AMERICAN UNIVERSITY OF BEIRUT

DIAGNOSTIC AND REMEDIAL WORK
IN
THE FUNDAMENTAL OPERATIONS OF ARITHMETIC

A Dissertation
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CONTENTS

INTRODUCTION

This paper is written for the purpose of improving the teaching of arithmetic in general and the four simple processes in particular. This can be done by presenting to the Elementary School teacher certain suggestions, principles and practices that have proved to be of great help in teaching arithmetic, and by making out of arithmetic a different course from what we have at present. A course which is easy to be taught, practical and more interesting to be studied and learned.

No one doubts the fact that arithmetic is poorly taught in many Elementary schools when he knows that the failures in those schools are caused more frequently by arithmetic than by any other subject in the curriculum. Leading writers declare that the subject is suffering because many teachers lack instruction in its theories, methods and devices.

The material of this paper is divided into four major parts. Part I treats very clearly the implications, need and different techniques of educational diagnosis. It thus serves to orient the teacher into the nature and purpose of educational diagnosis as compared with medical diagnosis. Part II gives careful consideration to the problem of diagnosis in arithmetic in regard to the following topics: aim, subject-matter and method. Part III acquaints the teacher with the techniques of measurement, i.e., evaluation of the results of teaching arithmetic. Items such as the construction, types and use of standardized diagnostic tests are fully discussed. Part IV presents and evaluates the most important techniques used in remedial and preventive work for the improvement of general and specific teaching methods of arithmetic.

The reader is requested to bear in mind that this paper is not an exhaustive treatment of the subject and that it is far from being complete. But if it furnishes some aid to those teachers who find difficulty in teaching arithmetic, and who are usually not efficient in their teaching, it will be serving its purpose.

Finally, I am deeply grateful to Professor G.D.Shahla who has supervised the whole work; to Professor S.C. Dodd who gave me encouragement, help and valuable suggestions from beginning to end; to Professors B. Khauli and L. Leavitt who have read the paper and offered helpful criticisms.

An Outline Form of the Thesis

On

DIAGNOSTIC AND REMEDIAL WORK

IN

THE FUNDAMENTAL OPERATIONS OF ARITHMETIC

This thesis is mainly concerned with the four simple processes of arithmetic. The mechanical as well as the reasoning phase of the subject are fully discussed under the following topics:-

I - Diagnosis

- A. Meaning:
    1. The determination of a disease by means of distinctive marks or characteristics.(Medical)
    2. Procedures and techniques which help to discover and evaluate both the strengths and weaknesses of the group as well as those of the individual. (Educational)
- B. Bases and scope
    1. A full knowledge of the different types of learning difficulties, their symptoms and causes are real bases of educational diagnosis.
    2. The complexity of the learning situation makes the scope of educational diagnosis an enormous one.
- C. Need and value
    1. Millions of pupils throughout the world are mentally and physically handicapped and are in bad need of diagnosis.
    2. The correction of deficiencies and the development of strengths constitute the value of diagnosis.
- D. Techniques of diagnosis
    1. The location of mal-adjusted pupils, errors and their causes, on the one hand;
    2. And the application of remedial and preventive procedures, on the other hand; are the most important techniques of diagnosis.

II- Aim or function of arithmetic instruction

- A. The most important ones are:
    1. Computational
    2. Social
    3. Informational
    4. Psychological

III- Subject-Matter

- A. The fundamental operations of arithmetic contained the following material:
    1. The basic combinations
    2. The four simple processes
    3. The special language of arithmetic
    4. Simple mathematical problems of ordinary life.

IV - Method

A. Bases of an efficient method of arithmetic instruction
  1. Good mastery of the combinations and the processes
  2. The knowledge of these in an automatic and a mechanical way.

B. Essential factors of problem-solving
  1. Problem comprehension
  2. Problem analysis

C. Drill
  1. Meaning: practice with interest in, and full attention to, the thing which is being repeated. This is the only sort of practice which makes perfect.

  2. Distribution: according to time, difficulty and applied at the point of error.

D. Motivation

  1. Ways of motivation
    a. Races with briefest time and fewest number of mistakes as limits.
    b. Pupils should keep a record of their own work and should try to surpass it.
    c. The use of diagnostic tests which leads to the discovery of difficulties and their correction.
    d. Pupils should be taught all sorts of number games.

  2. Ways to be avoided
    a. The use of the old practice of memorizing and repeating tables.
    b. Long drills on number combinations.
    c. Endless examples in the four simple processes.
    d. Solution of trifling problems.
    e. Problems which will never be needed by pupils.
    f. Problems which have for their only purpose mental discipline.

E. Other phases of method are:-
  1. Connection of arithmetic with concrete objects.
  2. Explanation of the operations inductively rather than deductively.
  3. Application of the operations to interesting situations.
  4. The use of simple language .
  5. The training of the pupils to analyze all the steps in problems.

V - Measurement

A. Principles of test construction
  1. Selection of material which measures things one sets out to measure.
  2. Provisions to measure most if not all important objectives should be made.
  3. Material should be moulded in more than one type of questions.
  4. Preliminary draft should include more items than the final form of the test.
  5. Draft should be submitted to the criticisms of teachers of the same subject.
  6. Items should be placed in order of ascending difficulty.

7. A regular sequence of responses should be avoided.
8. Test instructions are necessary. They should be clear, concise and complete.
9. Individual questions of the test should be scaled.
10. Age-and grade-norms of the test should be determined.

B. Characteristics of good tests:
   1. Validity
   2. Reliability
   3. Usability
   4. Norms
   5. Cost

VI- Measurement in arithmetic

A. Need for measurement
   1. Measurement in arithmetic is indispensable.

B. Purposes of measurement (page 53)

C. Methods of measurement
   1. Traditional method which can be made by:
      a. mere estimation of the pupil's ability or
      b. traditional examination
   2. Modern method by using standardized tests or teacher-created new objective tests. Samples of the latter type of tests are found on pages 57-68.

VII- Diagnostic tests

A. Kinds of tests
   1. Group diagnostic tests
   2. Individual diagnostic tests

B. When to test
   1. Before instruction
   2. During instruction
   3. After instruction

C. Parties concerned in diagnostic testing and the share of each
   1. The teacher must understand
      a. The mental processes of the pupils
      b. The complexity of the subject
      c. How to find errors, their causes and remedy
   2. The pupil must be accustomed from the beginning to good methods of performance, else:
      a. he adopts a great variety of round-about methods of procedure, or
      b. he continues to use the detailed methods employed in learning a new process.
         (1) Samples of the above mentioned inappropriate methods are found on PP. 77 - 81.

VIII-Remedial Work

A. Corrective procedures
   1. Correction can be planned when the cause or causes of the pupil's difficulties have been determined.
   2. The first principle of an effective remedial program should be the growth of the individual, and the elimination of those factors that interfere with this growth.

B. Preventive procedures

- 1. The seriousness of error
  - a. what is more important than corrective remedies is the application of preventive procedures. This is due to the nature and seriousness of error (PP.89-90)
- 2. Means of prevention
  - a. The teaching material
    - (1) Carefully graded exercises in which there is a step by step development are very necessary.
    - (2) The inclusion of the greatest number possible of the skills in a particular subject is important.
    - (3) A summary of the material built on the above mentioned principles is found on PP. 91-92.
  - b. Method of instruction
    - (1) Factors inherent in the method of instruction obviously contribute to the development of learning difficulty.
    - (2) To stress speed too much is dangerous. Hasty work inevitably leads to mistakes.
    - (3) Failure to adapt instruction to different pupils' needs leads to errors.
    - (4) Failure to maintain an adequate program of testing during teaching leads to the accumulation of mistakes which becomes a problem for remedial procedure.
    - (5) The result is the fixation of poor habits of work.
  - c. Individual differences
    - (1) Pupils belonging to the same class differ widely in acheivement, do not fail in a general sense and do not often need remedial work of a general nature.
    - (2) These different conditions make it necessary to provide for individual instruction else, serious difficulties are sure to arise in the work of the slow pupils.
    - (3) Remedies
      - (a) Give more difficult questions under the same topic to more advanced pupils.
      - (b) Let advanced pupils help to direct and check the work of the backward ones.
      - (c) Give maximun and minimum assignment.

PART I

EDUCATIONAL DIAGNOSIS

CHAPTER ONE

Diagnosis

A. Implications of diagnosis

1. Meaning of the term.

According to Webster's Dictionary, diagnosis means "the determination of a disease by means of distinctive marks or characteristics". This, of course, pertains to the medical diagnosis. Educationally, the term implies those procedures and techniques by means of which one can discover and evaluate both the strengths and the weaknesses of the group, as well as those of the individual, in anyone or more subjects. Once these strengths and weaknesses are discovered, the determination of their causes becomes possible.

2. Its bases.

"Diagnose before you dose"!,(1) is a rule that holds true in medicine as well as in education. It is self-evident that in medicine effective treatment must be suited to the ailment. So is the case in education.

Such

After having discovered the strong and weak points and found out their causes, the diagnostician, can then suggest the necessary developmental and remedial measures. Moreover, able diagnosticians can predict to a large extent the probable outcome of the condition under consideration. This is what is exactly meant by saying that "diagnosis leads to prognosis".(2)

Just as the modern physician should have, as a basis for his diagnosis, a full knowledge of symptoms of the various diseases and the causes underlying them, so the teacher should have a full knowledge of the different types of learning difficulties, their symptoms and causes.

3. Its scope.

Diagnosis in medicine has been greatly facilitated by the use of objective methods of analyzing and evaluating symptoms. The doctor, in order to make his observations more objective and more exact, can make use of such scientific instruments as the test tube, the clinical thermometer, the microscope, and of a large number of such instruments to aid the eye, the ear, and the hand, in making diagnosis of a physical ailment. The use of such instruments of precision insures accuracy of measurement, increases the reliability of diagnosis, and makes the physician better off than the teacher along this line.

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(1) C. C. Ross: "Measurements in Today's Schools", P. 392.
(2) National Society For The Study of Education: "Educational Diagnosis", Year Book XXXIV P. 2.

It is true that in education diagnosis is much more difficult than it is in medicine, because the learning situation is very complex rather than simple. Anyone single learning situation, no matter how simple it is, is conditioned by many factors which may be either classified as internal factors, such as physical, intellectual, emotional and educational; or, external factors, such as school and home environment.

In spite of this enormous scope of educational diagnosis, and the lack of such scientific instruments which are used by the doctor, one can see that diagnosis in education is rapidly becoming as accurate and as scientific as diagnosis in medicine, and that it seeks to reveal anyone cause or causes which tend to interfere with satisfactory progress in school.

There are at present several hundreds of refined statistical techniques; diagnostic tests, in the various subjects, which locate with a high degree of precision many of the types of faults made by pupils; diagnostic charts, which help the teacher to visualize deficiencies; charts for measuring vision, apparatus for measuring hearing plus many other devices and instruments which help to discover specific or general weaknesses in individual pupils in many fields of learning, and to determine their causes as a basis for their correction.

An outstanding educator recently observed that "experts in reading, arithmetic and spelling, can now make diagnosis no less valid and reliable than are most diagnoses in medicine".(1)

B. Need and Value of Diagnosis.

"The committee on special education of the White House Conference on Child Health and Protection estimated that there are 3,000,000 mentally and physically handicapped children in the schools of the nation. These individuals were found at all levels of schooling, from the nursery school through the college and university".(2)

Similarly, there are hundreds of children, in our schools at present, whose behavior reveals maladjustments of various kinds and who fail to master the subject-matter and skills that are being taught. Some of them do not seem to be able to compute with speed and accuracy; some of them fail to grow in ability to reason in solving verbal problems. Some encounter difficulty only in some single phase of the subject, while others are unable to do satisfactory work in any subject. These pupils are labeled, by their teachers, lazy, stupid or indifferent. The practice is to fail them and require them to repeat the work.

This failure on the part of the schools concerned to determine the causes of their difficulties and to use methods and materials that provide for individual differences, is quite similar to that treatment by a doctor who prescribes the same remedy to all ailments. Failure to diagnose these school ails and to do the necessary remedial work is a strong cause of much waste in the whole scheme of education. Naturally, the result is a high percentage of failure, a large amount of retardation, and elimination of many pupils from school.

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(1) C. C. Ross: "Measurement In Today's Schools", P. 393.
(2) National Society for the Study of Education: "Educational Diagnosis", Year Book XXXIV P. 2.

Recently, many attempts have been made to meet these conditions by making adequate provisions for diagnosis of pupils' difficulties, and on the basis of the information thus secured, remedial work is applied. The result was the availability of a large amount of information helpful in educational diagnosis. This scientific information helped many schools to become centers for both developmental and remedial instructions, and to give as much attention, on the one hand, to the correction of deficiencies of pupils who have learning disorders and who are definitely abnormal; and on the other hand, to the development of these strengths of the pupils who are regarded as perfectly normal, but who have various types of disorders.

In a word, diagnosis helps us to "put the oil where the squeak is"(1). For example, Baker found that four months special coaching of sixty nine-year-old pupils, resulted in a great gain. The coaching was devoted to the subject or subjects in which the pupils had shown weaknesses (2). Similarly, Stone found that pupils gained two to six times as much in ability to solve reasoning problems as other students of equal ability who had only the regular arithmetic work in school. In this experiment the pupils devoted not more than forty minutes a day for five weeks to diagnostic and practice tests (3).

CHAPTER TWO

Techniques of Diagnosis.

In the process of educational diagnosis there are five main techniques which are as follows:-

A. Locating maladjusted pupils.
B. Locating error.
C. Locating causes of error.
D. Remedial procedures.
E. Preventive procedures. (4)

We shall now proceed to the consideration of each technique alone.

A. Locating Maladjusted Pupils.

Who are the individuals needing diagnosis and how can we locate them? Those pupils who are not making satisfactory adjustment to the school situation are the ones needing diagnosis.

There are various ways and means of locating those pupils who require diagnostic study. Some of the considerable ones are: Survey and Group intelligence tests which are often employed to reveal those pupils whose level of achievement is below their level of intelligence and are worthy of special study. Other minor bases for locating the individuals needing diagnosis are as follows: Some believe that those pupils whose achievement in some school subject or subjects is much below

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(1) C. C. Ross: "Measurements In Today's Schools", P. 395.
(2) Ref. Ibid. P. 395.
(3) Ref. Ibid. PP. 395-396.
(4) Ref. Ibid. P. 396.

their general achievement level. Others, rely upon the judgment of the teachers concerned. Still others, justify special study and treatment for the lowest five or ten percent in a typical class. Finally, some believe that the most profitable cases are those whose achievement is average or even above, but it is much below what it should be.

"Sir William Osler argued that it was more important to know what kind of man has a certain disease than to know what disease the man had" (1). Accordingly, one can safely say that it is more important to know what kind of a pupil had a certain difficulty than to know what difficulty the pupil had.

All such suggestions have merit, but it seems to me that any individual who, for some reason or other, has some maladjustment to the school situation is worthy of diagnosis.

B. Locating Error.

After locating the pupils who have difficulties, the next step is to decide as to what phases of the human constitution or behavior should be diagnosed.

Hildreth suggests that the following five "areas of investigation" are important in diagnosis:-

1. Mental equipment of the learner.
   a. Aptitude for academic schoolwork.
   b. Language equipment.

2. Personality, temperament, and dynamic equipment.

3. Physical status, sensory and motor equipment, physical condition.

4. Environment and home history.

5. School situation, history and present status. (2)

To consider all these facts in any particular case, for locating error, is a hard task. The diagnosis of a relatively few factors, especially in the less serious cases, provides the diagnostician with satisfactory explanation. While the more serious cases will usually be found more complex to analyze as well as more difficult to remedy.

The methods that enable the teacher to carry on this phase of diagnosis may be grouped under six headings:-

1. Observation of the pupil at work.

2. Analysis of pupil's written work.

3. Analysis of oral work.

4. Objective devices.

5. The interview.

6. Laboratory procedures (3).

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(1) C. C. Ross: "Measurements in Today's Schools", P. 403.
(2) Ref. Ibid. P. 403.
(3) Ref. to National Society For the Study of Education: "Educational Diagnosis", Year Book XXXIV P. 142.

Now we proceed to discuss each of the above items in detail.

1. Observation of the pupil at work.

While the pupil is at work, the teacher may be able to get some hint from his behavior. This hint will be the beginning of diagnosing the work of that particular pupil. For example, the teacher may notice that the pupil is very slow in column addition. The reason for that can easily be detected if the teacher makes careful observation of the pupil while he is adding. The pupil may be using one or more of the inappropriate habits in addition. He may be counting with his fingers, toes, or tongue and usually whispering as he adds. Lapse of attention before the completion of the column is another cause of difficulty in adding. The teacher can easily detect this weakness and its cause when he notices pupils repeating partially completed work.

Some pupils may have slow reaction time and as a result they are slow in their work. There are still other pupil's study habits which may be also observed and analyzed.

The attitude of the pupil toward his school work should also be given due consideration. Some pupils have negative attitudes while others are indifferent, still others are uncooperative. All of these different sorts of attitudes are not conducive to learning. Many different factors, as a result of thorough diagnosis, may appear to be the cause of these attitudes. McCall says that accurate, detailed, trained and experienced observation of pupils in the process of normal work is one method of discovering the data upon which to base a diagnosis and prescribe corrective measures (1).

2. Analysis of pupil's written work.

The attempt to diagnose pupil's difficulties in arithmetic by analyzing written work is another method of the problem of diagnosis. Such analysis helps the diagnostician to detect what kinds of examples pupils can or cannot solve correctly, but it does not help him to determine the mental processes involved while the pupils are working the examples. The only solution for that is to have the teacher resort to what is known as "inference", and inferences as to those mental processes are not always correct.

After completing a thorough analysis of results from tests of pupils' ability to solve arithmetic problems, Monroe diagnosed many of the errors and came to the conclusion that the following were some of the causes of difficulty: Inability to calculate accurately, and inability to reason correctly. As a matter of fact, what Monroe has done is only mere inferences and these different sorts of inabilities are in turn due to still more fundamental causes.

There are many tests which may be used to facilitate diagnosis, specially those standard tests which have diagnostic value. To be able to determine the exact nature of the weakness of the pupils, an extensive and detailed analysis of the various kinds of errors made by pupils in these tests is necessary. Such analysis of the written work of the pupils reveals only the nature of the apparently unsatisfactory performances, and not the fundamental causes. The next item will help us in the solution of this difficulty.

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(1) Ref. to W. A. McCall: "Measurement", P. 389.

3. Analysis of oral work.

There are difficulties whose underlying causes would never come to light from an observation of the pupil at work, or from the analysis of his written work. To be able to discover some of the mental processes the pupils follow in solving arithmetic exercises is by interviewing these pupils individually and asking each one to do all his work aloud as he solves the various examples. Moreover, by following this method, the teacher can have a chance for additional questions to be asked when he is in doubt as to any point or method or procedure of work that the pupil is following.

Courtis has deviced a method of diagnosis in arithmetic that helps the teacher along the line of an analysis of oral statements of procedure. In working an example, the pupil is required to give aloud the mental steps by which he arrives at the solution. The examiner observes these responses and determines in what ways they are faulty. Buswell and John have also supplied us with a similar analysis, but it consists of a variety of unsatisfactory responses and methods of work, that have been used by a large group of pupils who are slow in arithmetic.

Finally, when the pupils are able to diagnose their own difficulties as such, they should be encouraged to indicate to the teacher points that are causing them difficulty. For example, the parts of an arithmetic process that they do not understand. In order to enable the pupils to do that, the teacher's efforts should be directed toward helping them to make such analysis of their own deficiencies by providing suitable diagnostic devices, by requiring them to give their analysis orally, by giving suggestions as to how to proceed in making a self-diagnosis, and by suggesting helpful corrective exercises. It is a waste of time and effort for the teacher to resort to the more elaborate methods, when it is possible for him to secure the pupils' help.

4. Objective devices.

Purely diagnostic tests of a very elaborate nature have been developed in arithmetic. These tests are of great value regarding the determination of the nature and significance of faults, since the aim of these tests is to reveal the specific location of the pupils' difficulties. Many other measuring devices have been constructed such as: intelligence tests, general survey tests, achievement tests, analytical tests, aptitude and prognostic tests, mechanical ability tests, tests of open-mindedness and many others. Even some of the informal tests may also be applied as well for the same purpose.

Buswell and John and Brueckner and still others have developed a series of analytical diagnostic charts plus many other devices containing illustrations of faults and types of responses which afford an excellent basis of diagnosis.

5. The interview.

When other methods of diagnosis fail, an interview by a skillful teacher with the pupil is of great importance. It will sometime help in locating the difficulty. In many cases it will be necessary to supplement the information secured by the use of the above methods with facts that can be secured through an interview. Such an interview can be made in the form of a test situation by the use of several standardized interview forms, check lists, questionaires and other forms of written responses. These ways and means of an interview are valuable aids to the personal interview and can be easily used by the teacher.

The following illustration makes clear the value of the interview as a supplement to the written test in locating the sources of difficulty in arithmetic:

A story is told of a sixth-grade negro girl who had an elaborate but usually ineffective system for solving reasoning problems. Her explanation was somewhat like this: "Whenever they is lots of numbers, I adds, but when they is only two numbers with lots of parts (digits) I subtracts, if they is just two numbers and one is littler than the other I divides when they comes out even, and multiplies when they don't" (1). It is most unlikely that any analysis of test papers, or observation of the pupil at work, would have resulted in a correct inference as to the real trouble in the above case.

6. Laboratory procedures.

The psychological laboratory with its more systematic and exact techniques may be employed when the information got by analytical tests and by the informal diagnostic techniques do not locate the difficulty. The following laboratory techniques such as the motion-picture camera, the dectaphone, and many other scientific devices furnish permanent records of the pupils' performance and make possible a much finer degree of analysis than can be secured by non-laboratory methods.

Laboratories are very expensive and require highly trained technicians. Though many school systems cannot provide for laboratory procedures, yet all schools can make and should make use of the findings of laboratory studies.

C. Locating causes of errors.

To know the cause or causes of errors is more important and far more difficult than knowing where they occur. One limitation of test scores in diagnosis, as has been mentioned before, is that they reveal the products of learning and not the learning process itself. Unless the real cause is known, and corrective measures are taken, there will be little hope in the improvement in the pupils' conditions.

One prominent author declares that basically, diagnosis involves two general steps. The first is measurement, or appraisal, and the second is interpretation, or inference (2). At times a reasonably safe inference can be made from the nature of the errors themselves. But, as a matter of fact, a complete explanation about the causes of errors or difficulties cannot be made without considering the child's past history, outside the school as well as inside.

"An illustration was recently observed in a third-grade class in which the teacher used the example 82 - 37 in introducing the new step of carrying in subtraction. The teacher carefully taught the pupils to say: 7 from 12 is 5. One little girl, however, also saw that 2 from 7 gave 5. She forthwith adopted the idea that in all cases one subtracts the smaller from the larger number, regardless of whether the larger number is in the minuend or in the subtrahend. Without

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(1) C. C. Ross: "Measurements in Today's Schools", P. 405.
(2) Ref. Ibid. P. 402.

determining the cause of her difficulty, the teacher simply assigned more practice on the new step. The child used this incorrect procedure for a whole month before a special examiner located the cause of her difficulty" (1)

D. Remedial Procedures.

The ultimate purpose of diagnosis is to enable the teacher to reach to an effective remedial procedure. When the cause or causes of the pupil's difficulties have been determined, correction can then be planned. Sometimes the same causes appear to operate in several pupils, in such a case group measures will be necessary. But, it is preferable to have remedial programs planned for each pupil individually.

There are three main essentials of a remedial program in arithmetic instruction. Namely, first, organization of instruction in the fundamental processes. Second, improving instruction in problem-solving. Third, improving teaching in other aspects of arithmetic. Each of these three topics will be fully discussed under part IV "Remedial Work", of this thesis. Briefly stated, the first and top most principle of an effective remedial program should be the growth of the individual and the elimination of those factors that interfere with this growth.

E. Preventive Procedures.

After all, the greatest value of a diagnostic and remedial program is: first, the finding out of those factors within the control of the school which are the cause of maladjustment; second, the application of remedial instruction to help pupils overcome their difficulties. To be able to meet this last point some proper adjustments should be made, such as, convenient modifications in school organizations, curriculum, instructional materials, and teaching methods.

There is another means which helps in the solution of this problem. It is preventive measures. Nobody ignores the fact that it is always better and easier to prevent errors than to correct them. "One ounce of prevention is worth a pound of cure". Difficulties in arithmetic should be checked so promptly and corrected during the teaching period rather than allowed to accumulate to make a problem for remedial progress. If the teacher can locate the breakdown in a given process as soon as it occurs, the subsequent difficulties due to that breakdown might be avoided entirely. For example, a third grade student was found to have errors on half the examples of an addition test. On checking this pupil's papers, it was found that his errors were due to mistakes in carrying. Apparently, this child had not learned how to carry when the process of carrying was taught. Instead, he was allowed to go on with this serious weakness in his arithmetic.

It is true that remedial work was very necessary instead of letting the child carry on with his difficulties. The most profitable treatment would have been to have located his difficulties while the topic of carrying was being taught. Moreover, the energy which was spent in diagnostic and remedial treatment might well be spent along more profitable lines of instruction. "Prevention is the highest level of diagnosis, its ultimate goal" (2).

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(1) Ref. National Society for the Study of Education : "Educational Diagnosis", Year Book, XXXIV P. 279.

(2) C. C. Ross! "Measurements in Today's School", P. 408.

PART II

DIAGNOSIS IN ARITHMETIC

CHAPTER THREE

Aims of Arithmetic Instruction.

Before teaching any subject, the teacher should have clearly in mind the purposes to be attained in teaching that particular subject. This holds true in arithmetic instruction. Any teacher who has a narrow view of the aims of arithmetic instruction may emphasize one or more of those aims to the neglect of the other important ones. In that case, what is actually taking place, is both aimless work and loss of time.

In the past, arithmetic was taught for its own sake, regardless of anything else. The emphasis was placed on the mastery of the computational skills. Even at present, one can realize this fact, if he only comes in contact with what is going on in many arithmetic classes in elementary schools. As a result of his classroom surveys in elementary schools, Steel found that approximately 86% of the time devoted to arithmetic is used for the presentation of new processes of computation in arithmetic, and practice in working abstract examples in new and previously taught processes. The remainder of the time, only 14% is used for problem work, applications of arithmetic projects, diagnostic work and other important aspects of instruction. It seems that the computational aim of number is considered to be the major aim of arithmetic and relatively little attention is given to the other aims.

At present arithmetic must be given broader aims and it should be taught in view of what life will require. In other words, arithmetic should not be taught only as a tool subject stressing the manipulation of number processes, but it should be regarded in a sense as a social subject which contributes social insight just as history and geography. Let us consider such topics as taxation, profit and loss, the metric system as a whole and the like. Are not these considered to be social institutions and at the same time they form a part of the field of arithmetic? There is no reason why a reasonable portion of the time in arithmetic should not be devoted to the consideration of these topics. The pupils should be led to understand their real meaning and be able to consider their social significance.

Some authors still emphasize another kind of aim of arithmetic instruction, namely, the informational aim. These authors believe that a good course in arithmetic today includes a consideration of the following topics from the social point of view: The saving and loaning of money, accounts, and public expenditures and insurance. Much of the work of these topics must be of the informational kind. The student should be offered the opportunity to learn in the arithmetic class something about the real reasons why interest is paid on borrowed money and why commercial discounts are given.

In addition to the above mentioned three aims of arithmetic instruction, Brueckner mentions a fourth one, namely, the psychological aim. This last one, as a matter of fact, underlies the preceding three. It centers around a good understanding and appreciation of the structure of the number system. In addition, it makes the pupil familiar with the many valuable contributions that number has made and is making to progress in the fields of business, industry, science and human relations (1).

What has all of this to do with the problem of diagnosis and remedial work? To be able to have a complete and fruitful diagnosis, the diagnostician of disabilities in arithmetic should bear in mind each one of the four aims mentioned above. Moreover, he should not concern himself only with a study of difficulties in computation, but he should take into consideration, while diagnosing a certain pupil, the other major aims as well.

At present, much more is known about computational difficulties than about those related to the other three aims. This does not justify the fact that the latter ones should be underevaluated or totally neglected. On the contrary, in view of their obvious importance, definite attempts should be made to improve the special techniques for diagnosing and evaluating the performances related to them.

A recognition of these four aims will lead to a reconstruction of the work our schools are offering in arithmetic, and their analysis, as such, shows that the desired outcomes of arithmetic instruction are numerous and varied.

CHAPTER FOUR

Subject-Matter

Many investigators have shown that arithmetic is a very complex subject. It is not made only of one general ability, but it is made up of specific abilities and skills. Some think of these abilities and skills as hierarchies of habits in which each new skill is based in part on simpler skills previously developed. For example, a large number of specific skills in addition, subtraction and multiplication is necessary for the solution of an example in long division. These specific skills should be thoroughly studied before the long division process be well taught. This arithmetical skill has two main phases: namely, computation and reasoning. We proceed now to the discussion of computation.

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(1) Ref. National Society for the Study of Education: "Educational Diagnosis", XXXIV Year Book. PP. 269-272.

A. Computation.

Skill in computation consists of computation of the number facts, and computation of the processes.

1. Number Facts.

What are the number facts, and how many kinds are they? By number facts is meant those combinations in arithmetic which the pupil should know at once without resorting to a series of steps to be able to find the answer. There are two kinds of these facts. The basic combinations which are made by adding one-digit number to another one-digit number such as 4 + 3 = 7, and the secondary or higher-decade combinations which are made by adding one-digit number to two-digit number such as 12 + 3 = 15.

a. Basic Combinations.

(1) Their number

The number of basic combinations in each of the four simple processes is as follows :-

1. 100 basic combinations in simple addition.
2. 100 basic combinations in simple subtraction.
3. 100 basic combinations in simple multiplication.
4. 90 basic combinations in simple division.

The sum total amounts to 390 basic combinations which should be known by every body. A discussion of each kind of these is given in the appendix on pages 98-102.

(2) Their relative difficulty

(a) Methods of determining difficulty.

There are different methods of determining the relative difficulty of number combinations. Only five well known methods will be mentioned very briefly hereafter. The first and second methods involve complicated laboratory techniques. The first consists in a study of the eye movements exhibited by pupils when adding columns of digits. The second consists in recording the precise time intervals between the successive operations in working an example. It is needless to say that we cannot afford to have either of these methods applied here.

The third method operates on the principle that the number of errors made by pupils in the various combinations is an index of difficulty of the combinations. The most difficult being the one on which the largest number of errors occurs.

The fourth method consists in measuring the degree of memory retention after a period of non-use.

The fifth method is by which the degree of difficulty is determined by introspection. The last two methods are open to question

(b) Order of difficulty

Formerly these various combinations were taught in the form of tables, as if all were of equal difficulty, but as a result of scientific investigations it was shown that they vary to a considerable extent in the degree of difficulty and memorization. The value to the teacher of knowing the relative difficulty of the number combinations, is invaluable. On the basis of this information drill can be properly distributed and it becomes more effective.

The following lists of the basic combinations of each of the four simple processes, arranged according to difficulty, and proceeding from the least to the most difficult, are the result of an investigation, based on the third of the above mentioned methods. I have made this investigation in the University Elementary School. A number of 175 pupils, ranging from the first grade up to the fifth grade, was included in the investigation. The basic combinations were given at two successive times, papers were corrected and errors recorded. The number of errors made by pupils in the various combinations was used as an index of difficulty, i.e., the most difficult being the one in which the largest number of errors occurred.

