ON VARIANTS AND CONVERSES OF TAYLOR'S THEOREM IN ABSTRACT ANALYSIS

By

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NOTATIONS

$\Longrightarrow$ it implies

$\not\Longrightarrow$ does not imply

$\exists$ there exists

$\in$ $x \in A$ (x is an element of the set A)

$\subset$ $A \subset B$ A is a subset of B

$\forall$ for every

iff if and only if

β $= o(\alpha)$ iff $\beta \to 0$ as $\alpha \to 0$

R reals

0 real number zero

θ vector zero

$\langle x, f \rangle$ denotes value of f at x.

$\therefore$ therefore

X, Y, Z Banach Spaces

A, B, C open subsets of X, Y, Z respectively

$\overline{x_1 x_2} = \{ x_t \mid x_t = x_1 + t(x_2 - x_1);\ t \in [0, 1] \} \subset X$

$f: J \to R$ where $J \subset R$ ordinary function

$f: X \to R$ functional

$f: J \to Y$ abstract function $J \subset R$

$f: X \to Y$ operator

$X_1 \times X_2 \times \ldots \times X_m = X^m$ if $X_1 = X_2 = \ldots = X_m = X$

$x^m = (x, x, \ldots, x) \in X^m$

$\mathcal{L}(X, Y) = \{ L \mid L: X \to Y;\ L$ linear continuous mapping$\}$

$\|x\|$ norm of x

G-variation, G-differential, F-differential denote Gateaux variation, Gateaux differential and Frechet differential respectively.

$Vf(x; h)$; $Df(x; h)$; $df(x; h)$ denote respectively first order, G-variation, G-differential, F-differential of f at x with incerement h

$f'(x; h)$ or $f'_x(h)$ may denote any first order differential of f at x with increment h the type meant should be clear from the context.

$\emptyset^n$ (t) n-th Riemann differential of $\emptyset$ at t.

$\emptyset^n$ (t) n-th ordinary differential of $\emptyset$ at t.

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ABSTRACT

This study deals with some problems of higher order differentiability of nonlinear operators on normed real linear spaces. Our main interest is the possibility and consequences of local approximations of nonlinear operators by abstract polynomials. In addition to the familier notions of n-th order Fréchet and Gâteaux differentials, we define in Chapter I the concepts of (Weak and Strong) Peano, Taylor, Riemann and direct n-th order differentials. Some implication relationships among these concepts are established.

Two different approaches to some problems of Calculus related to the mean value theorem and Taylor's theorem are discussed in Chapter II. The first method makes use of the results of the related problems from the Calculus of ordinary functions and generalizes them to operators by using a standard technique which is based on a corollary of the Hahn-Banach theorem. The second method treats the problems directly without recourse to the Calculus of ordinary functions.

In Chapter III, we consider the converse problem to Taylor's theorem. Stated in terms of ordinary functions, the problem is the following: If there exists constants a_0, a_1, ..., a_{n-1} and a function $M_n(t)$ defined in a deleted neighborhood of $t = 0$ by

$$f(x_o + t) = a_o + a_1 t + a_2 t^2 + \dots + a_{n-1} t^{n-1} + M_n(t),$$

what conditions should be imposed on $M_n(t)$ to guarantee the existence of $f^{(n)}(x_o)$. It is found that besides the natural conditions $M_n^{(n-1)}(t)$ exists in a neighborhood of $t = 0$ and $\lim_{t \to 0} M_n(t) = 0$, one needs

$\lim_{t \to 0} t^{(j)} M_n^{(j)}(t) = 0$ [1]. We shall treat this problem for operators using Gâteaux variations, Gâteaux differentials and Fréchet differentials.

CHAPTER I

SOME NOTIONS OF N-TH ORDER DIFFERENTIABILITY

In this chapter we review some notions of differentiability of operators on normed linear spaces that will be used in this paper. Several notions of n-th order differentials generalizing the definitions of Peano, Taylor, Riemann and direct differentials to nonlinear operators are introduced. We also prove some implication relationships among these n-th order differentials, and relate these differentials to the n-th order Frechet, Gateaux differentials.

For an exposition of calculus in Banach spaces and a study of fundamental properties of various notions of differentials we refer to Dieudonne [3], Nashed [9], Vainberg [13], Lusternik and Sobolev [8], Wiener [14] and to the older works of Graves and Hildebrandt [4, 5, 6].

I. Preliminaries

Definition 1. Let X and Y denote normed linear spaces a mapping $f: X \to Y$ is said to be linear if it is additive, $f(x+y) = f(x) + f(y)$ for all x, y in X and homogeneous, $f(\lambda x) = \lambda f(x)$ for all x in X, and scalar λ.

Definition 2. A mapping $L_m: X_1 \times X_2 \times \dots \times X_m \to Y$ where X_i ($i = 1, \dots, n$), Y are normed linear spaces, is called multilinear if it is additive and homogeneous in each argument.

Remarks 1. a. A linear operator is continuous on the whole space iff it is continuous at $x = \theta$.

b. A linear operator is bounded iff it is continuous.

c. Additivity and continuity of an operator imply boundedness and homogeneity of the operator.

Theorem 1. (See, for instance [3], p. 104 .) $\mathcal{L}(R; X)$ the space of all linear continuous operators from R into X is naturally identified with X, by the mapping $x \longrightarrow L_1^x$ which is a linear isometry (norm preserving) of X onto $\mathcal{L}(R; X)$.

Theorem 2. [3]. For each u in $\mathcal{L}(X_1 \times X_2; Y)$ and each x_1 in X_1, let u_{x_1} be the linear mapping $x_2 \rightarrow u(x_1, x_2)$. Then $\tilde{u}: x_1 \longrightarrow u_{x_1}$ is a linear continuous mapping of X_1 into $\mathcal{L}(X_2; Y)$ and the mapping $u \longrightarrow \tilde{u}$ is a linear isometry of $\mathcal{L}(X_1 \times X_2; Y)$ onto $\mathcal{L}(X_1; \mathcal{L}(X_2; Y))$.

Note. By induction on n it follows from the above theorem that $\mathcal{L}(X_1 \times X_2 \times \dots \times X_n; Y)$ can be naturally identified (with conservation of the norm) with $\mathcal{L}(X_1; \mathcal{L}(X_2; \dots, \mathcal{L}(X_n; Y) \dots)$.

Definition 3. Let $X_1 = X_2 = \dots = X_n = X$. A mapping $H_n: X^n \longrightarrow Y$ is called homogeneous of n-th degree in $x = (x_1, x_2, \dots, x_n)$ if $H_n(\lambda x) = \lambda^n H_n(x)$.

Definition 4. A multilinear mapping f is said to be symmetric if $X_1 = X_2 = \dots = X_n$ and

$$f(x_1, x_2, \dots, x_n) = f(x_{i_1}, x_{i_2}, \dots, x_{i_n})$$

where $i_1, \dots, i_n$ is an arbitrary permutation of the indices $1, 2, \dots, n$.

Remark 2. Let H_n and K_n be n-termed homogeneous linear forms from X^n to Y if $H_n(x^n) = K_n(x^n)$ for any x in X then $H_n = K_n$ i.e. $H_n(x_1, x_2, \ldots, x_n) = K_n(x_1, x_2, \ldots, x_n)$ for arbitrary $x_1, x_2, \ldots, x_n$.

Definition 5. Let $H_i: X^i \longrightarrow Y$ where H_i a homogeneous forms of ith degree. A sum $P_n(x) = \sum_{i=1}^{n} H_i(x^i)$ is called an abstract polynomial of n-th degree in h.

A detailed discussion on abtract polynomials is given in Hille and Phillips [7].

Before we proceed any further we state the Hahn-Banach theorem, which plays an essential role in the study of calculus of operators.

Theorem 3. [8] Each linear functional f defined on a linear manifold G of a normed linear space X can be extended onto the whole space with preservation of the norm.

Corollary. Let X be a normed linear space and x_o an arbitrary but fixed element in X. Then there exists a linear functional f defined on X such that $\|f\| = 1$ and $|f(x_o)| = \|x_o\|$.

Proof:

Let $G = \{x \mid x = tx_o, t \in R\}$; G is a linear manifold of X. Let ϕ be a mapping on G into R defined by $\phi(x) = t\|x_o\|$ where $x = tx_o$. Then $\phi(x_o) = \|x_o\|$ and $|\phi(x)| = |t|\|x_o\| = \|tx_o\| = \|x\|$ i.e. $\|\phi\| = 1$.

Applying the above theorem to ϕ we arrive at the desired functional $f: X \longrightarrow R$ with $\|f\| = 1$ and $|f(x_o)| = \|x_o\|$.

2. N-th Order Gateaux and Frechet Differentials

For convenience, we follow the definitions and notations used in [9].

<u>Definition 6</u>. Let $F: A \to Y$, where A is an open subset of X, Y are normed linear spaces: Let $x_0 \in A$ and h an arbitrary nonzero fixed element of X. Then $x_0 + th$ is in A for $|t| \leq \varepsilon(x_0; h)$. Let $\tau = \operatorname{Sup} \{ \varepsilon \mid |t| \leq \varepsilon \Rightarrow x_0 + th \in A \}$. Then $\phi(t) = F(x_0 + th)$ is defined for $|t| < \tau$. If

$$\lim_{t \to 0} \frac{\phi(t) - \phi(0)}{t} = \frac{d}{dt} F(x_0 + th)\Big|_{t=0}$$

exists, it is called the <u>Gateaux variation</u> (or the <u>weak differential</u>) of F at x_0 with increment h and is denoted by $VF(x_0; h)$.

If F has a Gateaux variations, at every point x in X, then F is said to have a first variation on X.

<u>Definition 7.</u> F has an <u>n-th variation</u> $V^nF(x_0; h)$ at a point x_0 if the function $F(x_0 + th)$ has an n-th derivative with respect to t at $t = 0$.

<u>Remarks 3.</u> a. $V^nF(x_0; h)$ is a homogeneous form of n-th degree in h.

b. If $VF(x_0; h)$ exists then F is continuous in the direction h, but not necessarily continuous at x_0.

c. $VF(x_0; h)$ is not necessarily linear in h.

<u>Definition 8.</u> If $VF(x_0, h)$ is linear and bounded in h, it is called the <u>Gâteaux differential</u> of F at x with increment h and is denoted by $DF(x_0; h)$.

<u>Theorem 4.</u> Let A be an open subset of X and let F be a nonlinear mapping from A into Y. A necessary and sufficient condition for F to be G-differentiable at x is that the following representation holds

$$F(x_0 + h) - F(x_0) = L(x_0; h) + R(x_0; h) \qquad (1)$$

for every h in E for which $x_0 + h$ is in X, where $L(x_0; h)$ is linear and continuous in h and

$$\lim_{\tau \to 0} \frac{R(x_0; \tau h)}{\tau} = 0 \text{ for every } h \qquad (2)$$

In this case we have $L(x_0; h) = DF(x_0; h)$.

Before we define the second order Gateaux differential let us note that

DF(;) is a function of two variables,

DF(x;): $X \to Y$ is a linear and continuous mapping for every fixed but arbitrary x in A,

DF(; h): $A \to Y$ where h is arbitrary but fixed element in X,

DF(;) itself could be thought of as a mapping from A into $\mathcal{L}(X, Y)$,

and DF(;): $x \to Df(x;\ )$.

<u>Definition 9.</u> F is said to have a <u>second order G-differential</u> at x if DF(x ; h) exists and DF(; h) has a first order G-differential at x.

Thus the second Gateaux differential has the following representation

$$DF(x + k; h) - DF(x;h) = D^2F(x; hk) + R(x; h, k) \qquad (3)$$

where

$$\lim_{t \to 0} \frac{R(x; h, tk)}{t} = 0 \qquad (4)$$

Similarly one can define higher order Gateaux differentials.

