AMERICAN UNIVERSITY OF BEIRUT

GRAPHICAL REPRESENTATION OF CERTAIN CLASSES OF GROUPS WITH DEFINING RELATIONS $a^m = b^n = (ab)^m = e$, $a^{-1}ba = b^k$

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Date of Thesis Presentation: May 20, 1968.

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Submitted in Partial Fulfilment for the

Requirements of the Degree Master of Science

in the Mathematics Department of the

American University of Beirut

Beirut, Lebanon,

1968

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ABSTRACT

This paper is a treatment of graphical representation of certain classes of groups having two generators with the following relations:

$$a^m = b^n = (ab)^m = e, \quad a^{-1}ba = b^k.$$

The emphasis is on representing graphically the groups of genus one when m = 3 and 4.

The illustrative examples, however, are chosen to clarify the techniques we use in representing the groups graphically.

The first chapter is an introduction to the graphical representation of groups in general. This forms the basis on which the method used for the groups in the next two chapters depend.

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CHAPTER I

INTRODUCTION

In this paper we shall study the graphical representation of the group $G = G(m, n, k)$ defined by $a^m = b^n = (ab)^m = e$, $ba = ab^k$ ($k \not\equiv 1 \pmod{n}$).

This group is generated by two distinct elements a and b of orders m and n respectively. The third generator ab, of order m, is redundant, but we will need it when we discuss the map of the group and the genus of the closed connected surface of the map.

The element $a^{-1}ba = b^k$, being conjugate to b, also has order n. Hence $(k,n) = 1$. Since every element of G may be uniquely expressed in the form $a^i b^j$ ($i = 0, 1, \ldots, m-1$; $j = 0, 1, \ldots, n-1$), the order of G is mn.

$ba = ab^k$, therefore $(ab)^r = a^r b^{k^{r-1} + k^{r-2} + \cdots + k + 1}$.

Thus substituting m for r we will have

$$k^{m-1} + k^{m-2} + \cdots + k + 1 \equiv 0 \pmod{n}.$$

In this paper we shall concider two special values of m, namely $m = 3$ and $m = 4$; that is: $G = G(3, n, k)$ and $G(4, n, k)$, where $k = 2, 3, \ldots$ For $k = 1$ the group is abelian and will not be discussed in this paper.

The groups defined by their generators and the defining relations between them may be represented graphically by maps.

A map is the decomposition of a surface into F non-overlapping regions (called faces) by E arcs (called edges) joining pairs of

V points (called vertices). The Euler-Poincaré characteristic $\chi = V - E + F$ is a property of the surface [1, p. 20].

The genus p of an orientable surface is defined by the equation $\chi = 2 - 2p$, or by the number of handles on a sphere. In particular a sphere is a surface of genus 0 and a torus is a surface of genus 1 [2, p. 381].

Sometimes different sets of generators give different maps of unequal genuses. However, there is a lower limit to the number p for any group of finite order, and this lower limit of p will be called the genus of the group [3, p. 397].

In order to have a clear idea about the mode of representation of groups we shall take some examples due to Burnside [3].

Example 1.

The cyclic group $\{a\}$ of order 5 may be represented as follows:

Take two circles C_1 and C_{-1} (C_{-1} may be taken as a straight line) intersecting at an angle $\pi/5 = 36^\circ$. Let C_2 be the inverse [2, c. 6] of C_{-1} in C_1 and C_{-2} the inverse of C_1 in C_{-1}. Similarly C_3 is the inverse of C_{-2} in C_1 (see Fig. 1-1). This gives five circles, each intersecting the two next to it at angle of 36°, thus dividing the plane into $2x5 = 10$ regions. If these regions are left white and colored as indicated on Fig. 1-1, then any white region can be transformed into another white region by an even number of inversions in the circles C_1 and C_{-1}. Let S denote the operation consisting of an inversion in C_{-1} followed by an inversion in C_1, then the white region between C_1 and C_{-1} will be transformed into another white region by an operation S^n, where $n = 1, 2, \ldots, 5$. Hence if we denote the white region between C_1 and C_{-1} by e, the identity element of the group, and make the correspondence $S^n \longrightarrow a^n$, then each white region will correspond to one and only one element of $\{a\}$, and the figure thus constructed gives a graphical representation of the group $\{a\}$.

This can be generalized for any cyclic group of order n. If n tends to infinity, then π/n (the angle between successive circles) will tend to 0. So, in the case of cyclic groups of infinite order the circles touch each other [3, p. 377].

If our circles are inverted in any circle with center at one of the intersection points, they all become straight lines through the inverse of the other intersection point. The center of inversion recedes to infinity. Fig. 1-1 then reduces to Fig. 1-2.

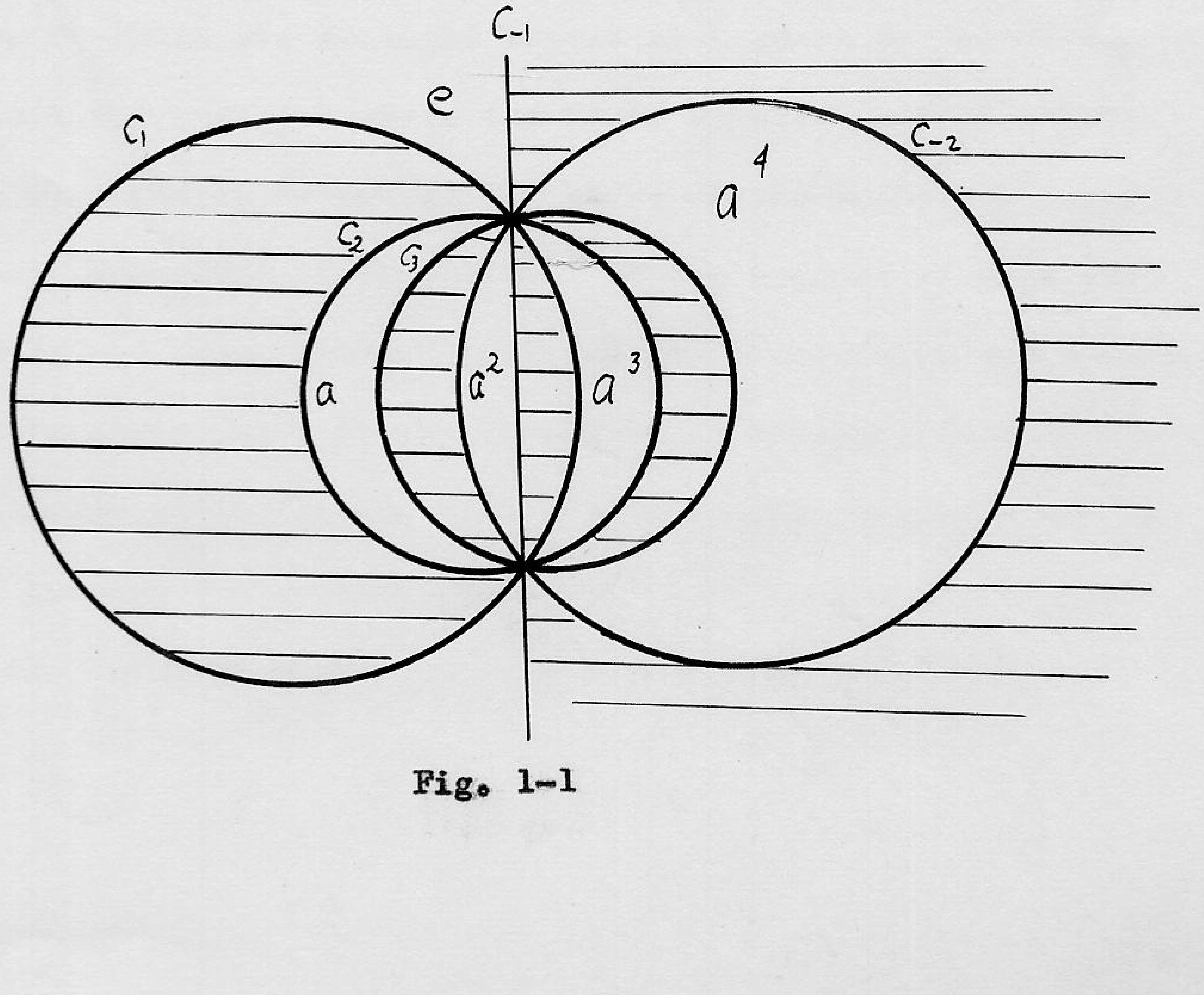


Fig. 1-1

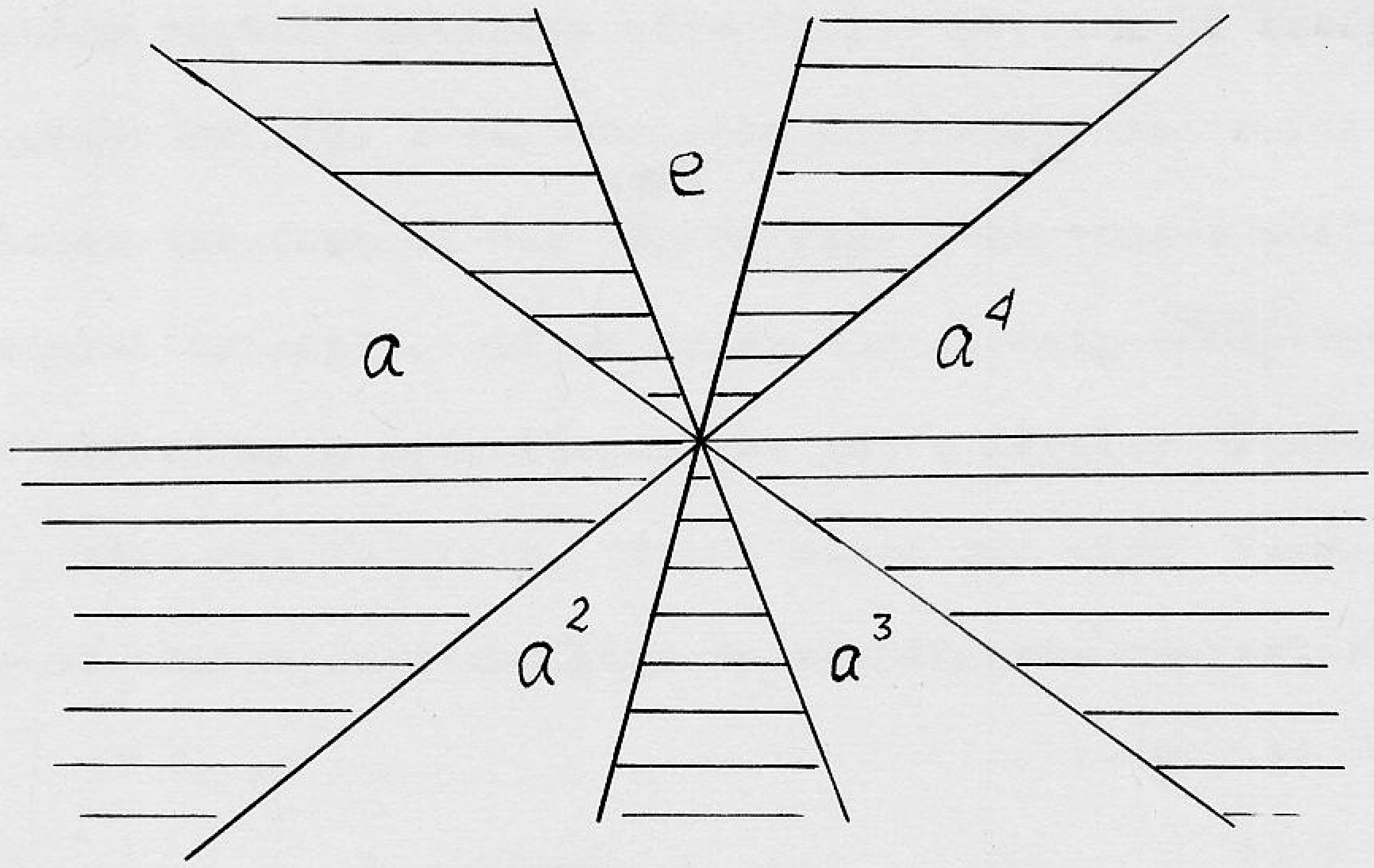


Fig. 1-2

Example 2.

The dihedral group D_n has order 2n.

The graphical representation of this group may be done as follows, taking D_2 as the first example: We take a straight line as a circle. On this line and to the same side of it we draw three circles (two of which are straight lines) orthogonal to the first. Then we invert the region between the three circles in each of its sides, then continually we invert the sides of the resulting regions till we obtain the graph of Fig. 1-3a. If the regions of this graph are left white and colored black in alternate succession, and if one of the white regions (or a black one) is taken to correspond to the identical element of the group, then a unique correspondence can be established between the white regions and the elements of the group.

Since the group is finite, we can topologically deform the figure into a regular division of a finite portion of the plane. Thus the graph of Fig. 1-3a, when the necessary stretchings are applied takes the form of the map of Fig. 1-3b, where each region is an equilateral triangle. If we bring the corresponding sides (numbered alike) into coincidence, we get a regular octahedron, Fig. 1-3c, with six vertices, twelve edges and eight faces. If p is the genus of the surface and χ the Euler-Poincaré characteristic, then

$$\begin{aligned} 2p &= 2 - \chi \\ &= 2 - (V - E + F) \\ &= 2 - (6 - 12 + 8) \\ &= 0 \end{aligned}$$

Therefore $p = 0$. That is, the genus of the map of the group is zero.

Similarly we get the map of the dihedral group of order 6. Fig. 1-4a.

If we bring into coincidence the corresponding edges (numbered alike) of Fig. 1-4b, we get a double pyramid on a hexagonal base. Fig. 1-4c.

Generalizing this, a dihedral group D_n, defined by $a^2 = b^n = (ab)^2 = e$, is mapped on the surface of a double pyramid on a 2n-sided regular polygonal base, which is of genus zero.

In the above example, we saw that, in order to construct a graphical representation of a group, generated by two distinct elements we need to take three circles touching each other externally and all the points of contact lying on a circle orthogonal to the others.

