for half-space.

CHAPTER VI

FORCE PARALLEL TO BOTH PLANES

1. Determination of Papkovitch Functions:

This problem is identical with the preceding problem, but the concentrated force this time is parallel to the y-axis.

The distribution of body forces is:

$$F_x = 0 \qquad \text{in } x \geq 0, \quad z \geq 0$$

$$\lim_{T \to 0} \int_T F_y \, d\tau = P_y \qquad " \qquad "$$

$$F_z = 0 \qquad " \qquad "$$

Let $B_z = 0$ in $x \geq 0$, $z \geq 0$, then we get from equations (1 - 6) by substitution:

$$\frac{\partial}{\partial z}(x B_x + y B_y + \beta) = 0 \qquad \ldots\ldots\ldots\ldots(25)$$

$$(1-2\nu)\frac{\partial B_y}{\partial z} - x\frac{\partial^2 B_x}{\partial y \partial z} - y\frac{\partial^2 B_y}{\partial y \partial z} - \frac{\partial^2 \beta}{\partial y \partial z} = 0 \qquad \ldots\ldots\ldots\ldots(26) \qquad \text{on } z = 0$$

$$(1-2\nu)\frac{\partial B_x}{\partial z} - x\frac{\partial^2 B_x}{\partial z \partial x} - y\frac{\partial^2 B_y}{\partial z \partial x} - \frac{\partial^2 \beta}{\partial z \partial x} = 0 \qquad \ldots\ldots\ldots\ldots(27)$$

$$2(1-\nu)\frac{\partial B_x}{\partial x} - y\frac{\partial^2 B_y}{\partial x^2} - \frac{\partial^2 \beta}{\partial x^2} + 2\nu\frac{\partial B_y}{\partial y} = 0 \qquad \ldots\ldots\ldots\ldots(28)$$

$$(1-2\nu)\left(\frac{\partial B_x}{\partial y} + \frac{\partial B_y}{\partial x}\right) - y\frac{\partial^2 B_y}{\partial x \partial y} - \frac{\partial^2 \beta}{\partial x \partial y} = 0 \qquad \ldots\ldots\ldots\ldots(29) \qquad \text{on } x = 0$$

$$(1-2\nu)\frac{\partial B_x}{\partial z} - y\frac{\partial^2 B_y}{\partial z \partial x} - \frac{\partial^2 \beta}{\partial z \partial x} = 0 \qquad \ldots\ldots\ldots\ldots(30)$$

Equations (25 - 30) are satisfied if we set:

$$\frac{\partial^2 B_y}{\partial x \partial z} = 0 \qquad \text{on } x = 0, \quad z = 0 \text{ .... } (31)$$

$$2(1 - \nu)\frac{\partial B_x}{\partial x} - y\frac{\partial^2 B_y}{\partial x^2} - \frac{\partial^2 \beta}{\partial x^2} + 2\nu\frac{\partial B_y}{\partial y} = 0 \quad \text{on } x = 0 \text{ ... } (32)$$

$$(1 - 2\nu)\, B_x - \frac{\partial \beta}{\partial x} = 0 \qquad \text{on } x = 0, \quad z = 0 \text{ .... } (33)$$

Applying Green's formula to $\frac{\partial^2 B_y}{\partial x \partial z}$, we get

$$\frac{\partial^2 B_y}{\partial x \partial z} = \frac{1}{4\pi\mu} \lim_{T\to 0} \int_T G_q \frac{\partial^2 F_y}{\partial \xi \partial \zeta}\, d\tau$$

$$= \frac{1}{4\pi\mu} \lim_{T\to 0} \left[\int_T \frac{\partial}{\partial \xi}\left(G_q \frac{\partial F_y}{\partial \zeta}\right) d\tau - \int_T \frac{\partial F_y}{\partial \zeta}\frac{\partial G_q}{\partial \xi}\, d\tau\right]$$

$$= \frac{1}{4\pi\mu} \lim_{T\to 0} \int_T \frac{\partial F_y}{\partial \zeta}\frac{\partial}{\partial x}\left(\frac{1}{r_1} - \frac{1}{r_2} + \frac{1}{r_3} - \frac{1}{r_4}\right) d\tau$$