The coefficients of correlation between the first and second trials in the combinations of each of the four simple processes were secured by the Spearman's Method of Rank Difference in order to ascertain reliability, and between my order and that of Professor Clapp in order to ascertain validity. These coefficients are given at the end of the lists.

Basic combinations of addition

2 + 2
1 + 3
4 + 4
1 + 4
4 + 6
8 + 1
4 + 1
0 + 2
6 + 0
1 + 7
2 + 3
3 + 7
0 + 5
3 + 0
2 + 4
1 + 1
1 + 8
3 + 3
5 + 0
1 + 9
7 + 0
6 + 3
2 + 6
9 + 2
5 + 4
7 + 7
9 + 9
7 + 1
0 + 8
0 + 4
3 + 6
0 + 1
1 + 0
4 + 3
1 + 2
3 + 2
8 + 0
5 + 1
5 + 3
3 + 1
8 + 2
5 + 2
7 + 2
6 + 1
9 + 1
8 + 3
3 + 4
7 + 4
7 + 3
0 + 7
1 + 6
2 + 5
5 + 5
4 + 6
2 + 1
4 + 0
4 + 2
3 + 5
0 + 6
2 + 7
2 + 0
4 + 8
8 + 4
5 + 9
0 + 9
0 + 0
4 + 7
1 + 5
6 + 2
8 + 8
9 + 0
5 + 6
0 + 3
6 + 6
2 + 8
6 + 7
9 + 6
5 + 7
2 + 9
7 + 6
7 + 9
9 + 4
8 + 7
3 + 9
3 + 8
9 + 8
6 + 4
7 + 8
6 + 5
8 + 9
9 + 5
9 + 7
4 + 9
8 + 5
6 + 9
6 + 8
9 + 5
8 + 6
5 + 8
7 + 5

Basic combinations of subtraction.

3 - 2
4 - 3
5 - 4
2 - 1
4 - 2
3 - 1
0 - 0
5 - 1
1 - 1
6 - 2
4 - 1
7 - 5
6 - 5
3 - 3
5 - 3
2 - 2
9 - 1
6 - 4
4 - 4
9 - 8
7 - 1
8 - 7
7 - 4
7 - 6
5 - 5
9 - 5
8 - 5
8 - 6
6 - 6
9 - 9
7 - 7
10 - 9
6 - 3
8 - 4
8 - 2
10 - 4
8 - 1
7 - 3
6 - 1
1 - 0
10 - 1
5 - 2
8 - 5
9 - 6
10 - 3
9 - 3
10 - 2
8 - 8
10 - 5
4 - 0
10 - 7
9 - 4
9 - 7
9 - 2
5 - 0
10 - 8
3 - 0
12 - 9
7 - 0
12 - 8
9 - 0
11 - 2
7 - 2
14 - 7
12 - 3
11 - 4
2 - 0
12 - 6
11 - 8
16 - 8
12 - 4
11 - 7
12 - 7
10 - 6
11 - 5
8 - 0
6 - 0
15 - 7
13 - 6
18 - 9
11 - 6
14 - 6
13 - 5
15 - 6
13 - 9
17 - 9
14 - 5
11 - 9
13 - 8
12 - 5
17 - 8
11 - 3
14 - 9
15 - 9
13 - 4
15 - 8
14 - 8
13 - 7
16 - 7
16 - 9

Basic combinations of multiplication

5 x 1
2 x 2
1 x 7
1 x 5
1 x 3
1 x 2
1 x 6
2 x 5
7 x 2
1 x 4
3 x 1
2 x 3
2 x 8
1 x 8
7 x 1
6 x 1
9 x 1
6 x 5
2 x 2
0 x 0
9 x 2
5 x 2
2 x 4
2 x 9
5 x 4
4 x 1
1 x 9
5 x 3
2 x 7
4 x 3
3 x 5
5 x 7
2 x 6
9 x 9

3 x 4
4 x 5
4 x 2
6 x 2
8 x 2
3 x 3
3 x 6
2 x 1
8 x 1
5 x 8
8 x 5
8 x 3
5 x 9
6 x 4
6 x 3
5 x 6
7 x 3
3 x 7
3 x 9
6 x 7
1 x 1
8 x 4
7 x 4
6 x 6
4 x 6
3 x 8
9 x 3
8 x 9
4 x 7
4 x 4
9 x 8
7 x 6
6 x 8

5 x 5
4 x 8
8 x 6
7 x 9
7 x 7
9 x 4
4 x 9
9 x 5
7 x 5
7 x 8
9 x 7
8 x 8
9 x 6
8 x 7
6 x 9
0 x 4
0 x 8
0 x 1
0 x 2
0 x 6
6 x 0
0 x 7
1 x 0
9 x 0
7 x 0
0 x 9
2 x 0
5 x 0
3 x 0
0 x 3
0 x 5
8 x 0
4 x 0

Basic combinations of division.

4 ÷ 2
25 ÷ 5
14 ÷ 2
15 ÷ 3
12 ÷ 2
20 ÷ 5
10 ÷ 2
6 ÷ 2
8 ÷ 4
9 ÷ 3
8 ÷ 2
72 ÷ 9
72 ÷ 8
36 ÷ 6
40 ÷ 8
24 ÷ 3
21 ÷ 3
18 ÷ 6
15 ÷ 5
16 ÷ 2
24 ÷ 4
12 ÷ 3
30 ÷ 5
14 ÷ 7
81 ÷ 9
49 ÷ 7
24 ÷ 8
12 ÷ 6
48 ÷ 6
10 ÷ 5

7 ÷ 1
40 ÷ 5
63 ÷ 7
21 ÷ 7
12 ÷ 4
6 ÷ 1
27 ÷ 9
28 ÷ 7
36 ÷ 9
42 ÷ 7
16 ÷ 4
5 ÷ 1
35 ÷ 5
64 ÷ 8
32 ÷ 8
45 ÷ 5
48 ÷ 8
63 ÷ 9
4 ÷ 1
6 ÷ 3
20 ÷ 4
24 ÷ 6
54 ÷ 6
18 ÷ 9
54 ÷ 9
45 ÷ 9
8 ÷ 1
16 ÷ 8
18 ÷ 3
42 ÷ 6

3 ÷ 1
4 ÷ 4
30 ÷ 6
9 ÷ 1
56 ÷ 7
35 ÷ 7
27 ÷ 3
0 ÷ 9
0 ÷ 5
7 ÷ 7
32 ÷ 4
8 ÷ 8
5 ÷ 5
56 ÷ 8
3 ÷ 3
0 ÷ 8
0 ÷ 2
0 ÷ 1
6 ÷ 6
36 ÷ 4
9 ÷ 9
0 ÷ 7
28 ÷ 4
0 ÷ 6
0 ÷ 4
18 ÷ 2
1 ÷ 1
2 ÷ 2
0 ÷ 3
2 ÷ 1

	Coefficient of reliability	Coefficient of validity.
Addition	0.90 ± .0128	0.61 ± .0423
Subtraction	0.97 ± .0039	0.89 ± .0140
Multiplication	0.93 ± .0091	0.92 ± .0104
Division	0.96 ± .0057	0.81 ± .0258

We generally see that all of these coefficients of reliability and validity, with the exception of the coefficient of validity of the addition combinations which is 0.61 ± .0423, are fairly high. They range between .90 and .97 in reliability and between .81 and .92 in validity.

Not being able to find the reliability coefficient of Professor Clapp's order of difficulty, I have applied only a part of the "correction for attenuation" Spearman's formula, which is $= \frac{r_{12}}{\sqrt{r_{11}}}$ to the coefficient of validity of the addition combinations. The result was 0.64, which is not very much higher.

b. Secondary combinations.

There are, in addition to the above mentioned basic combinations, several hundreds of secondary combinations of each of the four simple processes. Due to this large number, it becomes difficult for the pupils to learn them all, and if they can do so, they cannot retain them well. Students of arithmetic do believe that only the most important of these combinations should be learned and known as facts. Such as the higher-decade combinations which are used, in column addition up to 49 + 9, in multiplication with carrying; and the higher-decade subtraction combinations which are used in uneven short division.

As a result of an investigation I have made concerning these combinations, the following facts pertaining to each of the four operations were concluded:-

(1) Order of Difficulty

Using the number of errors made in each combination an an index of its difficulty some of these combinations were found to be more difficult than others. As a result, they were arranged in an order of ascending difficulty.

Addition

(a) The following list includes the addition secondary combinations, excluding the 0, the 1 combinations, and those which are used in "Carrying" in multiplication. They are arranged in an ascending order of difficulty, as they are met in the examples and not singly.

10 + 5
10 + 6
10 + 8
10 + 9
20 + 5
20 + 7
30 + 7
31 + 6
40 + 8
40 + 9
30 + 6
30 + 9
37 + 2
44 + 5
46 + 2
20 + 8
31 + 9
44 + 4
20 + 6
37 + 3
41 + 7
11 + 2
21 + 8
22 + 7
46 + 6
35 + 7
41 + 3
10 + 7
20 + 9

30 + 8
41 + 8
22 + 3
43 + 7
46 + 3
46 + 5
39 + 2
41 + 2
43 + 3
46 + 4
43 + 4
49 + 8
43 + 2
47 + 6
34 + 4
44 + 6
15 + 5
25 + 5
31 + 3
17 + 3
38 + 2
41 + 4
42 + 7
33 + 2
34 + 2
42 + 8
43 + 6
47 + 3
13 + 2

23 + 2
25 + 9
31 + 7
44 + 7
47 + 5
48 + 8
26 + 3
29 + 2
31 + 8
38 + 4
34 + 5
31 + 5
13 + 3
47 + 2
33 + 3
39 + 7
34 + 8
34 + 5
37 + 5
41 + 5
25 + 6
45 + 9
34 + 3
26 + 5
15 + 6
12 + 9
11 + 5
39 + 4
47 + 8

42 + 9
19 + 3
26 + 9
44 + 3
41 + 6
26 + 2
11 + 3
17 + 2
13 + 4
22 + 5
29 + 8
44 + 2
23 + 4
29 + 7
11 + 4
25 + 7
12 + 6
13 + 7
15 + 9
33 + 4
26 + 4
21 + 8
33 + 5
17 + 7
23 + 6
33 + 7
33 + 6
41 + 9
19 + 2

48 + 9
39 + 8
21 + 9
22 + 4
29 + 5
31 + 4
11 + 9
19 + 7
15 + 7
13 + 6
29 + 6
39 + 3
19 + 6
49 + 7
31 + 2
13 + 5
22 + 6
23 + 5
22 + 8
17 + 4
11 + 8
46 + 9
34 + 8
23 + 7
13 + 8
23 + 3
49 + 9
32 + 8
19 + 4

37 + 8
38 + 3
11 + 6
33 + 9
19 + 5
37 + 4
32 + 9
47 + 9
44 + 9
26 + 6
44 + 8
16 + 8
33 + 8
47 + 9
34 + 6
39 + 6
12 + 8
28 + 7
25 + 8
17 + 6
22 + 2
17 + 5
38 + 6
29 + 9
19 + 8
18 + 9
46 + 8
15 + 8
24 + 8

37 + 9
14 + 9
29 + 3
11 + 7
35 + 9
46 + 7
38 + 8
14 + 7
38 + 5
22 + 9
39 + 5
35 + 8
39 + 9
19 + 9
29 + 4
47 + 4
23 + 8
43 + 9
37 + 7
28 + 8
47 + 7
26 + 7
13 + 9
34 + 7
38 + 7
38 + 9
26 + 8
28 + 9
12 + 7

17 + 8, 36 + 9, 23 + 9, 16 + 9, 24 + 9, 14 + 8, 34 + 9, 37 + 6, 17 + 9.

(b) The following list includes the secondary addition combinations used in carrying in multiplication, excluding the 0 and the 1 combinations. They are arranged in an ascending order of difficulty, as they are met in the examples of column addition and not singly.

10 + 2
10 + 4
20 + 2
30 + 4
30 + 5
40 + 7
15 + 2
30 + 2
48 + 6
40 + 3
14 + 4
40 + 2
42 + 5
45 + 6
49 + 2
49 + 4
48 + 7
49 + 6
10 + 3
14 + 2
20 + 4
30 + 3

40 + 4
40 + 6
48 + 3
18 + 2
21 + 2
42 + 6
45 + 5
48 + 4
24 + 4
28 + 2
36 + 4
12 + 4
25 + 2
16 + 2
27 + 2
14 + 5
21 + 6
35 + 3
45 + 8
45 + 3
32 + 4
40 + 5

20 + 3
36 + 3
49 + 3
15 + 3
36 + 2
48 + 2
21 + 3
25 + 3
35 + 4
24 + 2
28 + 6
32 + 2
42 + 3
35 + 2
42 + 2
14 + 3
15 + 4
45 + 2
45 + 7
25 + 4
24 + 3

16 + 4
16 + 5
32 + 5
35 + 5
36 + 5
42 + 4
12 + 3
45 + 4
21 + 5
27 + 3
27 + 4
24 + 5
21 + 4
35 + 6
24 + 6
14 + 6
28 + 4
18 + 7
49 + 5
32 + 6
27 + 5

12 + 2
18 + 5
18 + 4
18 + 3
16 + 3
32 + 7
12 + 5
28 + 3
36 + 6
27 + 6
27 + 7
36 + 7
18 + 6
48 + 5
32 + 3
16 + 6
27 + 8
28 + 5
24 + 7
16 + 7
18 + 8
36 + 8

(c) The following are the rest of the addition secondary combinations used in carrying in multiplication. Being above 49 plus a digit, they are arranged according to the partial product, i.e., the number added to:-

54 + 2
54 + 3
54 + 4
54 + 5
54 + 6
54 + 7
54 + 8
56 + 2

56 + 3
56 + 4
56 + 5
56 + 6
56 + 7
63 + 2
63 + 3
63 + 4

63 + 5
63 + 6
63 + 7
63 + 8
64 + 2
64 + 3
64 + 4
64 + 5

64 + 6
64 + 7
72 + 2
72 + 3
72 + 4
72 + 5
72 + 6
72 + 7

72 + 8
81 + 2
81 + 3
81 + 4
81 + 5
81 + 6
81 + 7
81 + 8

Authorities on the subject have concluded that the above listed addition combinations are the ones which a child is almost sure to need after he leaves school. This means that, if a child fails to learn anyone of these combinations, both basic and secondary, there will be a corresponding gap in his ability to add integers.

Moreover, these combinations should be learned by the pupils as facts and not as processes. They should be known automatically and not through analysis. Example, 48 + 7 = 55, the answer 55 should be given at once orally or in writing and not by saying 8 + 7 = 15, put down 5 and carry 1, 1 + 4 is 5 etc. A fact is not a process.

Subtraction

There are 175 secondary combinations in subtraction which are prerequisite in uneven short division (1).

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(1) Ref. appendix p. 102

Multiplication

The following list includes the secondary combinations of multiplication involving carrying except those of 1, and arranged in an order of ascending difficulty:

3x0+2
3x6+2
6x0+2
8x5+2
5x3+3
6x0+3
6x8+3
5x4+4
9x1+4
8x0+2
9x0+2
5x8+4
8x0+4
6x0+5
6x6+5
7x0+5
9x0+5
7x6+6
4x6+3
6x9+3
5x7+4
6x0+4
6x6+4
7x0+4
8x5+4
7x2+5
8x5+6
6x1+5
8x1+5
9x0+6
9x5+6
4x4+2
4x7+2
5x1+2
7x0+2

8x4+5
7x4+2
8x3+2
5x2+3
5x5+3
7x5+5
7x8+3
8x5+3
9x5+3
9x6+3
9x7+4
8x4+5
6x8+5
9x1+5
7x7+6
7x9+6
9x2+7
7x8+2
9x5+2
9x6+6
8x8+7
7x9+2
3x9+2
6x7+2
5x7+3
5x8+3
5x9+3
7x9+4
6x9+5
7x5+5
9x3+5
7x6+6
9x2+6
8x1+6
3x2+2

8x9+3
9x3+3
5x0+2
8x1+2
9x1+2
5x0+3
5x6+3
6x2+3
7x1+3
9x2+3
5x0+4
8x2+4
8x0+5
9x1+8
3x1+2
3x7+2
4x0+2
4x5+2
4x8+2
6x6+2
7x1+2
7x2+2
7x6+3
8x2+5
9x9+5
9x2+8
6x2+4
8x6+5
3x4+2
6x1+2
6x5+2
7x3+2
9x6+2
9x9+2
4x0+3

5x4+3
7x3+3
8x0+3
5x1+4
6x1+4
7x1+4
8x1+4
8x0+6
9x0+7
5x5+2
5x7+2
8x2+2
8x9+2
9x5+3
8x4+7
7x6+5
6x4+4
8x5+4
9x1+6
8x1+7
9x1+7
3x5+2
4x6+2
5x3+2
6x2+2
7x8+2
4x1+3
4x8+3
6x1+3
6x4+3
6x5+3
6x7+3
8x4+3
5x2+4
5x6+4

6x5+4
7x7+4
8x7+4
9x8+8
8x4+2
9x3+8
8x6+4
8x8+2
8x7+2
8x7+5
9x3+2
9x4+2
9x7+2
9x8+7
5x1+3
7x4+3
8x1+3
8x3+3
8x7+3
9x0+3
5x5+4
6x7+4
7x3+4
7x1+5
7x1+6
9x6+8
4x1+2
5x9+2
7x6+4
9x4+5
6x9+4
8x9+6
8x2+6
7x5+4
8x3+4

8x9+2
8x6+7
9x5+7
3x3+2
5x6+2
6x9+2
8x6+2
9x2+2
4x7+3
9x9+3
5x3+4
8x5+5
4x2+2
6x4+2
7x5+2
6x6+3
8x2+3
5x9+4
7x8+4
9x7+5
8x2+7
8x6+3
7x2+4
4x2+3
9x7+3
9x3+4
6x2+5
7x9+5
9x2+5
7x4+6
9x9+8
4x3+2
4x4+3
4x5+3
7x2+3

9x8+3
6x8+4
9x5+4
9x9+4
6x7+5
7x5+5
8x6+6
9x4+6
8x8+6
9x5+8
8x8+3
8x3+7
9x7+8
8x4+4
6x3+5
7x8+5
8x8+5
8x9+5
9x8+5
7x2+6
7x8+6
9x7+6
9x8+6
9x6+7
9x7+8
9x9+7
9x4+8
6x3+2
6x8+2
7x6+2
4x3+3
7x7+3
7x4+4
9x9+6
8x7+7

4x9+3
7x7+5
7x9+3
9x4+3
9x4+4
9x8+4
7x4+5
7x3+6
7x5+6
8x3+6
8x8+4
3x4+2
4x8+2
4x9+2
9x8+2
6x3+3
9x1+3
6x3+4
9x6+4
6x5+5
8x3+5
8x7+6
9x3+7
8x4+6
9x3+6
8x9+7
9x2+7
9x4+7

Division

There are 324 facts in division with remainders (1).

(2) Percentage of error.

Addition

(a) By decades.

1. It was found out that 29% of the mistakes were made in those combinations whose sum is within the decade, such as 15 + 4 and 23 + 6 etc., and 71% in the combinations crossing the decade, such as 17 + 8 and 49 + 4. Apparently double time should be sent on teaching the latter type. Out of the 71%, 27% of errors were made in those combinations crossing the decade, but the unit digit of the first number is larger than

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(1) Ref. appendix page 103

the added digit, such as 37 + 5 and 48 + 3. 44% were made when the unit digit for the first number is smaller, such as 35 + 7 and 43 + 8. More emphasis should be placed on the latter type.

ii. The percentage of error decreases gradually as one proceeds from the second to the fourth decade, 36% of the errors were made in the second decade, 33% in the third, and 31% in the fourth. More automatization of the combinations in the first and second decades helps to decrease the percentage of error in the following decades.

(b) By Tables

With the exception of the tables whose unit digits are 0, 1, and 5, and their percentage of error is successively 0.49, 6.2 and 6.5, the rest of the tables are divided into two main categories. The first category contains those tables whose unit digits are 2, 3 and 4, and their percentage of error is around 11. The second contains those tables whose unit digits are 6, 7, 8 and 9; and their percentage of error is around 15.

Multiplication

(a) By Decade

i. 21% of all the errors are made in the basic combinations without carrying, while 79% are made in those involving carrying. 33% of the errors are made in carrying within the decade while 67% are made in carrying to a higher decade. In other words 16% of the errors are made in the basic combinations, i.e., without carrying, 33% in carrying within decade, and 51% in carrying to a higher decade. Consequently, pupils should be given for drill purposes multiplication examples of each of these types in proportion to their difficulty.

ii. Average number of mistakes per combination.

On the basis of the carried number

Carried Number	Within Decade	Higher Decade
1	6.7	-
2	7.5	6.6
3	8.5	9.4
4	9.4	11.9
5	11.9	12.2
6	13.3	13.8
7	12.6	14.5
8	10.3	14

Generally speaking, one can conclude that the larger the number to be carried the greater the average of mistakes. Moreover, the average of the mistakes in the higher decade is also greater than that which is within the decade with only one exception, namely, carrying of 2.

On the basis of each table.

Table	Within decade	Higher decade
2	4.7	
3	6.8	9
4	8.4	10
5	5.6	
6	8.1	9.5
7	10	10.75
8	10	14.25
9	9.2	13.75

With few exceptions the higher the table is, the greater the average of mistakes, specially so within the decade. Here again we find that the average of mistakes is greater in the higher decade than it is within the decade.

(3) The use of the secondary combinations.

Addition

The addition combinations, of all sorts, are used in the following manners:-

(a) The addition combinations are used, to begin with, as simple combinations or "singly". Such as 9 + 6, 5 + 4 etc.

(b) These combinations are also used in column addition, for example,

75
56
43
<u>88</u>
262

In solving the example at the left, the pupil must have a knowledge of the following combinations:

5 + 6 = 11
(11) + 3 = 14
(14) + 8 = 22

2 + 7 = 9
(9) + 5 = 14
(14) + 4 = 18
(18) + 8 = 26

We notice here an extra step, that the numbers within parenthesis are "thought of" not "seen", which is a new element of difficulty for the pupils. The same numbers in parenthesis plus the other numbers of these combinations are listed among either the basic addition combinations such as 5 + 6, 2 + 7 and 9 + 5 in the first and second columns; or among the secondary ones, such as 11 + 3, 14 + 8, 14 + 4 and 18 + 8 in the first and second also.

(c) Addition combinations are also used in a different way in the checking of addition examples, where the pupils get the benefit of using the combinations in the reverse form thus in adding down the column in the following example: 5
2
<u>6</u>
13 , the pupil has to add the combinations as such 5 + 2 + 6 = 13, while checking up, he has to add 6 + 2 + 5 i.e., in the reverse form, a thing which is very beneficial to the pupil.

(d) Addition combinations are also used in the checking of subtraction examples, where the difficulty consists chiefly in a knowledge of how to use the ability to add in checking the work 645
<u>326</u>
319. Here the pupil in checking the example adds 6 and 9 and compares the sum with the 5 in the minuend before carrying one to the one to check the second figure. The extra step of comparison here involved is an element which undoubtedly causes confusion. Thus in requiring pupils to check their work in subtraction, which is a very essential thing to do, they will get a double benefit: drill in a different way of using addition facts, and certainty of the correctness of the answer.

(e) Addition combinations are also used in "carrying" in multiplication. The number of addition facts involved in carrying in multiplication can easily be found. For example, the only number that is ever carried in multiplying by 2 is 1. In multiplying by 3, the numbers 1 and 2 are carried. In multiplying by 4, the numbers carried are 1, 2, 3; and so forth through 9.

There is no need to have this point worked over again since I have already worked it out carefully and the combinations have been included in the tables as indicated before. It should be clear to the teacher that practice on a combination, such as 48 + 5, does not involve the same difficulty or mental processes that arise in using this combination solving the example 67 x 8, where the number 5 must be carried "in mind" while the combination 8 x 6 is thought of; the pupil must then add two "thought of" numbers, 48 x 5, a process much more difficult than adding 48 + 5 where the figures are on the paper before the pupil. Of course, this difficult process can be made much easier to the pupils by giving them direct practice on the addition combinations involved in "carrying" in multiplication examples, or in "carrying" involved in the multiplication in uneven division.

Subtraction

The subtraction combinations, both basic and secondary, are met only in three ways as follows:-

(a) The basic combinations presented singly.
(b) The basic combinations presented in examples and problems.
(c) The secondary combinations are met in uneven short division.

The following is an analysis of the subtraction combinations on the basis of their appearance in examples and problems as well. Considered from this point of view, they can be divided into three main categories. The first one consists of those combinations where a digit is subtracted from an equal digit. These combinations can either be used without borrowing or they can be borrowed from and to. Example

```
7456
5466
1990  notice the 6 - 6 and the 4 - 4 combinations.
```

Arranged in an order of ascending difficulty, these combinations are as follows :-

a. Without borrowing	b. Borrowed from and to
7 - 7	2 - 2
0 - 0	4 - 4
6 - 6	5 - 5
4 - 4	3 - 3
8 - 8	6 - 6
9 - 9	7 - 7
3 - 3	9 - 9
5 - 5	0 - 0
1 - 1	1 - 1
2 - 2	8 - 8

It was found out that almost every combination was more difficult to the students when met in form (b). The consideration of the last four combinations of form (b) with the number of mistakes of each of these combinations in both forms is a good illustration:_

Combination	Number of mistakes	
	a.	b.
9 - 9	2	20
0 - 0	1	23
1 - 1	9	33
8 - 8	2	35

It is apparent from this number of mistakes that the degree of difficulty of the same combination differs in both forms. The student either forgets that he has borrowed or he does not know the combination.

The second category consists of those combinations where a smaller digit can be subtracted from a larger one. They can either be used without borrowing or they can be used to be borrowed from. Example,

857
376
481 Notice the 7 - 6 and the 8 - 3 combinations.

Arranged in an order of ascending difficulty, these combinations are as follows:-

a. Without borrowing

4-0
5-2
6-4
4-2
6-1
7-1
8-6
9-6
9-8
3-1
8-3
2-0
7-0
4-3
7-4
9-7
5-1
6-2
8-5
9-5
9-2
7-3
4-1
8-1
5-3
7-5
3-2
6-3
5-4
8-4
9-4
6-5
5-0
9-0
9-3
1-0
3-0
8-0
8-2
8-7
9-1
6-0
2-1
7-2
7-6

b. Borrowed from

3-0
6-0
4-2
5-0
7-0
5-3
8-4
9-4
7-6
9-5
8-2
9-2
8-7
4-3
8-5
5-2
7-2
6-3
2-0
6-2
5-1
9-3
6-1
8-6
2-1
6-5
9-8
8-3
8-0
5-4
7-4
1-0
7-3
7-5
9-7
8-1
3-2
4-0
7-1
6-4
9-6
3-1
4-1
9-1
9-0

It was also found out that almost every combination was more difficult to the students when met in form (b) than in form (a). Example, taking the last three combinations of form (b), the number of mistakes in both forms shows our point of view :-

Combination	Number of mistakes	
	a.	b.
4 - 1	7	33
9 - 1	13	39
9 - 0	0	43

This is due to the fact that the student forgets that he has borrowed from.

The third category consists of those combinations where a larger digit is subtracted from a smaller digit. They can be used either to be borrowed to, or borrowed from and to. Example,

654
365
289 notice the 4 - 5 and the 5 - 6 combinations.

Arranged in an order of ascending difficulty they are as follows :-

a. Borrowed to

0-7
4-9
0-6
2-7
2-4
2-8
0-3
7-8
1-5
1-6
2-6
5-8
8-9
1-2
0-4
3-5
6-7
1-9
1-4
4-6
1-7
5-9
0-2
5-7
4-8
2-9
2-3
4-5
5-6
3-7
1-8
6-9
1-3
3-6
0-8
0-9
7-9
3-9
0-5
3-8
2-5
4-7
3-4
0-1
6-8

b. Borrowed from and to

4-5
3-4
3-5
2-9
3-6
6-7
3-8
7-8
2-4
2-7
0-1
1-4
5-9
2-5
5-6
7-8
1-3
1-6
3-7
4-7
2-8
2-3
0-7
0-8
3-9
4-6
4-9
4-8
6-8
0-3
1-4
0-2
1-5
0-6
1-8
6-9
0-5
0-9
8-9
2-6
0-4
5-8
1-9
5-7
1-7

Apparently when a combination falls under form (b), it becomes more difficult to the pupils. The following examples illustrate this point:-

Combination	Number of mistakes	
	a.	b.
0 - 4	7	26
5 - 8	6	27
1 - 9	7	27
5 - 7	9	29
1 - 7	8	38

This is due to the fact that the student has to face one principle of subtraction in form (a), i.e., the borrowing to, and one additional principle in form (b), i.e., the borrowing from.

Of what use is the above mentioned analysis to the author and teacher? The following results of an investigation in the Elementary School will give the answer to this question: 40 subtraction examples containing each of the subtraction combinations once, were given to 211 pupils ranging from the second up to the sixth grade. Errors were classified on the basis of the above mentioned types and use of the subtractions combinations. The following results were concluded:-

1. 13.6% of the wrong answers were in no borrowing whatever.
2. 16.4% were in combinations to be borrowed to.
3. 30.1% were in combinations to be borrowed from.
4. 39.9% were in combinations to be borrowed from and to.

Thus we see that these combinations are of different degrees of difficulty as they appear in the examples. The author as well as the teacher should take note of this fact so that the pupils will be provided with, and drilled upon, subtraction examples containing these combinations in proportion to their difficulty.

2. Processes

A process means an operation which requires a combination of number facts. Example, 6 x 38 is a process because nobody is expected to learn it as a fact. On the contrary, one finds out the answer by performing the process of multiplying 6 x 8 = 48, 6 x 3 = 18 and 18 + 4 = 22. Thus reaching the conclusion 6 x 38 = 228.

If we look into the nature of arithmetical content of the four simple processes, from the standpoint of children's learning, that is from the psychological point of view, we see that each of the four processes is a complex or an organization of many units of skill, and not a single unitary affair. Thus the mastery of a process may be described as an organic whole composed of lesser abilities and each lesser ability composed of units of skill.

A child in mastering each of the four simple processes, for example, must learn many facts and master many skills needed in the proper manipulation of these facts. The following item will clarify this point.

a. Specific Skills.

Addition

There are eight such skills in addition to be mastered by the pupil. Each skill consists of several habits. Some authorities on the subject believe that the number of these habits amounts to 260, which the pupil must acquire and organize (1).

Subtraction

In teaching subtraction we are attempting to build a number of specific abilities or habits which are relatively distinct, and not a single arithmetical ability.

It is obvious that a child might become expert in subtracting examples without zeros, but not be able to subtract a problem where zeros are found. In a word, it has been said that there are as many different habits in subtraction as there are types of examples. (2)

Multiplication

The basic skills are used in many different ways in the solution of examples in multiplication. The chief concern of the teacher is to see to it that the steps in which these skills are taught present to the pupil only one new difficulty at a time (3).

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(1) Ref. appendix p. 117

(2) Ref. appendix p. 117-118

(3) The list of specific skills in multiplication is as follows:

1. The multiplication table facts or combinations.
2. One place multiplier, and 2 or more figures in the multiplicand with no carrying.
3. Zero in multiplier, multiplicand or product.
4. Carrying.
5. Multiplying by 10, 100 or any multiple of these.
6. Two or more figures in the multiplier.
7. Difficulties on the part of students in understanding the placing of digits in the partial product of ~~a~~ two or more figures, multiplicand and multiplier.

A detailed explanation should be given and several examples should be solved and explained on each kind. These are in brief the skills in multiplication.