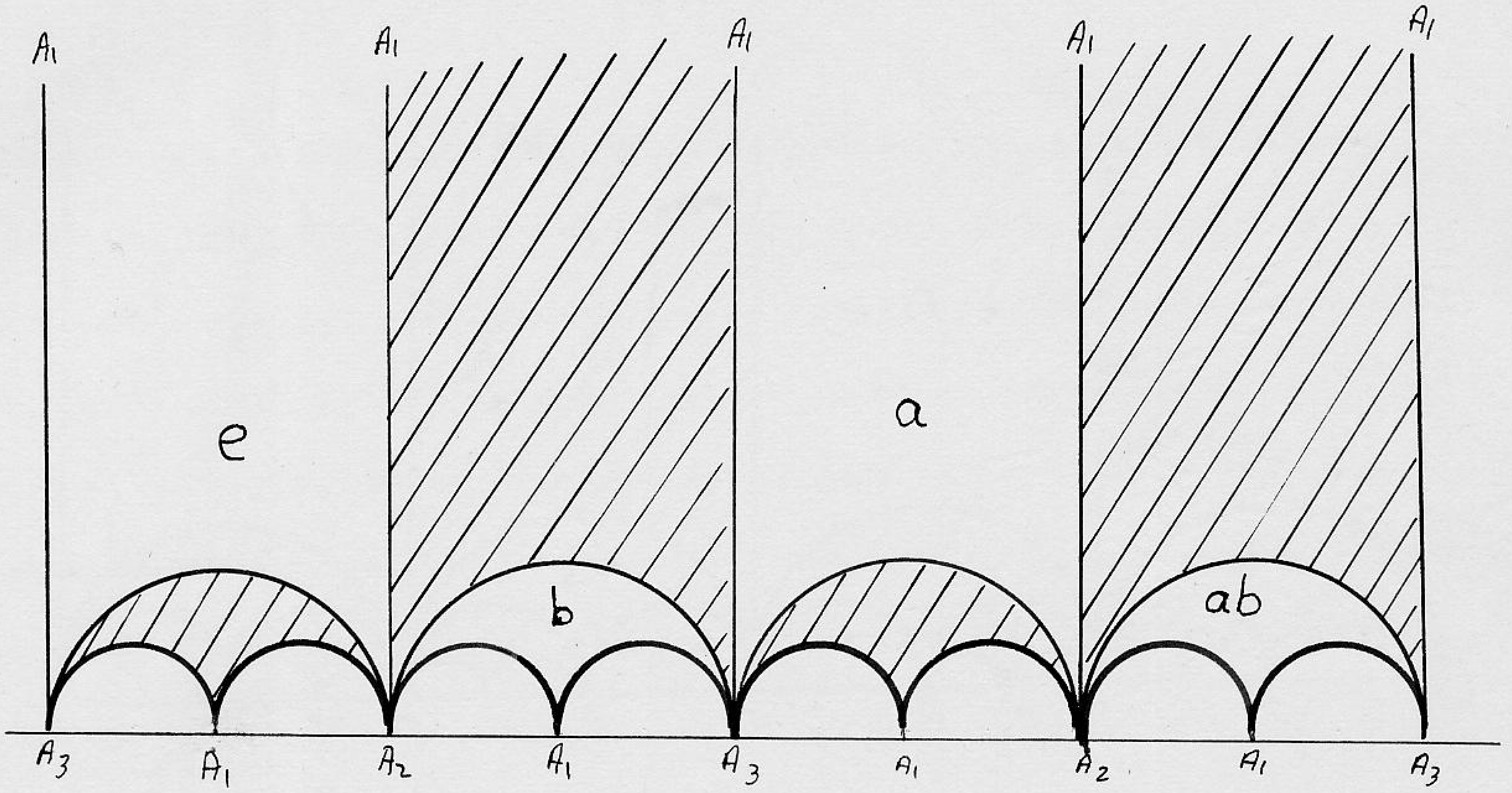


Fig. 1-3a

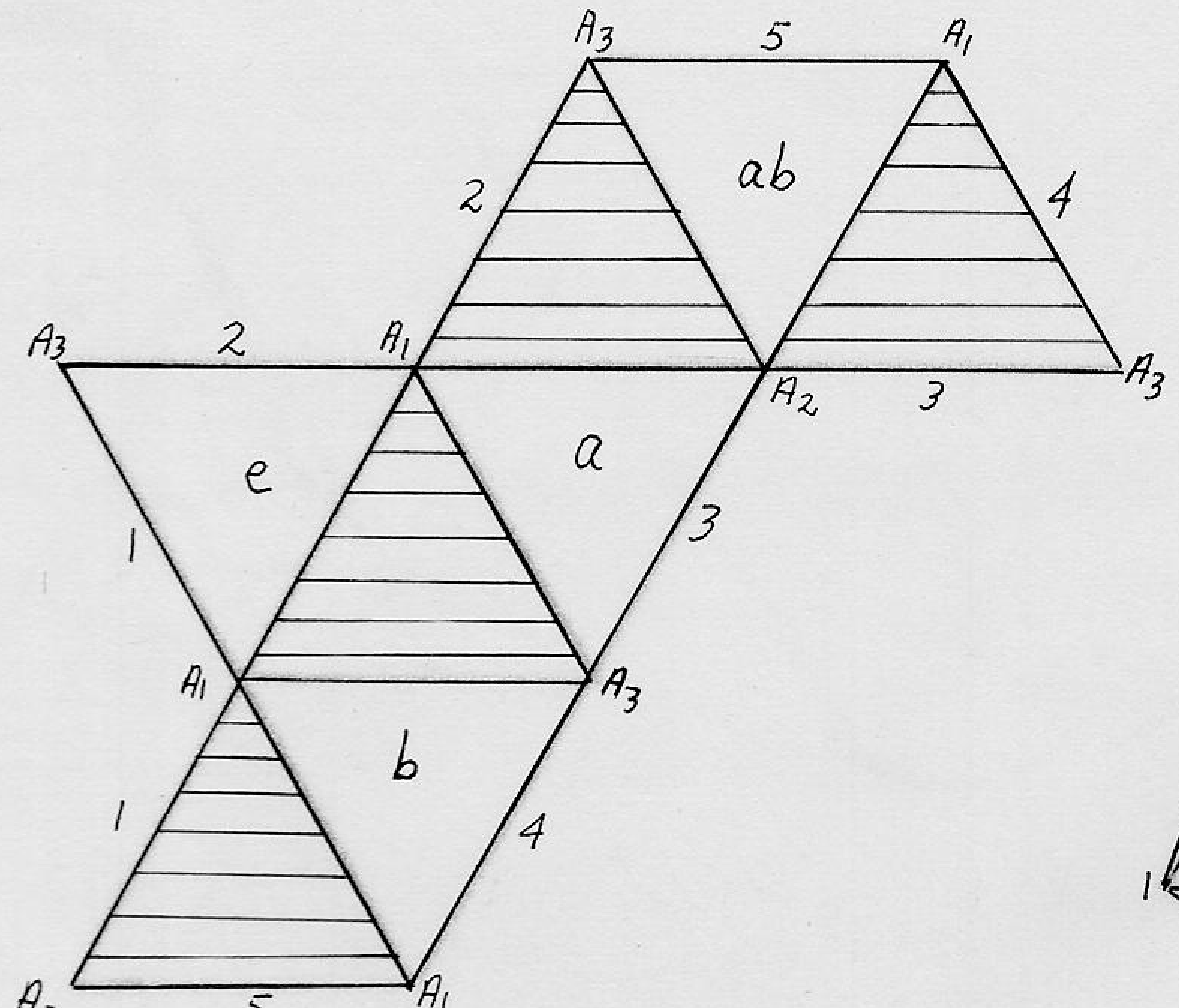


Fig. 1-3b

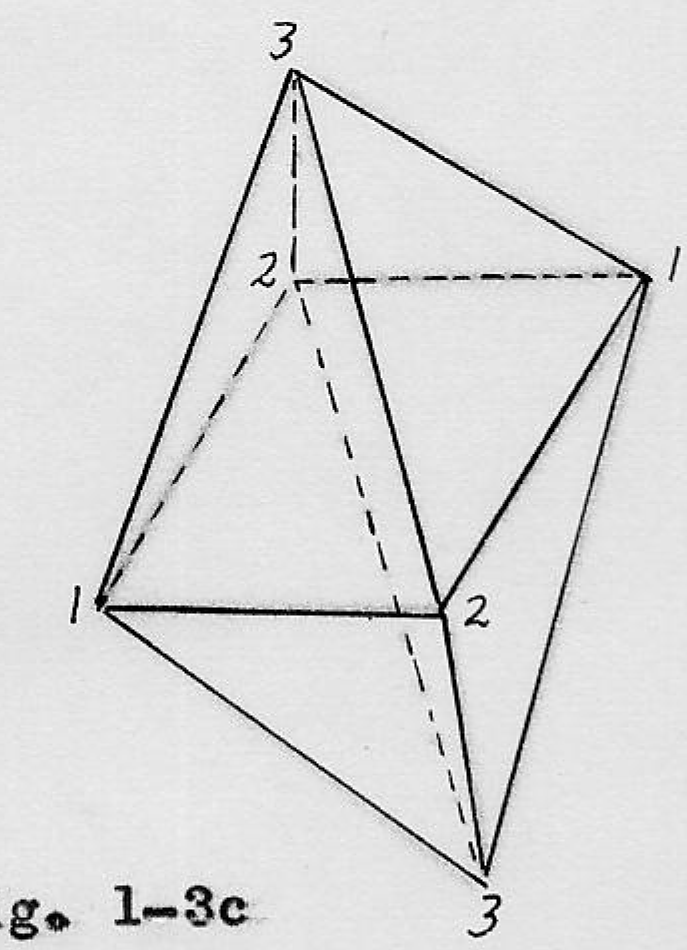


Fig. 1-3c

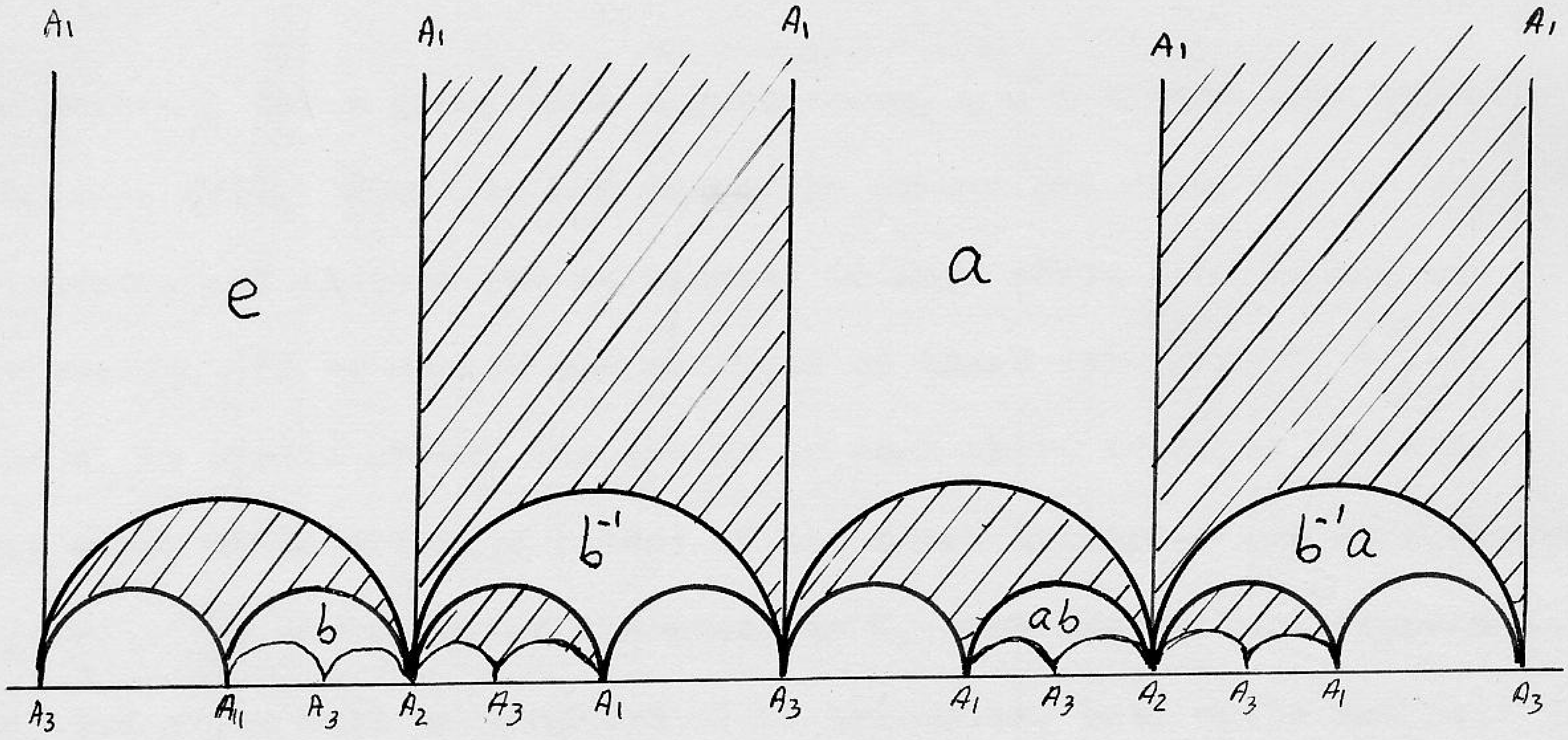


Fig. 1-4a

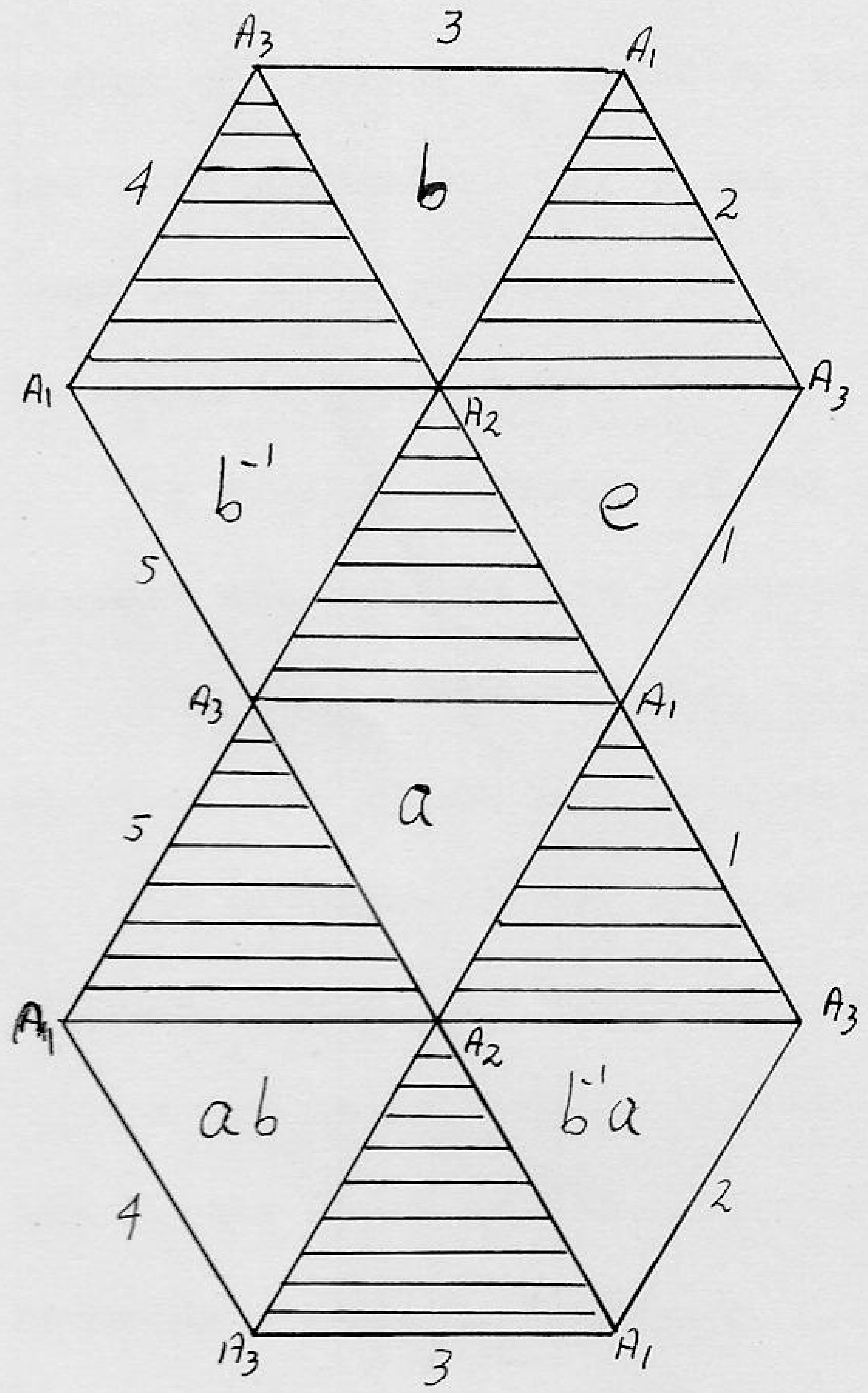


Fig. 1-4b

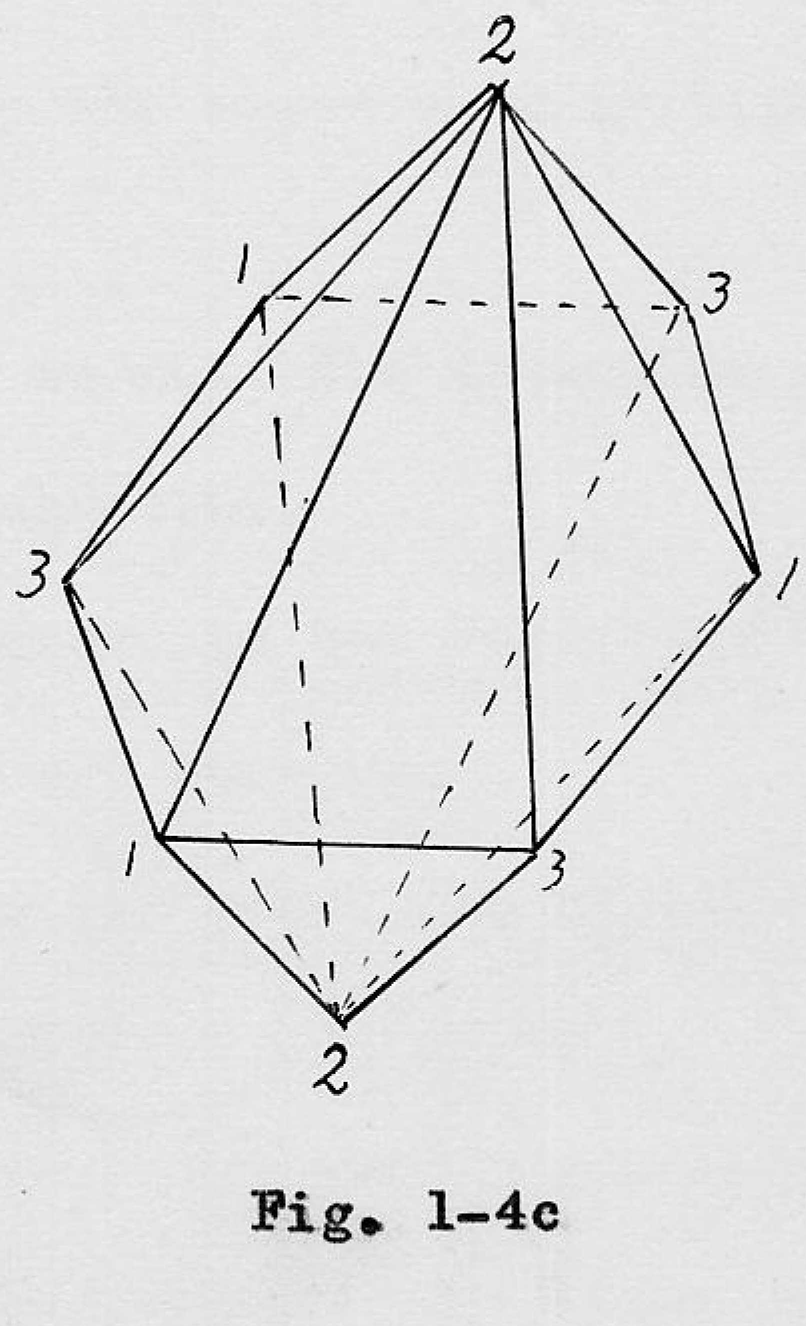


Fig. 1-4c

In general, for a group with n generators, n - 1 circles are needed [3, sec. 277]. Since in our case, the groups are generated by two elements, our figures can be reduced to maps where the regions are triangles. If we denote the vertices of these triangles by A_1, A_2 and A_3 in cyclic order, the cycles on each white triangle agree to any other white triangle (those of the black triangles are in opposite cycles). If the order of the group is N, then the surface representing the group will be composed of 2N triangles; half white and half black such that each pair (one white and one black) will correspond to a unique element of the group. This correspondence can be carried out as follows: A single positive rotation of a white triangle W around the corner A_r leads to the white triangle S_rW, where S_r denotes the r-th generator. If a and b are the first and second generators, then the third generator is ab. Usually the above rule applies to independent generators.

By regular division of the surface we mean the triangles which compose the surface are congruent to each other.

The maps of groups G(m, n, k) discussed in the following two chapters are of special interest. They are regular.

By an automorphism of a map we mean a permutation of its like elements preserving the relations of incidence [1, p. 101].

A map is said to be regular if it contains two automorphisms R and S, the first of which cyclically permutes the edges (and vertices) bounding a face F, while the other cyclically permutes the edges (and faces) which meet at a vertex V of F [1, p. 101].

If a face is p-sided and q edges meet at a vertex then the same is true for any face and any vertex. By Schläfli's notation such a map is denoted by the symbol $\{p,q\}$ [2, p. 61].

The map $\{q,p\}$ is the dual of $\{p,q\}$ in the following sense: The edges of one of these maps are the perpendicular bisectors of the edges of the other map, and vice versa. In particular Fig. 1-5a shows the maps of $\{6,3\}$ and $\{3,6\}$ drawn in black and red lines respectively. Fig. 1-5b shows the map of $\{4,4\}$ in black lines and its dual map $\{4,4\}$ in red lines [2, p. 62].

Maps of type $\{6,3\}$ or $\{3,6\}$ and $\{4,4\}$ on a torus are discussed by Coxeter and Moser [1, p. 103-8].

The vertices of a map of type $\{4,4\}$ may be taken as the points whose Cartesian coordinates are integers. The resulting figure on a plane surface is a square whose opposite sides are identified. If we take the coordinates of the end points of a side as (0,0) and (r,s) then the area of the square becomes equal to $r^2 + s^2$. So, on a torus we get a map of $n = r^2 + s^2$ vertices, 2n edges, and n faces, denoted by $\{4,4\}_{r,s}$.