$$= \frac{1}{4\pi\mu} \lim_{T\to 0} \left[\int_T \frac{\partial}{\partial \zeta}\left(F_y \frac{\partial}{\partial x}\left(\frac{1}{r_1} - \frac{1}{r_2} + \frac{1}{r_3} - \frac{1}{r_4}\right)\right) d\tau - \int_T F_y \frac{\partial^2}{\partial x \partial \zeta}\left(\frac{1}{r_1} - \frac{1}{r_2} + \frac{1}{r_3} - \frac{1}{r_4}\right) d\tau\right]$$

$$= \frac{1}{4\pi\mu} \lim_{T\to 0} \int_T F_y \frac{\partial^2}{\partial x \partial z}\left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4}\right) d\tau$$

$$= \frac{P_y}{4\pi\mu}\frac{\partial^2}{\partial x \partial z}\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4}\right), \text{ hence}$$

$$B_y = \frac{P_y}{4\pi\mu}\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4}\right) \text{ .... } (34)$$

Applying again Green's formula to $(1 - 2\nu)B_x - \frac{\partial \beta}{\partial x}$, we get:

$$(1-2\nu)B_x - \frac{\partial \beta}{\partial x} = \frac{1-2\nu}{4\pi\mu}\lim_{T\to 0}\int_T G_q F_x \, d\tau + \frac{1}{4\pi\mu}\lim_{T\to 0}\int_T G_q \frac{\partial(\eta F_y)}{\partial \xi} d\tau$$

$$= \frac{1}{4\pi\mu}\lim_{T\to 0}\int_T \frac{\partial(G_q \eta F_y)}{\partial \xi} d\tau - \frac{1}{4\pi\mu}\lim_{T\to 0}\int \eta F_y \frac{\partial G_q}{\partial \xi} d\tau$$

$$= \frac{1}{4\pi\mu}\lim_{T\to 0}\int_T \eta \, F_y \frac{\partial}{\partial x}\left(\frac{1}{r_1} - \frac{1}{r_2} + \frac{1}{r_3} - \frac{1}{r_4}\right) d\tau = 0$$

Hence:

$$(1-2\nu)B_x - \frac{\partial \beta}{\partial x} = 0 \qquad \text{therefore} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (35)$$

$$(1-2\nu)\frac{\partial B_x}{\partial x} - \frac{\partial^2 \beta}{\partial x^2} = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (36)$$

From equations (32), (34), (36), we get:

$$2(1-\nu)\frac{\partial B_x}{\partial x} - \frac{yP_y}{4\pi\mu}\frac{\partial^2}{\partial x^2}\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4}\right) - (1-2\nu)\frac{\partial B_x}{\partial x} +$$

$$\frac{2\nu P_y}{4\pi\mu}\frac{\partial}{\partial y}\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4}\right) = 0 \quad \text{on } x = 0$$

Hence

$$\frac{\partial B_x}{\partial x} = \frac{P_y}{4\pi\mu} y \frac{\partial^2}{\partial x^2}\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4}\right) - \frac{\nu P_y}{2\pi\mu}\frac{\partial}{\partial y}\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4}\right)$$

$$\text{on } x = 0$$

This equation may be written in the form:

$$\frac{\partial B_x}{\partial x} = \frac{P_y}{4\pi\mu} y \left(\frac{-2}{R_3{}^3} - \frac{2}{R_4{}^3} + \frac{6a^2}{R_3{}^5} + \frac{6a^2}{R_4{}^5}\right) - \frac{\nu P_y}{\pi\mu}\frac{\partial}{\partial y}\left(\frac{1}{R_3} + \frac{1}{R_4}\right) \quad \text{on } x = 0$$

or

$$\frac{\partial B_x}{\partial x} = \frac{2P_y}{4\pi\mu}\left[\frac{\partial}{\partial y}\left(\frac{1}{R_3} + \frac{1}{R_4}\right) + a\frac{\partial^2}{\partial x \partial y}\left(\frac{1}{R_3} + \frac{1}{R_4}\right)\right] - \frac{\nu P_y}{\pi\mu}\frac{\partial}{\partial y}\left(\frac{1}{R_3} + \frac{1}{R_4}\right) \ldots (37)$$

$$\text{on } x = 0$$

Therefore, we can let

$$\frac{\partial B_x}{\partial x} = \frac{P_y}{2\pi\mu}\left[\frac{\partial}{\partial y}\left(\frac{1}{R_3}+\frac{1}{R_4}\right) + a\frac{\partial^2}{\partial x\partial y}\left(\frac{1}{R_3}+\frac{1}{R_4}\right)\right] - \frac{\nu P_y}{\pi\mu}\frac{\partial}{\partial y}\left(\frac{1}{R_3}+\frac{1}{R_4}\right) \qquad \ldots\ldots (38)$$