Division

Division is more difficult than any other simple process. As a matter of fact, it is the climax of the simple processes. A child who has not mastered the first three simple processes will not be able to master division. That is why enough emphasis should be placed on the specific skills in division and definite instructions should be given regarding these skills (1).

b. Degree of difficulty (2).

"If arithmetic is to serve life, life must be examined". (3) An investigation was carried on by G.M. Wilson thru a campaign of 4068 persons, representing 155 occupations. The idea was to know from practical life what common uses of arithmetic there are in the daily activities of life. The total numbers of examples collected by the personnel of the campaign was about 14583. This total number was distributed among the four simple processes as follows:

Addition

The addition examples amounted to 4416. They were studied from the point of view of the size of numbers or the number of places that were used in the examples or problems. Generally speaking, it was found out that the numbers added were relatively small. There were less than 2% of additional problems with an addend of 5 places or above, most were either two-place or three-place addends. As to the number of addends in a problem, problems with two addends were most common. Three addends next. Problems involving 4 and 5 addends or more were very rare.

Drill work may be extended slightly beyond the above mentioned limits. The following table represents the size of addends in percentage.

Size of Addends	Percent
1 place	1.56
2 places	49.40
3 "	33.65
4 "	11.72
5 "	2.96
6 "	0.57
7 "	0.08 etc.

---

(1) 1. The form of long division and the technical terms pertaining to it.
2. Steps in the process:
   a- Find the first partial dividend
   b- Estimate the first figure in the quotient
   c- Multiply the divisor by the estimated quotient figure
   d- Compare the product with the first partial dividend
   e- Subtract the product from the partial dividend
   f- Bring down one figure from the dividend to form the next partial dividend and repeat the first five steps.
3. Examples which involve both borrowing and carrying in subtraction and multiplication.
4. Insertion of remainders.
5. Insertion of zeros and double zeros in the quotient as 507 and 6006.
6. Final zeros in the quotient as 460 and 530.
7. Estimation of the quotient.

(2) G.M.Wilson: "What Arithmetic shall we teach", ch. IV.

(3) Ibid. p. VII.

Subtraction

It was evident from the investigation that there were 1296 subtraction examples. The most common of these were problems of three-place minuends, next four-place minuends, followed in order by two, five, six, and one-place minuends.

The following table indicates the size of the minuend in percentage.

Size of Minuends	Percent
1 place	1.00
2 "	12.34
3 "	45.60
4 "	31.40
5 "	8.40
6 "	1.23 etc.

Multiplication

There were 2632 multiplication problems. These problems were studied in regard to the number of places in the multiplier. It was found out that the one-place multiplier makes up 53% of all multiplication. In the case of merchants it was observed that the two-place multiplier is more common.

The following list indicates the size of the number of places in the multiplier in percentage.

Size of Multiplier	Percentage
1 place	66.52
2 "	29.90
3 "	2.51
4 "	0.72
5 "	0.07 etc.

Division

6239 was the number of the division examples in the investigation. The two-place divisor is the most common, although nearly equalled in frequency by the one-place divisor.

The following table indicates the number of places in divisors in percentage.

Size of Divisor	Percent
1 place	43.48
2 "	48.13
3 "	6.83
4 "	1.24
5 "	0.31 etc.

c. Common Difficulties

As a result of a special study of errors in test papers I have classified 17320 wrong answers into the following groups:

1. Trouble with zero combinations in each of the four processes.
2. Failure to know number facts.
3. Difficulties in column addition.
4. Trouble in borrowing subtraction
5. Estimating the quotient and bringing down in long division.
6. Carrying in addition, subtraction and multiplication.
7. Copying.
8. Ignorance of the technical terms.

B. REASONING

Work in arithmetic may be divided into two parts, one of which consists in performing the whole fundamental operations, and the other in deciding how these operations are combined for the purpose of solving problems. In other words, the first phase is known as the mechanical and the second as the reasoning. Having considered the first we now come to consider the second.

1. Importance of problem-solving.

While it is true that skill in the fundamental operations occupies a large place in the practical applications of arithmetic, the ability to solve the variety of problems demanded of us every day is more important.

The problems of arithmetic are the most significant part of the subject. In solving a problem the pupil gets all the benefit that he can get from doing abstract examples, plus a benefit which is peculiar to the problems, namely, a training in thinking.

Problem-solving should begin at the beginning and proceed with the abstract work as long as the subject is taught. There has been a feeling that abstract work is fundamental and problem work is secondary, or first finish with the mechanical phase and then begin with the solution of problems. This is not true, Abstract figures and processes are by their name and nature generalizations from concrete expressions. Problems are fundamental, abstract processes are used in their solutions.

Outside the school there is no other use for abstract processes. It is only after a process has been shown to be useful through a problem approach, the process itself may be isolated for study and drill in order that it may be known automatically. Thus, we see that problems call for the processes in use, and as a result, the idea of each process is fortified when the pupil finds that it applies in an interesting situation.

2. Skills in problem-solving.

The following are some of the specific skills involved in problem-solving. In teaching this phase of the subject, the teacher should keep them in mind and make sure that the pupils will master them during the course.

1. Ability to tell:
   - a- What facts are given in a problem
   - b- What facts are required.
   - c- What process to use in solving one or more than one-step problems.
   - d- Estimate the answer.
   - e- How to check the answer.
2. Knowledge of:
   - a- Technical terms.
   - b- Essential denominate numbers.
   - c- Units of measure.
   - d- Essential principles and concepts.
3. Ability to:
   - a- Read accurately and exactly.
   - b- Follow directions.
   - c- Attack the solution of a problem in a systematic manner.
   - d- Detect cues in solving problems.
   - e- Restate the problem in the words of the pupils".

3. Classification and grading of problems.

By a problem we mean an exercise in which the pupil must determine the operation by which the problem can be solved. If the operation is indicated to the student this exercise is known as an example.

In the teaching of problem-solving an important device is classification. One way of classification is according to operation. The problem may be either addition, subtraction, multiplication or division.

Each of the operations may also, in turn, be classified further according to the type of solutions as follows: subtraction one-step problems, for example, may be divided into a number of types such as, how-many-are left problems; how many-are-taken-away problems; how many-more (or less) problems, and find the other-number-problems. Each of the other three operations can similarly be divided into types also.

There are eleven types of solutions in the four operations, each occurring in isolation in one-step problems. It will be noticed that these types depend not upon the operation, but upon an apprehension of the meaning of the problem. A broad and liberal treatment of one-step problems should be carried out during the first three grades, before attempting to introduce two step problems at the end of the third grade.

Next to one-step problems come the two-step problems. A two-step problem consists of two-operations in combination with each other. If we recognize four operations, there are about 16 types of two-step problems, since each operation may be associated with all four operations.

Generally speaking, two-step problems should not be given, in the full sense, before the fourth grade. When these problems are begun, they should be treated systematically and the students should be shown that a two-step problem is nothing more than two-one step problems of the kind they have been doing, but combined together.

Next to two-step problems naturally come three-step problems. A three-step problem consists of three operations in combination with one another.

As the higher grades are approached, the three-step problems should be presented to the pupils, perhaps by the end of the fourth grade. In the fifth grade the three-step problems should be mastered thoroughly well, after which the four-step problems should be introduced. In the sixth grade the pupils should master the four-step problems and be introduced to more complicated ones.

Of course, one cannot draw a sharp line as to where one type of problems ends and the other type begins. Moreover, the above-mentioned plan is a general one based upon experience, but it is not yet finally determined by scientific experimentation.

A teacher should always make sure that pupils do not move on from one type to another before it is well mastered, because as a matter of fact, the one type is a prerequisite to the other type following it.

CHAPTER FIVE

Method.

It goes without saying, that there is no one best way for teaching arithmetic. There cannot be found a clear-cut outline of all the essentials of method. It is, as a matter of fact, a collection of more or less separate problems many of which are results of scientific research along this line. There is no one single method which will make a person a good teacher of arithmetic.

Different methods do exist nowadays in teaching arithmetic in the schools. Some of these are good, others are bad and still others are indifferent. Moreover, subdivisions of each of these methods are almost as numerous as there are specific skills to be developed among elementary school pupils. Newcomb says that the one-method system belongs to the past. At present not only one method but many are necessary.

The teacher himself, in the light of his training, personal experience, and the experience of others, must choose his own methods. After all, he is the only one who can know conditions well enough, better than anybody else, to adjust teaching methods to the needs of his own pupils.

A. General Methods.

In teaching, as one can see, there are many procedures to follow, as in any other phase of education. The following items will be chiefly concerned with some of the most important teaching procedures.

1. Presentation of a new topic.

a. Gradual steps for good presentation.

Before teaching any topic in arithmetic, the teacher should ask himself the following question: How far should the class go into this topic? It is better to stress a few ideas about any subject, and get the students appreciate these few ideas and understand them well than it is to crowd a great many ideas in the brief time allotted and develop, as a result, only a confusion in the minds of the pupils. Not only is this true of class work but also of home work. Many teachers have the tendency to assign work more than the average pupil can do. Some teachers actually assign more work to pupils than they themselves can do in the time available.

In teaching a new topic, instruction should proceed according to difficulty. The teaching should progress in steps suited to the powers of the students and every step should be made thorough by working mentally a number of examples. This principle emphasizes the fact that the teacher should know the steps of difficulty in a process and he should proceed with his class thru the lesson avoiding, in the meantime, the too many difficulties and possibilities of error at a time. By doing so, the pupils will be able to attain a clear grasp of these new ideas and processes and can become familiarized with them before proceeding further. It is only in this way that the difficult and complex problems of higher grades can be met successfully later on. There is no harm to emphasize the fact again that, too rapid advance from one new thought to another, will only result in confusion.

In developing new topics, the numbers involved should first be kept so small and simple as to prevent, on the one hand, the mechanical difficulty from diverting attention from the new ideas to be grasped, and, on the other hand, to enable the students to perform the computations orally. Thus, the term "percent", can be made familiar much more easily and effectively by problems like, "find 50 percent of 80", than by "find 16 3/4 percent of 6387 2/3".

Following the teaching of a new principle, should be mental and blackboard work. This work should increase gradually in complexity in order to familiarize the pupils with applying these principles in many and varied ways. When by these means the pupils have gained a fair command of the new principle, they should advance to quite independent work and more difficult examples. These examples should be as varied in their nature as possible.

One of the poor causes of presenting new subject-matter in a bad way is lack of time. Some teachers delay this important point to the last minute of the class period, and then, while the pupils are collecting their copy-books and the confusion of changing classes is on, try to present the new subject to the class. Moreover, many teachers have often been content to make such assignments as "take from page 80-85", or "solve the next three problems", leaving the book alone to supply the explanation. Full explanations are very necessary for making arithmetic meaningful. Most arithmetic text-books lack them. Authors are sometimes proud of brevity. The following introduction to the addition of mixed numbers is typical of the prevailing custom:

Find the sum of 2 2/3 and 3 2/3

2 2/3 2/3 and 2/3 = how many thirds?

3 2/3 4/3 = 1 and how many thirds?

6 1/3 5 + 1 1/3 = 6 1/3

This amount of explanation is perhaps good enough for an adult who is familiar with arithmetic. It is doubtful if ever he would infer the necessity of using a common denominator in another problem of different denominators. How much more difficult it would be for the elementary school pupil to get the idea of the following steps on mixed fractions: finding a common denominator, adding the fractions, reducing them to a mixed number, and then adding the number to the integers. In fact, enough time should be given by the teacher for a clear explanation and illustration of all of these steps as has been mentioned before. Hurried instruction results in the loss of both time and energy.

b. Motivation of arithmetic teaching.

The usefulness of arithmetic is not a strong stimulus for pupils in the elementary school. As a matter of fact arithmetic is given a place in the curricula of schools not because of the present needs of the child, but largely because of his supposed future needs.

To the child this preparation for the future life, which to him it seems far off, is of little or no importance. Lennes says that the pupil of the elementary school is more or less a creature of the present. His dominant interests are in things that appeal to him because of immediate utility or pleasure. The purpose of making a living for wife and family is not a motivating force to the boy of ten. The fact that numbers are needed in the work of the house keeper does not make the girl in the 4th grade interested in arithmetic.

To get students interested in arithmetic, and to have them grasp very easily the arithmetical processes, the teacher should make them feel a definite need for the new knowledge. To be successful in creating this felt need among the elementary school pupils for the study of arithmetic, the teacher should appeal, thru different means, to the instincts of these pupils. For example, the instinct of play helps a great deal in making the subject attractive. A child climbs a number-ladder much better if he is urged to climb, as fast as he can, and if he makes a mistake, it means "falling" off the ladder and hurting himself. A boy in naming the sums of number combinations, increased his speed by playing that these combinations were stones in the mud and that he was stepping from one stone to the other. When he made a mistake he fell and got mud all over. Numbers arranged around a circle are added more rapidly and more accurately if it means running around the circle, and if making a mistake means "falling". These are few games, given as examples, out of several different games, which the teacher may think of, and bring with him to the class of arithmetic.

Not all number games are good. The following characteristics should be possessed by a good number game.

"1. It should keep all students at work during practically the whole period devoted to the game.
2. It should not lead to noise or confusion which distracts the attention from the subject.
3. It should create in the child a keen desire to learn.
4. It should make the element of group rivalry stand out sharply.
5. It should make the group exert pressure on the individual to do his best work.
6. It should make each pupil vitally interested in the work of the other members of his group". (1)

The purpose of such games is to make, as has been mentioned before, the child do his best throughout the whole period and to improve both, in speed and accuracy. The child should be made to do this cheerfully and when the game is over, he should be anxious to try it again with better results.

In addition to play, there are the following: such as the child's desire to construct things, personal distinction and rivalry, the fighting instincts, the group or community instinct, interest in things already known, interest in what is new, feelings of success and self-esteem. These are among the most effective means for motivation in the grades.

The question of interest is a very vital one. Teachers should always give it due consideration. Little of permenant value can be secured by a teacher in any subject, arithmetic as well as any other subject, unless pupils are attentive. A pupil to be attentive should be first interested. Therefore, every effort should be made on the part of the teacher to establish if possible, a permenant interest in the subject.

c. Inductive-deductive method.

In teaching a new process some of the present text books of arithmetic still give a number of abstract definitions, a description of the process according to these definitions, and illustrations of the process. That is they follow a deductive method. Beginning with the principle or generalization and ending down with the example.

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(1) N. J. Lennes: "The Teaching of Arithmetic", pp. 141-142.

In explaining subtraction with borrowing, one of these books, widely used, begins by defining subtraction as "The process of taking a certain number from another larger number". Then, concise definitions of minuend, subtrahend, remainder and the symbol of the dash, follow. In brief, the deductive method involves the statement of a principle and its application to specific problems.

From the standpoint of learning subtraction, one wonders as to what background experience has a nine or ten-years old child for understanding these definitions, and how do they help in mastering and using the process intelligently? Thus, we conclude that deductive teaching of arithmetic, which was and is still at present a characteristic of many tests, is too abstract, and consequently very difficult to be understood by the elementary school students.

Deductive teaching being as such, it gave rise to a strict demand for a transfer in method from deductive to inductive teaching. Inductive teaching, as we shall see, is more concrete, and consequently more easily understood. In his "Psychology of the Common Branches", F. N. Freeman points out that we may say as a matter of general principle, that it is highly desirable to develop in the pupil the ability to understand what he does in contrast with the habit of carrying on a process in a mechanical way.

Thorndike claims that the principle is then better understood and better remembered because it concerns something that the pupil is doing and has been doing. In careful inductive teaching, pupils can be lead up to useful generalizations (1).

The following is a specific example of an inductive lesson about "carrying" in addition. The process of "carrying" in addition is one of the topics which should be taught by the inductive method. Otherwise, the students, in carrying on the process of "carrying" in a mechanical way, right from beginning to end, will fall into much trouble and commit many mistakes whenever they are encountered by this difficulty in addition.

The problem may be raised through an attempt to add the points won by the basketball teams of the Eagles and Lions parties in the school on the preceding two weeks, in which 11 points were won by the Eagles and 17 points by the Lions in the first week. 14 points by the Eagles and 8 by the Lions on the following week.

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(1) According to Kendall and Mirick, in general the process of such a lesson, in the form of an outline, is as follows:

1. In teaching the new fact or process, begin with the fact or process that the pupils know.
2. Have them use these facts in familiar ways.
3. Introduce the new fact or process in its simplest form.
4. Suggest the new by means of illustrations of similar facts or processes with which they are familiar.
5. Apply the principle of the illustrations by the use of objects which the pupils themselves manipulate.
6. Have pupils apply the principles employed in their objective work to the fact or process that they have set out to learn.
7. Make several such applications.
8. Formulate a statement of what was done in each instance and the results that followed.
9. Formulate a general statement concerning all the instances or cases.
10. Make numerous applications of the new fact or process in oral and written exercises.

The pupils should be able to add 11 and 17, very readily, but they will encounter difficulty in adding 14 and 8.

A few short review questions at this point are necessary. These questions should remind students of the place-value of number which they should have taken in "notation and numeration". For instance, the number of units, and the number of tens, and the relative position occupied by the units and tens. The teacher, or some student should then proceed to find the sum of the two numbers by using splints, or toothpicks or small sticks, some of which have previously been tied in bundles of ten. For the Eagles, a list of 14 points equals one bundle of ten splints to be placed in the tens place and 4 splints to be placed in the units place. For the Lions, the 8 points should be placed in the units place. Adding the splints in units place, 12 are secured. Since ten units make one ten, ten of the splints in units place may be tied in a bundle and placed in the tens place. Adding the proper sum of 2 tens and 2 units, or 22 is secured.

This procedure may be repeated several times in the addition of other lists, the number of problems solved correctly in a previous lesson by two divisions of the same class may be a good incentive for the students to perform and the like.

The teacher should see that every boy is interested, active and asks all sorts of questions, as to why, how and when to take numbers from the units place and use them in the tens place.

After several illustrations of addition with the splints, one pupil should be sent to the blackboard and asked to add two numbers - such as 16 and 5, without the use of the splints, and explain each step. Another pupil may add 17 and 16, another 24 and 18. All these steps should be performed with full explanations. Each pupil may then be given 5 or 6 examples of a similar nature to add before the class period is over. The teacher may then explain that the process involved is called "carrying in addition", and he may state the rule in simple, concise language.

The next day's lesson should include a number of applications of the process of "carrying" in the form of different examples and problems in the book, or from outside sources.

d. Objective teaching.

(1) Begin with the concrete.

Teachers of arithmetic, especially those who teach in the lower grades of the elementary school, often fail to realize the difficulties which children encounter in understanding even the simplest arithmetical situations which are presented to them. Teachers fail to realize this fact, because they overestimate the student's abilities in beginning arithmetic. As a matter of fact, those arithmetical situations are naturally simple to the teachers but not to those beginners. Number work to the beginner is meaningless, something abstract and uninteresting. The teacher should, therefore, begin the course of study with some matter that has meaning and significance to the child. He should aid the pupil in his grasp of number by approaching it from the concrete.

The reason for the desirability of beginning with the concrete may be based upon the following psychological principle: everything that is learned should be related to previous experience. The new should be apprehended not as entirely new but as a modification and extension of the old. Thus 3 + 4 = 7 must be learned in the first place by placing a group of four objects with a group of three objects and have the child count the resulting group. The child must learn this fact in such a way that, in case of doubt, he will know how to obtain the result again from fundamental sources. This study of the concrete will not only furnish a method for finding the sum, in case it should be forgotten, but it will also develop a correct idea of the nature of addition. What is true with the learning of the addition combinations, holds true in learning the combinations of the other three operations.

Thus we see that it is often necessary for pupils to work with concrete material, for it is only thru actual contact with things that numbers acquire meaning. Moreover, concrete material provides for learning thru several senses and adds in interest to the learning processes.

It is a fact, that in many elementary schools the prevalent method is to teach numerical ideas and processes with scarcely any use of things, i.e., it is merely abstract figuring. The natural result of this kind of teaching is to see beginners hating, and failing in the course.

(2) The Real Concrete.

It has been pointed out by Dr. Dewey and others, that things present to the senses, objects that we can see, touch and handle, are not necessarily concrete. They are only concrete to us when there is an accompanying realization of their meaning and importance. A false conception regarding the concrete, has caused many teachers to be artificial in their teaching by asking primary children to count, add, etc. with things they will never be concerned with in life.

Objective material that is most usuable in the classroom should be selected. It should be also varied as much as possible. Unless it is varied, the pupil is apt to associate what is being learned with one object only, such as "Lentils and addition", or, "pebbles and subtraction", etc. and not to be able to apply the knowledge to other things or situations. The child also gets tired of one kind as a material.

Real objects are more interesting and possess the additional advantage that pupils are acquiring ideas and facts in connection with the kind of material with which they will be used.

Teachers often stick to the use of blocks, pebbles, splints, shoe-pegs, and grains of corn for the purpose of making quantitative numerical relations and processes concrete to the young. Example, two grains of corn and two grains of corn are four grains of corn. This situation is of no importance to any child, and, therefore, is abstract in spite of the fact that we are using grains of corn. But two piasters in a boy's purse today, and an imaginary two piasters to be received tomorrow, make the four piasters he needs to buy that lovely red ball, and represents the real concrete. "The three kittens, two white and one black, left at the farm months ago, may be more concrete to the little girl who loved them, then the colored disks arranged in groups of two and one before her eyes". (1)

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(1) A. E. Moore: "The Primary School", p. 283.

2. Drill

a. Need for drill.

Drill, means practice or frequent and regular repetitions of the act that is to be automatized.

The need for drill is twofold:
1. drill to build a new skill.
2. drill to maintain an old skill.

One of the psychological laws of learning is the law of exercise, and in so far as learning takes place in arithmetic, this law cannot be avoided in teaching the subject. On the other hand, it is agreed upon that arithmetic is a habit-forming subject, and that correct practice or drill is the foundation of habit-formation.

If all of this is true, it, therefore, follows that drill should be given an important place in the teaching of arithmetic, specially so in teaching the four simple processes.

The following four points should be kept in mind by the teacher in connection with the formation of any habit:

"1. That a clear understanding should be gained by the pupil of the habit to be formed.
2. A keen desire to form the habit should be present.
3. Frequent repetitions should be provided until the habit is formed.
4. All exceptions should be avoided". (1)

The first of the four above points suggests that the time for drill is immediately after an idea, process, principle or rule is taught and is thoroughly understood by the pupils. "Drill undertaken before the ideas involved are mastered, only piles up error, discomfort, and a resistive attitude" (2). Too often pupils are engaged in a speed drill without having first acquired a clear understanding of a right procedure and a correct start in forming a habit. In order to avoid that, the teacher should not separate between drill and instruction. It is a poor method to teach and explain large sections of a process and then drill upon them. It is better to explain a bit, then drill on it, explain a bit more and drill on it, and so on.

The second point suggests that in order to make drills more effective, is to motivate them by all means. To keep a record of one's progress is a good means for motivation. Experiments show that more gain in speed in arithmetic was made by pupils, after they began systematic drill and kept a careful record of progress from day to day. Another means for motivating drills, is a knowledge of the results and of the end sought. Peterson found that a class which was asked to copy a list of words from the blackboard and was told that those words would be called for later, remembered 50 percent more of them, than another class which was asked to copy the same list but was not told that they would be called later.

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(1) R. S. Newcomb: "Modern Method of Teaching Arithmetic" P. 72.
(2) "National Society for the Study of Education": 29th Year Book, P. 264.

The third point suggests that habits are formed and fixed by use or repetition. Thus, after the connection 4 + 3 = 7 has been made, it should be reviewed frequently until it becomes automatic.

The fourth and last point suggests that there should not be, on the part of the teacher, any leniency in, or allowance for, any exception in the drills undertaken in building a certain habit.

b. Techniques of drill.

Drill, as any other teaching procedure, involves many techniques. The following four are among the most important:-

(1) Distribution of drill according to time.
(2) Distribution of drill according to difficulty.
(3) Mixed rather than isolated drill.
(4) Standardized rather than unstandardized drill.

(1) Distribution of drill according to time.

The effectiveness of drill depends partly on the length of the drill periods and of the intervals between them. There are hundreds of experiments on the time arrangement of drill. (1)

It is not easy to say that any particular length of period is best for a practice period in arithmetic. "It is generally recognized that short periods of snappy drills, with concentrated attention are better than relatively long periods in which interest tends to lag. Drills which occur daily, or at least several times a week, are more effective than those with longer intervals. The ideal distribution of drill is to give, during the time of the first learning, enough drill to insure a reasonable degree of mastery of the process, and then to give practice in smaller and smaller amounts at longer and longer intervals". (2)

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(1) The following experiment, as has been carried on and described by Reed, appealed to me most. Reed divided a group of two hundred and three first-year and second-year college students into four groups. They were given problems of five three-place numbers to add. The time was distributed among the four groups as follows:
Group 1, sixty minutes continuously.
Group 2, twenty minutes a day for three days.
Group 3, ten minutes a day for six days.
Group 4, ten minutes every other day for twelve days.
The percentage of improvement was measured for all the groups in the last ten minutes over the first ten, both in the number of examples attempted and in the number right. The following were the results:

Group	Attempts	Rights
1	10.9	12.2
2	35.9	43.4
3	33.1	33.6
4	28.6	35.1

Twenty minutes a day for three days produced the greatest gains. Yet not much greater than those obtained from the ten-minute periods. All three are far superior to the gains produced by an hour's continuous work, which indicated fatigue effects.
This experiment suggests the following conclusion:
1. That drill periods be short enough to avoid fatigue, and close enough together to avoid forgetting from one period to the next.
2. That drill periods be long enough to get some work and be far enough apart for one period to give the next the benefit of stimulated nutrition and exercise.

(2) C. R. Stone: "Supervision of the Elementary School", p. 156.

(2) Distribution of drill according to difficulty.

Distributed drill is that type of drill which provides a number of practice to each combination involved in the process drilled upon. Thus, drills should be so built that each number combination appears with a calculated frequency, i.e., more practice on the harder combinations as on the easier. While in a haphazard drill, some combinations do not appear at all, or only a few times, or in the case of carrying, if thirty examples practiced the carrying of one, sixty-five times; the carrying of two, seventeen times; that of three, four times. Thus, it piled up practice on carrying one and slighted the carrying of two and three (1).

(3) Mixed rather than isolated drill.

This phase of drill pertains to the content of each drill unit. Units of review may be presented in such a way as to deal only with one single process, such as unit one on "addition of whole numbers", unit two on "subtraction of whole numbers", etc. On the other hand, each review unit may be a mixed drill in which examples of several or many processes may appear. Thus we see the two extremes of view on drill. Both forms of drill are perfectly possible but, apparently, for experimental purposes, isolated drill was tested as such against mixed drill. Experimental data on the two alternatives suggested a distinct advantage of the mixed type over the isolated type for purposed maintainance (2).

(4) Standardized rather than unstandardized drill material.

The underlying principle of standardized versus unstandardized drill material has a direct relation with the problem of motivation. By standardized material is meant that material which has standards of performance according to which each pupil can determine his own rating. He can know how well or how poorly he has performed a piece of work. Awareness of success or failure is by itself a very strong motivation (3).

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(1) Dr. Luse studied the effects of these two types of drill on six hundred fifth-grade pupils, equally divided into two groups, on a basis of general arithmetical ability. One of these two groups was given distributed type of drill, and the other a haphazard type. The results were as follows:
1. Both groups made a decided gain from the fifth periods of drill. The gain for the distributed drill was from 19.6% to 53.7% in attempts and 31.1% to 84.8% in rights. The gain from the haphazard type of drills was from 11.2% to 39.8% in attempts and from 13.3% to 60.8% in rights.
2. The distributed drill gave an excess over the non-distributed drill in examples solved correctly of 17.7% in addition, 18.8% in subtraction, 35% in multiplication, and 23.9% in division.

(2) There were 267 pupils in group M, and 263 in group I, who were given drill on the addition of fractions for a period of twenty-six twenty-minutes drills. The results were as follows: the mean of right answers of the mixed group in the pre-test was 93.75 and in the final test 150.89 while that of the isolated group was 29.75 in the pre-test and 133.51 in the final test. A difference in the pre-test of 0.82 and in the final test 17.37. In addition to this difference it was noticed that pupils using the mixed drills learned to work more accurately and more rapidly than pupils using the isolated drills.

(3) An experiment was carried on for one day a week, for twenty weeks. In this experiment two groups of fourth-grade arithmetic pupils were used. Only one of these groups was made aware of success or failure

3. Principles.

a. Written and oral work.

Oral work in arithmetic is invaluable specially when the teacher plans the material in advance, which is to be worked in class. Whether this material is taken from the text-book or is prepared by the teacher from outside the book, it should be simpler in, both, statement and calculation than the material assigned for written work.

The teacher should keep in mind that the material which he assigns for oral work should be of two kinds: Namely, abstract, in order to strengthen the pupils in calculation; and concrete, to strengthen them in interpretation and in using calculation in applied problems.

Oral work should proceed written work and be given daily, if possible, for about one fifth of the class period. Within this time limit so many examples and quite a good number of verbal problems can be solved. The habit of solving simple examples or easy problems orally should be built early enough in the pupils of the grades. Too often one notices that the pupils, sometimes resort to the use of the blackboard, or pencil and paper, even in answering questions about the simple number facts.

When pupils are assigned written work to be done in class or at home, the teacher should make sure that the pupils know the "What to do". Otherwise, he has to lead them, somehow, into these difficulties, because when pupils are assigned some work which they do not understand, they either waste a great deal of their time or they have to depend upon help given by others, usually their parents. Both phases of the question are disagreeable, because in addition to the lack of uniformity of this outside instruction, there is the generosity of the parents. The following incident indicates what, sometimes, happen to home work:

"One day a little girl took home her arithmetic lesson to learn. As usual, her mother helped her. The next day, on the child's return from school, her mother said, 'Did you have a successful day at school, Dorothy'?
'Yes, mamma,' was the reply.
'Were the problems all right'? Continued the mother.
'Oh, the problems', said Dorothy. 'No, none of them were right, but don't feel badly, mamma, none of the other mothers had them right either'." (1).

From this incident we conclude that no home work should

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by making pupils use a series of mixed fundamental drill units, which contain reference to standards of work, and provision for the class to keep an objective record of progress or failure in their work. The other group used exactly the same material and under the same conditions but without any reference to standards of work or provision for a record progress.
Lack of space prevents the insertion of data showing all sorts of comparison between the two groups, but suffice it to say, that while the mean of the two groups was the same, 86.41 in the initial test, it amounted to 285.14 for the experimental group, and 273.80 for control group. Thus, one can easily say that experimental evidence demonstrates the useful effect of awareness of success, as has already been made clear under motivation, and recommends highly the use of standardized material for drill purposes.

(1) Kendal and Mirick: "How to Teach the Fundamental Subjects", p. 316.

be given before the fourth grade, and when assigned it should be supplementary, or drill on easy examples. It should never consist of work in which the pupils will have to be helped.

b. Speed and accuracy.

To begin with, accuracy is of first importance, and speed is of a secondary one. If speed is accomplished only at the expense of accuracy, it is rather of little or of no value whatever. On the contrary, the teacher must continually insist upon one hundred percent accuracy to be set as a goal for every pupil.

The teachers, specially those of the early grades, should feel responsible to secure first accuracy and then speed in both oral and written work. In these grades, if inaccurate habits are formed, they stick to the pupils all thru life. The attempt , on the part of teachers, to get much speed at first inevitably leads to inaccuracy. But, later, when the work becomes more automatic and after a high degree of accuracy has been secured, "speed" exercises should be given. These exercises should be simple that practically all the attention can be given to the speed itself.