Similarly, the vertices of $\{3,6\}$ are taken as the lattice points whose axes form an angle of 60°. In this case the torus is derived by identifying the opposite faces of a rhombus with angle 60°. If we take (0,0) and (r,s) as two vertices of a side of the hexagon, then the map on the torus of type $\{6,3\}_{r,s}$ has 2t vertices, 3t edges and t hexagonal faces, where $t = r^2 + rs + s^2$. Its dual map $\{3,6\}_{r,s}$ has t vertices, 3t edges, and 2t triangular faces.

According to Sherk [4], any regular map of type $\{p,q\}$ with n edges and on an orientable surface is on a surface of genus $p = 1 - n(\frac{1}{p} + \frac{1}{q} - \frac{1}{2})$. If R and S are the automorphisms of the regular map then $R^p = S^q = (RS)^2 = E$, where E is the identity automorphism.

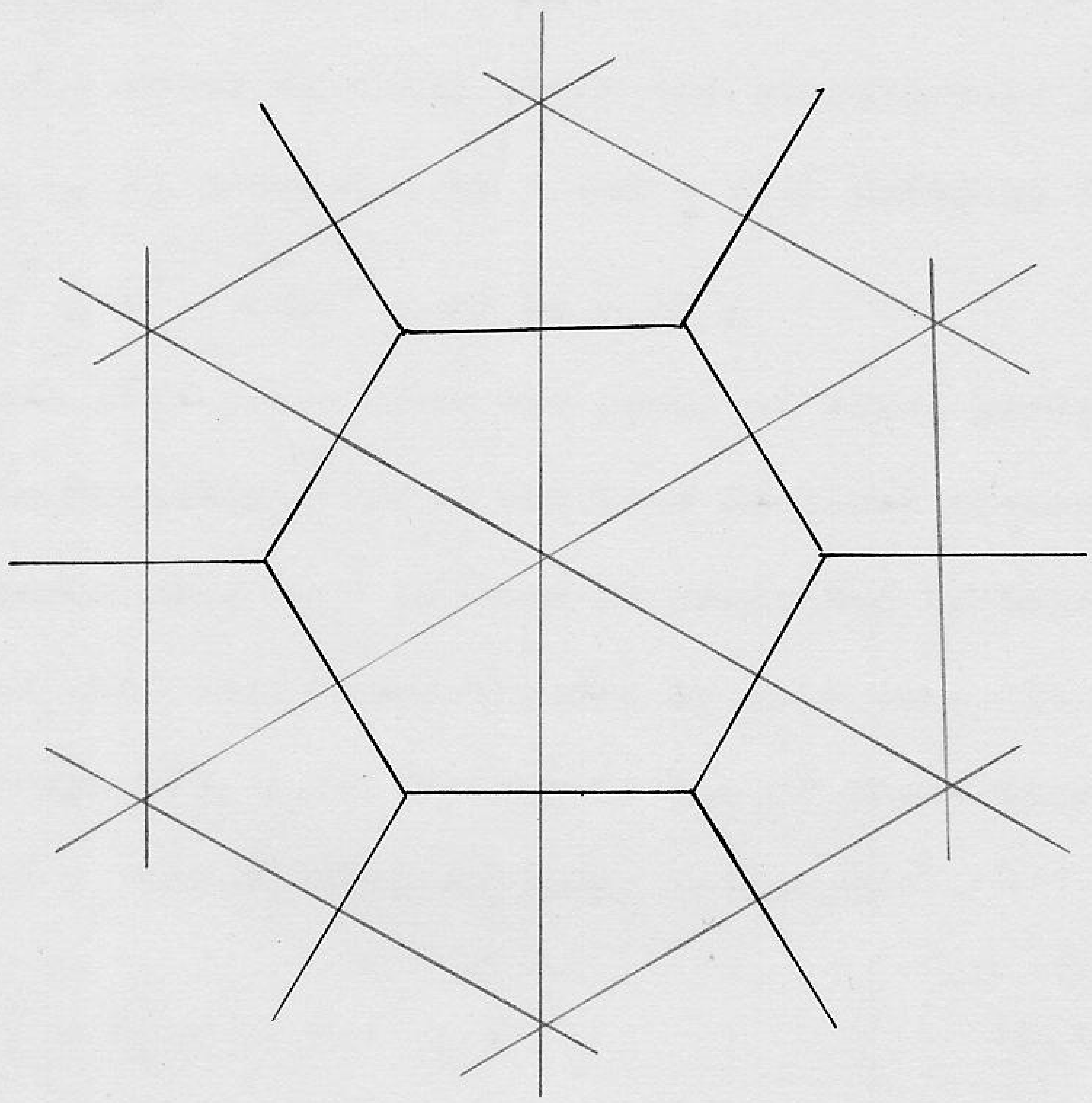


Fig. 1-5a

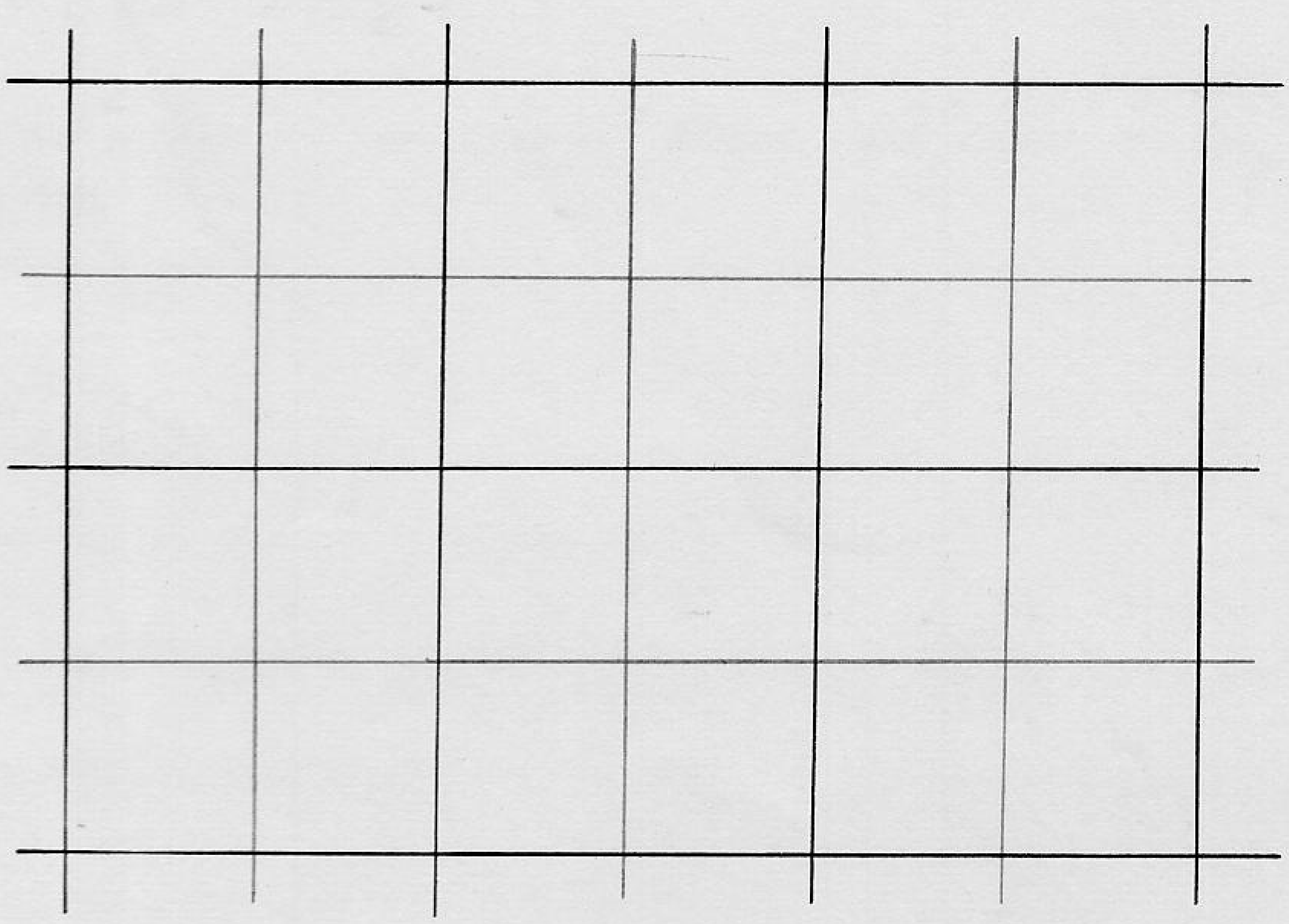


Fig. 1-5b

CHAPTER II

1. Introduction.

In this chapter we shall study the structure of graphs of groups G(3, n, k) generated by a and b with defining relations:

$$a^3 = b^n = (ab)^3 = e, \quad ba = ab^k.$$

We shall also prove that the genus of these groups is one. According to Burnside [1] there are four distinct classes of groups of genus one. One of them is generated by three distinct elements and the other three classes by only two. It turns out that the groups G(3, n, k) belong to one of these three classes generated by S_1 and S_2 with defining relations:

$$S_1^3 = S_2^3 = (S_1S_2)^3 = E. \tag{2.1}$$

$$(S_1S_2^2)^r(S_2S_1S_2)^s = E \tag{2.2}$$

$$N = 3(r^2 - rs + s^2) \tag{2.3}$$

N is the order of the group. Since this order is 3n,

$n = r^2 - rs + s^2$.

2. The Group G(3, 7, 2).

In the first chapter we saw that

$k^{3-1} + k^{3-2} + k^{3-3} \equiv 0 \pmod{n}$. If we replace k by 2 in the above relation, we will get $7 \equiv 0 \pmod{n}$. So $n = 7$.

If we represent this group graphically we get the graph shown in Fig. 2-1. where identical sides are numbered alike. In this map we have 17 vertices, 63 edges and 42 triangles (faces).

Since $2p = 2 - \chi$

$= 2 - V - E - S$

$= 2 - 17 - 63 - 42$

$= 6$

Therefore $p = 3$.

That is, the genus of the graph of this group is 3. But by choosing another set of generators we can decrease its genus down to 1.

We shall now attempt to generate the group as Burnside does, using a and ab as the first and second generators. Since the third generator is the product of the first two, the third generator is a^2b.

It is easy to verify that ab and a^2b are of order 3. The new defining relations are:

$$a^3 = (ab)^3 = (a^2b)^3 = e, \quad ba = ab^2$$

Now, from these new relations we may derive $b^7 = e$ (the product of the square of the first generator by the second generator gives us b, whose 7th power is the identity), which proves that they are equivalent to the old relations.