Since $\frac{\partial B_x}{\partial x}$ is harmonic (hence B_x is harmonic) and satisfies equations (32) and (Ch. II, 11).

Integrating Equation (38), we get:

$$B_x = \frac{P_y(1-2\nu)}{2\pi\mu}\int\frac{\partial}{\partial y}\left(\frac{1}{R_3}+\frac{1}{R_4}\right)dx + \frac{aP_y}{2\pi\mu}\frac{\partial}{\partial y}\left(\frac{1}{R_3}+\frac{1}{R_4}\right) \ldots\ldots (39)$$

from equation (35), we get:

$$\beta = (1-2\nu)\int B_x\, dx \qquad \text{or}$$

$$\beta = \frac{P_y(1-2\nu)^2}{2\pi\mu}\int\left[\int\frac{\partial}{\partial y}\left(\frac{1}{R_3}+\frac{1}{R_4}\right)dx\right]dx + \frac{aP_y(1-2\nu)}{2\pi\mu}\int\frac{\partial}{\partial y}\left(\frac{1}{R_3}+\frac{1}{R_4}\right)dx$$

Therefore the required Papkovitch functions are:

$$B_x = \frac{P_y(1-2\nu)}{2\pi\mu}\int\frac{\partial}{\partial y}\left(\frac{1}{R_3}+\frac{1}{R_4}\right)dx + \frac{aP_y}{2\pi\mu}\frac{\partial}{\partial y}\left(\frac{1}{R_3}+\frac{1}{R_4}\right)$$

$$B_y = \frac{P_y}{4\pi\mu}\left(\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}+\frac{1}{R_4}\right)$$

$$B_z = 0$$

$$\beta = \frac{P_y(1-2\nu)^2}{2\pi\mu}\int\left[\int\frac{\partial}{\partial y}\left(\frac{1}{R_3}+\frac{1}{R_4}\right)dx\right]dx + \frac{aP_y(1-2\nu)}{2\pi\mu}\int\frac{\partial}{\partial y}\left(\frac{1}{R_3}+\frac{1}{R_4}\right)dx$$

4. <u>Verification:</u>

on $x = 0$, we have

$$\frac{\partial B_y}{\partial x} = \frac{-P_y}{4\pi\mu}\left[\frac{x-a}{R_1^3} + \frac{x-a}{R_2^3} + \frac{x+a}{R_3^3} + \frac{x+a}{R_4^3}\right] = 0 \quad \text{hence}$$

$$\frac{\partial^2 B_y}{\partial x \partial z} = \frac{\partial^2 B_y}{\partial x \partial y} = 0 \quad \text{also}$$

$$(1-2\nu)B_x - \frac{\partial \beta}{\partial x} = 0 \qquad \text{and this implies that}$$

$$(1-2\nu)\frac{\partial B_x}{\partial y} - \frac{\partial^2 \beta}{\partial x \partial y} = 0$$

$$(1-2\nu)\frac{\partial B_x}{\partial z} - \frac{\partial^2 \beta}{\partial x \partial z} = 0$$

on $z = 0$

$$\frac{\partial B_x}{\partial z} = \frac{P_y}{2\pi\mu}(1-2\nu)\int \frac{\partial}{\partial y}\left(\frac{-(z-c)}{R_3^3} - \frac{(z+c)}{R_4^3}\right)dx - \frac{aP_y}{2\pi\mu}\frac{\partial}{\partial y}\left(\frac{z-c}{R_3^3} + \frac{z+c}{R_4^3}\right) = 0$$

It follows that $\dfrac{\partial^2 B_x}{\partial y \partial z} = \dfrac{\partial^2 B_x}{\partial x \partial z} = 0$

