SOME GROUPS WITH
DEFINING RELATIONS
$a^m = b^n = (ab)^m = e,$
$ba^2 = a^2 b^k$

By

Gaydzag Boyadjian

Submitted in Partial Fulfillment for the
Requirements of the Degree Master of Science
in the Mathematics Department of the
American University of Beirut
Beirut, Lebanon,
1967

SOME GROUPS WITH
DEFINING RELATIONS
$a^m = b^n = (ab)^m = e,$
$ba^2 = a^2 b^k$

Gaydzag Boyadjian

AKNOWLEDGEMENT

The writer wishes to express his deepest gratitude to Professor Peter Yff, whose encouragement and most valuable advice and guidence during the years 1966, 1967 made this work possible.

He is also indepted to Miss Mona Jabbour for her patience and skill in typing the manuscript.

ABSTRACT

This paper is a treatment of groups having two generators with the following relations:

$$a^m = b^n = (ab)^m = e, \; ba^2 = a^2b^k.$$

The emphasis is on finding some important subgroups, specifically the commutator subgroup.

The illustrative examples, however, are chosen to clarify the techniques we use to find the above subgroups.

The first chapter is an introduction to the Systematic Enumeration of Cosets, among other results, with an illustrative example. This forms the basis on which the properties of the groups in the succeeding chapters depend. Although the first part of the second chapter deals with finite groups, mainly the groups are infinite.

TABLE OF CONTENTS

CHAPTER I

INTRODUCTION

In this paper we shall begin a study of the structure of a non-abelian group G generated by a and b with defining relations $a^m = b^n = (ab)^m = e$, $ba^2 = a^2 b^k$.

In a defining relation such as $x^n = e$ it is assumed that the order of x is exactly n. Powers of a and b may always be reduced to their least non-negative residues (mod m and mod n respectively). However it is convenient to write $k = -1$ instead of $k = n - 1$.

Certain subgroups, such as commutator subgroups, will be investigated.

The case in which m is odd may be partially disposed of as follows. The order of a^2 is m, since $(2,m) = 1$, so $a^m = e$ may be replaced by $(a^2)^m = e$. Also, from $ba^2 = a^2b^k$ we obtain $ba^{2r} = a^{2r}b^{k^r}$ (r any integer), so that

$$b = ba^{2m} = a^{2m}b^{k^m} = b^{k^m},$$

which requires $k^m = 1 (\text{mod } n)$, since b has order n. This gives $k^m - 1 = (k - 1)(k^{m-1} \quad k^{m-2} \quad \dots \quad k \quad 1) = 0 (\text{mod } n)$. If $k = 1 (\text{mod } n)$ then b commutes with a^2, but $a = (a^2)^{\frac{m \quad 1}{2}}$, so $ba = ab$ and G is abelian, which will not be considered in this paper. Let k now be chosen so that

$$k^{m-1} \quad k^{m-2} \quad \dots \quad k \quad 1 = 0 (\text{mod } n).$$

Then $(a^2b)^m = a^{2m}\, b^{k^{m-1} + k + \dots \quad k + 1} = e$, and a^2b has order m because no smaller power of a^2 equals e. Therefore in this case G is generated by a^2 and b with the defining relations

$$(a^2)^m = b^n = (a^2b)^m = e,\ ba^2 = a^2b^k,$$

which is equivalent to a case studied by Basmaji 1 .

It may also happen that $k^m = 1 (\mathrm{mod}\ n)$ without either factor of k^{m-1} being congruent to zero (e.g., $m = 3$, $n = 26$, $k = 3$). However, since this paper will deal mainly with cases which can be generalized for all n, from now on m will always be even.

The condition that k should satisfy is

$$k^{m/2} = 1 (\mathrm{mod}\ n) \text{ since } b = ba^m = a^m b^{k^{m/2}} = b^{k^{m/2}} .$$

($\frac{m}{2}$ is always an integer since m is even). But $\frac{m}{2}$ is even or odd depending on whetehr $m = 0 (\mathrm{mod}\ 4)$ or $m = 2 (\mathrm{mod}\ 4)$ respectively. If $\frac{m}{2}$ is even, clearly $k = \quad 1$ is a solution of $k^{m/2} = 1 (\mathrm{mod}\ n)$ for all positive integer n. On the other hand if $\frac{m}{2}$ is odd then for all positive integers $n \quad 2$ $k = -1$ is not a solution of $k^{m/2} = 1 (\mathrm{mod}\ n)$ since $-1 \not\equiv 1 (\mathrm{mod}\ n)$. In case $n = 2$, $-1 = 1 (\mathrm{mod}\ 2)$ and hence they are equivalent. For some values of n there are other solutions of $k^{m/2} = 1 (\mathrm{mod}\ n)$. For example, if $m = 4$ and $n = 20$, there exist the additional solutions $k = 9$ and $k = 11$. However, it will be shown that the cases $k = \quad 1$ may be generalized for all relevant values of n, which is impossible for the other cases.

In this paper we will use two results, one of which is due to Dyck [2] which is: The effect of adding new relations to the abstract definition of a group G is to form a new group G' which is a factor group of G. By <u>abstract definition of G</u> we mean "A set of relations

$$g_k(s_1, s_2, \dots s_m) = e \quad (k = 1,2,3,\dots s)$$

satisfied by the generators $s_1, s_2 \dots s_m$ of G, is called an abstract definition of G if every relation satisfied by the generators is an algebraic consequence of these particular relations". (Coxeter [3,p.1]).

Let us specialize this definition to the case where the generators of the group G are a and b and let the abstract definition of G be the following set of relations.

$$a^m = b^n = (ab)^m = e, \ ba^2 = a^2b^k.$$

or

$$a^m = b^n = (ab)^m = a^{-2}\ ba^2b^{n-k} = e.$$

Now, according to Dyck's result, if we add the relation $ab = ba$ (or $a^{-1}b^{-1}ab = e$) to the above relations we get a factor group of G. Moreover, this factor group is abelian since the generators commute. Obviously, if no further relations are imposed, we will get the largest abelian factor group of G which is the commutator factor group of G.

So adding $ab = ba$ to the relations $a^m = b^n = (ab)^m = e$ $ba^2 = a^2b^k$ we get the defining relations of the commutator factor group of G.

The second result is due to Todd and Coxeter [3,p.12]. The Systematic Enumeration of Cosets which is described and illustrated below.

The Todd-Coxeter Method of Systematic Enumeration of Cosets.

If H is a subgroup of G and $a^m = e$ is a defining relation of G, there are at most m cosets of the type Ha^i. We indicate this by a table headed by the element a written m times. These a's serve to separate m + 1 columns from each other. The number 1, denoting the coset H, is entered at the top of the first column. Then the entry at the top of column $i(i = 2,3,\ldots,m + 1)$ should denote the coset Ha^{i-1}. If $Ha = H$, the number 1 will appear at the top of every column. If $Ha \neq H$, a number different from 1 will appear at the top of column 2 (2 may be used if it has not been used for a coset other than Ha). In any case, 1 must be entered at the top of the last column. For example, if $a^6 = e$, but there are only three cosets Ha^i, the first row may be filled in as follows:

$$\begin{array}{ccccccccccccc} & a & & a & & a & & a & & a & & a & \\ 1 & & 2 & & 3 & & 1 & & 2 & & 3 & & 1 \end{array}$$

If other cosets exist, a new row may be started with one of them, and this process may be repeated until there are no more cosets.

Simultaneously, the procedure is carried out for every defining relation, any information obtained from one table being entered whenever possible in other tables. Care must be taken to ensure that each coset appears in all possible non-equivalent columns. In the above example, all positions are equivalent. However, if $(a^2b)^2 = e$, the table headed by

a a b a a b

has three essentially different columns. Columns 1,4, and 7 are equivalent, columns 2 and 5 are equivalent, and columns 3 and 6 are equivalent. If $a^2bab = e$, all columns are different, except that the first and last columns are always regarded as identical.

When all the tables are complete, namely all the rows and columns are filled up, the process is at an end.

The interpretation of the above process is the following: If the tables are complete after m cosets have been entered, then the index of H is a divisor of m. If no coset has been inadvertently repeated, the index of H is exactly m. Hence if H is of order n, the order of G is mn. The process may even be applied when G is infinite, provided that H has finite index in G.

Example:

$$G = \{a, b\}$$

Let

$$a^4 = b^3 = (ab)^4 = e$$

and

$$ba^2 = a^2b^{-1}.$$

The above is equivalent to $a^4 = b^3 = (ab)^4 = ba^2ba^2 = e$ let $H = \{a^3ba, b\}$. It will be shown later that a^3ba and b commute and hence the order of H is $9 = 3^2$ and its elements are $a^3b^iab^j$ i, j = 0,1,2.

The tables with 1 = H inseted are

```
 a   a   a   a     b   b   b     a   b   a   b   a   b   a   b
1             1  1   1   1   1  1                          1   1


                    b   a   a   b   a   a
                   1  1                   1
                          1
```

Note that the symbol 1 appears in each table in every essentially different position. We define 2 = 1.a and insert this into the tables to obtain

```
   a   a   a   a      b   b   b      a   b   a   b   a   b   a   b
 1   2           1  1   1   1   1   1  2                       1   1
                    2           2   2                              2

                     b   a   a   b   a   a
                   1   1   2               1
                           1   2
                       2
```

Note that in the second, third and fourth tables we have placed a 2 in every essentially different position. Define $3 = 2.a = 1.a^2$.

	a		a		a		a	
1		2		3				1

	b		b		b	
1		1		1		1
2						2
3						3

	a		b		a		b		a		b		a		b	
1		2												1		1
2		3														2
3																3

	b		a		a		b		a		a	
1		1		2		3						1
				1		2						
		2		3								
		3										

Similarly here we have inserted 3 in the tables in every essentially different position.

Define 4 = 3a and hence from the first table we get 4.a = 1.

	a		a		a		a	
1		2		3		4		1

	b		b		b	
1		1		1		1
2						2
3						3
4						4

	a		b		a		b		a		b		a		b	
1		2										4		1		1
2		3														2
3		4														3

	b		a		a		b		a		a	
1		1		2		3		3		4		1
		4		1		2						
		2		3		4						
		3		4		1						

And from the last table we get 3b = 3.

Hence the tables will be

	a	a	a	a		b	b	b		a	b	a	b	a	b	a	b
1	2	3	4	1	1	1	1	1	1	2					4	1	1
					2			2	2	3	3	4					2
					3	3	3	3	3	4					2	3	3
					4			4									

	b	a	a	b	a	a
1	1	2	3	3	4	1
	4	1	2			
	2	3	4			
3	3	4	1	1	2	3

Now the elements of the coset 4 are $a^3b^iab^ja^3$ and the elements of 4b are $a^3b^kab^\ell a^3b$, where i,j,k, ℓ = 0,1,2, if i = j = 0. Then a^3 is in 4. Also if ℓ = 0 and k = 2 then a^3 is in 4b. Therefore 4 = 4b (since two cosets are either disjoint or identical).

Hence we will have

	a	a	a	a		b	b	b		a	b	a	b	a	b	a	b
1	2	3	4	1	1	1	1	1	1	2	2	3	3	4	4	1	1
					2			2	2	3	3	4	4	1	1	2	2
					3	3	3	3	3	4	4	1	1	2	2	3	3
					4	4	4	4									

	b	a	a	b	a	a
1	1	2	3	3	4	1
4	4	1	2			4
2	2	3	4	4	1	2
3	3	4	1	1	2	3

and it is obvious that 2b = 2. Hence the final situation is

	a		a		a		a			b		b		b			a		b		a		b		a		b		a		b	
1		2		3		4		1	1		1		1		1	1		2		2		3		3		4		4		1		1
									2		2		2		2	2		3		3		4		4		1		1		2		2
									3		3		3		3	3		4		4		1		1		2		2		3		3

	b		a		a		b		a		a	
1		1		2		3		3		4		1
4		4		1		2		2		3		4
2		2		3		4		4		1		2
3		3		4		1		1		2		3

so after defining 1 = H, 2 = 1.a, 3 = 2.a 4 = 3.a the tables close up and hence the order of G, which is generated by a and b with the defining relations

$$a^4 = b^3 = (ab)^4 = e\,,\ ba^2 = a^2b^{-1},$$

is the order of H times 4 (because H is of index 4); hence G is of order 4 x 9 = 36.

CHAPTER II

1. Introduction

In this chapter we shall study the structure of the non-abelian group G generated by a and b with defining relations:

$$a^4 = b^n = (ab)^4 = e \quad ba^2 = a^2b^k$$

where $k = \ 1$. The commutator subgroup of G will be investigated, and it turns out to be contained in the subgroup b, a^3ba (i.e, the subgroup generated by b, and a^3ba).

The tables of the systematic enumeration of cosets will be provided at the end of the chapter.

2. An Example

Let $n = 5$ and $k = -1$. Then G is finite and of order 100 since by the systematic enumeration of cosets, if we define $1 = \{b\}$, we need 20 cosets to close up the tables. (See Table 2.1).

The order of G is the order of b times its index. So G is of order $5 \times 20 = 100$. This result may be generalized as follows.

Theorem 2.1

If $m = 4$ and $k = -1$, then G is of order $4n^2$, where n is any positive integer.