<u>Definition 10.</u> The operator F is said to be <u>Fréchet differentiable</u> at x_0 if the representation (1) p. 5 holds where

$L(x_0; h)$ is linear and continuous in h and moreover

$$\lim_{h \to \theta} \frac{\| R(x_0; h)\|}{\| h\|} = 0 \qquad (5)$$

The following theorem exhibits an intrinsic difference between the G- and F-differentials.

Theorem 5. [9]. The operator F is F-differentials at x_0 iff the representation (1) holds where $L(x_0; h)$ is continuous and linear in h and

$$\lim_{\tau \to 0} \tau^{-1} \| R(x_0; \tau h)\| = 0$$

holds uniformly with respect to h on each set $\|h\| =$ constant.

Definition 11. Suppose for $m \geq 1$ the m-th order F-differential $d^m F(x_0; h_1, \ldots, h_m)$ of the mapping $F: A \longrightarrow Y$, has been defined for all $m + 1$ tuples $(x_0, h_1, \ldots, h_m)$ of element of $X \ni x_0 + \sum_{i=1}^{m} h_i$ in A. Then F is said to have an F-differential of order m + 1 if for every $(m + 2)$ tuples of elements $(x_0, h_1, \ldots, h_{m+1})$ of $X \ni x_0 + \sum_{i=1}^{m+1} h_i$ in A, the following representation holds

$$d^m F(x_0 + h_{m+1}; h_1, \ldots, h_m) - d^m F(x_0; h_1, \ldots, h_m)$$

$$= d^{m+1} F(x_0; h_1, \ldots, h_{m+1}) + R(x_0; h_1, \ldots, h_{m+1}) \qquad (6)$$

where the mapping $d^{m+1} F(x_0; h_1, \ldots, h_{m+1})$ is linear and continuous in h_{m+1} and

$$\lim_{h_{m+1} \to \theta} \| h_{m+1}\|^{-1} \| R(x_0; h_1, \ldots, h_m, h_{m+1})\| = 0 \qquad (7)$$

If such a representation exists, it is unique and $d^{m+1} F(x_0; h_1, \ldots, h_{m+1})$ is called the m + 1)-th F-differential of F at x_0. The operator $d^{m+1} F(x_0, \ldots)$ is called the $(m+1)$th Fréchet derivative.

Remarks 4. a. $d^n F(x; h_1, \ldots, h_n)$ is linear and continuous in $h_1, \ldots, h_n$ respectively.

b. If df is a continuous mapping from A into $\mathcal{L}(X; Y)$ and if it is twice differentiable at x_0 then the bilinear mapping $(h_1, h_2) \longrightarrow dF(x_0; h_1, h_2)$ is symmetric in h_1, h_2.

By induction we can prove the theorem for the m-th order differential.

c. If F has an m-th order F-differential at x_0, then the m-th order G-differential exists at x_0 and hence the m-variation exists as well and moreover

$$d^m F(x_0; h_1, \ldots, h_m) = \frac{\partial^m}{\partial t_1 \cdots \partial t_m} F(x_0 + \sum_{i=1}^{m} t_i h_i) \Big|_{t_1 = t_2 = \cdots = t_m = 0} \qquad (8)$$

d. Counter examples are available to show that

G-variation $\not\Longrightarrow$ G-differential $\not\Longrightarrow$ F-differential, (see, [3,9,14]).

e. If the n-th G-variation, G-differential, F-differential of a function f exists at a point x then all lower order differential exists as well at x.

We note that the Frechet differential enjoys most of the properties of the ordinary derivative [3] .

3. Taylor's Expansion for Operators

Proposition 1. If $f: X \longrightarrow Y$ has an expansion at x_0 of the form

$$f(x_0 + h) = H_0 + H_1(x_0; h) + \cdots + H_n(x_0; h) + \alpha_n(x_0; h)$$

where H_0 is a constant function H_i is homogeneous of i-th order in h, $1 \leq i \leq n$ and $\alpha_n(x;\ th) = o(t^n)$ where $t \in R$ then the expansion is unique.

Proof:

Assume there exists another expansion of $f(x_0 + h)$ with the properties stated in theorem, say,

$$f(x_0 + h) = K_0 + K_1(x_0;\ h) + \dots + K_n(x_0;\ h) + \beta_n(x_0;\ h)$$

where K_0 is constant K_i is homogeneous of i-th degree $1 \leq i \leq n$ and $\beta_n(x_0;\ th) = o(t^n)$.

Then

$$f(x_0 + th) = H_0 + tH(x_0;\ h) + \dots + t^n H_n(x_0;\ h) + \alpha_n(x_0;\ th)$$

$$= K_0 + tK_1(x_0;\ h) + \dots + t^n K_n(x_0;\ h) + \beta_n(x_0;\ th)$$

Letting $t \to 0$ it implies $H_0 = K_0$

$$\therefore\ tH_1(x_0;h) + \dots + t^n H_n(x_0;h) + \alpha_n(x_0;th) = tK_1(x_0;\ h) + \dots + t^n K_n(x_0;h) + \beta_n(x_0;\ th).$$

Dividing by t and taking limit as $t \to 0$ implies $H_1(x_0;h) = K_1(x_0;\ h)$ Continuing inductively we obtain desired result.

Proposition 2. A multilinear continuous form $H_n: X^n \longrightarrow Y$ has F-differential of all orders at every point of X^n.

Proof:

Let $x = (x_1, x_2, \dots, x_n)$ and $h = (h_1, \dots, h_n)$ then

$$VH_n(x;\ h) = \lim_{t \to 0} \frac{H_n(x + th) - H_n(x)}{t}$$

$$VH_n(x;\ h) = H_n(x_1, \dots, x_{n-1}, h_n) + H_n(x_1, \dots, x_{n-2}, h_{n-1}, x_n) + \dots + H_n(h_1, x_2, \dots, x_n) \quad . \qquad (9)$$

This is clearly linear and continuous in h, hence $VH_n(x; h) = DH_n(x; h)$. Moreover

$$\frac{R(x, th)}{t} = \frac{H_n(x_1, th_1, \ldots, th_n) + H_n(th_1, x_2, th_2, \ldots, th_n) + \ldots + H_n(th_1, \ldots, th_n, x_n) + H_n(th_1, \ldots, th_n, x_n)}{t}$$

$$= t^{n-2}H_n(x_1, h_2, \ldots, h_n) + \ldots + t^{n-2}H_n(h_1, \ldots, h_{n-1}, x_n) + t^{n-1}H_n(h_1, \ldots, h_n)$$

$\frac{R(x; th)}{t} \to 0$ as $t \to 0$ independent of h.

$$DH_n(x; h) = dH_n(x; h) .$$

Moreover since (9) is n-1 multilinear continuous operator we can differentiate it again and obtain $d^2f(x; h)$ as an n-2 multilinear continuous operator, continuing inductively we can prove that $d^nf(x; h)$ is independent of x and hence F-differentials of H_n, $n \geq 2$ at x exist and are the appropriate zero operators.

<u>Proposition 3.</u> Let F be an operator from the space X into Y whose n-th G-differentials exists and let e be an arbitrary but fixed element of $Y^* = \mathcal{L}(Y, R)$. Then $H = eF$ has an n-th differentials at x and $D^nH(x; h) = eD^nF(x;h)$.

<u>Proof</u>:

For $n = 1$

$$DH(x; h) = \lim_{t \to 0} \frac{H(x + th) - H(x)}{t}$$

$$= \lim_{t \to 0} \frac{eF(x + th) - eF(x)}{t}$$

$$= e \lim_{t \to 0} \frac{F(x + th) - F(x)}{t} = eDF(x;h).$$

Similarly for higher order differentials.

<u>Theorem 6.</u> Assume $F: A \to Y$ has Gateaux differentials up to

the n + 1-st order for every $x \in A$. Then for every $x, h \in A$ with $\overline{xh} = \{x + th \mid t \in [0, 1]\} \subset A$ the following expansion is valid

$$F(x + h) = F(x) + DF(x;h) + \frac{1}{2!} D^2F(x;h) + \dots + \frac{1}{n!} D^nF(x; h) + \frac{1}{(n+1)!} D^{n+1} F(x + \xi h; h), \text{with } 0 \le \xi \le 1 \quad (11)$$

Proof:

Define $\emptyset: [0, 1] \to A$ by $\emptyset(t) = x + th$.

Let e be an arbitrary element of Y^* and consider the composite mapping.

$$H(t) = e(F(\emptyset(t)) = e(F(x + th)).$$

Since this mapping is from the reals to the reals and has the n-th ordinary derivative at every point of $[0, 1]$, we can write

$$H(1) = H(0) + H^{(1)}(0) + \frac{1}{2!} H^{(2)}(0) + \dots + \frac{1}{n!} H^{(n)}(0) + \frac{1}{(n+1)!} H^{(n+1)}(\xi); 0 \le \xi \le 1.$$

or, using proposition 3,

$$e(F(x+h) = eF(x) + eDF(x, h) + \frac{1}{2!} eD^{(2)}F(x; h) + \dots + \frac{1}{n!} eD^{(n-1)}F(x;h) + \frac{1}{(n+1)!} eD^{(n)}F(x + \xi h; h)$$

$$e\left[F(x+h) - F(x) - \sum_{k=1}^{n} \frac{1}{k!} D^kF(x;h) - \frac{1}{(n+1)!} D^{n+1}F(x + \xi h; h)\right] = 0$$

applying (corollary to the Hahn-Banach theorem) $\Longrightarrow \exists, e^* \in Y^* \ni$

$$\left| e^*\left\{F(x+h) - F(x) - \sum_{k=1}^{n} \frac{1}{k!} D^kF(x;h) - \frac{1}{(n+1)!} D^{n+1}F(x + \xi h; h)\right\}\right|$$

$$= \left\| F(x+h) - F(x) - \sum_{k=1}^{n} \frac{1}{k!} D^kF(x;h) - \frac{1}{(n+1)!} D^{n+1}F(x + \xi h; h)\right\| = 0$$

hence

$$F(x+h) = F(x) + \sum_{k=1}^{n} \frac{1}{k!} D^kF(x;h) + \frac{1}{(n+1)!} D^{n+1}F(x + \xi h; h).$$

Note: The expansion could be obtained without the assumption that it is known for ordinary functions.

We shall now pass to introduce several notions of differentials of order n, m $\geqslant 2$ which are variants of the notions used for ordinary functions.

4. Riemann, Peano and Taylor n-th order Differentials

Definition 12. The n-th Riemann Differential

(a) Weak form

Let $f: A \longrightarrow Y$ and define

$$\triangle_h^n f(x) = \sum_{k=1}^{n} (-1)^{n-k} \binom{n}{k} f(x + (k - \tfrac{n}{2})h).$$

If $\lim\limits_{t \to 0} \frac{1}{t^n} \triangle_{th}^n (f(x))$ exists then we call it the n-th weak Riemann differential of f at x and we shall denote it by $R^n f(x; h)$

(b) Strong form

If $\exists$ a function $r^n f(x; h) \ni$

$$\frac{\| \triangle_h^n f(x) - r^n f(x; h) \|}{\| h \|^n} \to 0 \quad \text{as } h \to \theta$$

we call $r^n f(x; h)$ the n-th strong Riemann differentials of f at x.

Definition 13. n-th Peano differential

$f: X \longrightarrow Y$. If $\exists$ a polynomial

$P_n(h) = H_1(h) + \frac{1}{2!} H_2 + \dots + \frac{1}{n!} H_n(h)$ where H_i is homogeneous of i'th degree in h, $\ni$

$$f(x + h) = f(x) + P_n(h) + \alpha_n(h) \quad ,$$

where

$$\alpha_n(th) = O(t^n) \text{ then } H_n(h) = P^n f(x; h) \quad ,$$

is called the n-th weak Peano differential.