Also:

$$\frac{\partial \beta}{\partial z} = \frac{P_y(1-2\nu)^2}{2\pi\mu}\int\left[\int\frac{\partial}{\partial y}\left(\frac{z-c}{R_3^3} - \frac{z+c}{R_4^3}\right)dx\right]dx - \frac{aP_y(1-2\nu)}{2\pi\mu}\int\frac{\partial}{\partial y}\left(\frac{z-c}{R_3^3} + \frac{z+c}{R_4^3}\right)dx$$
$$= 0$$

Therefore, $\dfrac{\partial^2 \beta}{\partial x \partial z} = \dfrac{\partial^2 \beta}{\partial y \partial z} = 0.$ Finally

$$\frac{\partial B_y}{\partial z} = \frac{-P_y}{4\pi\mu}\left(\frac{z-c}{R_1^3} + \frac{z+c}{R_2^3} + \frac{z-c}{R_3^3} + \frac{z+c}{R_4^3}\right) = 0$$

Therefore,

$$\frac{\partial^2 B_y}{\partial y \partial z} = \frac{\partial^2 B_y}{\partial x \partial z} = 0$$

As the plane $x = 0$ recedes to infinity, we get:

$$B_x = 0$$

$$B_y = \frac{P_y}{4\pi\mu}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

$$B_z = 0$$

$$\beta = 0$$

These are identical with the solution of the problem for half-space.

CHAPTER VII

FORCE NORMAL TO PLANE WITH MIXED BOUNDARY CONDITIONS

1. Determination of Papkovitch Functions:

In this problem the concentrated force is taken to be parallel to the z-axis, having the same boundary conditions as in chapters V, VI. The distribution of body forces is given by:

$$F_x = 0 \qquad \text{in } x \geqslant 0,\ z \geqslant 0$$

$$F_y = 0 \qquad \text{"} \quad \text{"}$$

$$\lim_{\tau \to 0} \int F_z \, d\tau = P_z \qquad \text{"} \quad \text{"}$$

Let $B_y = 0$ in $x \geqslant 0$, $z \geqslant 0$, we get from equations (1 - 6) that:

$$(3 - 4\nu)B_z - \frac{\partial}{\partial z}(x B_x + \beta) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (40)$$

$$(1 - 2\nu)\frac{\partial B_z}{\partial y} - x\frac{\partial^2 B_x}{\partial y \partial z} - \frac{\partial^2 \beta}{\partial y \partial z} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (41) \quad \text{on } z = 0$$

$$(1 - 2\nu)\left(\frac{\partial B_z}{\partial x} + \frac{\partial B_x}{\partial z}\right) - x\frac{\partial^2 B_x}{\partial z \partial x} - \frac{\partial^2 \beta}{\partial z \partial x} = 0 \quad \ldots\ldots\ldots\ldots \quad (42) \quad \text{on } z = 0$$

$$2(1 - \nu)\frac{\partial B_x}{\partial x} - z\frac{\partial^2 B_z}{\partial x^2} - \frac{\partial^2 \beta}{\partial x^2} + 2\nu\frac{\partial B_z}{\partial z} = 0 \quad \ldots\ldots\ldots\ldots\ldots \quad (43)$$

$$(1 - 2\nu)\frac{\partial B_x}{\partial y} - z\frac{\partial^2 B_z}{\partial x \partial y} - \frac{\partial^2 \beta}{\partial x \partial y} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (44) \quad \text{on } x = 0$$

$$(1 - 2\nu)\left(\frac{\partial B_z}{\partial x} + \frac{\partial B_x}{\partial z}\right) - z\frac{\partial^2 B_z}{\partial z \partial x} - \frac{\partial^2 \beta}{\partial z \partial x} = 0 \quad \ldots\ldots\ldots\ldots \quad (45)$$

These equations are satisfied if we set:

$$\frac{\partial B_z}{\partial x} = 0 \qquad \text{on } x = 0, \quad z = 0 \ldots\ldots (46)$$

$$(1 - 2\nu)\frac{\partial B_x}{\partial z} - \frac{\partial^2 \beta}{\partial z \partial x} = 0 \qquad " \qquad " \ldots\ldots (47)$$

$$2(1 - \nu)\frac{\partial B_x}{\partial x} - z\frac{\partial^2 B_z}{\partial x^2} - \frac{\partial^2 \beta}{\partial x^2} + 2\nu\frac{\partial B_z}{\partial z} = 0 \quad \text{on } x = 0 \ldots\ldots (48)$$