David Eugene Smith says, "It is the loose manner of writing out solutions, tolerated by many teachers, that gives rise to half the mistakes in reasoning which the pupils work, and, teachers are coming to recognize that inaccuracies of statement tend to beget inaccuracy of thought and so should not be tolerated in the school room". (1)

This toleration on the part of the teachers, leads, as a matter of fact, some of the pupils to believe that accuracy in computation is of little importance. Teachers must continually insist upon accuracy, as has been mentioned before.

Even more than that, some teachers permit and encourage inaccuracy by giving high grades to the correct processes even if the answer is wrong.

c. Individual differences.

Any teacher of arithmetic can notice very easily many differences among pupils of the same class, no matter how exact the classification of these pupils has been. We are after those differences in ability to understand situations which call for a knowledge in arithmetic.

Since the existence of these differences is a fact, teachers should take them into account and provide for them, otherwise, there will be no hope for them to attain a high degree of efficiency in their teaching. No matter, whether the causes of these individual differences in ability to meet arithmetical situations are inborn or acquired ones, teachers of arithmetic are not excused from: first, finding out their causes; second, their remedies; if this is within their reach; third, making the best possible use of the abilities which these pupils possess.

If the causes of these individual differences are physical defects, such as inability to see or hear from a distance, why not give these pupils front seats? If need be, an examination of the eyes of these pupils should be made by a specialist, in order to provide suitable eye-glasses, which will help to remove this difficulty.

Some pupils may have missed a great deal because of an absence for illness or some other legal reason. In this case, those

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(1) Brown and Coffman, "How to Teach Arithmetic", p. 113.

pupils should be given individual help by the teacher or by some other member of the class, being capable of giving the necessary help.

Other pupils may do extremely well in one thing and very poorly in another thing. The latter may be due to the poor quality of learning which has preceded. In such a case, teachers should find out what each pupil needs and begin there. They should not pile up difficulties which result in an increase of differences among pupils, by continuing new work while they have a weak background.

Lack of interest in arithmetic also results in poor work and increases individual differences. The teacher has to resort to some sort of motivation, as has been mentioned previously under this item.

The following are other remedies for individual differences:

1. Give more difficult questions under the same topic to the more advanced pupils.
2. Let advanced pupils help to direct and check the work of the backward ones.
3. Give maximum and minimum assi gnments.

Finally, it was made clear in this discussion, that individual differences, no matter how great and numerous, can be grouped into types subject to particular methods of treatment. Otherwise, there should be as many methods of instruction as there are individuals in the class. A thing which is impossible.

d. The use of the text-book.

Under the old method, the text-book was all in all. Both, teacher and students had to stick to it, and any divergence from it, was considered to be a fatal mistake on the part of the teacher.

At present, the change in the method of teaching, as we have seen before, has brought with it a change in the use of the text-book. The text-book is no longer considered to be everything. It should serve only as a guide or reference for help. "It helps the teacher to organize his work, gives good teaching suggestions, and provides for a collection of graded drill, exercises and problems. It offers a chance for the study of arithmetical language, and some of the material can be assigned for silent reading". (1)

The teacher should feel free to omit problems and even topics from a text-book, provided care is taken that the omitted parts are not needed later in the course. Moreover, additional material, from other books and from community life and of the pupils' own making, should be used (2).

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(1) Almack and Lang: "The Beginning Teacher", p. 332.

(2) "What shall be said for the teacher who fears to omit certain problems which are not utilitarian and whose culture value is counter-balanced by the fact that they give a false notion of business, or to omit those traditional puzzles which depend for their difficulty upon their ambiguity of statement? Many a teacher, especially in our country schools, will confess to such a fear of omitting problems, lest he be accused of an inability to solve them.
It would be well for all teachers to assist in creating a sentiment in favor of omitting the unquestionably superfluous or dangerous, and thus to avoid this weak criticism. It should also be understood by timid teachers that it is no disgrace to be unable to solve every puzzle that may be sent in, or even every legitimate problem".

There are teachers who fear to omit a single exercise or problem. They insist upon the pupils to work everything in the book, and to do that by the books' methods. As a result, these pupils may solve every problem in the book, and yet be entirely unprepared to solve many of the problems required by real life situations.

B. Specific Methods

1. Special methods.

a. Number facts.

Students of arithmetic differ in opinion upon the number of facts to be taught. This difference of opinion has its cause in the theory of transfer, and it depends on the extent to which they rely upon this theory. Some believe that we should teach only forty-five of these facts, thus, leaving out of consideration the nineteen zero facts. Furthermore, they consider 4 + 7 = 11, and 7 + 4 = 11 as the same fact, depending to a large extent upon transfer. They believe that a student after being taught 7 + 4, should be able to know 4 + 7 accordingly. Thus, they also dispense with the teaching of thirty six more facts. What remains of the one hundred basic combinations are the fourty-five original basic facts.

Research has shown that not only forty-five, but one-hundred combinations must be regarded as basic. For example, 2 + 6 and 6 + 2 are not the same to the pupils of the grades. Clapp has shown that among 1154 third-grade pupils 220 got 2 + 6 wrong, whereas only 139 of the same pupils get 6 + 2 wrong. Evidently the learning of 2 + 6 is no assurance that 6 + 2 is likewise learned. This does not mean that there is no transfer, whatever, from 2 + 6 to 6 + 2, but it means that after one has been learned, the other will have to be learned separately, although, perhaps, with less effort especially if these facts are associated in the teaching and learning (1).

On the other extreme, other students of arithmetic believe that some of the secondary combinations should also be known as facts. The total of secondary combinations of the four operations amounts to several hundreds, which children cannot learn and if they can do so, they cannot retain them. As a matter of fact, only the most important of these combinations must be cared for, and for which provisions should be made. Such as the higher-decade combinations which are used in multiplication with carrying, and the higher-decade subtraction combinations which are used in uneven short division (2). We come next to the consideration of the third question, i.e., how should these number facts be taught?

In the first place, the number facts should be taught with mdaning. A pupil should not be taught numbers as separate entities by themselves, such as 3 or 7 or 9, in an abstract form, but should be taught as 3 something, 7 something, and 9 something, because one can never find in life these numbers standing alone without the "something".

In addition to that, as the pupils learn the facts of the four operations they should also be getting clear ideas of the meaning of these processes. As they learn the multiplication or division facts, they should be getting an intelligent idea of what multiplication or division is.

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(1) Ref. page 72
(2) Ref. appendix p. 102

A child does not know what the word "times" means by saying "three times five are fifteen". As a matter of fact, it is a meaningless phrase to him, which he recites parrot-like. If, on the other hand, this fact is taught in the form of three 5's are fifteen, the multiplication idea is clearly expressed in form which is easily understood. Three 5's and another five and still another five.

The most wrong answer to 1 times 4 is 5. This is because of the association made between 1 and 4 in the addition fact 1 + 4 = 5. This confusion comes about when the multiplication combination is read 1 x 4. But when the combination is taught in the form of one 4 is 4, it will be impossible for the child to say one 4 is 5.

Similarly, greater confusion is caused by the teaching of the division facts. Children are taught to say 7 is contained in 56, 8 times or 7 goes into 56, 8 times, or 36 divided by seven equals 8. None of these statements has much meaning for a young child. It is better to say the number of 7's in 56 is 8 or the 7's in 56 are 8.

Allied to teaching with meaning, is practice with meaning. Abstract number practice is relied upon too much. Initial learning, as has been mentioned before, should be charged with meaning.

It is practically impossible for a child to know his number facts in abstract form without being able to make any use of them. A child does not actually know a fact unless he can use it in a situation requiring it. To be able to say 6 x 7 are 42 is insufficient. The performance of a few good 6 x 7 problems is worth several times more than an equal number of practice on 6 x 7 equals 42, as an abstract fact.

In the second place, the number facts should be taught in groups. By teaching the number facts in groups of identical elements, transfer may be facilitated, and it will take care of a part of the learning. For example, a knowledge of 4 + 3 does not in general insure a knowledge of 14 + 3, or 24 + 3. The necessary higher-decade facts should be grouped with the basic combinations from which they are derived. On frequent occasions, the pupils should be asked to tell the whole story of about 7 + 6. This will provide the learning of 17 + 6, 27 + 6 and so on.

The pinciple of grouping facts should like-wise be applied in teaching the higher-decade subtraction facts. For example, 12 - 8, 22 - 8, 32 - 8, and so on. This also gives rise to another series of higher-decade facts. Example, 22 - 18, 32 - 8, 42 - 38, etc.

When we teach the basic facts themselves we should apply the same principle of grouping the number facts. Example, when we teach 6 and 8, we use the same teaching situation to present 8 and 6, as well as the corresponding subtraction facts 14 - 8.

In the third place, the basic combinations of each of the four operations should also be taught according to difficulty. The easy combinations should be presented first, and the difficult ones later. Moreover, instead of giving equal drill on all combinations, the teacher, in knowing their order of difficulty, can give the greatest amount of attention to those combinations which are most difficult. This is quite different from the old procedure of requiring pupils to learn these combinations in the form of tables, by repeating them over and over in order, and thus giving equal drill on all combinations regardless of their comparative difficulty (1).

In the fourth place, the number facts should be learned and drilled upon to the extent of being known by the pupils automatically. There are many grown up people who are uncertain when they are confronted with the multiplication combination 9 x 7.

b. Processes.

Addition

As a result of an oral diagnosis in column addition, I have found out that pupils can be sorted out into the following three groups. The first consists of those pupils who added by thinking the cummulative sums only, and thus they have added the numbers 6, 9, 5, 7, 8, as 15, 20, 27, 35. That is, they have used the methods known as addition by thinking results only.

The second group consists of those students who added the numbers as following: 6 and 9 are 15, and 5 are 20, and 7 are 27 and 8 are 35. The third group of pupils followed a more detailed method. They added as follows: 6 and 9 are 15, and 5 are 20, 20 and 7 are 27, 27 and 8 are 35. That is, pupils of the second and third groups added by thinking both results and numbers. There is no doubt that the first of the three groups has followed the more rapid method, i.e., addition by thinking results only. This method is the one recommended to be taught to pupils. (2)

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(1) Clapp reports the following frequencies of combinations in Book II of a certain series of arithmetic:

1 + 1,	434 times	2 + 1,	444 times,	1 + 2,	299 times
4 + 1,	474 "	6 + 8,	62 "	7 + 6,	74 "
8 + 7,	102 "	7 + 5,	76 "	7 + 9,	74 "

Osburn reports that "one hundred and eighty out of a total of 1,325 combinations do not occur at all in the book considered while some easy ones occur more than 300 times.
With the possible of texts published since the results of the first analysis have been available, it appears certain that the practice a pupil receives upon the combinations of arithmetic, both basic and secondary, will not be adjusted to the difficulty of the combinations, and that the amount of practice upon the different combinations will depend upon the text the pupil studies.

(2) "Conard and Arps studied the value of these methods with two equal classes, of high school students. There being thirty-two student in each class. Both classes received eight periods of drill in rapid work in each of the fundamental operations. One class added by thinking results only, and the other by thinking both results and numbers.
The following table gives the percentage of gain for operations combined for both classes in the Courtis tests, which were given at the beginning of the experiment and the end. Percentage of gain in addition by two methods (after Conard and Arps):

Methods	Attempts	Rights
Thinking results only	34.4	30.9
Thinking both results and numbers	8.5	2.5

Carrying in column addition.

"Carrying" in column addition should be taught inductively by means of splints tied in bundles of ten. Primarily the fundamental reasons for carrying should be explained as being based, upon the place value of numbers. This has been previously discussed under inductive-deductive methods.

In the beginning the teacher should confine his explanation to the carrying to ten. The carrying of hundreds can be done later in a similar way and perhaps without explanation.

Numbers to be carried should be added immediately to the first number in the succeeding column. Example:

57 We add as follows: 7units plus 6 units 13. 13 units 1 ten and
26 3 units. The 3 is written in the units column, and the 1 ten
83 is added directly to the ten's column. 1 ten plus 3 tens 6
tens, plus 2 tens 8 tens. Thus 57 26 8 tens and 3 units, or 83 ans.

This process of carrying in addition accounts for a considerable proportion of the total number of errors in addition, especially in the lower grades. Scott shows that in an elementary school class 34% of the errors in the fundamental arithmetical operations were due to mistakes in carrying and in another class of older pupils, the proportion was about 26%.

Among different causes, three main causes of error seem to be outstanding in this process. First,imperfect understanding of just what the carrying process implies. Second failure to remember the number to be carried until time to add it in the next column. Third,the difficulty of sustaining the attention to the fact that a number is to be carried.

Subtraction

There are four principle methods of subtraction in use throughout the world, namely: (1)

1. The complementary method.
2. The take-away-borrowing method.
3. The take-away-carrying method.
4. The additive method.

Two of these four different methods are more familiar and are still in use at present. These are the take-away-borrowing method, and the take-away-carrying method. The following example is an illustration of the first method:

84 In this example, we cannot take 6 from 4, and so, we
56 borrow 10 from the 8 tens, which, added to the 4 makes 14.
28 6 from 14 is 8. Since one ten was borrowed from the eight
tens, there are left only seven tens. Five tens from seven
tens is two tens.

The next example, is an illustration of the take-away-carrying method:

84 Since we cannot take 6 from 4, we add ten to the four,
56 making a sum of 14, and subtract 4 from 14. Since ten has
28 been added to the minuend we must add ten to the subtrahend,
which we do by adding one to the five. Then we subtract 6
from 8.

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(1) Ref. N. J. Lennes: "The Teaching of Arithmetic", pp. 227-234.

As to which subtraction method is the best, is still a matter of dispute. There are points in favor of each method, but evidence favours the take-away-borrowing method rather than the additive one.

To avoid interference and confusion it is very important that a school system should select one method and adhere to it throughout the course.

Whether the one method or the other is used, the students should be taught to subtract a digit from another larger digit directly, a digit from another smaller digit directly. They should avoid the practice of stating each step in detail.

Thus, in subtracting 28 from 56, they should say 8 from 16 at once, and a little later on, they should be taught to give the results merely without mentioning how each is attained. Because saying too much in the process is a waste of time.

Some children form the habit of writing the new form of a digit after "borrowing" or "carrying" is made, while they should learn to hold this in mind.

Multiplication

Kirk Patrick concludes that memorizing tables and using them later on, is a wasteful method of learning. Students should learn the multiplication combinations by derivation and computation. Children who memorized the multiplication tables made a gain of 10.1 percent. Those who learned them by using key and practice made a gain of 18.9 percent. The third group are the computers. They made a gain of 27.7 percent.

The results of this one experiment are very clearly against the traditional method of requiring pupils to memorize the multiplication tables and then expecting them to be efficient in the use of the combinations when presented in random order, as they occur in arithmetic work.

Division

Should the child say "5 into 20 how many times"? or "5 times what equals 20"? Mead and Sears trained two third-grade classes by these methods. The class that was trained by the first, the so-called into-method, made in five months a gain of 20 percent as measured by the number of combinations solved in one minute; the class that was trained by the multiplication method made a gain of 45 percent during the same interval. Later, the two authors found out that there is little relation between the ability to know the combinations and the ability to work the examples, for the multiplicative division class with high score in combinations, fell much below the old method division class in ability to work the examples. The multiplicative division class averaged 4.3 points higher than the traditional division class in ability in the simple combinations. In case of the longer examples the second class scored 0.7 point higher than the first.

Estimation of the Quotient.

Of all the difficulties involved in long division, probably the hardest one is the quotient difficulty. Sometimes the trial quotient figure has to be corrected from two to five times before the right quotient figure is found. These corrections are, without doubt, a source of waste in both time and energy.

There is no agreement as to the best method of estimating the quotient. However, two methods of estimating the quotient in long division are in use at the present time.

The first of these two methods is generally taught and it uses the first digit of the divisor as the trial divisor into the first one or two digits of the dividend. Thus in the following example:- $\begin{array}{r} 2 \\ 46\,\overline{)95} \end{array}$, 4 is the trial divisor, when divided into 9, it gives 2 as the quotient.

This method is called the apparent-quotient-method or rule. It is very often taught to the exclusion of the second rule.

The second method adds 1 to the tens' figure of the divisor. Generally, it is used for only those divisors in which the second digit is 7. 8. or 9, and sometimes 6, example, $\begin{array}{r} 3 \\ 48\,\overline{)174} \end{array}$. Here in estimating the quotient, some authorities advise the use of the first digit of the divisor increased by one. It, thus, gives the figure 5 as the trial divisor; because the correct quotient is not obtained when the 1st digit of the divisor is used as the trial divisor. This method is called the increase-by-one rule.

<u>Conclusion</u>:

1. If the unit figure of the divisor 1, 2, 3, 4, or 5, divide the partial dividend by the tens figure of the divisor to find each quotient figure. For example, if the divisor is 73, divide the partial dividend by 7.

2. If the divisor ends in 6, 7, 8, or 9, divide the partial dividend by one more than the first figure of the divisor to find each quotient figure. For example, if the divisor is 59, divide the partial dividend by 6.

If these rules are followed they give the correct quotient figure and the first trial is 75 percent of all cases. In practically all other cases the trial quotient figure have to be corrected only once to obtain the right quotient figure. These rules are the most efficient yet devised for finding the correct quotient figures with very few exceptions.

c. Reasoning.

Little is known on the basis of experiment as to what may be done to overcome specific or general difficulties in problem-solving. The results of measurement and the analysis of pupils' work both show clearly that many children have an adequate knowledge of the fundamental combinations as such, but they cannot comprehend the problem setting when the same numerical situations are presented in verbal form. It is a common complaint that the child will say, "30 divided by 6 is 5", without responding at all satisfactorily to the problem: "If you buy 6 apples for 30 piasters, what is the cost of each"?

Mastery of the fundamental processes does not automatically result in proficiency in applying them to the solution of problems. Often a pupil is ready to solve a problem if some one will tell him whether to add, subtract, multiply or divide. To know when to use and combine these operations in arithmetical situations is the essence of problem-solving. Moore says that skill in the fundamental processes only furnishes a tool, and how to use the tool is an entirely different subject.

Current text books in arithmetic lack any effective development of principles that underly the solution of problems. It is possible to organize the ideas and principles that underly the solution of problems in such manner as to unify and simplify the whole subject. The following discussion might be of some help along this line. The foundation of all solving problems is the solution of one-step problems as was stated before. These should for the most part and for most children involve small numbers and relate to familiar conditions. Many of these problems should also be solved orally.

The questions to be answered by the pupil are: What situation requires addition? subtraction? multiplication, or division? In the beginning all problems requiring the same operations should be grouped and presented to the pupil, after having studied formal work in the operation concerned. This arrangement makes the pupil show greater ability than he really possesses. Just because he has been studying addition. It is natural for him to add the numbers he finds in his problem. If problems requiring subtraction were presented to him at this time, we should find him attempting to solve them by adding. Similarly, after the formal work of each of the other three operations has been through, sets of problems should be presented according to the operation previously studied. Sets of problems requiring subtraction should follow the problem work of the subtraction operation and so on.

Sometimes in the second grade, groups of problems will be presented, some of which require addition and some subtraction. For the first few exercises of this sort the pupil's work will be a disappointment for both, to himself and to the teacher. The fact is, that pupils in the lower grades usually can solve certain simple problems while they are unable to solve others involving exactly the same principles.

In teaching pupils to solve problems, the teacher should train them according to the following general steps of the process.

The first step is to have the pupils read the statement of the problem carefully and to understand it well. This reading is a complex process and usually a higher degree of comprehension is required than in our reading of ordinary printed material. The teacher of arithmetic must assume some of the responsibility for training pupils to read the problems.

In the statement of the problem there are two kinds of words, namely, those which describe the setting of the problem or the particular environment in which it occurs, i.e., descriptive kind of words. The second kind of words are those which define quantities or quantitative relationships, i.e., the "technical kind". Investigations show that pupils use many words of either kind without being sufficiently acquainted with their meaning. These words define relationships which exist between the quantity and are cues for formulating the plan of solution. That is why pupils should know them fairly well.

It is said that one teacher frequently reminded her pupils that a "problem always tells you directly what you are to do if you understand the statement of it".

The second step is to find the material that the problem gives to be worked with, i.e., "what is given".

The third step is to decide what the pupil is asked to find, i.e., "what is required".

The fourth step is to decide what to do and to indicate the process or processes to be performed.

The fifth step is to estimate roughly the probable answer as a check upon the later and more careful conclusion.

The sixth step is to perform the processes decided on and to check the result with the estimated answer.

To illustrate these steps of the process of the solution of problems, the following simple example may be taken: A dealer buys 20 kgs. of grapes at 45 pts. a kilo. He sells it at 55 pts. a kilo. How much is the profit on the whole?

First, the problem is read carefully. The words "dealer" and "profit" are perhaps explained.

Second, the problem states that one kilo of grapes cost 45 pts. and is sold for 55 pts.

Third, the problem calls for the profit on all the grapes 20 kilos.

Fourth, the profit on one kilo is found by subtracting 45 pts. from 55 pts. The profit on 20 kilos is the profit on one kilo multiplied by 20 kilos.

Fifth, if one kilo is sold for half a pound, the profit on the whole would be one pound, but as the profit per kilo is double that, the whole profit would be approximately two pounds.

Before ending the subject about problem-solving by analysis, I should like to mention another method quite different from the above-mentioned, proposed by Washburne and Osborne. According to these authors training in the seeing of analogies (stating a problem in simpler terms or with simpler number) appears to be equal or slightly superior to training in formal analysis for the superior half of the children; analysis appears to be decidedly superior to analogy for the lower half; but merely giving many problems, without any special technique of analysis or the seeing of analogies, appears to be decidedly the most effective method of all.

The general recommendations, then, growing out of the investigation are as follows: Problems should be so constructed as to present real situations familiar to the child. Children should be given many such problems to solve without special training in any generalized, formal technique of analysing problems. Concentration on practice in solving practical problems will yield gratifying results. (1)

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(1) Ref. L.J. Brueckner & E.O. Melby, "Diagnostic and Remedial Teaching", pp. 224-225.

2. Verification

a. Value of verification

It is very important that pupils be taught some means of verification, and that they be asked to employ at least one of its methods in every example solved. Everyone is, likely, apt to make errors, and the habit of verifying results is a good guarantee of the detection and correction of mistakes.

We often fail to realize that teaching pupils to verify their work is a very vital part of the successful teaching of arithmetic. As a result of this failure, on the part of the teacher, inaccuracies creep into the pupils' work until these pupils form the habit of being satisfied with inaccurate results.

Verification develops in the pupil a critical attitude toward his own work; it results in a strong feeling of self-correction; it helps to diminish the number of wrong answers, and finally, it doubles practice on the combinations or their reverse.

b. Ways of verification

Verification can be made in two ways. Namely, by rough estimation of the answer before an operation is performed, and by actual checking, after the full performance of the operation. We shall now consider the first way of verification, that is verification by rough estimation.

(1) Estimating results.

Closely allied to training for skill in calculation is training in "estimation". Estimation is one of the most helpful methods of verification, and one can be used very easily in arithmetic by the pupils. Training for estimation should begin early in the grades in estimating lengths, areas, weights, etc. Later on, have the pupils read the problem or example, estimate the answer, and give reasons for their estimates. Then let them work the example to see how nearly right their answers are.

All kinds of problems which help along this line should be given in all grades until pupils establish the habit of estimating the answer and of doing their mathematical work with an expectation of arriving at a reasonable result. Following, is the second type of verification, that is actual checking.

(2) Actual checking.

Much of what has been said about estimation holds true about actual checking. However, there are still the following few points to be mentioned in particular to checking proper.

Actual checking of an operation should be taught when that operation is being taught. Sometimes, a single operation may be checked by many methods, but not all of the checks should be taught to the pupils. One good check, which is well known, and frequently used until it becomes automatic to the pupils, is of more value to them than three or more checks but notone of which has been thoroughly mastered.

Each of the four operations may be checked by one known method as follows:-

Checking addition.

Addition can be checked by combining the addends in the reverse order. To combine in the same order the second time as the first is practically no guarantee of the correctness of the result. The mind tends to repeat its own mistakes. If two numbers are incorrectly combined the first time, the chances are that the same mistake will be made if the order of the addends is not changed.

Checking subtraction.

The well-known check for subtraction is by adding the difference to the subtrahend to produce the minuend. This method can be easily applied.

Checking multiplication.

Multiplication can be checked in a number of ways. Probably, the method most familiar to the majority of teachers is by casting out the nines and is easily applied.

The following example will illustrate the method of applying this check:

4836	3 (excess of nines in 4836)
285	6 (excess of nines in 285)
24180	
38688	
9672	
1378260	18 = Product of the excesses.

Excess in the product (1378260) is 0, which is also the excess in product of the excesses.

Checking division.

Division may be checked by determining whether the result obtained by multiplying the divisor by the quotient and adding the remainder, if any, to this product equals the dividend.

Checking reasoning problems.

Reasoning problems may be checked in different ways. The pupil may be taught to solve a problem in two possible ways, and then compare the results. Instead of using the unitary method, for example, he may be able to use proportion etc. The pupil may also begin the solution of the problem backwards, i.e., beginning with what is to be found until he reaches in the solution what was given, and so, he can compare the figures and find out whether the problem is correct or wrong.

Checking, in general whether the mechanical or the reasoning phase of the subject, in an occasional way is of small value. The pupils should build the habit of getting correct results and should also know that the answers they have got are correct.

PART III

DIAGNOSTIC TESTING

CHAPTER SIX

Construction of Tests.

A. General Principles of Test Construction.

1. Preparation of the test material.

In preparing the test, the teacher or test maker should always keep in mind, among other things, one underlying principle. This principle is the selection of that particular material which, as a matter of fact, measures the things one sets out to measure. This is known as the validity of the material which is more basic than anything else along the line of test construction. Validity of the test will be treated fully in a later paragraph. (See page 51).

In the meantime, enough provisions should be made which help to measure most if not all the important objectives of that subject for which the test is especially made. To be able to do that, some teachers follow the habit of jotting down items to be included in the test day by day as they proceed along the subject. It seems to me that this is a good policy to follow since no important point in the course will be omitted in the test.

When the material is thus ready, it should be moulded in more than one type of question, so as to have the test be more interesting and less monotonous, specially if it is a long one. The preliminary draft of the test should include more items than will be needed in the final form. In this connection, Ruch suggests that from 25 to 50% more items be prepared than are likely required.

When the material of this draft is organized, somehow, it should be submitted to the criticisms of teachers of the same subject. As a result, all items which are of doubtful importance, or not clearly stated, or other items which are ambiguous in their wordings, all of these unreasonable obstacles in the pupils' ways will be avoided. On the other hand, the inclusion of clues is as bad because obstacles as well as clues defeat the purpose for which the test is made.

There is another intention of the submission of the test than criticism. All items of a particular type will be placed together in an order of ascending difficulty. This arrangement results in an easy way of scoring the test, it enables the pupils to take full advantage of a particular mind-set at a time, and it will have a wholesome effect upon the moral of the pupils. Placing very difficult items at the beginning, without doubt, produces discouragement in the pupils, specially those who are average or below.

A regular sequence of responses is another essential thing to be avoided in test construction. It should rather be a chance order rather than otherwise. If items are arranged alternately true and false, or two true and two false, the pupils are apt to discover the arrangement and respond accordingly.

Space for the pupils' responses should also be provided for. Specially so, when pupils are asked to number the responses in multiple - choice items, or, to fill in blanks in completion items. Space for such responses should be arranged in column form rather than scattered all over the page in an irregular way. This, without doubt, helps to save time and produce the danger of error of scoring.

This is a brief description of the process of the preparation and organization of the test material proper. The following item will include a discussion about construction of the test instructions.

2. Preparation of test instructions.

Test instructions, or as they are sometimes called "directions" form a vital and an indispensable part of the test. Neither the examiner nor the examined will be able to tell exactly what to do or how to proceed in administering or taking the test without its particular instructions.

The directions to the pupil should be clear, concise and complete. The major aim is to make the weakest pupil in the class know what he is expected to do, regardless of the fact as to whether he is able to do or not. On the other hand, too long test instructions will cause confusion to both examiner and examinee. McCall says that it is well to remember that the primary function of the instruction is to give a pupil adequate, but not necessarily complete information about the test. Patterson said, "Test, don't teach".

Some sort of a demonstration and a preliminary test is sometimes necessary as clarifying means to students, specially if the form of the test is unfamiliar or complicated. It is said that an ounce of demonstration is worth a pound of words. Children are more able to imitate than to follow directions.

Finally, if need be, a blackboard demonstration is sometimes permissible if the test is unfamiliar or complicated.

The general order of instructions should be the order of doing. The following directions may be considered satisfactory for a class unfamiliar with objective tests:

<u>Directions</u>: Below are thirty statements about measurement in arithmetic. ~~Examine each~~ statement and decide whether it is true or false. Draw the circle around the letter T. before each statement you think is true, or around the letter F. before each statement you think is false. Your score will be the number right minus the number wrong. Study the samples below, they are answered correctly.

<u>Samples</u>:

[✓ T.] F. The area of a square is equal to the square of its side.
T. [✓ F.] There are 100 cu. cm. in a cubic dec.

These directions will be shortened when the student becomes familiar with true-false tests. This holds true in other types of objective tests.

3. Scaling the test.

There are different methods of scaling tests. The following are some of the well known and mostly used: The percentile scale, Age Scale, Grade scale, Product scale and T-scale. The following materials describe briefly the method of constructing the T-scale:

a. Scale the individual questions.

After the test has been prepared as such, it is ready to be given a trial in actual use. Since it is impossible in advance to know exactly how good the test is or to locate all the poor items, the try-out should be considered a necessary step in constructing the test in its final form. The test should be applied to a few hundred pupils so as to yield a rough approximation to a normal frequency distribution. The answers should be considered either right or wrong. All questions which prove to be ambiguous, unscorable, overlapping etc. should be eliminated. The percentage of correct answer should be calculated. From prepared tables in statistics this percentage can be changed into S.D. value. This leads to a re-arrangement of the questions in order of actual difficulty. The material thus selected and arranged should be printed along with instructions, and other advice, in its final booklet form.

b. Scaling the total number of questions right.

The total number of questions which are answered correctly is thus determined. The next step is to compute the percent of all the pupils exceeding plus half those reaching all total number of test elements correct. Finally, these percents should be converted into scale scores by the use of tables after which the scale will be finished. This leads directly to the determination of age norms.

Determination of age norms.

When we know the number of pupils of each age answering correctly a defined number of questions, and at the same time, we know the scale score for each question, all that one has to do is to multiply each score by its frequency and divide by the number of cases in order to get the norms for that particular age.

Determination of grade norms.

In a similar method, but on a different basis, that is putting together all pupils of the same grade and not the same age, as has been done before, we shall be able to determine the norms for the different grades. This completes in full "scaling the test".

B. Characteristics of Good Tests.

To begin with, some criteria should be used for the choice of good tests. There are at least 5 indispensable ones in any satisfactory measuring instrument. These are: Validity, Reliability, Usability, Norms and Cost. There are in addition several minor characteristics which will be mentioned in a summary form at the end of this section. We shall now consider each of the above mentioned criteria very briefly.

1. Validity

The examiner should choose a test because it is valid. The first concern of one who wishes to choose a standardized test is to make sure that the test really measures what it claims to measure. For example, a reasoning test in arithmetic is valid if it really succeeds in measuring reasoning ability in arithmetic rather than other things such as reading ability and the like. If a test lacks validity it is worthless.

Validity can be measured or determined by the correlation of scores on the test with some independent criteria of the school subject in question, such as school marks or teachers' estimates etc.

2. Reliability.

The examiner should choose a test because it is reliable. Reliability means the extent to which a test gives the same results when repeated. In a word, reliability means consistency. One way of measuring the reliability of a test is by the correlation of scores of two different trials of the same test. If the correlation is high the test is considered to be reliable, otherwise, it is useless. There are also two other ways, namely, use of parallel forms of the test and split-half method.