The mode of representation of this group is as follows: Take a regular hexagon, draw those three diagonals which join the opposite vertices to obtain six equilateral triangles. Then leave three of them white and color the rest black in alternate succession such that no two triangles of the same color share a common edge. Now number the

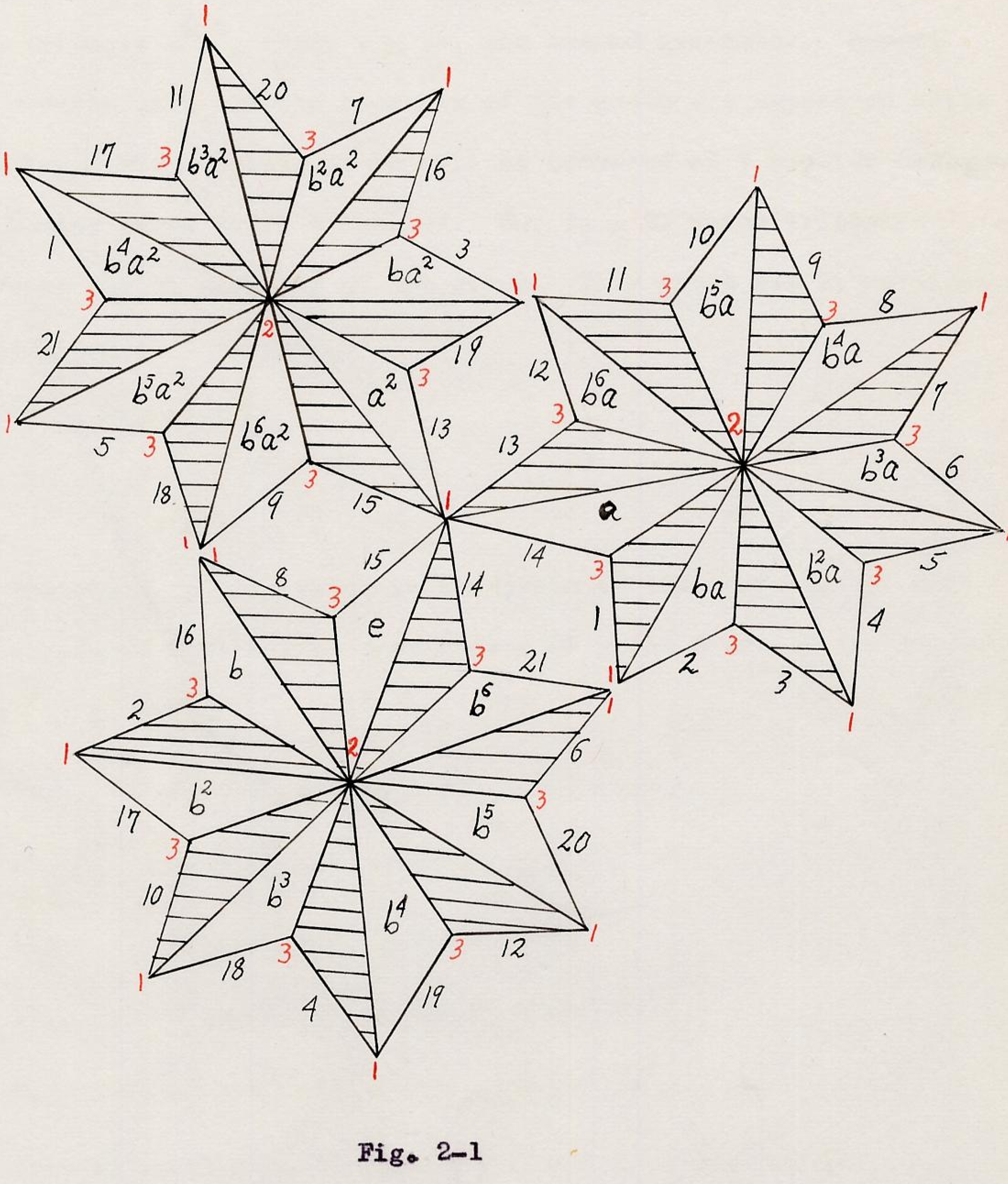

b^3a^2
b^2a^2
b^4a^2
ba^2
b^5a^2
b^6a^2
a^2
b^5a
b^4a
b^6a
b^3a
a
b^2a
ba
e
b
b^6
b^2
b^5
b^3
b^4

Fig. 2-1

vertices of these triangles by 1, 2 and 3 (1, the vertex of the middle of the hexagon. Fig. 2-2.) and denote one of the white triangles by e, the identity of the group. Then any 120^o counter-clockwise rotation of a white triangle, say w, around the vertex 1, leads to the next white triangle aw. Around the vertex 2, it leads to the next white triangle cw and around the vertex 3, to the next white triangle a^2c^2, where c = ab, the second generator. Repeat this process till all the elements of the group are mapped on white triangles. The complete graph will be composed of 7 regular hexagons, each having three white triangles. So, 3n = 21 white triangles represent the 21 elements of the group. This graph has 21 vertices, 63 edges and 42 triangles.

$$\text{So,} \quad 2p = 2 - \chi$$
$$= 2 - 21 + 63 - 42$$
$$= 2$$

Therefore, p = 1, which is the genus of the surface. It can not be reduced further, because groups of genus 0 are of the following types [3, p. 408].

I. $S_1^n = S_2^n = S_1S_2 = E;$

II. $S_1^2 = S_2^2 = S_3^n = S_1S_2S_3 = E;$

III. $S_1^2 = S_2^3 = S_3^3 = S_1S_2S_3 = E;$

IV. $S_1^2 = S_2^3 = S_3^4 = S_1S_2S_3 = E;$

V. $S_1^2 = S_2^3 = S_3^5 = S_1S_2S_3 = E.$

Next we shall find the integers r and s to satisfy the relations (2.2) and (2.3).

$N = 3n.$

Since $n = 7 = 3^2 - 3.1 + 1^2$

$$N = 3(r^2 - rs + s^2)$$
$$= 3(3^2 - 3.1 + 1^2)$$

That is: $r = 3$ and $s = 1$.

These values of r and s satisfy the second condition. For

$$(ac^2)^r(cac)^s = (aabab)^3.(abaab)^1$$
$$= (b^3)^3(b^5)$$
$$= b^{9+5}$$
$$= b^{14} = e.$$

Since the group G(3, 7, 2) is of genus 1, its representative graph, which is also of genus 1, can be mapped on a torus. This map is of type $\{3,6\}$ and is regular. So it can be represented by a rhombus with an angle of 60°. To do this we take any vertex of Fig. 2-2 as a vertex of the rhombus, then moving 4 units horizontally and one unit in an oblique line inclined at 60°, we get the second vertex of the rhombus. By similar steps we get the other vertices. So the graph of Fig. 2-2 is transformed to the rhombus of Fig. 2-3. Of course, the precise geometry is not essential, since we are dealing with a topological property. In fact, a rhombus or any plane figure must suffer some distortion when mapped onto an actual torus, but it enables us to express the regular map in a clear manner.

In order to have a clear idea about the position of the elements we represent the group G(3, 13, 3), expressing the elements in terms of a and b (Fig. 2-4). The elements in any standard hexagon are b^j, ab^j, a^2b^j for a fixed j. The rhombus in Fig. 2-5 represents the same graph as Fig. 2-4.

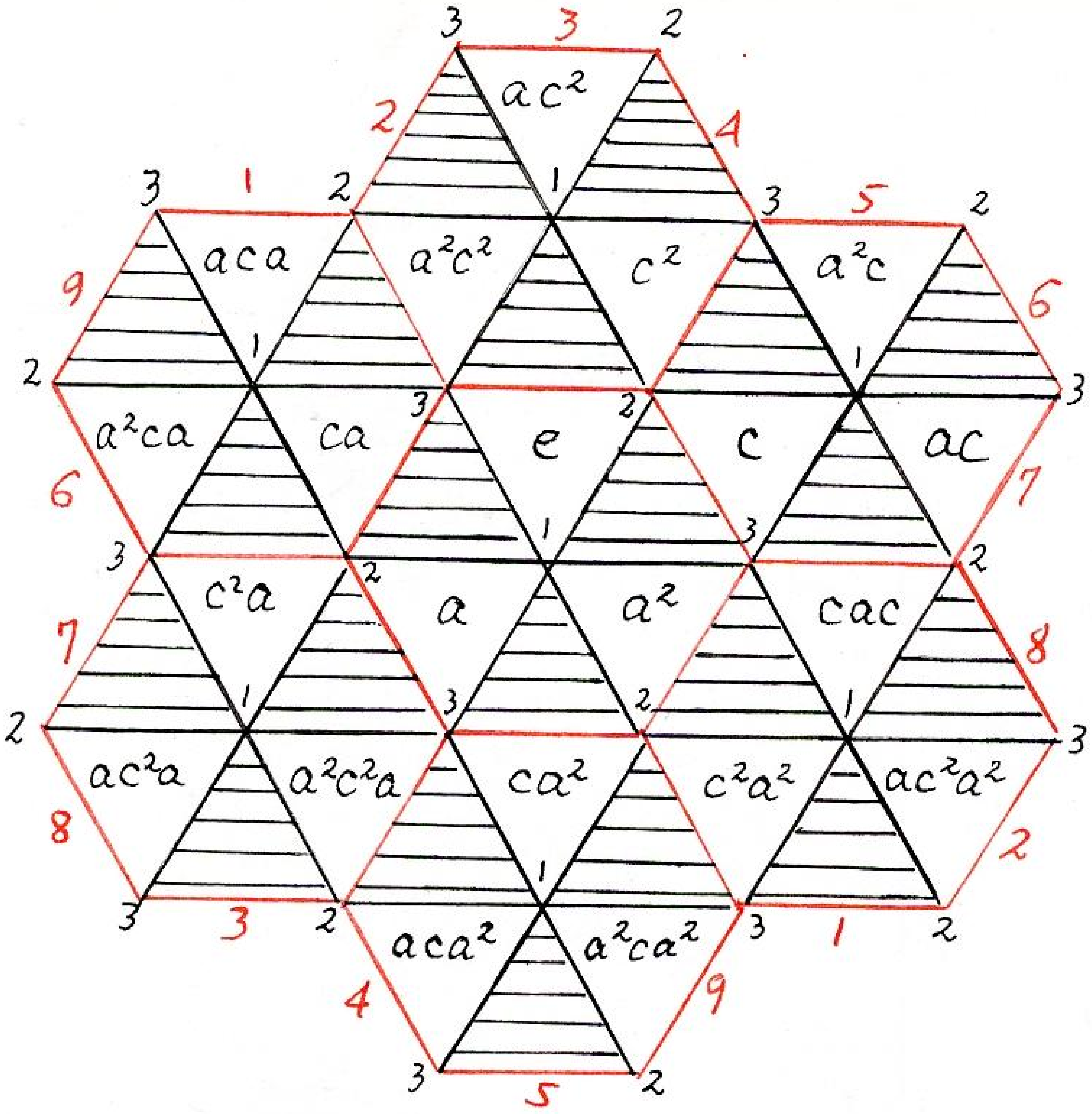

ac²
a c a
a²c²
c²
a²c
a²ca
ca
e
c
ac
c²a
a
a²
cac
ac²a
a²c²a
ca²
c²a²
ac²a²
aca²
a²ca²

Fig. 2-2

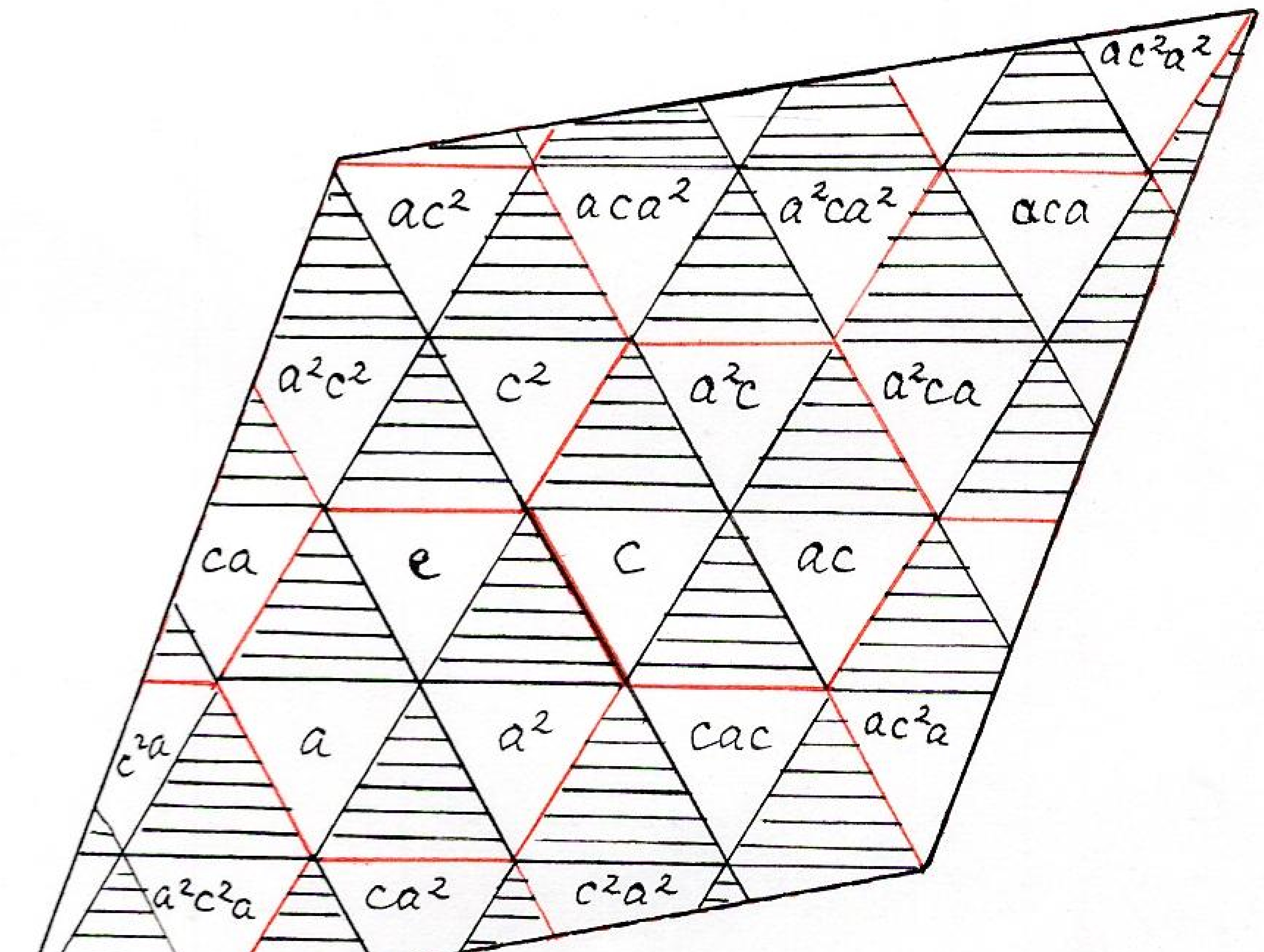
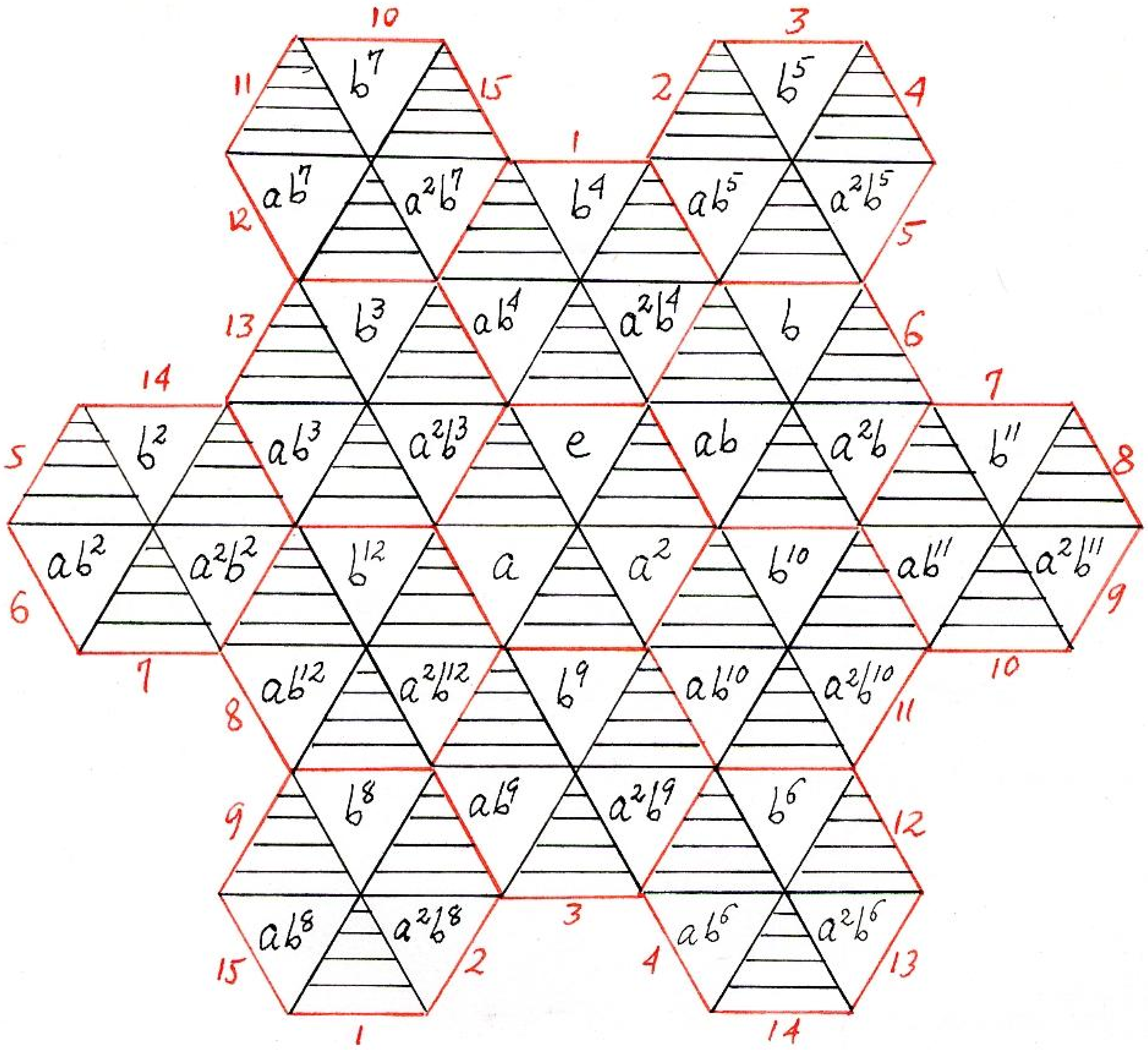

ac²a²
ac²
aca²
a²ca²
aca
a²c²
c²
a²c
a²ca
ca
e
c
ac
c²a
a
a²
cac
ac²a
a²c²a
ca²
c²a²