Applying Green's formula to $\frac{\partial B_z}{\partial x}$, we get

$$\frac{\partial B_z}{\partial x} = \frac{P_z}{4\pi\rho}\frac{\partial}{\partial x}\left(\frac{1}{R_1} - \frac{1}{R_2} + \frac{1}{R_3} - \frac{1}{R_4}\right)$$

Hence

$$B_z = \frac{P_z}{4\pi\rho}\left(\frac{1}{R_1} - \frac{1}{R_2} + \frac{1}{R_3} - \frac{1}{R_4}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (49)$$

also, applying Green's formula to $(1 - 2\nu)\frac{\partial B_x}{\partial z} - \frac{\partial^2 \beta}{\partial z \partial x}$, we get:

$$(1 - 2\nu)\frac{\partial B_x}{\partial z} - \frac{\partial^2 \beta}{\partial z \partial x} = \frac{(1-2\nu)}{4\pi\rho}\lim_{T\to 0}\int_T G_q \frac{\partial F_x}{\partial \zeta}d\tau + \frac{1}{4\pi\rho}\lim_{T\to 0}\int_T G_q \frac{\partial^2(\xi F_z)}{\partial \xi \partial \zeta}d\tau$$

$$= \frac{1}{4\pi\mu}\lim_{T\to 0}\int_T G_q \frac{\partial^2(\xi F_z)}{\partial \xi \partial \zeta}d\tau$$

but

$$\int_T G_q \frac{\partial^2(\xi F_z)}{\partial \xi \partial \zeta}d\tau = \int_T \frac{\partial}{\partial \xi}\left[G_q \frac{\partial(\xi F_z)}{\partial \zeta}\right]d\tau - \int_T \frac{\partial(\xi F_z)}{\partial \zeta}\frac{\partial G_q}{\partial \xi}d\tau$$

$$= -\int_T \frac{\partial(\xi F_z)}{\partial \zeta}\cdot\frac{\partial G_q}{\partial \xi}d\tau$$

$$= \int_T \frac{\partial(\xi F_z)}{\partial \zeta}\frac{\partial}{\partial x}\left(\frac{1}{r_1} - \frac{1}{r_2} + \frac{1}{r_3} - \frac{1}{r_4}\right)d\tau$$

$$= \int_T \frac{\partial}{\partial \xi}\left[\xi F_z \frac{\partial}{\partial x} \left(\frac{1}{r_1} - \frac{1}{r_2} + \frac{1}{r_3} - \frac{1}{r_4}\right)\right] d\tau$$

$$- \int_T \xi F_z \frac{\partial}{\partial x}\left[\frac{\partial}{\partial \xi} \left(\frac{1}{r_1} - \frac{1}{r_2} + \frac{1}{r_3} - \frac{1}{r_4}\right)\right] d\tau$$

$$= \int_T \xi F_z \frac{\partial^2}{\partial x \partial z} \left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4}\right) d\tau$$

Hence:

$$(1 - 2\nu)\frac{\partial B_x}{\partial z} - \frac{\partial^2 \beta}{\partial z \partial x} = \frac{cP_z}{4\pi\mu} \frac{\partial^2}{\partial x \partial z} \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4}\right) \ldots\ldots (50)$$

Integrating equation (50) with respect to z, we get:

$$(1 - 2\nu)B_x - \frac{\partial \beta}{\partial x} = \frac{cP_z}{4\pi\mu} \frac{\partial}{\partial x} \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4}\right) \ldots\ldots\ldots\ldots (51)$$

from which we get by differentiation with respect to x:

$$-\frac{\partial^2 \beta}{\partial x^2} = \frac{cP_z}{4\pi\mu} \frac{\partial^2}{\partial x^2} \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4}\right) - (1 - 2\nu)\frac{\partial B_x}{\partial x}$$

substituting in equation (48), we get:

$$2(1 - \nu)\frac{\partial B_x}{\partial x} - \frac{zP_z}{4\pi\mu} \frac{\partial^2}{\partial x^2} \left(\frac{1}{R_1} - \frac{1}{R_2} + \frac{1}{R_3} - \frac{1}{R_4}\right) - (1 - 2\nu)\frac{\partial B_x}{\partial x}$$