Before proving this theorem we prove a Lemma.

Lemma 2.2

The subgroup N generated by b and a^3ba is of order n^2

Proof:

The order of b is n. Also the order of a^3ba is n, since a^3ba is the conjugate of b. Moreover, a^3ba and b commute, because

$$a^3bab = a^2.abab$$

$$= a^2.b^{-1}a^{-1}b^{-1}a^{-1} \qquad \text{since } (ab)^4 = e.$$

$$= ba^2a^3b^{-1}a^3$$

$$= ba^2aba^2a^3$$

$$= ba^3ba.$$

Hence N is commutative since its generators commute. This commutativity means that $N = \{b^i(a^3ba)^j / i,j = 0,1,\dots,n-1\}$. Therefore the order of N is n^2 This completes the proof of the Lemma.

Proof of Theorem 2.1

Let $1 = N = \{b, a^3ba\}$. Then after defining four cosets the tables will close up. (See table 2.2).

Therefore the order of G is $4n^2$. This completes the proof of the theorem. Note that the cosets above form a cyclic group $\{Na\}$ of order four.

Corollary 2.3

The group G (Theorem 2.1) consists of all elements of the form $b^ia^3b^ja^r$ ($r = 0,1,2,3$; $i,j = 0,1,\dots n-1$).

Proof

Every element of G is in a coset Na^{r-1}. Since an element of N may be written $b^i(a^3ba)^j = b^ia^3b^ja$, the result follows.

Theorem 2.4

$N = \{b, a^3ba\}$ is a normal subgroup of G when $m = 4$, $k = -1$.

Proof:

We need to prove only that $aN = Na$, since b commutes with N and every element of G is of the form $b^iab^ja^r$. But every element of N is of the form $b^ia^3b^ja$ and hence an element of Na is of the form $b^ia^3b^ja^2$.

$$b^ia^3b^ja^2 = b^ia^3.a^2b^{-j} = b^iab^{-j}.$$

Also, an element of aN is of the form $ab^pa^3b^qa$. But $ab^pa^3b^qa = a.a^3b^qab^p$, since a^3ba and b commute, and $a.a^3b^qab^p = b^qab^p$ is an element in Na. Hence $aN = Na$, and the result follows.

Let C be the commutator subgroup of G. Then $x,y \in G$ imply $x^{-1}y^{-1}xy \in C$. But $x,y \in G$ implies that

$$x = b^ia^3b^ja^{r_1}\ , \quad y = b^ka^3b^\ell a^{r_2};$$

where

$$r_1,\ r_2 = 0,1,2,3;$$

$$i,j,k,\ell = 0,1,\ \dots\ n-1.$$

Also ,
$$x^{-1} = a^{-r_1}b^{-j}ab^{-i}\ , \quad y^{-\ell} = a^{-r_2}\,b^{-\ell}ab^{-k}.$$

Hence,
$$x^{-1}y^{-1}xy = a^{-r_1}b^{-j}ab^{-i}a^{-r_2}b^{-\ell}ab^{-k}b^ia^3b^ja^{r_1}b^ka^3b^\ell a^{r_2}.$$

Note that b^i and a^3b^ia commute (Lemma 2.2) so:

If

$$r_1 = 0,\ r_2 = 0,$$

then

$$x^{-1}y^{-1}xy = b^{-j}ab^{-i-\ell}ab^{i-k}a^3b^{j+k}a^3b^{\ell}$$

$$= b^{-j}ab^{-i-\ell}ab^{i-k}a^3b^{j+k}ab^{-\ell}a^2$$

$$= b^{-j}ab^{-i-\ell}ab^{i-k-\ell}a^3b^{j+k}a^3$$

$$= b^{-j}ab^{-i-\ell}a^3b^{k+\ell-i}ab^{j+k}a^3$$

$$= b^{-j}ab^{j+k-i-\ell}a^3b^{k+\ell-i}$$

$$= b^{k+1-i-j}a^3b^{i+\ell-j-k}a.$$

If

$$r_1 = 0,\ r_2 = 1$$

then

$$x^{-1}y^{-1}xy = b^{-j}ab^{-i}a^3b^{-\ell}ab^{-k}b^{i}a^3b^{j+k}a^3b^{\ell}a$$

$$= b^{-j-\ell}ab^{k-\ell}a^3b^{k+j}$$

$$= b^{-j-\ell}a^3b^{\ell-k}ab^{k+j}$$

$$= b^{k-\ell}a^3b^{\ell-k}a.$$

Similarly, we have

If $r_1 = 0, r_2 = 2$, then $x^{-1}y^{-1}xy = b^{i-j-k-\ell}a^3b^{i-\ell+j+k}a.$

$r_1 = 0,\ r_2 = 3$ $\quad x^{-1}y^{-1}xy = b^{\ell-k-j-i}a^3b^{2i-\ell-k}a$

$r_1 = 1,\ r_2 = 0$ $\quad x^{-1}y^{-1}xy = b^{j-i}a^3b^{-i-j}a$

$r_1 = 1,\ r_2 = 0$ $\quad x^{-1}y^{-1}xy = b^{j-i}a^3b^{-i-j}a$

$r_1 = 1,\ r_2 = 1$ $\quad x^{-1}y^{-1}xy = e$ (identity element of G)

$r_1 = 1,\ r_2 = 2 \qquad x^{-1}y^{-1}xy = b^{-i-2\ell-j}a^3b^{i-j}a$

$r_1 = 1,\ r_2 = 3 \qquad x^{-1}y^{-1}xy = b^{-2i}a^3b^{-2j}a$

$r_1 = 2,\ r_2 = 0 \qquad x^{-1}y^{-1}xy = b^{j+i-k+\ell}a^3b^{j-i-\ell-k}a$

$r_1 = 2,\ r_2 = 1 \qquad x^{-1}y^{-1}xy = b^{\ell+k}a^3b^{\ell-k}a$

$r_1 = 2,\ r_2 = 2 \qquad x^{-1}y^{-1}xy = b^{j+k-i-\ell}a^3b^{\ell-i-j+k}a$

$r_1 = 2,\ r_2 = 3 \qquad x^{-1}y^{-1}xy = b^{2j+\ell-k}a^3b^{k-\ell-2i}$

$r_1 = 3,\ r_2 = 0 \qquad x^{-1}y^{-1}xy = b^{2\ell+i}a^3b^{i+\ell-2k}a$

$r_1 = 3,\ r_2 = 1 \qquad x^{-1}y^{-1}xy = b^{2k}a^3b^{2\ell}a$

$r_1 = 3,\ r_2 = 2 \qquad x^{-1}y^{-1}xy = b^{i+j-2\ell}a^3b^{j-i+2k}a$

$r_1 = 3,\ r_2 = 3 \qquad x^{-1}y^{-1}xy = b^{2i-2k}a^3b^{2j-2\ell}a$

Clearly the commutator subgroup of G is contained in N, since for all possible values of r_1 and r_2 $x^{-1}y^{-1}xy$ is in N.

Now we shall use Dyck's result which is mentioned in the first chapter.

Add $ab = ba$ to the defining relations of G. We get

$$a^4 = b^n = (ab)^4 = e,\ ba^2 = a^2b^{-1} \text{ and } ab = ba$$

or

$$a^4 = b^n = a^4 b^4 = e, \quad b = b^{-1}$$

or

$$a^4 = b^n = b^4 = e, \quad b^2 = e.$$

Now, if $n \equiv 1 (\bmod 4)$ we get

$$a^4 = b = e \quad \text{or} \quad a^4 = e \ .$$

If $n \equiv 3 (\mathrm{mod}\ 4)$ we get

$$a^4 = b^3 = b^2 = e \quad \text{or} \quad a^4 = e \ .$$

If $n \equiv 0 (\mathrm{mod}\ 4)$ we get

$$a^4 = b^n = b^4 = e \ , \ b^2 = e \ ,$$

or

$$a^4 = b^4 = b^2 = e \quad \text{or} \quad a^4 = b^2 = e \ .$$

If $n \equiv 2 (\mathrm{mod}\ 4)$ we get

$$a^4 = b^2 = e.$$

Hence, if n is odd (i.e. $n \equiv 1, 3 (\mathrm{mod}\ 4)$), the defining relation of G/C, the commutator factor group, is

$$(Ca)^4 = C \qquad ; \qquad (1)$$

and if n is even (i.e. $n \equiv 0, 2 (\mathrm{mod}\ 4)$), the defining relations of G/C are

$$(Ca)^4 = (Cb)^2 = e, \ Ca\ Cb = Cb\ Ca \ . \qquad (2)$$

Now we are ready to state

Theorem 2.5

The commutator subgroup C of G ($m = 4$, $k = -1$) is N if n is odd, and is $\{b^2, a^3b^2a, a^{-1}b^{-1}ab\} = Q$ if n is even.

Proof:

We know that C is contained in N. We also know from above, that the index of C is 4 if n is odd. Moreover the index of N is

4. (See table 2.2)

Now

$$G/N = \{Na^i \,/\, i = 0,1,2,3\},$$

and

$$G/C = \{Ca^i \,/\, i = 0,1,2,3\}. \qquad (1)$$

Since G/N and G/C are both cyclic and of order 4 if n is odd, then G/N is isomorphic to G/C. Therefore we conclude that $C = N$ if n is odd; otherwise the cosets of C would be properly contained in the respective cosets of N and would not contain all the elements of G.

If n is even:

$$G/C = \{Ca^i b^i \,/\, i = 0,1,2,3,\ j = 0,1\}.$$

Obviously $Q = \{b^2, a^3b^2a, a^{-1}b^{-1}ab\}$ is a subgroup of $N = \{b, a^3ba\}$.

Moreover, the index of Q is 8, and the set of right cosets of Q is $\{Qa^ib^j \,/\, i = 0,1,2,3, j = 0,1\}$. (See table 2.3). Obviously the right cosets of Q and C are in one-to-one correspondence. Moreover, $a^{-1}b^{-1}ab$ is in C; and $(Cb)^2 = C$, in the defining relations of G/C, means b^2 is in C. Also $a^{-1}b^2a = a^3b^2a$ is in C since C is normal. So the generators of Q are elements of C. Therefore Q is contained in C.

Hence we conclude that $C = Q$ if n is even. Otherwise the cosets of Q would be properly contained in the respective cosets of C and would not contain all elements of G. This completes the proof of the theorem.

<u>Corollary 2.6</u>

The order of C is n^2 if n is odd and $\frac{n^2}{2}$ if n is even. The proof is obvious.

3. G (m = 4, k = 1)

Let us consider the group G generated by a and b so that the following relations hold:

$$a^4 = b^n = (ab)^4 = e, \quad ba^2 = a^2b.$$

Theorem 2.7

$G(m = 4, k = 1)$ is infinite for every $n > 3$.

Before giving a proof we introduce a result due to Dyck (Coxeter [3, p.61]).

Let S and T be the two generators of a group G, and let $S^m = T^n = (ST)^{\ell} = e$.

Then, if $\frac{1}{n} + \frac{1}{m} + \frac{1}{\ell} > 1$, G is finite; otherwise G is infinite.

Proof of Theorem 2.7

Let $H = \{b, a^3ba\}$ be a subgroup of G. The order of b is n , so $b^n = e$. Also $(a^3ba)^n = e$ since a^3ba is a conjugate of b. The order of a^3bab is 2, since

$$a^3baba^3bab = a^4abababab = e.$$

But

$$\frac{1}{n} + \frac{1}{n} + \frac{1}{2} < 1 \quad \text{if } n > 3.$$

Hence H is infinite if $n > 3$. Therefore G is infinite since H is a subset of G. This completes the proof of the theorem.

Theorem 2.8

The order of $G(m = 4, k = 1)$ is 48 when $n = 3$.

Proof:

Let $1 = \{a\}$, then after defining 12 cosets the tables close up; and since the order of $\{a\}$ is 4, the order of G is 48. (See table 2.4).

4. Normal Subgroups of G(m = 4, k = 1)

Theorem 2.9

$H = \{b, a^3ba\}$ is normal in G and is of index 4.

Proof:

To prove that $H = \{b, a^3ba\}$ is normal in G, note that $b^{-1}Hb = H$, since $b \in H$; $a^{-1}ba \in H$;

$$a^{-1}(a^3ba)a = a^2ba^2 = b \in H;$$

therefore $a^{-1}Ha$ contains the generators of H, so it contains H. Similarly, H contains $a^{-1}Ha$. For any $x \in G$, the operation $H \rightarrow x^{-1}Hx$ consists of taking successive conjugates of H with respect to powers of a and b. Therefore $x^{-1}Hx = H$ and H is normal in G. Moreover, H is of index 4, (see table 2.5) where the cosets of H form a cyclic group.

$$G/H = \{Ha^i \ / \ i = 0,1,2,3\}.$$

This completes the proof of the theorem.

Proposition 2.10

The subgroup $\{a^2, b, a^3ba\}$ is normal.

Proof:

The index of $\{a^2, b, a^3ba\}$ is 2. Hence the result follows. (See table 2.6).

Note that if n is odd, the index of $\{b, aba\}$ is 2 (see table 2.6a). (Hence normal).

