A STUDY OF HEREDITARY AND PRODUCTIVE TOPOLOGICAL PROPERTIES

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ABSTRACT

In this thesis, we discuss the heredity and productivity of topological properties. We show that the basic separation properties are hereditary and productive with the exception of normality (which is neither) and complete normality (which is only hereditary). We next show that first and second countability are hereditary and weakly productive. Compactness and countable compactness are non-hereditary, but they are productive. The Lindelöf property and paracompactness are non-hereditary and non-productive. Local compactness is non-hereditary and is only finitely product invariant. Separability is non-hereditary, but it is product invariant for c factor spaces. The property of being developable is hereditary and weakly productive. The basic connectivity properties (connectedness, local connectedness, "connected im Kleinen at x" and arcwise connectedness) are all non-hereditary and are productive except for "connected im Kleinen at x" and local connectedness which are finitely product invariant. Finally, we show that metrizability, Moore space, semi-metrizability and a-metrizability are all hereditary and weakly productive.

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I. INTRODUCTION

In this thesis, we discuss the heredity and productivity of the basic topological properties. A property P is a topological property iff P is invariant under homeomorphisms (1 - 1, bicontinuous mapping). A property P is hereditary iff it is inherited by every subspace of any space possessing the property. P is productive iff the product space of an arbitrary number of spaces, each having property P, also has property P. P is weakly productive iff P is countably "product invariant".

We denote by $\langle S, \mathcal{J} \rangle$ the topological space consisting of $S \neq \emptyset$ and the topology $\mathcal{J}$ on S. $\langle A, A \cap \mathcal{J} \rangle$ denotes the subspace of $\langle S, \mathcal{J} \rangle$ consisting of $A \subset S$ and the relative topology $A \cap \mathcal{J}$ on A. An arbitrary index set is usually denoted by the letter A, but we use I^+ (the set of positive integers) for a countable index set.

Let $\langle S_\alpha, \mathcal{J}_\alpha \rangle$ be a topological space $\forall \alpha \in A$. The topological product $\langle \mathbb{P}_A S_\alpha, \mathbb{P}_A \mathcal{J}_\alpha \rangle$ of these spaces consists of the cartesian product set $\mathbb{P}_A S_\alpha$ and the product topology $\mathbb{P}_A \mathcal{J}_\alpha$. Thus, $x \in \mathbb{P}_A S_\alpha$ iff $x = \{x_\alpha\}_{\alpha \in A}$ where $x_\alpha \in S_\alpha \; \forall \alpha \in A$. A basis for $\mathbb{P}_A \mathcal{J}_\alpha$ is the collection:

$\{\mathbb{P}_A U_\alpha \mid U_\alpha \in \mathcal{J}_\alpha$ and $U_\alpha \neq S_\alpha$ for only finitely many $\alpha \in A\}$.

The elements of $\mathbb{P}_A \mathcal{J}_\alpha$ are obtained by taking arbitrary unions and finite intersections of these basic sets. Moreover, the projection mappings $\Pi_\alpha : \mathbb{P}_A S_\alpha \rightarrow S_\alpha$, given by $\Pi_\alpha(x) = x_\alpha$, are both continuous and open $\forall \alpha \in A$. Thus, an equivalent basis for $\mathbb{P}_A \mathcal{J}_\alpha$ is the collection of all sets of the form $\bigcap_{i=1}^{n} \Pi_{\alpha_i}^{-1}(U_{\alpha_i})$ where $U_{\alpha_i} \in \mathcal{J}_{\alpha_i}$, $i = 1, \ldots, n$ and $n \in I^+$.

We begin our exposition with a discussion of the separation axioms. The properties T_0, T_1, T_2 (Hausdorff), $T_{5/2}$, T_3 (regularity) and $T_{7/2}$ (Tychonoff) are all hereditary and productive. T_4 (Normality) is neither hereditary nor productive, and T_5 (complete normality) is hereditary but not productive.

Chapter three is concerned with the various covering properties. The first and second axioms of countability are hereditary and weakly productive. Separability is neither hereditary nor productive, but it is invariant under the product of $\mathfrak{c}$ spaces (the cardinality of the reals). Compactness is productive by the Tychonoff theorem, but it is not hereditary. Local compactness is neither hereditary nor productive, but it is invariant under the product of a finite number of spaces. Countable compactness is productive, but it is not hereditary. Paracompactness and the Lindelöf property are neither hereditary nor productive. Finally, the developable property is hereditary and weakly productive.

Chapter four treats the connectivity properties: connectedness, local connectedness, connected im Kleinen at x, and arcwise connectedness. None of them is hereditary, but all are productive with the exception of local connectedness and connected im Kleinen which are only finitely "product invariant".

In the final chapter, we consider the generalized metrizability properties: metrizability, Moore space, semi-metrizability, and a-metrizability. All are hereditary and weakly productive.

II. SEPARATION PROPERTIES

We consider in this chapter the "Trennungsaxioms" T_i ($i = 0, 1, 2, 3, 4, 5$) of Alexandroff and Hopf (see Hocking and Young [2], pp. 37, 40-42) and related separation properties. All are shown to be both hereditary and productive with the exception of normality (T_4) and complete normality (T_5) which are not productive. Complete normality is hereditary, but normality is not.

<u>Definition 2.1.</u> $\langle S, \mathfrak{J} \rangle$ is a T_0-space iff given any two points of S $\exists$ an open set containing one of them but not the other.

<u>Theorem 2.1.</u> If $\langle S, \mathfrak{J} \rangle$ is a T_0-space and $\langle A, A \cap \mathfrak{J} \rangle$ is any subspace of $\langle S, \mathfrak{J} \rangle$, then $\langle A, A \cap \mathfrak{J} \rangle$ is also T_0.

Proof. Let $p, q \in A$. Since $p, q \in S$ also, $\exists\, U \in \mathfrak{J} \ni p \in U$, $q \notin U$ or $q \in U$, $p \notin U$. Thus, either $p \in U \cap A$, $q \notin U \cap A$ or $q \in U \cap A$, $p \notin U \cap A$, where $U \cap A \in A \cap \mathfrak{J}$. By Definition 2.1, the space $\langle A, A \cap \mathfrak{J} \rangle$ is T_0.

<u>Theorem 2.2.</u> If $\langle S_\alpha, \mathfrak{J}_\alpha \rangle$ is a T_0-space $\forall \alpha \in A$, then the product space $\langle \mathbb{P}_A S_\alpha, \mathbb{P}_A \mathfrak{J}_\alpha \rangle$ is T_0.

Proof. Let $p, q \in \mathbb{P}_A S_\alpha$. For some $\beta \in A$, $p_\beta \neq q_\beta$. Hence, $\exists\, U_\beta \in \mathfrak{J}_\beta \ni$ (i) $p_\beta \in U_\beta$, $q_\beta \notin U_\beta$ or (ii) $p_\beta \notin U_\beta$, $q_\beta \in U_\beta$. Because of the similarity of the two cases, we consider only case (i). Now $\pi_\beta^{-1}(U_\beta) \in \mathbb{P}_A \mathfrak{J}_\alpha$ since π_β is continuous. Also, $p \in \pi_\beta^{-1}(U_\beta)$ and $q \notin \pi_\beta^{-1}(U_\beta)$ since $q_\beta \notin U_\beta$. Hence, $\langle \mathbb{P}_A S_\alpha, \mathbb{P}_A \mathfrak{J}_\alpha \rangle$ is T_0.

<u>Definition 2.2.</u> $\langle S, \mathcal{T} \rangle$ is a T_1-space iff given any two points of S, each of them lies in an open set not containing the other.

<u>Theorem 2.3.</u> Any subspace $\langle A, A \cap \mathcal{T} \rangle$ of a T_1-space $\langle S, \mathcal{T} \rangle$ is T_1.

Proof. If $p, q \in A$, then $p, q \in S$ also. Thus, $\exists$ $U, V \in \mathcal{T} \ni p \in U$, $q \notin U$ and $q \in V$, $p \notin V$. Consider $A \cap U, A \cap V \in A \cap \mathcal{T}$. $p \in A \cap U$, $q \notin A \cap U$ and $q \in A \cap V$, $p \notin A \cap V$. Hence, $\langle A, A \cap \mathcal{T} \rangle$ is T_1.

<u>Theorem 2.4.</u> If $\langle S_\alpha, \mathcal{T}_\alpha \rangle$ is a T_1-space $\forall \alpha \in A$, then the product space $\langle \mathbb{P}_A S_\alpha, \mathbb{P}_A \mathcal{T}_\alpha \rangle$ is T_1.

Proof. Let $p, q \in \mathbb{P}_A S_\alpha$. For some $\gamma \in A$, $p_\gamma \neq q_\gamma$. $\exists$ $U_\gamma, V_\gamma \in \mathcal{T}_\gamma \ni p_\gamma \in U_\gamma$, $q_\gamma \notin U_\gamma$ and $q_\gamma \in V_\gamma$, $p_\gamma \notin V_\gamma$. The sets $\pi_\gamma^{-1}(U_\gamma)$, $\pi_\gamma^{-1}(V_\gamma) \in \mathbb{P}_A \mathcal{T}_\alpha$ since π_γ is continuous. Also, $p \in \pi_\gamma^{-1}(U_\gamma)$, $q \notin \pi_\gamma^{-1}(U_\gamma)$ and $q \in \pi_\gamma^{-1}(V_\gamma)$, $p \notin \pi_\gamma^{-1}(V_\gamma)$. Thus, the product space is also T_1.

<u>Definition 2.3.</u> $\langle S, \mathcal{T} \rangle$ is a T_2 (Hausdorff)-space iff for any two points of S, $\exists$ two disjoint open sets each containing one of the points.

<u>Theorem 2.5.</u> Any subspace $\langle A, A \cap \mathcal{T} \rangle$ of a Hausdorff space $\langle S, \mathcal{T} \rangle$ is Hausdorff.

Proof. If $p, q \in A$, then $p, q \in S$ also. $\exists$ $U, V \in \mathcal{T} \ni p \in U$, $q \in V$ and $U \cap V = \emptyset$. Now $A \cap U, A \cap V \in A \cap \mathcal{T}$, and $p \in A \cap U$, $q \in A \cap V$. Also, $(A \cap U) \cap (A \cap V) = \emptyset$ since $U \cap V = \emptyset$. Hence, $\langle A, A \cap \mathcal{T} \rangle$ is Hausdorff.

<u>Theorem 2.6.</u> If $\langle S_\alpha, \mathcal{T}_\alpha \rangle$ is a Hausdorff space $\forall \alpha \in A$, then the product space $\langle \mathbb{P}_A S \ , \mathbb{P}_A \mathcal{T}_\alpha \rangle$ is Hausdorff.

Proof. Let $p, q \in \mathcal{P}_A S$. For some $\beta \in A$, $p_\beta \neq q_\beta$. Hence, $\exists\ U_\beta, V_\beta \in \mathcal{T}_\beta \ni p_\beta \in U_\beta$, $q_\beta \in V_\beta$ and $U_\beta \cap V_\beta = \emptyset$. The sets $\pi_\beta^{-1}(U_\beta)$, $\pi_\beta^{-1}(V_\beta) \in \mathcal{P}_A \mathcal{T}_\alpha$. Also, $p \in \pi_\beta^{-1}(U_\beta)$, $q \in \pi_\beta^{-1}(V_\beta)$, and $\pi_\beta^{-1}(U_\beta) \cap \pi_\beta^{-1}(V_\beta) = \emptyset$ since $U_\beta \cap V_\beta = \emptyset$. Thus, the product space is Hausdorff.

We next investigate a separation property which is stronger than Hausdorff but weaker than T_3. We label this property $T_{5/2}$. R.H. Bing gave in [1] an example of a countable, connected, Hausdorff space which is not $T_{5/2}$ since the closures of each two basic sets have a nonempty intersection. B.T. Sims gave in his Thesis [6] an example of a $T_{5/2}$-space which is not regular.

<u>Definition 2.4</u>. $\langle S, \mathcal{T} \rangle$ is a $T_{5/2}$-space iff for any two points of S, $\exists$ two open sets, each containing just one of the points, with disjoint closures.