3. Usability.

This term means practicability, i.e., the degree to which the test can be used by classroom teachers with the least amount possible of expenditure of time and energy. There are many factors which determine usability of a test. Ease of administration ranks first. It is said that examination should be easy to give, easy to take, and easy to score. As far as the first two qualities are concerned, enough has been said all through the chapter, but a word about scoring needs to be mentioned here. A test should not be primarily selected because it is easy to score, but a good test ought to be as easy to score as possible. There should be a key for scoring. The key should be printed as near to the margin of the paper as possible so that it can be brought close enough to the answers of the pupils. The key should also have the same spacing as that of the material.

4. Norms.

A standardized test without norms is practically useless, because norms are very essential for the interpretation of the results of the test. They can also be used for comparison purposes, so that the scores of individual pupils, or of a class, or of an entire school, may be compared with the average scores attained by other pupils representing various ages and school grades.

5. Cost.

It is true that the purchaser of a test is more concerned with what he gets with his money than with what he has to pay, yet he should not pay for tests more than necessary. This is not to mention anything about the type of print which should be very clear and finally the quality of pictures and illustrations used. (1)

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(1) The following material gives a clear idea about good characteristics of standardized tests put in an outline form:

SCALE FOR RATING STANDARDIZED TESTS

I. Preliminary Information.
  1. Exact name of test
  2. Name of publisher
  3. Cost
  4. Date of copyright
  5. Purpose of test

Remark: The above scale is adapted from N. L. Bossing, "Progressive Methods of Teaching in Secondary Schools", P.728-729

CHAPTER SEVEN

MEASUREMENT IN ARITHMETIC

A. Need for Measurement in Arithmetic.

If it is true that education is growth, it naturally follows that this growth brings about changes in the abilities of the learners. The pupil may increase the speed and accuracy with which he is able to add number facts or the reasoning ability in the solution of verbal problems. To prove that this increase has taken place, it is necessary to compare the pupil's ability before he was taught, with his ability after being taught. This comparison of results is one kind of measurement. The teacher who corrects and grades a number of arithmetic papers is also doing some measurement.

E. L. Thorndike says that education is one form of human engineering and will profit by measurements of human nature and achievement, as mechanical and electrical engineering have profited by using the foot, pound, volt, and ampere.

If measurements are necessary in the field of education, what are some of the purposes for which they are used?

B. Purposes of Measurements.

Measurements are being used in arithmetic, according to Symond, as in any other school subject, for the following purposes:-

1. To inform pupils of their achievements.
2. To promote competition between groups, individuals and with one's past record.
3. To determine promotion, demotion and classification of new students.
4. To provide reports to parents.
5. To diagnose weak points in the pupils' achievements.
6. To determine the quality of arithmetic instruction of the school.

Thus, we see that the value of measurement is indispensable in the field of arithmetic, and it is no longer a matter of doubt.

II. Validity.
  1. Curricular
    a. Exact field which test measures
    b. Ages and grades for which intended
    c. Criteria with which material was correlated
  2. Statistical
    a. Size of correlation coefficient & probable error.
    b. Size and representativeness of sampling.
    c. Proof of validity of items used in test.

III. Reliability.
  1. Reliability coefficient.
  2. How was reliability established.

IV. Ease of Administration.
  1. Manual of directions
  2. Simplicity of directions

C. Methods of Measurement.

1. Traditional Method.

There are two types of the traditional method, which have been employed for measurement. The first type is the estimation of the capacity of the pupils by the teachers. This method is nothing more nor less than an off-hand judgement. According to this method, some pupils have been classified as "bright", others as "average" and still others as "stupid". The second type is the traditional examination. According to this method the results are being expressed by means of grades, such as, 75, 80, 81, etc., or by means of letters, such as, A, B, C, etc.

These two types give relative rather than an exact measure of the pupil's ability. They have been employed since school began, and are still in use, mostly the second, in this part of the world.

a. Defect of the system.

Many a student in our schools is considered a failure if the result of his work shows that he is a few points, or in some instances a fraction of a point, below the "passing mark"; otherwise, he will be promoted.

It is a well known fact that some teachers give a larger percent of "high grades" than other teachers do. Moreover, a whole department in a big institution becomes well-known, because of the "high grades" or "low grades" which are given to the students. It is commonly understood that such "high grades" are the result of superior achievements and that "low grades" are the result of poor achievements. But in some cases it may be that it is only because the teacher likes to boast that all members of his class have made very high grades, or he merely intends to give "high" or "low" grades, or he will be afraid of being considered a poor instructor if his pupils get "low grades".

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a. Amount of explanation needed for pupils by examiner.
b. Are directions clear and simple ?
c. Is arrangement of test convenient for pupils ?
d. Are samples and fore-exercises given when needed?
3. Alternate forms.
a. Number
b. Evidence of equivalency
4. Time needed for giving.
V. Ease of Scoring.
1. Degree of objectivity.
2. Are adequate directions given?
3. Is scoring key adjusted to size of test?
4. Time needed for scoring.
VI. Ease of Interpretation.
1. Norms - age, grade percentile, etc.
2. Are there provisions for graphing results?
3. Is interpretation of raw scores easy or hard?
4. Are directions and suggestions given for application of results.
5. Are tests survey or diagnostic?
a. If diagnostic - of what?
b. Is a remedial program provided?
VII. Miscellaneous.
Typography, legibility, cost, service and kind of new type question used.

b. Unreliability of the marking system.

We conclude from the previous discussion that the marking of examination papers is subjective. The following investigation by Starch and Elliott is a typical one along this line: A final examination paper in geometry, written by a student in one of the largest schools in Wisconsin, was reproduced and a set of the questions were sent to one hundred and sixteen high schools in the North Central Association. It was requested that this paper be graded according to the practice and standards of the school by the principal teacher of mathematics. It is strange to know that the grades differed so much that one might think that some of the papers were corrected with no care or attention whatsoever. The following are some of the results: out of the 116 marks two were above 90, while one was below 30. Twenty were 80 and above, while twenty were below 60. Forty-seven teachers assigned a mark passing or above, while sixty-nine teachers considered the papers not worthy of a passing mark.

Even more than that, it is said that the same paper corrected by the same person at different times was graded differently.

This unreliability is more true of marks in the case of languages and social sciences.

c. Factors producing the variability of marks.

Authorities in the field of measurement give four major factors which account for such wide range of differences. These factors are:
1. Differences among the standards of different schools.
2. Differences among the standards of different teachers.
3. Differences in the relative values placed by different teachers upon various elements in a paper.
4. Differences due to the inability to distinguish between closely allied degrees of merit.

2. Modern Methods.

a. Standardized tests.

Foote says that the day of guess work must give way to definite facts. There, thus came a time, when the traditional written examination was considered as an imperfect measuring instrument. Consequently, a need was felt for a refined, definite, and precise instrument which would help to do away with all the limitations of the traditional method of measuring the abilities of students, and to eliminate or reduce to a minimum the defects of written examinations.

The name given to these improved measuring instruments is "Standardized Tests".

Standardized tests in arithmetic may be divided into two groups: the formal tests for computation and the tests for ability to solve-problems.

(1) Formal Tests.

Samples of the more important of the available standardized tests (not in this part of the world) for measuring abilities in the operations of arithmetic are: (1)

(a) The Courtis Standard Research Tests. (2)
(b) The Cleveland-Survey Arithmetic Tests.

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(1) Ref. W. S. Monroe and J. C. DeVoss: "Educational Tests and Measurements", pp. 26-68.
(2) Ref. appendix pp. 104-109

- (c) Monroe's General Survey Scales in Arithmetic.
- (d) The Woody Arithmetic Scales.
- (e) The Luceford Diagnostic Tests in Addition.
- (f) Monroe's Diagnostic Tests in Arithmetic.

(2) Tests for Measuring the ability to solve problems.

- (a) Monroe's Standardized Measuring Test in Arithmetic.
- (b) Buckingham's Scale for Problems in Arithmetic.
- (c) Stone Reasoning Tests.

It is not intended to describe every one of the above mentioned tests, but, as I have made use of the Courtis Standardized Research Tests, a word concerning these will be mentioned on page.71. Only items 2 and 6 of the "Instructions to Examiners", by Courtis, will be quoted hereafter, as I have found them to be most interesting and instructive. (1)

Item 2. Begin by saying, "My purpose this morning is to measure how well this school teaches its children how to add, subtract, multiply, and divide. I have here some printed tests. They are not examinations, because exactly these same tests are given to all the grades from the third through high school. They are also being given in other schools in this city, and in other cities all over the country. It is the school that is being examined to-day. If you treat the tests as though they were a game, you will enjoy them and do your best for the honor of your school. I am going to give each of you a set of these papers, but do not look at them until I tell you to do so. Will the boys and girls in the front seats please distribute them for me?"

Item 6. "Now please listen closely. In these tests it is important that we all start at the same time and stop at the same time. We can do this easily, if you follow my instructions exactly. Lay your papers on your desks in position to work the examples, but close the cover with your left hand, keeping it between your thumb and finger, like this (illustrate), so that you can open it quickly when I tell you to start. Take your pencil in your right hand, and when I say 'Get ready', raise your pencil hand in the air as if you were going to ask a question. (Illustrate, by suiting the action to the words.) Then when I say 'Start', you can bring your pencil down as you turn the cover back, and every one will start at the same time. When I say 'Stop', I want you all to stop at once, and to raise your hands again so that I can see that you have stopped. Now I think we are ready to try the test."

When the second hand of the watch reaches the 55-second mark say "Get ready for the addition test. Hands up." Exactly at the 60 mark say "Start".

In addition to the above mentioned, the teacher should be, at least, acquainted with some sample tests. He should also have a comprehensive understanding of their nature and use, manuals of directions, scoring keys and record forms.

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(1) Adapted by G.M. Wilson and K. J. Hoke, "How to Measure", pp. 6 2-63.

b. Informal, or Teacher-Created Tests.

Informal Tests are also known as teacher-created objective tests. The lack of a full adaptability of the standardized tests to all phases of the school curriculum, on the one hand, and the non-objectivity of the essay examinations, as has been mentioned above, caused a felt need for the teacher-created objective tests.

These tests should represent the objectivity of the standardized tests without being so much refined and standardized. A good teacher who teaches the same course over and over again should be prepared to construct informal objective tests for use in his own class.

Items of the test should be turned into true-false, multiple choice, completion or matching forms, etc. These are the most common used types of objective tests. Responses should be indicated by single word, or underscoring, or by a circle, or by similar brief devides.

This is a brief discussion which, of course, does not cover completely all that can be said concerning this subject, but it covers more or less, its important aspects. Entire volumes have been written concerning the subject.

I have prepared the following four series of tests. They are supposed to be of the informal diagnostic type:

1. To measure the ability in number facts---Series A.
2. Series B are designed to measure work in the four fundamental operations.
3. Series C are of a general nature.
4. Series D measure ability in reasoning.

Series A - Number facts.

Add the following number facts as quickly as possible:

2	2	7	3	8	1	2	9	7	9	1	3	6	0	5	2	4	5
2	3	0	6	2	6	0	0	9	5	3	7	3	1	2	5	8	6
9	9	4	0	2	1	7	5	8	0	8	4	1	3	9	4	6	4
4	7	4	5	6	0	2	5	4	3	7	9	4	0	2	3	1	6
5	6	3	8	4	2	5	1	9	2	0	2	3	6	8	1	7	3
9	6	9	5	5	4	4	2	1	1	9	8	8	9	1	1	7	2
8	4	0	6	9	6	4	1	9	8	3	4	4	9	6	9	0	3
3	0	0	7	8	8	1	8	9	0	4	2	7	6	4	3	2	3
7	5	7	3	1	5	7	8	6	5	0	5	7	0	6	2	6	5
1	1	4	5	5	7	8	6	0	0	8	3	3	6	2	9	5	8
1	1	0	3	0	2	8	7	8	7								
7	9	4	1	7	7	8	6	9	5								

Subtract the following number facts as quickly as possible:

3	4	7	7	10	10	9	12	11	17	4	7	8	10	5	9
2	1	1	7	1	7	0	4	6	8	3	5	7	9	2	4
11	11	14	11	5	6	7	6	8	9	7	12	13	14	2	3
2	7	6	3	4	5	4	3	5	7	2	7	5	9	1	3
7	8	9	9	14	10	15	15	4	5	5	8	10	5	12	11
6	4	6	2	7	6	6	9	2	3	5	2	3	0	3	5
13	13	3	2	9	10	9	10	11	8	17	15	0	9	8	8
9	4	1	2	5	4	3	8	4	0	9	8	0	1	3	1
10	3	2	6	14	14	5	6	8	7	8	12	12	15	11	13
2	0	0	0	5	8	1	4	6	3	8	9	6	7	9	7
1	4	6	6	10	7	11	13	13	16	6	9	9	1	4	12
1	4	6	1	5	0	8	6	8	7	2	8	9	0	0	8
16	18	12	16												
8	9	5	9												

Multiply the following number facts as quickly as possible:

5	3	9	3	3	7	9	7	8	9	2	2	5	5	2	3	8	7
1	1	2	5	6	3	3	9	7	0	2	3	2	7	1	7	9	7
6	7	1	2	2	2	8	3	4	9	0	0	1	1	2	9	5	6
9	0	7	8	4	6	1	9	7	4	4	9	5	8	9	9	8	7
4	4	0	2	1	7	5	3	8	1	9	9	0	5	1	6	4	4
4	9	8	0	3	1	4	4	5	1	8	5	1	0	2	1	1	5
8	8	7	7	0	3	1	9	1	4	5	7	6	7	0	0	2	6
3	4	6	5	2	0	6	1	9	2	9	4	8	8	6	3	5	5
5	6	6	6	5	9	6	0	7	3	2	8	6	4	4	8	0	8
3	2	4	6	5	7	0	5	2	2	7	2	3	6	8	8	7	0
1	0	4	3	5	3	8	9	1	4								
4	0	3	3	6	8	6	6	0	0								

Divide the following number facts as quickly as possible:

2/4 2/8 4/24 1/7 4/16 4/20 1/3 4/32 9/9 5/25

9/72 3/12 5/40 1/5 6/24 4/4 8/8 0/7 2/14 8/72

5/30 7/63 5/35 6/54 6/30 5/5 4/28 3/15 6/36

7/14 7/21 8/64 9/18 1/9 8/56 6/0 2/12 8/40

9/81 4/12 8/32 9/54 7/56 3/3 4/0 5/20 3/24

7/49 1/6 5/45 9/45 7/35 8/0 2/18 2/10 3/21

8/24 9/27 8/48 1/8 3/27 2/0 1/1 2/6 6/18

6/12 7/28 9/63 8/16 9/0 1/0 2/2 4/8 5/15

6/48 9/36 1/4 3/18 5/0 6/6 3/0 3/9 2/16

5/10 7/42 3/6 6/42 7/7 4/36 1/2

Series B - processes.

Addition.

The following are 16 column addition examples containing the basic and the secondary addition combinations, from 0 + 0 to 49 + 9. Each secondary combination comes only once. They are also arranged in an order of ascending difficulty. I gave these examples to 211 pupils, from the second to the sixth grade inclusive. They range from 65.87 to 35.54 right percent.

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
400	570	830	431	42	180	420	538
714	119	980	512	27	857	836	309
700	671	132	506	97	441	470	649
716	220	306	645	89	732	306	260
522	129	253	723	58	723	583	402
160	702	404	502	17	309	441	537
411	535	643	365	65	211	215	126
322	246	201	414	31	170	250	142
920	873	769	993	97	573	996	928

(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)
457	209	990	919	290	420	230	291
602	359	181	848	558	623	885	985
432	446	973	876	987	660	781	819
756	167	567	311	675	895	794	987
305	548	403	136	736	685	007	609
486	482	144	304	708	807	314	154
719	263	475	551	460	228	599	440
830	304	526	622	226	255	808	353
986	695	983	309	589	268	742	963
					796		

Subtraction

The following are 40 examples in subtraction containing all subtraction combinations. Each of these combinations comes only once. They are arranged in an order of ascending difficulty. I gave these examples to 212 pupils, from the second to the sixth grade inclusive. They range from 90.09 to 58.98 correct percent.

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
28832	98204	92541	84334	55740	98455	98732	65421
8407	57604	32055	32456	15692	94772	46784	20346

(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)
36586	41668	57513	99230	85346	52435	43805	51224
32557	11569	39691	3736	23283	13884	32018	24992

(17)	(18)	(19)	(20)	(21)	(22)	(23)	(24)
81425	42666	75027	79710	94619	67036	68367	73211
35426	5309	34506	2603	46827	63844	19778	50123

(25)	(26)	(27)	(28)	(29)	(30)	(31)	(32)
99118	18607	90217	29089	91370	70372	88354	71685
15341	2947	76871	16856	26681	23508	71999	49859

(33)	(34)	(35)	(36)	(37)	(38)	(39)	(40)
67950	64407	90139	74102	30018	89803	71236	90576
5977	47194	87878	20123	14586	1805	17691	9898

Multiplication

The following are 96 examples in multiplication containing all the multiplication combinations. Each of these combinations comes only once, particularly with carrying. They are arranged in an order of ascending difficulty. I gave these to 184 students, from the third to the sixth grade. They range from 97.28 to 2.71 right percent.

1) 84 5	2) 69 6	3) 33 9	4) 829 5	5) 39 2	6) 908 8	7) 25 38
8) 115 7	9) 13 14	10) 99 3	11) 58 7	12) 904 9	13) 154 6	14) 94 6
15) 64 9	16) 55 9	17) 964 3	18) 28 7	19) 29 7	20) 456 90	21) 99 9
22) 650 75	23) 805 83	24) 39 7	25) 33 28	26) 35 29	27) 823 8	28) 863 9
29) 29 62	30) 746 7	31) 69 45	32) 678 91	33) 72 56	34) 502 68	35) 560 780
36) 879 7	37) 84 69	38) 548 517	39) 987 6	40) 604 650	41) 762 59	42) 768 8
43) 9107 59	44) 344 16	45) 509 73	46) 74 89	47) 705 197	48) 214 24	49) 5785 23
50) 87 98	51) 8402 71	52) 785 7	53) 57 89	54) 542 327	55) 8064 46	56) 9508 25
57) 4030 495	58) 907 708	59) 829 235	60) 24 69	61) 854 98	62) 5008 907	63) 9014 96
64) 728 79	65) 906 90	66) 634 807	67) 844 48	68) 3908 64	69) 819 689	70) 3436 95
71) 967 15	72) 9523 97	73) 367 369	74) 42009 896	75) 80104 708	76) 977 487	77) 2775 98
78) 3617 346	79) 12325 465	80) 4316 789	81) 2938 98	82) 9218 789	83) 4629 132	84) 45653 645

85) 8619
457

86) 6736
872

87) 80312
238

88) 17147
598

89) 9693
798

90) 59278
654

91) 3988
985

92) 97489
436

93) 6589
789

94) 82649
789

95) 24513
897

96) 47958
968

Short Division

The following are 85 examples in short division containing all division combinations. Each of these combinations comes only once. These examples are arranged in an order of ascending difficulty. I gave them to 165 students, from the fourth to the sixth grade inclusive. They range from 98.1 to 87.9 correct percent.

1) 65436 ÷ 2
2) 203468 ÷ 8
3) 40263 ÷ 3
4) 929936 ÷ 8
5) 177660 ÷ 5
6) 23645 ÷ 5
7) 411820 ÷ 5
8) 516584 ÷ 8
9) 948760 ÷ 8
10) 130176 ÷ 8
11) 37500 ÷ 4
12) 170408 ÷ 8
13) 11370 ÷ 2
14) 627952 ÷ 8
15) 300424 ÷ 8
16) 192078 ÷ 9
17) 42958 ÷ 2
18) 225184 ÷ 8
19) 554336 ÷ 8
20) 890892 ÷ 9
21) 20208 ÷ 6
22) 763410 ÷ 6
23) 79208 ÷ 2
24) 925044 ÷ 6
25) 459474 ÷ 6
26) 91344 ÷ 4
27) 379512 ÷ 8
28) 293856 ÷ 4
29) 150376 ÷ 8
30) 334458 ÷ 9
31) 236958 ÷ 6
32) 290817 ÷ 9
33) 597864 ÷ 6
34) 237195 ÷ 9
35) 165951 ÷ 9
36) 346962 ÷ 7
37) 890533 ÷ 7
38) 517266 ÷ 9
39) 257265 ÷ 9
40) 710736 ÷ 6
41) 74838 ÷ 3
42) 692875 ÷ 5
43) 59604 ÷ 3
44) 994890 ÷ 5
45) 410212 ÷ 4
46) 111489 ÷ 7
47) 557816 ÷ 7
48) 72505 ÷ 5
49) 467253 ÷ 9
50) 791896 ÷ 8
51) 919326 ÷ 6
52) 16917 ÷ 3
53) 172956 ÷ 7
54) 579432 ÷ 4
55) 743820 ÷ 5
56) 430857 ÷ 9
57) 566361 ÷ 9
58) 265872 ÷ 8
59) 23109 ÷ 3
60) 613216 ÷ 8
61) 312135 ÷ 5
62) 577332 ÷ 7
63) 692895 ÷ 7
64) 821172 ÷ 6
65) 138825 ÷ 9
66) 847965 ÷ 5
67) 410877 ÷ 9
68) 226429 ÷ 7
69) 867868 ÷ 4
70) 795160 ÷ 4
71) 82755 ÷ 3
72) 625446 ÷ 9
73) 962871 ÷ 7
74) 478758 ÷ 7
75) 992411 ÷ 7
76) 717363 ÷ 9
77) 609894 ÷ 9
78) 904032 ÷ 9
79) 357888 ÷ 6
80) 3103506 ÷ 7
81) 200564 ÷ 7
82) 466944 ÷ 8
83) 744138 ÷ 9
84) 212508 ÷ 9
85) 7445305 ÷ 7

Long Division

The following are 30 examples in long division containing most of the skills, if not all, which the pupils should know in this process. They are arranged in an order of ascending difficulty. I gave them to 165 students, from the fourth to the sixth grade inclusive. They range from 94.8 to 49 correct percent.

1) 95 ÷ 19
2) 384 ÷ 96
3) 701 ÷ 80
4) 8500 ÷ 17
5) 83 ÷ 19
6) 136 ÷ 17
7) 548 ÷ 69
8) 840 ÷ 21
9) 5936 ÷ 53
10) 5792 ÷ 25
11) 14623 ÷ 423
12) 7800 ÷ 25
13) 774231 ÷ 387
14) 569837 ÷ 518
15) 8970 ÷ 41
16) 3956 ÷ 43
17) 1266 ÷ 63
18) 712 ÷ 34
19) 967 ÷ 16
20) 1000 ÷ 26
21) 7392 ÷ 46
22) 259342 ÷ 263
23) 557389 ÷ 516
24) 83795 ÷ 341
25) 70627 ÷ 617
26) 4147 ÷ 25
27) 417643 ÷ 347
28) 7616 ÷ 32
29) 170472 ÷ 416
30) 65404 ÷ 387

Series C - General nature.

This series contains different items pertaining to the language of arithmetic, tables of measurement and general principles. I have arranged these items into three main categories. The first is of the true-false type, the second is of the completion type and the third is of the matching type.

In order to ascertain the reliability of these tests, I have tried them on the upper three grades of the University Elementary School. Coefficients of correlation, which I have got, by using the split-half method and the product moment method for correlation, will be given at the end of this item.

Part I - True-False Test.

Directions: Each of the following statements is either true or false. If the statement is true, encircle the T in front of it; if it is false, encircle the F. If you do not know which is the correct answer, skip the item, and continue with the next one. Do not guess. Work as rapidly and as accurately as you can.

T F 1. A meter is used to weigh things.
T F 2. Gain = selling price - cost price.
T F 3. One hektoliter = 10 liters.
T F 4. All sides of a rectangle are equal.
T F 5. 25.18 x 100 = 0.2518.
T F 6. Loss equals cost price - selling price.
T F 7. The decimal point is placed under the decimal point in the addition of decimal fractions.
T F 8. One kilogram equals 100 hektograms.
T F 9. All sides of a square are equal.
T F 10. A meter equals 100 decimeters.
T F 11. The perimeter of a square equals one side multiplied by itself.
T F 12. The decimal point is not placed under the decimal point in the decimal subtraction.
T F 13. One kilometer equals 100 dekameter.
T F 14. The area of a rectangle equals length x width.
T F 15. One hektogram equals 100 dekagram.
T F 16. The perimeter of a square ÷ 4 equals the side of the square.
T F 17. One kiloliter equals 100 dekaliter.
T F 18. The area of a rectangle ÷ its length equals its width.
T F 19. A square meter is used to measure areas.
T F 20. The decimal point should be placed under the decimal point in the decimal multiplication.
T F 21. The liter is used to measure capacity.
T F 22. The hektoliter equals 10 dekaliters.
T F 23. 25.18 x 100 equals 2518.
T F 24. The gram is used to measure volume.
T F 25. The dekaliter equals 100 liter.
T F 26. The decimal point of the divisor should be done away with before dividing.
T F 27. The cubic meter is used to measure volume.
T F 28. A square meter equals 100 square decimeters.
T F 29. The volume of a cube equals the product of its side by itself three times.
T F 30. The Ar equals one square meter.
T F 31. The area of a cube equals the product of its side by itself x 4.
T F 32. The perimeter of a triangle equals the sum of its sides.
T F 33. The denominator of the common fraction is the number which is above the line.

T F 34. The square hektometer equals 10 square dekameters.
T F 35. The area of a parallelogram ÷ its altitude equals its base.
T F 36. The hektar equals one square hektometer.
T F 37. The square kilometer equals 1000 square hektometers.
T F 38. The area of a triangle is equal ~~to~~ the base x altitude.
T F 39. The square meter equals 10,000 square centimeters.
T F 40. The centar is a square dekaliter.
T F 41. The square decimeter equals 100 square centimeters.
T F 42. The volume of one cubic decimeter equals one liter.
T F 43. 3/4 + 2/5 = 5/9
T F 44. To divide one fraction by another, we multiply that fraction by the reverse of the other fraction.
T F 45. One liter of water weighs one gram.
T F 46. Rate equals distance ÷ time.
T F 47. The percent of profit or loss is based on the selling price and not on the cost price.
T F 48. When the numerator and denominator are both divided or multiplied by the same number, the value of the fraction does not change.
T F 49. 3/8 - 2/6 = 1/2.
T F 50. A common fraction cannot be changed to a decimal fraction.
T F 51. A cubic decimeter of water weighs one kilogram.
T F 52. A decimal fraction can be changed into a common fraction.
T F 53. The area of a triangle ÷ 1/2 base equals altitude.
T F 54. The yard equals 91.44 centimeters.
T F 55. The area of a parallelogram ÷ 1/2 base equals altitude.
T F 56. The meter equals 39.37 kirats.
T F 57. 5/8 x 3/8 = 15/8
T F 58. The okiyyah equals 213 2/3 grams.
T F 59. One cubic meter of water weighs one tonne.
T F 60. 20 shillings equal one sterling.
T F 61. The perimeter of a circle equals $1/_2$ diameter x 3.1416.
T F 62. The sign % is used in percentage.
T F 63. The area of a parallelogram ÷ 1/2 base equals altitude.
T F 64. 100 kilograms equal 39 ratles.
T F 65. The denominator is the number which is under the line in a common fraction.
T F 66. The circumference ÷ 3.1416 equals diameter.
T F 67. The area of a trapezoid ÷ altitude equals the sum of the bases.
T F 68. Area of a trapezoid ÷ 1/2 sum of two bases equals altitude.
T F 69. 5200 feet equal one mile.
T F 70. Interest equals principle x rate/100.
T F 71. The volume of a sphere equals radius squared x 3.1416 x 4/3.
T F 72. The mason dra' equals 68 centimeters.
T F 73. The ratle equals 2.564 kilograms.
T F 74. Specific gravity equals weight - volume.
T F 75. The commercial dra' equals 75 centimeters.

Grade	Coefficients of correlation.
IV	0.69
V	0.74
VI	0.68

The following numbers have been omitted from the fifth grade test: 54-56, 58, 60, 64, 67-69, 71-75.
The following numbers were also omitted from the fourth grade test: 29-31, 33, 35-36, 38, 40, 43-44, 47-50, 52-75.

Part II - Completion Test.

Directions: Write the answer in the blank that you think best answers the question.

1. The answer of an addition example is called ...
2. The sign of equality is ...
3. The answer of a division example is called ...
4. The sign of multiplication is ...
5. The answer of a multiplication example is called ...
6. The sign of addition is ...
7. The answer of a subtraction example is called ...
8. The sign of division is ...
9. The sign of subtraction is ...
10. The upper number of subtraction is called ...
11. The lower number of subtraction is called ...
12. The upper number in multiplication is called ...
13. The lower number in multiplication is called ...
14. What do you call the first number in the following example, $37 \div 3 = 12\frac{1}{3}$
15. What do you call the 3 in No. 14.
16. What do you call the 12 in No. 14.
17. What do you call the 1 in No. 14.
18. The kiloliter equals ... liter.
19. The area of a rectangle ÷ width equals ...
20. A square kilometer equals ... square dekameters.
21. A square kilometer equals ... square meters.
22. Kilogram equals ... grams.
23. The hektogram equals ... grams.
24. The decimeter equals ... milimeters.
25. The kilogram equals ... dekagram.
26. The decigram equals ... centigram.
27. The hektogram equals ... dekagrams.
28. The dekagram equals ... grams.
29. A square dekameter equals ... square meters.
30. The gram equals ... centigrams.
31. The decimeter equals ... milimeter.
32. The dekameter equals ... meter.
33. The cubic meter of water equals .... kilograms.
34. The cubic meter equals .... cubic centimeters.
35. The centimeter equals ... milimeter.
36. The meter equals ... milimeters.
37. The hektometer equals ... dekameters.
38. The decigram equals ... miligram.
39. The liter equals ... centiliters.
40. The square hektometer equals ... square meters.
41. The centigram equals ... miligrams.
42. The liter equals .... centiliters.
43. The kilometer equals ... hektometers.
44. The gram equals ... miligrams.
45. The liter equals ... deciliters.
46. The decimeter equals ... centimeters.
47. The deciliter equals ... centiliters.
48. The meter equals ... centimeters.
49. The gram equals ... decigrams.
50. The centiliter equals ... milliliters.
51. This quadrilateral ▭ is called .....
52. This quadrilateral □ is called .....
53. This shape △ is called .....
54. This shape ∟ is called ....
55. This shape ○ is called ....
56. This shape ▱ is called ....

57. This shape  is called ....
58. 7/8 is a ..... fraction.
59. 0.77 is a ..... graction.

60. $\frac{\frac{5}{6} + \frac{3}{7}}{1 - 2\ 1/2}$ is a..... fraction.

Grade	Coefficients of correlation.
IV	0.80
V	0.79
VI	0.83

Part III - Matching Test.

Directions: In each exercise on the left, the unfinished statement is completed by one of the seventeen items on the right. You are to place the number of the item that completes the exercise correctly on the line at the left of the exercise number. Work as rapidly and as accurately as you can.

Exercise No. 1.