(b) Strong form.

f is said to have an n-th strong Peano differentials at x if $\exists$ a polynomial

$$P_n(h) = L_1(h) + \frac{1}{2!} L_2(h) + \dots + \frac{1}{n!} L_n(h)$$

L_i is a linear continuous form in h $\ni$

$$f(x + h) = f(x) + P_n(h) + \alpha_n(x; h)$$

where

$$\frac{\| \alpha_n(x; h) \|}{\| h \|^n} \to 0 \text{ as } h \to \theta$$

then

$$L_n h^n = p^n f(x; h).$$

<u>Definition 14.</u> <u>The n-th Taylor Differentials</u>

(a) Weak form $f: X \longrightarrow Y$.

If the (n - 1)st. Gateaux variations exist then the n-th weak Taylor differentials is defined by

$$T^n f(x; h) = \lim_{t \to 0} \frac{n!}{t^n} \left[f(x + th) - \sum_{k=0}^{n-1} \frac{1}{k!} V^k f(x; th) \right]$$

whenever this limit exists.

(b) Strong form

If $\exists$ a function of h $\tau^n f(x; h)$ $\ni$

$$\frac{\left\| n!\, f(x + h) - \sum_{h=0}^{n-1} \frac{1}{K!} d^k f(x; h) - \tau^n f(x; h) \right\|}{\| h \|^n} \to 0 \text{ as } h \to \theta$$

where $d^k f(x; h)$ is the k-th Frechet differentials of f at x.

<u>Definition 15.</u> If $f(x + th) = \phi(t)$ has <u>an n-th Riemann differential</u> at every point $t \in [0, 1]$

and

$$\lim_{\tau\to 0} \frac{1}{\tau^n} \Delta^n_{\tau h} f(x+th) = \lim_{\tau\to 0} \frac{1}{\tau^n} \Delta^n_h \phi(t)$$

is uniform in the neighborhood of every point t then $R^n f(x+th;h)$ is called the weak <u>Uniform Riemann differential of order n.</u>

We shall prove that we have the following order for the above defined n-th order differentials (listed in order of descending generality).

Weak Riemann	Weak Peano	Weak Taylor	G-variation.
W.R	W.P.	W.T.	G-V

<u>Theorem 7.</u> If the n-th Peano differential of f at x exists then the n-th Riemann differential of f at x exists and the two differentials are equal i.e. (W.P. $\Longrightarrow$ W.R.).

Since f has an n-th Peano differential at x, then

<u>Proof</u>:

$$f(x+th) = f(x) + tH(h) + \dots + \frac{t^n}{n!} H_n(h) + \alpha_n(x;\, th) \tag{12}$$

where

$$\lim_{t\to 0} \frac{\alpha_n(x;\, th)}{t^n} = 0$$

now

$$\frac{\Delta^n_\tau \ \alpha_n(th)\big|_{t=0}}{\tau^n} = \frac{1}{\tau^n} \sum_{k=0}^{n} (-1)^{n-k}\binom{n}{k} \alpha_n(x;\, th + h(k-\tfrac{n}{2})\tau)\Big|_{t=0}$$

$$= \frac{1}{\tau^n} \sum_{k=0}^{n} (-1)^{n-k}\binom{n}{k} \alpha_n(x;\, h(k-\tfrac{n}{2})\tau)$$

But

$$\frac{\alpha_n(x;\, h(k-\frac{n}{2})\tau}{\tau^n} \to 0 \text{ as } \tau \to 0,$$

$$\therefore \lim_{\tau\to 0} \frac{1}{\tau^n} \sum_{k=0}^{n} (-1)^{n-k}\binom{n}{k} \alpha_n(x;\, h(k-\tfrac{n}{2})\tau) = 0,$$

hence taking the n-th Riemann differential of (12) at x we obtain

$$\lim_{\tau\to 0} \frac{1}{\tau^n} \Delta^n_\tau f(x+th)\Big|_{t=0} = 0 + \dots + 0 + \frac{n!}{n!} H_n(h) + \lim_{\tau\to 0} \frac{\Delta^n_\tau \ \alpha_n(th)}{\tau^n}$$

$$= H_n(h).$$

Hence the n-th weak-Riemann differential of f at x = n-th weak Peano differential of f at x.

Lemma. $T^n f(x; h)$ is homogeneous in h of n-th degree

Proof:

$$T^n f(x; \lambda h) = \lim_{t \to 0} \frac{n!}{t^n}\left[f(x+t\lambda h) - \sum_{k=0}^{n-1} \frac{1}{k!} V^k f(x; t\lambda h)\right]$$

$$= \lambda^n \lim_{t \to 0} \frac{n!}{\lambda^n t^n}\left[f(x+t\lambda h) - \sum_{k=0}^{n-1} \frac{1}{k!} V^k f(x; t\lambda h)\right]$$

$$= \lambda^n \lim_{\tau \to 0} \frac{n!}{\tau^n}\left[f(x+\tau h) - \sum_{k=0}^{n-1} \frac{1}{k!} V^k f(x; \tau h)\right]$$

$$= \lambda^n T^n f(x; h).$$

Theorem 8. If the n-th weak Taylor differential exists and is finite then the n-th weak Peano differential exists. i.e (W.T. $\Longrightarrow$ W.P.)

Proof:

Definition (14a) can actually be written as

$$f(x+h) = f(x) + Vf(x; h) + \dots + \frac{1}{(n-1)!} V^{n-1} f(x; h) + \frac{T^n f(x;h)}{n!} + \alpha_n(x;h)$$

where

$$\lim_{t \to 0} \frac{\alpha_n(x; th)}{t^h} = 0.$$

Applying the above Lemma it follows that the n-th weak Peano differential at x exists and is equal to the n-th Taylor differential at x.

Theorem 9. If the n-th Gateaux variation exists at a point x then the n-th weak Taylor differential exists and the two are equal.

Proof:

See Chapter III, Theorem 1.

We shall show by counter examples that in general

(W.R.) $\not\Longrightarrow$ (W.P.) $\not\Longrightarrow$ (W.T.) $\not\Longrightarrow$ (G-V)

1. Let $f(x) = x\,|x|$,

where $x \in R$

then the second Riemann differential exists at $x = 0$.

$$R_2 f[0;\, h] = \lim_{\tau \to 0} \frac{h\tau\,|h\tau| - \tau h\,|-\tau h|}{\tau^2} = 0.$$

The first order ordinary differential of f at $x = 0$ exist and is equal to zero hence the first order Peano differential exists and is equal to zero. However the second order Peano differential does not exist at $x = 0$ because otherwise,

$$t\,|t| = a_1(0)t + a_2(0)\frac{t^2}{2!} + o(t^2)$$

but $a_1(0)\, t = 0$, hence $t\,|t| = a_2(0)\frac{t^2}{2!}$

which implies that $a_2(0)$ depends on t

$a_2(0) = 2$ for $t > 0$

$a_2(0) = -2$ for $t < 0$

hence the second order Peano differential at $x = 0$ does not exist

i.e. W.R. $\not\Rightarrow$ W.P.

2. Let

$$f(x) = \begin{cases} e^{-x^{-2}} \sin e^{x^{-2}} & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Then the third order Peano differential at $x = 0$ exists and is equal to zero. Because we can have

$$e^{-h^{-2}} \sin e^{h^{-2}} = 0 + 0h + \frac{0h^2}{2!} + \frac{0h^3}{3!} + o(h^3)$$

hence $a_3(0) = 0$.

However the third order Taylor differential does not exist at $x = 0$ because the second order ordinary differential does not exist at $x = 0$.

$\therefore$ WP $\not\Rightarrow$ WT

3. Let

$$f(x) = \begin{cases} x^3 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Then it is easy to show that $f'(0) = 0$. Define $M_2(h)$ by

$$f(h) = f(0) + f'(0)h + M_2(h)h^2 , \; h \neq 0$$

i.e.,

$$f(h) = 0 + 0.h + M_2(h)h^2$$

Then

$$\lim_{h\to 0} M_2(h) = \lim_{h\to 0} h \sin \frac{1}{h} = 0$$

exists; hence the second order Taylor differential $x = 0$ exists and is equal to 0.

But $f''(x)$ does not exist at 0.

Hence W.T. $\not\Rightarrow$ G.V.

Sufficient conditions under which

W.R. $\Longrightarrow$ G.-V., W.P. $\Longrightarrow$ W.T., W.T. $\Longrightarrow$ G.-V.

are given in Theorem 10, 11, below and Theorem 1 Chapter III, respectively.

<u>Theorem 10.</u> If $f: I \longrightarrow Y$ has a uniform n-th Riemann differential at $t_o \in I$ and is uniformly continuous in a neighborhood of t_o, then the ordinary n-th G-variation exists at t_o and the two differentials are equal.

Proof:

Let e be an arbitrary but fixed element in Y^* define the composite mapping

$$\phi(t) = ef(t)$$

then

$$R^n\phi(t_o) = \lim_{\tau\to 0} \frac{1}{\tau^n} \sum_{k=0}^{n} (-1)^{n-k}\binom{n}{k}\phi(t_o - (k - \tfrac{n}{2})\tau)$$

$$= e \lim_{\tau\to 0} \frac{1}{\tau^n} \sum_{k=0}^{n} (-1)^{n-k}\binom{n}{k} f(t_o - (k - \tfrac{n}{2})\tau)$$

because e is linear and continuous

$$R^n \phi(t_o) = e R^n f(t_o).$$

The n-th ordinary differential of ϕ at t is equal to $e(f^{(n)}(t_o))$ as well but by ([8] p. 216).

$$\phi^{(n)}(t_o) = \phi^{[n]}(t_o) =$$

$$e(f^{(n)}(t)) = e(f^{[n]}(t))$$

applying Hahn Banach theorem we can find an $e^* \in Y^* \ni$

$$\| f^{[n]}(t) - f^{(n)}(t) \| = 0.$$

Theorem 11. If $f: X \longrightarrow Y$ has an n-th order weak Peano differential at x and if the n-th G-variations of f at x exists then the n-th order weak Taylor differential at x exists.

Proof:

This follows from Proposition 1.

5. Direct and Difference Differentials

The Riemann differential is one form of a difference differential.

Another form is the direct differential defined below.

Let f be a mapping from A an open subset of X into Y. Let

$$\Delta'\Delta\, f(x;h,h') = f(x+h+h') - f(x+h') - f(x+h) + f(x).$$

<u>Definition 16.</u> We shall say that f is <u>directly second order differentiable in Gateaux sense</u> at x if

$$\lim_{t,t'\to 0} \frac{\Delta'\Delta\, f(x;\, th,\, t'h')}{tt'}\; ;\; t,\, t' \in R$$

exists for any h and h'. Assuming this limit exists, we shall denote it by $\delta'\delta\, f(x;\, h,\, h')$. Clearly $\delta'\delta\, f(x;\, h,\, h')$ is homogeneous in h and h'. If $\delta'\delta\, f(x;\, h,\, h')$ is linear and continuous in h and h' we shall denote it by $DD'f(x;\, h,\, h')$.

If $\exists$ a continuous bilinear transformation ϕ: $X \times X \longrightarrow Y$ depending on f and x $\ni$

$$\lim_{h',h\to\theta} \frac{\|\Delta'\Delta\, f(x;\, h,\, h') - \phi(x;\, h,\, h')\|}{\|h\| \quad \|h'\|} = 0,$$

then we shall say f is <u>directly second order differentiable in Frechet's sense</u> at the point x and by definition $\phi(x;\, h,\, h')$ is the direct second order differential in Frechet's sense. We shall put $\phi(x;\, h,\, h') = d'd\, f(x;\, h,\, h')$.

Let $\langle f, e\rangle$ denote the value of e at f

<u>Remark 6.</u> If $f: X \longrightarrow Y$ has a direct second order G-differential at x then the following mean value theorem holds

$$\langle \Delta'\Delta f(x;h,h'),\, e\rangle = \langle \delta'\delta\, f(x+\theta h+\theta' h;h,h'),\, e\rangle,\ 0 \leq \theta,\, \theta' \leq 1\ \forall e \in Y^*$$

using the Hahn-Banach theorem we can actually write

$$\| \Delta'\Delta f(x;\ h,\ h') \| \leq \| \delta'\delta\ f(x+\theta h+\ \theta' h';h,h') \|$$

If $\delta'\delta f(x;\ h,\ h')$ is linear and continuous in h and h' we can moreover write

$$\| \Delta'\Delta\ f(x;\ h,h') \| \leq \|h\| \|\ h'\| \| D'Df(x+\theta h+\theta' h';\ldots) \|$$

<u>Theorem 12.</u> If $\delta'\delta f(x;\ h,h')$ exists in a neighborhood of a point x_0 and is continuous in x at x_0 and continuous in h and h' at θ, then $\delta'\delta\ f(x,\ h,\ h')$ is bilinear and continuous in h and h'.