<u>Theorem 2.7</u>. If $\langle S, \mathcal{T} \rangle$ is a $T_{5/2}$-space, then any subspace $\langle A, A \cap \mathcal{T} \rangle$ of $\langle S, \mathcal{T} \rangle$ is also $T_{5/2}$.

Proof. If $p, q \in A$, then $p, q \in S$ also. Hence, $\exists\ U, V \in \mathcal{T} \ni p \in U$, $q \in V$, $\overline{U} \cap \overline{V} = \emptyset$. Thus, $p \in U \cap A$, $q \in V \cap A$, and $U \cap A$, $V \cap A \in A \cap \mathcal{T}$. Also, $\overline{U \cap A} \subset \overline{U} \cap \overline{A} \subset \overline{U}$ and $\overline{V \cap A} \subset \overline{V} \cap \overline{A} \subset \overline{V}$. Since $\overline{U} \cap \overline{V} = \emptyset$, $(\overline{U \cap A} \cap A) \cap (\overline{V \cap A} \cap A) = \emptyset$. Thus, $\langle A, A \cap \mathcal{T} \rangle$ is also $T_{5/2}$.

<u>Theorem 2.8</u>. If $\langle S_\alpha, \mathcal{T}_\alpha \rangle$ is a $T_{5/2}$-space $\forall \alpha \in A$, then the product space $\langle \mathcal{P}_A S_\alpha, \mathcal{P}_A \mathcal{T}_\alpha \rangle$ is $T_{5/2}$.

Proof. If $p, q \in \mathcal{P}_A S_\alpha$, then for some $\beta \in A$, $p_\beta \neq q_\beta$. Hence, $\exists\ U_\beta, V_\beta \in \mathcal{T}_\beta \ni p_\beta \in U_\beta$, $q_\beta \in V_\beta$, and $\overline{U}_\beta \cap \overline{V}_\beta = \emptyset$. Thus, $\pi_\beta^{-1}(U_\beta)$, $\pi_\beta^{-1}(V_\beta) \in \mathcal{P}_A \mathcal{T}_\alpha$, and $p \in \pi_\beta^{-1}(U_\beta)$, $q \in \pi_\beta^{-1}(V_\beta)$. We have to show that $\overline{\pi_\beta^{-1}(U_\beta)} \cap \overline{\pi_\beta^{-1}(V_\beta)} = \emptyset$.

Since π_β is continuous, we have that $\overline{\pi_\beta^{-1}(V_\beta)} \subset \pi_\beta^{-1}(\overline{V}_\beta)$ and $\overline{\pi_\beta^{-1}(U_\beta)} \subset \pi_\beta^{-1}(\overline{U}_\beta)$. Moreover, $\pi_\beta^{-1}(\overline{V}_\beta) \cap \pi_\beta^{-1}(\overline{U}_\beta) = \emptyset$ since $\overline{U}_\beta \cap \overline{V}_\beta = \emptyset$. Hence $\overline{\pi_\beta^{-1}(U_\beta)} \cap \overline{\pi_\beta^{-1}(V_\beta)} = \emptyset$.

Definition 2.5. $\langle S, \mathcal{T} \rangle$ is regular iff given any closed subset C of S and a point $p \in S - C$, $\exists$ two disjoint open sets $U, V \ni C \subset U$, $p \in V$. $\langle S, \mathcal{T} \rangle$ is a T_3-space iff it is a regular T_1-space.

Theorem 2.9. Any subspace $\langle A, A \cap \mathcal{T} \rangle$ of a regular space $\langle S, \mathcal{T} \rangle$ is regular.

Proof. Let $p \in A$ and let C be a closed subset of A with $p \notin C$. Hence, $\exists$ C^* closed in $S \ni C = C^* \cap A$. We have $p \in S - C^*$ since $p \notin C$ and $p \in A$. Now $\exists\, U, V \in \mathcal{T} \ni C^* \subset U$, $p \in V \Rightarrow C \subset U$, $p \in V$ with $U \cap V = \emptyset$. The sets $U \cap A$, $V \cap A \in A \cap \mathcal{T}$. Also, $p \in A \cap U$, $C \subset U \cap A$ since $C \subset C^* \subset U$ and $C \subset A$. Moreover, $(U \cap A) \cap (V \cap A) = \emptyset$ since $U \cap V = \emptyset$. Hence, $\langle A, A \cap \mathcal{T} \rangle$ is regular.

Corollary 2.1. Since T_1 is a hereditary property, it follows that every subspace of a T_3-space is also T_3.

We shall make use of the following characterization of regularity in proving that regularity is a productive property. This result is proved as Theorem 2-4 in Hocking and Young [2], p. 41. $\langle S, \mathcal{T} \rangle$ is regular iff $\forall_p \in S$ and $\forall\, U \in \mathcal{T} \ni p \in U$, $\exists\, V \in \mathcal{T} \ni p \in V \subset \overline{V} \subset U$.

Theorem 2.10. If $\langle S_\alpha, \mathcal{T}_\alpha \rangle$ is regular $\forall \alpha \in A$, then the product space $\langle P_A S_\alpha, P_A \mathcal{T}_\alpha \rangle$ is regular.

Proof. Let $p \in P_A S_\alpha$ and $W \in P_A \mathcal{T}_\alpha \ni p \in W$. $\exists\, U \subset W \ni p \in U$ and $U = \bigcap_{i=1}^{n} \pi_{\alpha_i}^{-1}(U_{\alpha_i})$. Since $p \in \pi_{\alpha_i}^{-1}(U_{\alpha_i})$ for $i = 1, 2, \ldots, n$, $\ni$

$p_{\alpha_i} \in U_{\alpha_i}$ for $i = 1, 2, \ldots, n$. Since each component space is regular, $\exists\, V_{\alpha_i} \in \mathcal{J}_{\alpha_i} \ni p_{\alpha_i} \in V_{\alpha_i} \subset \overline{V}_{\alpha_i} \subset U_{\alpha_i}$ for $i = 1, 2, \ldots, n$. Each $\pi_{\alpha_i}^{-1}(\overline{V}_{\alpha_i})$ is closed since π_{α_i} is continuous. Thus, $\bigcap_{i=1}^{n} \pi_{\alpha_i}^{-1}(\overline{V}_{\alpha_i})$ is closed. Moreover,

$$\bigcap_{i=1}^{n} \pi_{\alpha_i}^{-1}(V_{\alpha_i}) \subset \bigcap_{i=1}^{n} \pi_{\alpha_i}^{-1}(\overline{V}_{\alpha_i}) \subset \bigcap_{i=1}^{n} \pi_{\alpha_i}^{-1}(U_{\alpha_i})$$ since

$V_{\alpha_i} \subset \overline{V}_{\alpha_i} \subset U_{\alpha_i}$, $i = 1, 2, \ldots, n$. Also,

$$\bigcap_{i=1}^{n} \pi_{\alpha_i}^{-1}(V_{\alpha_i}) \subset \overline{\bigcap_{i=1}^{n} \pi_{\alpha_i}^{-1}(V_{\alpha_i})} \subset \bigcap_{i=1}^{n} \pi_{\alpha_i}^{-1}(\overline{V}_{\alpha_i})$$ since $\bigcap_{i=1}^{n} \pi_{\alpha_i}^{-1}(\overline{V}_{\alpha_i})$ is closed. Thus, $p \in \bigcap_{i=1}^{n} \pi_{\alpha_i}^{-1}(V_{\alpha_i}) \subset \overline{\bigcap_{i=1}^{n} \pi_{\alpha_i}^{-1}(V_{\alpha_i})} \subset U \subset W$.

Corollary 2.2. Since T_1 is a productive property, it follows that any product of T_3-space is a T_3-space.

In between the separation properties of regularity and normality, we have a separation property known as complete regularity, which seems to have first been investigated by Tychonoff (Hocking and Young [2], p.74). A completely regular T_1-space is usually called a Tychonoff space and sometimes labeled a $T_{7/2}$-space.

Definition 2.6. $\langle S, \mathcal{J} \rangle$ is completely regular iff given any closed subset C of S and a point $p \in S - C$, $\exists$ a continuous function $f: S \rightarrow [0,1] \ni f(C) = 1$ and $f(p) = 0$. $\langle S, \mathcal{J} \rangle$ is a $T_{7/2}$ (Tychonoff)-space iff $\langle S, \mathcal{J} \rangle$ is a completely regular T_1-space.

Theorem 2.11. If $\langle A, A \cap \mathcal{J} \rangle$ is subspace of a completely regular space $\langle S, \mathcal{J} \rangle$, then $\langle A, A \cap \mathcal{J} \rangle$ is completely regular.

Proof. Let C be a closed subset of A and $p \in A - C$. $\exists\, C^* \subset S \ni C^*$ is closed and $C^* \cap A = C$. Also, $p \notin C^*$ since $p \in A$ and $p \notin C$.

Since $\langle S, \mathcal{J}\rangle$ is completely regular, $\exists$ a continuous function $f: S \to [0,1]$ $\ni$ $f(C^*) = 1$ and $f(p) = 0$. Let $g(x) = f(x)\ \forall_{x \in} A$; i.e., $g: A \to [0,1]$. Also, $g(C) = f(C) \subset f(C^*) = 1$ and $g(p) = f(p) = 0$. It remains to show that g is continuous. If V is open in $[0,1]$, then $U = f^{-1}(V)$ is open in S since f is continuous. Hence, $g^{-1}(V) = U \cap A$ which is open in A. Thus, $\langle A, A \cap \mathcal{J}\rangle$ is completely regular.

Corollary 2.3. Any subspace of a Tychonoff space is Tychonoff, since property T_1 is hereditary.

We will need some auxiliary results in order to show that complete regularity is productive. These are given below as Lemmas 2.1, 2.2.

Lemma 2.1. Let $\langle S, \mathcal{J}\rangle$ be a space and $f_i: S \to [0,1]$ be a continuous function $\ni$ if $x_o \in S$ and $x_o \in U_i \in \mathcal{J}$, then $f_i(x_o) = 0$ and $f_i(S - U_i) = 1, i = 1,2$. Then the function g defined by $g(x) = \sup\{f_i(x) \,/\, i = 1,2\}$ has the following properties: (i) $g(x_o) = 0$, (ii) $g[S - (U_1 \cap U_2)] = 1$, (iii) g is continuous.

Proof. (i) $g(x_o) = 0$ since $f_1(x_o) = f_2(x_o) = 0$.
(ii) $g[S - (U_1 \cap U_2)] = 1$. Let $p \in S - (U_1 \cap U_2)$. Then $p \in S - U_1$ or $p \in S - U_2$. Thus, $f_1(p)=1$ or $f_2(p) = 1$, which implies $g(p) = 1$.
(iii) g is continuous. Let $x_1 \in S$ be arbitrary. We distinguish two cases.

(a) $g(x_1) = f_1(x_1) = f_2(x_1)$. Let W be any open set containing $g(x_1)$; $\exists$ open sets U_{x_1}, V_{x_1}, containing $x_1 \ni f_1(U_{x_1}) \subset W$ and $f_2(V_{x_1}) \subset W$, since f_1 and f_2 are continuous. Thus, $g(U_{x_1} \cap V_{x_1}) \subset W$, and g is continuous at x_1.

(b) $g(x_1) = f_1(x_1) > f_2(x_2)$ (the third case, $g(x_1) = f_2(x_1) > f_1(x_1)$, is similar). Let $f_1(x_1) - f_2(x_1) = 3r > 0$, and let N_1, N_2 be open intervals centered at $f_1(x_1)$ and $f_2(x_2)$, respectively, of radius r. $\exists$ open sets U_{x_1}, V_{x_1}

containing $x_1 \ni f_1(U_{x_1}) \subset N_1$ and $f_2(V_{x_1}) \subset N_2$ since f_1 and f_2 are continuous. Since $N_1 \cap N_2 = \emptyset$, $f_1(U_{x_1}) \cap f_2(V_{x_1}) = \emptyset$, and every element in $f_1(U_{x_1} \cap V_{x_1})$ is greater than every element in $f_2(U_{x_1} \cap V_{x_1})$. Hence, $g(U_{x_1} \cap V_{x_1}) = f_1(U_{x_1} \cap V_{x_1}) \subset N_1$, and g is continuous at x_1.