____ 1. Perimeter of a rectangle equals
____ 2. Area of a square equals
____ 3. A kilometer equals
____ 4. A hektometer equals
____ 5. A kiloliter equals
____ 6. A cubic meter equals
____ 7. A cubic decimeter equals
____ 8. A cubic centimeter equals
____ 9. A cubic dekameter equals
____ 10 A cubic centimeter of water weighs
____ 11 A cubic decimeter of water weighs
____ 12 A cubic hektoliter equals
____ 13 The volume of a cylinder equals

1. Gram
2. Base x altitude
3. Radius squared x 3.1417 x altitude.
4. 1000 cubic decimeters.
5. Mason dra'
6. 1,000,000 cubic meters.
7. Dimension x 4.
8. 1,000 cubic milimeter.
9. Length x width x height.
10. 1,000 cubic hektometer.
11. 1,000 cubic meter.
12. 1,000 cubic centimeters.
13. Kilogram
14. Dimension by itself.
15. 100 meters.
16. 1,000 meters.
17. The sum of dimensions.

Remark: The following numbers were omitted from the fourth grade test: 27, 33, 34, 38, 41, 44, 50, 56, 57, and 60.
The following number was omitted from the fifth grade test: 60.

Exercise No. 2.

____ 1. The lateral area of a cylinder equals
____ 2. The area of a triangle ÷ altitude x 2 gives
____ 3. 12 inches equal
____ 4. A cubic meter of water weighs
____ 5. The area of a circle equals
____ 6. The area of a trapezoid equals
____ 7. The weight ÷ specific gravity equals
____ 8. The whole area of a cylinder equals
____ 9. 1760 yards equal
____ 10. The area of a sphere equals
____ 11. 3 feet equal
____ 12. Volume x specific gravity equal

1. A yard
2. Foot
3. Weight ÷ specific gravity
4. One mile
5. One tonne.
6. Volume x specific gravity
7. Weight ÷ volume.
8. Hektoliter
9. Half the sum of the two bases x altitude.
10. The base.
11. r^2 x 3.1416 x altitude.
12. r^2 x 3.1416 .
13. diameter of the base x 3.1416 x altitude.
14. commercial dra'.
15. altitude x the sum of the two bases.
16. 1,000,000 cubic dekameter
17. kantar.

Grade	Coefficients of correlation.
V	0.71
VI	0.72

Series D - Reasoning.

The following eighteen problems form a scale for measuring the ability of pupils in reasoning. The scale begins with one-step problem and ends with a four-step problem. The problems which are between the first and the last increase in difficulty as one approaches the end of the scale. A student who can solve the first four problems is eligible for admission to the third grade. A knowledge of the second four problems, i.e., from 5 to 8 inclusive, entitles the student for admission to the fourth grade. Ability to solve the problems nos. 9 to 13 inclusive entitles the student for admission to the fifth grade. Similarly, ability to solve the problems nos. 14 to 18 inclusive entitles the student for admission to the sixth grade.

Solve as many of the following problems as quickly and accurately as you can:

1. Jamil distributed a sum of money among three of his friends, giving the first 428 pts. the second 532 pts, and the third 650 pts. Find the amount which he distributed.

2. A farmer had 460 kgs. of corn. If he sold a certain number of kgs. and kept for himself 62 kgs; how many kgs. did he sell?

3. If you read 98 pages of a book which contains 287 pages, how many pages remain to be read?

4. A kg. of flour costs 85 pts. Find the cost of 12 kgs.

5. Usama is 12 years old, and his father is 18 years older than he is. What is the father's age.

Remark: The following numbers were omitted from the fifth grade test: 3, 6, 7, 9-12 inclusive.

6. The speed of a car is 21 miles per hour. How many hours does it take to go 273 miles?

7. Selma wanted to save 25 pts. but she had already saved 13 pts., how many pts. remain to be saved?

8. If you have 25 pts. and your brother has 7 times as many, how many pts. does your brother have?

9. In order to buy a bicycle whose price is 145 pounds I need 56 pounds more than what I have. How many pounds do I have?

10. Kamel bought a number of arithmetic books for 731 pts. If he then sold the books for 50 pts. each with a profit of 7 pts. How many books did he buy.

11. A worker earns 1620 piasters in 36 days of work. How many days must he work to earn 64,400 piasters?

12. Amin and Jamil bought 32 kantals of potatoes for 19500 pts. How many piasters should Amin pay if his share is 15 kantals?

13. What is the price of each of two pieces of cloth of the same stuff, if the first is 12m long and the second is 10m long, and the price of the first piece is 50 piasters more than the price of the second?

14. If I bought a small box of oranges for 83 piasters, for how much should I sell the number of boxes which I bought for the sum of 1494 piasters, in order to make a profit of 15 piasters in every box?

15. A farmer sold his horse, and with its price he bought a cow for 2950 francs and two sheep for 135 francs each. How much did he pay for the horse, if he sold it at a profit of 850 francs?

16. A lamp-dealer sold 10 small lamps for £.Syr. 7 each, and 7 large lamps. He cashed a total sum of £.Syr. 154 for all. For how much did he sell the large lamp?

17. Find the total number of pen-nibs in 25 boxes, if 14 boxes contain 150 pen-nibs each, and the rest of the boxes 144 pen-nibs each?

18. A man bought 60 plates for 150 pts. each, but 4 of them were broken during handling. If he then sold the plates and made a profit of 388 pts., find the selling price of each plate.

General Characteristics.

These informal tests are still to a great extent "raw" so to speak. They are all meant to be diagnostic in their nature. Their different items, with the exception of the three items of the objective test, are arranged in an ascending order of difficulty. The degree of difficulty of each item is ascertained. None of them have been timed, and so, the tests on the mechanical phase of the four simple processes can be used only to measure accuracy and not speed. They still lack a special key for scoring. Age - and grade-norms have not yet been made. Directions for some of them have already been made, for others not yet.

In a word, these tests are about half-way towards standardization which I hope will be reached before very long.

CHAPTER EIGHT

Measuring Students' Achievements.

A. Diagnostic Testing.

"It is a fortunate school in which no one of the pupils has difficulty in arithmetic" (1).

Pupils, who do not get their arithmetic lessons well, are usually kept in after school, or are assigned more problems to solve at home. The prevailing idea is that, these pupils can fulfill their duties, as far as the arithmetic lesson is concerned, if they can only have more time. This is a wrong idea and a wrong procedure too.

The teacher's aim should be to find out the cause of difficulty which these pupils have, remove it if possible, and apply corrective measures. Diagnostic tests are the instruments which help the teacher along this line. By using these tests the teacher can know specifically just where the pupils' difficulties lie, and, in addition to that, he can find out the cause of these difficulties. Consequently, he will be able to prescribe the remedy.

In order to be able to make a successful diagnosis of a pupil's inefficiency in arithmetic, the teacher should have a clear idea about the skills that constitute the processes (pages 22-24), the complexity of these processes for the learner (pages 23-24), and finally, a knowledge of the most common causes of types of difficulties shown by the study of the work of the pupils in the grades (page 24).

In addition to what has been mentioned above, the teacher has to use certain techniques of analysis to be able to determine causes of error (pages 3-7). The techniques of analysis of the written and oral responses of pupils are the most common ones for discovering faults and their causes in the arithmetic processes. First, we shall consider the written phase of the subject.

1. When to Test.

Measurement in the past has been carried on at the end of a quarter or a semester of the academic year. This traditional method is still followed by many schools at present. The chief purpose was only to check what has been taught during that particular period.

In a modern school, on the contrary, testing takes place before instruction, during instruction and after instruction. The chief purpose is not only to check the ability of the pupils, but to point out also the pupils' weaknesses which are to be remedied by the teacher before moving on with his class from one unit to another new unit.

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(1) Almack and Lang: "The Beginning Teacher", P. 334.

a. Testing before instruction.

When a teacher first begins his work with a new class, he does not have a full idea about the abilities of his pupils. So, he should begin the teaching of arithmetic by giving a diagnostic test at the beginning of the work. In doing so, he will be able to plan his work in such a way as not to waste time in teaching things which the pupils have already known, nor to begin where they have not yet reached.

b. Testing during instruction.

The teacher may see it fit to test during the teaching of a unit. He may notice that some pupils fail to grasp the process which is being taught, or the underlying principle, and he needs to know just where the difficulty lies. In such a case he should carry on diagnosis. This will prevent pupils from going astray and using inefficient ways of work until the end of the teaching unit.

c. Testing after instruction.

When a teacher has spent enough time in teaching a given unit, such as long division, and the class has finished it, he should give a diagnostic test in that unit in order to find out whether it has been sufficiently mastered by the class, before moving on to the next unit. If not, further drill will be carried on until the class masters the unit.

B. Kinds of Diagnosis.

There are two inseparable kinds of diagnosis: namely, 1. Group, 2. Individual diagnosis. There is no conflict between the two and the former is usually used as the basis for the latter, i.e., one usually begins with group diagnosis and ends with the individual.

1.Group Diagnosis (Analysis of the pupils' written work).

Group diagnosis has, at least, three different functions. First, it enables the teacher to find those pupils who are having difficulties in the subject, second, to know what items are most difficult, and third, it helps him to discover the causes of these difficulties, specially when they are further diagnosed by detailed individual tests.

Group diagnosis can be done by giving a diagnostic test in any unit of the subject to a class or to a group of students of the same or different classes. After the test is scored, a diagnostic chart should be constructed on which the names of the pupils will be placed at the left, and the items (or their numbers) of the test will be listed at the top of the chart from the left to the right. Then on the chart under the number of each item, and in the same row of the pupils' row, will be placed a zero for the omitted items, and a check mark for the incorrect ones. Blank spaces indicate correct answers. At the bottom of the chart one can usually find the sum of the items solved correctly, i.e., the pupils' scores. The following table will be used as an illustration; using the wrong scores:

	0÷9	9÷9	18÷9	27÷9	36÷9	45÷9	54÷9	63÷9	72÷9	81÷9	
1.					x					x	2
2.	x		x				x	x		x	5
3.									x	x	2
4.	x									x	2
5.								x		x	2
6.	x				x			x			3
7.	x							x		x	3
8.	x	x	x		x		x	x		x	7
9.		x	x						x	x	4
	5	2	3		3		2	5	2	8	

The table of 9 in short division given as a group diagnostic test to 9 fourth grade pupils.

The table on the previous page reveals clearly what combinations were most difficult, such as 81÷9 and 0÷9, and which ones were solved correctly by the whole class, such as 27÷9 and 45÷9. A look at the right side of the table points out the students who have difficulty, such as nos. 8 and 9. Such an analysis enables the teacher to adapt drill on the difficult number combinations to the needs of the class without waste of time and energy both on his part and on the part of the pupils. Efficient teachers have always used this technique of analysing the written work of the pupils, and have in many cases succeeded in locating the sources of difficulty.

a. Diagnostic Testing of Pupils in Number Facts.

In order to be able to diagnose the pupils' ability in the computation of number facts, I have used the Courtis Standardized Research Tests. These are the most used of the standardized tests. They are arranged in two groups: Series A and Series B. Series A are designed to measure performance in the simple facts of the four simple processes (1). Series B is more extensively used and consists of four tests which are designed to test the ability of the pupils in the four simple processes (2). I have used both series twice in measuring the ability of the students in the Elementary School, from the III to the VI grades inclusive for the first trial on December 24, 1941. After a period of about 3 months of remedial work they were given for a second trial.

(1) Speed and accuracy.

The following table shows the different standard norms of the tests for each of the grades, i.e., the number of correct combinations, and also the norms attained by the Elementary School grades in the first trial:

Addition	Grade III	Grade IV	Grave V	Grade VI
Courtis Norms	26	34	42	60
Elem. School Norms.	30	33	37	46
Subtraction				
Courtis Norms	19	25	31	38
Elem. School Norms	19	19	23	27
Multiplication				
Courtis Norms	16	23	30	37
Elem School Norms	22	21	30	32
Division				
Courtis Norms	16	23	30	37
Elem. School Norms	12	15	19	27

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(1) Ref. appendix pp. 104-107
(2) Ibid pp. 107-109

(2) Common Difficulties & Their Causes.

As a result of the diagnosis concerning the basic combinations of the four simple processes, the following facts were also concluded:-

These combinations were found to be of different levels of difficulty. Consequently, their relative difficulty was ascertained, and the coefficients of reliability and validity were found out, as was shown before on page 13. The frequency of each combination in the course of study, should therefore be relative to the degree of its difficulty (1).

What has just been said holds true in all the four simple processes, but what follows pertains to each operation separately.

Addition

1. Both forms of each combination, the original such as 3 + 9, and its reverse 9 + 3 should be taught:

a. With more emphasis on the original ones, because 57% of the pupils were mistaken in 39 original combinations and only 43% were mistaken in the reverse form of the same combinations.

b. Five third grade students were wrong in 3 + 9, while only two of the same students were wrong in 9 + 3.

2. Excluding the 0, 2, and 6 tables, each of which contains greater number of errors than the one following it, the number of mistakes increases regularly as the table of 9 is approached. Special emphasis should be placed on these three tables.

a. The 6 and 2 (in Arabic ٦ و ٢) should be differentiated right from the very beginning. About 21% of all the mistakes were due to the confusion of students by these two figures. 11% were due to the use of 2 instead of 6; such as 8 + 6 = 10 (١٠ = ٦ + ٨). 10% used 6 instead of 2, such as 4 + 2 = 10 (١٠ = ٢ + ٤).

b. The zero concept should also be made very clear. About 6% of all the mistakes were made in the zero combinations, such as 0 + 7 = 0 and 8 + 0 = 9.

3. The combinations whose sum is 10 and above, such as 7 + 4 or 9 + 8 should be emphasized more than those combinations whose sum is below 10, such as, 6 + 3 and 4 + 2. 71% of the mistakes were made in the former type, while only 29% in the latter.

4. 46% of the errors were over estimated (above the correct answer), while 54% were under estimated (below the correct answer).

Subtraction

1. The 12 - 3 and the 12 - 9 types. It was found out that 50.7% of the errors were in the first type, and 49.3% in the second type of combinations. Both types should be taught and emphasized since they are nearly equal in difficulty.

2. Excluding the 9 and 0 tables, which contain more errors than the other tables, the number of mistakes increases regularly as table 9 is approached. Each of these two contains about 16.5% of all the mistakes committed in all the tables, such as 8 - 0 = 0 and 17 - 9 = 7. Special emphasis should be placed on both of them.

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(1) Ref. appendix pages 110-112

3. 72% of the errors were made in those combinations whose minuend is 10 and above, such as 10 - 3 and 13 - 4, while 28% of the errors were made in those combinations whose minuend is below 10, such as 9 - 6 and 7 - 2. Emphasis should be placed on the former type.

4. 45% of the errors were over estimated, while 55% were under estimated.

Multiplication

1. Both forms of each combination, the original, such as 3 x 8 and its reverse, 8 x 3 should be taught.

a. 51% of the mistakes were made in the former, and 49% in the latter.

b. Four fourth grade students were wrong in 7 x 9, while all of them were right in 9 x 7.

2. Difficulty increases as one approaches the table of 9 with the exception of the table of 5, which seems to be comparatively easy, and the 0 table, which alone includes 30% of all the errors. Moreover, taking the 19 zero combinations alone, we find that 62% of all the errors are made in these combinations, while 38% pertain to the remaining 81 combinations. The right zero concept should be emphasized from the beginning.

3. The concept of the 1 digit should also be made clear, because 6% of all the errors were distributed as follows:- 4% in combinations beginning with the 1, such as 1 x 7, and 2% in combinations ending with the 1, such as 8 x 1.

4. 65% of the errors were over estimated and 35% were under estimated. This high percentage of over estimation is due to the number of errors in over estimating the value of zero, such as 0 x 6 = 6 and 4 x 0 = 4.

Division

1. The 8 ÷ 2 and 8 ÷ 4 types. It was found out that 57% of the errors made in 28 combinations were made in the former type, and 43% in the same number of combinations were made in the latter type. More emphasis should be put on the former type.

2. With the exception of the 0, 1 and 9 tables, which include more errors than the other tables, the difficulty increases as one approaches the table of 9.

a. 23% of all the errors in the division combinations were made in the zero table. 90% of these errors are of the type 0 ÷ 7 = 7.

b. 12% of all the errors in the division combinations were made in table one. 50% of these errors are of the type 9 ÷ 1 = 1. Pupils should be drilled enough upon the 0, 1 and 9 tables.

3. 54% of the errors were over estimated, and 46% were under estimated.

b. Diagnostic testing of pupils in the processes.

(1) Speed and accuracy in the processes.

The following table shows the different standard norms of the tests, i.e., number of correct examples, and the norms attained by the Elementary School grades in the first trial of the processes:

Addition	Grade III	Grade IV	Grade V	Grade VI
Courtis Norms	2	3	4	5
Elem. School Norms	1.75	2	3	4.5
Subtraction				
Courtis Norms	1	3	5.5	7
Elem. School Norms	2.8	4.4	6	6.7
Multiplication				
Courtis Norms		1.5	4	5.5
Elem. School Norms		3.3	5	6
Division				
Courtis Norms		1	3	5
Elem. School Norms		2.6	4.3	6

(2) Common difficulties and their causes.

The following are four out of hundreds of examples which I have found out in diagnosing the papers of pupils. Each of these examples pertains to one of the simple processes.

Addition	Subtraction	Multiplication	Division
28	32	425	3422
11	17	73	32)9864
2	25	2955	
14			

Of course, it is very easy to have the teacher mark each of these examples incorrect and give the pupils zeros. With little patience on the part of the teacher, he will be able to discover that the boys had followed in each example a wrong principle, and unless, these boys are taught the correct kind of procedure they will continue to commit such mistakes in spite of the zeros they receive for their work.

What the pupils actually did is as follows: in the first example the pupil added continually from the first column on to the second. The pupil had correctly performed a process in addition considerably more difficult than that required in the example.

In the second example, the student not understanding the borrowing process, he subtracted the minuend from the subtrahend.

In the third example, the pupil not understanding how to multiply by two digits consequently, he multiplied by the first digit only once and then used the second digit of the multiplier. He naturally got a wrong answer.

In the fourth example, the pupil not understanding the correct procedure in the process of long division, he solved the example as follows: 9÷3 = 3, 8 ÷ 2 = 4, 6 ÷ 3 = 2, 4 ÷ 2 = 2. So, he naturally got the wrong answer.

As a result of diagnostic work in arithmetic for a period of about nine years, I could collect hundreds and even thousands of such samples of work done by pupils of different levels and ability in the subject. These samples are classified and filed for reference, diagnostic and remedial purposes.

c.Diagnostic Testing of Pupils in Reasoning.

As a result of several diagnosis of errors in written problems, it was found out that these errors fall into types. Some of these types are so general that they cover a large number of cases. I shall cite only the following four types which I have found to be most common in the lower grades of the University Elementary School.

1. Pupils use the wrong operation, and naturally the answers they give indicate very clearly that they have used any one of the other three operations besides the one which is needed for the solution of the problem on hand. Example, "A man bought 100 eggs for 700 pts. for how much did he buy an egg"? The commonest wrong answers by third graders were 800, 600, 70,000 indicating respectively addition, subtraction and multiplication.

2. Failure to comprehend the statement of the problem. This is sometimes shown by the absurdity of the pupils' answers. Example, "A butcher bought three sheep. He paid for the first £.Syr. 90, and for each of the other two £.Syr. 125. How much did he pay in all?" Fourth grade pupils gave fourteen different wrong answers to this single problem. Some of these answers are as follows: 214, 375, 3750, 5175, 12385, etc. .

Sometimes pupils decide upon the right operation but carry it only part way. They lose confidence or encounter some obstacle. Example, "A man sold 1/3 of a piece of cloth, then 1/4, then 1/5; if there remained 15 m. what was its length?" The commonest wrong answer by the fifth graders was 1m. They have added first the parts 1/3 + 1/4 + 1/5 = 47/60 + 13 = 60/60 = 1 answer.

4. Errors in the fundamental processes are very common, as every teacher knows. Osburn found that about 1/5 of the errors made in solving verbal problems were attributable to this cause.

Even more than that, I have noticed that many times pupils solve verbal problems by the trial and error method, and they reach the correct answers, which they have known somehow before, inspite of the fact that they have used a wrong method. The following are two concrete illustrations worked out by pupils:

First, Yusuf is 15 years old. Tawfic is 48. In how many years will the age of Yusuf be 3/4 of that of Tawfic? Some of the pupils solve it in the following way:

$$15 + 48 = 63 \text{ years}$$
$$3/4 = 63 \text{ "}$$
$$4/4 = 84 \text{ " ans.}$$

Thus we see a wrong method of solution resulting in the correct answer.

Second, A merchant bought a certain number of chairs at 7.5 sh. each. A reduction in the price of 0.25 sh. each would have effected a saving of 15 sh. How many chairs were bought? Solution -

$$1/4 = 15$$
$$4/4 = 60$$

Again, we see that the students followed a wrong method and got the right answer.

A fundamental principle which should be recognized, is that pupils must be trained to solve problems rather than to secure correct answers. The emphasis should constantly be placed upon learning how to solve problems rather than upon the answers obtained.

This technique of diagnosis takes time and it is based, as one can see, to a large extent on inferences. Inferences are not always correct. Moreover it does not give any information with regard to the pupil's difficulties so long as his answers are correct, in spite of the fact that the pupil secures the correct answers by round-about methods. The most common among which is counting. A fourth grade student multiplied 6 x 6 by counting six 6's on his fingers and he secured the right answer.

2. Individual diagnosis (Observation of the pupil's methods of work)

Quite often group diagnosis does not reveal the causes of the errors, and in order to prevent errors from being made the source of the trouble must be known. It is by individual diagnosis that we can locate the cause of errors.

A unique method of diagnozing individual difficulties in the fundamental processes in arithmetic has been developed by Buswell & John. It consists of a pupil's Work Sheet containing examples in each of the four operations; the teacher's Diagnostic Chart containing all the examples which appear on the pupil's Work Sheet plus a list of numerous inappropriate habits of work which lead pupils to make errors (1). This list of faulty habits was determined by the authors as a result of an analysis of the errors made by 329 pupils in grades III to VI. The results of their analysis of bad habits which they found in the four operations are given elsewhere (2).

By means of some good arithmetic test or tests the teacher can locate those pupils who need diagnosis (as mentioned above). Then, he takes one pupil at a time, gives him the Work Sheet, tells him the purpose of the examination and asks him to work aloud. The teacher sits where he can observe the child at work. He should not suggest anything to the child about his work. He has only to discover how the pupil works, and to check on the Diagnostic Chart the habits which the child uses.

a. Oral diagnosis of pupils' abilities in column addition.

An oral diagnostic study of 90 pupils in column addition was made. These pupils ranged from the 2nd to the 6th grades, all of whom ranked average or below in a written group diagnostic test. The sixteen examples of the informal test on the process of addition, on pp.59-60,were used for the purpose.

Each pupil was called individually to a room in order to add only eight of the sixteen examples from the blackboard. The pupil was asked to add orally and loudly. He was told that the purpose of the test is not to quizz him but to help him, in order to determine the cause and nature of his difficulties. He was also encouraged very much to cooperate in the work. Of course, the primary purpose was to locate the faulty methods and other possible causes of inefficiency of work.

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(1) Ref. appendix p. 116
(2) Ibid, pP. 113-115

As the pupil was adding I had my seat in a convenient place in the room where I could see the pupil's total behaviour. I had on hand a blank paper on which I marked the types of faults that have been discovered. An average time of 15 minutes was given to every pupil at a time.

When the work of the test had been completed I had carefully analyzed the notes taken during the examination and summarized the findings of the diagnosis. These are recorded and filed together to be used whenever needed.

The following statement summarizes the findings of this diagnostic study:

1- Counting (by fingers, lips, feet and tongue).
2- Carrying:
    (a) Forgot to add carried number.
    (b) Added carried number last.
    (c) Added carried number irregularly.
    (d) Carried wrong number.
    (e) Wrote number to be carried in partial sum.
3- Procedure in adding:
    (a) Added downwards.
    (b) Added upwards.
    (c) Added columns downwards and other columns upwards.
    (d) Proceded irregularly in the column.
4- Grouped two or more numbers, particularly those whose sum is ten or five, or, when the same digit is to be added twice.
5- Split numbers (5 + 9, was added as 5 + 5 + 4).
6- Short attention span.
    (a) Retraced work after partly done, 52 cases.
    (b) Omitted digits (95 cases)
    (c) Added same number twice.
7- Dropped back one decade.
8- Skipped one or more decades.
9- Individual difference in speed. (Some third and fourth graders proved to be quicker than many fifth and sixth graders.)
10- Difference in methods of adding.
    (a) 12.5% added by thinking results only.
    (b) 50% added by thinking numbers and results.
    (c) 37.5% added by mentioning every number to be added.

The first group proved to be quick and accurate. The second, average. The third, slow and under average.

Before bringing this discussion to an end, I should like to mention two very interesting instances. The first is that I have gathered six examples with correct answers. Although the answers were correct, yet the students committed two mistakes in the same column of each of the six examples which were brought to light by the oral diagnosis. It so happened that one mistake corrected the other. I shall mention here only two pairs of these mistakes: 5 + 7 = 13 and 33 + 4 = 36. The second pair was: 4 + 7 = 12 and 12 + 3 = 14. As one can see the first mistake was overvalued by one and the second was under valued by one, so that the first mistake was corrected by the second, or vice versa. Had it not been for the oral diagnosis I should not have been able to discover the pupils' difficulties in as far as the sums of the examples are correct.

The second instance is the following: As the pupil A.K., a six grader, was adding, I noticed that he sometimes used to mention combinations which were not included on the diagnostic sheet. A thing which surprised me very much. As a result of a short interview with the pupil I found out that he could not add easily certain combinations, such as 32 + 9 and 43 + 8 etc. Whenever this pupil met such combinations he used to change them to 39 + 2 and 48 + 3 and to add them correctly. Teachers can never realize the great number of ways and means which poor pupils resort to in doing their work in arithmetic until they come into close contact with those pupils while doing their work.

b. Oral Diagnosis of Pupils' Ability in Problem-Solving.

(1) Varied ways of procedure in problem-solving: A little observation of pupils at work on verbal problems leads inevitably to the conclusion that the following varied ways are used by the pupils:

First, one can notice that a pupil is ready to solve a problem if someone will tell him whether to add, subtract, multiply or divide.

Second, some pupils think of it as a duplication of a procedure illustrated in a problem worked out in the text. In spite of the fact that these two problems are of two different types.

Third, some students think of it as a mechanical application of certain rules in a haphazard way.

Fourth, some pupils think of it as a trial-and-error process in which they perform certain operations in the hope that they will be accepted as correct.

Fifth, still others have wrong attitudes towards the subject because of the following reasons:

1. Arithmetic is a very difficult subject.
2. Daddy and Mamy cannot do well in it.
3. So many pupils fail in it.
4. Mr. so and so teaches it.
5. It has little practical value.
6. The family is not mathematical.

One child when asked how he could tell whether to add, subtract, multiply or divide said: "If there is more than two numbers, I adds, if there is two numbers, I subtracts, unless one of them is a little number, then I multiplies, but if it will go even, I divides".

Such students are not necessarily "born short" in arithmetic when they fail to meet difficulties in solving problems with success. Their failure may be due to one or more of the following difficulties which the teacher himself should discover and remedy on the moment.

(2) Difficulties encountered by pupils in problem-solving: Banting's study of pupil difficulties in problem solving is one of the most comprehensive that has been undertaken. It contains information concerning the faults revealed by questioning boys and girls as to their difficulties in problem-solving. The following are the causes, in brief, of inability to solve problems. (1)

1. Failure to comprehend the problem in whole or in part.
2. Lack of the ability to perform accurately and readily the fundamental operations.
3. Lack of the knowledge of facts essential to the solution of a problem. For example, lack of knowledge of the tables of weights, measures, etc.
4. Lack of the ability to identify the proper process or processes with the situations indicated in the problem.
5. Lack of sufficient interest in the problems to inspire the required mental effort.
6. Failure to form the habit of verifying the results.

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(1) Ref. L.J. Brueckner, "Diagnostic & Remedial Teaching", pages 230-233.

7. The habit of focussing the attention upon the numbers and being guided by them instead of by the conditions of the problem.
8. Pupils are sometimes completely perplexed by large numbers. They are perplexed by numbers larger than those of every-day experience.
9. The habit of being guided by some verbal sign instead of making an analysis of the problem.
10. Lack of ability or care to properly arrange the written work in orderly, logical form.
11. The failure to recognize the mathematical similarity to type-problems.
12. Lack of ability to understand quantitative relations such as:
    a. cost, loss or gain, selling price.
    b. interest, rate, time, principle, amount, etc.
13. The pupil may fail because the problem required exertion beyond his span of attention.
14. The pupil may fail because of absolute inability to do reflective thinking (1).

C. Miscellaneous

From time to time, I used to investigate through the different grades for testing the pupils' abilities in one subject or the other in which they proved to be inefficient. The following are some of the investigations which I have made:

The first investigation was about testing the pupils' abilities in understanding the different steps in problem solving. Only one-step problems were used.

Fifty-four Fifth grade pupils shared in this investigation. The following table shows very clearly the above mentioned steps with the correct percentage of each:

Steps in Problem Solving	Correct percentage
1. Problem comprehension	51
2. What is given	30
3. What is called for	25
4. Probable answer	15
5. Process analysis	80
6. Relationships in problems	65

The above results show very clearly the weaknesses of the pupils in problem solving, particularly in the second, third and fourth steps.

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(1) Osburn classified 30,000 errors made by 6,000 children on Buckingham problem test as follows:-

1. Total failure to comprehend the problem	30%
2. Procedure partly correct but with the omission of one or two essential elements	20%
3. Ignorance of fundamental quantitative relations	10%
4. Errors of fundamentals	20%
5. Miscellaneous errors	2%
6. Errors whose cause could not be discovered	18%

This information is taken from C. R. Stone, "Supervision of the Elementary School", p. 175.

The second investigation was about the comprehension of the meaning of words and technical terms as they are met in problems. Some of these technical terms are the following: area, centar, radius, liter, gram and volume. These were put in the multiple choice form, of five answers to each. They were given to fifty-six fourth grade pupils. The pupils had to check the correct answer.

The following table is a good illustration of how capable these fourth graders were:

Term	Correct prcentage
Area	41
Centar	43
Radius	80
Litre	92
Gram	71
Volume	30

Here, again, one can clearly see the weaknesses of the pupils, specially as indicated by the first two items and the last.

The third investigation was mainly about the technical terms of the four simple processes. One-hundred ninety-one students ranging from the third to the sixth grades inclusive shared in the investigation. They were given one example on each of the four simple processes and were asked to give the technical terms for each part of the examples.

The following table gives a clear idea about the pupils' ignorance in knowing the terms.

Grade	Correct percentage
III	9.5
IV	31.5
V	51.0
VI	59.0

One is disappointed in getting these results because a third grader is expected to know these terms thoroughly well.

It is interesting to know that the pupils invented eighteen different names for the product, seventeen for the quotient, fourteen for the divisor and eight for the multiplier which was the least.

As a result of this investigation, these technical terms were grouped in descending order of difficulty as follows: Quotient, multiplier, product, remainder, minuend, multiplicant, subtrahend, divident, divisor, sum and the remainder of division.

The fourth investigation was about testing the pupils' abilities in checking each of the four simple processes. An example of each of these processes was given to one-hundred ninety-three pupils, ranging from the third to the sixth grade inclusive. These pupils were asked to do the examples and check their answers.

Books in the west usually contain the following statements: "Add and check", "Subtract and check", etc. This is to remind the pupils to, and to build in them the habit of checking the work. For further information about this item see pages 47 & 48.

The following table indicates the correct percentage in checking the four simple processes by the upper four grades in the Elementary School.

Grade	Addition	Subtraction	Multiplication	Division
III	100	100	78	33
IV	83	100	77	73
V	82	100	79	87
VI	68	100	84	82

Apparently, all the pupils knew how to check subtraction examples, but many of them failed to be as efficient in checking the examples of the other three processes, as is seen in the above table.