<u>Proof</u>:

Since $\delta'\delta\ f(x_0;\ h,\ h')$ is continuous at $h = \theta$, $h' = \theta$ $\exists\ m > 0$, $m' > 0$ and $M \ni \|h\| \leq m$ and $\|h'\| \leq m'$ imply $\| \delta'\delta f(x_0;\ h,\ h') \| \leq M$. But since $\delta'\delta f(x_0;\ h,\ h')$ is homogeneous in h and h',

$$\| \delta'\delta f(x_0,\ h,h') \| = \left\| \frac{\|h\|}{m} \frac{\|h'\|}{m'}\ \delta'\delta f(x_0;\ \frac{mh}{\|h\|}\ \frac{m'h'}{\|h'\|}) \right\| \leq \frac{M}{m'm} \|h\| \|h'\|$$

Therefore $\delta'\delta f(x_0;\ h,h')$ is a bounded operator.

We shall prove that $\delta'\delta\ f(x_0;\ h,\ h')$ is additive in each of h and h'. We carry the proof for h'. We shall put $\delta'\delta\ f(x_0;h,h')$ $= \delta'\delta\ f_{x_0}(h,\ h')$. *Then*

$$\delta'\delta f_{x_0}(h_1,h') = \frac{\Delta'\Delta f(x_0;\ th,\ t'h')}{tt'} + \alpha_1$$

$$\delta'\delta f_{x_0}(h_2,\ h') = \frac{\Delta'\Delta f(x_0;\ th_2,\ t'h')}{tt'} + \alpha_2$$

$$\delta'\delta f_{x_0}(h_1 +\ h_2;\ h') = \frac{\Delta'\Delta f(x_0,\ th_1 +\ th_2,\ t'h')}{tt'} + \alpha_3$$

where $\lim_{t,t' \to 0} \alpha_i = 0$ for $i = 1, 2, 3$.

$$\therefore \| \delta'\delta f_{x_0}(h_1+h_2, h') - \delta'\delta f_{x_0}(h_1,h') - \delta'\delta f_{x_0}(h_2,h')\| \leq \varepsilon/4 +$$

$$\frac{1}{tt'}\| f(x_0 + th_1 + th_2 + t'h') - f(x_0 + th_1 + th_2) - f(x_0 + th_2 + t'h)$$

$$+ f(x_0 + th_2) - f(x_0 + th_1 + t'h') + f(x_0 + th_1) + f(x_0 + t'h') - f(x_0)\|$$

We shall show that the last term $\to 0$ as $t', t \to 0$.

Let e be an arbitrary but fixed element in Y^* then making use of the mean value theorem.

$$\frac{1}{tt'}\langle f(x_0 + th_1 + th_2 + t'h') - f(x_0 + th_1 + th_2) - f(x_0 + th_2 + t'h') +$$

$$f(x_0 + th_2) - f(x_0 + th_1 + t'h') + f(x_0 + th_1) + f(x_0 + t'h') - f(x_0), e\rangle$$

$$= \langle \delta'\delta f(x_0 + th_2 + \theta_1 h' + \theta_2 h', h_1 h_1') - \delta'\delta f(x_0 + \phi_1 h_1 + \phi_2 h'; h_1 h'), e\rangle$$

where $0 \leq \theta_1, \theta_2 \leq t$ and $0 \leq \phi_1, \phi_2 \leq t'$.

Applying the Hahn-Banach theorem we can find an $e \ni$

$$\left| \frac{1}{tt'} e\{ f(x_0 + th_1 + th_2 + t'h') - f(x_0 + th_1 + th_2) - f(x_0 + th_2 + t'h') + \right.$$

$$\left. f(x_0 + th_2) - f(x_0 + th_1 + t'h') + f(x_0 + th_1) + f(x_0 + t'h') - f(x_0)\}\right|$$

$$= \frac{1}{tt'}\| f(x_0 + th_1 + th_2 + t'h') - f(x_0 + th_1 + th_2) - f(x_0 + th_2 + t'h') +$$

$$f(x_0 + th_2) - f(x_0 + th_1 + t'h') + f(x_0 + th_1) + f(x_0 + t'h') - f(x_0)\|$$

$$\leq \| \delta'\delta f(x_0 + th_2 + \theta_1 h_1 + \theta_2 h'; h_1, h') - f(x_0 + \phi_1 h_1 + \phi_2 h'; h_1, h')\| \to 0$$

as $t \to 0$ because $\delta'\delta f(x; hh')$ is continuous in x.

Similarly we can prove that $\delta'\delta f(x; h,h')$ is addetive in h'.

<u>Proposition 4.</u> Under the conditions of Theorem 12, $d'df(x_0;h,h')$ exists.

<u>Proof</u>:

Let $f: X \longrightarrow Y$

$$f(x_o + h + h') - f(x_o + h) - f(x_o + h') + f(x_o) = D'Df(x_o; h, h') + R(x_o, h, h')$$

we shall show that $\frac{\| R(x_1; h, h')\|}{\| h\| \| h'\|} \rightarrow 0$ as $h, h' \rightarrow \theta$.

Let e be an arbitrary element of Y^*

then,

$$\langle R(x_o; h, h'), e\rangle = \langle f(x_o + h + h') - f(x_o + h) - f(x_o + h') + f(x_o), e\rangle$$

$$- \langle D'D f(x_o; h, h'), e \rangle$$

$$= \langle D'D f(x_o + \theta h + \theta' h'; h, h'), e\rangle - \langle D'Df(x_o; h, h'), e\rangle$$

$$0 \leqslant \theta, \theta' \leqslant 1$$

$$\therefore \| R(x_o; h, h') \| \leqslant \| D'Df(x_o + \theta h + \theta' h', \ldots) - D'Df(x_o, \ldots) \| \, \| h \| \, \| h' \|$$

But since $D'Df$ is continuous at x

$$\Longrightarrow \quad \frac{\| R(x_o; h, h')\|}{\| h\| \, \| h'\|} \rightarrow 0.$$

as $h, h' \rightarrow \theta$.

The following simple example shows that the direct second order Gateaux differential does not imply the existence of the first order variation of the function: let $f: R \rightarrow R$ be defined by $f(x) = |x|$ then one can easily check that $\delta' \delta f(0; h, h') = 0$ where as $f'(0)$ does not exist.

CHAPTER II

MEAN VALUE THEOREM AND VARIANTS OF TAYLOR'S THEOREM IN GENERAL ANALYSIS

We shall discuss first various forms of the mean value theorem which is "probably the most useful theorem in analysis" It should not be astonishing that the mean value theorem will appear in the form of an inequality rather than an equality as in the case of ordinary functions, where we have $f(b) - f(a) = (b - a)f'(c)$ where $c \in [a, b] \subset R$. The reason for this is that division is not allowed once we are dealing with vectors. At any rate that is not a disadvantage on the contrary, "... the real nature of the mean value theorem is exhibited by writing it as an inequality and not as an equality ..." for the equality "... completely concels the fact that nothing is known on the number c except that it lies between a and b, and for most purposes, all one needs to know is that $f'(c)$ is a number which lies between the g. l. b. and l. u. b. of f' in the interval a, b (and not the fact that it actually is a value of f') [3, p. 142]. We then apply the mean value theorem to prove a generalization of the difference quotient differential. In the second part of this chapter we consider Taylor's expansion for operators and show how one can prove it without recourse to the expansion for ordinary functions.

1. Mean Value Theorem and Difference Quotients

Theorem 1. [13] Let A be a convex subset of X. If the operator $f: A \longrightarrow Y$ has a Gateaux differential $Vf(x; h)$ at each point of A, then for every x, x + h in A we have $\| f(x+h) - f(x) \| \leq \| Vf(x + \tau h; h) \|$ where $0 \leq \tau \leq 1$.

Proof:

Let $\phi: I \longrightarrow A$ be defined by $\phi(t) = x + th$ and e be an arbitrary element in Y^*. Then $e(f(\phi(t)) = H(t)$ is a function whose domain and range are the re al numbers and

$$H'(t) = \lim_{t \to 0} \frac{H(t + \Delta t) - H(t)}{t} = e(Vf(x + th, h)).$$

But $H(1) - H(0) = H'(\tau)$ for some τ, $0 \leq \tau \leq 1$, by the mean value theorem for ordinary functions. Hence,

$$e(f(x+h)) - e(f(x) = eVf(x + \tau h; h).$$

By the Hahn-Banach theorem an $e \in Y^*$ can be found $\ni$

$$| ef(x+h) - ef(x) | = \| f(x+h) - f(x) \| \leq \| Vf(x + \tau h; h) \|.$$

Note: Stating the same theorem replacing Vf by Df we can write above result in the following form

$$\| f(x+h) - f(x) \| \leq \| Df(x + \tau h; \cdot) \| \, \|h\|,$$

which looks closer to the mean value theorem for ordinary functions.

In the sequel, we shall use the following notations interchangably; $df(x; h)$, $f'(x; h)$, $f'_x(h)$. The choice of a particular notations depends on what we think fits best the situation at hand.

Dieudonne proves the same theorem without preassuming the knowledge of the mean value theorem for ordinary functions his proof depends on the following theorem.

Theorem 2. [3] Let $J = [\alpha, \beta]$ be a compact interval in R, f a continuous mapping of J into X, ϕ a continuous mapping of J into R. We suppose that there is a denumerable subset $D \ni, \forall \xi \in J - D$, f and ϕ have both a derivative at ξ with respect to J and that

$$\| f'(\xi; \cdot) \| \leq \phi'(\xi; 1)$$

Then

$$\| f(\beta) - f(\alpha) \| \leq \phi(\beta) - \phi(\alpha).$$

Corollary. If there is a denumerable subset D of $J \ni, \forall \xi \in J - D$, f has at ξ a derivative with respect to $J \ni \| f'(\xi; \cdot) \| \leq M$ then

$$\| f(\beta) - f(\alpha) \| \leq M(\beta - \alpha).$$

Note: Dieudonne considers only F-differentials but since $f: J \to X$ then the F-differentials, G-differentials coincide in this case.

Theorem 3. [3] Let $f: A \to Y$ be a continuous mapping of A a neighborhood of the segment S joining two points x_o and $x_o + h$ of A. If f is differentiable at every point of S, then

$$\| f(x_o + h) - f(x_o) \| \leq \|h\| \sup_{0 \leq \xi \leq 1} \| f'(x_o + \xi t; \cdot) \|$$

Proof:

Let $\phi: I \to A$ be defined by $\phi(t) = x_o + th$ and let $g(t) = f(\phi(t))$ then $g: R \to R$. Differentiating this composite mapping $g_t'(\cdot) = f'_{\phi(t)}(\phi_t'(\cdot))$ see, [3, p. 145]

$$g_t'(\cdot) = f'_{x_o + th}(h(\cdot)).$$

Hence

$$\| g_t'(\cdot)\| \leq \sup_{0 \leq t \leq 1} \| f'_{x_o + th}(h(\cdot)\| \leq \| h\| \sup_{0 \leq t \leq 1} \| f'_{x_o + th}(\cdot)\|$$

By corollary to Theorem 2 it follows that

$$|g(1) - g(0)| = \| f(x_o + h) - f(x_o)\| \leq (1-0)\| h\| \sup_{0 \leq t \leq 1} \| f'_{x_o + th}(\cdot)\|.$$

The mean value theorem could be expressed in integral form as well. To do this we shall define and briefly discus the notion or integration for abstract functions.