Lemma 2.2. If $f_1, \ldots, f_n$ are continuous functions mapping S into $[0,1]$, as defined in Lemma 2.1, then the function g defined by $g(x) = \sup\{f_i(x)/i = 1,\ldots,n\}$ has properties (i), (ii) and (iii) of Lemma 2.1, $\forall n \in I^+$.

Proof. By mathematical induction. For $n = 2$, it is true by Lemma 2.1. Assume the Lemma is true for $n = k$; i.e., $g_k(x) = \sup\{f_i(x)/i = 1,\ldots,k\}$ has properties (i), (ii) and (iii). We show that $g_{k+1}(x) = \sup\{f_i(x)/i = 1,2,\ldots,k+1\}$ also has these three properties. This follows since

$$g_{k+1}(x) = \sup\{\sup\{f_i(x)/i = 1,\ldots,k\}, f_{k+1}(x)\} = \sup\{g_k(x), f_{k+1}(x)\}.$$

By the induction hypothesis, $g_k(x)$ has these three properties, and $f_{k+1}(x)$ has the same properties by hypothesis. Thus, we apply Lemma 2.1 with $f_1 = g_k$ and $f_2 = f_{k+1}$, and we have that g_{k+1} has properties (i), (ii) and (iii).

Notation. Any function $f: S \to [0,1]$ as defined in Lemma 2.1 will be called a function for the pair (x_o, U_{x_o}).

Theorem 2.12. If $\langle S_\alpha, \mathfrak{I}_\alpha \rangle$ is completely regular $\forall \alpha \in A$, then the product space $\langle \mathbb{P}_A S_\alpha, \mathbb{P}_A \mathfrak{I}_\alpha \rangle$ is completely regular.

Proof. Let $x \in \mathbb{P}_A S_\alpha$, and let U be any basic open set containing x. Since $U = \bigcap_{i=1}^{n} \pi_{\alpha_i}^{-1}(U_{\alpha_i})$, we have that $x_{\alpha_i} \in U_{\alpha_i} \in \mathfrak{I}_{\alpha_i}$, $i = 1, \ldots, n$. Since $\langle S_{\alpha_i}, \mathfrak{I}_{\alpha_i} \rangle$ is completely regular, $\exists$ a function

f_i for $(x_{\alpha_i}, U_{\alpha_i})$, $i = 1,\ldots,n$. Hence, $f_i \cdot \pi_{\alpha_i}$ will be a function for $(x, \pi_{\alpha_i}^{-1}(U_{\alpha_i}))$, $i = 1,\ldots,n$. According to Lemma 2.2, $\sup\{f_i \cdot \pi_{\alpha_i} / i = 1,\ldots,n\}$ is a function for (x,U). Hence, the product space is completely regular.

Corollary 2.4. Any product of Tychonoff spaces is Tychonoff, since property T_1 is productive.

Definition 2.7. $\langle S, \mathcal{J} \rangle$ is normal iff given any two disjoint closed subsets of S, $\exists$ two disjoint open sets each containing just one of them. $\langle S, \mathcal{J} \rangle$ is a T_4 - space iff $\langle S, \mathcal{J} \rangle$ is a normal T_1-space.

We give now an example of a normal space which is not hereditarily normal. The construction technique is due to Tychonoff, although this particular example is found in Pervin [4], pp. 93-94.

Example 2.1. Let $S_1 = X_1 \cup \{a\}$ be the one-point compactification of an uncountable discrete space X_1, and let $S_2 = X_2 \cup \{b\}$ be the one-point compactification of an infinite discrete space X_2. The spaces S_1, S_2 are compact Hausdorff, since X_1, X_2 are Hausdorff and locally compact (see [2], Theorem 2.54, p.73). Since compactness and the Hausdorff property are productive, $S_1 \times S_2$ is compact Hausdorff (hence, normal).

The subspace $S_1 \times S_2 - \{\langle a, b \rangle\}$ is not normal.

Proof. Let $A = \{\langle a, y \rangle / y \in X_2\}$ and $B = \{\langle x, b \rangle / x \in X_1\}$. Clearly, A and B are disjoint subsets of $S_1 \times S_2 - \{\langle a, b \rangle\}$. We show that they are also closed in $S_1 \times S_2 - \{\langle a, b \rangle\}$. Let $\langle r, s \rangle$ be any point of $S_1 \times S_2 - \{\langle a, b \rangle\}$ which is not in A (hence, $r \neq a$). The set $\{r\} \times S_2$ is open in $S_1 \times S_2 - \{\langle a, b \rangle\}$ and contains no point of A. Therefore, $\langle r, s \rangle$ is not a limit point of A. Thus, A is closed. Similarly, B is closed in $S_1 \times S_2 - \{\langle a, b \rangle\}$.

Assume $\exists$ disjoint open subsets U,V of $S_1 \times S_2 - \{\langle a, b\rangle\}$ $\ni$ $A \subset U$ and $B \subset V$. Let $\{y_1, y_2, \ldots\}$ be a denumerable set of distinct points of X_2. Thus, $\{\langle a,y_1\rangle, \langle a,y_2\rangle, \ldots\}$ is a denumerable set of distinct points of A. U must contain all but a finite number of the points in each set $\{\langle x,y_i\rangle / x \in X_1\}$, since U is a neighborhood of $\langle a,y_i\rangle$, $\forall\, i = 1, 2, \ldots$. It follows that at most a finite number of sets of the form $\{\langle x,y\rangle / y \in S_2\}$ can be completely contained in V. Similarly, at most a finite number of sets of the form $\{\langle x,y\rangle / y \in S_2\}$ can be completely contained in V, except for exactly one point each. Continuing this argument by induction, we see that for each $n \in I^+$, only a finite number of sets of the form $\{\langle x,y\rangle / y \in S_2\}$ can be contained in V, except for exactly n points each. Thus, for only a countable number of points $x \in X_1$, can V contain all but a finite number of the points in each set $\{\langle x,y\rangle / y \in S_2\}$. However, since V is an open set containing B, V must contain all but a finite number of the points of the set $\{\langle x,y\rangle / y \in S_2\}$ for each $x \in X_1$, since V is a neighborhood of $\langle x,b\rangle$ $\forall\, x \in X_1$. There are uncountably many such points $\langle x,b\rangle$, $x \in X_1$. Contradiction.

We give now an example of a normal space, whose topological product with itself is not normal. This example is due to Sorgenfrey (See Kelley [3], pp. 133-134).

<u>Example 2.2</u>. Let R be the space of real numbers with the lower limit topology. R is regular since if $x \in V$, an open neighborhood, then $\exists$ a basic open set $B \ni x \in B \subset V$. Since B is also closed the regularity is established. Now we show that R is Lindelöf. Let C be an arbitrary covering of R with basic open sets. We

delete as unnecessary each member of C which is contained in the union of two or more members of C. We need to show two things: (i) Every disjoint subfamily of C is countable; (ii) those subfamilies of C which have overlapping members are also countable. Hence, C contains a countable subcovering of R, and R is Lindelöf. (i) Since the end points of a basic set are real numbers, we associate with each basic set in C a rational number contained in it. In this manner, we obtain a one-one correspondence between the disjoint members of C and a subset of the rationals, which implies that this subfamily is countable. (ii) We repeat the same argument as above, associating a rational number with the non-empty intersection of any two basic sets in C. Thus, any subfamily of C having overlapping members is countable also.

Since this argument is true for any covering of R with basic open sets, it is also true for any open covering of R, since every open set contains a basic open set. Thus, R is normal since every regular Lindelöf space is normal (Pervin [4], p. 92).

We now consider the space $R \times R$. Let
$A = \{\langle x,y\rangle \in R \times R \, / \, x + y = 0, \ x \text{ rational}\}$ and
$B = \{\langle x,y\rangle \in R \times R \, / \, x + y = 0, \ x \text{ irrational}\}$. Clearly, the sets A and B are disjoint. We show that they are closed. Let $\langle p,q\rangle \in R \times R - A$. If $\langle p,q\rangle \in B$, then the open set $[p,p+1) \times [q,q+1)$ contains $\langle p,q\rangle$ and is disjoint from A. If $\langle p,q\rangle \notin B$, then it is either on the right of the line $x + y = 0$ or on its left. In the first case, the open set $[p,p+1) \times [q,q+1)$ contains $\langle p,q\rangle$ and is disjoint from A. In the second case, let $|p + q| = r$, where $r > 0$. The open set $[p, p+\frac{r}{2}) \times [q, q+\frac{r}{2})$ contains $\langle p,q\rangle$, and is disjoint

from A. A similar argument shows that B is closed. Let V be any open set containing A. For each $\langle p,q\rangle \in A$, $\exists$ a basic open set $N = [p, p+\varepsilon) \times [q, q+\varepsilon)$ containing $\langle p,q\rangle$ and contained in V. For each $n \in I^+$, let $N_n = \{\langle p,q\rangle \in A \,/\, \text{diameter } (N) > \frac{1}{n}\}$. Clearly $A = \bigcup_{n\in I^+} N_n$. A is dense in $\{\langle x,y\rangle \in R \times R \,/\, x + y = 0\}$ with respect to the usual topology of the plane. Thus, $\exists\, m \in I^+ \ni N_m$ is dense in a segment $(\langle s, -s\rangle, \langle t, -t\rangle)$ of the line $\{\langle x,y\rangle \in R \times R \,/\, x + y = 0\}$. We see that every point of the open rectangle D, bounded by $x + y = 0$, $x + y = \frac{1}{m}$, $x - y = 2s$ and $x - y = 2t$, is in some basic open set N about some point $\langle p,q\rangle \in A$. Hence, $D \subset V$. Since D intersects B, $\overline{V}$ intersects B also. It follows that B is intersected by the closure of every open set containing A. Hence, $R \times R$ is not normal. The last assertion makes use of the following theorem in Hocking and Young [2], p.41. If H and K are disjoint closed subsets of a normal space S, then there exist disjoint open sets with disjoint closures, one containing H and the other containing K.

<u>Definition 2.8</u>. $\langle S, \mathcal{T}\rangle$ is completely normal iff given any two separated subsets of S, $\exists$ two disjoint open sets, each containing just one of them. $\langle S, \mathcal{T}\rangle$ is a T_5-space iff $\langle S, \mathcal{T}\rangle$ is a completely normal T_1-space.

<u>Theorem 2.13</u>. If $\langle A, A \cap \mathcal{T}\rangle$ is any subspace of a completely normal space $\langle S, \mathcal{T}\rangle$, then $\langle A, A \cap \mathcal{T}\rangle$ is completely normal.

Proof. Let B,D be separated subsets of A. Thus, B, D are separated subsets of S. Since $\langle S, \mathcal{T}\rangle$ is completely normal, $\exists$ disjoint open sets U, V $\ni B \subset U$, $D \subset V$. It follows that $A \cap U$, $A \cap V$

are disjoint open subsets of A containing B and D, respectively.

Corollary 2.4. Any subspace of a T_5-space is T_5, since property T_1 is hereditary.

Example 2.3. We reconsider the space R of Example 2.2 which is completely normal. The product space $R \times R$ is not completely normal since it is not normal.

In showing that R with the lower limit topology is completely normal, we shall make use of the following theorem in Pervin [4], p.92. A topological space $\langle S, \mathcal{T} \rangle$ is completely normal iff every subspace of $\langle S, \mathcal{T} \rangle$ is normal.

Let A be any subspace of R. A is easily shown to be Lindelöf by the same sort of argument we used to show that R is Lindelöf. Since R is hereditarily Lindelöf, and regularity is hereditary, we have that R is hereditarily normal (hence, completely normal), using the following theorem: Every regular Lindelöf space is normal. ([4], p. 92).