Fig. 2-3


10
3
11
b^7
15
2
b^5
4
1
ab^7
a^2b^7
b^4
ab^5
a^2b^5
12
5
13
b^3
ab^4
a^2b^4
b
6
14
7
5
b^2
ab^3
a^2b^3
e
ab
a^2b
b^{11}
8
ab^2
a^2b^2
b^{12}
a
a^2
b^{10}
ab^{11}
a^2b^{11}
6
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7
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ab^{12}
a^2b^{12}
b^9
ab^{10}
a^2b^{10}
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11
9
b^8
ab^9
a^2b^9
b^6
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15
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13
1
14

Fig. 2-4

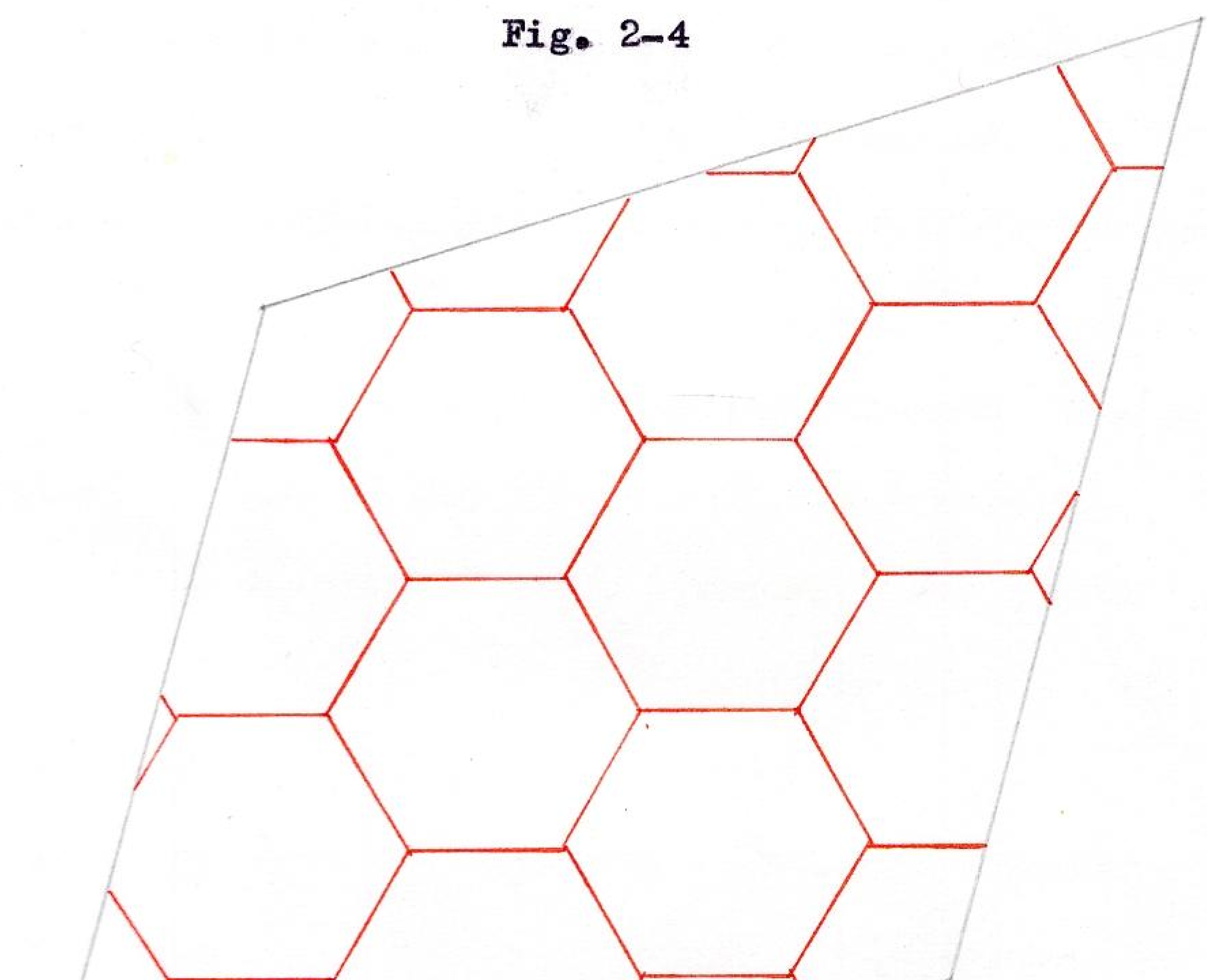


Fig. 2-5

3. The group $G(3, n, k)$.

In order to discuss this group some lemmas are needed.

Lemma 2-1.

The congruence $k^2 + k + 1 \equiv 0 \pmod{n}$ has no solution if n is divisible by 3^2 or by any prime not congruent to 0 or 1 modulo 3 [5, Lemma, p. 179].

The proofs of the following two lemmas are adapted from Niven and Zuckerman [6, Theorem 5.8 and 5.9, pp. 106-8].

A solution x, y of $x^2 + xy + y^2 = n$ is called primitive if $(x,y) = 1$.

Lemma 2-2.

Suppose $n > 2$. Each non-negative primitive solution of $x^2 + xy + y^2 = n$ determines a unique k modulo n such that $ky \equiv x \pmod{n}$. Furthermore $k^2 + k + 1 \equiv 0 \pmod{n}$, and different non-negative primitive solutions determine different k modulo n.

Proof:

If x, y is a non-negative primitive solution, then $(y,n) = 1$ and hence $ky \equiv x \pmod{n}$ determines a unique k modulo n. Furthermore if y' is a solution of $yy' \equiv 1 \pmod{n}$, then $k \equiv xy' \pmod{n}$, and we have $k^2 \equiv x^2y'^2 \equiv y'^2(-y^2 - xy) \equiv yy'(-xy' - yy') \equiv -k - 1 \pmod{n}$, or $k^2 + k + 1 \equiv 0 \pmod{n}$.

In order to show that different solutions determine different k, assume that u, v is another non-negative primitive solution such that $kv \equiv u \pmod{n}$. We thus have $xv \equiv kyv \equiv yu \pmod{n}$. But $1 \leq x < \sqrt{n}$ and $1 \leq v < \sqrt{n}$ so we have $1 \leq xv < n$ and, similarly, $1 \leq yu < n$. Therefore $xv = yu$ and hence $x = u$, $y = v$, since

$(x,y) = (u,v) = 1$ and all the numbers are positive.

Lemma 2-3.

Suppose $n > 2$, $k^2 + k + 1 \equiv 0 \pmod{n}$. There is a non-negative primitive solution x, y of $x^2 + xy + y^2 = n$ such that $ky \equiv x \pmod{n}$.

Proof:

Consider the set of integers $u - kv$ where u and v are integers such that $0 \leq u \leq \sqrt{n}$, $0 \leq v \leq \sqrt{n}$. There are $(1 + [\sqrt{n}])^2 > n$ different pairs u, v. Therefore there are two pairs u_1, v_1 and u_2, v_2 such that $u_1 - kv_1 \equiv u_2 - kv_2 \pmod{n}$. Let $v_2 - v_1 = v_o$ and $u_2 - u_1 = u_o$, then $kv_o \equiv u_o \pmod{n}$ and $|u_o| \leq \sqrt{n}$, $|v_o| \leq \sqrt{n}$. Moreover u_1, v_1 and u_2, v_2 being different pairs, u_o and v_o cannot both be zero and since $kv_o \equiv u_o \pmod{n}$, neither can be zero. Also we can show that at least one of $|u_o|$ and $|v_o|$ is less than $\sqrt{n}$. This is obvious if n is not a square. If n is a square and $|u_o| = |v_o| = \sqrt{n}$, we have $k\sqrt{n} \equiv \pm n \pmod{n}$, and hence $k \equiv \pm 1 \pmod{\sqrt{n}}$, or $k^2 \equiv 1 \pmod{\sqrt{n}}$. But $k^2 \equiv - k - 1 \pmod{n}$, so $k^2 \equiv - k - 1 \pmod{\sqrt{n}}$. Therefore $1 \equiv - k - 1 \pmod{\sqrt{n}}$ or $k + 2 \equiv 0 \pmod{\sqrt{n}}$, hence $(k + 2)^2 \equiv 0 \pmod{\sqrt{n}}$, but $(k + 2)^2 \equiv (k^2 + k + 1) + 3(k + 1) \equiv 0 \pmod{\sqrt{n}}$. Since n is a square, $(3,n) = 1$, by lemma 2-1. So, $k + 1 \equiv 0 \pmod{\sqrt{n}}$. But $k + 1 \equiv - k^2 \pmod{n}$, so $k^2 \equiv 0 \pmod{n}$, or $k \equiv 0 \pmod{\sqrt{n}}$, therefore $1 \equiv 0 \pmod{\sqrt{n}}$. This means that $n = 1$, which cannot occur since $n > 2$.

Now, u_o and v_o are bounded above by $\sqrt{n}$, and since one of them at least is less than $\sqrt{n}$, we have the inequality $1 \quad u_o^2 - u_o v_o - v_o^2 \quad 3n$.

We also have
$$u_o^2 + u_o v_o + v_o^2 \equiv k^2 v_o^2 + kv_o^2 + v_o^2$$
$$\equiv v_o^2(k^2 + k + 1)$$
$$\equiv 0 \pmod{n}.$$

This congruence together with the above inequality imply that $u_o^2 + u_ov_o + v_o^2 = 2n$ or $u_o^2 + u_ov_o + v_o^2 = n$. The first equality implies that u_o and v_o are both even. Therefore 2n is divisible by 4 or n is divisible by 2. This contradicts lemma 2-1. So only the second equality holds.

Let $g = (u_o, v_o)$. Then g^2 divides n and $k(v_o/g) \equiv (u_o/g)$ (mod n/g) and hence

$$n/g^2 \equiv (u_o^2 + u_ov_o + v_o^2)/g^2 \equiv (k^2v_o^2 + kv_o^2 + v_o^2)/g^2$$
$$\equiv (v_o^2/g^2)(k^2 + k + 1) \equiv 0 \pmod{n/g}.$$

This is possible only if $g = 1$, so we have $(u_o, v_o) = 1$.

Finally, if u_o and v_o have the same sign we let $x = |u_o|$, $y = |v_o|$. So we have $ky \equiv k(\pm v_o) \equiv \pm u_o \equiv x \pmod n$. If u_o and v_o have opposite signs, say, $u_o > 0$ and $v_o < 0$, then we shall consider two cases.

(i) $|v_o| > u_o$.

Let $x = -(u_o + v_o)$. Since $x^2 + xy + y^2 = u_o^2 + u_ov_o + v_o^2$, $y = u_o$. Then $ky \equiv ku_o \equiv k^2v_o \equiv -(k + 1)v_o \equiv x \pmod n$.

(ii) $|v_o| < u_o$.

Let $x = -v_o$. Then $y = u_o + v_o$. $ky \equiv k(u_o + v_o) \equiv k(kv_o + v_o)$ $\equiv v_o(k^2 + k) \equiv -v_o \equiv x \pmod n$.

The last two lemmas show that there is one to one correspondence between the non-negative primitive solutions of $x^2 + xy + y^2 = n$ and the solutions of the congruence $k^2 + k + 1 \equiv 0 \pmod n$.

<u>Theorem 2-4.</u>

The group G(3, n, k) is a group of genus 1 with defining relations (2.1), (2.2), (2.3).

Proof:

$G(3, n, k)$ is defined by $a^3 = b^n = (ab)^3 = e$, $ba = ab^k$, $k \not\equiv 1 \pmod n$ and $k^2 + k + 1 \equiv 0 \pmod n$.

The same group can also be generated by another set of generators of order 3.

a and b also generate the group. Therefore, if we take $S_1 = a$, $S_2 = ab$, then $S_3 = a^2b$. These generators are all of order 3. Among the new defining relations are:

$$a^3 = (ab)^3 = (a^2b)^3 = e.$$

Therefore the relations (2.1) are satisfied.

$$S_1S_2^2 = aabab = b^{k+1}.$$

$$S_2S_1S_2 = abaab = b^{k^2+1}.$$

If r and s satisfy relation (2.2), that is,

$(b^{k+1})^r(b^{k^2+1})^s = e$, then

$r(k + 1) + s(k^2 + 1) \equiv 0 \pmod n$, or $k(r - s) + r \equiv 0 \pmod n$.

By lemma 2-3, there exist two non-negative integers x and y such that $x^2 + xy + y^2 \equiv n$, $(x,y) = 1$ and $ky \equiv x \pmod n$. By taking $x = r$ and $x + y = s$, relations (2.2) and (2.3) are satisfied. Therefore the group $G(3, n, k)$ is of genus one.

But the groups with defining relations (2.1), (2.2) and (2.3) are not all of this type. That is, the groups $G(3, n, k)$ form a proper subset of the Burnside groups. This fact can easily be seen, if we consider the following counterexample.

Example 2-5.

Consider the group with defining relations:

$a^3 = b^3 = (ab)^3 = e$, of order 27[1, p. 135]. This is a special case of the Burnside groups, where in relations (2.2), r is taken as 3 and s as 0. So by relation (2.3) the order N is equal to $3(r^2 - rs + s^2) = 3(3^2) = 27$.

By lemma 2-1, the orders of G(3, n, k) are not divisible by 3^2 or, by higher powers of 3. But this is not the case for G, whose order, in fact, is 3^3. The map of G is given by Fig. 2-6.

4. Regular Maps of G(3, 91, 9) and G'(3, 91, 16).

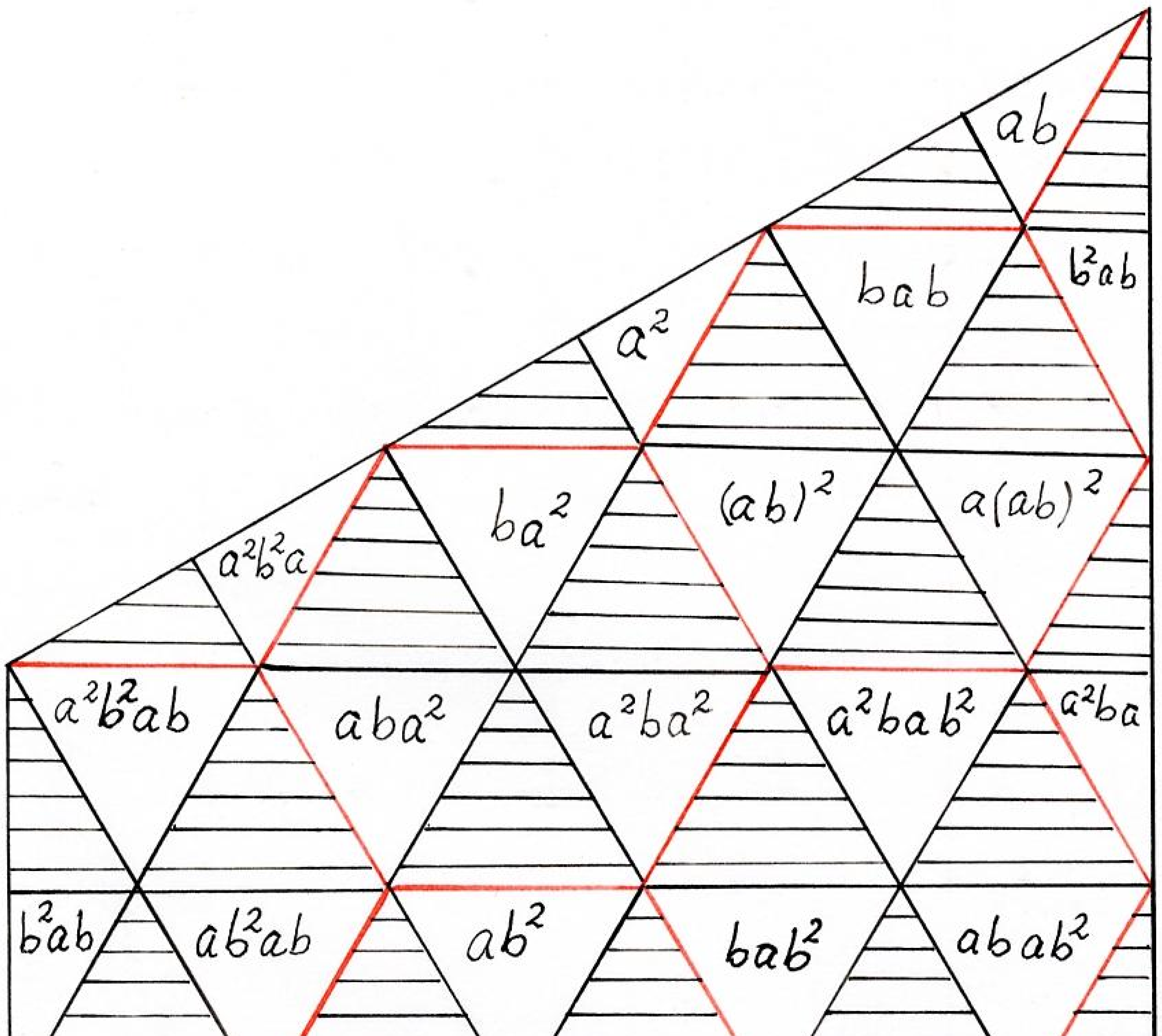
Theorem 2-6.