Definition 1. A continuous mapping $g: I \longrightarrow Y$ is a primitive of f in I if there exists a denumerable set $D \subset I \ni, \forall \xi \in I - D$, g is differentiable at ξ and $g'(\xi) = f(\xi)$.

It follows immediately that if g_1, g_2 are two primitives of f in I, then $g_1 - g_2$ is a constant in I.

Theorem 4. [3]. Any continuous mapping of I into R has a primitive in I, and this primitive has at every $\xi \in I$ a derivative with respect to I equality $f(\xi)$.

Definition 2. If g is any primitive of a continuous functions, the difference $g(\beta) - g(\alpha)$ for any $\alpha, \beta \in I$ is independent of the particular primitive g which is considered; it is written $\int_\alpha^\beta f(\xi)d\xi$ and called the integral of f between α and β.

Any formal rule of derivation can be translated into this notation and yields a corresponding formula of "integral calculus", for details see [3, p. 159-166].

In particular the mean value theorem has the following translation.

<u>Theorem 5.</u> [3]. For any continuous function f in a compact interval $[\alpha, \beta]$

$$\left\| \int_{\alpha}^{\beta} f(\xi)\, d\xi \right\| \leq \int_{\alpha}^{\beta} \| f(\xi) \| d\xi \leq (\beta - \alpha) \sup_{\xi \in I} \| f(\xi) \| .$$

<u>Proof:</u>

Let $F'(\xi) = f(\xi)$ and $\phi(\xi) = \xi \sup_{\xi \in I} f(\xi)$ then $\phi'(\xi) = \sup_{\xi \in I} \| f(\xi) \|$.

Hence

$$\| F'(\xi) \| = \| f(\xi) \| \leq \| \phi'(\xi) \| = \sup_{\xi \in I} \| f(\xi) \| , \forall \xi \in I$$

Therefore

$$\| F(\beta) - F(\alpha) \| \leq | \phi(\beta) - \phi(\alpha) | ,$$

$$\left\| \int_{\alpha}^{\beta} f(\xi) d\xi \right\| \leq (\beta - \alpha) \sup_{\xi \in I} \| f(\xi) \| .$$

We shall apply the mean value theorem now to prove a generalization of a difference quotient differential.

Let f be an n times F-differentiable mapping of A into Y. Let $x_o \in A$, $h_i \in X (1 \leq i \leq n)$ be $\ni$ $x_o + \sum_{i=1}^{n} \xi_i h_i \in A$ for $0 \leq \xi_i \leq 1$, $1 \leq i \leq n$. We define by induction on $k (1 \leq k \leq n)$

$$\Delta^1 f(x_o; h_1) = f(x_o + h_1) - f(x_o)$$

$$\Delta^k f(x_o; h_1, \ldots, h_k) = \Delta^{k-1} g_k(x_o; h_1, \ldots, h_{k-1})$$

with

$$g_k(x) = f(x + h_k) - f(x)$$

using the above terminology we state.

<u>Theorem 6.</u> 1. $\| \Delta^n f(x_o; h_1, \ldots, h_n) \| \leq \| h_1 \| \ldots \| h_n \| \sup_{z \in P_n} \| d^n f(z) \|$

where

$$P_n = \left\{ z \mid z = x_o + \sum_{i=1}^{n} \xi_i h_i ;\ 0 \leq \xi_i \leq 1 \right\}$$

and

2. $\| \Delta^n f(x_o; h_1, \ldots, h_n) - d^n f(x_o; h_1, \ldots, h_n) \|$

$$\leq \|h_1\| \ldots \|h_n\| \sup_{z \in P_n} \| d^n f(z) - d^n f(x_o) \| .$$

<u>Proof</u>:

1. By induction. If n = 1 it immediately follows from the mean value theorem that

$$\| \Delta^1 f(x_o; h_1) \| = \| f(x_o + h_1) - f(x_o) \| \leq \|h_1\| \sup_{z \in P_1} \| d' f(z; \quad) \|$$

$$P_1 = \left\{ z \,\middle|\, z = x_o + \xi_1 h_1; \; 0 \leq \xi_1 \leq 1 \right\}$$

assume statement true for n = k i.e. for every function $g: A \longrightarrow Y$ which is k-times F-differentiable one has

$$\| \Delta^k g(x_o; h_1, \ldots, h_k) \| \leq \|h_1\| \, \|h_2\| \ldots \|h_k\| \sup_{z \in P_k} \| d^k g(z, \ldots) \| .$$

Let

$$g(x) = f(x + h_{k+1}) - f(x)$$

where h_{k+1} in X satisfies the condition

$$x_o + \sum_{i=1}^{k+1} \xi_i h_i \in A; \qquad 1 \leq \xi_i \leq k + 1$$

The function $g(x) = f(x + h_{k+1}) - f(x)$ has a k-th differential at every point of A,

$$d^k g(x_o; h_1, \ldots, h_k) = d^k f(x_o + h_{k+1}; h_1, \ldots, h_k) - d^k f(x_o; h_1, \ldots, h_k),$$

and

$$\Delta^k g\,(x_o; h_1 \ldots, h_1) = \Delta^{k+1} f(x_o; h_1, \ldots, h_k, h_{k+1})$$

But

$$\| d^k g(z; h_1, \ldots, h_k) \| = \| d^k f(z + h_{k+1}; h_1, \ldots, h_k) - d^k f(z; h_1, \ldots, h_k) \| ,$$

$$\leq \|h_1\| \, \|h_2\| \ldots \|h_{k+1}\| \sup_{z \in P_{k+1}} \| d^{k+1} f(z; \ldots \quad \| ,$$

$$\| h_1\| \cdots \| h_k\| \sup_{z \in P_k} \| d^k g(z;\ldots)\| \leq \| h_1\| \cdots \| h_{k+1}\| \sup_{z \in P_{k+1}} \| d^{k+1} f(z,\ldots)\| .$$

The proof is again by induction.

2. For $n = 1$, we have to prove

$$\| f(x_o + h_1) - f(x_o) - df(x_o; h_1)\| \leq \| h_1\| \sup_{z \in \overline{x_o\, h_1}} \| df(z; \quad) - df(x_o, \quad)\|$$

Let $\mu(z) = f(z) - f'_{x_o}(z)$ then $\Delta\mu(z;h_1) = \Delta(f(z; h_1) - f'_{x_o}(h))$ since $f'_{x_o}(z)$ is linear in z and

$$\mu'(z; h_1) = f'(z; h_1) - f'_{x_o}(h_1)$$

the result then follows by applying the mean value theorem.

Assume the relation true for $n = k - 1$

$$\| \overset{k-1}{\Delta} f(x_o; h_1 \ldots h_{k-1}) - d^{k-1} f(x_o; h_1,\ldots,h_{k-1})\|$$

$$\leq \| h_1\| \cdots \| h_{k-1}\| \sup_{z \in P_{k-1}} \| d^{k-1} f(z;\ldots) - d^{k-1} f(x_o,\ldots)\| .$$

Define

$$\mu(z) = \overset{k-1}{\Delta}[f(z;h_1,\ldots,h_{k-1}) - f^k_{x_o}(h_1 \ldots h_{k-1}, z)],$$

$$\Delta\mu(z;h_k) = \overset{k}{\Delta}[f(z; h_1 \ldots h_{k-1}, h_k) - f^k_{x_o}(h_1 \ldots h_k)],$$

$$\mu'(z;h_k) = d\left[\overset{k-1}{\Delta}(f(z;h_1,\ldots h_{k-1}); h_k) - f^k_{x_o}(h_1,\ldots h_k)\right],$$

and

$$g(x) = Df(x; h_k),$$

then

$$\mu'(z;h_k) = \overset{k-1}{\Delta} g(z; h_1, \ldots, h_{k-1}) - g^{k-1}_{x_o}(h_1,\ldots h_{k-1}).$$

Using the first part of the theorem, we can write

$$\| \Delta\mu(z;h_k)\| = \| \overset{k}{\Delta} f(z;h_1 \ldots h_k) - f^k_{x_o}(h_1 \ldots h_k)\| \leq \| h_k\| \sup_z \| \mu'(z; \quad)\|$$

$$= \| h_k \| \sup_{z} \| \overset{k-1}{\Delta} g(z; h_1 \dots h_{k-1}) - g_{x_0}^{k-1}(h_1 \dots h_{k-1}) \|$$

$$\leq \| h_k \| \; \| h_1 \| \dots \| h_{k-1} \| \sup_{z \in P_{k+1}} \| d^{k-1} g(z; \dots) - d^{k-1} g(x_0, \dots) \|$$

$$= \| h_1 \| \dots \| h_k \| \sup_{z \in P_{k+1}} \| d^k f(z; \dots) - d^k f(x_0, \dots) \|$$

<u>Remarks 1</u>. a. If $n = 1$, then statement (2) would read

$$\| f(x_0 + h_1) - f(x_0) - df(x_0; h_1) \| \leq \| h_1 \| \sup_{z \in \overline{x_0 \; h_1}} \| df(z; \dots) - df(x_0, \dots) \|$$

which is another form of the mean value theorem for operators.

b. Statement (2) could be written as

$$\frac{\| \Delta^n f(x_0; h_1 \dots h_n) - d^n f(x_0; h_1 \dots h_n) \|}{\| h_1 \| \dots \| h_n \|} \leq \sup_{z \in P} \| d^n f(z; \;\;) - d^n f(x_0; \;\;) \|$$

Hence if $d^n f(x; \;\;)$ is continuous in x one obtains

$$\frac{\| \Delta^n f(x_0; h_1 \dots h_n) - d^n f(x_0; h_1 \dots h_n) \|}{\| h_1 \| \dots \| h_n \|} \rightarrow 0 \quad \text{as } h_i \rightarrow \Theta.$$

In this case $d^n f(x_0; h_1 \dots h_n)$ is the n-th direct differential in Frechet sense [11].

2. <u>Taylor's Expensions for Frechet Differentiable Operators of Order n</u>

We shall show how that one can actually obtain Taylor's expansion without the assumption that such an expansion is known for ordinary functions.

<u>Lemma 1.</u> Let f, g be two n-times continuously differentiable mappings of I into X, Y respectively and let $[.] : X \times Y \rightarrow Z$ be a continuous bilinear mapping. Let $h(\xi) = [f(\xi) . g(\xi)] \; \forall \xi \in I$. Then

$$dh(\xi; 1) = [f(\xi) . g'(1)] + [f'_{\xi}(1) . g(\xi)].$$

Proo ;

We first note that the mapping $g' \in \mathcal{L}(I, Y)$ could be identified with the vector $g'_\xi(1)$ of Y Theorem 1, so above equation is meaningful.

Define the composite mapping h by

$$h: I \xrightarrow{\theta} (f(\xi), g(\xi)) \xrightarrow{\phi} [f(\xi).g(\xi)]$$

Hence

$$dh(\xi ; 1) = \phi'_{\theta(\xi)} (\theta'(1))$$

$$dh(\xi ; 1) = \phi'_{(f(\xi), g(\xi))} (f'_\xi(1), g'_\xi(1))$$

$$= [f(\xi).g'_\xi(1)] + [f'_\xi(1).g(\xi)] .$$

Lemma 2. Let f, g be two n-times continuously differentiable mapping of I into X, Y respectively and let $[\,.\,] : X \times Y \to Z$ be a continuous bilinear mapping. Define

$$h_1(\xi) = [f(\xi).\ d^{n-1}g_\xi(1)]$$

$$h_2(\xi) = [df_\xi(1).\ d^{n-2}g_\xi(1)]$$

$$\vdots$$

$$h_n(\xi) = (-1)^{n-1}[d^{n-1}f_\xi(1).g(\xi)]$$

Then

$$[f(\xi).d^n_\xi g(1)] - (-1)^{n-1}[d^n f_\xi(1).g(\xi)]$$

$$= dh_1(\xi;1) - dh_2(\xi;1) \ldots (-1)^{n-1}\ dh_n(\xi ; 1).$$

Proof:

Follows easily by straight application of Lemma 2.