III. COVERING PROPERTIES

In this chapter, we discuss the basic covering properties of point-set topology. First and second countability are seen to be hereditary and weakly productive. Separability is neither hereditary nor productive, but it is product invariant for $\mathfrak{c}$ factor spaces. Compactness, local compactness, and countable compactness are all non-hereditary. Compactness and countable compactness are productive, but local compactness is product invariant for only a finite number of factor spaces. Paracompactness and the the Lindelöf Property are neither hereditary nor productive. Finally, we show that the property of being developable is hereditary and weakly productive.

<u>Definition 3.1</u>. A space $\langle S, \mathcal{T} \rangle$ is first countable iff $\mathcal{T}$ has a countable local basis at each $x \in S$.

<u>Theorem 3.1</u>. Any subspace $\langle A, A \cap \mathcal{T} \rangle$ of a first countable space $\langle S, \mathcal{T} \rangle$ is first countable.

Proof. If $p \in A$, then $p \in S$ also. Hence, $\exists$ a countable local basis $\{B_n(p) / n \in I^+\}$ for $\mathcal{T}$ at p. The collection $\{O_n(p) / n \in I^+\}$, where $O_n(p) = A \cap B_n(p)$ $\forall$ $n \in I^+$, is a countable local basis for $A \cap \mathcal{T}$ at p.

We give an example to show that the property of being first countable is not productive.

<u>Example 3.1</u>. Let $\langle S_\alpha, \mathcal{T}_\alpha \rangle$ be a countable discrete space $\forall \alpha \in A$ (uncountable). $\langle S_\alpha, \mathcal{T}_\alpha \rangle$ is first countable $\forall \alpha \in A$,

since $\{x_\alpha\}$ is a countable local basis for $\mathcal{J}_\alpha$ at x_α $\forall$ $x_\alpha \in S_\alpha$. Let $x \in \prod_A S_\alpha$, and suppose $\prod_A \mathcal{J}_\alpha$ has a countable local basis $\{B_n(x) / n \in I^+\}$ at x. $\forall_{n \in I^+}$, there must exist a basic open set $\prod_A U_\alpha \ni x \in \prod_A U_\alpha \subset B_n(x)$, where $U_\alpha \in \mathcal{J}_\alpha$ $\forall \alpha \in A$ and $U_\alpha = S_\alpha$ for all but a finite number of values of α. Thus, the collection A_1 of all such $\alpha \in A$, $\ni U_\alpha \neq S_\alpha$ is countable, since a countable collection of finite sets is countable. Let $\alpha \in A - A_1$, and choose $U_\alpha \in \mathcal{J}_\alpha \ni x_\alpha \in U_\alpha \neq S_\alpha$. This is possible since each $\langle S_\alpha, \mathcal{J}_\alpha \rangle$ is discrete. Thus, $x \in \pi_\alpha^{-1}(U_\alpha) \in \prod_A \mathcal{J}_\alpha$, but $\pi_\alpha^{-1}(U_\alpha)$ does not contain any set $B_n(x)$. Contradiction.

Theorem 3.2. If $\langle S_i, \mathcal{J}_i \rangle$ is first countable $\forall_{i \in I^+}$, then the product space $\langle \prod_{I^+} S_i, \prod_{I^+} \mathcal{J}_i \rangle$ is first countable.

Proof. Let $x = \langle x_1, x_2, \ldots \rangle \in \prod_{I^+} S_i$. Let $\{B_n^i(x_i) / n \in I^+\}$ be a countable local basis for $\mathcal{J}_i$ at $x_i \in S_i$ $\forall$ $i \in I^+$. The family $\{\pi_i^{-1}(B_n^i(x_i)) / i, n \in I^+\}$ is a countable collection of open sets in the product space. The set of all finite intersections of members of this collection is a countable local basis for $\prod_{I^+} \mathcal{J}_i$ at x.

Definition 3.2. A space $\langle S, \mathcal{J} \rangle$ is second countable (completely separable) iff $\mathcal{J}$ has a countable basis.

Theorem 3.3. Any subspace $\langle A, A \cap \mathcal{J} \rangle$ of a second countable space $\langle S, \mathcal{J} \rangle$ is second countable.

Proof. Let $\{B_n / n \in I^+\}$ be a countable basis for $\mathcal{J}$. The Collection $\{O_n / n \in I^+\}$, where $O_n = A \cap B_n$ $\forall n \in I^+$, is a countable basis for $A \cap \mathcal{J}$.

We give an example to show that the property of being second countable is not productive.

Example 3.2. Let $\langle S_\alpha, \mathcal{T}_\alpha \rangle$ be a countable discrete space $\forall \alpha \in A$ (uncountable). $\langle S_\alpha, \mathcal{T}_\alpha \rangle$ is second countable $\forall \alpha \in A$, since $\{\{x_\alpha\} / x_\alpha \in S_\alpha\}$ is a countable basis for $\mathcal{T}_\alpha$. Since we previously showed that $\langle \prod_A S_\alpha, \prod_A \mathcal{T}_\alpha \rangle$ is not first countable, it can't be second countable.

Theorem 3.4. If $\langle S_i, \mathcal{T}_i \rangle$ is second countable $\forall i \in I^+$, then the product space $\langle \prod_{I^+} S_i, \prod_{I^+} \mathcal{T}_i \rangle$ is second countable.

Proof. Let $\{B_n^i / n \in I^+\}$ be a countable basis for $\mathcal{T}_i \; \forall i \in I^+$. The family $\{\pi_i^{-1}(B_n^i) / i, n \in I^+\}$ is a countable collection of open sets in the product space. The set of all finite intersections of members of this collection is a countable basis for $\prod_{I^+} \mathcal{T}_i$.

Definition 3.3. A space $\langle S, \mathcal{T} \rangle$ is separable iff S contains a countable dense subset.

We give an example of a non-separable space which is a subspace of a separable space.

Example 3.3. Let S be an uncountable set with the discrete topology on S. Let a be an element not belonging to S, and let $S \cup \{a\}$ have a topological basis consisting of $\emptyset$, $\{a\}$, and $\{x, a\} \; \forall x \in S$. $S \cup \{a\}$ is separable, because $\{a\}$ is a countable, dense subset of $S \cup \{a\}$. However, S is a non-separable subspace of $S \cup \{a\}$, since it's discrete and uncountable.

We give an example to show that separability is not productive.

Example 3.4. Let $S_\alpha = \{x_\alpha, y_\alpha\}$ have the discrete topology $\mathcal{T}_\alpha = \{\emptyset, \{x_\alpha\}, \{y_\alpha\}, S_\alpha\}$ $\forall \alpha \in A$, where card $(A) > \mathfrak{c}$. $\langle S_\alpha, \mathcal{T}_\alpha \rangle$ is separable $\forall \alpha \in A$. Assume $\langle \prod_A S_\alpha, \prod_A \mathcal{T}_\alpha \rangle$ is also separable, and let C be a countable dense subset of $\prod_A S_\alpha$. $\forall \beta \in A$, let $\prod_A V_\alpha$ be the basic open set with $V_\beta = \{x_\beta\}$ and $V_\alpha = S_\alpha$ if $\alpha \neq \beta$. Also, $\forall \beta \in A$ define a function $f_\beta : C \rightarrow \{0,1\}$ as follows:

$$f_\beta(x) = \begin{cases} 0 & \text{if } x \in \prod_A V_\alpha \text{ where } V_\beta = \{x_\beta\} \text{ and } V_\alpha = S_\alpha \text{ for } \alpha \neq \beta \\ 1 & \text{otherwise.} \end{cases}$$

Now card $\{f_\beta \,/\, f_\beta : C \rightarrow \{0,1\}\} = 2^{\aleph_0} = \mathfrak{c}$, since card $(C) = \aleph_0$. We show that card $(A) \leq$ card $\{f_\beta \,/\, \beta \in A\} = \mathfrak{c}$, which contradicts our assumption that card $(A) > \mathfrak{c}$. Let $\beta_1, \beta_2 \in A \ni \beta_1 \neq \beta_2$. Consider the basic open set $\prod_A V_\alpha$, where $V_{\beta_1} = \{x_{\beta_1}\}$ and $V_\alpha = S_\alpha$ for $\alpha \neq \beta_1$, and $\prod_A U_\alpha$, where $U_{\beta_2} = \{y_{\beta_2}\}$ and $U_\alpha = S_\alpha$ for $\alpha \neq \beta_2$. Since the intersection of these two basic open sets is a nonempty member of $\prod_A \mathcal{T}_\alpha$, it must contain a point Z of C. Since $Z \in \prod_A V_\alpha \cap \prod_A U_\alpha$, we have $f_{\beta_1}(Z) = 0$ and $f_{\beta_2}(Z) = 1$. Thus, $\beta_1 \neq \beta_2$ implies $f_{\beta_1} \neq f_{\beta_2}$. Hence, card $(A) \leq \mathfrak{c}$. Contradiction. Hence, $\langle \prod_A S_\alpha, \prod_A \mathcal{T}_\alpha \rangle$ is not separable if card $(A) > \mathfrak{c}$.

The following theorem of Pondiczery establishes that separability is product invariant for $\mathfrak{c}$ factor spaces. The proof we give is due to Marczewski and is found in [5], pp. 398-399.

Theorem 3.5. If $\langle S_\alpha, \mathcal{T}_\alpha \rangle$ is separable $\forall \alpha \in R^+$ (the set of positive reals), then the product space $\langle \prod_{R^+} S_\alpha, \prod_{R^+} \mathcal{T}_\alpha \rangle$ is separable.

Proof. Let $D_\alpha = \{x_k^\alpha / k \in I^+\}$ be a countable dense subset of S_α $\forall \alpha \in R$. Let T be the set of all tuples $t = \{r_1, \ldots, r_{n-1} ; k_1, \ldots, k_n\}$, where $r_1 < \ldots < r_{n-1}$ are positive rationals, $k_1, \ldots, k_n \in I^+$ and $n \geq 2$. The set T is countable, being a countable collection of countable sets. Define a function $f : T \to \prod_{R^+} S_\alpha$ as follows:

$$t \in T \xrightarrow{f} x^t = \{x_\alpha^t\}_{\alpha \in R^+} \in \prod_{R^+} S_\alpha \quad \text{where}$$

$$x_\alpha^t = \begin{cases} x_{k_1} & \text{if } \alpha \leq r_1 , \\ x_{k_m} & \text{if } r_{m-1} < \alpha \leq r_m, \\ x_{k_n} & \text{if } \alpha > r_{n-1} . \end{cases}$$

$f(T)$ is countable since T is countable. We show that $f(T)$ is also dense in $\prod_{R^+} S_\alpha$. Let $\prod_{R^+} U_\alpha$ be any basic open set in the product space. Thus, $U_\alpha \neq S_\alpha$ only for $\alpha = \alpha_1, \ldots, \alpha_n$ say. Choose positive rationals $r_1, \ldots, r_{n-1} \ni \alpha_1 < r_1 < \alpha_2 < r_2 < \ldots < r_{n-1} < \alpha_n$. This is possible since, the rationals are dense in the reals. Since $U_{\alpha_i} \in \mathcal{J}_{\alpha_i}$ and D_{α_i} is dense in S_{α_i}, $\exists$ $k_1, \ldots, k_n \in I^+ \ni x_{k_i}^{\alpha_i} \in U_{\alpha_i}$, for $i = 1, \ldots, n$. Let $t = \{r_1, \ldots, r_{n-1}; k_1, \ldots, k_n\}$, the point x^t, where x_α^t is defined as above, is in $\prod_{R^+} U_\alpha$. Hence, $\langle \prod_{R^+} S_\alpha , \prod_{R^+} \mathcal{J}_\alpha \rangle$ is separable.

<u>Definition 3.4</u>. A space $\langle S, \mathcal{J} \rangle$ is compact iff every open covering of S contains a finite subcovering of S.

We give an example to show that compactness is not hereditary.

<u>Example 3.5</u>. The Heine-Borel theorem states that a subset of

the reals, under the interval topology, is compact iff it is closed and bounded. Thus, under the interval topology for the reals, the closed interval $[0,1]$ is compact and the open interval $(0,1)$, which is a subset of $[0,1]$, is not compact.

