A TWO DIMENSIONAL MANIFOLD
IMMERSED IN A FOUR DIMENSIONAL EUCLIDEAN SPACE

By

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ABSTRACT

The purpose of this paper is to investigate canonicality in a 2- dimensional Riemannian manifold M^2 immersed in a 4- dimensional Euclidean space E^4, to develop formulas and prove translation theorems for two manifolds.

A combination of tensor analysis and exterior differential forms is used, according as one method or the other seems to be a more natural tool.

The equations of Gauss and Codazzi are developed in chapter I. In chapter II, a necessary condition for canonicality is established. Chapters III and IV deal with parallel transformations.

Recently many authors have been able to find conditions on H_{rp} and H^*_{rp}, the p-th mean curvatures of two compact oriented manifolds M^n and M^{*n} relative to corresponding unit normals e_r and e^*_r respectively, under which f is a translation. The pioneers were Hopf and Voss [3] who studied the case $m = p = 1$, $n = 2$. The most general case was studied by Hsiung and Nassar [5]. In this paper the case $m = n = p = 2$ is investigated.

INTRODUCTION

The purpose of this paper is to investigate canonicality in a 2- dimensional Riemannian manifold M^2 immersed in a 4- dimensional Euclidean space E^4, to develop formulas and prove translation theorems for 2 manifolds.

A combination of tensor analysis and exterior differential forms is used, according as one method or the other seems to be a more natural tool.

The equations of Gauss and Codazzi are developed in chapter II. In chapter III, a necessary condition for canonicality is established. Chapters IV and V deal with parallel transformations. A parallel transformation $f: M \to M^*$ between two manifolds M and M^* is a diffeomorphism such that the line joining every pair of corresponding points P and P^* of M and M^* respectively is parallel to a fixed unit vector E of E^4.

Recently many authors have been able to find conditions on H_{rp} and H^*_{rp}, the p-th mean curvatures of two compact oriented manifolds M^n and M^{*n} relative to corresponding unit normals e_r and e^*_r respectively, under which f is a translation. The pioneers were Hopf and Voss [3] who studied the case $m=p=1$, $n=2$. Later Hsiung [4] and Voss [7] independently extended this result to the case where $m = 1$, $p = 1$, and the case where $m = 1$ respectively. Very recently Stong [6] dealt with the case where $p = 1$. The most general case where m, n, p are arbitrary, was studied by Hsiung and Nassar [5]. In this paper the case $m = n = p = 2$ is investigated.

CHAPTER I

PRELIMINARIES

Let M^2 be a two dimensional Riemannian manifold of class C^3 immersed in a four dimensional Euclidean space E^4. Let $Y=(y^1, y^2, y^3, y^4)$ denote the position vector of a point P with respect to a fixed orthonormal frame $(OI_1I_2I_3I_4)$- If x^1, x^2 are local coordinates of P on M^2, then the tangent space is the space spanned by:

$$Y_1 = \frac{\partial Y}{\partial x^1} \quad \text{and} \quad Y_2 = \frac{\partial Y}{\partial x^2}$$

In what follows it is assumed that the x^1's take real values in a simply connected domain D of the two dimensional real number space such that the matrix $\| \partial Y / \partial x^i \|$ is of rank two at all points of D. Since g, the determinant of the metric tensor $g_{ij}=Y_i \cdot Y_j$, is not zero, there exist (see for example [1] pp. 141-146) two orthonormal unit vectors e_1 and e_2 none of which is zero normal to M^2.

Unless otherwise indicated, tensor notation will be used throughout this paper, so that the repetition of a letter as a subscript and superscript indicates summation on that letter from 1 to 2.

The following definitions and results will be needed later. They are special cases of the corresponding ones for $M^n \subset E^{m+n}$ in [5].

Definition 1: If E is any vector at an ordinary point P of the manifold M^2 and e_1, e_2 are mutually orthogonal unit vectors normal to M^2 at P such that $E \cdot e_1 \neq 0$ at a dense set, and $e_2 \cdot E = 0$, then e_1 is called the normal vector of M^2 associated with the vector E at P.

Definition 2: e_1 is said to be canonical if $e_{1,i} = \frac{\partial e_1}{\partial x^i}$ (i = 1,2) are in the tangent space of the manifold M^2 at the point P.

Definition 3: With respect to the fixed orthonormal frame $(OI_1I_2I_3I_4)$ in E^4, the scalar product of two vectors A_1 and A_2 is:

$$A_1 \cdot A_2 = \sum_{k=1}^{4} A_1^j \, A_2^j$$

where A_i^j are the components of A_i in the direction I_j.

Definition 4: The vector product of three vectors A_1, A_2, A_3 through the point O in E^4 is the vector A_4 through O given by:

$$A_4 = A_1 \times A_2 \times A_3$$

satisfying the following 3 conditions:

(i) A_4 is normal to the 3-dimensional subspace of E^4 spanned by A_1, A_2, A_3.

(ii) Its magnitude is equal to the volume of the parallelepiped whose edges are A_1, A_2, A_3.

(iii) Its sense is such that $(OI_1I_2I_3I_4)$ and $(OA_1A_2A_3A_4)$ have the same orientation.

The scalar product of A_1 and $A_2 \times A_3 \times A_4$ is given by:

$$A_1.(A_2 \times A_3 \times A_4) = (-1)^3 |A_1, A_2, A_3, A_4|$$

the right hand side being the negative of the determinant of the contravariant components of the A_i's.

Definition 5: The element of area of M^2 at the point P is given by:

$$dA = \sqrt{g}\, dx^1 \wedge dx^2$$

where the wedge $\wedge$ denotes exterior product of differentials, which is associative, distributive and anti-symmetric (see for example [2]).

Definition 6: The combined operator of the vector product of vectors and exterior product of differentials in M^n E^{m+n} is defined by:

$$A^1 \otimes \ldots \otimes A^a \otimes dA^{a+1} \otimes \ldots \otimes dA^{n+m-1} =$$

$$(A^1 \times \ldots \times A^a \times A^{a+1}{}_{,i_{a+1}} \times \ldots \times A^{n+m-1}{}_{,i_{n+m-1}})$$

$$.dx^{i_{a+1}} \wedge \ldots \wedge dx^{i_{n+m-1}}, \qquad (a = 1, \ldots, n+m-1).$$

It is obvious that this product vanishes for $n + m - 1 - a > n$ and is independent of the order of $dA^{a+1}, \ldots, dA^{n+m-1}$.

Result 1:

$$|Y_1, Y_2, e_1, e_2| = \sqrt{g},$$

where e_1, e_2 are unit normals to M^2 and their senses chosen so that $PY_1Y_2e_1e_2$ and $OI_1I_2I_3I_4$ have the same orientation.