$$+ \frac{cP_z}{4\pi\mu} \frac{\partial^2}{\partial x^2} \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4}\right) + \frac{2\nu P_z}{4\pi\mu} \frac{\partial}{\partial z} \left(\frac{1}{R_1} - \frac{1}{R_2} + \frac{1}{R_3} - \frac{1}{R_4}\right)$$

$$= 0 \quad \text{on } x = 0$$

or:

$$\frac{\partial B_x}{\partial x} = \frac{P_z}{4\pi\mu} \frac{\partial^2}{\partial x^2} \left(\frac{z - c}{R_1} - \frac{z - c}{R_2} + \frac{z - c}{R_3} - \frac{z + c}{R_4}\right)$$

$$- \frac{2\nu P_z}{4\pi\mu} \frac{\partial}{\partial z} \left(\frac{1}{R_1} - \frac{1}{R_2} + \frac{1}{R_3} - \frac{1}{R_4}\right) \quad \text{on } x = 0$$

or:

$$\frac{\partial B_x}{\partial x} = \frac{P_z}{4\pi\mu}\left[-\frac{(z-c)}{R_1^3} + \frac{(z+c)}{R_2^3} - \frac{(z-c)}{R_3^4} + \frac{z+c}{R_4^3} + \frac{3(x-a)^2(z-c)}{R_1^5} - \frac{3(z+c)(x-a)^2}{R_2^5}\right.$$

$$\left. + \frac{3(z-c)(x+a)^2}{R_3^5} - \frac{3(z+c)(x+a)^2}{R_4^5}\right] - \frac{2\nu P_z}{4\pi\mu}\frac{\partial}{\partial z}\left(\frac{1}{R_1} - \frac{1}{R_2} + \frac{1}{R_3} - \frac{1}{R_4}\right)$$

on x = 0

This may be written in the form:

$$\frac{\partial B_x}{\partial x} = \frac{2P_z}{4\pi\mu}\left[-\frac{(z-c)}{R_3^3} + \frac{z+c)}{R_4^3} + \frac{3a^2(z-c)}{R_3^5} - \frac{3a^2(z+c)}{R_4^5}\right]$$

$$- \frac{\nu P_z}{\pi\mu}\frac{\partial}{\partial z}\left(\frac{1}{R_3} - \frac{1}{R_4}\right) \quad \text{.................on } x = 0$$

$$\frac{\partial B_x}{\partial x} = \frac{P_z}{2\pi\mu}\left[\frac{\partial}{\partial z}\left(\frac{1}{R_3} - \frac{1}{R_4}\right) + a\frac{\partial^2}{\partial x \partial z}\left(\frac{1}{R_3} - \frac{1}{R_4}\right)\right] - \frac{\nu P_z}{\pi\mu}\frac{\partial}{\partial z}\left(\frac{1}{R_3} - \frac{1}{R_4}\right)$$

on x = 0

we can let:

$$\frac{\partial B_x}{\partial x} = \frac{P_z}{2\pi\mu}\left[\frac{\partial}{\partial z}\left(\frac{1}{R_3} - \frac{1}{R_4}\right) + a\frac{\partial^2}{\partial x \partial z}\left(\frac{1}{R_3} - \frac{1}{R_4}\right)\right] - \frac{\nu P_z}{\pi\mu}\frac{\partial}{\partial z}\left(\frac{1}{R_3} - \frac{1}{R_4}\right)$$

in x 0, z 0

Since $\frac{\partial B_x}{\partial x}$ (and hence B_x) is harmonic in the region and satisfies equations (48), and (Ch. I, 11).