Theorem 7. Let I be an open interval in R, f a function of $\mathcal{E}^n_X(I)$ then for any pair of points $\alpha, \xi \in I$

$$f(\xi) = f(\alpha) + (\xi - \alpha) f'(\alpha;1) + \frac{(\xi-\alpha)^2}{2!} f''(\alpha\ ;\ 1,\ 1) + \dots + \frac{(\xi-\alpha)^{n-1}}{(n-1)!} f^{n-1}(\alpha;1..) + \int_\alpha^\xi \frac{(\xi-\alpha)^{n-1}}{(n-1)!} f^{(n-1)}(\mathcal{S})\,d\mathcal{S}.$$

__Proof__:

Let $f: I \longrightarrow X$ and $g: I \longrightarrow R$, defined by $g(\mathcal{S}) = \frac{(\xi - \mathcal{S})^{n-1}}{(n-1)!}$ and let $[\cdot] : I \times X \longrightarrow X$, be defined by $[\cdot](\lambda, x) = \lambda x$. To get desired result, apply Lemma 2 and integrate between α and ξ.

__Theorem 8.__ Let f be an n-times continuously differentiable mapping of A into Y. Then if the segment joining x and $x + h$ is in A, we have

$$f(x+h) = f(x) + f'(x;h) + \frac{1}{2} f''(x;\ h,\ h) + \dots + \frac{1}{(n-1)!} f^{(n-1)}(x;\ h \dots h) + \int_0^1 \frac{(1-\mathcal{S})^{n-1}}{(n-1)!} f^{(n)}(x + \mathcal{S}h;\ h,\ \dots\ h)\,d\mathcal{S}.$$

__Note:__

$$\int_0^1 \frac{(1-\mathcal{S})^{n-1}}{(n-1)!} f^{(n)}(x + \mathcal{S}h; h \dots h)\,d\mathcal{S} = \left(\int_0^1 \frac{(1-\mathcal{S})^{n-1}}{(n-1)!} f^{n}(x + \mathcal{S}h)\,d\right) \cdot h^n$$

Define $\phi: I \rightarrow A$ by $\phi(\xi) = f(x + \xi h)$. The result follows by applying Theorem 7, to this function ϕ.

__Theorem 9.__ With the same given as in Theorem 8, we can obtain the following result : $\forall \varepsilon > 0\ \exists\ r > 0\ \ni \|h\| \leq r;\ r \in R$

$$\left\| f(x+h) - f(x) - \sum_{i=0}^{n} \frac{1}{k!} f_x^{(k)}(h) \right\| \leq \varepsilon \|h\|^n.$$

__Proof__:

With the result of Theorem 8, all that we need to prove is that

$$\left\| \int_0^1 \frac{(1-\mathcal{S})^{n-1}}{(n-1)!} f_{x+\mathcal{S}h}^{(n)}(\ \ )\,d\mathcal{S} - \frac{1}{n!} f_x^{(n)}(\ .\ ) \right\| \leq \varepsilon$$

From the continuity of $f^{(n)}_x(\ )$ in $x \Longrightarrow \exists\, r \geq 0 \ni$

$$\left\| f^{(n)}_{x+\rho h}(\ ) - f^{(n)}_x(\ ) \right\| \leq n!\,\varepsilon \text{ for } 0 < \rho < 1 \text{ and } \|h\| \leq r.$$

since $\frac{1}{n!} f^{(n)}_x(\ )$ is independent of ρ

$$\int_0^1 \frac{1}{n!} f^n_x(\ )\,d\rho = \frac{1}{n!} f^n_x(\ ) .$$

Making use of Theorem 5 we can write,

$$\left\| \int_0^1 \frac{(1-\rho)^{n-1}}{(n-1)!} f^{(n)}_{x+\rho h}(\ ) - \frac{1}{n!} f^{(n)}_x(\ )\,d\rho \right\| \leq \sup_x \left\| \frac{(1-\rho)^{n-1}}{(n-1)!} f^{(n)}_{x+\rho h} - \frac{1}{n!} f^{(n)}_x(\ ) \right\| \leq \frac{1}{n!} \sup_x \left\| (1-\rho)^{n-1}\, n\, f^{(n)}_{x+\rho h}(\ ) - f^{(n)}_x(\ ) \right\| \leq \varepsilon$$

<u>Proposition 1.</u> Theorem 8 and 9 could be obtained by employing a method similar to the one used in (Chapter I, Theorem 6) we illustrate for Theorem 9. Let

$$\phi(\xi) = f(x + \xi h)\ ,\ \xi \in [0, 1]$$

and let $e \in Y^*$. The mapping H, $H(\xi) = e\,\phi(\xi) = ef(x+\xi h)$ is continuous on $[0, 1]$ and has derivatives up to the n-th order; moreover $H^{(k)}(\xi) = e\left[f^{(k)}_{x+\xi h}(h)\right]$ for $0 \leq k \leq n$.

Applying Taylor's expansion with integral remainder for the ordinary function H, and noting that e commutes with the integral sign we can apply the Hahn-Banach theorem and get the result namely

$$f(x+h) = f(x) + \sum_{k=1}^{n-1} \frac{1}{k!} f^{(k)}_x(h) + \frac{1}{n!} \int_0^1 (1-\rho)\, f^{(n)}_{x+\rho h}(h)\,d\rho\ ,\quad 0 < \rho < 1.$$

So far we have been imposing gloval conditions on the n-th differential. However we can obtain a variant which require the existence of the n-th order differential at a point.

<u>Theorem 10.</u> Let f be an operator form X into Y. If $D^n f(x_0; h)$ exists then the following expansion holds.

$$f(x_0 + h) = f(x_0) + Df(x_0; h) + \frac{1}{2!} D^2 f(x_0; h) + \dots + \frac{1}{n!} D^n f(x_0; h) + R_n(x_0; h)$$

where $\lim_{t \to 0} \frac{R_n(x_0; th)}{t^n} = 0.$

<u>Proof:</u>

Define $H(t) = ef(x_0 + th)$, where e is an artistrary element in Y^*. Then H is an ordinary function whose n-th derivative at 0 exists. Hence we can write

$$H(0 + t) = H(0) + H^{(1)}(0) + \dots + \frac{1}{n!} H^{(n)}(0) + \alpha_n(0; t)$$

where $\lim_{t \to 0} \frac{\alpha_n(0; t)}{t^n} = 0.$

Replacing $H(t)$ by $ef(x_0 + th)$ we obtain,

$$ef(x_0 + th) = ef(x_0) + eDf(x_0; th) + \dots + \frac{1}{n!} eD^n f(x_0; th) + \alpha_n(0; t)$$

Hence

$$\alpha_n(0; t) = e\left\{ f(x_0 + th) - \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(x_0; h) \right\} = eR_n(x_0; h).$$

Applying the Hahn-Banach theorem we obtain

$$f(x_0 + th) = f(x_0) + Df(x_0; th) + \dots + \frac{1}{n!} D^n f(x_0; th) + R_n(x_0; th)$$

where $\lim_{t \to 0} \frac{R_n(x_0; th)}{t^n} = 0.$

<u>Theorem 11.</u> Let f be an abstract function whose n-th Frechet differential at $\alpha \in I$ exists. Then the following expansion holds:

$$f(\xi) = f(\alpha) + (\xi - \alpha) f^{(1)}(\alpha) + \frac{1}{2!}(\xi - \alpha)^2 f^{(2)}(\alpha) + \dots + \frac{1}{n!}(\xi - \alpha)^n f^{(n)}(\alpha) + R_n(\xi)$$

where $\lim\limits_{\xi \to \alpha} \dfrac{R_n(\xi)}{(\xi - \alpha)^n} = 0.$

Proof:

For $n = 1$, this is simply the definition of the first order Frechet differential at α. Assume that the theorem is valid for every function whose k-th differential at α exists, i.e. $f^{(k)}(\alpha)$ exists then

$$\lim_{\xi \to \alpha} \frac{\| R_k(\xi) \|}{(\xi - \alpha)^k} = 0$$

where $R_k(\xi) = f(\xi) - f(\alpha) - (\xi - \alpha) f^{(1)}(\alpha) - \dots - \dfrac{(\xi - \alpha)^k}{k!} f^{(k)}(\alpha).$

Hence $\forall \varepsilon \geq 0 \; \exists$ a $\delta \ni$

$$\frac{\| R_k(\xi) \|}{(\xi - \alpha)^k} \leq \varepsilon \quad \text{whenever} \quad |\xi - \alpha| < \delta ,$$

i.e. $\| R_k(\xi) \| < \varepsilon \, (\xi - \alpha)^k$ whenever $|\xi - \alpha| < \delta$.

Let f be $k+1$ times differentiable at α, then $f^k, f^{k-1}, \dots, f$ exist and are continuous in a neighborhood of α. Let

$$R_{k+1}(x) = f(x) - f(\alpha) - (x - \alpha) f'(\alpha) - \dots - \frac{(x - \alpha)^k}{k!} f^{(k)}(\alpha) - \frac{(x - \alpha)^{k+1}}{(k+1)!} f^{k+1}(\alpha)$$

where $\xi \leq x \leq \alpha$, then

$$R'_{k+1}(x) = f'(x) - f'(\alpha) - \dots - \frac{(x - \alpha)^{k-1}}{(k-1)!} f^{(k)}(\alpha) - \frac{(x - \alpha)^k}{k!} f^{k+1}(\alpha).$$

But $f'(x)$ is k times differentiable at α, hence $\| R'_{k+1}(x) \| \leq \varepsilon' (x - \alpha)^k$, by induction hypothesis. Applying Theorem 2 we obtain

$$\| R_{k+1}(\xi) - R_{k+1}(\alpha) \| \leq \frac{(\xi - \alpha)^{k+1}}{(k+1)} \varepsilon'$$

$$\therefore \| f(\xi) - f(\alpha) - (\xi - \alpha) f'(\alpha) - \dots - (\xi - \alpha)^{k+1} f^{k+1}(\alpha) \| \leq \frac{(\xi - \alpha)^{k+1}}{k+1} \varepsilon'$$

<u>Theorem 11.</u> Let f be an operator form X into Y. If the n-th order F-differential at x_o exists, then the following expansion holds,

$$f(x_o + h) = f(x_o) + f'(x_o; h) + \dots + \frac{1}{n!} f^{(n)}(x_o; h) + R_n(x_o; h)$$

where $\lim\limits_{h \to \theta} \frac{\| R_n(x_o; h)\|}{\|h\|^n} = 0.$

<u>Proof</u>:

For $n = 1$, this is simply the definition of the first order F-differential of f at x_o. Assume that the theorem is valid for $n = s$. Then if $f^{s+1}(x_o; h)$ exists $f^s(x_o; h)$, $f^{s-1}(x_o; h), \dots, f(x_o)$ exist and are continuous in a neighborhood of x_o. Let

$$R_{s+1}(x_o; \xi h) = f(x_o + \xi h) - f(x_o) - f'(x_o; \xi h) - \dots - \frac{1}{(s+1)!} f^{(s+1)}(x_o; \xi h),$$

$$0 \leq \xi \leq 1$$

Suppressing the x_o and differentiating again

$$R'_{s+1}(\xi h; h) = df(x_o + \xi h; h) - df'(\xi h; h) - \dots - \frac{1}{(s+1)!} df^{s+1}(\xi h; h).$$

but $df(x_o + \xi h; \cdot)$ is s times differentiable at x_o hence

$$\| R'_{s+1}(\xi h; \cdot) \| \leq \|h\|^s \varepsilon .$$

Applying Theorem 3, we obtain

$$\| R_{s+1}(h) - R_{s+1}(0) \| \leq \|h\| \sup_{0 \leq \xi \leq 1} \| R'_{s+1}(\xi h, \ ) \| \leq \varepsilon \|h\|^{s+1}.$$

But $R_{s+1}(0) = 0$, hence the result.