Proof:

$$\text{Let } D^2 = D^tD = \begin{vmatrix} y_1^1 & y_1^2 & y_1^3 & y_1^4 \\ y_2^1 & y_2^2 & y_2^3 & y_2^4 \\ e_1^1 & e_1^2 & e_1^3 & e_1^4 \\ e_2^1 & e_2^2 & e_2^3 & e_2^4 \end{vmatrix} \begin{vmatrix} y_1^1 & y_2^1 & e_1^1 & e_2^1 \\ y_1^2 & y_2^2 & e_1^2 & e_2^2 \\ y_1^3 & y_2^3 & e_1^3 & e_2^3 \\ y_1^4 & y_2^4 & e_1^4 & e_2^4 \end{vmatrix}$$

where y_k^i is the i-th component of $\partial Y/\partial x^k$ and e_k^i is the i-th component of e_k with respect to $(OI_1I_2I_3I_4)$, $(k = 1,2$, $i = 1,\ldots,4)$.

Then

$$D^2 = \begin{vmatrix} Y_1\cdot Y_1 & Y_1\cdot Y_2 & 0 & 0 \\ Y_2\cdot Y_1 & Y_2\cdot Y_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} g_{11} & g_{12} & 0 & 0 \\ g_{21} & g_{22} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = g$$

Hence $D = \pm\sqrt{g}$, and because the orientations of $PY_1Y_2e_1e_2$ and $OI_1I_2I_3I_4$ are the same, $D = \sqrt{g}$.

Result 2:

$$Y_1 \times Y_2 \times e_2 = -\sqrt{g}\, e_1$$

Proof:

Since $Y_1 \times Y_2 \times e_2$ is normal to the space spanned by Y_1, Y_2, e_2, it is in the direction of e_1, i.e. $Y_1 \times Y_2 \times e_2 = Ae_1$. To find A, form the scalar product of both sides of this equation and e_1 :

$$\begin{aligned} A &= e_1 \cdot (Y_1 \times Y_2 \times e_2) \\ &= -|e_1 , Y_1 , Y_2 , e_2| \\ &= -|Y_1 , Y_2 , e_1 , e_2| \\ &= -\sqrt{g} \end{aligned}$$

CHAPTER II

FUNDAMENTAL FORMS

In what follows subscripts denote ordinary differentiation and subscripts after a comma denote covariant differentiation with respect to g_{ij}. Therefore:

$$Y_i = Y_{,i}$$

$$Y_{,i} \cdot Y_{,j} = g_{ij} \quad \ldots\ldots\ldots\ldots\ldots (2.1)$$

Differentiating (2.1) covariantly:

$$Y_{,ik} \cdot Y_{,j} + Y_{,i} \cdot Y_{,jk} = 0 \quad (2.2)$$

A cyclic permutation of i, j, k gives the two similar relations:

$$Y_{,ji} \cdot Y_{,k} + Y_{,j} \cdot Y_{,ki} = 0 \quad (2.3)$$

$$Y_{,kj} \cdot Y_{,i} + Y_{,k} \cdot Y_{,ij} = 0 \quad (2.4)$$

Now

$$Y_{,ij} = Y_{i,j} = \frac{\partial Y_i}{\partial x^j} - Y_k \begin{Bmatrix} k \\ ij \end{Bmatrix}$$

and

$$Y_{,ji} = Y_{j,i} = \frac{\partial Y_j}{\partial x^i} - Y_k \begin{Bmatrix} k \\ ji \end{Bmatrix}$$

Since g_{ij} is symmetric, the christoffel symbol $\begin{Bmatrix} k \\ ij \end{Bmatrix}$ is symmetric in i and j. Since Y is of class C^2, $Y_{,ij} = Y_{,ji}$. In this case (2.3) + (2.4) - (2.2) gives:

$$Y_{,ij} \cdot Y_k = 0$$

which means that $Y_{,ij}$ are normal to $Y_{,k}$ and hence they are linear combinations of the e_r's $(r = 1,2)$.

$$Y_{,ij} = \sum_{r=1}^{2} b_{r|ij}\, e_r \quad \ldots\ldots\ldots\ldots\ldots\ldots (2.5)$$

The first fundamental form for M^2 is given by:

$$I = dY \cdot dY = \frac{\partial Y}{\partial x^i} dx^i \cdot \frac{\partial Y}{\partial x^j} dx^j$$

$$= \frac{\partial Y}{\partial x^i} \cdot \frac{\partial Y}{\partial x^j} dx^i\, dx^j$$

$$= \sum_{k=1}^{4} (y_i^k\, y_j^k)\, dx^i\, dx^j.$$

Hence $$g_{ij} = \sum_{k=1}^{4} y_i^k\, y_j^k$$

The second fundamental form for M^2 is given by

the quantity $$II_r = - de_r \cdot dY.$$

Differentiating $e_r \cdot Y_{,i} = 0$, one gets:

$$e_r \cdot Y_{,ij} = - e_{r,j} \cdot Y_{,i} \quad (2.6)$$

Hence $$II_r = - de_r \cdot dY = - e_{r,j} \cdot Y_{,i}\, dx^i\, dx^j$$

$$= e_r \cdot Y_{,ij}\, dx^i\, dx^j$$

and by (2.5)

$$II_r = \sum_{s=1}^{2} b_{s|ij}\, e_s \cdot e_r\, dx^i\, dx^j \quad \ldots\ldots\ldots (2.7)$$

$$= b_{r|ij}\, dx^i\, dx^j$$

i.e. the $b_{r\,ij}$ are the coefficients of II_r.

(2.5) and (2.6) imply:

$$b_{r|ij} + Y_{,i} \cdot e_{r,j} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (2.8)$$

Let $$e_r \cdot e_{s,i} = \theta_{sr\,i} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2.9)$$

$$e_r \cdot e_s = \delta_{rs} \quad \text{(Kronecker delta)}$$

Hence $e_{r,i} \cdot e_s + e_r \cdot e_{s,i} = 0$

and by (2.9): $\theta_{rs|i} + \theta_{sr|i} = 0$ and $\theta_{rr|i} = 0$ (2.10)

(2.8) and (2.9) show that $e_{r,i}$ are linear combinations of the $Y_{,i}$'s and the e_r's.

Let $e_{r,i} = A^k_{r|i} Y_{,k} + B^s_{r|i} e_s$.

Substitution in (2.9) gives:

$$e_r \cdot (A^k_{s|i} Y_{,k} + B^r_{s|i} e_r) = \theta_{sr|i}.$$

Hence $B^r_{s|i} = \theta_{sr|i}$.