From equation (51), we get:

$$\frac{\partial \beta}{\partial x} = (1 - 2\nu)B_x - \frac{cP_z}{4\pi\mu}\frac{\partial}{\partial x}\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4}\right)$$

or

$$\frac{\partial \beta}{\partial x} = (1-2\nu)\left\{ \frac{P_z}{2\pi\mu}\left[\frac{\partial}{\partial z}\left(\frac{1}{R_3}-\frac{1}{R_4}\right)dx + a\frac{\partial}{\partial z}\left(\frac{1}{R_3}-\frac{1}{R_4}\right)\right] - \right.$$

$$\left. \frac{\nu P_z}{\pi\mu}\int \frac{\partial}{\partial z}\left(\frac{1}{R_3}-\frac{1}{R_4}\right)dx \right\} - \frac{cP_z}{4\pi\mu}\frac{\partial}{\partial x}\left(\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}+\frac{1}{R_4}\right)$$

Therefore, we get by integration with respect to x:

$$\beta = \frac{(1-2\nu)P_z}{2\pi\mu}\int\left[\int \frac{\partial}{\partial z}\left(\frac{1}{R_3}-\frac{1}{R_4}\right)dx\right]dx + \frac{a(1-2\nu)P_z}{2\pi\mu}\int \frac{\partial}{\partial z}\left(\frac{1}{R_3}-\frac{1}{R_4}\right)dx$$

$$- \frac{\nu P_z(1-2\nu)}{\pi\mu}\int\left[\int \frac{\partial}{\partial z}\left(\frac{1}{R_3}-\frac{1}{R_4}\right)dx\right]dx - \frac{cP_z}{4\pi\mu}\left(\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}+\frac{1}{R_4}\right)$$

Finally the Papkovitch functions for this problem are:

$$B_x = \frac{P_z(1-2\nu)}{2\pi\mu}\int \frac{\partial}{\partial z}\left(\frac{1}{R_3}-\frac{1}{R_4}\right)dx + \frac{aP_z}{2\pi\mu}\frac{\partial}{\partial z}\left(\frac{1}{R_3}-\frac{1}{R_4}\right) \quad \ldots\ldots\ (52)$$

$$B_y = 0 \quad \ldots\ldots\ (53)$$

$$B_z = \frac{P_z}{4\pi\mu}\left(\frac{1}{R_1}-\frac{1}{R_2}+\frac{1}{R_3}-\frac{1}{R_4}\right) \quad \ldots\ldots\ (54)$$

$$\beta = \frac{P_z}{2\pi\mu}(1-2\nu)^2\int\left[\int \frac{\partial}{\partial z}\left(\frac{1}{R_3}-\frac{1}{R_4}\right)dx\right]dx + \frac{a(1-2\nu)P_z}{2\pi\mu}\int \frac{\partial}{\partial z}\left(\frac{1}{R_3}-\frac{1}{R_4}\right)dx$$

$$- \frac{cP_z}{4\pi\mu}\left(\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}+\frac{1}{R_4}\right) \quad \ldots\ldots\ (55)$$

<u>2. Verification:</u>

on $z = 0$

$$\frac{\partial B_x}{\partial z} = \frac{P_z}{2\pi\mu}\left[\int \frac{\partial^2}{\partial z^2}\left(\frac{1}{R_3}-\frac{1}{R_4}\right)dx + a\frac{\partial^2}{\partial z^2}\left(\frac{1}{R_3}-\frac{1}{R_4}\right)\right] - \frac{\nu P_z}{\pi\mu}\int \frac{\partial^2}{\partial z^2}\left(\frac{1}{R_3}-\frac{1}{R_4}\right)dx = 0$$

Since

$$\frac{\partial^2}{\partial z^2}\left(\frac{1}{R_3} - \frac{1}{R_4}\right) = -\frac{1}{R_3^3} + \frac{1}{R_4^3} + \frac{3(z-c)^2}{R_3^5} - \frac{3(z+c)^2}{R_4^5} = 0 \quad \text{on } z = 0$$

it follows that

$$\frac{\partial^2 B_x}{\partial y \partial z} = \frac{\partial^2 B_x}{\partial x \partial z} = 0$$

also we have:

$B_z = 0$, hence $\dfrac{\partial B_y}{\partial y} = \dfrac{\partial B_z}{\partial x} = 0$

$$\frac{\partial \beta}{\partial z} = \frac{P_z}{2\pi\mu}(1-2\nu)^2 \int \left[\int \frac{\partial^2}{\partial z^2}\left(\frac{1}{R_3} - \frac{1}{R_4}\right) dx \right] dx + \frac{a(1-2\nu)P_z}{2\pi\mu} \int \frac{\partial^2}{\partial z^2}\left(\frac{1}{R_3} - \frac{1}{R_4}\right) dx$$

$$- \frac{cP_z}{4\pi\mu}\frac{\partial}{\partial z}\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4}\right)$$

or

$$\frac{\partial \beta}{\partial z} = \frac{P_z}{2\pi\mu}(1-2\nu)^2 \int \left[\int \left(-\frac{1}{R_3^3} + \frac{1}{R_4^3} + \frac{3(z-c)^2}{R_3^5} - \frac{3(z+c)^2}{R_4^5} \right) dx \right] dx$$

$$+ \frac{a(1-2\nu)P_z}{2\pi\mu} \int \left(-\frac{1}{R_3^4} + \frac{1}{R_4^3} + \frac{3(z-c)^2}{R_3^5} - \frac{3(z+c)^2}{R_4^5} \right) dx$$

$$- \frac{cP_z}{4\pi\mu}\left(-\frac{(z-c)}{R_1^3} - \frac{(z+c)}{R_2^3} - \frac{(z-c)}{R_3^3} - \frac{(z+c)}{R_4^3} \right) = 0$$

Therefore, $\dfrac{\partial^2 \beta}{\partial y \partial z} = \dfrac{\partial^2 \beta}{\partial z \partial x} = 0$

on $x = 0$

from equation (51), we have:

$$(1-2\nu)B_x - \frac{\partial\beta}{\partial x} = \frac{cP_z}{4\pi\mu}\frac{\partial}{\partial x}\left(\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}+\frac{1}{R_4}\right)$$

$$= -\frac{cP_z}{4\pi\mu}\left(\frac{x-a}{R_1^{\,3}}+\frac{x-a}{R_2^{\,3}}+\frac{x+a}{R_3^{\,3}}+\frac{x+a}{R_4^{\,3}}\right) = 0$$

on $x = 0$, it follows that

$$(1-2\nu)\frac{\partial B_x}{\partial y} - \frac{\partial^2\beta}{\partial x\partial y} = (1-2\nu)\frac{\partial B_x}{\partial z} - \frac{\partial^2\beta}{\partial x\partial z} = 0$$

Finally

$$\frac{\partial B_z}{\partial x} = \frac{-P_z}{4\pi\mu}\left(\frac{x-a}{R_1^{\,3}} - \frac{x-a}{R_2^{\,3}} + \frac{x+a}{R_3^{\,3}} - \frac{x+a}{R_4^{\,3}}\right) = 0$$

hence

$$\frac{\partial^2 B_z}{\partial x\partial y} = \frac{\partial^2 B_z}{\partial x\partial z} = 0$$

Now as the plane $x = 0$ recedes to infinity, we get that limiting values of Papkovitch functions are:

$$B_x = 0$$

$$B_y = 0$$

$$B_z = \frac{P_z}{4\pi\mu}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

$$\beta = -\frac{cP_z}{4\pi\mu}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

Which are the Papkovitch functions as given by Hijab for half-space.

REFERENCES

1. S. Timoshenko and J. N. Goodier: "Theory of Elasticity", second edition, McGraw-Hill, New York, 1951.

2. W. A. Hijab: "Application of Papkovitch Functions to a Three-Dimensional Problem of Elasticity, "Proceedings, Mathematical and Physical Society of Egypt (U.A.R.), No. 23, 1959, pp. 99 - 115.

3. W. A. Hijab: "Concentrated Force in Eighth-Space with Mixed Boundary Conditions", scheduled to appear in Journal fur reine und angewandte Mathematik (West Germany).

4. R. D. Mindlin: Bull. Am. Math. Soc., vol. 42, 1936 pp. 373 - 376.

5. L. Rongved: "Force at a Point in The Interior of a Semi-Infinite Solid with Fixed Boundary", journal of applied Mechanics, Dec. 1955, pp. 545 - 546.

6. C.E. Weatherburn: Advanced vector Analysis. Bell, London, 1928.

7. J. G. Coffin: Vector Analysis. Wiley, New York, 1909.

8. H. B. Philips: Vector Analysis. Wiley, New York, 1949.

9. R. D. Mindlin: "Force at an Interior Point of a Semi-Infinite Solid", Proceedings of the First Mid-Western Conference on Solid Mechanics. The Engineering Experiment Station, University of Illinois, Urbana, Illinois, 1953.