Also,

$$b \in \{a^2, b, a^3ba\},$$

and

$$aba = a^2(a^3ba) \in \{a^2, b, a^3ba\}.$$

so

$\{b, aba\}$ is a subgroup of $\{a^2, b, a^3ba\}$.

Moreover, both subgroups have isomorphic (abelian) factor groups. (See tables 2.6 and 2.6a). Hence, we conclude that

$$\{a^2, b, a^3ba\} = \{b, aba\}$$

if n is odd. However, if n is even the index of $\{b, aba\}$ is 4. (See table 2.6b).

So $\{b, aba\}$ is properly contained in $\{a^2, b, a^3ba\}$ if n is even.

The Commutator Subgroup of $G(m = 4, k = 1)$.

Lemma 2.11

The commutator subgroup C of G is a subgroup of $H = \{b, a^3ba\}$.

Proof:

G/H is abelian, (see table 2.5) so C is contained in H.

Let us apply Dyck's result, mentioned in the first chapter, to decide the commutator subgroup of $G(m = 4, k = 1)$.

Adjoin the further relation

$$ab = ba \text{ to } a^4 = b^n = (ab)^4 = e, \ ba^2 = a^2b.$$

We get

$$a^4 = b^n = a^4b^4 = e,$$

which is equivalent to

$$a^4 = b^n = b^4 = e \quad . \qquad (1)$$

If n is odd ,

$$b^n = b^4 = e \text{ implies } b = e.$$

So (1) will reduce to

$$a^4 = e \quad , \qquad (2)$$

and (2) will become the defining relation $(Ca)^4 = C$ of the factor group of the commutator group. Hence

$$G/C = \left\{ Ca^i \;/\; i = 0,1,2,3 \right\}.$$

Also if we consider $H = \left\{ b, a^3ba \right\}$ which is normal and is of index 4, (see table 2.5), the factor group G/H is $\left\{ Ha^i/i = 0,1,2,3 \right\}$ and obviously its defining relation is $(Ha)^4 = H$.

Clearly G/H is isomorphic to G/C since both are cyclic of order 4.

Theorem 2.12

If n is odd, then the commutator subgroup C of G ($m = 4$, $k = 1$) is equal to $H = \left\{ b, a^3 ba \right\}$.

Proof:

G may be partitioned into C, Ca, Ca^2, Ca^3 or H, Ha, Ha^2, Ha^3. Also C is contained in H Lemma 2.11. If $C \neq H$, part of H must lie in a coset $Ca^i \neq C$. But $Ca^i \subset Ha^i \neq H$. Therefore H would intersect Ha^i, which is impossible. Hence $C = H$.

Theorem 2.13

If H and K are subgroups of G, and H is a subgroup of K, and if there is a one-to-one correspondence

$$Hx_i \longrightarrow Kx_i \quad (i = 1,2,\dots t)$$

between the sets of right cosets of H and K then

$H = K$.

<u>Proof:</u>

$H \subset K$ implies $Hx_i \subset Kx_i$ $(i = 1, 2, \dots, t)$. If $H \nleq K$, part of K must lie in a coset $Hx_i \nleq H$. But $Hx_i \subset Kx_i$, so part of K would be in Kx_i, which is disjoint from K. Contradiction. Hence $H = K$.

Since $a^{-1}b^{-1}ab$ and $ab\,a^{-1}b^{-1}$ are in C, so is their product:

$$\begin{aligned}
a^{-1}b^{-1}ababa^{-1}b^{-1} &= a^3b^{-1}ababa^3b^{-1}\\
&= a^3b^{-2}bababa^3b^{-1}\\
&= ab^{-2}a^2bababaa^2b^{-1}\\
&= ab^{-2}babababb^{-1}a^2b^{-1}\\
&= ab^{-2}a(ab)^4b^{-1}a^2b^{-1}\\
&= ab^{-2}a^3b^{-2}\\
&= (ab^{-2}a^3b^2)b^{-4}.
\end{aligned}$$

Since $ab^{-2}a^3b^2$ is a commutator, and since C is a subgroup, b^{-4} and its inverse b^4 are in C.

So for any positive integer n, where n is the order of b, b^4 is in C. If n is odd then b is in C and $C = \{b, a^3ba\}$. Now, if n is even we have to consider two cases;(i) $n \equiv 2(\mathrm{mod}\ 4)$ and (ii) $n \equiv 0(\mathrm{mod}\ 4)$. (i) If $n \equiv 2(\mathrm{mod}\ 4)$ and b^4 is in C, then all the powers of b^4 are elements of C. Among these powers comes also $b^{n+2} = b^2$. Hence b^2 is an element of C. Let us apply Dyck's result on case (i).

Add

$$ab = ba \quad \text{to} \quad a^4 = b^n = (ab)^4 = e,\ ba^2 = a^2b.$$

We get

$$a^4 = b^n = b^4 = e,$$

but

$$b^n = b^4 = e \quad \text{implies} \quad b^2 = e.$$

So we have;

$$a^4 = b^2 = e,\ ab = ba \qquad (3)$$

Hence (3) will become the defining relations $(Ca)^4 = (Cb)^2 = C$, $CaCb = Cb\,Ca$ of the commutator factor group C of G $(m = 4,\ k = 1)$. Therefore $G/C = \{Ca^ib^j \ /\ i = 0,1,2,3,\ j = 0,1\}$ which is of order 8, so the index of C is 8 ,

Theorem 2.14

If $n \equiv 2(\mathrm{mod}\ 4)$, then the commutator subgroup C of $G(m = 4,\ k = 1)$ is $\{b^2, a^3b^2a, a^{-1}b^{-1}ab\} = K$.

Proof:

We showed that b^2 is an element of C. Since C is normal, $a^{-1}b^2a = a^3b^2a$ is also an element of C. Also $a^{-1}b^{-1}ab$ is an element of C. Since each of the generators of K is an element of C, then K is a subgroup of C. But the index of K is 8. (See table 2.7).

Moreover, the set of right cosets of K is

$$\{Ka^ib^j \;/\; i = 0,1,2,3;\; j = 0,1\} .$$

Also

$$G/C = \{Ca^ib^j \;/\; i = 0,1,2,3;\; j = 0,1\}.$$

Obviously, there is a one-to-one correspondence between the right cosets of K and C. So, by theorem 2.13, K = C

namely

$$C = \{b^2,\; a^3b^2a,\; a^{-1}b^{-1}ab\}.$$

This completes the proof of the theorem.

(ii) If $n \equiv 0 \pmod 4$ then b^4 is in C.

Now add

$$ab = ba \quad\text{to}\quad a^4 = b^n = (ab)^4 = e,\; ba^2 = a^2b.$$

We get

$$a^4 = b^n = b^4 = e,\quad ab = ba$$

or

$$a^4 = b^4 = e,\quad ab = ba \tag{4}$$

Hence (4) will become the defining relations $(Ca)^4 = (Cb)^4 = C$, $Ca\, Cb = Cb\, Ca$ of the commutator factor group of $G(m = 4,\, k = 1)$.

Therefore

$$G/C = \{Ca^ib^j \;/\; i,\; j = 0,1,2,3\}$$

which is of order 16, so the index of C is 16.

Theorem 2.15

If $n \equiv 0 \pmod 4$, then the commutator subgroup C of $G(m = 4,\, k= 1)$ is

$$\{b^4, a^3b^4a,\; a^{-1}b^{-1}ab,\; ab\, a^{-1}b^{-1}\} = Q$$

Proof:

By a similar argument as in the proof of the previous theorem we have $b^4 \in C$, $a^3 b^4 a \in C$, $a^{-1} b^{-1} ab \in C$, and also $aba^{-1}b^{-1} \in C$ since they are commutators. Hence Q is a subgroup of C. But the index of Q is 16. (See table 2.8).

Moreover, the set of right cosets is

$$\{ Qa^ib^j \ / \ i, \ j = 0,1,2,3.]$$

Also

$$G/C = \{Ca^ib^j \ / \ i, \ j = 0,1,2,3\}.$$

Obviously, there is a one-to-one correspondence between the right cosets of Q and C.

So by theorem 2.12 $Q = C$,

namely

$$C = \{b^4, \ a^3b^4a, \ a^{-1}b^{-1}ab, \ aba^{-1}b^{-1}\}.$$

This completes the proof of the theorem.

To summarize the results of these three theorems, we may state the following:

Let $G(m = 4, k = 1)$ be the group generated by a and b, with defining relations:

$$a^4 = b^n = (ab)^4 = e, \ ba^2 = a^2b.$$

Then for $n \equiv 1, 3 (\text{mod } 4)$, $n \equiv 2 (\text{mod } 4)$, $n \equiv 0 (\text{mod } 4)$ the commutator subgroup of G is

$$C = \{b, \ a^3ba\}$$

$$C = \{b^2, \ a^3 \ b^2a, \ a^{-1}b^{-1}ab\}$$

$$C = \{b^4, \ a^3b^4a, \ a^{-1}b^{-1}ab, \ aba^{-1}b^{-1}\}$$

respectively.

5. A Note on the Indices of the Subgroups of $G(m = 4)$.

What the systematic enumeration of cosets tells us about the index of a subgroup H of G is the following: If after defining m cosets of H (where $1 = H$) the tables close up, then the index of H is a divisor of m. On the other hand, through Dyck's result, we can decide the exact index of the commutator subgroup C. Moreover, if H is a subset of C then the index of H is greater than or equal to the index of C.

Throughout the chapter we have manipulated the multiple of the index of N, H, K, Q, etc. and for each case we have shown, independent of the tables, that the commutator subgroup contains it, and it has happened that the multiple of the index we have found coincides with the exact index of the commutator subgroup. Hence we conclude that the index of the commutator subgroup is the proper number sought. For example, through table 2.5 we have the multiple of the index of $H = \{b, a^3ba\}$ is 4, where H is a subgroup of $G(m = 4, k = 1)$. And we have shown that if n is odd the commutator subgroup C of G contains H. But the index of C is 4. Hence we conclude that the index of H is not merely a divisor of 4 but is exactly 4.

TABLE 2.1

1 ={b}= 4.a
2 = 1.a = 7.b
3 = 2.a = 3.b
4 = 3.a = 11 b
5 = 4b = 8.a
6 = 5a = 6b

7 = 6a = 15b
8 = 7.a = 8b
9 = 2b = 12a
10 = 9a = 10b
11 = 10a = 17.b
12 = 11a = 12b

13 = 5b = 18a
14 = 13a = 14b
15 = 14a = 16b
16 = 9b = 19a
17 = 13b = 20a
18 = 15a = 18b
19 = 17a = 19b
20 = 16a = 20b

$$a^4 = b^5 = e.$$

A

	a		a		a		a	
1		2		3		4		1
5		6		7		8		5
9		10		11		12		9
13		14		15		18		13
16		20		17		19		16
17		19		16		20		17

B

	b		b		b		b		b	
1		1		1		1		1		1
2		9		16		15		7		2
3		3		3		3		3		3
4		5		13		17		11		4
6		6		6		6		6		6
8		8		8		8		8		8
10		10		10		10		10		10
12		12		12		12		12		12
14		14		14		14		14		14
18		18		18		18		18		18
19		19		19		19		19		19
20		20		20		20		20		20

C

	a		b		a		b		a		b		a		b	
1		2		9		10		10		11		4		1		1
2		3		3		4		5		6		6		7		2
3		4		5		6		6		7		2		3		3
4		1		1		2		9		10		10		11		4
8		5		13		14		14		15		7		8		8
14		15		7		8		8		5		13		14		14
12		9		16		20		20		17		11		12		12
11		12		12		9		16		20		20		17		11
18		13		17		19		19		16		15		18		18
15		18		18		13		17		19		19		16		15
19		16		15		18		18		13		17		19		19

D

	b		a		a		b		a		a	
1		1		2		3		3		4		1
11		4		1		2		9		10		11
7		2		3		4		5		6		7
3		3		4		1		1		2		3
8		8		5		6		6		7		8
15		7		8		5		13		14		15
6		6		7		8		8		5		6
12		12		9		10		10		11		12
9		16		20		17		11		12		9
10		10		11		12		12		9		10
18		18		13		14		14		15		18
13		17		19		16		15		18		13
14		14		15		18		18		18		14
16		15		18		13		17		19		16
15		7		8		5		13		14		15
19		19		16		20		20		17		19
20		20		17		19		19		16		20

$$a^4 = b^5$$

$$a^4 = b^5 = e.$$

TABLE 2.2

				A				
	a		a		a		a	
1		2		3		4		1

						B						
	b		b		b	...	b		b		b	
1		1		1		1		1		1		1
2		2		2		2		2		2		2
3		3		3		3		3		3		3
4		4		4		4		4		4		4

								C								
	a		b		a		b		a		b		a		b	
1		2		2		3		3		4		4		1		1
2		3		3		4		4		1		1		2		2
3		4		4		1		1		2		2		3		3

						D						
	b		a		a		b		a		a	
1		1		2		3		3		4		1
4		4		1		2		2		3		4
2		2		3		4		4		1		2
3		3		4		1		1		2		3