Substitution in (2.8) gives:

$$b_{r|ij} + Y_{,i} \cdot (A^k_{r|j} Y_{,k} + B^s_{r|j} e_s) = 0,$$

or $$b_{r|ij} + A^k_{r|j} g_{ik} = 0,$$

multply by g^{jk}:

$$A^k_{r|j} g_{ik} g^{jk} = - b_{r|ij} g^{jk}$$

$$A^k_{r|j} \delta^j_i = - b_{r|ij} g^{jk}$$

$$A^k_{r|i} = - b_{r|ij} g^{jk}.$$

Hence:

$$e_{r,i} = -b_{r|ij} g^{jk} Y_{,k} + \sum_{s=1}^{2} \theta_{rs|i} e_s \quad \ldots\ldots\ldots\ldots (2.11)$$

CHAPTER III

A NECESSARY CONDITION FOR CANONICALITY

Let $E = (y_3^1, y_3^2, y_3^3, y_3^4)$ be a fixed unit vector at the point P, the normal vector to M^2 associated with the vector E at P is the vector e_1 such that:

$$E \cdot e_2 = 0$$

$$E \cdot e_1 \neq 0 \quad \text{almost everywhere.}$$

e_1 and e_2 are determined by:

$$e_2 \cdot E = 0 \qquad e_1 \cdot e_2 = 0 \quad \ldots\ldots\ldots\ldots (3.1)$$

$$e_2 \cdot Y_1 = 0 \qquad e_1 \cdot Y_1 = 0 \quad \ldots\ldots\ldots\ldots (3.2)$$

$$e_2 \cdot Y_2 = 0 \qquad e_1 \cdot Y_2 = 0 \quad \ldots\ldots\ldots\ldots (3.3)$$

$$|e_2| = 1 \qquad |e_1| = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (3.4)$$

(3.1), (3.2) and (3.3) imply $\sum_{k=1}^{4} y_4^k y_i^k = 0$ $(i = 1, 2, 3)$, where y_4^k are the components of e_2 in E^4, giving three linear homogeneous equations in four unknowns y_k^4 $(k = 1,\ldots,4)$.

$$\frac{y_4^1}{C_4^1} = \frac{y_4^2}{C_4^2} = \frac{y_4^3}{C_4^3} = \frac{y_4^4}{C_4^4} = t,$$

where C_4^k is the cofactor of y_4^k in the determinant $|y_j^i|$ $(i, j, k = 1,\ldots,4)$.

We observe that C_4^k $(k = 1,\ldots,4)$ cannot vanish simultaneously almost everywhere, and hence e_2 is defined.

At points where E is a linear combination of Y_1 and Y_2, e_2 is then defined by continuity.

Furthermore (3.4) implies:

$$\sum_{k=1}^{4}(y_4^k)^2 = 1,$$

which implies:

$$t^2\sum_{k=1}^{4}(C_4^k)^2 = 1,$$

$$t = \frac{\pm 1}{\sqrt{\sum_{k=1}^{4}(C_4^k)^2}},$$

and we choose sign of t so that $Y_1 \times Y_2 \times E$ has the same direction as e_2.

Result 2 implies: $e_1 = -\frac{Y_1 \times Y_2 \times e_2}{\sqrt{g}}$.

Let $e_1 = (a_4^1, a_4^2, a_4^3, a_4^4)$

$Y_1 = (a_1^1, a_1^2, a_1^3, a_1^4) = (y_1^1, y_1^2, y_1^3, y_1^4)$

$Y_2 = (a_2^1, a_2^2, a_2^3, a_2^4) = (y_2^1, y_2^2, y_2^3, y_2^4)$

$e_2 = (a_3^1, a_3^2, a_3^3, a_3^4) = (y_4^1, y_4^2, y_4^3, y_4^4)$.

Then:

$$e_1 = -\frac{1}{\sqrt{g}}(A_4^1\, i_1 + A_4^2\, i_2 + A_4^3\, i_3 + A_4^4\, i_4)$$

$$= -\frac{1}{\sqrt{g}} A_4^j\, i_j \quad (j = 1,\ldots,4),$$

where i_k $(k = 1,\ldots,4)$ are the base vectors of E^4 and A_4^k is the cofactor of a_4^k in $|a_j^i|$.

Let us now consider the second fundamental form relative to e_1.

(2.7) implies $\quad II_1 = b_{1|ij}\, dx^i\, dx^j.$

Since
$$Y_{,ij} = \frac{\partial Y_i}{\partial x^j} - Y_k \left\{ {k \atop ij} \right\}$$
$$= \frac{\partial^2 Y}{\partial x^i \partial x^j} - Y_k \left\{ {k \atop ij} \right\},$$

and by (2.5) $\quad Y_{,ij} = b_{1|ij}\, e_1 + b_{1|ij}\, e_2\, ;$

then
$$b_{1|ij} = Y_{,ij} \cdot e_1$$
$$= \frac{\partial^2 Y}{\partial x^i \partial x^j} \cdot e_1$$
$$= Y_{ij} \cdot e_1$$
$$= - \frac{Y_1 \times Y_2 \times e_2}{\sqrt{g}} \cdot Y_{ij}\, ,$$

and
$$II_1 = - \frac{Y_1 \times Y_2 \times e_2}{\sqrt{g}} \cdot Y_{ij}\, dx^i\, dx^j.$$

Now since $\quad e_2 = \dfrac{Y_1 \times Y_2 \times E}{\sqrt{g_E}}\, ,$

where $\sqrt{g_E} = \left| Y_1, Y_2, E, e_2 \right|$, it may be possible to symplify a vector product of the form:

$$Y_1 \times Y_2 \times (Y_1 \times Y_2 \times E).$$

In general, consider the vector:

$$V = A \times B \times (C \times D \times E).$$

V is normal to A, b and $C \times D \times E$, therefore:

$$V = uC + vD + wE,$$

i.e. V is in the space spanned by C, D, E.

Since $V.A = V.B = 0$,

$uC.A + vD.A + wE.A = 0$,

and $uC.B = vD.B + wE.B = 0$.

Hence

$$\frac{u}{(D.A)(E.B)-(D.B)(E.A)} = \frac{v}{(C.B)(E.A)-(C.A)(E.B)}$$

$$= \frac{w}{(C.A)(D.B)-(D.A)(C.B)} = k$$

An argument based on the dimension of V shows that k is a scalar i.e. of dimension zero.

To calculate k, let:

$A = i_1$, $B = i_3$, $C = i_1$, $D = i_2$, $E = i_3$, then:

$$V = A \times B \times (C \times D \times E)$$

$$= i_1 \times i_3 \times (i_1 \times i_2 \times i_3)$$

$$= i_1 \times i_3 \times i_4$$

$$= i_2$$

$$= - k\, i_2.$$

Therefore $k = -1$.