The fifth or the last investigation was about diagnosing the pupils' abilities in estimating the quotient in long division. Five examples were given to fifty-four fourth grade pupils. They were asked to do these examples, but to leave on paper every digit of the quotient they thought fit which turned to be either greater or smaller than the correct one.

The following table shows us some of the partial dividents of the five examples with the number of trials made for some of the digits of the quotients:

Partial dividents	Correct Percentage	First Trial	Second Trial	Third Trial	Fourth Trial	Fifth Trial
176 ÷ 42	70	29	20	4	1	-
429 ÷ 63	-	36	13	3	2	-
510 ÷ 63	66	40	10	2	-	1
218 ÷ 24	66	37	12	3	2	-
693 ÷ 95	68	39	13	1	-	1
104 ÷ 36	68	34	18	2	-	1
324 ÷ 36	-	44	5	4	-	1

I shall cite only two concrete examples which I have extracted from the work of two fourth grade pupils. They are:

```
665 = 7              9
      9 7            8
      8 6          2 7
445 = 7 4          3 3
      5 5          5 5
    ---------    ---------
  95)69350     36)10440
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It is needless to say, that these two students should have worked out seven other examples within the time wasted on these two examples had they been taught the rules for the estimation of the quotient. Details about these rules are given on pages 43-44.

It is very interesting to find out that the first pupil tried digit seven twice. Apparently, he was not satisfied with the first trial and so moved on to higher digits and back to seven again. Further diagnosis showed that the pupil was mistaken in multiplying 95 x 7 for which he got 445 as answer, a chance mistake. But he got the correct product which is 665 in the second trial and so he was satisfied.

PART IV

REMEDIAL WORK

Remedial work in arithmetic can be considered from both the corrective and the preventive points of view. The former one will be discussed in the following chapter.

CHAPTER NINE

Corrective Procedures.

A. Need for Corrective Procedures.

All that has been mentioned in part III forms only the first step in the process of diagnosis. Testing the pupils, marking their papers and tabulating the results, all of this is very good work on the part of the teacher and is very necessary, but, it will lead nowhere until the whole thing is interpreted in terms of pupil needs. It is only then that the whole process of instruction will be modified accordingly.

It is not enough to say, after one has made a careful diagnostic test, that pupil x or class y in the school is average, below, or above average, because in doing this only, we will be neglecting the most valuable information obtainable from testing. As a matter of fact, this is what usually happens. Very often téachers give tests, and after these tests have served the needs of the teachers, perhaps only grading the pupils, they are thrown away. Thus, even not half the use of the tests is madé of the responses of pupils to these tests as could or should be made.

A good teacher, after testing and grading his pupils, if he wishes to do so, should have, first of all, for the pupils tested a specific list of habits of work which needs attention later on after diagnosis is over. Second, he should work out, in a systematic way, for each habit remedial procedures which seem to be needed. Third, these should be carefully made use of in actual school work. Finally, a cumulative record of each remedial procedure with the result secured through its use should be kept for future work.

Such a record makes possible a systematic review of the steps that have been taken, and furnishes valuable information concerning methods and materials to be used with other cases. This is not to mention the many important facts which it reveals about the learning process itself. In a word, successful methods of remedial treatment for common difficulties can thus be located and consequently inserted in instructional materials.

Otherwise, the work of this teacher will be similar to the work of that physician who, after examining his patient, tells him that he has ulcer in his stomach, or that one of his kidneys is not functioning properly well and he ends his prescription at this point. This physician's diagnosis may be correct and, therefore, a very good one, but as far as the patient is concerned it is, of course, insufficient. To be of some real help to his patient the physician must interpret the results of the examination in terms of the patient's needs. It is then that he will accordingly prescribe the medicine.

The scores of the pupils and their errors correspond to the maladies of the patient, and in interpreting these results of diagnosis, it is necessary to make similar prescriptions of remedial work as has just been mentioned before. After determining the particular difficulty which pupils have, remedial work must begin. In order to be effective, it must begin with a specific attack on that difficulty which was revealed by the diagnostic study. It is useless, for example, to attempt to improve the work of a pupil scoring low in long division examples, when he is weak in the multiplication process involved. But, instead of that, if the pupil splits number in adding rather than adds by the complete number combinations, or if he continually forgets to add the number which is carried, the question is how to make him stop splitting numbers and add the digits as a whole as they appear in the example, and how to eliminate the other difficulty. It is only by such means of procedure can any advantage be gained from diagnostic testing.

It is very significant to have remedial measures follow individual or group diagnosis, but it seems that no standard procedures have been so far developed to that effect. Moreover, the arithmetic drill devices which have really been based on experimental trial and which are known to be effective and constructed particularly for remedial purposes are not so numerous. For this reason, every teacher has to try out different methods of corrective procedures. The details of these procedures may grow out of the teacher's previous experience with similar cases, or they may be based on others' reports of remedial work, or may be the result of the trial and error process.

We come next to consider two of these methods of corrective procedures as they were applied in the Elementary School.

B. Practical Use of Corrective Procedures.

1. Number facts.

As a result of the application of the Courtis Tests, as was mentioned on page 71, all the difficult combinations of each of the four simple processes were singled out and drilled upon from five to eight minutes daily for six weeks. At the end of this period of remedial work, the same tests were given again in order to ascertain the results of the reteaching period. The following table found on page 71 will be reproduced again with the addition of the grade norms which resulted from the second trial. This table being as such, will enable the reader to make comparisons, first, between results of the first and second trials of the test itself, and second, between each of these results and the standard norms of the tests.

What has just been said in the above paragraph holds true of the first three simple processes but not of division which was only given to the grades for the first trial because of lack of time. The same condition is true in the processes, as we shall see later, for the same reason.

The following table shows first, the different standard norms of the tests for each of the grades, and second, the norms attained by the Elementary School grades successively in the first and second trials.

Addition	Grade III	Grade IV	Grade V	Grade VI
Courtis Norms	26	34	42	50
El. Sch. 1st trial	30	33	37	46
" " 2nd "	36	36.5	42	50

Subtraction				
Courtis Norms	19	25	31	38
El. Sch. 1st trial	19	19	23	27
" " 2nd "	21	23	31.5	33

Multiplication				
Courtis Norms	16	23	30	37
El. Sch. 1st trial	22	21	30	32
" " 2nd "	21	27	29	33

Division				
Courtis Norms	16	23	30	37
El. Sch. 1st trial	12	15	19	27

Apparently, in almost all the processes, as one can easily see, there is a progress and a good one too on the one hand, and in many cases the Elementary School grades proceeded beyond the standard norms of the test on the other hand.

2. The simple processes.

Here again after the first trial of the test as was indicated before, remedial work was carried on for six weeks at the end of which the pupils were retebted. The following table is reproduced again with the result of the second trial of the test.

Addition	Grade III	Grade IV	Grade V	Grade VI
Courtis Norms	2	3	4	5
El. Sch. 1st trial	1.75	2	3	4.5
" " 2nd "	2	3	4.3	4.7

Subtraction				
Courtis Norms	1	3	5.5	7
El. Sch. 1st trial	2.8	4.4	6	6.7
" " 2nd "	3.5	5.2	6	8

Multiplication	Grade III	Grade IV	Grade V	Grade VI
Courtis Norms	-	1.5	4	5.5
El. Sch. 1st trial	-	3.3	4.75	5.75
" " 2nd "	-	3.75	5.	6
Division				
Courtis Norms	-	1	3	5
El. Sch. 1st trial	-	2.6	4.3	6

One can conclude from the above table that there was a general improvement in all the grades except in one or two instances at most. In addition to that, the Elementary School norms are higher in a general sense than the standard norms of the test.

One wonders as to what kind of remedial procedures were used and how? The following details give the answer to these questions:

a. General procedures(briefly stated).

1. After giving the diagnostic test to the grades, a discussion of the most common errors was made with each. The purpose is to draw the attention of the pupils to their methods of work rather than to the examples themselves.

2. The attention of the pupils was drawn to one or two of these errors at a time. Different children worked out examples on the board using the correct terms and forms.

3. Errors have been presentedin such order that one new difficulty and only one was presented at a time.

4. Each pupil was asked to keep a record of his errors in a book after they have been corrected for the sake of review.

5. This corrective work was given an abundance of motivated drill by the use of intrinsic number games and activities.

6. It was always remembered that practice could be carried beyond the point of recall.

b. Specific procedures.

In the following paragraphs, only some samples of the remedial methods which have been used are given, partly because of the direct value which they may have for other teachers, and partly to illustrate the type of remedial measures which may be used by teachers in similar cases.

From the diagnostic results on pages 72 and 73 it was apparent that we should go back and teach the fundamental processes that should have been properly taught in the lower grades.

The most common difficulty is with the zero in all operations, in both the number facts and the processes. The real cause of this difficulty was found to be the lack of correct zero concept. This was shown by such answers as 7 + 0 = 8, 7 - 0 = 6, 7 x 0 = 7 and 0 ÷ 7 = 7.

Plenty of practice with objective material was given in order to develop in the pupils the right zero concept. Most of the work was done on the play level.

The students were taught by words the meaning of zero, such as zero means "not any". This was followed immediately by much practice, such as "If I give you zero pencils, how many pencils do I give you"?, or the use of such simple concrete examples as the following: "If you have no apples and you divide them among 6 boys, how many would each get"? "If George had no money and you had five times as much, how much would you have"?

To avoid writing rows of zeros, the pupils were told that this is a long method and a great waste of time, and that instead they should follow a short cut. Then, followed several examples having a zero in the multiplier and the work was done before them writing the rows of zeros. Then the work was done omitting this time the row of zeros and showed that the answers were all the same. This procedure appealed to them very much and they were so pleased that they asked for more examples of the same type to solve. Finally, pupils were taught that whenever zero is multiplied by a number the result is always zero, and whenever a number is multiplied by zero the result is always zero.

The second sample is about the teaching of both the original and the reverse forms of the addition and multiplication combinations, such as 6 + 8 and 8 + 6; 6 x 8 and 8 x 6. Diagnosis showed us that transfer is very limited and that both forms of each combination should be taught.

The third sample is about the confusion of students by the digits 6 and 2 (in the vernacular ٢,٦). This difficulty is found most frequently in the primary grades, although in some cases children in the Intermediate and even in the upper grades have not learned to write them correctly. This is either due to the absence of certain specific habits, or in certain cases there has been a lack of sufficient practice, or it may be due to an incorrect mental picture of these numbers.

The teacher was asked to make each number slowly and carefully for the pupils to call attention as to how each looks, where to begin when writing each and how does each curve; then to erase it and have pupils try. The pupils were also given special emphasis to the difference of these two digits in form. Plenty of practice in discriminating, making and reading them was made. These processes were repeated again and again before satisfactory results were obtained.

The fourth sample is when pupils have difficulty in multiplying where the multiplier has two or more figures. They were given a type and were asked to memorized every phaze of it so that they can reproduce it from memory at any time.

The fifth sample is to overcome the habit of adding the carried numbers irregularly. Children should form the habit of adding the number to be carried immediately at the beginning of the second columr Use was made of such examples as the following:

38 The class was told to add the first column, put down the one
84 and remember the two. When all the pupils got that far, then
59 they were asked to proceed with the second column adding the two
181 first. By repeating this orally for a few minutes every day, the
habit of adding the carried number seems to be fairly impressed.

After mastering the addition of two columns the class went on to three and four columns.

The sixth sample is where a tendency was found to subtract the minuend from the subtrahend which was larger. A review of the borrowing method of subtraction was necessary. Examples with the borrowing from the ten's unit were only presented first. Then examples where pupils had to borrow only twice were given; later on three and four times and so on. Sometimes, the pupils were also asked to write the letter b where they borrowed and the letter n where they did not borrow. This was only followed in order to remind them of the borrowing principle in subtraction.

3. Reasoning.

It is possible to bring about a big improvement in the ability of pupils to solve verbal problems in arithmetic. The best policy is to have the class teacher keep in mind the list of difficulties most pupils meet in the solution of problems as indicated on pages 78 and 79, and have him think of remedies for each of them.

The following general suggestions might be of some help in the development of power to solve verbal problems. (1)

(1) Failure to comprehend the problem.
    (a) Drill pupils in silent reading.
    (b) Acquaint them with arithmetical terms.
    (c) Put a premium upon accuracy in reading and in copying figures.
    (d) Select problems from various sources.
(2) Lack of the knowledge of essential facts.
    (a) Never give problems which require for solution facts which pupils do not know and cannot secure.
    (b) Teach pupils to analyze a problem and to state what they need to know in order to solve it.
(3) Lack of interest.
    (a) Discard the bookish problems.
    (b) New problem material should be used. Everything possible should be made to make the work vivid and interesting.
    (c) Give the practical problems of daily life.
    (d) Give well graded problems each of which is a challenge to the pupils.
(4) Failure to verify the results.
    (a) Answers should be labeled by pupils.
    (b) They should be estimated, and later on their reasonableness checked.
(5) Large numbers; in teaching a new principle, only simple numbers should be used.
(6) Arrangement of the work in an orderly logical form.
    (a) Pupils must be impressed with the importance of neatness, care and logical order in all written work in arithmetic.
    (b) Samples of good written solutions to problems should be exhibited on the bulletin board.
(7) Failure due to problems which require exertion beyond the span of attention of pupils.
    (a) Pupils should not be given problems beyond their powers of comprehension and span of attention.

Finally, a sample of types of remedial materials designed to give the pupil practice in reading verbal problems and training in

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(1) Ref. L.J. Brueckner, "Diagnostic & Remedial Teaching", pages 233-240.

their solution is given below. (1) Some samples contain the following:

(1) Exercises stressing vocabulary.
(2) Exercises stressing problem comprehension.
(3) Exercises stressing what is given in the problem.
(4) Exercises stressing what is called for.
(5) Exercises stressing the estimation of answers.
(6) Exercises stressing choice of procedure.
(7) Exercises stressing relationships in problems.

Excerpt From Compass Diagnostic Tests in Arithmetic, Test XVII, Problem Analysis. (2)

Problems	Part I - Comprehension	Part 2 - What is given
Read each problem below. Then work across the two facing pages to the right, doing all the Parts for one problem before going to the next. Do not go back and work on a Part after you have completed the one following. Read the sample below.	Put a cross (x) on the line before the one statement below which is true for each problem.	Put a cross (x) on the line before ever statement below whic tells a fact given in the problem.
Remember: Work across the page to the right.		
(Read the problem)	(Check true statement)	(Check what is given
Problem I Our baseball team played 7 games this summer. We lost 2 and tied none. How many games did we win?	___ Team won all games played. ___ Team lost all games played. ___ Team won more than it lost. ___ Team lost half of the games played. ___ Team won about half of the games played.	___ Number of games played. ___ Number of boys on team. ___ Number of games tied. ___ Number of games won. ___ Number of games lost.

Part 3-What is called for.	Part 4-Probable Answer.	Part 5-Correct Solution.
Put a cross (x) on the line before the one statement below which tells what is called for in the problem.	Put a cross (x) on the line before the one statement below which gives the nearest probable answer to the problem. Do not take time to work the problems.	Put a cross (x) on the line before the one correct solution given for each problem. Figure in the margin if you want to.

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(1) Ref. The National Council of Teachers of Mathematics, "The Second Yearbook", pages 67-72.

(2) As adapted by Greene, Jorgensen and Gerberich, "Measurement and Evaluation in the Elementary School", pages 316 and 317.

Remember: Work across the page to the right.

(Check what is called for)

___ Number of games lost.
___ Number of boys on team.
___ Number of games won.
___ Number of games where score was tied.
___ Number of games played.

(Check probable answer)

___ 9 boys.
___ 6 schools.
___ About 7 boys.
___ 9 games.
___ About 5 games.

(Check correct solution)

___ 7 + 2 = 9
___ 7 - 2 = 5
___ 7 ÷ 3 = 3 1/2
___ 7 x 2 = 14
___ 7 x 2 = 14: 14-9=5

Now start problem 2.

CHAPTER TEN

Preventive Procedures

"Prevention is the climax of diagnosis". Preventive measures are the most influential means which help in the solution of the problem under discussion, as we shall see later on. No two people disagree upon the fact that it is always better and easier to prevent errors than to correct them. Educators are right when they declare that preventive teaching is always preferable to corrective teaching and that the goal toward which the teaching process should work is prevention rather than remedy or correction. This declaration coincides very well with the well known proverb, "One ounce of prevention is worth one pound of cure". If all of this is true, one might ask what is the reason for that? The following lines answer the question.

A. The Seriousness of Error.

The teacher should take care and be on the alert lest his pupils make mistakes. Because, first, a mistake once committed, it becomes easy to be made again in just the same way. Second, it will never be entirely done away with. Third, it might become very serious.

In order to realize these facts, it is worth while trying to record the successive responses which a pupil makes for a mistake and then see how it always leaves its traces in those responses. I have followed up a third grade pupil who made an error on the combination 2 + 4. The following were the successive responses which he made: 7, 7, 6, 6, 7, 6, 6, 6, 6, 6, 7, 7, 6, 7, 6, 6, 6, 6, 6, 6, 7, 6, etc. We thus see how the wrong answer persists. Notice that the initial error of seven, although it becomes relatively less frequent as time passes, can never with any confidence be said to have disappeared.

It is not strange to say that the nervous system of many of us is full of such mistakes which spring up at any unexpected time, and that as far as our nervous systems are concerned, we receive no help, because the wrong answer is as positive as a right one. There will only be a difference between the two if we resort to the psychological law of learning, namely, the law of effect. If the satisfactions arising from the two are different, i.e., if a mistake is quickly associated with some unpleasant result, it is not as likely to recur as would be the case if satisfaction or neutral feeling were present. The old principle of "Learning by one's mistakes" thus only holds true, if such mistakes are associated with unpleasant and annoying consequences.

Knight believes that much of the present thinking about error assumes that error is a much simpler phenomenon than it is. Errors may not be single in their nature.... Treating all errors alike is analogous to treating all fevers alike.... Much of our treatment of error, be it ever so heroic, is never the less superficial and unsuccessful because it is based on too little understanding of the psychological nature of the error and the processes leading up to it.

It is very unfortunate to have most of the errors some of our pupils make, especially in their homework, not to be associated with anything. Poor pupils, at the same time, are not conscious of them. To be on the safe side, the teacher should neither assign home-work nor seat work unless he is sure that the pupils concerned are in full mastery of the required information about the subject on hand.

B. The Prevention of Error.

If what has been mentioned in the last item in relation to the seriousness of error holds true, it then naturally follows that the prevention of error should become more important as a school problem than the removal of error.

Many difficulties of arithmetic will be eliminated by anticipating them, for example, a teacher may learn by diagnostic testing that pupils able in most phases of multiplication have difficulty with the zero of the multiplier. Preventive work in teaching multiplication, will accordingly place special stress on zeros in the multiplier. There are hundreds of similar situations in the field of arithmetic. Moreover, this anticipation saves both parties, the pupils as well as the teacher, the trouble of double work on any single fact learned in the wrong way. Because, to correct a mistake, the teacher has first to put down, to destroy so to speak, what the students have built, and then he has to rebuild again. Thus, we see how errors once committed become a real source of waste of both time and energy.

Finally, sincere effort should be made to remedy those mistakes which a person has not been able to prevent. The analysis, and the study of causes and types of error will be the best means of making these efforts fruitful.

C. Means of Prevention.

If prevention of error is preferable to its correction, and if it is so important in the teaching process, it, therefore, follows that a knowledge and application of its different ways and means is necessary. Apparently, there are at least three most important means the discussion of which will follow:

1. The teaching material.

An authority on the subject says that the quality of teaching material of instruction has a marked influence on the success with which pupils master arithmetic.

The use of teaching material will certainly result in the development of error if this material is constructed without careful consideration of the results of scientific research studies or the nature of the arithmetical processes. Sorry to say that most of the available material of arithmetic instruction in the hands of the pupils is of this sort.

The presence of carefully graded exercises in which there is a step by step development of the processes new to the grades is almost rare. The inclusion of properly distributed drill material which is very effective in preventing errors is lacking. On analysis, one can easily find out that the greater part of the teaching material includes only very few of the skills which should be maintained in the processes. It thus overemphasizes those skills and causes loss of time and energy, while other skills are, at the same time, neglected altogether causing ignorance of these skills and leading to error. This material is apparently constructed on the assumption that there is a high degree of transfer among the various skills of which a process is composed.

In addition, one can also find out that some material contain subject-matter that is too difficult for the pupils because it has been constructed without definite specifications as to structure and content and without due consideration of the difficulties presented to the pupils.

Because I felt the need for more systematically constructed materials than those which are now ordinarily used, I set out to work in my spare moments towards this goal since nine years.

In constructing this material I have carefully considered the scientific information which is available concerning the learning processes, such as, the difficulties the learner encounters in mastering the essential skills, the relative difficulty of number facts, the learning steps and the difficult points in the development of a process, the factors that interfere with successful learning and the level of maturity at which the particular skills can be mastered most easily and adequately.

The following items include briefly the material which I have worked out and referred to above:

(1) A small pamphlet on the number concept.
This helps to develop a vivid realization of the meaning of numbers involved. Some of the items discussed center around the following: reproduction, identification and discrimination of number. In mastering these topics the pupil will be able to count, measure, compare, group and separate things.
(2) Three books on the number facts and the simple processes a book for each of the lower three grades.
(3) Four different scales, one for each of the simple processes, containing the following number of examples:

Addition	1152
Subtraction	974
Multiplication	944
Short Division	2263
Long Division	1746
Total	7079

(4) A collection of verbal problems.

a. One-step problems, classified and graded as follows:

addition	210	problems
subtraction	260	"
multiplication	190	"
division	210	"

The following to be classified and graded:

b. 280 two-step problems.
c. 133 three-step problems.
d. 60 four-step problems.
e. 25 five-step problems.
f. 15 six-step problems.
g. 500 verbal problems for grades V and VI arranged on topical bases.

By using efficiently such organized material, errors will be for sure reduced to a minimum. But in case of unavoidable mistakes steps must be taken on the moment to correct them so as not to have them accumulate lesson after lesson, and lead to a complete break-down in the ability of the pupils to learn.

2. Method of instruction.

In addition to what has been mentioned on pages 27-48 in regard to method, I should like to add the following information:

An authority on the subject said that factors inherent in the methods of instruction obviously contribute to the development of learning difficulty, although little is known as to what extent they do so.

Some teachers stress speed too much, and so assign practice on new step in a process before they have determined whether or not the pupils understand the step and know the procedures to use. When the pupil does not understand the step, his practice is hit or miss and may result in uneconomical and even incorrect procedures. Other teachers fail to adapt instruction to the different pupil needs. Some teachers assign further practice in spite of the fact that pupils show low performance on a test, instead of attempting to determine the nature of the difficulty which caused the poor work.

The result is the fixation of poor habits of work. Still other teachers may fail to recognize the need of an adequate program for maintaining the skills in arithmetic. This is not to mention anything about the failure on the part of some teachers to use legitimate types of motivation, in both building a skill and maintaining it; failure to teach the pupils efficient and economical methods of work; failure to use practice well distributed and sufficient in amount. All these latter suggestions have been dealt with in detail on pages 28-35.

Efficient methods of work should be taught from the beginning. The faulty round-about procedures used by some pupils should not be allowed to become established habits. Simpler methods should be taught when such fault is discovered in the work of pupils not making satisfactory progress.

Finally, if the school maintains an adequate program of testing during teaching, difficulties in arithmetic can be located so promptly that they will be corrected during the teaching period rather than allowed to accumulate to make a problem for remedial procedure.

3. Individual differences.

In addition to what has been mentioned on pages 37-38 in regard to individual differences, I should like to add the following information.

The individual differences of pupils furnish another problem in the teaching process. As a result of a careful diagnosis we have seen that pupils belonging to the same class differ widely in achievement, that they do not fail in a general sense, and that they do not often need remedial work of a general nature. We have also seen that pupils' errors fall into types and that they are more or less specific in their nature. The most frequent of these types should receive special emphasis so that they can be eliminated. These different conditions make it necessary to provide for individual instruction. Whereas in many schools at present, these differences are ignored in classes and the whole group of pupils is dealt with as a unit and the same exercides are prescribed for all. When methods of instruction thus disregard individual differences, serious difficulties are sure to arise in the work of slow pupils.

In order to adapt instruction to individual differences the teacher should devise or discover in other sources learning exercises in addition to those provided by the text. Even when the test is an ideal one, it will not solve the situation when it is prescribed for all members of the class. Because their needs will thus differ with respect to both number and kind of exercises.

The pupils of superior ability can learn more rapidly, and because of their greater ability, they are much more able to generalise than inferior pupils. They can analyze their difficulties and are able in most cases to devise methods of work without direct help of the teacher. Nor they do require learning units so carefully graded as seem necessary for pupils of inferior ability. Where as the pupils of inferior ability must in most cases be given careful instruction in efficient study habits and should be assigned well graded tasks that they can master. Hence, it is important to use teaching materials and methods adapted to various degrees in ability to learn.

I have made an analysis for all four of the simple processes. When a school subject has thus been analyzed into "unit skills", the use of analysis for individual instruction becomes possible. After the weaknesses of each child have been determined, the teacher may then provide him with practice material corresponding to the step to which one of his difficulties belongs. This material he will use until he gives evidence of having mastered the difficulty in question. He will then take a test pertaining to this difficulty, and if he passes it, he will go on to the material covering the next difficulty.