$1 = \{a^3ba, b\} = 4.a = 1.b$

$2 = 1.a = 2b$

$3 = 2.a = 3.b$

$4 = 3.a = 4.b$

TABLE 2.3

$n \equiv 2(\mathrm{mod}\ 4)$

				A				
	a		a		a		a	
1		2		3		4		1
5		6		7		8		5
6		7		8		5		6

						B						
	b		b		b	...	b		b		b	
1		5		1		5...5		1		5		1
2		6		2		6...6		2		6		2
3		7		3		7...7		3		7		3
4		8		4		8...8		4		8		4

								C								
	a		b		a		b		a		b		a		b	
1		2		6		7		3		4		8		5		1
4		1		5		6		2		3		7		8		4
2		3		7		8		4		1		5		6		2
3		4		8		5		1		2		6		7		3

						D						
	a		a		b		a		a		b	
4		1		2		6		7		8		4
5		6		7		3		4		1		5
1		2		3		7		8		5		1
2		3		4		8		5		6		2
8		5		6		2		3		4		8
6		7		8		4		1		2		6

$1 = \{a^3b^2a, b^2, a^{-1}b^{-1}ab\} = 4.a = 5.b$
$2 = 1.a = 2b^2. = 6.b$
$3 = 2a = 3.b^2 = 7.b$
$4 = 3.a = 4.b^2 = 8.b$
$5 = 1.b = 8a$
$6 = 2b = 5a$
$7 = 3.b = 6.a$
$8 = 4.b = 7.a$

To show that $8a = 5$

$8a = 4.ba = 10^3ba = 1.a^3ba^2.\ a^3b^{-1}a.$
$= 1.a^3b^{-1}a.$
$8ab = 1.a^3b^{-1}ab$
$= 1.a^{-1}b^{-1}ab$
$= 1. = 5.b$
$8a = 5.$

TABLE 2.2

```
        A                           B                                 C                                 D
  a   a   a   a         b   b   b ... b     b   b       a   b   a   b   a   b   a   b         b   a   a   b   a   a
1   2   3   4   1     1   1   1   1       1   1   1   1   2   2   3   3   4   4   1   1     1   1   2   3   3   4   1
                      2   2   2   2       2   2   2   2   3   3   4   4   1   1   2   2     4   4   1   2   2   3   4
                      3   3   3   3       3   3   3   3   4   4   1   1   2   2   3   3     2   2   3   4   4   1   2
                      4   4   4   4       4   4   4                                         3   3   4   1   1   2   3
```

$1 = \{a^3ba, b\} = 4.a = 1.b$

$2 = 1.a = 2b$

$3 = 2.a = 3.b$

$4 = 3.a = 4.b$

TABLE 2.3

n ≡ 2(mod 4)

```
        A                           B                                 C                                 D
  a   a   a   a         b   b   b ... b     b   b       a   b   a   b   a   b   a   b         a   a   b   a   a   b
1   2   3   4   1     1   5   1   5...5   1   5   1   1   2   6   7   3   4   8   5   1     4   1   2   6   7   8   4
5   6   7   8   5     2   6   2   6...6   2   6   2   4   1   5   6   2   3   7   8   4     5   6   7   3   4   1   5
6   7   8   5   6     3   7   3   7...7   3   7   3   2   3   7   8   4   1   5   6   2     1   2   3   7   8   5   1
                      4   8   4   8...8   4   8   4   3   4   8   5   1   2   6   7   3     2   3   4   8   5   6   2
                                                                                            8   5   6   2   3   4   8
                                                                                            6   7   8   4   1   2   6
```

$1 = \{a^3b^2a, b^2, a^{-1}b^{-1}ab\} = 4.a = 5.b$
$2 = 1.a = 2b^2. = 6.b$
$3 = 2a = 3.b^2 = 7.b$
$4 = 3.a = 4.b^2 = 8.b$
$5 = 1.b = 8a$
$6 = 2b = 5a$
$7 = 3.b = 6.a$
$8 = 4.b = 7.a$

To show that 8a = 5

$8a = 4.ba = 10^3ba = 1.a^3ba^2.\ a^3b^{-1}a.$
$= 1.a^3b^{-1}a.$
$8ab = 1.a^3b^{-1}ab$
$= 1.a^{-1}b^{-1}ab$
$= 1. = 5.b$
$8a = 5.$

TABLE 2.4

A

	a	a	a	a
1	1	1	1	1
2	4	2	4	2
3	5	3	5	3
6	7	6	7	6
8	10	8	10	8
9	11	9	11	9
12	12	12	12	12

B

	b	b	b
1	2	3	1
4	6	9	1
5	8	7	5
10	11	12	10

C

	a	b	a	b	a	b	a	b
1	1	2	4	6	7	5	3	1
4	2	3	5	8	10	11	9	4
7	6	9	11	12	12	10	8	7

D

	b	b	a	a	b	a	a
1	2	3	5	3	1	1	1
2	3	1	1	1	2	4	2
3	1	2	4	2	3	5	3
6	9	4	2	4	6	7	6
4	6	9	11	9	4	2	4
8	7	5	3	5	8	10	8
5	8	7	6	7	5	3	5
9	4	6	7	6	9	11	9
7	5	8	10	8	7	6	7
11	12	10	8	10	11	9	11
10	11	12	12	12	10	8	10
12	10	11	9	11	12	12	12

1 = {a} = 3.b
2 = 1.b
3 = 2b = 5a
4 = 2.a = 9.b
5 = 3a = 7.b
6 = 4.b = 7.a
7 = 6.a = 8.b

8 = 5.b = 10.a
9 = 6.b = 11.a
10 = 8.a = 12.b
11 = 10.b = 9.a
12 = 11.b = 12a

TABLE 2.5

															n factors																					
				A											B														C							
	a		a		a		a			b		b		b	...	b		b		b			a		b		a		b		a		b		a	b
1		2		3		4		1	1		1		1		1...1		1		1		1	1		2		2		3		3		4		4		1 1
									2		2		2		2...2		2		2		2	2		3		3		4		4		1		1		2 2
									3		3		3		3...3		3		3		3															
									4		4		4		4...4		4		4		4															

											D									
											n - 1 factors									
	b		b		b		...		b		a		a		b		a		a	
1		1		1		1	...	1		1		2		3		3		4		1
2		2		2		2	...	2		2		3		4		4		1		2
3		3		3		3	...	3		3		4		1		1		2		3
4		4		4		4	...	4		4		1		2		2		3		4

$1 = \}\, b,\ a^3ba\, \{ = 4.a$

$2 = 1a = 2.b$

$3 = 2a = 3.b$

$4 = 3a = 4b$

TABLE 2.6

```
        A                         B                                 C
  a   a   a   a       b   b   b  ...  b   b   b       a   b   a   b   a   b   a   b
1   2   1   2   1   1   1   1  1...  1  1   1   1   1   2   2   1   1   2   2  1 1
                    2   2   2  2...  2  2   2   2
```

```
                            D
  b   b   b   ...   b   b   b   a   a   b   a   a
1   1   1   1...  1   1   1   1   2   1   1   2   1
2   2   2   2...  2   2   2   2   1   2   2   1   2
```

$1 = \{a^2, b, a^3ba\} = 2a$

$2 = 1a = 2b$

TABLE 2.6a

A

	a		a		a		a	
1		2		2		1		1

B
n factors

	b		b		b	...	b		b		b	
1		1		1		1...1		1		1		1
2		2		2		2...2		2		2		2

C
n-1 factors

	a		b		a		b		a		b		a		b	
1		2		2		1		1		2		2		1		1

D
n-1 factors

	b		b		b	...	b		b		b		a		a		b		a		a	
1		1		1		1...1		1		1		1		2		1		1		2		1
2		2		2		2...2		2		2		2		1		2		2		1		2

$1 = \{b, aba\}$

$2 = 1.a = 2b$

TABLE 2.6b

n is even

A

	a		a		a		a	
1		2		4		3		1

B
n factors

	b		b		b	...	b		b		b	
1		1		1		1...1		1		1		1
2		2		2		2...2		2		2		2
3		3		3		3...3		3		3		3
4		4		4		4...4		4		4		4

C

	a		b		a		b		a		b		a		b	
1		2		3		1		1		2		3		1		1
2		4		4		3		2		4		4		3		2

D
n-1 factors

	b		b		...		b		b		a		a		b		a		a	
1		1		1	...	1		1		1		2		4		4		3		1
2		3		2	...	3		2		3		1		2		3		1		2
4		4		4	...	4		4		4		3		1		1		2		4
3		2		3	...	2		3		2		4		3		2		4		3

$1 = \{aba, b\} = 3a$

$2 = 1.a = 3b$

$3 = 2b = 4a$

$4 = 2a = 4b$

TABE 2.7

A

	a		a		a		a	
1		2		3		4		1
5		6		7		8		5
6		7		8		5		6

B

	b		b		...		b		b	
1		5		1	...	1		5		1
2		6		2	...	2		6		2
3		7		3	...	3		7		3
4		8		4	...	4		8		4

C

	a		b		a		b		a		b		a		b	
1		2		6		7		3		4		8		5		1
4		1		5		6		2		3		7		8		4
2		3		7		8		4		1		5		6		2
3		4		8		5		1		2		6		7		3

D

	a		a		b		a		a		b		b		...		b		b	
4		1		2		6		7		8		4		8	...	4		8		4
5		6		7		3		4		1		5		1	...	5		1		5
1		2		3		7		8		5		1		5	...	1		5		1
3		4		1		5		6		7		3		7	...	3		7		3
7		8		5		1		2		3		7		3	...	7		3		7
6		7		8		4		1		2		6		2	...	6		2		6
2		3		4		8		5		6		2		6	...	2		6		2
8		5		6		2		3		4		8		4	...	8		4		8

$1 = \left\{b^2,\ a^3b^2a,\ a^{-1}b^{-1}ab\right\} = 4.a = 5.b$

$2 = 1.a = 2b^2 = 6.b$

$3 = 2.a = 3.b^2 = 7.b$

$4 = 3.a = 4b^2 = 8.b$

$5 = 1.b = 8.a$

$6 = 2.b = 5.a$

$7 = 3.b = 6a$

$8 = 4.b = 7a$

To show that $8a = 5$:

$$8a = 4ba = 1a^3ba = 1a^3b^2aa^3b^{-1}a$$

$$= 1a^3b^{-1}a$$

$$8ab = 1a^3b^{-1}ab$$

$$= 1a^{-1}b^{-1}ab = 1$$

$5b = 1$

$8a = 5$

TABLE 2.8

		A		
a	a	a	a	
1	2	3	4	1
5	8	11	14	5
6	9	12	15	6
7	10	13	16	7

					B					
b	b	b	b		...	b	b	b	b	
1	5	6	7	1	...	1	5	6	7	1
2	8	9	10	2	...	2	8	9	10	2
3	11	12	13	3	...	3	11	12	13	3
4	14	15	16	4	...	4	14	15	16	4

				C				
a	b	a	b	a	b	a	b	
1	2	8	11	12	15	16	7	1
4	1	5	8	9	12	13	16	4
2	3	11	14	15	6	7	10	2
3	4	14	5	6	9	10	13	3
5	6	7	10	2	3	11	14	15

$1 = \{b^4, a^3b^4a, a^{-1}b^{-1}ab, aba^{-1}b^{-1}\} = 4a = 7b$

$2 = 1.a = 10.b$

$3 = 2a = 13.b$

$4 = 3.a = 16b$

$5 = 1.b = 14a$

$6 = 5.b = 15a$

$7 = 6.b = 16a$

$8 = 2.b = 5.a$

$9 = 8.b = 6.a$

$10 = 9.b = 7a$

$11 = 3.b = 6a$

$12 = 11b = 9a$

$13 = 12b = 10a$

$14 = 4.b = 11a$

$15 = 14.b = 12a$

$16 = 15b = 13a$

To show that $14a = 5$

$14 = 4b = 1a^3b.$ $\quad b^4, a^3b^4a, a^{-1}b^{-1}ab, aba^{-1}b^{-1}$

$14a = 1a^3ba$

$14ab = 1.a^3bab = 1aba^3b = 1aba^{-1}b^{-1}.b^2$

$14ab = 1.b^2$

$14a = 1.b = 5$

$14a = 5$

D

a a b a a b b b b ... b b b b

1	2	3	11	14	5	6	7	1	5	...	1	5	6	7	1
7	10	13	3	4	1	5	6	7	1	...	7	1	5	6	7
6	9	12	13	16	7	1	5	6	7	...	6	7	1	5	6
5	8	11	12	15	6	7	1	5	6	...	5	6	7	1	5
4	1	2	8	11	14	15	16	4	14	...	4	14	15	16	4
3	4	1	5	8	11	12	13	3	11	...	3	11	12	13	3
13	16	7	1	2	3	11	12	13	3	...	13	3	11	12	13
10	13	16	4	1	2	8	9	10	2	...	10	2	8	9	10
16	7	10	2	3	4	14	15	16	4	...	16	4	14	15	16
9	12	15	16	7	10	2	8	9	10	...	9	10	2	8	9
8	11	14	15	6	9	10	2	8	9	...	8	9	10	2	8
2	3	4	14	5	8	9	10	2	8	...	2	8	9	10	2
12	15	6	7	10	13	3	11	12	13	...	12	13	3	11	12
11	14	5	6	9	12	13	3	11	12	...	11	12	13	3	11
15	6	9	10	13	16	4	14	15	16	...	15	16	4	14	15
14	5	8	9	12	15	16	4	14	15	...	14	15	16	4	14

CHAPTER III

1. Introduction

The difference between this chapter and the previous one is that now the order of a (where a is one of the two generators of the non-abelian group G) is 6 instead of 4. That is, we shall begin to study the structure of the non-abelian group G generated by a and b with defining relations:

$$a^6 = b^n = (ab)^6 = e, \ ba^2 = a^2b^k.$$

Since

$$b = ba^6 = b(a^2)^3 = (a^2)^3 b^{k^3} = b^{k^3},$$

it follows that $k^3 - 1 \equiv 0 \pmod{n}$,

or

$$(k - 1)(k^2 + k + 1) \equiv 0 (\mathrm{mod}\ n).$$

For some n there are values of k for which $k^2 + k + 1 \equiv 0(\mathrm{mod}\ n)$; for example, $k = 2$ or 4 when $n = 7$. However, in this chapter we shall let $k = 1$, which is valid for every n.