Thus $V = -[(D.A)(E.B)-(D.B)(E.A)]C-[(C.B)(E.A)-(C.A)(E.B)]D$
$-[(C.A)(D.B)-(D.A)(C.B)]E.$ ....(3.5)

Hence:

$$e_1 = - \frac{Y_1 \times Y_2 \times (Y_1 \times Y_2 \times E)}{\sqrt{g} \qquad \sqrt{g_E}}$$

$$e_1 = \frac{1}{\sqrt{gg_E}} [(Y_2.Y_1)(E.Y_2)-(Y_2.Y_2)(E.Y_1)]Y_1$$

$$+ [(Y_1.Y_2)(E.Y_1)-(Y_1.Y_1)(E.Y_2)]Y_2+[(Y_1)^2(Y_2)^2-(Y_1.Y_2)^2]E.$$

$$e_1 = \frac{1}{\sqrt{gg_E}} [(g_{21}E\cdot Y_2 - g_{22}E\cdot Y_1)Y_1 + (g_{12}E\cdot Y_1 - g_{11}E\cdot Y_2)Y_2 + gE] \quad (3.6)$$

The coordinates x^1, x^2 can be chosen such that Y_1 is orthogonal to Y_2 or $g_{12} = g_{21} = 0$. In this case:

$$e_1 = \frac{1}{\sqrt{g_E}} [-\frac{g_{22}}{\sqrt{g}} (E\cdot Y_1)Y_1 - \frac{g_{11}}{\sqrt{g}} (E\cdot Y_2)Y_2 + g\,E]$$

$$N_1 = \frac{\sqrt{g_E}}{\sqrt{g}} e_1 = E - (E\cdot Y_i)\, g^{ij}\, Y_j \ , \quad (i,j = 1,2) \ \ldots\ldots\ldots (3.7)$$

where $g^{ij} = \frac{G^{ij}}{g}$ and G^{ij} is the cofactor of g_{ij} in g.

$$|e_1|^2 = \frac{1}{g_E} [g - g_{11}(E\cdot Y_2)^2 - g_{22}(E\cdot Y_1)^2] = 1.$$

Hence:

$$\sqrt{g_E} = [g - g_{11}(E\cdot Y_2)^2 - g_{22}(E\cdot Y_1)^2]^{\frac{1}{2}} \ \ldots\ldots\ldots\ldots (3.8)$$

If $\quad e_1 = a^1 Y_1 + a^2 Y_2 + bE$

$$de_1 = Y_1 da^1 + Y_2 da^2 + a^1 dY_1 + a^2 dY_2 + E db.$$

$$de_1 = Y_1 da^1 + Y_2 da^2 + a^1(\frac{\partial Y_1}{\partial x^1}dx^1 + \frac{\partial Y_1}{\partial x^2}dx^2) + a^2(\frac{\partial Y_2}{\partial x^1}dx^1 + \frac{\partial Y_2}{\partial x^2}dx^2) + E db$$

$$= Y_1 da^1 + Y_2 da^2 + a^1(Y_{11}dx^1 + Y_{12}dx^2) + a^2(Y_{21}dx^1 + Y_{22}dx^2) + E db$$

$$= Y_1 da^1 + Y_2 da^2 + a^1[(Y_{,11} + Y_k\left\{{k \atop 11}\right\})dx^1 + (Y_{,12} + Y_k\left\{{k \atop 12}\right\})dx^2]$$

$$+ a^2[(Y_{,21} + Y_k\left\{{k \atop 21}\right\})dx^1 + (Y_{,22} + Y_k\left\{{k \atop 22}\right\})dx^2] + E db$$

$$de_1 = Y_1 da^1 + Y_2 da^2 + \left(a^1 Y_k \left\{ {k \atop 11} \right\} + a^2 Y_k \left\{ {k \atop 21} \right\}\right) dx^1 + \left(a^1 Y_k \left\{ {k \atop 12} \right\} + a^2 Y_k \left\{ {k \atop 22} \right\}\right) dx^2$$

$$+ E db + (a^1 Y_{,11} + a^2 Y_{,12}) dx^1 + (a^1 Y_{,12} + a^2 Y_{,22}) dx^2.$$

The component of E along e_2 is zero (since $E \cdot e_2 = 0$). Also de_1 has no component in the direction of e_1, since $e_1 \cdot e_1 = 1$ implies $e_1 \cdot de_1 = 0$.

Therefore e_1 is canonical if and only if the component of de_1 along e_2 is zero. This component is $de_1 \cdot e_2$.

Since $Y_{,ij} = b_{1|ij}\, e_1 + b_{2|ij}\, e_2$,

$$de_1 \cdot e_2 = (a^1 b_{2|11} + a^2 b_{2|12}) dx^1 + (a^1 b_{2|12} + a^2 b_{2|22}) dx^2$$

$$e_{1,1} \cdot e_2 = a^1 b_{2|11} + a^2 b_{2|12} = \theta_{21|1}$$

$$e_{1,2} \cdot e_2 = a^1 b_{2|12} + a^2 b_{2|22} = \theta_{21|2} ,$$

a^1 and a^2 are not zero at points where E is not in the tangent space since g_{11} and g_{22} are not zero.

Hence $b_{2|11} b_{2|22} - (b_{2|12})^2 = 0$,

i.e. $\left| b_{2|ij} \right| = 0$ at a dense set and by continuity

$\left| b_{2|ij} \right| = 0$ everywhere.

In general we have the following

THEOREM: Let M^n be an n-dimensional Riemannian manifold of class C^2 immeresed in a Euclidean space E^{m+n} of dimension $m+n$. Let E be a unit fixed direction in E^{m+n}, and $e_1, \ldots, e_m$ the unit normals of M^n. If $E.e_r = 0$, $r=2,\ldots,m$, e_1 is canonical and does not coincide with E except possibly at a set of measure zero, then $|b_{r|ij}| = 0$, $r=2,\ldots,m$, where $b_{r|ij}$ are the coefficients of the second fundamental form relative to e_r.

<u>Proof</u>:

At points where $e_1 \neq E$, one can write:

$$e_1 = a^i \, Y_i + bE,$$

where $a^i \neq 0$ for all i.

Differentiation with respect to x^j gives:

$$e_{1j} = a^i \, Y_{ij} + a^i{}_j \, Y_i + b_j \, E$$

$$= a^i \, (Y_{,ij} + \left\{ {k \atop ij} \right\} Y_k) + a^i{}_j \, Y_i + b_j \, E$$

$$e_{1j} \cdot e_r = a^i \, b_{r|ij} \, .$$

Since e_1 is canonical, $e_{1j} \cdot e_r = 0$ and hence :

$$a^i \, b_{r|ij} = 0,$$

which implies $\left| b_{r|ij} \right| = 0$ at points where $e_1 \neq E$, and hence by continuity $\left| b_{r|ij} \right| = 0$ everywhere.