Theorem 3.6. Any product of compact spaces is compact by the Tychonoff theorem. ([2] , p. 25).

Definition 3.5. $\langle S, \mathcal{T} \rangle$ is countably compact iff every infinite subset of S has a limit point in S.

We give an example to show that countable compactness is non-hereditary.

Example 3.6. The Bolzano-Weierstrass theorem implies that on the real line compactness and countable compactness are equivalent. Thus, the closed interval $[0,1]$ is countably compact since it is compact. However, the open interval $(0,1)$ is a subset of $[0,1]$ which is not countably compact since it is not compact.

The following characterization of countable compactness is given in [2] , p. 21. A space $\langle S, \mathcal{T} \rangle$ is countably compact iff given any countable collection $\{C_i \, / \, i \in I^+\}$ of closed subsets of S satisfying the finite intersection hypothesis, then $\bigcap_{i \in I^+} C_i \neq \emptyset$.

Theorem 3.7. If $\langle S_\alpha, \mathcal{T}_\alpha \rangle$ is countably compact $\forall \alpha \in A$, then the product space $\langle \mathbb{P}_A S_\alpha, \mathbb{P}_A \mathcal{T}_\alpha \rangle$ is countably compact.

Proof. Let $\{C_i \, / \, i \in I^+\}$ be any countable collection of closed subsets of $\mathbb{P}_A S_\alpha$ satisfying the finite intersection hypothesis. Thus, the collection $\{\pi_\alpha(C_i) \, / \, i \in I^+\}$ satisfies the finite intersection hypothesis $\forall \alpha \in A$. Moreover, the collection $\{\overline{\pi_\alpha(C_i)} \, / \, i \in I^+\}$ of closed subsets of S_α satisfies the finite intersection hypothesis $\forall \alpha \in A$. Since $\langle S_\alpha, \mathcal{T}_\alpha \rangle$ is countably compact $\forall \alpha \in A$, $\exists\ x_\alpha \in \bigcap_{i \in I^+} \overline{\pi_\alpha(C_i)}\ \forall \alpha \in A$. Let $U_\alpha \in \mathcal{T}_\alpha \ni x_\alpha \in U_\alpha\ \forall \alpha \in A$. Since $U_\alpha \cap \pi_\alpha(C_i) \neq \emptyset$ for each $i \in I^+$, it follows that $\pi_\alpha^{-1}(U_\alpha) \cap C_i \neq \emptyset\ \forall i \in I^+$. Let p be that point of $\mathbb{P}_A S_\alpha$ with coordinate $x_\alpha \in S_\alpha$ where x_α is as above. Let $V = \mathbb{P}_A V_\alpha$ be any basic open set containing p. Since $V_\alpha \neq S_\alpha$ for at most a finite number of values of α (say $\alpha_1, \ldots, \alpha_k$), we have $V = \bigcap_{n=1}^{k} \pi_{\alpha_n}^{-1}(V_{\alpha_n})$. Since $\pi_{\alpha_n}^{-1}(V_{\alpha_n}) \cap C_i \neq \emptyset\ \forall i \in I^+$ and for $n = 1, 2, \ldots, k$, it follows that $V \cap C_i \neq \emptyset, \forall i \in I^+$. Since V is an arbitrary basic open set, this implies that p is a limit point of $C_i\ \forall i \in I^+$. Hence, $p \in C_i$, since C_i is closed $\forall i \in I^+$. Thus, $p \in \bigcap_{i \in I^+} C_i$.

<u>Definition 3.6</u>. A space $\langle S, \mathcal{T} \rangle$ is locally compact iff $\forall x \in S$ $\exists$ an open set which contains x and has a compact closure.

We give an example to show that local compactness is non-hereditary.

<u>Example 3.7</u>. Let $S = \{x / x \geq 0\}$ with a topological basis consisting of the family of half-open intervals $[a,b)$, $a,b \in S$ with $a < b$. Let $R = S \cup \{\infty\}$ be the one-point compactification of S.. Clearly, R is locally compact since it is compact. Consider the point 1 in S. The closure of every basic open set $[a,b)$ containing 1 is the set $[a,b)$ itself. Also, $[a,b)$ is not compact, since $\{[a, b - \frac{1}{n}) / n \in I^+\}$ is an open covering of $[a,b)$ which contains no finite subcovering of $[a,b)$. Hence, no open set containing 1 has a compact closure, and S is not locally compact.

We give an example of a non-locally compact space which is the topological product of infinitely many locally compact spaces.

Example 3.8. Let $S_i = (0,1)$ with the interval topology $\forall\, i \in I^+$, and let $p = \langle p_1, p_2, \ldots \rangle \in \mathcal{P}_{I^+} S_i$. Suppose $\exists$ a set U containing p and having compact closure. Then $\exists$ a basic open set $B \ni p \in B \subset U$, where $B = \mathcal{P}_{I^+} V_i$ and $V_i = S_i$ for all but a finite number of values of i. Since $\overline{U}$ is compact and $\overline{B} \subset \overline{U}$, $\overline{B}$ is compact also. Since π_i is continuous, $\pi_i(\overline{B})$ is compact $\forall\, i \in I^+$. This is impossible, since for infinitely many values of i, $\pi_i(\overline{B}) = V_i = S_i = (0,1)$, which is not compact by the Heine-Borel theorem. Contradiction. Hence, no such set U exists, and $\mathcal{P}_{I^+} S_i$ is not locally compact.

Theorem 3.8. The product of two locally compact spaces is locally compact.

Proof. Let $\langle S_1, \mathcal{T}_1 \rangle$, $\langle S_2, \mathcal{T}_2 \rangle$ be locally compact spaces. Let $\langle p,q \rangle \in S_1 \times S_2$. Since $\langle S_1, \mathcal{T}_1 \rangle$ is locally compact, $\exists\, U_1 \in \mathcal{T}_1 \ni p \in U_1$ and $\overline{U}_1$ is compact. Similarly, $\exists\, U_2 \in \mathcal{T}_2 \ni q \in U_2$ and $\overline{U}_2$ is compact. Now $\langle p,q \rangle \in U_1 \times U_2$ and $\overline{U_1 \times U_2} = \overline{U}_1 \times \overline{U}_2$, which is compact by the Tychonoff theorem. Hence, $\langle S_1 \times S_2, \mathcal{T}_1 \times \mathcal{T}_2 \rangle$ is locally compact.

Corollary 3.1. The topological product of finitely many locally compact spaces is locally compact.

Proof. By mathematical induction using Theorem 3.8.

Definition 3.7. $\langle S, \mathcal{T} \rangle$ is paracompact iff every open covering of S has an open, locally finite refinement.

We give two examples to show that paracompactness is non-hereditary and non-productive.

Example 3.9. Let S be the set of real numbers with a topological basis defined as follows: A basic neighborhood of $x \in S$ consists of x and the set of rationals in an open interval around x. The one-point compactification $R = S \cup \{a\}$ of S is clearly paracompact. However, the subspace S is not paracompact, since it is Hausdorff but not regular. It is Hausdorff since for any two distinct points of S, we can always find two disjoint basic neighborhoods containing them. It is not regular, since the only open set containing the closed set of irrationals is the set S itself.

Example 3.10. The space R of Example 2.2 is paracompact since it is regular and Lindelöf ([4] , p. 167). However, $R \times R$ was shown to be Hausdorff but not normal. Hence, $R \times R$ is not paracompact.

Definition 3.8. A space $\langle S, \mathcal{T} \rangle$ is Lindelöf iff every open covering of S contains a countable subcovering of S.

We give two examples to show that the property of being Lindelöf is non-hereditary and non-productive.

Example 3.11. Let S be an uncountable space with the discrete topology. Let $S \cup \{a\}$ be the one-point compactification of S. Thus, $S \cup \{a\}$ is compact; hence, Lindelöf. However, the subspace S is not Lindelöf, since it is uncountable and has the discrete topology.

Example 3.12. The space R of Example 2.2 was shown to be regular and Lindelöf and $R \times R$ not normal. Thus, $R \times R$ is not Lindelöf since a regular Lindelöf space is normal ([4] , p. 92).

<u>Definition 3.9</u>. $\langle S, \mathcal{T}\rangle$ is developable iff $\exists$ a sequence $\{G_i\}_{i \in I^+}$ of open coverings of $S \ni G_{i+1}$ is a refinement of G_i $\forall\, i \in I^+$; and $\forall\, x \in S$ and $U \in \mathcal{T} \ni x \in U$ $\exists\, n = n(x,U) \in I^+$ $\ni G_n^*(x) \subset U$. $G_n^*(x)$ is the union of all those members of G_n which contain x. The collection $\{G_i \,/\, i \in I^+\}$ is called a development of $\langle S, \mathcal{T}\rangle$.

<u>Theorem 3.9</u>. Any subspace $\langle A, A \cap \mathcal{T}\rangle$ of a developable space $\langle S, \mathcal{T}\rangle$ is developable.

Proof. Let $\{G_i \,/\, i \in I^+\}$ be a development of $\langle S, \mathcal{T}\rangle$. Let $O_i = A \cap G_i$ $\forall\, i \in I^+$. Thus, $\{O_i\}_{i \in I^+}$ is a sequence of open coverings of A and O_{i+1} is a refinement of O_i $\forall\, i \in I^+$, since G_{i+1} is a refinement of G_i $\forall\, i \in I^+$. Let $x \in A$ and $U \in A \cap \mathcal{T} \ni$ $x \in U$. Thus, $x \in S$ and $U = V \cap A$, where $V \in \mathcal{T}$. Since $\langle S, \mathcal{T}\rangle$ is developable, $\exists$ $n = n(x,V) \in I^+ \ni G_n^*(x) \subset V$. Hence, $O_n^*(x) \subset A \cap V = U$. Thus, $\langle A, A \cap \mathcal{T}\rangle$ is developable.

We give an example to show that the property of being developable is not productive.

<u>Example 3.13</u>. Let $\langle S_\alpha, \mathcal{T}_\alpha\rangle$ be a finite nondegenerate discrete space $\forall \alpha \in A$ (uncountable). Trivially, $\langle S_\alpha, \mathcal{T}_\alpha\rangle$ is first countable and developable $\forall \alpha \in A$. Let $x \in \mathbb{P}_A S_\alpha$, and suppose $\mathbb{P}_A \mathcal{T}_\alpha$ has a countable local basis $\{B_n(x) / n \in I^+\}$ at x. $\forall\, n \in I^+$, there must exist a basic open set $\mathbb{P}_A U_\alpha \ni$ $x \in \mathbb{P}_A U_\alpha \subset B_n(x)$, where $U_\alpha \in \mathcal{T}_\alpha$ $\forall \alpha \in A$ and $U_\alpha = S_\alpha$ for all but a finite number of values of α. Thus, the set A_1 of all such $\alpha \in A \ni U_\alpha \neq S_\alpha$ is countable. Let $\alpha \in A - A_1$, and choose $U_\alpha \in \mathcal{T}_\alpha \ni x_\alpha \in U_\alpha \neq S_\alpha$. This is possible since $\langle S_\alpha, \mathcal{T}_\alpha\rangle$ is

discrete. Thus, $x \in \pi_\alpha^{-1}(U_\alpha) \in P_A \mathcal{T}_\alpha$. However, $\pi_\alpha^{-1}(U_\alpha)$ does not contain any set $B_n(x)$. It follows that $\langle P_A S_\alpha, P_A \mathcal{T}_\alpha \rangle$ is not first countable. Hence, it is not developable, since every developable T_1-space is semi-metrizable and, consequently, first countable ([6] , p. 40).

<u>Theorem 3.10</u>. If $\langle S_i, \mathcal{T}_i \rangle$ is developable $\forall i \in I^+$, then the product space $\langle P_{I^+} S_i, P_{I^+} \mathcal{T}_i \rangle$ is developable.