Let $n = p_1 p_2$, where p_1 and p_2 are distinct primes congruent to 1 modulo p. If k_1 and k_2 are independent solutions of $x^{p-1} + x^{p-2} + \ldots + x + 1 \equiv 0 \pmod{n}$, the groups $G_1(p, n, k_1)$ and $G_2(p, n, k_2)$ are not isomorphic [5, Theorem, p. 180].

By independent solution we mean a solution which is not a power (mod n) of k. Its existence is indicated by lemma 2-1.

The groups G(3, 91, 9) and G"(3, 91, 81) are isomorphic, since $k_2 = k_1^2$. This isomorphism may be obtained by replacing a by a^2.

Since k = 9 and k = 16 are independent solutions of $k^2 + k + 1 \equiv 0 \pmod{91}$, the groups G(3, 91, 9) and G'(3, 91, 16) are not isomorphic. Their rhombic maps, shown in Figures 2-7 and 2-8 respectively, clearly show this fact.


ab
a^2
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ba^2
$(ab)^2$
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a^2b^2
b^2
a^2b
ab^2a
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Fig. 2-6

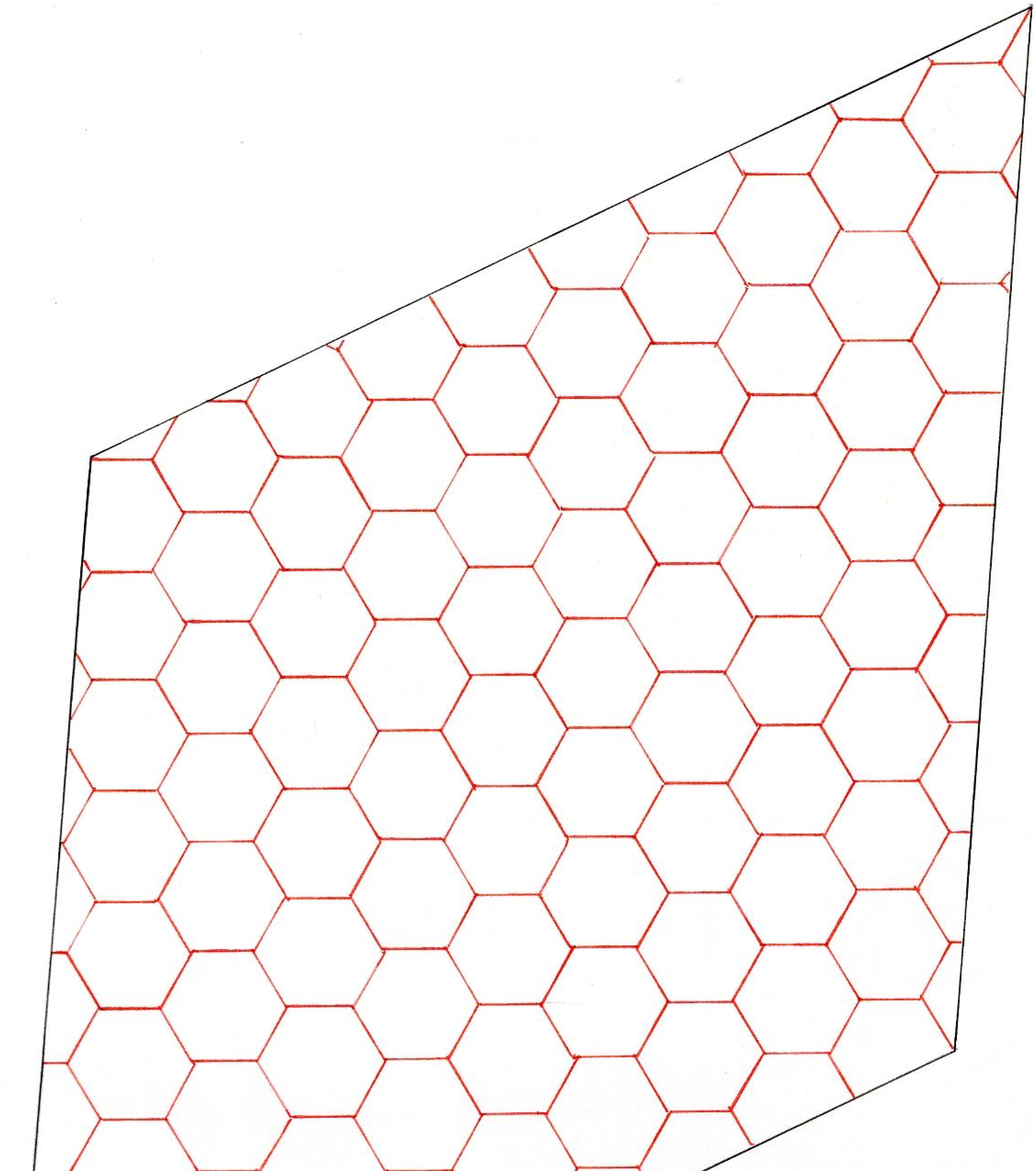


Fig. 2-7

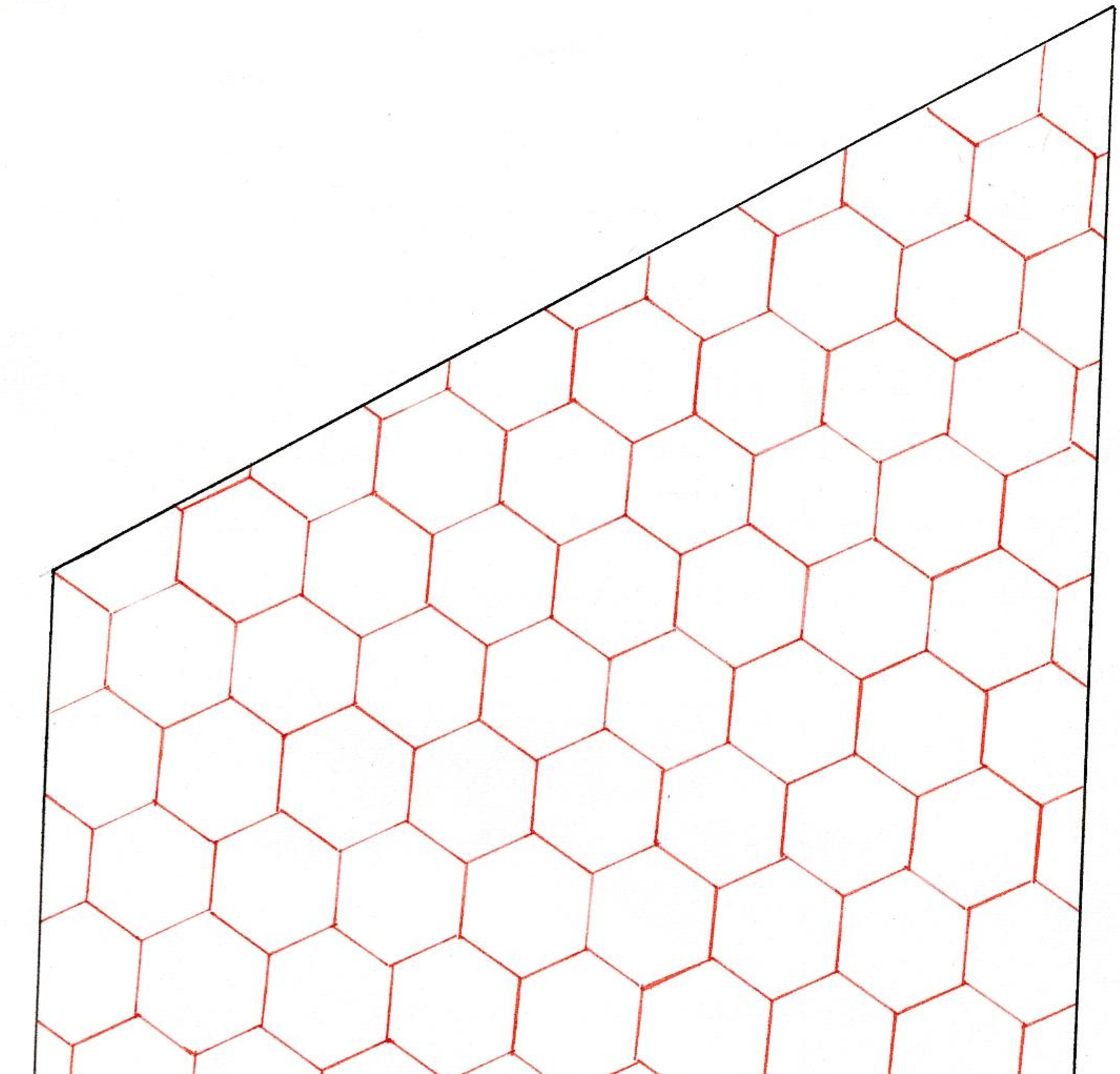


Fig. 2-8

CHAPTER III

1. Introduction.

In this chapter we shall study the structure of graphs of groups $G(4, n, k)$ generated by a and b with defining relations:

$$a^4 = b^n = (ab)^4 = e, \; ba = ab^k.$$

Since $m = 4$, $k^3 + k^2 + k + 1 = (k + 1)(k^2 + 1) \equiv 0 (\mathrm{mod}\ n)$. Here the quadratic factor interests us most and we shall prove that for $k^2 + 1 \equiv 0(\mathrm{mod}\ n)$ the groups $G(4, n, k)$ belong to one of the four classes of genus one, generated by S_1 and S_2 with defining relations:

$$S_1^2 = S_2^4 = (S_1S_2)^4 = E \qquad (3.1)$$

$$(S_1S_2)^r(S_2S_1S_2)^s = E \qquad (3.2)$$

$$N = 4(r^2 + s^2) \qquad (3.3)$$

N is the order of the group. Since this order is 4n,

$n = r^2 + s^2$.

2. The Group G(4, 5, 2).

If we take the quadratic factor $k^2 + 1 \equiv 0(\mathrm{mod}\ n)$ and substitute 2 for k we get $5 \equiv 0(\mathrm{mod}\ n)$. So $n = 5$.

If we represent this group graphically we get the graph shown in Fig. 3-1, where identical sides are numbered alike. In this map we have 14 vertices, 60 edges and 40 triangles.

If p is the genus of this map, then

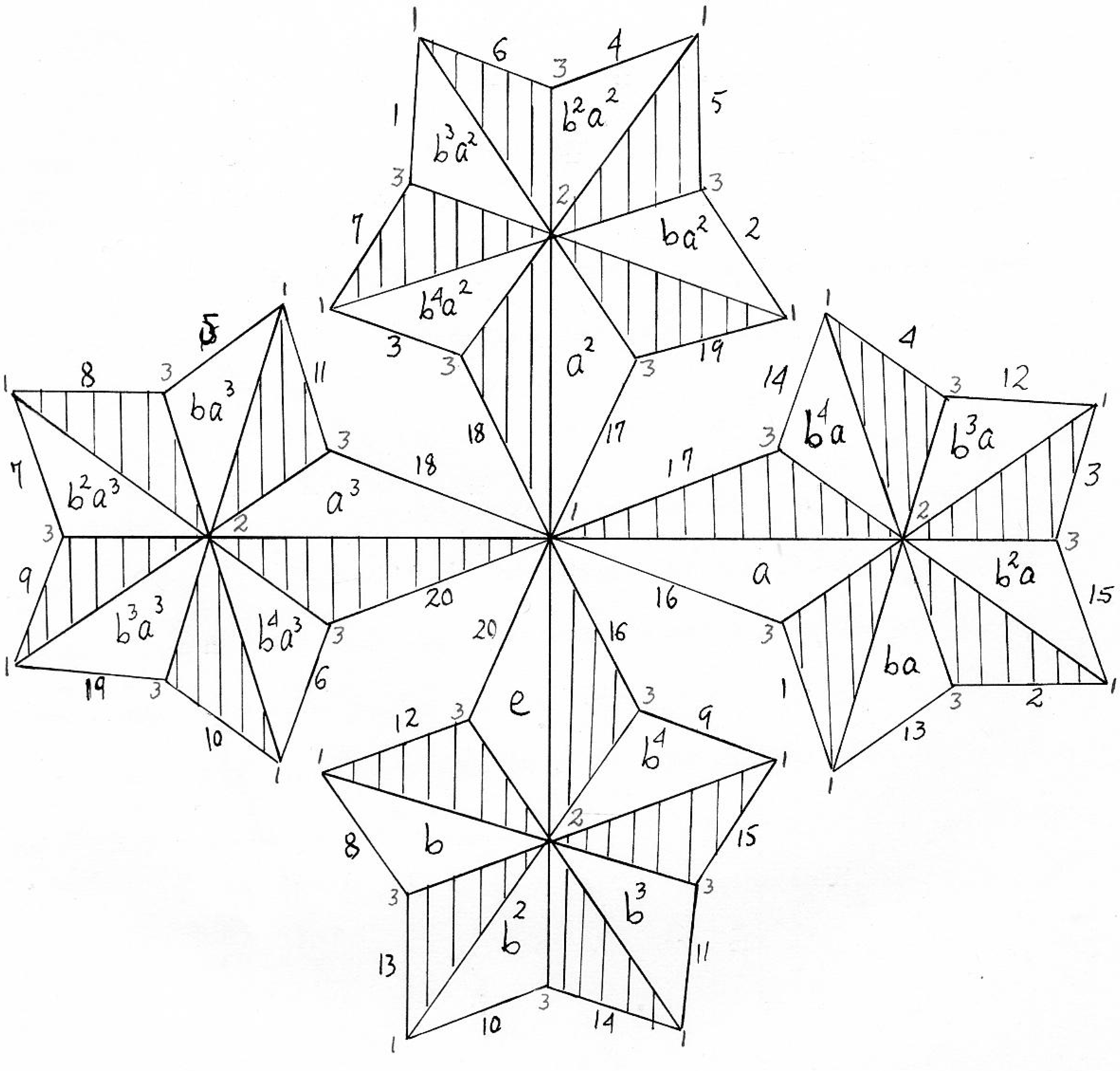

e
a
a^2
a^3
b
b^2
b^3
b^4
ba
b^2a
b^3a
b^4a
ba^2
b^2a^2
b^3a^2
b^4a^2
ba^3
b^2a^3
b^3a^3
b^4a^3

Fig. 3-1

$$2p = 2 - \chi$$
$$= 2 - 14 + 60 - 40$$
$$= 8$$

Therefore $p = 4$.

We shall now attempt to generate the group as Burnside does, using a^2b and a as the first and second generators. The third generator is $a^2ba = a^3b^2$.