The commutator subgroup of G will be investigated, and it turns out to be contained in the subgroup $\{b, a^5ba\}$. The tables will be provided at the end of the chapter.

Theorem 3.1

The group $G(m = 6, k = 1)$ is infinite if $n > 2$.

Proof:

Consider the subgroup $H = \{b, a^5ba\}$ of G. The order of b is n, so $b^n = e$. Also $(a^5ba)^n = e$ since a^5ba is a conjugate of b. The order of a^5bab is 3 since

$$a^5baba^5baba^5bab = a^{12}(ab)^6 = e.$$

But

$$\frac{1}{n} + \frac{1}{n} - \frac{1}{3} \leq 1 \text{ if } n > 2.$$

Hence H is infinite if $n > 2$ [4, p.61]. Therefore G is infinite since H is a subgroup of G. This completes the proof of the theorem.

Remark: When $n = 2$, the order of G is 36. (See table 3.1)

2. Normal Subgroups of $G(m = 6, k = 1)$.

Theorem 3.2

$H = \{b, a^5ba\}$ is normal in G and is of index 6.

Proof:

To prove that H is normal in G note that $b^{-1}Hb = H$ since b is in H;

$$a^{-1}ba = a^5ba \text{ is in } H;$$

$$a^{-1}(a^5ba)a = a^4ba^2 = b \text{ is in } H;$$

therefore $a^{-1}Ha$ contains the generators of H, so it contains H. Similarly, H contains $a^{-1}Ha$. For any $x \in G$, the operation $H \to x^{-1}Hx$ consists of taking successive conjugates of H with respect to powers of a and b. Therefore $x^{-1}Hx = H$ and H is normal in G.

Moreover, H is of index 6. (See table 3.2). This completes the proof of the theorem. Note that the factor group of G with respect to H is cyclic of order 6.

$$G/H = \{Ha^i \ / \ i = 0,1,\ldots 5\}. \text{ (Cf Th. 2.4, Th 2.9)}.$$

Proposition 3.3

The subgroup $\{a^2, b, a^5ba\}$ is normal.

Proof

The index of $\{a^2, b, a^5ba\}$ is 2. Hence the result follows. (See table 3.3).

3. The Commutator Subgroup of $G(m = 6, k = 1)$

Lemma 3.4

The commutator subgroup C of G is a subgroup of $H = \{b, a^5ba\}$.

Proof

G/H is abelian, (see table 3.2) so C is contained in H. This completes the proof of the lemma.

Now, to determine the commutator subgroup of G, we apply Dyck's result, mentioned in the first chapter.

Adjoin the further relation

$$ab = ba \quad \text{to} \quad a^6 = b^n = (ab)^6 = e, \ ba^2 = a^2b$$

we get

$$a^6 = b^n = a^6b^6 = e,$$

which is equivalent to $a^6 = b^n = b^6 = e$ (1)

Now, we have to consider six possibilities:

$n \equiv 1 (\text{mod } 6)$, $n \equiv 2 (\text{mod } 6)$, $n \equiv 3 (\text{mod } 6)$, $n \equiv 4 (\text{mod } 6)$, $n \equiv 5 (\text{mod } 6)$, $n \equiv 0 (\text{mod } 6)$.

If $n \equiv 1 (\text{mod } 6)$, then (1) reduces to

$$a^6 = b = e \quad \text{or simply} \quad a^6 = e.$$

Also, if $n \equiv 5 (\text{mod } 6)$, then (1) reduces to

$$a^6 = b^5 = b^6 = e \quad \text{or} \quad a^6 = e.$$

So if $n \equiv 1,5 (\text{mod } 6)$ the defining relations of the commutator factor group is:

$$(Ca)^6 = C \, , \qquad (2)$$

Similarly, if $n \equiv 2,4 (\text{mod } 6)$ (1) reduces to

$$a^6 = b^2 = e,$$

so the defining relations of the commutator factor group are:

$$(Ca)^6 = (Cb)^2 = C, \; CaCb = CbCa \qquad (3).$$

Also, if $n \equiv 3 (\text{mod } 6)$ and $n \equiv 0 (\text{mod } 6)$ the defining relations of the commutator factor group are

$$(Ca)^6 = (Cb)^3 = C, \; CaCb = CbCa \qquad (4),$$

and

$$(Ca)^6 = (Cb)^6 = C, \; CaCb = CbCa \qquad (5)$$

respectively.

Theorem 3.5

If $n \equiv 1,5 (\text{mod } 6)$, then the commutator subgroup C of

$G(m = 6, k = 1)$ is equal to $H = \{b, a^5ba\}$.

Proof

The factor group G/H is cyclic and obviously its defining relation is $(Ha)^6 = H$. (See table 3.2) clearly, there is a one-to-one correspondence

$$Ha^i \longrightarrow Ca^i$$

between the set of right cosets of H and C, since

$$G/C = \left\{Ca^i \;/\; i = 0,1,\ldots 5\right\}.$$

Moreover, by **Lemma** 3.4, C is contained in H. Therefore, by Theorem 2.13, $H = C$; that is,

$$C = \left\{b,\ a^5\ ba\right\}.$$

Theorem 3.6

If $n = 2,4 \pmod 6$, then the commutator subgroup C of $G(m = 6, k = 1)$ is equal to K

$$K = \left\{b^2,\ a^5b^2a,\ a^{-1}b^{-1}\ ab\right\}.$$

Proof

$(Cb)^2 = C$ implies b^2 is in C. Also a^5b^2a is in C since C is normal, and clearly $a^{-1}b^{-1}ab$ is in C. So, each generator of K is in C. Hence we conclude that K is contained in C. Also, the index of K is 12, and the set of right cosets of K is

$$\left\{Ka^ib^j \;/\; i = 0,1,\ldots,5,\ \ j = 0,1\right\}.$$

(See table 3.4).

$$G/C = \left\{ Ca^i b^j \ / \ i = 0,1,\ldots j, \ j = 0,1 \right\}.$$

Obviously there is a one-to-one correspondence

$$Ka^i b^j \longrightarrow Ca^i b^j$$

between the sets of right cosets of K and C. Then by Theorem 2.13,

$$C = K.$$

or

$$C = \left\{ b^2, \ a^5 b^2 a, \ a^{-1} b^{-1} ab \right\}.$$

<u>Remark</u>: This theorem is valid even when G is finite, that is, when $n = 2$. Then $b^2 = e$, so $C = \left\{ a^{-1}b^{-1}ab \right\}$, which is cyclic of order 3.

<u>Theorem 3.7</u>

If $n \equiv 3 (\mathrm{mod}\ 6)$, then the commutator subgroup C is $G(m = 6, k = 1)$ is

$$Q = \left\{ b^3, \ a^5 b^3 a, \ a^{-1} b^{-1} ab, \ aba^{-1} b^{-1} \right\}.$$

<u>Proof</u>:

By a similar argument as in the previous theorem we can show that Q is contained in C. Now, the index of Q is 18 and the set of right cosets of Q is

$$\left\{ Qa^i b^j \ / \ i = 0,1,\ldots 5, \ j = 0,1,2 \right\}.$$

(See table 3.5)

$$G/C = \left\{ Ca^i b^j \ / \ i = 0,1, \ldots, 5, \ j = 0,1,2 \right\}$$

Obviously there is a one-to-one correspondence

$$Qa^ib^j \longrightarrow Ca^ib^j$$

between the sets of right cosets of Q and C. Thus C = Q by theorem 2.13. That is

$$C = \{b^3,\ a^5b^3a,\ a^{-1}b^{-1}ab,\ aba^{-1}b^{-1}\}.$$

Theorem 3.8

If $n \equiv 0 \pmod 6$, then the commutator subgroup C of $G(m = 6, k = 1)$ is equal to

$$M = \{b^6, a^5b^6a, a^{-1}b^{-1}ab, aba^{-1}b^{-1}, ab^2a^{-1}b^{-2}, ab^3a^{-1}b^{-3}, ab^4a^{-1}b^{-4}\}.$$

Proof

By a similar argument as in the previous theorems we can show that M is contained in C. Also, the index of M is 36 (see table 3.6) and the set of right cosets of M is

$$\{Ma^ib^j \ / \ i,\ j = 0,1,\dots,\ 5\}.$$

$$G/C = \{Ca^ib^j \ / \ i,\ j = 0,1,\dots,5\}.$$

Moreover there is a one-to-one correspondence

$$Ma^ib^j \longrightarrow Ca^ib^j$$

between the sets of right cosets of M and C. Then by theorem 2.13 C = M.

Namely

$$C = \{b^6, a^5b^6a, a^{-1}b^{-1}ab, aba^{-1}b^{-1}, ab^2a^{-1}b^{-2}, ab^3a^{-1}b^{-3}, ab^4a^{-1}b^{-4}\}.$$

TABLE 3.1

A

	a	a	a	a	a	a
1	1	1	1	1	1	1
2	3	2	3	2	3	2
4	5	4	5	4	5	4
6	6	6	6	6	6	6

B

	b	b
1	2	1
3	4	3
5	6	5

C

	a	b	a	b	a	b	a	b	a	b	a	b
1	1	2	3	4	5	6	6	5	4	3	2	1

D

	a	a	a	a	b	a	a	b
1	1	1	1	1	2	3	2	1
2	3	2	3	2	1	1	1	2
4	5	4	5	4	3	2	3	4
3	2	3	2	3	4	5	4	3
5	4	5	4	5	6	6	6	5
6	6	6	6	6	5	4	5	6

1 = a = 2.b

2 = 1b = 3.a

3 = 2.a = 4.b

4 = 3.b = 5a

5 = 4.a = 6.b

6 = 5.b = 6.a

TABLE 3.2

```
      A              B                    C
 a a a a a a   b b b...b b b   a b a b a b a b a b a b
1 2 1 2 1 2 1 1 1 1 1...1 1 1 1 2 2 1 1 2 2 1 1 2 2 1 1
              2 2 2 2...2 2 2
```

```
              D
 a a a a b a a b b b ... b b b
1 2 1 2 1 1 2 1 1 1 1...1 1 1 1
2 1 2 1 2 2 1 2 2 2 2...  2 2 2
```

$1 = a^2, b, a^5ba = 2a$

$2 = 1.a = 2b$

TABLE 3.3

A

$1 ^{a} 2 ^{a} 3 ^{a} 4 ^{a} 5 ^{a} 6 ^{a} 1$

B

$n \equiv 1.5 (\mathrm{mod}\ 6)$

	b		b		b		b		b		b	...		b		b		b	
1		1		1		1		1		1		...	1		1		1		1
2		2		2		2		2		2		...	2		2		2		2
3		3		3		3		3		3		...	3		3		3		3
4		4		4		4		4		4		...	4		4		4		4
5		5		5		5		5		5		...	5		5		5		5
6		6		6		6		6		6		...	6		6		6		6

C

$1 ^{a} 2 ^{b} 2 ^{a} 3 ^{b} 3 ^{a} 4 ^{b} 4 ^{a} 5 ^{b} 5 ^{a} 6 ^{b} 6 ^{a} 1 ^{b} 1$

D

	a		a		b		a		a		a		a		b		b		b		b		...		b		b		b	
1		2		3		3		4		5		6		1		1		1		1		1	...	1		1		1		1
6		1		2		2		3		4		5		6		6		6		6		6	...	6		6		6		6
5		6		1		1		2		3		4		5		5		5		5		5	...	5		5		5		5
4		5		6		6		1		2		3		4		4		4		4		4	...	4		4		4		4
3		4		5		5		6		1		2		3		3		3		3		3	...	3		3		3		3
2		3		4		4		5		6		1		2		2		2		2		2	...	2		2		2		2

$1 = b, a^5ba$

$2 = 1.a = 2b$

$3 = 2.a = 3b$

$4 = 3.a = 4.b$

$5 = 4.a = 5b$

$6 = 5.a = 6.b$

To show that 2b = 2

$a^5ba = 1$

$a^5ba^2 \quad 1.a$

$a^5ba^2 = ab \quad 1.a$

$ab \quad 2$

But $a \quad 1.a = 2$

$ab \quad 2.b$

$2.b = 2$

To show that 3b = 3

$a^5bab \quad 1$

$a^5baba \quad 2$

$2.ab = 3$

$2 = 3.b^{-1}a^{-1}$

$2.b = 3.b^{-1}a^{-1}b$

$= 3a^{-1}b$

TABLE 3.2 4

A

	a		a		a		a		a		a	
1		2		3		4		5		6		1
7		8		9		10		11		12		7
8		9		10		11		12		7		8