Proof. By hypothesis each space $\langle S_i, \mathcal{T}_i \rangle$ has a development $\{G_n^i \,/\, n \in I^+\}$ where G_n^i is an open covering of $\langle S_i, \mathcal{T}_i \rangle \ni G_{n+1}^i$ is a refinement of G_n^i $\forall n \in I^+$. Let $C_k = \{\pi_i^{-1}(G_k^i) \,/\, i \in I^+\}$. $\forall k \in I^+$, C_k is an open covering of $P_{I^+} S_i$. We show that C_{k+1} is a refinement of C_k. Let $U^{k+1} \in C_{k+1}$. For some $i \in I^+$, $U^{k+1} \in \pi_i^{-1}(G_{k+1}^i)$ and thus, $\pi_i(U^{k+1}) \in G_{k+1}^i$. Since G_{k+1}^i is a refinement of G_k^i, $\exists$ $V \in G_k^i \ni V \supset \pi_i(U^{k+1})$. Hence, $U^{k+1} \subset \pi_i^{-1}(V) \in C_k$. Thus, C_{k+1} is a refinement of C_k $\forall k \in I^+$.

Now let $p \in P_{I^+} S_i$ and $U \in P_{I^+} \mathcal{T}_i \ni p \in U$. Since U is open, $\exists$ a basic open set $P_{I^+} V_i \subset U$ containing p with $V_i \neq S_i$ only for $i = i_1, \ldots, i_n$. For each i_k, $k = 1, \ldots, n$, $V_{i_k} \in \mathcal{T}_{i_k}$ and $p_{i_k} \in V_{i_k}$. By hypothesis, $\exists$ $N_{i_k} \in I^+ \ni G_{N_{i_k}}^{i_k *}(p_{i_k}) \subset V_{i_k}$. Let $N = \max_{1 \leq k \leq n} N_{i_k}$. Clearly, $G_N^{i_k *}(p_{i_k}) \subset V_{i_k}$, $k = 1, \ldots, n$. Hence, $\pi_{i_k}^{-1}\{G_N^{i_k *}(p_{i_k})\} \subset P_{I^+} V_i$, $k = 1, \ldots, n$. Let $U^N \in C_N \ni p \in U^N$. For $i \neq i_1, \ldots, i_n$, we have $p_i \in \pi_i(U^N) \in G_N^i$. Hence, $p_i \in \pi_i(U^N) \subset G_N^{i *}(p_i) \subset S_i = V_i$. Therefore, $p \in \pi_i^{-1}(p_i) \subset U^N \subset \pi_i^{-1}(G_N^{i *}(p_i)) \subset P_{I^+} V_i \subset U$. It follows that $C_N^*(p) \subset P_{I^+} V_i \subset U$.

IV. CONNECTIVITY PROPERTIES

In this chapter, we consider the basic connectivity properties: connectedness, local connectedness, connected im Kleinen at x, and arcwise connectedness. All are shown to be non-hereditary. Connectedness and arcwise connectedness are shown to be productive, and examples are given to show that local connectedness and connected im Kleinen at x are not productive. However, the latter two properties are shown to be product invariant for a finite number of factor spaces.

<u>Definition 4.1</u>. $\langle S, \mathcal{T} \rangle$ is separated iff $S = A \cup B$, where $A, B \in \mathcal{T} - \{\emptyset\}$ and $A \cap B = \emptyset$. $\langle S, \mathcal{T} \rangle$ is connected iff it is not separated.

We give an example to show that connectedness is not a hereditary property.

<u>Example 4.1</u>. The open interval $(0,2)$ with the interval topology is connected. However, the subspace $(0,1) \cup (1,2)$ of $(0,2)$ is not connected.

<u>Definition 4.2</u>. Let $\langle S, \mathcal{T} \rangle$ be a space. Two points of S are connected iff $\exists$ a connected subspace of $\langle S, \mathcal{T} \rangle$ containing the two points.

<u>Lemma 4.1</u>. $\langle S, \mathcal{T} \rangle$ is connected iff each two points of S are connected.

Proof. The necessity is trivial. For the sufficiency, assume each two points of S are connected, and suppose $\langle S, \mathcal{T} \rangle$ is separated.

$\exists$ U, V $\in \mathfrak{T} - \{\emptyset\} \ni S = U \cup V$ and $U \cap V = \emptyset$. Let $x \in U$, $y \in V$ and let $\langle C, C \cap \mathfrak{T} \rangle$ be any connected subspace of $\langle S, \mathfrak{T} \rangle$ containing x and y. Thus, $(U \cap C) \cup (V \cap C) = C$ and $(V \cap C) \cap (U \cap C) = \emptyset$. Hence, $\langle C, C \cap \mathfrak{T} \rangle$ is not connected. Contradiction. Hence, $\langle S, \mathfrak{T} \rangle$ is connected.

Theorem 4.1. If $\langle S_1, \mathfrak{T}_1 \rangle$ and $\langle S_2, \mathfrak{T}_2 \rangle$ are connected, then the product space $\langle S_1 \times S_2, \mathfrak{T}_1 \times \mathfrak{T}_2 \rangle$ is connected.

Proof. Let $\langle p_1, q_1 \rangle$, $\langle p_2, q_2 \rangle \in S_1 \times S_2$. The subspace $\{p_1\} \times S_2$, being homeomorphic to $\langle S_2, \mathfrak{T}_2 \rangle$, is connected and contains the points $\langle p_1, q_1 \rangle$ and $\langle p_1, q_2 \rangle$. Thus, $\langle p_1, q_1 \rangle$ and $\langle p_1, q_2 \rangle$ are connected by Definition 4.2. Similarly, $S_1 \times \{q_2\}$ is a connected space containing $\langle p_1, q_2 \rangle$ and $\langle p_2, q_2 \rangle$. Thus, $\langle p_1, q_2 \rangle$ and $\langle p_2, q_2 \rangle$ are connected by Definition 4.2. This implies that $\langle p_1, q_1 \rangle$ and $\langle p_2, q_2 \rangle$ are connected. Hence, $\langle S_1 \times S_2, \mathfrak{T}_1 \times \mathfrak{T}_2 \rangle$ is connected by Lemma 4.1.

Corollary 4.1. The topological product of finitely many connected spaces is connected.

Proof. By mathematical induction using Theorem 4.1.

Theorem 4.2. If $\langle S_\alpha, \mathfrak{T}_\alpha \rangle$ is connected $\forall \alpha \in A$, then the product space $\langle \prod_A S_\alpha, \prod_A \mathfrak{T}_\alpha \rangle$ is connected.

Proof. Let $p \in \prod_A S_\alpha$ and C be the component of $\prod_A S_\alpha$ which contains p. We shall show that C is dense in $\prod_A S_\alpha$. Let $U = \prod_A U_\alpha$ be a basic open set in $\prod_A S_\alpha$, where $U_\alpha \neq S_\alpha$ only for $\alpha_1, \ldots, \alpha_n$. Let $x_i \in U_{\alpha_i}$, $i = 1, \ldots, n$, be fixed but arbitrary. Let q be the point in $\prod_A S_\alpha$ whose α - coordinate is the same as that of p if $\alpha \neq \alpha_i$, and equal to x_i if $\alpha = \alpha_i$ for some

i $(1 \leq i \leq n)$. Thus, $q \in U$. Let $X = \{x \,/\, x_\alpha = p_\alpha$ for all $\alpha \in A = \{\alpha_1, \ldots, \alpha_n\}\}$. The subspace X is homeomorphic to $\langle \prod_{i=1}^{n} S_{\alpha_i}, \prod_{i=1}^{n} \mathcal{J}_{\alpha_i} \rangle$ which is connected by Corollary 4.1. Since connectedness is a topological property, X is connected and contains both p and q. Since C is the component containing p, $X \subset C$ and $q \in C$. Hence, $q \in U \cap C$ and every basic open set meets C. This implies that $\overline{C} = \prod_A S_\alpha$. Moreover, $\overline{C} = C$ since C is a component. Hence, $\langle \prod_A S_\alpha, \prod_A \mathcal{J}_\alpha \rangle$ is connected.

<u>Definition 4.3.</u> $\langle S, \mathcal{J} \rangle$ is locally connected at $x \in S$ iff for every open set U containing x, $\exists$ a connected open set $V \ni x \in V \subset U$. $\langle S, \mathcal{J} \rangle$ is locally connected iff $\langle S, \mathcal{J} \rangle$ is locally connected at x, $\forall x \in S$.

The following space is a locally connected space which is not hereditarily locally connected.

<u>Example 4.2.</u> Let $[0,1]$ and $[-1,1]$ have the interval topology and $S = [0,1] \times [-1,1]$ with the product topology. Since $[0,1]$ and $[-1, 1]$ are locally connected, S is locally connected by Theorem 4.3, which follows this example. Consider the subspace $X = \{\langle x,y \rangle \in S \,/\, y = \sin \frac{1}{x}, 0 < x \leq 1\} \cup \{\langle 0,0 \rangle\}$. The open set $C = \{\langle x,y \rangle \in S \,/\, x^2 + y^2 < 10^{-4}\} \cap X$, in the subspace topology of X, contains $\langle 0, 0 \rangle$. Moreover, the only connected subset of C which contains $\langle 0, 0 \rangle$ is the singleton set $\{\langle 0,0 \rangle\}$. However, $\{\langle 0, 0 \rangle\}$ is not a member of the subspace topology of X. Hence, X is not locally connected.

<u>Theorem 4.3</u>. If $\langle S_1, \mathcal{T}_1 \rangle$ and $\langle S_2, \mathcal{T}_2 \rangle$ are locally connected, then the product space $\langle S_1 \times S_2, \mathcal{T}_1 \times \mathcal{T}_2 \rangle$ is locally connected.

Proof. Let $\langle p, q \rangle \in S_1 \times S_2$ and $U \in \mathcal{T}_1 \times \mathcal{T}_2 \ni \langle p, q \rangle \in U$. $\exists$ a basic open set $V_1 \times V_2 \ni \langle p, q \rangle \in V_1 \times V_2 \subset U$. Since $\langle S_1, \mathcal{T}_1 \rangle$ and $\langle S_2, \mathcal{T}_2 \rangle$ are locally connected, $\exists\ U_1 \in \mathcal{T}_1$, $U_2 \in \mathcal{T}_2 \ni U_1$ and U_2 are connected and $p \in U_1 \subset V_1$, $q \in U_2 \subset V_2$. Thus, $U_1 \times U_2$ is connected by Theorem 4.1, and $\langle p, q \rangle \in U_1 \times U_2 \subset U$.

<u>Corollary 4.2</u>. The topological product of finitely many locally connected spaces is locally connected.

Proof. By mathematical induction using Theorem 4.3.

We give an example to show that local connectedness is not infinite product invariant.

<u>Example 4.3</u>. Let $S_i = \{x_i, y_i\}$ have the discrete topology $\mathcal{T}_i = \{\emptyset, \{x_i\}, \{y_i\}, S_i\}\ \forall i \in I^+$. Let $p \in \mathbb{P}_{I^+} S_i$ and $U \in \mathbb{P}_{I^+} \mathcal{T}_i \ni p \in U$. Suppose U is connected. Hence, $\exists$ a basic open set $\mathbb{P}_{I^+} V_i$ in the product topology $\ni p \in \mathbb{P}_{I^+} V_i \subset U$. Thus, for infinitely many values of i, we have $V_i = S_i$. Let $i = k$ be one such value. Thus, $\pi_k(\mathbb{P}_{I^+} V_i) = S_k$. Since U contains $\mathbb{P}_{I^+} V_i$, we have $\pi_k(U) = S_k$ also. Since U is connected and π_k is continuous, S_k must be connected. However, S_i is disconnected $\forall i \in I^+$ because it has the discrete topology. Contradiction. Hence, there does not exist any connected member of the product topology containing p. Hence $\langle \mathbb{P}_{I^+} S_i, \mathbb{P}_{I^+} \mathcal{T}_i \rangle$ is not locally connected.