CHAPTER IV

PARALLEL TRANSFORMATIONS

Let M^2 and M^{*2} be two Riemannian manifolds of dimension 2 immersed in a four-dimensional Euclidean space E^4. Consider a diffeormorphism $f: M^2 \rightarrow M^{*2}$, f is said to be a parallel transformation from M^2 into M^{*2} in the direction of a fixed unit vector E in E^4 if the line joining every pair of corresponding points of M^2 and M^{*2} is parallel to E. It is assumed that the set of points of M^2 for which E is in the tangent space has measure zero.

Let Y denote the position vector of any point P in M^2, then the corresponding position vector for M^{*2} is

$$Y^* = f(Y) = Y + wE = Y + W \quad \text{.................}(4.1)$$

where w is a scalar, and suppose further that M^2 and M^{*2} have the same local coordinates x^1, x^2.

Define a linear variation between M^2 and M^{*2} as follows:

$$M^2(t): \; Y(t) = Y + twE \; , \; (0 = t = 1) \quad \text{......}(4.2)$$

Hence $M^2(0) = M^2$ and $M^2(1) = M^{*2}$,

and $$Y(t)_i = Y_i + tw_iE \; , \; (i = 1,2) \quad \text{............}(4.3)$$

Let $[Y_1, Y_2, E](t)$ be the subspace of E^4 spanned by $Y(t)_1$, $Y(t)_2$ and E. It is obvious that

$$[Y_1, Y_2, E](t) = [Y_1, Y_2, E](0) = [Y_1, Y_2, E].$$

The dimension of $[Y_1, Y_2, E]$ is 3 almost everywhere.

Thus as in chapter III, we can define e_2 unit normal to $[Y_1, Y_2, E](t)$, and it is independent of t since

$$\begin{aligned} N &= Y(t)_1 \times Y(t)_2 \times E \\ &= (Y_1 + tw_1E) \times (Y_2 + tw_2E) \times E \\ &= Y_1 \times Y_2 \times E. \end{aligned}$$

Now the subspace $[Y_1, Y_2, e_2](t)$ of E^4 has a unit normal $e(t)_1$ which in general depends on t since

$$e(t)_1 = \frac{N(t)}{|N(t)|} \qquad \text{...............(4.4)}$$

where

$$\begin{aligned} N(t) &= Y(t)_1 \times Y(t)_2 \times e_2 \\ &= (Y_1 + tw_1E) \times (Y_2 + tw_2E) \times (Y_1 \times Y_2 \times E) \\ &= Y_1 \times Y_2 \times (Y_1 \times Y_2 \times E) \\ &\quad + (tw_1)E \times Y_2 \times (Y_1 \times Y_2 \times E) \\ &\quad + (tw_2)Y_1 \times E \times (Y_1 \times Y_2 \times E) \\ &\quad + t^2w_1w_2E \times E \times (Y_1 \times Y_2 \times E). \end{aligned}$$

Using (3.5) we get:

$$N(t) = [g_{21}(E\cdot Y_2)-g_{22}(E\cdot Y_1)]Y_1+[g_{12}(E\cdot Y_1)-g_{11}(E\cdot Y_2)]Y_2+gE$$
$$+tw_1[(E\cdot Y_2)^2-g_{22}]Y_1+tw_1[g_{12}-(E\cdot Y_1)(E\cdot Y_2)]Y_2$$
$$+tw_1[g_{22}(E\cdot Y_1)-g_{12}(E\cdot Y_2)]E$$
$$+tw_2[g_{21}-(E\cdot Y_2)(E\cdot Y_1)]Y_1+tw_2[(E\cdot Y_1)^2-g_{11}]Y_2$$
$$+tw_2[g_{11}(E\cdot Y_2)-g_{21}(E\cdot Y_1)]E \quad \ldots\ldots\ldots\ldots\ldots\ldots (4.5)$$

The above expression may be simplified further if the coordinate curves are chosen to be orthogonal so that $g_{12} = g_{21} = 0$.

Notice that $e(0)_1 = e_1$ is given in (3.6).

LEMMA 4.1: If $e'(t)_1$ denotes the derivative of $e(t)_1$ with respect to t then,

$$e'(t)_1 = -\, g(t)^{ij}(W_i \cdot e(t)_1)Y(t)_j.$$

<u>Proof</u>: Differentiation of $e(t)_1 \cdot e(t)_1 = 1$ with respect to t gives:

$$e(t)_1 \cdot e'(t)_1 = 0.$$

Therefore $e'(t)_1$ is normal to $e(t)_1$ and hence ,

$$e'(t)_1 = ae_2 + b^kY(t)_k \quad , \quad (k = 1,2) \quad (4.6)$$

where a, b^1, b^2 are determined as follows:

Differentiation of $e(t)_1 \cdot e_2 = 0$ with respect to t gives:

$$e'(t)_1 \cdot e_2 = 0.$$

Scalar multiplication of (4.6) by e_2 and $Y(t)_i$ respectively implies

$$a = 0,$$

and

$$e'(t)_1 \circ Y(t)_i = b^k \, Y(t)_k \circ Y(t)_i$$

$$= b^k \, g(t)_{ki},$$

multiply both sides by $g(t)^{ij}$ and sum on i :

$$(e'(t)_1 \cdot Y(t)_i) g(t)^{ij} = b^k g(t)_{ki} g(t)^{ij}$$

$$= b^k \delta_k^j$$

$$= b^j \quad \ldots\ldots\ldots\ldots\ldots (4.7)$$

From differentiation of $e(t)_1 \circ Y(t)_i = 0$ and $Y(t)_i = Y_i + tw_iE = Y_i + tW_i$ with respect to t, it follows that:

$$e'(t)_1 \cdot Y(t)_i + e(t)_1 \circ Y'(t)_i = 0,$$

and

$$Y'(t)_i = w_iE = W_i.$$

Hence

$$e'(t)_1 \cdot Y(t)_i = - e(t)_1 \cdot Y'(t)_i$$

$$= - e(t)_1 \cdot W_i.$$

Substitution of this expression in (4.7) and (4.6) gives:

$$e'(t)_1 = - g(t)^{ij}(W_i \circ e(t)_1) \, Y(t)_j.$$

LEMMA 4.2:

$$[g(t)]^{\frac{1}{2}}b(t)_{ij} = (1-t)[g]^{\frac{1}{2}}b_{ij} + t[g^*]^{\frac{1}{2}}b^*_{ij},$$

where the subscript 1 has been dropped from $b_{1|ij}$ since this will not lead to any confusion.