It is easy to verify that a^2b and a^3b^2 are of orders 2 and 4 respectively. The new defining relations are:

$$(a^2b)^2 = a^4 = (a^3b^2)^4 = e, \quad ba = ab^2.$$

These relations are equivalent to the old, because they generate all the elements and relations of the group. To prove this, we shall show that they generate a and b and the previous relations.

From $(a^2b)^2 = a^4 = e$ and $ba = ab^2$, we obtain $a^2ba^2b = a^4b^5 = e$, so $b^5 = e$. Also $(ab)(a^3b^2) = a^4b^{10} = e$, so $ab = (a^3b^2)^{-1}$. From $(a^3b^2)^4 = e$, it follows that $(ab)^4 = e$. Conversely, the new generators and relations were derived from the old, so the equivalence is proved.

The mode of representation of this group is as follows: Draw the maps of $\{4,4\}$ and its dual on the same sheet and join the vertices of $\{4,4\}$ to the vertices of its dual (Fig. 3-2). So each square of $\{4,4\}$ which is drawn in red is composed of 8 right angled isoceles triangles. Now, as before, we leave four of them white and color the rest black in alternate succession such that no two triangles of the same color share a common edge. Then we number the vertices of $\{4,4\}$ by 3, those of its dual by 2 and the intersections of these two maps by 1. If we denote one of the white triangles by e, the identity of the group, then any half turn around the vertex 1 of a white triangle, say, w, leads to the next white triangle a^2bw. A quarter turn around the vertex 2 leads to the next white triangle aw, and around the vertex 3 to the next white triangle abw. a^2b and a are the first and second generators while ab is the inverse of the third generator. The complete graph will be composed of 5 large squares (in red), each having four white triangles. So $4n = 20$ white triangles represent the 20 elements of the group. This map has 20 vertices, 60 edges and 40 triangles.

$$\begin{aligned} \text{So} \quad 2p &= 2 - \chi \\ &= 2 - 20 + 60 - 40 \\ &= 2 \end{aligned}$$

Therefore $p = 1$. It cannot be reduced further because it is not one of the groups of genus 0, mentioned in chapter II. So by definition the genus of this group is 1.

Next we shall find the integers r and s to satisfy relations (3.2) and (3.3).

$N = 4n$, $n = 5 = 2^2 + 1^2$.

Let $r = 2$ and $s = 1$.

These values of r and s satisfy relation (3.2). For

$$(S_1S_2^2)^r(S_2S_1S_2)^s = (a^2ba^2)^2(aa^2ba)^1$$
$$= (b^4)^2(b^2)$$
$$= b^{10}$$
$$= e.$$

Since the group G(4, 5, 2) is of genus 1, its representative graph, which is also of genus 1, can be mapped on a torus. This map is of type {4,4} and is regular. So it can be represented by a square. To do this we take any vertex of Fig. 3-2 as a vertex of the square. Then, moving four units to the right and two units upwards, we obtain the second vertex of the square. By similar steps we get the other vertices. So the graph of Fig. 3-2 is transformed to the square of Fig. 3-3. Of course, as in the case of the rhombus, this process is not essential, but it enables us to express the regular map in a clear manner.

3. The Group G(4, n, k). $k^2 + 1 \equiv 0 \pmod{n}$.

Lemma 3-1.

Suppose $n > 1$. Each non-negative primitive solution of $x^2 + y^2 = n$ determines a unique k modulo n such that $ky \equiv x \pmod{n}$. Furthermore $k^2 + 1 \equiv 0 \pmod{n}$, and different non-negative primitive solutions determine different k modulo n [6, Theorem 5.8, p. 106].

A solution x, y of $x^2 + y^2 = n$ will be called primitive if $(x,y) = 1$.

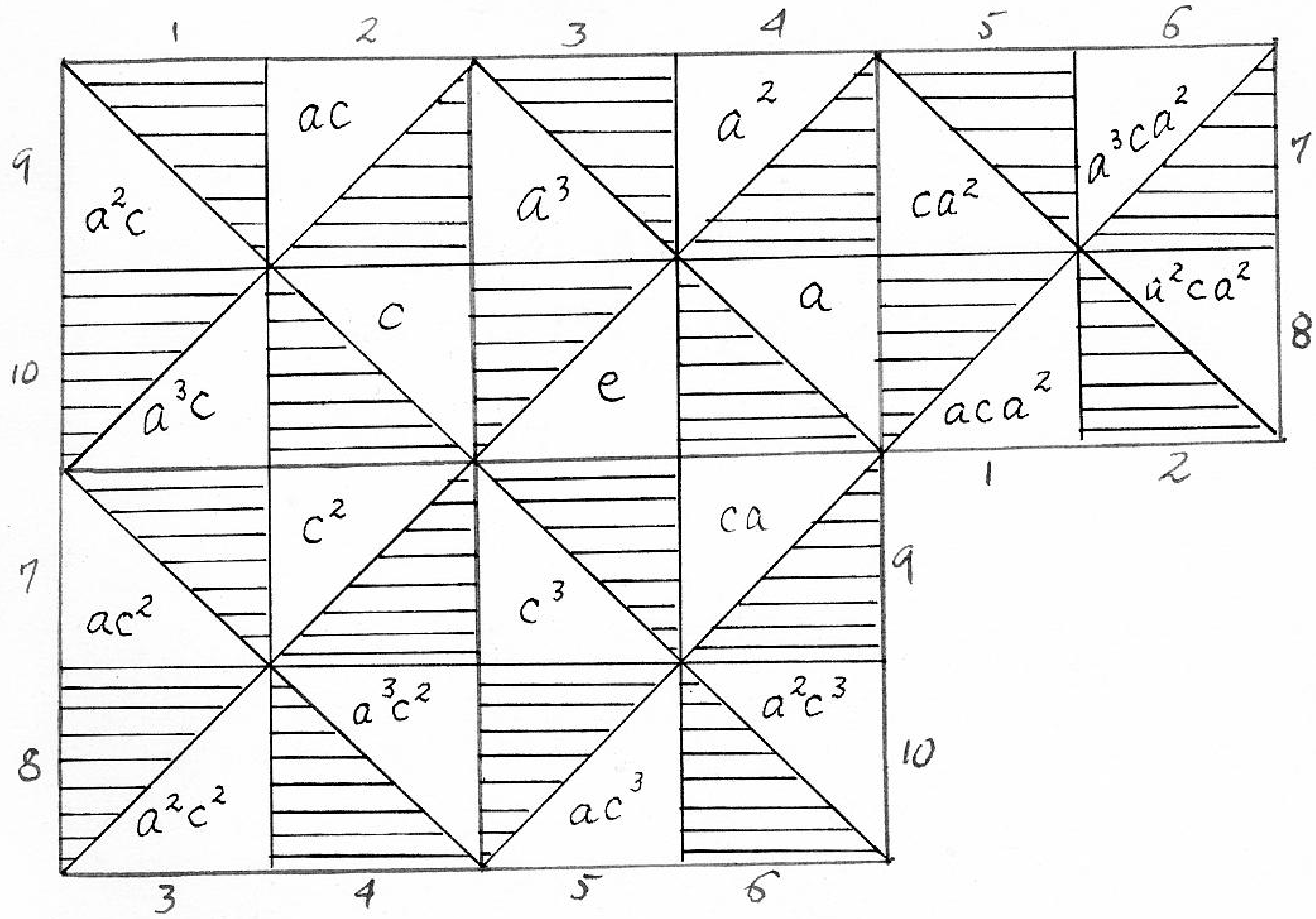

1
2
3
4
5
6
9
10
7
8
ac
a²
a²c
a³
ca²
a³ca²
c
a
a²ca²
a³c
e
aca²
c²
ca
ac²
c³
a³c²
a²c³
a²c²
ac³

Fig. 3-2

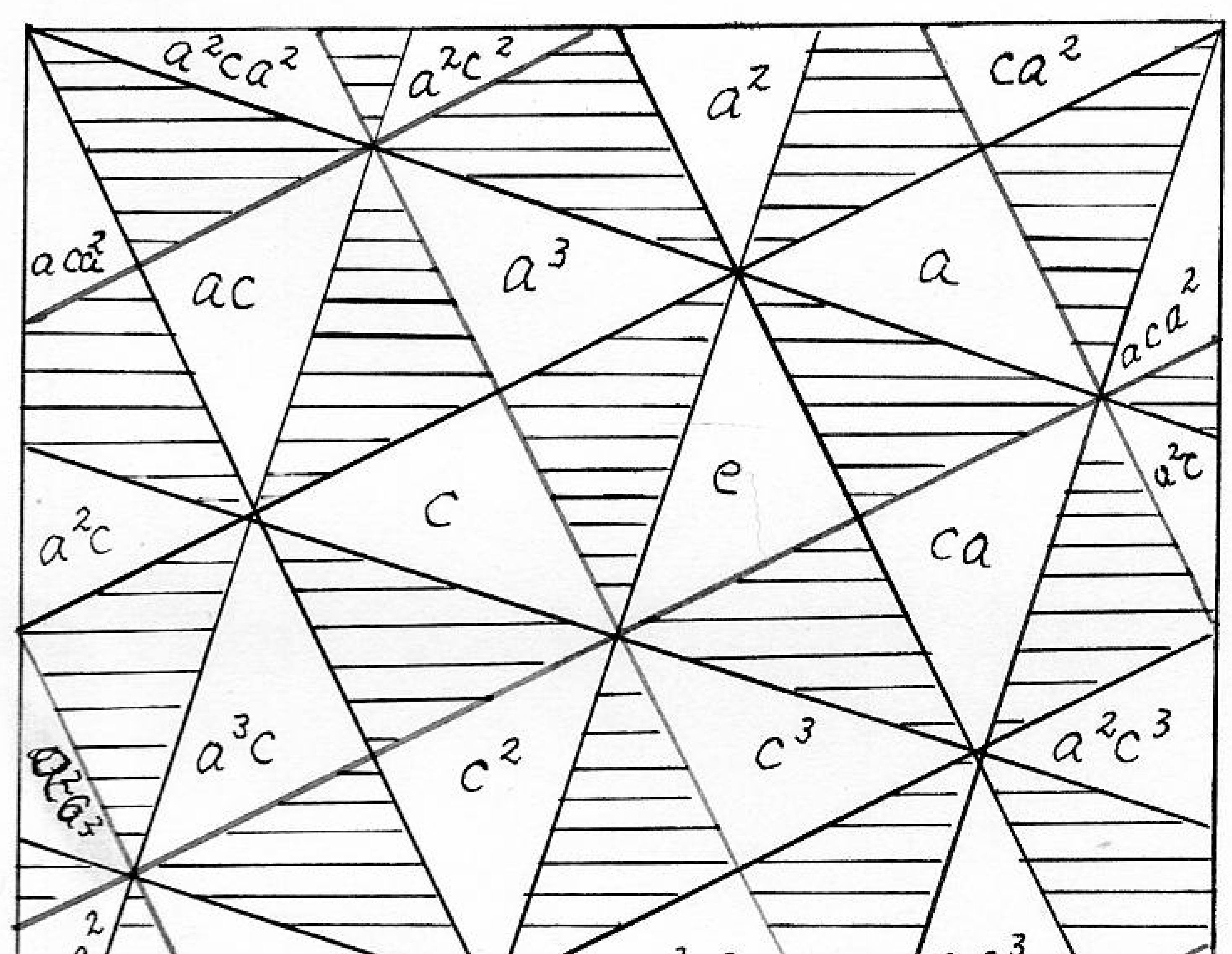

a²ca²
a²c²
a²
ca²
aca²
ac
a³
a
e
a²c
c
ca
a³c
c²
c³
a²c³
a³ca²
ac²
a³c²
ac³

Fig. 3-3

Lemma 3-2.

Suppose $n > 1$, $k^2 + 1 \equiv 0 \pmod{n}$. There is a non-negative primitive solution x, y of $x^2 + y^2 = n$ such that $ky \equiv x \pmod{n}$ [6, Theorem 5.9, p. 107].

Theorem 3-3.

If $k^2 + 1 \equiv 0 \pmod{n}$ then the group $G(4, n, k)$ is of genus one with defining relations (3.1), (3.2) and (3.3).

Proof:

The group $G(4, n, k)$ is defined by $a^2 = b^n = (ab)^4 = e$, $ba = ab^k$, $k \not\equiv 1 \pmod{n}$ and $k^2 + 1 \equiv 0 \pmod{n}$.

The same group can also be generated by another set of generators. a^2b and a generate the same group since they can generate b. If we take $S_1 = a^2b$ and $S_2 = a$, then $S_3 = a^2ba$ or $S_3 = a^3b^k$. The first generator is of order two while the others are of order four. Some of the new defining relations therefore are: $a^2b)^2 = a^4 = (a^3b^k)^4 = e$.

Thus the relations (3.1) are satisfied.

Next we shall determine r and s in such a way to satisfy relations (3.2) and (3.3). For this purpose

$$S_1S_2^2 = a^2ba^2 = b^{k^2}$$

$$S_2S_1S_2 = aa^2ba = b^k.$$

If r and s satisfy relation (3.2), that is,

$$(b^{k^2})^r(b^k)^s = e,$$

then $rk^2 + sk \equiv 0 \pmod{n}$, or $sk \equiv r \pmod{n}$. But by lemma 3-2, there exist two non-negative integers x and y such that $x^2 + y^2 = n$, $(x,y) = 1$ and $ky \equiv x \pmod{n}$. By taking $x = r$ and $y = s$, relations (3.2) and (3.3) are satisfied. Therefore the groups $G(4, n, k)$ are of genus one, if $k^2 + 1 \equiv 0 \pmod{n}$.

But not all groups of genus one defined by (3.1)-(3.3) are of this type. To prove this we take a counterexample.

Let the group $G = \{a, b\}$ be defined by the following relations:

$$a^4 = b^3 = (ab)^4 = e, \quad ba^2 = a^2b^2.$$

This group is of order 36 [7, p. 9]. It can also be defined by the following relations:

$$(a^2b)^2 = a^4 = (b^2a^3)^4 = e, \quad ba^2 = a^2b^2.$$

So relations (3.1) are satisfied.

Relation (3.3) is also satisfied, since $N = 4(3^2 + 0^2)$.

For relation (3.2) we consider

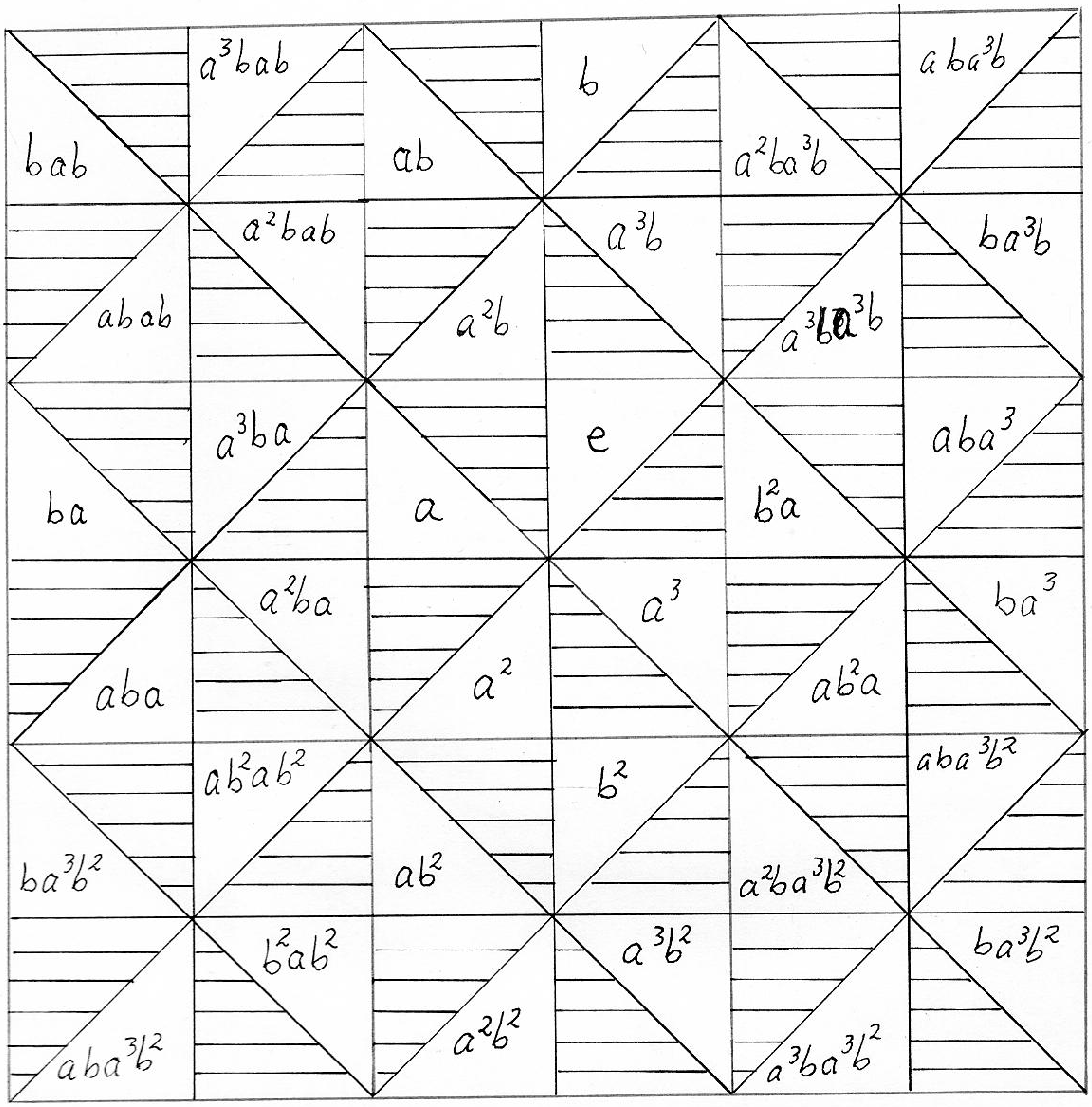
$$S_1S_2^2 = a^2ba^2 = b^2$$

$S_2S_1S_2 = a^3ba$. Hence

$$(S_1S_2^2)^3(S_2S_1S_2)^0 = (b^2)^3(a^3ba)^0 = e, \text{ or}$$

$$(S_1S_2^2)^0(S_2S_1S_2)^3 = (b^2)^0(a^3ba)^3 = e.$$

Therefore this group is of genus one. Its graphical representation is shown in Fig. 3-4 by a square with opposite sides identified.


a^3bab
b
aba^3b
bab
ab
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a^2bab
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a^2b
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ba^3b^2
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$a^2ba^3b^2$
b^2ab^2
a^3b^2
ba^3b^2
aba^3b^2
a^2b^2
$a^3ba^3b^2$

Fig. 3-4

4. Regular Maps of G(4, 65, 8) and G'(4, 65, 18) for $k^2 + 1 \equiv 0 \pmod{n}$.

If k is a solution, then $k^3 \equiv n - k \pmod{n}$ is also a solution yielding an isomorphic group [5, p. 181].