<u>Definition 4.4</u>. $\langle S, \mathcal{T} \rangle$ is connected im Kleinen at $x \in S$ iff for every open set U containing x, there is an open set V,

containing x and contained in $U \ni$ if $y \in V$, then $\exists$ a connected subset of U containing $\{x, y\}$. $\langle S, \mathcal{T} \rangle$ is connected im Kleinen iff $\langle S, \mathcal{T} \rangle$ is connected im Kleinen at x $\forall x \in S$.

Remark. Since local connectedness of $\langle S, \mathcal{T} \rangle$ is equivalent to $\langle S, \mathcal{T} \rangle$ being connected im Kleinen, we shall only consider the weaker property, "connected im Kleinen at x".

Example 4.2 shows that "connected im Kleinen at x" is not a hereditary property. S is locally connected at $\langle 0, 0 \rangle$. Hence, S is connected im Kleinen at $\langle 0,0 \rangle$. However, X is not connected im Kleinen at $\langle 0, 0 \rangle$. If X were connected im Kleinen at $\langle 0, 0 \rangle$, then X would be connected im Kleinen at every point of X. By our remark above, X would be locally connected. However, this is not the case, as we saw previously. Thus, X is not connected im Kleinen at $\langle 0, 0 \rangle$.

Theorem 4.4. If $\langle S_1, \mathcal{T}_1 \rangle$ is connected im Kleinen at $x_1 \in S_1$ and $\langle S_2, \mathcal{T}_2 \rangle$ is connected im Kleinen at $x_2 \in S_2$, then $\langle S_1 \times S_2, \mathcal{T}_1 \times \mathcal{T}_2 \rangle$ is connected im Kleinen at $\langle x_1, x_2 \rangle \in S_1 \times S_2$.

Proof. Let U be any open set containing $\langle x_1, x_2 \rangle$. $\exists$ a basic open set $U_1 \times U_2 \ni \langle x_1, x_2 \rangle \in U_1 \times U_2 \subset U$. Since $\langle S_1, \mathcal{T}_1 \rangle$ is connected im Kleinen at x_1, $\exists V_1 \in \mathcal{T}_1 \ni x_1 \in V_1 \subset U_1$, and $\forall y_1 \in V_1$, $\exists$ a connected subset C_1 of U_1 containing $\{x_1, y_1\}$. Similarly, $\exists V_2 \in \mathcal{T}_2 \ni x_2 \in V_2 \subset U_2$, and $\forall y_2 \in V_2$, $\exists$ a connected subset C_2 of U_2 containing $\{x_2, y_2\}$. Let $\langle y_1, y_2 \rangle \in V_1 \times V_2 \in \mathcal{T}_1 \times \mathcal{T}_2$. $\exists$ a connected set $C_1 \subset U_1$ and containing $\{x_1, y_1\}$ and a connected set $C_2 \subset U_2$ and containing $\{x_2, y_2\}$. Thus, $C_1 \times C_2 \subset U_1 \times U_2 \subset U$. Also, $C_1 \times C_2$ is connected by

Theorem 4.1 and contains $\{\langle x_1, x_2\rangle, \langle y_1, y_2\rangle\}$. Hence, $\langle S_1 \times S_2, \mathcal{T}_1 \times \mathcal{T}_2\rangle$ is connected im Kleinen at $\langle x_1, x_2\rangle$.

Corollary 4.3. If $\langle S_i, \mathcal{T}_i\rangle$ is connected im Kleinen at $x_i \in S_i$, $i = 1, 2, \ldots, n$, then the product space $\langle \prod_{i=1}^{n} S_i, \prod_{i=1}^{n} \mathcal{T}_i\rangle$ is connected im Kleinen at $\langle x_1, \ldots, x_n\rangle \in \prod_{i=1}^{n} S_i$.

Proof. By mathematical induction using theorem 4.4.

Since each space $\langle S_i, \mathcal{T}_i\rangle$ of Example 4.3 is locally connected, we have by a previous remark that $\langle S_i, \mathcal{T}_i\rangle$ is connected im Kleinen $\forall\, i \in I^+$. If $\langle \prod_{I^+} S_i, \prod_{I^+} \mathcal{T}_i\rangle$ were connected im Kleinen, then it would be locally connected. However, we have already seen that this is not the case. Hence, $\langle \prod_{I^+} S_i, \prod_{I^+} \mathcal{T}_i\rangle$ cannot be connected im Kleinen at all of its points. Thus, we see that "connected im Kleinen at x" is not infinite product invariant.

Definition 4.5. $\langle S, \mathcal{T}\rangle$ is arcwise connected iff any two points of S are the end points of an arc in S. (Recall that an arc is any homeomorphic image of $[0, 1]$).

The following example shows that arcwise connectedness is not hereditary.

Example 4.4. Let $S = [0, 2]$ with the interval topology. Clearly, S is arcwise connected. However, the subspace $[0, 1) \cup (1, 2]$ is not arcwise connected, since the points 0 and 2 are not the end points of any arc in $[0, 1) \cup (1, 2]$.

<u>Theorem 4.5</u>. If $\langle S_\alpha , \mathcal{T}_\alpha \rangle$ is arcwise connected $\forall \alpha \in A$, then the product space $\langle \mathcal{P}_A S_\alpha , \mathcal{P}_A \mathcal{T}_\alpha \rangle$ is arcwise connected.

Proof. Let $p, q \in \mathcal{P}_A S_\alpha$. For each $\alpha \in A$, p_α, q_α $\in S_\alpha$, and $\langle S_\alpha , \mathcal{T}_\alpha \rangle$ is arcwise connected. Hence, $\exists$ a homeomorphism $f_\alpha : I \rightarrow S_\alpha \ni f_\alpha(0) = p_\alpha$ and $f_\alpha(1) = q_\alpha$ $\forall \alpha \in A$. We must show that $\exists$ a homeomorphism $f : I \rightarrow \mathcal{P}_A S_\alpha$, with $f(0) = p$ and $f(1) = q$. This follows immediately from above, if we let $f = \mathcal{P}_A f_\alpha$ where the α-coordinate of $f(t)$ is $f_\alpha(t)$ $\forall$ $t \in [0, 1]$. Thus $f(0) = \mathcal{P}_A f_\alpha(0) = \mathcal{P}_A \{p_\alpha\} = p$ and $f(1) = \mathcal{P}_A f_\alpha(1) = \mathcal{P}_A \{q_\alpha\} = q$. f is a homeomorphism, since the coordinate mapping f_α is a homeomorphism $\forall \alpha \in A$. Thus, $\langle \mathcal{P}_A S_\alpha , \mathcal{P}_A \mathcal{T}_\alpha \rangle$ is arcwise connected.

V. METRIZATION PROPERTIES

In this final chapter, we consider the generalized metrization properties: metrizability, Moore space, semi-metrizability, and a-metrizability. All of these are easily seen to be hereditary. None of them is productive, but all are shown to be weakly productive.

<u>Definition 5.1.</u> $\langle S, \mathcal{T} \rangle$ is metrizable iff $\exists$ a metric d on $S \ni$ the collection $\{S_d(x;r) \,/\, r > 0\}$ is a local basis for $\mathcal{T}$ at x $\forall$ $x \in S$. (d is a metric on S iff d: $S \times S \longrightarrow R^+ \cup \{0\}$ and satisfies the following conditions: (i) $d(x,y) = 0$ iff $x = y$; (ii) $d(x,y) = d(y,x)$; (iii) $d(x,z) \leq d(x,y) + d(y,z)$ $\forall$ $x,y, z \in S$.)

<u>Theorem 5.1.</u> Every subspace $\langle A, A \cap \mathcal{T} \rangle$ of a metrizable space $\langle S, \mathcal{T} \rangle$ is metrizable.

Proof. Let d: $S \times S \longrightarrow R^+ \cup \{0\}$ be any admissible metric for $\langle S, \mathcal{T} \rangle$. The restriction $d_1 = d \,/\, A$ of the domain of d to $A \times A$ is clearly a metric on A. We show that the collection $\{S_{d_1}(x;r)/r > 0\}$ is a local basis for $A \cap \mathcal{T}$ at x $\forall$ $x \in A$. Let $V \in A \cap \mathcal{T} \ni x \in V$. Thus, $\exists$ $U \in \mathcal{T} \ni V = A \cap U$. By definition of a local basis, $\exists$ an open d-sphere $S_d(x;r)$ about $x \ni x \in S_d(x;r) \subset U$. Hence, $x \in A \cap S_d(x;r) = S_{d_1}(x;r) \subset A \cap U = V$.

We give an example to show that the product of uncountably many metrizable spaces is not necessarily metrizable.

<u>Example 5.1.</u> We reconsider the space of Example 3.13, which we used previously to show that the property of being developable is

not productive. Each factor space is metrizable, and the product space is not first countable. Hence, the product space is not metrizable.

Although metrizability is not productive, we are able to show that it is weakly productive.

Theorem 5.2. If $\langle S_i, \mathcal{T}_i \rangle$ is metrizable $\forall i \in I^+$, then the product space $\langle P_{I^+} S_i, P_{I^+} \mathcal{T}_i \rangle$ is metrizable.

Proof. Let d_i be any admissible metric for $\langle S_i, \mathcal{T}_i \rangle$ $\forall i \in I^+$. For each $i \in I^+$, let ρ_i be the admissible, bounded metric on $\langle S_i, \mathcal{T}_i \rangle$ given by $\rho_i(x_i, y_i) = \dfrac{d_i(x_i, y_i)}{1 + d_i(x_i, y_i)}$ $\forall x_i, y_i \in S_i$. $\forall x, y \in P_{I^+} S_i$, we let $d(x,y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \cdot \rho_i(x_i, y_i)$. We show that d is a metric on $P_{I^+} S_i$ and induces the product topology $P_{I^+} \mathcal{T}_i$ on $P_{I^+} S_i$.

(i) $d(x,y) \geq 0$. Since $\rho_i(x_i, y_i) \geq 0$, each term of the series $\sum_{i=1}^{\infty} \frac{1}{2^i} \cdot \rho_i(x_i, y_i)$ is non-negative. Thus, $d(x,y) \geq 0$.

(ii) $d(x,y) = 0$ iff $x = y$. If $x = y$, then $x_i = y_i$ $\forall i \in I^+$. Hence, $\sum_{i=1}^{\infty} \frac{1}{2^i} \cdot \rho_i(x_i, y_i) = \sum_{i=1}^{\infty} \frac{1}{2^i} (0) = 0$. If $d(x,y) = 0$, then $\sum_{i=1}^{\infty} \frac{1}{2^i} \cdot \rho_i(x_i, y_i) = 0$. Hence, each term of the series is zero. Thus, $\rho_i(x_i, y_i) = 0$ and $x_i = y_i$ $\forall i \in I^+$, which implies $x = y$.

(iii) $d(x,y) = d(y,x)$. Since $\rho_i(x_i, y_i) = \rho_i(y_i, x_i)$, $\sum_{i=1}^{\infty} \frac{1}{2^i} \cdot \rho_i(x_i, y_i) = \sum_{i=1}^{\infty} \frac{1}{2^i} \cdot \rho_i(y_i, x_i)$, which implies $d(x,y) = d(y,x)$. (iv) $d(x,y) + d(y,z) \geq d(x,z)$. Since $\rho_i(x_i, y_i) + \rho_i(y_i, z_i) \geq \rho_i(x_i, z_i)$, it follows that

$$\sum_{i=1}^{\infty} \frac{1}{2^i} \cdot \rho_i(x_i, y_i) + \sum_{i=1}^{\infty} \frac{1}{2^i} \rho_i(y_i, z_i) = \sum_{i=1}^{\infty} \frac{1}{2^i} \{ \rho_i(x_i, y_i) + \rho_i(y_i, z_i) \}$$

$$\geq \sum_{i=1}^{\infty} \frac{1}{2^i} \cdot \rho_i(x_i, z_i),$$ all of these series being convergent.