__Proof__: Using (2.5), (4.1), (4.3) and the fact that

$$\left|Y(t)_1,\ Y(t)_2,\ e(t)_1,\ e_2\right| = [g(t)]^{\frac{1}{2}},$$

and

$$Y(t)_{,ij} = Y(t)_{ij} - Y(t)_k \left\{ {k \atop ij} \right\}(t),$$

one obtains:

$$\begin{aligned}
[g(t)]^{\frac{1}{2}}b(t)_{ij} &= \left|Y(t)_1,\ Y(t)_2,\ e(t)_1,\ e_2\right| b(t)_{ij} \\
&= \left|Y_1+tw_1E,\ Y_2+tw_2E,\ b(t)_{ij}e(t)_1,\ e_2\right| \\
&= \left|Y_1+tw_1E,\ Y_2+tw_2E,\ Y(t)_{,ij},\ e_2\right| \\
&= \left|Y_1+tw_1E,\ Y_2+tw_2E,\ Y(t)_{ij},\ e_2\right| \\
&= \left|Y_1+tw_1E,\ Y_2+tw_2E,\ Y_{ij}+tw_{ij}E,\ e_2\right| \\
&= (1-t)\left|Y_1,\ Y_2,\ Y_{ij},\ e_2\right| \\
&\quad +t\left|Y_1,\ Y_2,\ Y_{ij},\ e_2\right| \\
&\quad +t\left|Y_1,\ Y_2,\ w_{ij}E,\ e_2\right| \\
&\quad +t\left|Y_1\ ,\ w_2E,\ Y_{ij},\ e_2\right| \\
&\quad +t\left|w_1E,\ Y_2,\ Y_{ij},\ e_2\right| \\
&= (1-t)[g]^{\frac{1}{2}}b_{ij} + t[g^*]^{\frac{1}{2}}b^*_{ij}.
\end{aligned}$$

LEMMA 4.3: If e_1 and e_1^* are canonical, then $e(t)_1$ is also canonical.

<u>Proof</u>: By (2.10) if e_1 is canonical, then e_2 is also canonical. Therefore:

$$e_{2,i} = - b_{2|ik}\, g^{kj}\, Y_j = - b_{2|i}{}^{j} Y_j \quad (i = 1,2),$$

and similarly

$$e_{2,i} = - b^*_{2|i}{}^{j} Y^*_j \,.$$

Hence by (4.1),

$$b_{2|i}{}^{j} Y_j = b^*_{2|i}{}^{j}\,(Y_j + w_j E).$$

Since Y_1 and Y_2 are linearly independent, this relation implies:

$$b_{2|i}{}^{j} = b^*_{2|i}{}^{j} \,,$$

and $$0 = b^*_{2|i}{}^{j}\, w_j = b_{2|i}{}^{j}\, w_j \quad \ldots\ldots\ldots\ldots\ldots\ldots (4.8)$$

Because $e_{2,i}$ has the same value on M^2 and $M^2(t)$, we have:

$$-b_{2|i}{}^{j} Y_j = -b(t)_{2|i}{}^{j} Y(t)_j + \theta_{21|i}\, e(t)_1 + \theta(t)_{22|i}\, e_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (4.9)$$

From (4.8), (4.3) and (2.10), (4.9) becomes:

$$-b_{2|i}{}^{j} Y(t)_j = - b(t)_{2|i}{}^{j} Y(t)_j + \theta(t)_{21|i}\, e(t)_1 .$$

Scalar multiplication by $e(t)_1$ shows that: $\theta(t)_{21|i} = 0$.

This completes the proof.

LEMMA 4.4: $(E.e(t)_1)[g(t)]^{\frac{1}{2}} = (E.e_1)[g]^{\frac{1}{2}}$ ....(4.10)

$(E.e(t)_1)dA(t) = (E.e_1)dA$ ....(4.11)

Proof:

$$E.e(t)_1 = E.\frac{Y(t)_1 \times Y(t)_2 \times e_2}{|Y(t)_1 \times Y(t)_2 \times e_2|}.$$

But $e(t)_1.(Y(t)_1 \times Y(t)_2 \times e_2) = |e(t)_1||Y(t)_1 \times Y(t)_2 \times e_2|$,

or $|Y(t)_1, Y(t)_2, e(t)_1, e_2| = |Y(t)_1 \times Y(t)_2 \times e_2|$

$$[g(t)]^{\frac{1}{2}} = |Y(t)_1 \times Y(t)_2 \times e_2|,$$

Similarly $[g]^{\frac{1}{2}} = |Y_1 \times Y_2 \times e_2|$

Therefore $E.e(t)_1 = \dfrac{|Y(t)_1, Y(t)_2, E, e_2|}{[g(t)]^{\frac{1}{2}}}$

$$\begin{aligned}[E.e(t)_1][g(t)]^{\frac{1}{2}} &= |Y_1 + tw_1E, Y_2 + tw_2E, E, e_2| \\ &= |Y_1, Y_2, E, e_2| \\ &= E.(Y_1 \times Y_2 \times e_2) \\ &= E.\frac{Y_1 \times Y_2 \times e_2}{|Y_1 \times Y_2 \times e_2|}|Y_1 \times Y_2 \times e_2| \\ &= (E.e_1)[g]^{\frac{1}{2}}.\end{aligned}$$

(4.11) follows immediately by defintion 5.

CHAPTER V

TRANSLATION THEOREMS

Since g_{ij} is symmetric in i and j, $g_{ij}u^iu^j$ is positive definte and $b_{1|ij}$ is symmetric in i,j, the roots k_1 and k_2 of $\left| b_{1|ij} - kg_{ij} \right| = 0$ are real. They are called the principal curvatures of M^2 relative to e_1. The first and second mean curvatrues are denoted by:

$$H_1 = \tfrac{1}{2}(k_1 + k_2)$$

$$H_2 = k_1 k_2 \;.$$

Consider:

$$de_1 \otimes de_1 \otimes e_2 = (e_{1,i} \times e_{1,j} \times e_2) dx^i \wedge dx^j$$

$$= 2(e_{1,1} \times e_{1,2} \times e_2) dx^1 \wedge dx^2$$

$$= a e_1 \;,$$

where a is to be determined.