If $G_1(4, n, k)$ is defined as $a^4 = b^n = (ab)^4 = e$, $ba = ab^k$, then $G_2(4, n, k^3)$ is defined as $c^4 = d^n = (cd)^4 = e$, $dc = cd^{k^3}$. Then G_1 is isomorphic to G_2 under the correspondence $c \dashrightarrow a^3$, $d \dashrightarrow b$.

Since in G(4, 65, 8) and G'(4, 65 18), $8^3 \not\equiv 18 \pmod{65}$, the two groups are not isomorphic. The aquare maps of these two groups are shown by Figures 3-5 and 3-6 respectively, where the non-isomorphism is clearly seen.

5. The Group G(4, n, k), $k^2 + 1 \not\equiv 0 \pmod{n}$.

Suppose that a group is represented graphically by a regular division of a surface. If N is the order of the group generated by h elements, the surface then will be divided into 2N h-sided polygons. Let $A_1, A_2, \ldots, A_h$ be the vertices of these polygons and suppose that C_1 of them are A_1, C_2 of them are A_2, and so on. Around each corner A_i there are m_i white polygons, where m_i is the order of the i-th generator. Then

$$N = m_1C_1 + m_2C_2 + \ldots + m_hC_h, \text{ or } \sum_1^h C_i = N \sum_1^h 1/m_i$$

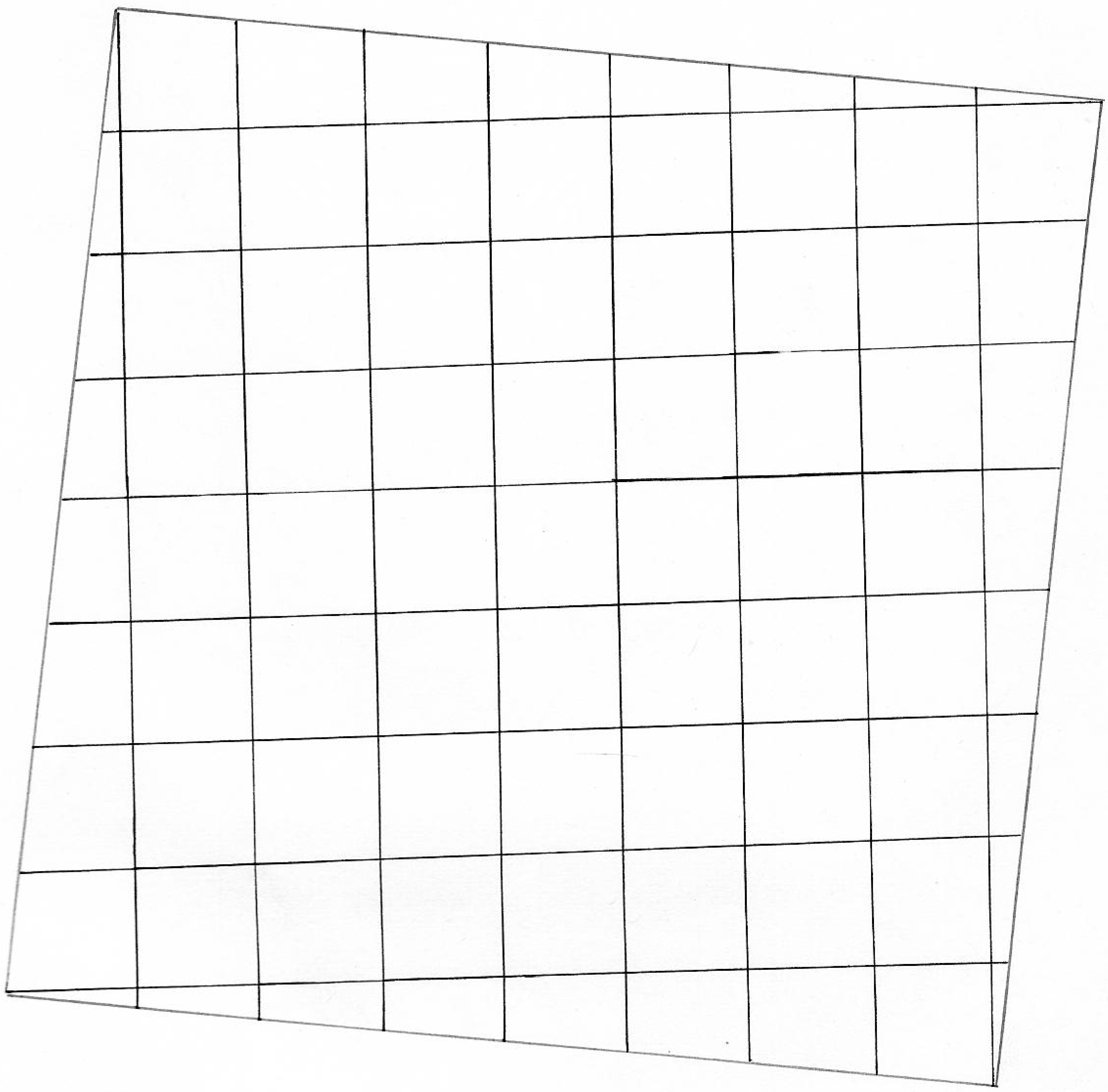

Fig. 3-5

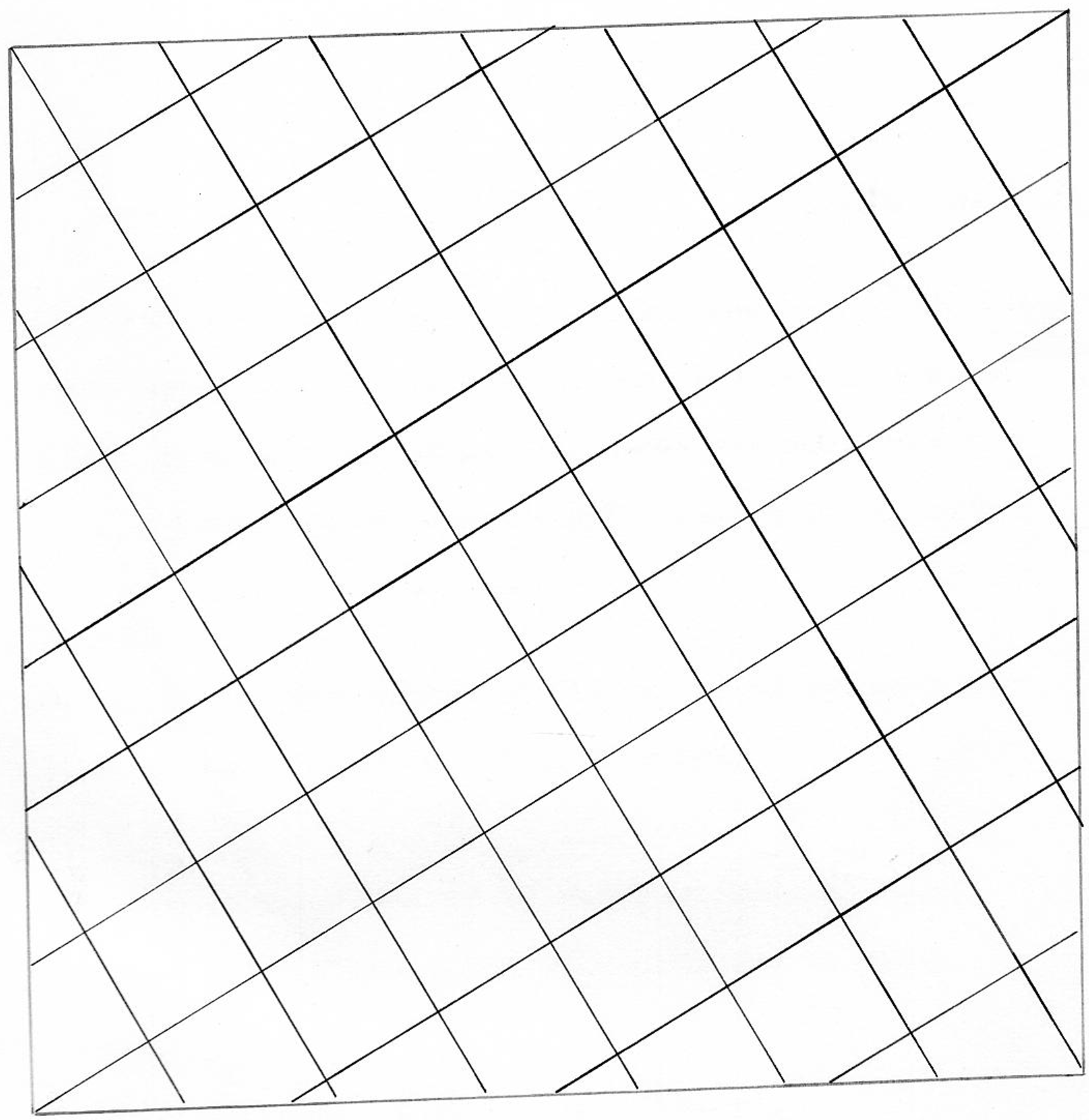

Fig. 3-6

which is the total number of vertices.

Each edge belongs to two and only two polygons. therefore the number of the edges $E = 2Nh/2 = Nh$. 2N is the total number of polygons.

$$
\begin{aligned}
2(p-1) &= \chi \\
&= E - 2N - V \\
&= Nh - 2N - N\sum_{1}^{h} 1/m_i \\
&= N(h - 2 - \sum_{1}^{h} 1/m_i) \qquad (3.4)[3, \text{ sec. } 290].
\end{aligned}
$$

Burnside uses this formula to discuss the groups of genus 0 and 1, and proves that the order of a group that can be represented by the regular division of a surface of genus p cannot exceed $84(p - 1)$, p being grater than unity[3, Theorem II, p. 399].

<u>Theorem 3-4.</u>

If $k + 1 = n$, then the group $G(4, n, k)$ is of genus n if k is even, and of genus $n - 1$ if k is odd.

<u>Proof:</u>

The proof of this theorem is based on relation (3.4).

In this formula N and h for this group are 4n and 3 respectively.

$$
\begin{aligned}
k^2 + 1 &= (n-1)^2 + 1 \\
&= n^2 - 2n + 2 \\
&\equiv 2 \pmod{n}.
\end{aligned}
$$

The elements a, ab^x and a^3b^x are of order 4 for any x.

$(a^2b^x)^2 = b^{x(k^2+1)} = b^{2x}$. If we take a^2b^x and a as the two generators, then $(n,x) = 1$. So take $x = 1$. Then the order of a^2b is n if k is odd and 2n if k is even. So for any set of generators, two of the generators are of order 4 and a third one of order n or 2n. If we take 4, 4 and 2n as the orders of a set of generators for an even k, and substitude them for m_i in relation (3.4), we get $p = n$. And if we replace m_i by 4, 4 and n(the orders of a set of generators when k is odd), then $p = n - 1$.

Lemma 3-5.

$$(k+1,\ k^2+1) \leq 2.$$

Proof:

Let $1 < r | k+1$. Then $r \nmid k$. If $k = 2m$, then $k + 1 = 2m + 1$ and $k^2 + 1 = 4m^2 + 1$ are both odd. $k^2 + 1 = (k+1)^2 - 2k$, if $r | k^2 + 1$, then $r | 2k$. Therefore r is even.

So, if k is even then $(k+1,\ k^2+1) = 1$.

Let $k = 2m + 1$. Then $k + 1 = 2(m+1)$ and $k^2 + 1 = 2(2m^2 + 2m + 1)$. Therefore $(k+1,\ k^2+1) = 2$ when k is odd.

Theorem 3-6.

If $(k+1)(k^2+1) \equiv 0 \pmod{n}$ and $n = n_1n_2$ such that $n_1 | k+1$ and $n_2 | k^2+1$ but not the other way round, then the group G(4, n, k) is of genus

1. $n - 2n_2 + 1$, if k and n_2 are odd and n_1 is even.

2. $n - n_2 + 1$ otherwise.

Proof:

If we consier the defining relations $a^4 = b^n = (ab)^4 = e$, and substitute 4, n, 4, the orders of the three generators for m_1 in relation (3.4), we get $p = n$. But it is possible to decrease the value of p by choosing another set of generators. If we choose a and ab^x as the first and second generators, then they should generate all the elements.

Now, $a^3(ab^x) = b^x$. b^x generates all the b's if x is relatively prime to n, so there is no loss of generality to take $x = 1$. The second generator then becomes ab and the third generator a^2b. The order of a^2b depends on k, n_1 and n_2, so we shall take each case seperately.

1. If $2 \nmid n_2$, then $2n_2 \mid k^2 + 1$. n_1 is even, so $(a^2b)^2 = b^{k^2+1}$ is of order $n_1/2$, therefore a^2b is of order n_1.

$$\text{Hence } 2(p - 1) = 4n\left[3 - 2 - \tfrac{1}{4} - \tfrac{1}{4} - 1/n_1\right]$$

$$p - 1 = 2n(\tfrac{1}{2} - 1/n_1)$$

$$= n - 2n_2$$

$$\text{or } p = n - 2n_2 + 1.$$

2. (i) If k is even, then by lemma (3.5) $(n_1, k^2 + 1) = 1$. $(a^2b)^2 = b^{k^2+1}$. So the order of a^2b is $2n_1$.

(ii) If k and n_1 are odd then again $(n_1, k^2 + 1) = 1$, Therefore the order of a^2b is $2n_1$.

(iii) If k is odd, so $k^2 + 1$ is divisible by 2 and not by 4, that is, $(n_1, (k^2 + 1)/n_2) = 1$. So again the order of a^2b is $2n_1$.

Thus for all these cases the group can be defined as:

$$a^4 = (ab)^4 = (a^2b)^{2n_1} = e.$$

Substituting the orders of these generators for m_i in relation (3.4), we get

$$2(p - 1) = 4n\left[3 - 2 - 1/4 - 1/4 - 1/2n_1\right]$$

$$p - 1 = 2n\left[\tfrac{1}{2} - \tfrac{1}{2}n_1\right].$$

$$\text{So } p = n - n_2 + 1.$$

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