B

$n \equiv 2.4 \pmod 6$

	b		b		b		b		b	...	b		b		b		b	
1		7		1		7		1		7...1		7		1		7		1
2		8		2		8		2		...2		8		2		8		2
3		9		3		9		3		...3		9		3		9		3
4		10		4		10		4		...4		10		4		10		4
5		11		5		11		5		...5		11		5		11		5
6		12		6		12		6		...6		12		6		12		6

C

	a		b		a		b		a		b		a		b		a		b		a		b	
1		2		8		9		3		4		10		11		5		6		12		7		1
6		1		7		8		2		3		9		10		4		5		11		12		6

$1 \equiv b^2,\ a^5b^2a,\ a^{-1}b^{-1}ab \equiv 6.a\ 7.b$

$2 \equiv 1.a \equiv 2b^2 \equiv 8.b$

$3 \equiv 2.a \equiv 3b^2 \equiv 9.b$

$4 \equiv 3.a \equiv 4b^2 \equiv 10.b$

$5 \equiv 4.a \equiv 5b^2 \equiv 11.b$

$6 \equiv 5.a \equiv 6b^2 \equiv 12.b$

$7 \equiv 1.b \equiv 11.a^2 \equiv 12.a$

$8 \equiv 2.b \equiv 12.a^2 \equiv 7.a$

$9 \equiv 3.b \equiv 7.a^2 \equiv 8.a$

$10 \equiv 4.b \equiv 8.a^2 \equiv 9.a$

$11 \equiv 5.b \equiv 9.a^2 \equiv 10a$

$12 \equiv 6.b \equiv 10a^2 \equiv 11a$

To show that $12a \equiv 7$

$12 \equiv 6b \equiv 1.a^5b$

$12a \equiv 1.a^5ba \equiv 1.a^5b^2aa^5b^{-1}a$

$\equiv 1.a^5b^{-1}a$

$- 1.a^{-1}b^{-1}a$

$12ab \equiv 1.a^{-1}b^{-1}ab \equiv 1$

$12ab \equiv 7.b$

$12a \equiv 7.$

TABLE 3.4 (Cont.)

D_{n-1}

	a	a	b	a	a	a	a	b	b	b	b	b	...		b	b	b	b
1	2	3	9	10	11	12	7	1	7	1	7	1	...	1	7	1	7	1
7	8	9	3	4	5	6	1	7	1	7	1	7	...	7	1	7	1	7
6	1	2	8	9	10	11	12	6	12	6		6	...	6		6		6
5	6	1	7	8	9	10	11	5	11	5	11	5	...	5	11	5	11	5
10	11	12	6	1	2	3	4	10	4	10	4	10	...	10	4	10	4	10
9	10	11	5	6	1	2	3	9	3	9	3	9	...	9	3	9	3	9
8	9	10	4	5	6	1	2	8	2	8	2	8	...	8	2	8	2	8
2	3	4	10	11	12	7	8	2	8	2	8	2	...	2	8	2	8	2
11	7	8	2	3	4	5	6	12	6	12	6	12	...	12	6	12	6	12
3	9	5	11	12	7	8	9	3	9	3	9	3	...	3	9	3	9	3
4	5	6	12	7	8	9	10	4	10	4	10	4	...	4	10	4	10	4
11	12	7	1	2	3	4	9	11	5	11	5		...	11	5	11	5	11

TABLE 3.5

A

	a	a	a	a	a	a
1	2	3	4	5	6	1
7	9	11	13	15	17	7
8	10	12	14	16	18	8
9	11	13	15	17	7	9
10	12	14	16	18	8	10

B

$n \equiv 3 \bmod 6$

	b	b	b	b ... b		b	b	b
1	7	8	1	...	1	7	8	1
2	9	10	2	...	2	9	10	2
3	11	12	3	...	3	11	12	3
4	13	14	4	...	4	13	14	4
5	15	16	5	...	5	15	16	5
6	17	18	6	...	6	17	18	6

C

	a	b	a	b	a	b	a	b	a	b	a	b
1	2	9	11	12	14	4	5	15	17	18	8	1
2	3	11	13	14	16	5	6	17	7	8	10	2
3	4	13	15	16	18	6	1	7	9	10	12	3

$1 = b^3, a^5b^3a, a^{-1}b^{-1}ab, aba^{-1}b^{-1} = 6.a = 8.b$
$2 = 1.a = 2.b^3 = 10.b$
$3 = 2.a = 3b^3 = 12.b$
$4 = 3.a = 4b^3 = 14.b$
$5 = 4.a = 5b^3 = 16.b$
$6 = 5.a = 6b^3 = 18.b$
$7 = 1.b = 15a^2 = 17a$
$8 = 7.b = 16a^2 = 18a$
$9 = 2.b = 17a^2 = 7a$

$10 = 9.b = 18a^2 = 8a$
$11 = 3.b = 7.a^2 = 9.a$
$12 = 11.b = 8.a^2 = 10a$
$13 = 4.b = 9a^2 = 11a$
$14 = 13b = 10a^2 = 12a$
$15 = 5.b = 11a^2 = 13.a$
$16 = 15.b = 12a^2 = 14a$
$17 = 6.b = 13a^2 = 15a$
$18 = 17.b = 14.a^2 = 16a$

To show that $17a = 7$ and $18a = 8$

$17 = 6b = 1.a^5b$
$17a = 1.a^5ba$
$17ab = 1.a^5bab = 1aba^5b$
$= 1.aba^{-1}b^{-1}b^2$
$= 1.b^2$
$17a = 1.b = 7$

$18a = 1.a^5b^2a = 1.a^5b^3a.a^5b^{-1}a$
$18ab = 1.a^5b^3aa^5b^{-1}ab$
$= 1.a^{-1}b^{-1}ab = 1$
$18ab = 8b$
$18a = 8$

TABLE 3.5 (cont.)

D_{n-1}

a	a	b	a	a	a	a	b	b	b	b	b	...	b	b	b	b
1	2	3	11	13	15	17	7	8	1	7	8	1 ...	1	7	8	1
8	10	12	3	4	5	6	1	7	8	1	7	8	8	1	7	8
7	9	11	12	14	16	18	8	1	7	8	1	7	7	8	1	7
6	1	2	9	11	13	15	17	18	6	17	18	6	6	17	18	6
5	6	1	7	9	11	13	15	16	5	15	16	5	5	15	16	5
16	18	8	1	2	3	4	5	15	16	5	15	16	16	5	15	16
14	16	18	6	1	2	3	4	13	14	4	13	14	14	4	13	14
12	14	16	5	6	1	2	3	11	12	3	11	12	12	3	11	12
10	12	14	4	5	6	1	2	9	10	2	9	10	10	2	9	10
9	11	13	14	16	18	8	10	2	9	10	2	9	9	10	2	9
2	3	4	13	15	17	7	9	10	2	9	10	2	2	9	10	2
18	8	10	2	3	4	5	6	17	18	6	17	18	18	6	17	18
11	13	15	16	18	8	10	12	3	11	12	3	11	11	12	3	11
3	4	5	15	17	7	9	11	12	3	11	12	3	3	11	12	3
13	15	17	18	8	10	12	14	4	13	14	4	13	13	14	4	13
4	5	6	17	7	9	11	13	14	4	13	14	4	4	13	14	4
15	17	7	8	10	12	14	16	5	15	16	5	15	15	16	5	15
17	7	9	10	12	14	16	18	6	17	18	6	17	17	18	6	17

TABLE 3.6

A

	a	a	a	a	a	a
1	2	3	4	5	6	1
7	12	17	22	27	32	7
8	13	18	23	28	33	8
9	14	19	24	29	34	9
10	15	20	25	30	35	10
11	16	21	26	31	36	11

B

$n \equiv 0 \pmod 6$

	b	b	b	b	b	b	b ... b		b	b	b	b	b	b
1	7	8	9	10	11	1	...	1	7	8	9	10	11	1
2	12	13	14	15	16	2	...	2	12	13	14	15	16	2
3	17	18	19	20	21	3	...	3	17	18	19	20	21	3
4	22	23	24	25	26	4	...	4	22	23	24	25	26	4
5	27	28	29	30	31	5	...	5	27	28	29	30	31	5
6	32	33	34	35	36	6	...	6	32	33	34	35	36	6

C

	a	b	a	b	a	b	a	b	a	b	a	b
11	2	12	17	18	23	24	29	30	35	36	11	1
6	1	7	12	13	18	19	24	25	30	31	36	6
2	3	17	22	23	28	29	34	35	10	11	16	2
3	4	22	27	28	33	34	9	10	15	16	21	3
4	5	27	32	33	8	9	14	15	20	21	26	4
5	6	32	7	8	13	14	19	20	25	26	31	5

TABLE 3.6 (cont.)

If $n \equiv 0 (\mathrm{mod}\ 6)$

$1 \equiv b^6, a^5b^6a, a^{-1}b^{-1}ab, aba^{-1}b^{-1}, ab^2a^{-1}b^2, ab^3a^{-1}b^{-3}, ab^4ab^{-4} \equiv 6.a \equiv 11.b$
$2 \equiv 1.a \equiv 16.b$
$3 \equiv 2a \equiv 21.b$
$4 \equiv 3.a \equiv 26.b$
$5 \equiv 4.a \equiv 31b$
$6 \equiv 5.a \equiv 36b$
$7 \equiv 1.b \equiv 27a^2 \equiv 32.a$
$8 \equiv 7.b \equiv 28a^2 \equiv 33a$
$9 \equiv 8.b \equiv 29a^2 \equiv 34a$
$10 \equiv 9.b \equiv 30a^2 \equiv 35a$
$11 \equiv 10.b \equiv 31a^2 \equiv 36a$
$12 \equiv 2.b \equiv 32a^2 \equiv 7a$
$13 \equiv 12.b \equiv 33a^2 \equiv 8a$
$14 \equiv 13.b \equiv 34a^2 \equiv 9a$
$15 \equiv 14.b \equiv 35a^2 \equiv 10a$
$16 \equiv 15.b \equiv 36a^2 \equiv 11a$
$17 \equiv 3.b \equiv 7a^2 \equiv 12a$
$18 \equiv 17.b \equiv 8a^2 \equiv 13a$
$19 \equiv 18.b \equiv 9a^2 \equiv 14a$
$20 \equiv 19.b \equiv 10a^2 \equiv 15a$
$21 \equiv 20b \equiv 11a^2 \equiv 16a$
$22 \equiv 4.b \equiv 12a^2 \equiv 17a$
$23 \equiv 22b \equiv 13a^2 \equiv 18a$
$24 \equiv 23.b \equiv 14a^2 \equiv 19a$
$25 \equiv 24.b \equiv 15a^2 \equiv 20a$

$26 \equiv 25b \equiv 16a^2 \equiv 21a$
$27 \equiv 5.b \equiv 17a^2 \equiv 22a$
$28 \equiv 27.b \equiv 18a^2 \equiv 23a$
$29 \equiv 28.b \equiv 19a^2 \equiv 24a$
$30 \equiv 29.b \equiv 20a^2 \equiv 25a$
$31 \equiv 30.b \equiv 21.a^2 \equiv 26a$
$32 \equiv 6.b \equiv 22a^2 \equiv 27a$
$33 \equiv 32.b \equiv 23a^2 \equiv 28a$
$34 \equiv 33b \equiv 24a^2$
$35 \equiv 34b \equiv 25a^2 \equiv 30a$
$36 \equiv 35b \equiv 26a^2 \equiv 31a$

TABLE 3.6 (cont.)

To show that $36a = 11$

$36a = 6b^5a = 1.a^5b^5a$
$36ab = 1.a^5b^5\ ab$
$\quad = 1.a^5b^6a\ a^5b^{-1}ab$
$\quad = 1.a^{-1}b^{-1}ab = 1 = 11.b$
$36a = 11$

To show that $33a = 8$

$33a = 6b^2a = 1.a^5b^2a$
$33ab = 1.a^5b^2ab = 1.ab^2a^5b$
$\quad = 1.ab^2a^{-1}b^{-2}.b^3$
$\quad = 1.b^3 = 8$
$33a = 32ba = 6b^2a = 1.a^5b^2a$
$33ab = 1.a^5b^2ab = 1ab^2a^5b$
$\quad = 1.ab^2a^{-1}b^{-2}b^3$
$\quad = 1.b^3$
$33a = 1.b^2 = 8$

To show that $34a = 9$

$34a = 33ba = 32b^2a = 6b^3a = 1.a^56^3a$
$34ab = 1.a^5b^3ab = 1.ab^3a^5b = 1.ab^3a^{-1}b^{-1}b^{-3}.b^4 = 1.b^4$
$34a = 1.b^3 = 9$

To show that $35a = 10$

$35a = 34ba = 33b^2a = 32b^3a = 6b^4a = 1.a^5b^4a$
$34ab = 1.a^5b^4ab = 1.ab^4a^5b = 1.ab^4a^{-1}b^{-4}b^5 = 1.b^5$
$35a = 1.b^4 = 10$

To show that $32a = 7$

$32 = 1a^5b$
$32a = 1.a^5ba$
$32ab = 1.a^5bab = 1.aba^5b$
$\quad = 1.aba^{-1}b^{-1}\ b^2 = 1.b^2$
$32a = 1.b = 7$

TABLE 3.6 (cont.)