Scalar multiplication by e_1 of both sides of

$$a e_1 = 2(e_{1,1} \times e_{1,2} \times e_2) dx^1\, dx^2 \;,$$

gives:

$$a = -2 \left| b_{1j} g^{jk} Y_k \,,\; b_{2i} g^{il} Y_l \,,\; e_1 \,,\; e_2 \right| dx^1 \wedge dx^2$$

$$= -2\, b_{1j} g^{jk} b_{2i} g^{il} \left| Y_k \,,\; Y_l \,,\; e_1 \,,\; e_2 \right| dx^1 \wedge dx^2$$

$$a = -2b_{1j}g^{j1}b_{2i}g^{i2}\left|Y_1 , Y_2 , e_1 , e_2\right|dx^1{\wedge}dx^2$$

$$- 2b_{1j}g^{j2}b_{2i}g^{i1}\left|Y_2 , Y_1 , e_1 , e_2\right|dx^1{\wedge}dx^2$$

$$= -2(b_{11}g^{11}b_{21}g^{12}+b_{11}g^{11}b_{22}g^{22}+b_{12}g^{21}b_{21}g^{12}+b_{12}g^{21}b_{22}g^{22}$$

$$-b_{11}g^{12}b_{21}g^{11}-b_{11}g^{12}b_{22}g^{21}-b_{12}g^{22}b_{21}g^{11}-b_{12}g^{22}b_{22}g^{21})dA$$

$$= \frac{-2}{g^2}(-b_{11}g_{22}b_{21}g_{12}+b_{11}g_{22}b_{22}g_{11}+b_{12}g_{12}b_{21}g_{12}-b_{12}g_{12}b_{22}g_{11}$$

$$+b_{11}g_{12}b_{21}g_{22}-b_{11}g_{12}b_{22}g_{12}-b_{12}g_{11}b_{12}g_{22}+b_{12}g_{11}b_{22}g_{12})dA$$

$$= \frac{-2}{g^2}[b_{11}b_{22}(g_{11}g_{22}-g_{12}^2) - b_{12}^2(g_{11}g_{22}-g_{12}^2)]dA$$

$$= -2\,\frac{\left|b_{ij}\right|}{g}\,dA.$$

Therefore

$$de_1 \otimes de_1 \otimes e_2 = -2H_2e_1dA.$$

Hence

$$de(t)_1 \otimes de(t)_1 \otimes e_2 = - 2H(t)_2e(t)_1dA(t).$$

Scalar multiplication by W gives:

$$\left|W , de(t)_1 , de(t)_1 , e_2\right| = 2H(t)_2[W.e(t)_1]dA(t).$$

From (4.11) it is easely seen that

$$[W.e(t)_1]dA(t) = (W.e_1)dA .$$

Hence:

$$\left| W \ , \ de(t)_1 \ , \ de(t)_1 \ , \ e_2 \right| = 2H(t)_2 \ (W \cdot e_1) \ dA.$$

Differentiation with respect to t gives:

$$H'(t)_2 \ (W \cdot e_1 \ dA = \left| W \ , \ de'(t)_1 \ , \ de(t)_1 \ , \ e_2 \right| \quad (5.1)$$

Since e_1 and e_1^* are canonical it follows by lemma (4.3) that $e(t)_1$ is canonical, and since (by lemma (4.1)) $e'(t)$ is in the tangent space, then:

$$\left| W \ , \ e'(t\ )_1 \ , \ de(t)_1 \ , \ e_2 \right| = 0.$$

This together with Poincaré's theorem ($d^2e(t) = 0$) leads to:

$$d\left| W \ , \ e'(t)_1 \ , \ de(t)_1 \ , \ e_2 \right| + \left| e'(t)_1 \ , \ dW \ , \ de(t) \ , \ e_2 \right|$$

$$= \left| W \ , \ de'(t)_1 \ , \ de(t)_1 \ , \ e_2 \right| \ \ldots\ldots\ldots\ldots\ldots\ldots (5.2)$$

By lemma (4.1)

$$\left| e'(t)_1 \ , \ dW \ , \ de(t)_1 \ , \ e_2 \right| =$$

$$-g(t)_{ij}(W \cdot e(t)_1) b(t)_{km} g^{ml} \left| W_h \ , \ Y(t)_j \ , \ Y(t)_l \ , \ e_2 \right| dx^h{}_\wedge dx^k \quad (5.3)$$

$$\left| W_h \ , \ Y(t)_j \ , \ Y(t)_l \ , \ e_2 \right| dx^h{}_\wedge dx^l =$$

$$= W_h \cdot \ (Y(t)_j \times Y(t)_l \times e_2) \ dx^h{}_\wedge dx^l.$$

Let $Y(t)_j \times Y(t)_l \times e_2 = a \ e(t)_1$,

where a is to be determined.

$$a = e(t)_1 \cdot (Y(t)_j \times Y(t)_l \times e_2)$$

$$= \left| Y(t)_j \, , \, Y(t)_l \, , \, e(t)_1 \, , \, e_2 \right| .$$

Substitution in (5.2) gives:

$$\left| e'(t)_1 \, , \, dW \, , \, de(t)_1 \, , \, e_2 \right|$$

$$= -g(t)^{ij}(W_i \cdot e(t)_1)(W_h \cdot e(t)_1)b(t)_{km}g^{ml} \left| Y(t)_j, Y(t)_l, e(t)_1, e_2 \right| dx^h \wedge dx^k$$

$$i \, , \, j \, , \, k \, , \, l \, , \, m = 1 \, , \, 2.$$

Notice that terms for which **k = h** or j = l are zero.

to facilitate the following derivations, the parameter t will be dropped where convenient from here on through (5.4).

$$\left| e(t)_1 \, , \, dW \, , \, de(t)_1 \, , \, e_2 \right|$$

$$= -g^{i1}(w_iE \cdot e_1)(w_1E \cdot e_1)b_{2m}g^{m2} \left| Y_1 \, , \, Y_2 \, , \, e_1 \, , \, e_2 \right| dx^1 \wedge dx^2$$

$$-g^{i1}(w_iE \cdot e_1)(w_2E \cdot e_1)b_{1m}g^{m2} \left| Y_1 \, , \, Y_2 \, , \, e_1 \, , \, e_2 \right| dx^2 \wedge dx^1$$

$$-g^{i2}(w_iE \cdot e_1)(w_1E \cdot e_1)b_{2m}g^{m1} \left| Y_2 \, , \, Y_1 \, , \, e_1 \, , \, e_2 \right| dx^1 \wedge dx^2$$

$$-g^{i2}(w_iE \cdot e_1)(w_2E \cdot e_1)b_{1m}g^{m1} \left| Y_2 \, , \, Y_1 \, , \, e_1 \, , \, e_2 \right| dx^2 \wedge dx^1$$

$$= (Ew_i \cdot e_1)(Ew_1 \cdot e_1)(g^{i2}b_{2m}g^{m1} - g^{i1}b_{2m}g^{m2})dA$$