CHAPTER III

CONVERSES OF TAYLOR'S THEOREM

In this chapter we consider the converse problem to the Taylor's expansion for operators. For ordinary functions we know that if $f^{(n)}(a)$ exists and if the function M_n is defined in a suitable deleted neighborhood $N(0)$ by

$$f(a+h) = f(a) + f'(a)h + \dots + \frac{f^{n-1}(a)}{(n-1)!} h^{n-1} + M_n(h)\, h^n \tag{1}$$

then

$$\lim_{h \to 0} M_n(h) = \frac{f^{(n)}(a)}{n!}$$

The converse problem [1] is to find conditions on $M_n(h)$ so that $f^{(n)}(a)$ would exist in case f has an expansion of the form (1). The example

$$f(x) = \begin{cases} x^3 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases} \tag{2}$$

shows that the conditions $\lim_{h \to 0} M_n$ exists; M_n', M_n'', ... M_n^{n-1} exists are not sufficient for the existence of the n-th differential of f at a.

It is of some interest to note that Graves [4] mentions that W. H. Young in a paper published in the proceedings of the London Mathematical Society Vol. 7 (1909) p. 158 states that if a function has an expansion similar to (1) with the condition that $\lim_{h \to 0} M_n(h)$ exists and is finite then the n-th derivative of f at x exists Graves gave (2) as a counter example to Young's statement. For a discussion of the problem for ordinary functions consider [1] . We shall carry a similar proof with a slight modification in the form. At the end of this discussion we shall prove that both forms [1] and the form that we shall use are equivalent.

<u>Theorem 1.</u> Let f be an operator from A(an open subset of X) into Y where X, Y are Banach spaces and h be any fixed element in X. Choose $(-\tau,\tau) \subset R \ni \forall t \in (-\tau,\tau)$ $x_o + th \in A$. Define $\phi(t) = f(x_o + th)$, $t \in (-\tau,\tau)$. If ϕ is continuous at $t = 0$ (i.e. f continuous along the ray h), then $V^n f(x_o; h)$ exists iff $\exists$ H_o, $H_1(x_o; h)$, ..., $H_n(x_o; h)$ where H_i is a homogeneous form of i-th degree in h, $i = 0, 1, \ldots, n$ and a function $\alpha_n(x_o; h)$ where $\alpha_n(x_o; th) = \beta_n(t)$ is defined in a deleted neighborhood of $t = 0$ by

$$f(x_o + th) = H_o + H_1(x_o; th) + \ldots + H_n(x_o; th) + \beta_n(t)$$

and where $\beta_n(t)$ satisfies the following conditions:

I. $\beta_n^{n-1}(t)$ exists in some deleted neighborhood of $t = 0$, $\tilde{N}(0)$

II. $\dfrac{\beta^n(t)}{t^n} = o(t)$ $\forall$ $t \in \tilde{N}(0)$

III. $\dfrac{\beta_n^{(j)}(t)}{t^{n-j}} = o(t)$ $\forall$ $t \in \tilde{N}(0)$ $j = 1, \ldots, n-1.$

Moreover in this case H_o, H_1, ..., H_n, $\alpha_n(x_o; h)$ are uniquely determined and

$$H_k(h) = \frac{V^k f(x_o; h)}{k!} \quad k = 0, 1, \ldots, n$$

<u>Proof</u>:

We prove both parts by induction, proving first the sufficiency of the conditions. If $n = 1$ we thus assume that f is continuous at

x_o (along the ray h), and that $\exists$ H_o and H_1 and a function $\alpha_1(x_o;h)$ where $\alpha_1(x_o;\ th) = \beta_1(t)$ is defined in a deleted neighborhood of $t = 0$ by

$$f(x_o + th) = H_o + tH_1(x_o;\ h) + \beta_1(t).$$

Now $\lim_{t\to 0} \beta_1(t) = \lim_{t\to 0} \frac{\beta_1(t)}{t} \cdot t = 0$. Hence letting $t \to 0$ it follows that $f(x_o) = H_o$.

Hence

$$\lim_{t\to 0} \frac{f(x_o + th) - f(x_o)}{t} = H_1(x_o;\ h) + \lim_{t\to 0} \frac{\beta_1(t)}{t} = H_1(x_o;\ h).$$

But $H_1(x_o;\ h) = Vf\,(x_o;\ h)$ by definition.

Assume conditions sufficient for $n = k$, we shall prove that they are sufficient fo $n = k + 1$ i.e. we assume that $\exists$,

$$H_o,\ H_1(x_o;\ h)\ \ldots,\ H_{k+1}(x_o;\ h),\ \alpha_{k+1}(x_o;\ h)$$

with

$$f(x_o + h) = H_o + \sum_{i=1}^{k} H_i(x_o;\ h) + \alpha_{k+1}(x_o;\ h) \tag{3}$$

where $\alpha_{k+1}(x_o;\ th) = \beta_{k+1}(t)$ satisfies, I, II, III.

We shall prove that $f^{(k+1)}(x_o)$ exists.

$$f(x_o + th) = H_o + \sum_{i=1}^{k} t^i\, H_i(x_o;\ h) + \beta_{k+1}(t) \tag{4}$$

Since the first k G-variations of $\beta_{k+1}(t)$ exist we can differentiate (4).

$$\frac{d}{dt} f(x_o + th) = H_1(h) + \ldots + (k+1)t^k\, H_{k+1}(x_o;\ h) + \frac{d}{dt}\beta_{k+1}(t). \tag{5}$$

We shall prove that the induction hypothesis apply to (5). $\frac{d}{dt} f(x_o\ \ th)$ is continuous at $t = 0$ because $\lim_{t\to 0} \frac{d}{dt} f(x_o + th) = H_1(h)$ from (5) and from (3) $H_o = f(x_o)$.

Hence

$$H_1(h) = \lim_{t\to 0} \frac{f(x_o + th) - f(x_o)}{t} = \frac{d}{dt} f(x_o + th)\Big|_{t=0} = Vf(x_o;h)$$

We shall show that $\gamma_k(t) = \frac{d}{dt} \beta_{k+1}(t)$ satisfies I, II, III.

Since $\beta_{k+1}(t)$ was assumed to have a k-th G-variation $\gamma_k^{k-1}(t)$ exists.

The $\lim_{t\to 0} \frac{1}{t^k} \gamma_k(t) = \lim_{t\to 0} \frac{1}{t^k} \frac{d}{dt} \beta_{k+1}(t) = 0$ by III hence II holds for γ_k. Finally the $\lim_{t\to 0} \frac{1}{k-j} \frac{d^j}{dt^j} \gamma_k = \frac{1}{t^{k-j}} \frac{d^{j+1}}{dt^{j+1}} \beta_{k+1}$ $= 0$ for $j = 1, 2, \ldots, k - 1$.

Hence induction hypothesis apply to $\frac{d}{dt} f(x_o + th)$. Hence $\frac{d}{dt} f(x_o + th)$ has a k-th G-variation at $t = 0$ which means that $V^{k+1} f(x_o; h)$ exists.

If $Vf(x_o; h)$ exists then f is continuous at x_o along the h-ray. If we define $H_o = f(x_o)$, $H_1(x_o; h) = Vf(x_o; h)$ and $\alpha(x_o; h)$ by $f(x_o + h) = f(x_o) + Vf(x_o; h) + \alpha(x_o; h)$ then $\beta(t) = \alpha(x_o; th)$ is defined in a deleted neighborhood of $t = 0$, and $\lim_{t\to 0} \frac{\beta(t)}{t} = 0$ from the definite of $Vf(x_o; h)$. Hence the conditions are necessary for $n = 1$. Assume that the conditions are necessary for $n = k$. If $V^{k+1} f(x_o; h)$ exists then $V^k f(x_o; h)$, $V^{k-1} f(x_o;h)$, $\ldots$, $f(x_o)$ are defined and continuous in a neighborhood of x_o along the h ray.

Hence $\alpha_{k+1}(x_o; h)$ is well defined by,

$$f(x_o + h) = f(x_o) + Vf(x_o; h) + \ldots + \frac{1}{k!} V^k f(x_o; h) + \frac{1}{k+1!} V^{k+1} f(x_o;h) + \alpha_{k+1}(x_o; h). \quad (6)$$

Recall that $V_k f(x_o; h)$ is homogeneous of k-th degree in h, II follows easily. III had been already treated see chapter II Theorem 10.

We shall prove III. Differentiating (6) we obtain

$$\frac{d}{dt} f(x_o + th) = Vf(x_o; h) + \ldots + \frac{t^k}{k!} V^{k+1} f(x_o; h) + \frac{d}{dt} \alpha_{k+1}(x_o; th).$$

But the k-th variation of $\frac{d}{dt} f(x_o + th)$ exists in a neighborhood of $t = 0$. Hence $\gamma_k(t) = \frac{d}{dt} \alpha_{k+1}(x_o; th)$ satisfies conditions I, II, III in particular.

$$\lim_{t \to 0} \frac{1}{t^{k-j}} \frac{d^j}{dt^j} \gamma_k(t) = 0 \text{ for } t \in \tilde{N}(0)$$

$$\lim_{t \to 0} \frac{1}{t^{k+1-j-1}} \frac{d^{j+1}}{dt^{j+1}} \alpha_{k+1}(x_o; th) = 0.$$

<u>Theorem 2.</u> Let f be an operator from A into Y. Choose $(-\tau, \tau) \subset R \ni \forall t \in (-\tau, \tau)$, $x_o + th \in A$ where $h \in X$. If $f(x_o + th)$ is continuous at $t = 0 \ \forall h \in X$, then f has an n-th order G-differential at $x_o \in A$, iff $\exists H_o, H_1, \ldots, H_n$ where H_i is a multilinear continuous operator of i-th degree, $i = 0, 1, 2, \ldots, n$ and a function $\alpha_n(x_o; h)$ where $\alpha_n(h)$ (suppressing x_o) is defined in a deleted neighborhood of θ_x by

$$f(x_o + h) = H_o + H_1(x_o; h) + \ldots + H_n(x_o; h) + \alpha_n(h)$$

and where $\alpha_n(h)$ satisfies the following conditions as well.

I. $D^{n-1} \alpha_n(h; \ldots)$ exists $\forall h \in \tilde{S}(\theta)$ a deleted neighborhood of (θ).

II. $$\lim_{t \to 0} \frac{\| \alpha_n(x_o; th) \|}{t^n} = 0, \forall h \in X$$

III. $$\lim_{t \to 0} \frac{\| D^j \alpha_n(th; \ldots) \|}{t^{n-j}} = 0$$

for $j = 1, \ldots, n-1$ and $\forall\ h \in X$.

Then H_i and α_n are uniquely determined.

$$H_i = \frac{D^i f(x_0; \ldots)}{i!} \quad \text{for } i = 0, 1, \ldots n$$

<u>Proof</u>:

Similar to theorem (1).

<u>Theorem 3.</u> Let f be a mapping from A into Y, $x_0 \in A$. If f is continuous at x_0, then f has an n-th order F-differential at $x_0 \in A$ iff the following representation holds $\forall\ h \in X \ni x_0 + h \in A$

$$f(x_0 + h) = H_0 + H_1(x_0; h) + \ldots + H_n(x_0; h) + \alpha_n(x_0; h) \tag{7}$$

where H_0 is constant in h; H_i is an i-th continuous linear operator in h and $\alpha_n(h)$ (suppressing x_0) is defined in a deleted neighborhood $\tilde{S}(\theta)$ of $h = \theta$ by (7) and satisfies the following conditions,

I. $d^{n-1} \alpha_n(h; \ldots)$ exists $\forall\ h \in \tilde{S}(\theta)$

II. $$\lim_{h \to \theta} \frac{\| \alpha_n(x_0; h) \|}{\| h \|^n} = 0$$

III. $$\lim_{h \to \theta} \frac{\| d^j \alpha_n(h; \ldots) \|}{\| h \|^{n-j}} = 0, \quad j = 1, 2, \ldots, n-1$$

in this case $H_0, \ldots, H_n, \alpha_n$ are uniquely determined and

$$H_i(\cdot) = \frac{d^i f(x_0; \ldots)}{i!} \quad i = 1, \ldots n.$$

<u>Proof</u>:

The conditions are sufficient for $n = 1$, see [9].