(n - 1) n ≡ 0(mod 6)
D

a	a	b	a	a	a	a	b	b	b	b	b	b	b	...	b	b	b	b	b	b	b	b
1	2	3	17	22	27	32	7	8	9	10	11	1	7	8	11	1	7	8	9	10	11	1
7	12	17	18	23	28	33	8	9	10	11	1	7	8	9	1	7	8	9	10	11	1	7
8	13	18	19	24	29	34	9	10	11	1	7	8	9	10	7	8	9	10	11	1	7	8
9	14	19	20	25	30	35	10	11	1	7	8	9	10	11	8	9	10	11	1	7	8	9
10	15	20	21	26	31	36	11	1	7	8	9	10	11	1	9	10	11	1	7	8	9	10
11	16	21	3	4	5	6	1	7	8	9	10	11	1	7	10	11	1	7	8	9	10	11
16	21	26	4	5	6	1	2	12	13	14	15	16	2	12	15	16	2	12	13	14	15	16
21	26	31	5	6	1	2	3	17	18	19	20	21	3	17	20	21	3	17	18	19	20	21
26	31	36	6	1	2	3	4	22	23	24	25	26	4	22	25	26	4	22	23	24	25	26
31	36	11	1	2	3	4	5	27	28	29	30	31	5	27	30	31	5	27	28	29	30	31
5	6	1	7	12	17	22	27	28	29	30	31	5	27	28	31	5	27	28	29	30	31	5
6	1	2	12	17	22	27	32	33	34	35	36	6	32	33	36	6	32	33	34	35	36	6
2	3	4	22	27	32	7	12	13	14	15	16	2	12	13	16	2	12	13	14	15	16	2
36	11	16	2	3	4	5	6	32	33	34	35	36	6	32	35	36	6	32	33	34	35	36
15	20	25	26	31	36	11	16	2	12	13	14	15	16	2	14	15	16	2	12	13	14	15
14	19	24	25	30	35	10	15	16	2	12	13	14	15	16	13	14	15	16	2	12	13	14
13	18	23	24	29	34	9	14	15	16	2	12	13	14	15	12	13	14	15	16	2	12	13
12	17	22	23	28	33	8	13	14	15	16	2	12	13	14	2	12	13	14	15	16	2	12
20	25	30	31	36	11	16	21	3	17	18	19	20	21	3	19	20	21	3	17	18	19	20
19	24	29	30	35	10	15	20	21	3	17	18	19	20	21	18	19	20	21	3	17	18	19
18	24	28	29	34	9	14	19	20	21	3	17	18	19	20	17	18	19	20	21	3	17	18
17	22	27	28	33	8	13	18	19	20	21	3	17	18	19	3	17	18	19	20	21	3	17
3	4	5	27	32	7	12	17	18	19	20	21	3	17	18	21	3	17	18	19	20	21	3
25	30	35	36	11	16	21	26	4	22	23	24	25	26	4	24	25	26	4	22	23	24	25
24	29	34	35	10	15	20	25	26	4	22	23	24	25	26	23	24	25	26	4	22	23	24
23	28	33	34	9	14	19	24	25	26	4	22	23	24	25	22	23	24	25	26	4	22	23
22	27	32	33	8	13	18	23	24	25	26	4	22	23	24	4	22	23	24	25	26	4	22
4	5	6	32	7	12	17	22	23	24	25	26	4	22	23	26	4	22	23	24	25	26	4
30	35	10	11	16	21	26	31	5	27	28	29	30	31	5	29	30	31	5	27	28	29	30
29	34	9	10	15	20	25	30	31	5	27	28	29	30	31	28	29	30	31	5	27	28	29
28	33	8	9	14	19	24	29	30	31	5	27	28	29	30	27	28	29	30	31	5	27	28
27	32	7	8	13	18	23	28	29	30	31	5	27	28	29	5	27	28	29	30	21	5	27
35	10	15	16	21	26	31	36	6	32	33	34	35	36	6	34	35	36	6	32	33	34	35
34	9	14	15	20	25	30	35	36	6	32	33	34	35	36	33	34	35	36	6	32	33	34
33	8	13	14	19	24	29	34	35	36	6	32	33	34	35	32	33	34	35	36	6	32	33
32	7	12	13	18	23	28	33	34	35	36	6	32	33	34	6	32	33	34	35	36	6	32

CHAPTER IV

1. Introduction

Now some of the theorems of the previous chapters will be generalized. One notes that the proofs are almost the same for these theorems. However, Theorem 2.13 is already in a generalized form.

Also, we are going to consider some specific examples in which $k \neq 1$ $(ba^2 = a^2b^k)$. Of course, these examples cannot be generalized since any $k \neq 1$ is valid for only finitely many values of ns in the congruence $k^{m/2} \equiv 1 (\mathrm{mod}\ n)$.

2. The Index of $\{b, a^{-1}ba\}$

The argument below will hold for both cases $k = 1$ and $k = -1$ (if $k = -1$ is a solution of $k^{m/2} \equiv 1(\mathrm{mod}\ n)$).

Define $1 = b, a^{-1}ba$; clearly $1.b = 1$. Then define $q = 1a^{q-1}$. Now, from the tables (part A) we get $ma = 1$, i.e. $1a^{m-1} = 1$ where m is the order of a. Now $q(= 1a^{q-1})$ is either (i) even or (ii) odd.

(i) q even implies $q - 1$ is odd. But $a^{-1}b^ia \quad 1$, so $a^{-1}b^ia^q \quad q$ so $a^{q-1}\ b^j \quad q$, where i is any integer and

$$j = i \quad \text{or} \quad j = -i$$

Obviously $a^{q-1}\ b^{j+1} \in q\ b$, but j is arbitrary. Hence,

$$qb = q \ , \ q = 2,4,\dots m.$$

(ii) q is odd implies $q - 1$ is even.

But

$$b^i a^{q-1} = a^{q-1} b^j \in q \, , \quad j = i \quad \text{or} \quad j = -i.$$

Also $a^{q-1} \, b^{j+1} \in qb$ and j is arbitrary.

Hence $qb = q$, $q = 1, 3, \ldots m - 1$. So we get

$$q = 1a^{q-1} = qb, \quad 1 = 1a^{m-1}.$$

We claim that the tables will close up. Clearly part A of the tables will close up with one row. Part B also closes up since $qb = q$, all columns will be equal. ly. Part C will begin as follows: the first row begins with 1, then $1a = 2$, $2b = 2$, $2a = 3$ and so on. Now, there are m a's and m b's at the head of part C. since $(ab)^m = e$. Since the presence of b's does not affect the process (because $qb = q$). Then obviously the rows of part C will be in a way equivalent to part A. Moreover there are at most 2m rows so we are finished with part C.

Similarly, in part D there are m a's and the presence of b's does not affect the process. Moreover, there are $m + 2$ essentially different positions and so $m(m - 2)$ rows. Also, every row is equivalent to a row in part A. So we are done with part D. Hence we conclude that the index of

$$\{b, a^{-1} ba\} \quad \text{is} \quad m,$$

where

$$a^m = b^n = (ab)^m = e, \qquad ba^2 = a^2 b$$

or

$$a^m = b^n = (ab)^m = e, \qquad ba^2 = a^2 b^{-1}.$$

3. Normal Subgroup of G

Theorem 4.1

$N = \{b, a^{-1} ba\}$ is a normal subgroup of $G(k = -1)$.

Proof

To prove that $N = \{b, a^{-1} ba\}$ is normal in G note that $b^{-1} N b = N$, since $b \in N$.
Also $a^{-1} ba \in N$;

$$a^{-1}(a^{-1} ba) a = a^{m-2} ba^2 = b^{-1} \in N.$$

Therefore $a^{-1}Na$ contains the generators of N, so it contains N. Similarly N contains $a^{-1}Na$. For any x G, the operation $N \rightarrow x^{-1} N x$ consists of taking successive conjugates of N with respect to powers of a and b.
Therefore $x^{-1} Nx = N$ and N is normal in $G(k = -1)$. This completes the proof of the theorem.

Since N is normal we can construct the factor group G/N. Obviously

$$G/N = \{N a^i / i = 0,1, \dots m - 1\}.$$

(See table 4.1)

So we can state a Lemma.

Lemma 4.2

The commutator subgroup C of $G(k = -1)$ is contained in N.

Proof

G/N is abelian, since it is cyclic.

Hence C is contained in N.

Let us apply Dyck's result to determine the commutator subgroup C of $G(k = - 1)$. Adjoin

$$ab = ba \quad \text{to} \quad a^m = b^n = (ab)^m = e, \ ba^2 = a^2b^{-1}.$$

We get

$$a^m = b^n = a^m b^m = e, \ b = b^{-1}$$

or

$$a^m = b^n = b^m = = e, \ b^2 = e.$$

We have two cases

(i) n is odd,

(ii) n is even.

(i) If n is odd

$$a^m = b^n = b^m = e, \quad b^2 = e$$

becomes

$$a^m = e \quad \text{since} \quad b = e.$$

Hence the defining relation of G/C is

$$(Ca)^m = C.$$

(ii) If n is even

$$a^m = b^n = b^m = e, \ b^2 = e$$

becomes

$$a^m = b^2 = e, \quad ab = ba .$$

Hence the defining relations of G/C are

$$(Ca)^m = (Cb)^2 = C, \quad Ca \ Cb = Cb \ Ca.$$

Theorem 4.3

If n is odd the commutator subgroup C of $G(K = -1)$ is equal to $N = \{b, a^{-1} ba\}$

Proof

C is contained in N by the previous Lemma.

$$G/N = \{Na^i \ / \ i = 0, 1, \ldots m - 1\};$$

$$G/C = \{Ca^i \ / \ i = 0, 1, \ldots m - 1\}.$$

Obviously there is a one-to-one correspondence between the sets of right cosets of C and N. Hence by theorem 2.13

$$C = N$$

or

$$C = \{b, a^{-1} ba\}$$

This completes the proof of the theorem.

If n is even we have

$$(Ca)^m = (Cb)^2 = C, \ Ca \ Cb = Cb \ Ca$$

as the defining relations of G/C.

Now, we cannot immediately determine generators for the commutator subgroup, because we need some information from the tables to decide the index of a certain subgroup. However, one might investigate the index of the subgroup $\{b^2, a^{-1}b^2a\}$ which is a subgroup of the commutator subgroup C because $(Cb)^2 = Cb^2 = C$ implies $b^2 \in C$, and therefore $a^{-1}b^2a \in C$. One can do the following through the electronic computers or otherwise: Calculate the index of $\{b^2, a^{-1}b^2a\}$. If its index is not equal to that of C, add a suitable commutator to the generators, and repeat the process till the index of the subgroup at hand is equal to that of C. In this way one can decide the commutator subgroup C of G.

When G has the defining relations

$$a^m = b^n = (ab)^m = e, \; ba^2 = a^2b,$$

then $H = b, a^{-1}ba$ is of index m. The argument is almost the same as for N. The only difference is that part D of the table will be headed by $a^{m-2}b\, a^2b^{n-1}$ instead of $a^{m-2}ba^2b$. But we said that the presence of the b's does not affect the table.

Theorem 4.4

$H = \{b, a^{-1}ba\}$ is a normal subgroup of $G(k = 1)$.

Proof

The proof is essentially the same as that for N. (Theorem 4.1)

Since H is normal we can construct the factor group G/H .

$$G/H = \{Ha^i \,/\, i = 0, 1, \ldots m - 1\},$$

which is cyclic of order m.

Hence we can state a lemma.

Lemma 4.5

The commutator subgroup C of $G(k = 1)$ is contained in H.

Proof

G/H is abelian; hence C is contained in H. Let us apply

Dyck's result to determine the commutator subgroup C of $G(k = 1)$. Add

$$ab = ba \text{ to } a^m = b^n = (ab)^m = e,\ ba^2 = a^2b.$$

We get

$$a^m = b^n = b^m = e, \quad ab = ba.$$

But $b^n = b^m = e$ implies that $b^d = e$, where $d = (m,n)$. Hence the defining relations of the commutator factor group of $G(k = 1)$ become

$$(Ca)^m = (Cb)^d = C,\ Ca\ Cb = Cb\ Ca \qquad (1)$$

Theorem 4.6

If $d = (m,n) = 1$ then the commutator subgroup C of $G(k = 1)$ is $H = \{b, a^{-1}ba\}$.

Proof

H contains C by lemma 4.5.

$$G/H = \{Ha^i \ / \ i = 0,1,\ldots\ m-1\}.$$

(See table 4.1)

$$G/C = \{Ca^i \ / \ i = 0,1,\ldots\ m-1\}$$

since the defining relation of G/C is $(Ca)^m = C$ from (1).

Clearly there is a one-to-one correspondence between the sets of right cosets of H and C. Therefore $C = H$ by theorem 2.13, that is,

$$C = \{b, a^{-1}\ ba\}.$$

This completes the proof of the theorem.

Now, consider the case where $d = (n,m) \neq 1$.