We now show that d induces the product topology $\mathbb{P}_{I^+}\mathcal{T}_i$. Let $p \in \mathbb{P}_{I^+}S_i$ and let $\mathbb{P}_{I^+}U_i$ be any basic open set containing p, where $U_i \neq S_i$ only for $i = i_1, \ldots, i_k$. Since ρ_i is an admissible metric on $\langle S_i, \mathcal{T}_i\rangle$ $\forall\, i \in I^+$, $\exists\, S_{\rho_i}(p_i;r_i) \ni p_i \in S_{\rho_i}(p_i;r_i) \subset U_i$, $i = i_1, \ldots, i_k$. Let $r = \min\left\{\frac{1}{2^i}\cdot r_i \,/\, i = i_1, \ldots, i_k\right\}$. If $q \in S_d(p;r)$, then $\sum_{i=1}^{\infty} \frac{1}{2^i}\rho_i(p_i,q_i) < r$. Thus, $\frac{1}{2^i}\rho_i(p_i,q_i) < r \leq \frac{1}{2^i}r_i$ and $\rho_i(p_i,q_i) < r_i$ $\forall\, i = i_1, \ldots, i_k$. Hence, $q_i \in S_{\rho_i}(p_i;r_i) \subset U_i$, and $q \in \mathbb{P}_{I^+}U_i$. Thus, $p \in S_d(p;r) \subset \mathbb{P}_{I^+}U_i$, and d induces the product topology $\mathbb{P}_{I^+}\mathcal{T}_i$ on $\mathbb{P}_{I^+}S_i$.

The original axioms for what is now known as a Moore space were first given by R.L. Moore and later investigated by F.B. Jones, R.H. Bing *et al.* Moore showed that this concept is a generalization of a metrizable space, and Bing showed that the two concepts are equivalent for the class of collectionwise normal (paracompact) spaces.

Definition 5.2. $\langle S, \mathcal{T}\rangle$ is a Moore space iff $\langle S, \mathcal{T}\rangle$ is a developable T_3-space.

Theorem 5.3. Every subspace $\langle A, A \cap \mathcal{T}\rangle$ of a Moore space $\langle S, \mathcal{T}\rangle$ is a Moore space.

Proof. This follows immediately since T_3 and the property of being developable are both hereditary.

We give an example to show that the property of being a Moore space is not productive, and then we prove that the property of being a Moore space is weakly productive.

Example 5.2. We consider once again the space of Example 3.13, which we used to show that the property of being developable is not productive. Each factor space, being metrizable, is a Moore space. The product space is not a Moore space since it is not developable.

Theorem 5.4. If $\langle S_i, \mathcal{T}_i \rangle$ is a Moore space $\forall i \in I^+$, then the product space $\langle \prod_{I^+} S_i, \prod_{I^+} \mathcal{T}_i \rangle$ is a Moore space.

Proof. Since T_3 is productive (Corollary 2.2) and the product of a countable number of developable spaces is developable (Theorem 3.10), the result follows.

K. Menger first introduced the notion of a semi-metrizable space as a generalization of a metrizable space. Various topological characterizations of semi-metrizability have been given in the literature by McAuley, Cedar, Heath, Boyd, et al. Sims gave in his Thesis [6] an "indexed neighborhood" characterization of semi-metrizability and showed that every developable T_1-space is semi-metrizable.

Definition 5.3. $\langle S, \mathcal{T} \rangle$ is semi-metrizable iff $\exists$ a semi-metric d on S $\ni$ the collection $\{S_d(x;r) \,/\, r > 0\}$ is a local basis for $\mathcal{T}$ at x $\forall$ x $\in$ S. (d is a semi-metric on S iff $d: S \times S \to R^+ \cup \{0\}$ and satisfies the following conditions: (i) $d(x,y) = 0$ iff $x = y$; (ii) $d(x,y) = d(y,x)$.)

Theorem 5.5. Every subspace $\langle A, A \cap \mathcal{T} \rangle$ of a semi-metrizable space $\langle S, \mathcal{T} \rangle$ is semi-metrizable.

Proof. Let $d: S \times S \to R^+ \cup \{0\}$ be any admissible semi-metric for $\langle S, \mathcal{T} \rangle$. The restriction $d_1 = d/A$ of the domain of d to A x A is clearly a semi-metric on A. As in the proof of Theorem 5.1, the collection $\{S_{d_1}(x;r) \,/\, r > 0\}$ is easily seen to be a local basis for $A \cap \mathcal{T}$ at x $\forall$ x $\in$ A.

Example 5.3. We reconsider the space of Example 3.13, which we used to show that the property of being developable is not productive. Each factor space is metrizable (hence, semi-metrizable). However, the product space is not semi-metrizable since it is not first countable.

Theorem 5.6. If $\langle S_i, \mathcal{T}_i \rangle$ is semi-metrizable $\forall\, i \in I^+$, then the product space $\langle \mathbb{P}_{I^+} S_i, \mathbb{P}_{I^+} \mathcal{T}_i \rangle$ is semi-metrizable.

Proof. Let d_i be any admissible semi-metric on $\langle S_i, \mathcal{T}_i \rangle$ $\forall\, i \in I^+$, and let ρ_i be the admissible, bounded semi-metric on $\langle S_i, \mathcal{T}_i \rangle$ $\forall\, i \in I^+$ given by $\rho_i(x_i, y_i) = \dfrac{d_i(x_i, y_i)}{1 + d_i(x_i, y_i)}$ $\forall\, x_i, y_i \in S_i$.

As in the proof of Theorem 5.2, one can easily show that

$$d(x,y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \rho_i(x_i, y_i) \quad \forall\, x, y \in \mathbb{P}_{I^+} S_i$$

is a semi-metric on $\mathbb{P}_{I^+} S_i$ which induces the product topology $\mathbb{P}_{I^+} \mathcal{T}_i$.

B.T. Sims has defined in his Thesis [6] the notion of a-metrizability and shown it to be a generalization of semi-metrizability.

Definition 5.4. $\langle S, \mathcal{T} \rangle$ is a-metrizable $(a > 0)$ iff $\exists$ a metric d on $S \ni$ the collection $\{S_d(x;r) \,/\, r > a\}$ is a local basis for $\mathcal{T}$ at x $\forall\, x \in S$. d is called an a-metric. In particular, $\langle S, \mathcal{T} \rangle$ is 0-metrizable iff it is metrizable.

Theorem 5.7. Every subspace $\langle A, A \cap \mathcal{T} \rangle$ of an a-metrizable space $\langle S, \mathcal{T} \rangle$ is a-metrizable.

Proof. Let $d: S \times S \longrightarrow R^+ \cup \{0\}$ be any admissible a-metric for $\langle S, \mathcal{T} \rangle$. The restriction $d_1 = d/A$ of the domain of d to $A \times A$ is clearly a metric on A, since d is a metric on S. As in the proof of Theorem 5.1, the collection $\{S_{d_1}(x;r) \,/\, r > a\}$ is seen to be a local basis for $A \cap \mathcal{T}$ at x $\forall\, x \in A$. Hence, d_1 is an admissible a-metric for $\langle A, A \cap \mathcal{J} \rangle$.

Example 5.4. We again consider the space of Example 3.13, which we used to show that the property of being developable is not productive. Each factor is 0-metrizable since it is metrizable. However, $\forall\, a \geqslant 0$ the product space is not a-metrizable since it is not first countable ([6] , p. 13).

In proving that a-metrizability is weakly productive, we shall make use of the following lemma.

Lemma 5.1. $\langle S, \mathcal{J} \rangle$ is a-metrizable $(a > 0)$ iff $\langle S, \mathcal{J} \rangle$ is 1-metrizable.

Proof. Let d be an admissible a-metric $(a > 0)$ for $\langle S, \mathcal{J} \rangle$ Thus, $\{S_d(x;r) \;/\; r > a\}$ is a local basis at x $\forall\, x \in S$. Let $\rho(x,y) = \frac{1}{a}\, d(x,y)\ \forall\, x,y \in S$. ρ is a metric on S since d is . Clearly, $\{S_d(x;r) \;/\; r > a\} = \{S_{d(\frac{1}{a})}(x;\frac{r}{a}) \;/\; \frac{r}{a} > \frac{a}{a}\} = \{S_\rho(x;r_1) \;/\; r_1 > 1\}$, where $r_1 = \frac{r}{a}$. Hence, ρ is an admissible 1-metric for $\langle S, \mathcal{J} \rangle$.

Theorem 5.8. If $\langle S_i, \mathcal{J}_i \rangle$ is 1-metrizable $\forall\, i \in I^+$, then the product space $\langle \mathbb{P}_{I^+} S_i, \mathbb{P}_{I^+} \mathcal{J}_i \rangle$ is 1-metrizable.

Proof. For each $i \in I^+$, let ρ_i be an admissible 1-metric for $\langle S_i, \mathcal{J}_i \rangle$, and let d_i be the admissible, bounded $\frac{1}{2}$-metric for $\langle S_i, \mathcal{J}_i \rangle$ given by $d_i(x_i,y_i) = \dfrac{\rho_i(x_i,y_i)}{1+\rho_i(x_i,y_i)}$ $\forall\, x_i,y_i \in S_i$. Let $d(x,y) = \sup\,\{d_i(x_i,y_i) \;/\; i \in I^+\}$ $\forall\, x,y \in \mathbb{P}_{I^+} S_i$. We show that d is a metric on $\mathbb{P}_{I^+} S_i$. (i) $d(x,y) = 0$ iff $\sup\,\{d_i(x_i,y_i) \;/\; i \in I^+\} = 0$ iff $x_i = y_i\ \forall\, i \in I^+$ iff $x = y$. (ii) $d(x,y) = \sup\,\{d_i(x_i,y_i) \;/\; i \in I^+\} = \sup\,\{d_i(y_i,x_i) \;/\; i \in I^+\} = d(y,x)$. (iii) $d(x,y) + d(y,z) = \sup\,\{d_i(x_i,y_i) \;/\; i \in I^+\}$ +

$\sup\{d_i(y_i,z_i) \ / \ i \in I^+\} \geq \sup\{d_i(x_i,y_i) + d_i(y_i,z_i) \ / \ i \in I^+\} \geq$
$\sup\{d_i(x_i,z_i) \ / \ i \in I^+\} = d(x,z)$.

We next show that $\{S_d(p;r) \ / \ r > \frac{1}{2}\}$ is a local basis for $\mathbb{P}_{I^+} \mathcal{T}_i$ at $p \ \forall \ p \in \mathbb{P}_{I^+} S_i$. Let $p \in \mathbb{P}_{I^+} S_i$ and let $\mathbb{P}_{I^+} U_i$ be a basic open set containing p. Thus, $U_i \neq S_i$ only for $i = i_1, \ldots, i_n$. Since d_i is an admissible $\frac{1}{2}$-metric for $\langle S_i, \mathcal{T}_i \rangle$, $\exists \ r_i > \frac{1}{2} \ni p_i \in S_{d_i}(p_i;r_i) \subset U_i$, $i = i_1, \ldots, i_n$. Let $r = \min\{r_i \ / \ i = i_1, \ldots, i_n\}$. Clearly, $r > \frac{1}{2}$. If $q \in S_d(p;r)$, then $d_i(p_i,q_i) \leq d(p,q) < r \leq r_i$, $i = i_1, \ldots, i_n$. It follows that $q_i \in S_{d_i}(p_i;r_i)$ for $i = i_1, \ldots, i_n$, and $q \in \mathbb{P}_{I^+} U_i$. Thus, $p \in S_d(p;r) \subset \mathbb{P}_{I^+} U_i$, and $\{S_d(p;r) \ / \ r > \frac{1}{2}\}$ is a local basis for $\mathbb{P}_{I^+} \mathcal{T}_i$ at $p \ \forall p \in \mathbb{P}_{I^+} S_i$. Thus, $\langle \mathbb{P}_{I^+} S_i, \mathbb{P}_{I^+} \mathcal{T}_i \rangle$ is $\frac{1}{2}$-metrizable. Hence, it is 1-metrizable by Lemma 5.1. Indeed, if we let $\rho(x,y) = 2\,d(x,y) \ \forall \ x,y \in \mathbb{P}_I + S_i$, then ρ is an admissible 1-metric for $\langle \mathbb{P}_{I^+} S_i, \mathbb{P}_{I^+} \mathcal{T}_i \rangle$.

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