$$+(Ew_i \cdot e_1)(Ew_2 \cdot e_1)(g^{i1}b_{1m}g^{m2} - g^{i2}b_{1m}g^{m1})dA$$

$$\left| e(t)_1 \ , \ dW \ , \ de(t)_1 \ , \ e_2 \right|$$

$$=(Ew_i \circ e_1)(Ew_1 \circ e_1)b_{2m}(g^{i2}g^{m1} - g^{i1}g^{m2})dA$$

$$+(Ew_i \circ e_1)(Ew_2 \cdot e_1)b_{1m}(g^{i1}g^{m2} - g^{i2}g^{m1})dA$$

$$=(Ew_1 \cdot e_1)(Ew_1 \circ e_1)[b_{21}(g^{12}g^{11}-g^{11}g^{12})+b_{22}(g^{12}g^{21}-g^{11}g^{22})]dA$$

$$+(Ew_1 \circ e_1)(Ew_2 \cdot e_1)[b_{21}(g^{22}g^{11}-g^{21}g^{12})+b_{22}(g^{22}g^{21}-g^{21}g^{22})]dA$$

$$+(Ew_2 \circ e_1)(Ew_1 \circ e_1)[b_{11}(g^{11}g^{12}-g^{12}g^{11})+b_{12}(g^{11}g^{22}-g^{12}g^{21})]dA$$

$$+(Ew_2 \cdot e_1)(Ew_2 \circ e_1)[b_{11}(g^{21}g^{12}-g^{22}g^{11})+b_{12}(g^{21}g^{22}-g^{22}g^{21})]dA$$

$$=[(Ew_1 \circ e_1)(Ew_2 \cdot e_1)gb_{21} - (Ew_1 \circ e_1)^2 gb_{22}$$

$$+(Ew_1 \cdot e_1)(Ew_2 \cdot e_1)gb_{12} - (Ew_2 \circ e_1)^2 gb_{11}]dA$$

$$=g(E \circ e_1)^2(2w_1w_2b_{12} - w_1^2b_{22} - w_2^2b_{11})dA$$

$$=-g(E \circ e_1)^2 b^{*ij}w_iw_j dA,$$

where b^{*ij} is the general element of the adjoint matrix of b_{ij}.

Thus we have:

$$\left| e'(t)_1, dW, de(t)_1, e_2 \right| = -g(t)b(t)^{*ij}w_iw_j(E \cdot e(t)_1)^2 dA(t).$$

Using (4.10) and (4.11) this becomes

$$\left| e'(t)_1 , dW , de(t)_1 , e_2 \right| = - g^{\frac{1}{2}} g(t)^{\frac{1}{2}} b(t)^{*ij} w_i w_j (E.e_1)^2 dA.$$

This result together with (5.1) and (5.2) gives:

$$H'(t)_2 (W.e_1) dA = d\left| W, e'(t)_1 , de(t)_1 , e_2 \right|$$

$$-g^{\frac{1}{2}} g(t)^{\frac{1}{2}} b(t)^{*ij} w_i w_j (E.e_1)^2 dA \quad \ldots\ldots\ldots (5.4)$$

Integrating both sides with respect to t over the interval [0,1], and interchanging the integration and differentiation signs because of the continuity of the integrand, (5.4) becomes:

$$(H_2^* - H_2)(W.e_1) dA = - g^{\frac{1}{2}} \left[\int_0^1 g(t)^{\frac{1}{2}} b(t)^{*ij} w_i w_j dt \right] (E.e_1)^2 dA$$

$$+ d \int_0^1 \left| W, e'(t)_1 , de(t)_1 , e_2 \right| dt \quad \ldots\ldots (5.5)$$

Let M^2 and M^{*2} be two compact oriented Riemannian manifolds of class C^3 and f a parallel transformation between them in the direction of a fixed unit vector E in E^4. Let e_1, e_1^* be the unit vectors normal to M^2, M^{*2} associated with E, and suppose that the two manifolds have positive definite second fundamental forms relative to e_1 and e_1^* which are assumed to be canonical. Under such conditions, (5.5) can be used to prove the following two important theorems.

These theorems in their most general form i.e.

M^n E^{n+m} and H_{rp}, the p-th $(1 = p = n)$ mean curvature of of M^n relative to e_r are due to Hsiung and Nassar [5]. Cases where $n = 2$, $m = p = 1$ are due to Hopf and Voss [7], $m = p = 1$ due to Hsiung [4] and $m = 1$ due to Stong [6].

THEOREM 1: If M^2 and M^{*2} have empty boundaries and $H_2 = H_2^*$, then f is a translation.

<u>Proof</u>: Since $H_2 = H_2^*$ the left-hand side of (5.5) reduces to zero. Integration of (5.5) over M^2 and using Stokes Theorem give:

$$0 = \int_{M^2} -g^{\frac{1}{2}}\left[\int_0^1 g(t)^{\frac{1}{2}} b(t)^{*ij} w_i w_j \, dt\right](E.e_1)^2 dA$$
$$+ \int_C \left[\int_0^1 \left| W , e'(t)_1 , de(t)_1 , e_2 \right| dt\right] \quad \ldots\ldots\ldots(5.6)$$

where C is the boundary of M^2 which is assumed to be empty. Therefore the integral over C is zero. Hence (5.6) reduces to

$$\int_{M^2} -g^{\frac{1}{2}}\left[\int_0^1 g(t)^{\frac{1}{2}} b(t)^{*ij} w_i w_j \, dt\right](E.e_1)^2 dA = 0 \quad \ldots\ldots\ldots(5.7)$$

Now since $b_{ij}u^i u^j$ and $b^*_{ij}u^i u^j$ were assumed to be positive definite, lemma (4.2) shows that $b(t)_{ij}u^i u^j$ is also positive definite. Hence

$$\left[\int_0^1 g(t)^{\frac{1}{2}} b(t)^{*ij} w_i w_j \, dt\right]$$

is positive definite, and since $E.e_1 \neq 0$ almost everywhere, (5.7) implies that $w_1 = 0$ and $w_2 = 0$ almost everywhere. Therefore W = constant and f is a translation.

THEOREM 2: If M^2 and M^{*2} have nonempty boundaries, $H_2 = H_2^*$ and if the normal vectors e_1 and e_1^* are equal at all corresponding points of the boundaries, then f is a translation.

<u>Proof</u>: On the boundary of M^2 $e_1^* = e_1$, (4.3) and

$$Y_i \cdot e_1 = 0 \quad \text{and} \quad Y_i^* \cdot e_1^* = 0,$$

imply $$W_i \cdot e_1 = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(5.8)}$$

(5.8) and (4.10) imply

$$W_i \cdot e(t)_1 = (W_i \cdot e_1) g^{\frac{1}{2}} g(t)^{-\frac{1}{2}} = 0.$$

Using lemma (4.1), one can easily obtain $e'(t)_1 = 0$. Thus the second term on the right-hand side of (5.6) vanishes. The rest of the proof is essentially the same as in the previous theorem.

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