The defining relations of G/C are

$$(Ca)^m = (Cb)^d = C, \ Ca\ Cb = Cb\ Ca.$$

But $(Cb)^d = Cb^d = C$ implies $b^d \in C$. Therefore $a^{-1}b^d a \in C$. Hence $\{b^d, a^{-1}b^d a\}$ is a subgroup of C, since its generators are in C. Then we investigate, through the electronic computer or otherwise, the subgroup

$$\{b^d, a^{-1}b^d a\},$$

if necessary, add to its generators suitable commutators till the index of the subgroup at hand is equal to that of C. If the index of $\{b^d, a^{-1}b^d a, C_1, C_2, \dots C_q\}$, where C_i are commutators, then by theorem 2.13

$$C = \{b^d, a^{-1}b^d a, C_1, C_2 \dots C_q\}.$$

So we can state a general theorem.

Theorem 4.7

The commutator subgroup C of $G(k = 1)$ is equal to $\{b^d, a^{-1}b^d a, C_1, C_2 \dots C_q\}$, where $d = (m,n)$ $C_i (i = 1,2,\dots q)$ are commutators.

Note that, if $\{b^d, a^{-1}b^d a\}$ is contained in C, then $\{b^d, a^{-1}b^d a, C_1, C_2 \dots C_q\}$ is also contained in C where C_i are commutators.

Theorem 4.8

$\{a^2, b, a^{-1} ba\}$ is a normal subgroup of G.

Proof

The index of $\{a^2, b, a^{-1}ba\}$ is 2. (See table 4.2). Hence the result follows.

4. Examples where $k \not\equiv \pm 1$

In the first chapter we mentioned that $k \equiv 1$ is a solution of the congruence $k^{m/2} \equiv 1 (\mathrm{mod}\ n)$ for all integers n and for all even integers m, $k \equiv -1$ is a solution for all integers n if $\frac{m}{2}$ is even. However, for some values of n and m there are values of k where $k \not\equiv \pm 1$ and yet satisfies the congruence $k^{m/2} \equiv 1 (\mathrm{mod}\ n)$. For example, if $n \equiv 24$, $m \equiv 4$ then

$$k \equiv 1, 5, 7, 11, 13, 17, 19, 23$$

$$(\text{or } k \equiv \pm 1, \pm 5, \pm 7, \pm 11)$$

are solutions of $k^2 \equiv 1 (\mathrm{mod}\ 24)$.

It turns out that in every case the subgroup $H \equiv \{b, a^{-1}ba\}$ is of index 4, and is normal.

Also
$$G/H \equiv \{Ha^i \ / \ i \equiv 0, 1, 2, 3\}$$

which is abelian for all values of k. (See table 4.3) (Note that table 4.3 serves to determine the index of H for all values of k which are mentioned above, since the only difference between tables for different values of k is the excess of b's in part D of the tables which does not affect the tables). Hence H contains the commutator subgroup C of G.

For $k = 5, 13, 17$, the relations of G/C turn out to be:

$$(Ca)^4 = (Cb)^4 = C, \quad Ca\ Cb = Cb\ Ca,$$

while for $k = 7, 11, 19$ the defining relations of G/C are

$$(Ca)^4 = (Cb)^2 = C, \quad Ca\ Cb = Cb\ Ca.$$

For $k = 5,13,17, \quad C = \{b^4, a^3b^4a, a^{-1}b^{-1}ab, a^{-1}b^{-2}ab^2\}.$

(See table 4.4, and apply theorem 2.13).

For $k = 7,11,19, \quad C = \{b^2, a^{-1}b^2a, a^{-1}b^{-1}ab\}.$

(See table 4.5 and apply Theorem 2.13).

Note that the tables marked out are for $k = 17$ (Table 4.4) and $k = 7$ (Table 4.5) but with slight modifications Table 4.4 will serve for $k = 5.13$, and table 4.5 will serve for $k = 11,19$.

5. The Order of $G(k = 1)$

Consider $H = \{b, a^{-1}ba\}$.

The order of b is n. The order of $a^{-1}ba$ is n since it is a conjugate of b.

Also the order of $a^{-1}bab$ is $\frac{m}{2}$ because

$$\begin{aligned}(a^{-1}bab)^q &= (a^{m-2}abab)^q \\ &= a^{q(m-2)}((ab)^2)^q, \\ &= a^{q(m-2)}(ab)^{2q}.\end{aligned}$$

Therefore the order of $a^{-1}bab$ is greater or equal to $2q$. Let $q = \frac{m}{2}$. Then

$$
\begin{aligned}
(a^{-1}bab)^{\frac{m}{2}} &= a^{\frac{m}{2}(m-2)} \ (ab)^{m} \\
&= a^{m.\frac{m}{2} - m} \ (ab)^{m} \\
&= a^{m\frac{m}{2}} \ . \ a^{-m} \ (ab)^{m} \\
&= e.
\end{aligned}
$$

But

$$
\frac{1}{n} + \frac{1}{n} + \frac{1}{m/2} = \frac{2}{n} + \frac{2}{m} \leq 1 \quad \text{if } m \geq 4
$$

$$
n \geq 4.
$$

Hence the order of H is infinite. [4, p. 61]. Therefore G is infinite since H is a subgroup of G.

TABLE 4.1

A

	a	a	a …	a	a	a
1	2	3	4…	m-1	m	1

B

	b	b	b …	b	b	b	b
1	1	1	1…	1	1	1	1
2	2	2	…	2	2	2	2
3	3	3	…	3	3	3	3
⋮						⋮	
k	k	k	…	k	k	k	k
⋮						⋮	
m	m	m	…	m	m	m	m

C

	a	b	a	b … a	b	a	b
1	2	3	3	3…m-1	m	m	1	1
2	3	3	4	4…m	1	1	2	2
⋮				⋮	⋮	⋮	⋮	⋮
m	1	1	2	2…m-1	m-1		m	m

D

m - 2

	a	a	… a	a	b	a	a	b
1	2	3…	m-2	m-1	m-1	m	1	1
2	3	4…		m	m	1	2	2
3	4	5…	m	1	1	2	3	3
⋮			⋮				⋮	

$1 = a^{-1}ba,\ b = ma$

$2 = 1.a = 2b$

⋮

$k = 1a^{k-1} = kb$

⋮

$m = 1a^{m-1} = mb$

TABLE 4.2

A B C

a a ... a a b b ... b b a b a b ... a b a b

1 2 1...1 2 1 1 1 ... 1 1 1 2 2 1 1...1 2 2 1 1

2 2 ... 2 2

D

m - 2

a a ... a a b a a b b ... b b

1 2 1 ... 2 1 1 2 1 1 1 ... 1 1

2 1 2 ... 1 2 2 1 2 2 2 ... 2 2

1

$1 = a^2, b, a^{-1}ba = 2a$

$2 = 1a = 2b$

TABLE 4.3

A

	a		a		a		a	
1		2		3		4		1

B

	b		b...		b		b		b	
1		1		1...		1		1		1
2		2		2...		2		2		2
3		3		3...		3		3		3

C

	a		b		a		b		a		b		a		b	
1		2		2		3		3		4		4		1		1

D

	a		a		b		a		a		b		b	...		b		b		
1		2		3		3		4		1		1		...	1		1		1	
2		3		4		4		1		1		2		...	2		2		2	
3		4		1		1		2		3		3		...	3		3		3	
4		1		2		2		3		4		4		...	4		4		4	

1 = b, a^{-1} ba = 4a

2 = 1.a = 2b

3 = 2a = 3b

4 = 3a = 4b

TABLE 4.4

		A								B										C				
	a	a	a	a		b	b	b	b	...	b	b	b	b	b		a	b	a	b	a	b	a	b
1	2	3	4	1	1	5	6	7	1	...	1	5	6	7	1	1	2	8	11	12	15	16	7	1
5	8	11	14	5	2	8	9	10	2	...	2	8	9	10	2	4	1	5	8	9	12	13	16	4
6	9	12	15	6	3	11	12	13	3	...	3	11	12	13	3	2	3	11	14	15	6	7	10	2
7	10	13	16	7	4	14	15	16	4	...	4	14	15	16	4	3	4	14	5	6	9	10	13	3

$1 = b^4, a^{-1}b^4a, a^{-1}b^{-1}ab, a^{-1}b^{-2}ab^2 = 4a = 7b$

$2 = 1.a = 2b^{20} = 2b^4 = 10b$

$3 = 2a = 3b^4 = 3b$

$4 = 3a = 4b^4 = 16b$

$5 = 1.b = 14a$

$6 = 5b = 15a$

$7 = 6b = 16a$

$8 = 2b = 5a$

$9 = 8b = 6a$

$10 = 9b = 7a$

$11 = 3b = 8a$

$12 = 11b = 9a$

$13 = 12b = 10a$

$14 = 4b = 11a$

$15 = 14b = 12a$

$16 = 15b = 13a$

$$
\begin{aligned}
16a &= 4b^3a = 1a^3b^3a \\
&= 1a^3b^4a\ a^3b^{-1}a \\
&= 1a^3b^{-1}a
\end{aligned}
$$

16ab =

$16ab = 1a^{-1}b^{-1}ab = 1 = 7b$

$16a = 7.$

$$
\begin{aligned}
15a = 4b^2a = 1a^3b^2a &= 1a^3b^4a\ a^3b^{-2}a \\
&= 1a^3b^{-2}a
\end{aligned}
$$

$15ab^2 = 1a^3b^{-2}ab = 1 = 6b^2$

$15a = 6.$

TABLE 4.4 (cont.)

D

	a	a	b	a	a	b	b	b	b	b	b	b
1	2	3	11	14	5	6	7	1	5	6	7	1
5	8	11	12	15	6	7	1	5	6	7	1	5
6	9	12	13	16	7	1	5	6	7	1	5	6
7	10	13	3	4	1	5	6	7	1	5	6	7
10	13	16	4	1	2	8	9	10	2	8	9	10
13	16	7	1	2	3	11	12	13	3	11	12	13
3	4	1	5	8	11	12	13	3	11	12	13	3
4	1	2	8	11	14	15	16	4	14	15	16	4
2	3	4	14	5	8	9	10	2	8	9	10	2
8	11	14	15	6	9	10	2	8	9	10	2	8
9	12	15	16	7	10	2	8	9	10	2	8	9
16	7	10	2	3	4	14	15	16	4	14	15	16
12	15	6	7	10	13	3	11	12	13	3	11	12
11	14	5	6	9	12	13	3	11	12	13	3	11
14	5	8	9	12	15	16	4	14	15	16	4	14
15	6	9	10	13	16	4	14	15	16	4	14	15

TABLE 4.5

A

	a	a	a	a		b	b	b	b	...
1	2	3	9	1	1	5	1	5	1	...
5	6	7	8	5	2	6	2	6	2	...
6	7	8	5	6	3	7	3	7	3	...
					4	8	4	8	4	...

B

b	b	b	b
5	1	5	1
6	2	6	2
7	3	7	3
8	4	8	4

C

	a	b	a	b	a	b	a	b
1	2	6	7	3	4	8	5	1
4	1	5	6	2	3	7	8	4
2	3	7	8	4	1	5	6	2
3	4	8	5	1	2	6	7	3

D

	a	a	b	a	a	b	b	...	b	b	b
1	2	3	7	8	5	1	5	...	1	5	1
5	6	7	3	4	1	5	1	...	5	1	5
6	7	8	4	1	2	6	2	...	6	2	6
7	8	5	1	2	3	7	3	...	7	3	7
3	4	1	5	6	7	3	7	...	3	7	3
4	1	2	6	7	8	4	8	...	4	8	4
2	3	4	8	5	6	2	6	...	2	6	2
8	5	6	2	3	4	8	4	...	8	4	8

$1 = b^2, a^{-1}b^2a, a^{-1}b^{-1}ab = 4a = 5b$

$2 = 1.a = 2b^{14} = 2b^{10} = 2b^4 = 2b^2 = 6b$

$3 = 2a = 3b^2 = 7b$

$4 = 3.a = 8b$

$5 = 1.b = 8a$

$6 = 2b = 5a$

$7 = 3b = 6a$

$8 = 4b = 7a$

$$
\begin{aligned}
8a = 4ba = 1a^3ba &= 1a^3b^2a\ a^3b^{-1}a \\
&= 1a^3\ b^{-1}\ a \\
8ab &= 1a^3\ b^{-1}\ ab \\
&= 1\ a^{-1}b^{-1}ab = 1. = 5b \\
8a &= 5
\end{aligned}
$$

BIBLIOGRAPHY

1. Basmaji, B., Some groups having two generators. Unpublished master's thesis. American University of Beirut, 1964.

2. Dyck, W., Gruppentheoretische Studien, Mathematische Annalen, Vol. 20 (1882), pp. 1 - 44.

3. Coxeter, H. S. M. and Moser, W. O. J., Generators and relations for discrete groups, Ergebnisse der Mathematik und iihrer Grenzgebiete 14, Berlin-Gottingen-Heidelberg, Springer-Verlag (1957).

4. Coxeter, H.S.M., The groups determined by the relations, $s^{\ell} = T^{m} = (S^{-1} T^{-1} S T)^{p} = 1$, Duke Mathematical Journal, Vol. 2 (1936), pp. 61-73.