Assume conditions are sufficient for $n = s$ and suppose that

H_o, $H_1(x_o, h)$, ..., $H_{s+1}(x_o; h)$, $\alpha_{s+1}(x_o; h)$ where H_i is an i-th continuous linear operator for i = 0, 1, 2, ..., $s+1$, $\ni$

$$f(x_o + h) = H_o + H_1(h) + \dots + H_{s+1}(h) + \alpha_{s+1}(h) \tag{8}$$

where $\alpha_{s+1}(h)$ satisfies I, II, III we want to prove that $d^{s+1}f(x_o, \dots)$ exists. Since $\alpha_{s+1}(h)$ has the s - first F-differential for $h \in \tilde{S}(\theta)$ we can differentiate (8) to obtain

$$df(x_o + h; k) = 0 + H_1(k) + \dots + dH_{s+1}(h; k) + d\alpha_{s+1}(h; k) \tag{9}$$

but by (Chapter I Proposition 1) we know that $dH_2(h; k)$ is linear in h; $dH_3(h; k)$ is bilinear in h etc. Moreover $df(x_o + h; k)$ is continuous at x_o. Because $\lim_{h \to \theta} df(x_o + h; k) = H_1(k)$ from (9) and $H_1(k)$ $= df(x_o; k)$ from (8). Moreover let $\gamma_s(h) = d\alpha_{s+1}(h; k)$ then $d^{s-1}\gamma_s(h; \dots) = d^s\alpha_{s+1}(h; k, \dots)$ exists.

$$\lim_{h \to \theta} \frac{\|\gamma_s(h)\|}{\|h\|^s} = \lim_{h \to \theta} \frac{\|d\alpha_{s+1}(h; k)\|}{\|h\|^s} = 0$$

because of III. And

$$\lim_{h \to \theta} \frac{\|d^j\gamma_s(h; \dots\dots)\|}{\|h\|^{s-j}} = \lim_{h \to \theta} \frac{\|d^{j+1}\alpha_{s+1}(h; k, \dots\dots)\|}{\|h\|^{s-j}}$$

$$= \lim_{h \to \theta} \frac{d^i\alpha_{s+1}(h; k \dots\dots)}{\|h\|^{s+1-i}} = 0 \quad \text{by III.}$$

Therefore $df(x_o + h; k)$ has an s - order F-differential at x_o. i.e. $d^{s+1} f(x_o; \dots)$ exists.

The conditions are necessary for n = 1 see [9]. Assume that the conditions are necessary for n = s and assume $d^{s+1} f(x_o; \dots)$ exists then $d^s f(x_o, \dots)$, $d^{s-1} f(x_o, \dots)$, ... $df(x_o, \ )$ and $f(x_o)$

are defined and continuous in a neighborhood of x_0. Hence we define $H_0 = f(x_0)$; $H_1 = df(x_0, \ )$, ... $H_{s+1}(\ ) = d^{s+1}f(x_0, \dots)$ and therefore $\alpha_{s+1}(h)$ is well defined in a deleted neighborhood of $h = \theta$ by

$$f(x_0 + h) = f(x_0) + df(x_0; h) + \dots + d^{s+1}f(x_0; h) + \alpha_{s+1}(h). \tag{10}$$

$d^s \alpha_{s+1}(h; \dots)$ clearly exists because we assumed that f has the $s+1$ F-differential existing at x_0. I follows easily.

II. is satisfied see (Theorem 12 Chapter 2).

To prove III differentiate (10)

$$df(x_0 + h, k) = df(h; k) + \dots + \frac{d^{s+2}}{(s+1)!} f(h; k) + d\alpha_{s+1}(h; k)$$

since the s-th F-differential exists at $x_0 \Longrightarrow d\alpha_{s+1}(h; k)$ satisfies in particular IV

$$\lim_{h \to \theta} \frac{\| d^{j+1}\alpha_{s+1}(h; k \dots) \|}{\| h \|^{s-j}} = \lim_{h \to \theta} \frac{\| d^{i}\alpha_{s+1}(h; k \dots) \|}{\| h \|^{s-i-1}} = 0.$$

Hence conditions are necessary for $s+1$.

We shall show, that if we consider the conditions of Theorem 2 and require that they hold uniformly then they imply conditions of Theorem (3) and conversely.

<u>Theorem 4.</u> (1) $f(x_0 + th) \to f(x_0)$ as $t \to 0$ uniformly $\forall\ h$ $\| h \| = 1$ is equivalent to $f(x_0 + h) \to f(x_0)$ as $h \to \theta$.

(2) $\lim_{t \to \theta} \frac{\| \alpha_n(x_0; th) \|}{t^n} = 0$ uniformly $\forall\ h$

$\| h \| = 1$ is equivalent to $\lim_{h \to \theta} \frac{\| \alpha_n(x_0; h) \|}{\| h \|^n}$

(3) $\lim_{t \to 0} \frac{\| d^j\alpha_n(th; \dots) \|}{t^{n-j}} = 0$

uniformly $\forall\ h$, $\| h \| = 1$, is equivalent to

$$\lim_{h \to \theta} \frac{\| d^j \alpha_n(h; \ldots) \|}{\| h \|^{n-j}}$$

<u>Proof:</u>

1. Assume $\forall\ \varepsilon > 0\ \exists$ a $\delta > 0\ \ni$

$$\| f(x + th) - f(x) \| \leq \varepsilon \text{ for } |t| < \delta(\varepsilon) \text{ and every } h \ni \|h\| = 1$$

Let k be any vector $\in X \ni \|k\| \leq \delta$ then $k = \|k\| \frac{k}{\|k\|}$.

holds for $t = \|k\|$ and $h = \frac{k}{\|k\|}$.

$$\therefore \left\| f\left(x + \|k\| \frac{k}{\|k\|}\right) - f(x) \right\| = \| f(x + k) - f(x) \|$$

whenever $\|k\| \leq \delta$

$$\therefore \lim_{k \to \theta} f(x + k) = f(x).$$

<u>Conversely</u>. If $\lim_{h \to \theta} f(x + h) - f(x) = 0$

(2) (i.e. $\forall\ \varepsilon > 0\ \exists\ \delta \ni \| f(x + h) - f(x) \| \leq \varepsilon$ for $\|h\| \leq \delta$).

Consider a subset of X whose element are of the form tk where $|t| < \delta$ and $\|k\| = 1$ then (2) holds for all element of this subset of X

i.e. $\| f(x + tk) - f(x) \| < \varepsilon$ for $\|tk\| = |t| < \delta$.

Proof of (2) assume

$$\lim_{t \to 0} \frac{\alpha_n(x; th)}{t^n} = 0 \text{ uniformly } \forall\ h; \|h\| = 1$$

i.e. $\forall\ \varepsilon > 0,\ \exists\ \delta > 0,\ \delta(\varepsilon) \ni$ if $0 < t < \delta$

then

$$\frac{1}{|t^n|} \| \alpha_n(x; th) \| < \varepsilon\ \forall\ h\ \|h\| = 1.$$

Let $k \in A$, $\|k\| < \delta$ then

$$k = \|k\| \frac{k}{\|k\|} = t'h'.$$

with

$$t' = \|k\| < \delta \quad \text{and} \quad \left\| \frac{k}{\|k\|} \right\| = \|h'\| = 1$$

hence (3) holds for this t' and h'

$$\frac{1}{\|k\|^n} \|\alpha_n(x;\ k)\| \leq \varepsilon \quad \forall\ k;\ 0 \leq \|k\| < \delta.$$

Conversely. Consider a subset of A whose elements are of the form tk; $|t| < \delta$ and $\|k\| = 1$ then $\|tk\| < \delta$.

$$\frac{\|R(x_o;\ tk)\|}{\|t^n k^n\|} = \frac{\|R(x_o;\ tk)\|}{t^n} < \varepsilon \text{ for } |t| < \delta \text{ and } \forall k;\ \|k\| = 1$$

(3) could be done similarly.

Proposition. For ordinary functions Theorem 1, 2, 3 reduce to:

If f is continuous at a, then $f^{(n)}(a)$ exists (finite) iff:

1. $\exists$ constants $a_o, a_1, \ldots, a_{n-1}, a_n$ and a function α_n defined in some deleted neighborhood $\tilde{N}(0)$ of 0 by:

$$f(a+h) = a_o + a_1h + \ldots + a_{n-1}h^{n-1} + a_nh^n + \alpha_n(h).$$

for $h \in \tilde{N}(0)$ where

(2) α_n^{n-1} exists in $\tilde{N}(0)$

(3) $\lim_{h \to 0} \frac{\alpha_n(h)}{h^n} = 0$

(4) $\lim_{h \to 0} \frac{\alpha_n(h)}{h^{n-j}} = 0$ for $j = 1, 2, \ldots, n - 1$

Moreover in this case $a_o, a_1, \ldots, a_n$ and $_n(h)$ are uniquely determined and $a_k = \frac{f^k(a)}{k!}$ for $k = 0, 1, \ldots, n$

We shall prove that these conditions are equivalent to:

if f is continuous at a, then $f^{(n)}(a)$ exists (finite) iff:

1.* $\exists$ constants $a_o, a_1, \ldots, a_{n-1}$ and a function M_n defined in some deleted neighborhood $\tilde{N}(0)$ by

$$f(a+h) = a_o + a_1 h + \ldots + a_{n-1} h^{n-1} + M_n(h)h^n \quad \text{for } h \in \tilde{N}(0)$$

where

2.* $M_n^{(n-1)}$ exists in $\tilde{N}(0)$

3.* $\lim_{h\to 0} M(h)$ exists

4.* $\lim_{h\to 0} h^j M_n^{(j)}(h) =$ for $j = 1, 2, \ldots, n-1$.

Moreover in this case, $a_o, a_1, \ldots, a_{n-1}$ and M_n are uniquely determined and $a_k = \frac{f^{(k)}(a)}{k!}$ for $k = 0, 1, \ldots, n-1$. While $\lim_{h\to 0} M_n(h) = \frac{f^{(n)}(a)}{n!}$.

<u>Proof</u>:

Let $M(h)\ h^n = a_n h^n + \alpha_n(h)$ then conditions 1*, 2*, 3* are clearly equivalent to 1, 2, 3 we shall prove that $4^* = 4$

$4^* \Longrightarrow 4$

$$\lim_{h\to 0} h^j \left(\frac{\alpha_n(h)}{h^n}\right)^{(j)} = 0 \quad \text{for } j = 1, 2, \ldots, n-1.$$

Let $w = h^{-n} \alpha_n(h)$.

$$w^{(j)} = (-1)^j \frac{(n+j)!}{(n-1)!} h^{-n-j} \alpha_n + \binom{n}{j}(-1)^{j-1} \frac{(n+j-1)!}{(n-1)!} h^{-n-j+1} \alpha_n^{(1)} + \ldots + h^{-n} \alpha_n^{(j)}$$

and

$$h^j w^{(j)} = h^{-n} \left[(-1)^j \frac{(n+j)!}{(n-1)!} \alpha_n + \binom{n}{j}(-1)^{j-1} \frac{(n+j-1)}{(n-1)!} h \alpha_n^{(1)} + \ldots + h^j \alpha_n^{(j)} \right]$$

$\lim_{h\to 0} h^j w^{(j)} = 0$; $j = 1, 2, \ldots n-1$ implies that $\lim_{h\to 0} h^{-n}\left[h^j \alpha_n^{(j)}\right] = 0$ $j = 1, 2, \ldots, n$

$4 \Longrightarrow 4^*$

$$\lim_{h\to 0} h^j M_n^{(j)}(h) = \lim_{h\to 0} h^j w^{(j)}(h) = 0.$$

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