This material should be abundant, so that at any given time one pupil may be using one type of material and another, another. Moreover each pupil will advance at his own rate.

Finally, I end the discussion by the following quotation which appeals to me very much:"Arithmetic must be taught in terms of the pupil, not simply in terms of the subject itself, and not until teachers understand intimately the nature of the difficulties which children encounter in arithmetic can the school expect very much change in the high percentage of failures in this subject". (1)

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(1) G. T. Buswell, "Diagnostic Studies in Arithmetic", page 197.

CHAPTER ELEVEN

Conclusion.

This thesis is mainly concerned with the four simple processes of arithmetic. The mechanical as well as the reasoning phase of the subject are fully discussed under the following topics: aim or function of arithmetic instruction, subject-matter, method, measurement by diagnostic testing and remedial work.

To many pupils arithmetic is the hardest subject in the school. This is indicated by the large percentage of failures in it than in any other school subject. This is partly because,in many elementary schools at present, arithmetic is merely considered as a tool subject and emphasis is only placed on the computational function on the expense of the other functions. This is being done in spite of the fact that pupils never care for abstract, meaningless, numbers, and if they do learn 8 + 7 = 15 and 6 x 9 = 54, etc. it is only because the book or the teacher says so.

In modern schools arithmetic is no longer considered to be only a tool subject, but as a social one of major importance too. Being considered as such, these schools emphasize the computational as well as the other functions of arithmetic, namely, the social, informational and psychological functions. A recognition of these four functions by our schools will lead without doubt, to a reconstruction of the work in arithmetic.

As far as subject-matter is concerned there are all sorts of text-books in arithmetic containing endless amounts of material. This material includes many obsolete topics and types of isolated problems which have long been in general use, and are inapplicable in present-day practice. It is, therefore, needless to say that there is much waste of time and effort in the study of the arithmetic course which is as such. This waste cannot be avoided until some elimination of all the obsolete topics and isolated problems is made, and new material, scientifically constructed is incorporated in the course of arithmetic.

The fundamental operations of arithmetic contain the following material: the basic combinations, the four simple processes, the special language of arithmetic and finally, simple mathematical problems of ordinary life.

An efficient method of arithmetic instruction should result in a good mastery of the combinations and the processes which is the basis of success in arithmetic. Their application to problem solving is very important. It is this latter phase of arithmetic which is the real test of one's ability. Emphasis should, therefore, be placed on both the fundamentals and problem-solving, and both must be carefully taught.

Pupils should know the combinations, both basic and secondary, of the processes with high speed and accuracy. They should also know them automatically and in a mechanical way, so that their brain is set free when the difficulty of problem solving comes in. To be able to do that much drill is necessary..

Drill, here, does not mean mere repetition, but practice with interest in, and full attention to the thing which is being repeated. This is the only sort of practice which makes perfect. Elementary schools do not develop enough speed and accuracy in the simple processes of arithmetic as they ought to. Speed and accuracy should be developed to a great extent in this phase of the work. The grades are usually half as rapid as they ought to be.

The secret of high speed and accuracy in arithmetic lies in adequate drill, distributed according to time and difficulty and applied at the point of error. Moreover, by properly distributed drill the rate of forming habits will be increased and the permanent retention of them is greatly improved.

The more important work of arithmetic is reasoning. Problem comprehension and problem analysis are very essential factors that determine success in problem-solving. The following type of training is useful for the development of pupils' abilities in the solution of verbal problems:

(1) Asking a series of questions, the answers to which would make the meaning of a problem clear.
(2) Making a story out of the problem.
(3) Training in problem analysis.
    a) analyzing the meaning of words used in problems
    b) finding the facts given, the question asked, the processes to be used, estimating the answer in round numbers and verifying it with the real answer.
(4) Solving problems collected by pupils from life situations.
(5) Solving problems without numbers by indicating processes to be used.

The problem of motivation in arithmetic is as important as that of drill. In order to motivate the pupils and keep them interested in thě simple processes the teacher should avoid the use of the old practice which is still being used in many schools. It consists of first, memorizing and repeating tables all through the course, second, long drills on number combinations, third, endless examples in the four simple processes particularly in multiplication and long division.

Similarly to keep pupils interested in problem-solving the teacher should not waste the time of the pupils in trying to solve first, trifling problems, second, problems which will never be needed by pupils and third, problems which have for their only purpose mental discipline.

This old method of teaching arithmetic should be avoided because it segregates arithmetic from the rest of experience and directs it from the interest of the learners. Moreover, it makes arithmetic a formal activity which is done only under compulsion. Instead of all that the teacher should use the following means for motivation: first, races with briefest time and fewest number of mistakes as limits; second, have pupils keep a record of their work and try to surpass it; third, the use of diagnostic tests which leads to the discovery of difficulties and their correction; and fourth, number games of all sorts should be taught to pupils.

Mr. Safford says that in arithmetic, throughout, time must be taken from riddles, puzzles, operations with enormous numbers, long sums, and given to solid work, to practical applications of arithmetic which really mean something.

There are other phases of method than drill and motivation. These should also be known by the efficient teacher and applied. Accordingly the teacher should connect arithmetic with concrete objects, should explain operations inductively rather than deductively, should apply the operations, to interesting situations and connect them with interests coming with the child's experience, should use arithmetic to solve real problems, should use simple language and should train to analyze all the steps in problems. (1)

In all that has been mentioned before, the pupils unless carefully looked after, they adopt a great variety of round-about and clumsy ways of procedure or continue to use the detailed methods employed in learning a process. They should be accustomed from the beginning to good methods of performance. Because for the child whose methods of work are wasteful, to give him more drill is simply to fasten upon him habits of work which are not conducive to good results.

Before the teacher can adequately do that he must understand the mental processes of the pupil on the one hand, and the complexity of the subject for them on the other hand. In addition to that, he should analyze these mental processes in order to discover exactly how pupils carry on their work and to determine the precise difficulties which they experience. It is this full acquaintance with the mental processes of the pupils which is most helpful to the teacher. It enables him to lay out a plan of teaching to help the pupils over-come their difficulties.

One of the means which enables the teacher to do that, is diagnostic testing, which can be given before, during and after instruction, i.e., when the teacher sees it fit to test diagnostically. This paper has given objective evidence of the great value of diagnostic and remedial work in the four simple processes. By using diagnostic tests it is possible for the teacher to analyze the errors made by the pupils in arithmetic and to devise a type of drill adopted to their correction. An advantage of such a procedure is saving many useless and misapplied repetitions and making these used more effective.

Analysis of pupils' errors is not all the job but it should be supplemented by observations of the methods actually used by pupils while solving problems orally. The first technique of diagnosis gives us the general type of mistakes that require general treatment. While the second, shows us the peculiar type of mistakes that require specific treatment, after which we are enabled to devise corrective remedies.

Corrective remedies are necessary and important specially for these mistakes which have been made before. But, what is more important is the application of preventive procedures rather than the corrective ones. After realizing the nature and seriousness of error (its persistence and tendency to recur), the teacher's effort should first be to prevent the occurence of mistakes. The same effort on the part of the teacher should likewise be made to remedy the mistakes which he has not been able to prevent.

This effort results in the adoptation of accurately adjusted methods of procedure. Such work will make out of the teacher a research worker, will develop in him the attitude of studentship. It will also enable him to grow in service and to become truly expert.

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(1) Ref. H.B. Reed, "Psychology of Elementary School Subjects", p. 155.

APPENDIX

Types and Number of the basic combinations of each of the four simple processes.

Addition

The 100 basic combinations are divided into the following types.

The 45 fundamental combinations:

1	1	1	1	1	1	1	1	1
1	2	3	4	5	6	7	8	9
	2	2	2	2	2	2	2	2
	2	3	4	5	6	7	8	9
		3	3	3	3	3	3	3
		3	4	5	6	7	8	9
			4	4	4	4	4	4
			4	5	6	7	8	9
				5	5	5	5	5
				5	6	7	8	9
					6	6	6	6
					6	7	8	9
						7	7	7
						7	8	9
							8	8
							8	9
								9
								9

The 36 reverse combinations:

2					
1					
3	3				
1	2				
4	4	4			
1	2	3			
5	5	5	5		
1	2	3	4		
6	6	6	6	6	
1	2	3	4	5	
7	7	7	7	7	7
1	2	3	4	5	6

$$\begin{array}{ccccccc} 8 & 8 & 8 & 8 & 8 & 8 & 8 \\ \underline{1} & \underline{2} & \underline{3} & \underline{4} & \underline{5} & \underline{6} & \underline{7} \end{array}$$

$$\begin{array}{cccccccc} 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 \\ \underline{1} & \underline{2} & \underline{3} & \underline{4} & \underline{5} & \underline{6} & \underline{7} & \underline{8} \end{array}$$

The zero combinations:

Fundamental form:

$$\begin{array}{cccccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \underline{0} & \underline{1} & \underline{2} & \underline{3} & \underline{4} & \underline{5} & \underline{6} & \underline{7} & \underline{8} & \underline{9} \end{array}$$

Reverse form:

$$\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \end{array}$$

This leads to 100 basic combinations in simple addition.

Subtraction.

The 100 basic combinations in simple subtraction are divided into the following types:

The 45 basic combinations (without the zero combinations and borrowing):

$$\begin{array}{ccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \underline{1} & \underline{1} & \underline{1} & \underline{1} & \underline{1} & \underline{1} & \underline{1} & \underline{1} & \underline{1} \\
 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ & \underline{2} & \underline{2} & \underline{2} & \underline{2} & \underline{2} & \underline{2} & \underline{2} & \underline{2} \\
 & & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ & & \underline{3} & \underline{3} & \underline{3} & \underline{3} & \underline{3} & \underline{3} & \underline{3} \\
 & & & 4 & 5 & 6 & 7 & 8 & 9 \\ & & & \underline{4} & \underline{4} & \underline{4} & \underline{4} & \underline{4} & \underline{4} \\
 & & & & 5 & 6 & 7 & 8 & 9 \\ & & & & \underline{5} & \underline{5} & \underline{5} & \underline{5} & \underline{5} \\
 & & & & & 6 & 7 & 8 & 9 \\ & & & & & \underline{6} & \underline{6} & \underline{6} & \underline{6} \\
 & & & & & & 7 & 8 & 9 \\ & & & & & & \underline{7} & \underline{7} & \underline{7} \\
 & & & & & & & 8 & 9 \\ & & & & & & & \underline{8} & \underline{8} \\
 & & & & & & & & 9 \\ & & & & & & & & \underline{9}
\end{array}$$

The 45 basic combinations involving borrowing:-

$$\begin{array}{ccccccccc}
10 & & & & & & & & \\
\underline{1} & & & & & & & & \\
10 & 11 & & & & & & & \\
\underline{2} & \underline{2} & & & & & & & \\
10 & 11 & 12 & & & & & & \\
\underline{3} & \underline{3} & \underline{3} & & & & & & \\
10 & 11 & 12 & 13 & & & & & \\
\underline{4} & \underline{4} & \underline{4} & \underline{4} & & & & & \\
10 & 11 & 12 & 13 & 14 & & & & \\
\underline{5} & \underline{5} & \underline{5} & \underline{5} & \underline{5} & & & & \\
10 & 11 & 12 & 13 & 14 & 15 & & & \\
\underline{6} & \underline{6} & \underline{6} & \underline{6} & \underline{6} & \underline{6} & & & \\
10 & 11 & 12 & 13 & 14 & 15 & 16 & & \\
\underline{7} & \underline{7} & \underline{7} & \underline{7} & \underline{7} & \underline{7} & \underline{7} & & \\
10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & \\
\underline{8} & \underline{8} & \underline{8} & \underline{8} & \underline{8} & \underline{8} & \underline{8} & \underline{8} & \\
10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 \\
\underline{9} & \underline{9} & \underline{9} & \underline{9} & \underline{9} & \underline{9} & \underline{9} & \underline{9} & \underline{9}
\end{array}$$

The 10 zero combinations:-

$$\begin{array}{cccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0}
\end{array}$$

This leads to 100 basic combinations in simple subtraction, from 0 - 0 to 18 - 9.

Multiplication.

The 100 basic combinations in simple multiplication are divided into the following types:

$$\begin{array}{ccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\underline{1} & \underline{1} & \underline{1} & \underline{1} & \underline{1} & \underline{1} & \underline{1} & \underline{1} & \underline{1} \\
 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
 & \underline{2} & \underline{2} & \underline{2} & \underline{2} & \underline{2} & \underline{2} & \underline{2} & \underline{2} \\
 & & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
 & & \underline{3} & \underline{3} & \underline{3} & \underline{3} & \underline{3} & \underline{3} & \underline{3} \\
 & & & 4 & 5 & 6 & 7 & 8 & 9 \\
 & & & \underline{4} & \underline{4} & \underline{4} & \underline{4} & \underline{4} & \underline{4} \\
 & & & & 5 & 6 & 7 & 8 & 9 \\
 & & & & \underline{5} & \underline{5} & \underline{5} & \underline{5} & \underline{5}
\end{array}$$

$$\begin{array}{cccc} 6 & 7 & 8 & 9 \\ \underline{6} & \underline{6} & \underline{6} & \underline{6} \\ & 7 & 8 & 9 \\ & \underline{7} & \underline{7} & \underline{7} \\ & & 8 & 8 \\ & & \underline{8} & \underline{8} \\ & & & 9 \\ & & & \underline{9} \end{array}$$

The 36 reverse multiplication combinations:-

$$\begin{array}{cccccccc} 1 \\ \underline{2} \\ 1 & 2 \\ \underline{3} & \underline{3} \\ 1 & 2 & 3 \\ \underline{4} & \underline{4} & \underline{4} \\ 1 & 2 & 3 & 4 \\ \underline{5} & \underline{5} & \underline{5} & \underline{5} \\ 1 & 2 & 3 & 4 & 5 \\ \underline{6} & \underline{6} & \underline{6} & \underline{6} & \underline{6} \\ 1 & 2 & 3 & 4 & 5 & 6 \\ \underline{7} & \underline{7} & \underline{7} & \underline{7} & \underline{7} & \underline{7} \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \underline{8} & \underline{8} & \underline{8} & \underline{8} & \underline{8} & \underline{8} & \underline{8} \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \underline{9} & \underline{9} & \underline{9} & \underline{9} & \underline{9} & \underline{9} & \underline{9} & \underline{9} \end{array}$$

The zero combinations:

Fundamental form:

$$\begin{array}{cccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \end{array}$$

Reverse form:

$$\begin{array}{ccccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \underline{1} & \underline{2} & \underline{3} & \underline{4} & \underline{5} & \underline{6} & \underline{7} & \underline{8} & \underline{9} \end{array}$$

This leads to 100 basic combinations in simple multiplications, from 0 x 0 to 9 x 9.

Division

The following are the 90 basic combinations in simple division, from 0 - 0 to 81 - 9 arranged on the basis of the divisor:

0÷1	0÷2	0÷3	0÷4	0÷5	0÷6	0÷7	0÷8	0÷9
1÷1	2÷2	3÷3	4÷4	5÷5	6÷6	7÷7	8÷8	9÷9
2÷1	4÷2	6÷3	8÷4	10÷5	12÷6	14÷7	16÷8	18÷9
3÷1	6÷2	9÷3	12÷4	15÷5	18÷6	21÷7	24÷8	27÷9
4÷1	8÷2	12÷3	16÷4	20÷5	24÷6	28÷7	32÷8	36÷9
5÷1	10÷2	15÷3	20÷4	25÷5	30÷6	35÷7	40÷8	45÷9
6÷1	12÷2	18÷3	24÷4	30÷5	36÷6	42÷7	48÷8	54÷9
7÷1	14÷2	21÷3	28÷4	35÷5	42÷6	49÷7	56÷8	63÷9
8÷1	16÷2	24÷3	32÷4	40÷5	48÷6	56÷7	64÷8	72÷9
9÷1	18÷2	27÷3	36÷4	45÷5	54÷6	63÷7	72÷8	81÷9

Subtraction

There are 175 secondary combinations in subtraction which are prerequisite in uneven short division. Arranged according to the subtrahand in an ascending order, they are as follows:

11	12	13	14	13	14	15	16	17	15	16	17	18	19	20	16
10	10	10	10	12	12	12	12	12	14	14	14	14	14	14	15
17	18	19	17	18	19	20	21	22	23	19	20	21	22	23	24
15	15	15	16	16	16	16	16	16	16	18	18	18	18	18	18
25	26	21	22	23	24	22	23	24	25	26	27	25	26	27	28
18	18	20	20	20	20	21	21	21	21	21	21	24	24	24	24
29	30	31	26	27	28	29	28	29	30	31	32	33	34	35	29
24	24	24	25	25	25	25	27	27	27	27	27	27	27	27	28
30	31	32	33	34	31	32	33	34	35	33	34	35	36	37	38
28	28	28	28	30	30	30	30	30	30	32	32	32	32	32	32
39	36	37	38	39	40	41	37	38	39	40	41	42	43	44	41
32	35	35	35	35	35	35	36	36	36	36	36	36	36	36	40
42	43	44	45	46	47	43	44	45	46	47	48	46	47	48	49
40	40	40	40	40	40	42	42	42	42	42	42	45	45	45	45
50	51	52	53	49	50	51	52	53	54	55	50	51	52	53	54
45	45	45	45	48	48	48	48	48	48	48	49	49	49	49	49
55	55	56	57	58	59	60	61	62	57	58	59	60	61	62	63
49	54	54	54	54	54	54	54	54	56	56	56	56	56	56	56
64	65	66	67	68	69	70	71	65	66	67	68	69	70	71	73
63	63	63	63	63	63	63	63	64	64	64	64	64	64	64	72
74	75	76	77	78	79	80	82	83	84	85	86	87	88	89	
72	72	72	72	72	72	72	81	81	81	81	81	81	81	81	

The above mentioned combinations, of all kinds, should be learned by the pupils as facts and not as processes. They should be known automatically and not through analysis.

Division

There are 324 facts in division with remainders.

These facts are grouped according to the divisor.

3÷2
5÷2
7÷2
9÷2
11÷2
13÷2
15÷2
17÷2
19÷2

4÷3
5÷3
7÷3
8÷3
10÷3
11÷3
13÷3
14÷3
16÷3

17÷3
19÷3
20÷3
22÷3
23÷3
25÷3
26÷3
28÷3
29÷3

5÷4
6÷4
7÷4
9÷4
10÷4
11÷4
13÷4
14÷4
15÷4

17÷4
18÷4
19÷4
21÷4
22÷4
23÷4
25÷4
26÷4
27÷4

29÷4
30÷4
31÷4
33÷4
34÷4
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Courtis Standardized Research Tests.

The following are Courtis Standardized Research Tests, Series A and Series B, as they are found in D. Starch "Educational Measurements" pp. 116-124.

I have included these tests here for the following reasons:

1. Because they are the ones which I have used.
2. To give the reader a concrete idea and full acquaintance with these tests.
3. To give chance to those who like to use them.
4. Because this is the only complete set of standardized tests which I could find in the available sources on the subject.

Arithmetic Test No. 1 Speed Test.

Addition - Series A.

The standard June scores, based on over 60,000 pupils, are as f follows:

Grade :	3	4	5	6	7	8
Rights:	26	34	42	50	58	63

Direction: Write on this paper, in the space between the lines, the answers to as many of these addition examples as possible in the time allowed, (one minute is allowed):

1	7	9	3	2	1	3	6	0	3	1	4	8	0	2	1	8	6
3	7	6	0	4	5	8	9	7	2	6	7	9	5	7	9	4	7
0	5	9	2	5	0	6	2	4	5	1	6	4	8	5	0	7	1
2	4	1	8	7	4	3	1	8	9	0	2	4	2	6	9	3	6
4	9	0	4	8	9	7	8	2	3	4	7	7	3	1	2	5	6
7	5	1	2	1	9	6	0	5	1	6	9	8	5	4	9	8	0
7	2	9	7	4	5	3	7	9	0	4	5	9	6	7	5	1	6
2	2	3	8	0	2	3	4	8	6	5	5	2	8	0	3	8	4
7	0	2	1	7	9	3	2	1	6	9	0	4	5	8	6	9	4
5	3	6	3	7	6	0	8	2	6	5	1	2	1	3	5	0	3
1	3	8	2	3	4	8	9	5	3	7	4	8	0	3	6	9	8
7	9	5	0	7	1	8	7	7	6	1	9	6	0	4	1	4	7
1	2	8	9	7	8	5	1	2	6	0	1						
1	3	1	9	6	0	2	6	7	9	7	2						

Arithmetic Test No. 2 - Speed Test.

Subtraction - Series A.

The standard June scores, based on over 60,000 pupils are as follows:

Grade :	3	4	5	6	7	8
Rights:	19	25	31	38	44	49

Write on this paper, in the space between the lines, the answers to as many of these subtraction examples as possible in the time allowed, (one minute is allowed):

8	11	12	5	10	1	9	13	4	12	4	9	6	7	11	2
0	9	7	1	2	0	7	8	3	6	0	6	9	1	3	0

10	11	91	4	8	13	12	8	13	3	8	14	6	10	8	11
6	5	8	7	8	9	5	7	4	0	4	9	5	4	0	9

12	5	10	5	8	17	6	11	9	7	11	8	12	4	10	3
7	1	2	5	6	9	4	8	9	3	6	1	3	4	7	5

10	9	1	6	15	4	8	9	7	14	15	11	2	12	15	3
1	4	1	3	9	2	3	0	2	8	4	2	2	8	7	1

10	7	6	13	10	9	3	9	14	4	18	1	9	13	4	12
3	7	2	6	8	3	3	5	6	1	9	0	7	8	3	6

6	11	16	10	12	2	7	13	3	10	7	8	16	9	11	5
0	7	8	9	4	1	5	7	2	5	0	5	7	1	4	0

12	15	5	16	6	7	17	6	9	0	5	14	7	8	6	11	15
9	6	3	8	6	4	8	1	2	0	2	5	6	2	0	7	8

10	12	9	7	11	8	12
9	4	9	3	6	1	3

Arithmetic Test No. 3 - Speed Test

Multiplication - Series A

The standard June scores, based on over 60,000 pupils are as follows:

Grade :	3	4	5	6	7	8
Rights :	16	23	30	37	41	45

Write on this paper, in the space between the lines, the answers to as many of these multiplication examples as possible in the time allowed (one minute is allowed):

4	2	7	4	9	9	5	4	7	6	1	2	8	1	5	6	2	8
1	9	6	0	5	1	2	8	0	5	9	5	7	1	3	1	7	4

9	5	1	4	8	0	4	1	6	8	0	9	1	6	9	1	7	9
0	7	5	4	9	3	5	4	2	8	7	3	8	2	4	0	2	1
5	4	7	6	2	3	9	0	7	1	2	7	0	8	2	5	6	0
2	8	0	5	1	3	6	5	4	6	8	7	6	3	3	5	9	8
7	3	2	6	0	8	1	3	6	0	3	2	5	4	3	7	1	9
3	1	4	7	1	5	7	4	8	0	9	2	8	9	0	5	3	2
8	0	3	4	2	7	4	9	3	4	9	0	5	8	2	7	5	4
6	4	7	1	9	6	0	5	2	7	8	2	6	1	6	9	0	6
1	3	7	6	5	4	3	9	8	6	7	3	9	2	4	5	8	6
2	5	8	0	9	2	6	7	0	4	1	8	9	0	3	1	2	3
0	5	3	4	9	0	5	2	3	9	0	7						
9	4	2	7	8	2	6	1	3	6	5	4						

Arithmetic Test - Speed Test

Division - Series A.

The standard June scores, based on over 60,000 pupils, are as follows:

Grade :	3	4	5	6	7	8
Rights :	16	23	30	37	44	49

Write on this paper in the space between the lines, the answers to as many of these division examples as possible in the time allowed, (one minute is allowed):

9)9 3)21 6)48 1)1 5)10 3)9 4)32 6)36 2)0 7)28 1)8

5)30 8)72 1)0 9)36 2)6 4)24 7)63 6)0 8)32 1)4 5)35

9)45 2)2 3)12 8)8 4)28 5)40 2)2 8)16 5)5 4)36 8)45

0)8 4)12 1)5 2)16 8)48 1)2 9)27 3)6 4)20 7)49 1)3

2)8 1)7 2)10 7)42 1)1 6)18 5)0 3)24 9)63 2)4 8)24

6)6 3)27 8)64 1)2 4)16 7)7 2)18 6)42 3)0 7)21 5)5

2)14 8)40 9)0 5)15 4)4 3)15 9)81 7)0 6)12 4)4 6)30

8)56 1)0 7)14 1)5 3)18 9)72 4)0 6)24 1)4 2)12 5)25

3)3 8)8 1)6 7)35 6)54 1)3 5)20 1)9 5)25 7)56 3)3

9)18 1)8 5)30 8)72 1)6 9)36 9)9 3)21 6)48 1)1 5)10

3)9 4)32 6)36 2)0 7)28 2)5 4)25 7)35 6)0 8)32

Courtis Tests - Series B

Addition.

The tests are scored by determining the number of problems done correctly. The following are the standard scores for the various grades, derived from approximately 25,000 pupils.

Grade :	3	4	5	6	7	8
Attempts :	4.0	6.0	7.0	9.0	10.5	12.0
Rights :	2.0	3.0	4.0	5.0	6.0	8.0

You will be given 8 minutes to find the answer to as many of these addition examples as possible. Write the answers on this paper directly underneath the example. You will be marked for both speed and accuracy, but it is more important to have your answers right than to try a great many examples.

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
127	996	237	386	186	474	877	537
375	320	949	463	775	787	845	685
953	778	489	827	684	591	981	452
333	886	987	240	260	106	693	904
325	913	354	616	372	869	184	511
911	164	600	261	846	451	772	988
554	897	744	755	595	336	749	559
167	972	195	833	254	820	256	127
554	119	234	959	137	533	258	323

(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)
237	564	632	674	421	258	326	267
492	278	263	158	988	885	770	854
679	947	318	745	465	600	753	684
513	522	949	121	114	874	199	358
468	989	746	437	676	726	469	938
731	243	653	426	729	142	643	333
856	334	428	953	235	355	698	493
302	669	456	674	190	947	186	775
925	142	532	329	406	351	173	239

(17)	(18)	(19)	(20)	(21)	(22)	(23)	(24)
873	622	485	172	236	537	648	584
168	479	871	426	578	227	396	157
332	283	524	951	877	725	389	617
419	791	919	537	916	598	374	624
934	808	722	989	543	906	859	467
493	253	456	565	593	763	191	369
529	419	216	230	956	195	423	511
156	952	862	673	459	480	849	245
224	524	424	258	309	108	342	233

Courtis Tests - Series B.

Subtraction.

The tests are scored by determining the number of problems done correctly. The following are the standard scores for the various grades, derived from approximately 25,000 pupils:

Grade :	3	4	5	6	7	8
Attempts:	4	6	8	10	11.5	12.5
Rights :	1	3	5.5	7	8.5	10

You will be given four minutes to find the answers to as many of these subtraction examples as possible. Write the answers on this paper directly underneath the example. You are not expected to be able to do them all. You will be marked for both speed and accuracy, but it is more important to have your answers right than to try a great many examples.

114957187 90271797	94752808 67349640	106089449 16915390	99833978 73160227	115171700 63087381	82484740 48207825
115916913 55536329	72229470 45049173	146246252 52160891	80650266 68164329	123385018 73098624	107419373 65348405
37953635 23913884	137825921 62789490	152695030 85612816	178976226 93060305	97089301 20203267	93994413 54783938
108051861 73463849	163130569 91061255	168354186 70537861	188545364 92471259	120981427 64188045	105755782 90863147

Courtis Tests - Series B

Multiplication.

The tests are scored by determining the number of problems done correctly. The following are the standard scores for the various grades, derived from approximately 25,000 pupils:

Grade :	4	5	6	7	8
Attempts:	4.5	7.0	8.5	10.0	11.5
Rights :	1.5	4.0	5.5	6.5	8.0

You will be given 6 minutes to work as many of these multiplication examples as possible. You are not expected to be able to do them all. Do your work directly on this paper; use no other. You will be marked for both speed and accuracy, but it is more important to have your answers right than to try a great many examples.

$\begin{array}{r}8259\\ \underline{28}\end{array}$	$\begin{array}{r}3467\\ \underline{93}\end{array}$	$\begin{array}{r}4637\\ \underline{82}\end{array}$	$\begin{array}{r}2859\\ \underline{47}\end{array}$	$\begin{array}{r}7436\\ \underline{65}\end{array}$	$\begin{array}{r}5289\\ \underline{39}\end{array}$	$\begin{array}{r}6437\\ \underline{740}\end{array}$	$\begin{array}{r}8529\\ \underline{56}\end{array}$	$\begin{array}{r}8632\\ \underline{206}\end{array}$
$\begin{array}{r}5947\\ \underline{62}\end{array}$	$\begin{array}{r}3268\\ \underline{95}\end{array}$	$\begin{array}{r}4795\\ \underline{83}\end{array}$	$\begin{array}{r}7954\\ \underline{74}\end{array}$	$\begin{array}{r}2386\\ \underline{38}\end{array}$	$\begin{array}{r}9745\\ \underline{59}\end{array}$	$\begin{array}{r}6283\\ \underline{47}\end{array}$	$\begin{array}{r}9624\\ \underline{503}\end{array}$	$\begin{array}{r}7853\\ \underline{35}\end{array}$
$\begin{array}{r}4926\\ \underline{620}\end{array}$	$\begin{array}{r}5873\\ \underline{49}\end{array}$	$\begin{array}{r}2964\\ \underline{94}\end{array}$	$\begin{array}{r}8357\\ \underline{87}\end{array}$	$\begin{array}{r}6249\\ \underline{78}\end{array}$	$\begin{array}{r}3875\\ \underline{35}\end{array}$			

Courtis Tests - Series B

Division.

These tests are scored by determining the number of problems done correctly. The following are the standard scores for the various grades derived from approximately 25,000 pupils:

Grade :	4	5	6	7	8
Attempts:	3.5	5.0	6.5	8.5	10.5
Rights :	1.0	3.0	5.0	7.0	9.0

You will be given 8 minutes to work as many of these division examples as possible. You are not expected to be able to do them all. Do your work directly on this paper; use no other, you will be marked for both speed and accuracy, but it is more important to have your answers right than to try a great many examples:

$24\overline{)5984}$	$95\overline{)85880}$	$36\overline{)10440}$	$87\overline{)81867}$	$78\overline{)62968}$	$42\overline{)17682}$
$63\overline{)26460}$	$59\overline{)50799}$	$36\overline{)16236}$	$87\overline{)61161}$	$95\overline{)69350}$	$24\overline{)10800}$
$63\overline{)42903}$	$42\overline{)28560}$	$59\overline{)29913}$	$78\overline{)44538}$	$29\overline{)34679}$	$57\overline{)51642}$
$38\overline{)32300}$	$64\overline{)61504}$	$46\overline{)34086}$	$75\overline{)55500}$	$92\overline{)27784}$	$83\overline{)26643}$

Sample Representing Division into mixed Lessons, Order of Difficulty and Frequency of the Basic Combination of Addition and Subtraction of the Simple Processes.

Lesson	Rank No.	Combination	Frequency
I	1	0 + 0	7
	2	5 + 5	"
	3	2 + 2	"
	4	2 + 1	"
	5	1 + 7	"
II	6	1 + 6	9
	7	1 + 3	"
	8	1 + 1	"
	9	3 + 3	"
	10	5 + 4	"
III	11	6 + 1	11
	12	2 + 0	"
	13	1 + 8	"
	14	4 + 0	"
	15	3 + 1	"
IV	16	4 + 1	13
	17	4 + 4	"
	18	2 + 8	"
	19	2 + 5	"
	20	7 + 1	"
VII	21	1 + 5	15
	22	4 + 3	"
	23	3 + 2	"
	24	5 + 0	"
	25	9 + 1	"
VIII	26	1 + 4	17
	27	6 + 4	"
	28	8 + 2	"
	29	6 + 0	"
	30	0 + 8	"
XIX	31	4 + 5	19
	32	3 + 0	"
	33	2 + 4	"
	34	0 + 2	"
	35	3 + 6	"
XII	36	0 + 4	21
	37	6 + 2	"
	38	8 + 1	"
	39	0 + 5	"
	40	1 + 0	"

Lesson	Rank No.	Combination	Frequency
V	1	0 - 0	7
	2	5 - 1	"
	3	7 - 7	"
	4	3 - 2	"
	5	3 - 3	"
	6	6 - 6	"
VI	7	4 - 2	9
	8	8 - 1	"
	9	1 - 1	"
	10	5 - 4	"
	11	5 - 3	"
	12	9 - 9	"
X	13	9 - 1	11
	14	4 - 4	"
	15	9 - 8	"
	16	6 - 5	"
	17	8 - 4	"
	18	5 - 5	"
XI	19	4 - 3	13
	20	7 - 1	"
	21	5 - 2	"
	22	6 - 3	"
	23	5 - 2	"
	24	2 - 1	"
XV	25	6 - 1	15
	26	6 - 4	"
	27	7 - 5	"
	28	2 - 2	"
	29	4 - 1	"
	30	8 - 7	"
XVI	31	9 - 5	17
	32	7 - 6	"
	33	10 - 2	"
	34	6 - 2	"
	35	10 - 1	"
	36	11 - 2	"
XX	37	8 - 6	19
	38	10 - 7	"
	39	12 - 6	"
	40	7 - 2	"
	41	9 - 0	"
	42	6 - 0	"

Lesson	Rank No.	Combination	Frequency
XIII	41	7 + 2	23
	42	0 + 6	"
	43	0 + 7	"
	44	7 + 0	"
	45	5 + 1	"
XIV	46	0 + 9	25
	47	0 + 5	"
	48	5 + 3	"
	49	1 + 2	"
	50	4 + 2	"
XVII	51	5 + 2	27
	52	1 + 0	"
	53	8 + 0	"
	54	4 + 0	"
	55	8 + 4	"
XVIII	56	2 + 7	29
	57	7 + 3	"
	58	6 + 3	"
	59	3 + 5	"
	60	2 + 3	"
XIX	61	3 + 9	31
	62	3 + 4	"
	63	3 + 8	"
	64	6 + 5	"
	65	0 + 1	"
XXII	66	9 + 3	33
	67	2 + 6	"
	68	9 + 0	"
	69	3 + 7	"
	70	9 + 8	"
XXIII	71	7 + 4	35
	72	7 + 6	"
	73	4 + 6	"
	74	5 + 6	"
	75	6 + 7	"
XXIV	76	9 + 2	37
	77	6 + 6	"
	78	8 + 3	"
	79	2 + 9	"
	80	7 + 7	"
XXVII	81	4 + 9	39
	82	9 + 4	"
	83	9 + 5	"
	84	7 + 5	"
	85	4 + 7	"

Lesson	Rank No.	Combination	Frequency
XXI	43	7 - 3	21
	44	9 - 3	"
	45	8 - 8	"
	46	10 - 4	"
	47	8 - 2	"
	48	9 - 6	"
XXV	49	5 - 0	23
	50	3 - 0	"
	51	4 - 0	"
	52	9 - 4	"
	53	9 - 7	"
	54	2 - 0	"
XXVI	55	8 - 0	25
	56	10 - 6	"
	57	10 - 5	"
	58	12 - 9	"
	59	7 - 0	"
	60	8 - 3	"
XXX	61	1 - 0	27
	62	12 - 8	"
	63	7 - 4	"
	64	10 - 8	"
	65	12 - 4	"
	66	14 - 7	"
XXXI	67	10 - 9	29
	68	10 - 3	"
	69	9 - 2	"
	70	11 - 8	"
	71	11 - 9	"
	72	8 - 5	"
XXXIII	73	15 - 8	31
	74	13 - 6	"
	75	11 - 7	"
	76	11 - 9	"
	77	18 - 9	"
	78	16 - 8	"
XXXIV	79	11 - 6	33
	80	11 - 4	"
	81	12 - 7	"
	82	12 - 5	"
	83	14 - 8	"
	84	15 - 6	"
XXXV	85	12 - 3	35
	86	14 - 6	"
	87	11 - 3	"
	88	13 - 8	"
	89	13 - 7	"
	90	16 - 7	"

Lesson	Rank No.	Combination	Frequency
XXVIII	86	8 + 6	41
	87	8 + 9	"
	88	5 + 9	"
	89	9 + 6	"
	90	8 + 7	"
XXIX	91	7 + 8	43
	92	5 + 7	"
	93	6 + 9	"
	94	6 + 8	"
	95	9 + 7	"
XXXII	96	5 + 8	45
	97	7 + 9	"
	98	8 + 5	"
	99	9 + 9	"
	100	8 + 8	"

Lesson	Rank No.	Combination	Frequency
XXXVI	91	13 - 5	37
	92	13 - 9	"
	93	15 - 7	"
	94	17 - 8	"
	95	15 - 9	"
XXXVII	96	17 - 9	39
	97	14 - 5	"
	98	16 - 9	"
	99	13 - 4	"
	100	14 - 9	"

The above plan gives a clear idea about the organization of the instruction material concerning the number facts. The following points are clearly shown by the plan:

(1) The presentation of the number facts in order of difficulty the easiest comes first.

(2) The content of each lesson is clearly indicated.

(3) The frequency of each combination is proportional to its difficulty.

(4) Sufficient material is provided to afford ample practice

(5) Finally, addition and subtraction are presented alternately. Teaching each process alone is an old practice.

The number facts of multiplication and division form the other part of the plan and are treated similarly.

INAPPROPRIATE HABITS

As a result of extended diagnostic studies in arithmetic, Professor Buswell says that there is a great variety of habits of work employed by children in dealing with the four fundamental operations. This fact leads to the conclusion that many of these habits are the result of the Child's own inventiveness, necessitated by the lack of adequate teaching on the part of the school. Left to his own devices, the child has stumbled upon many methods which are uneconomical and which might have been avoided if the teacher had understood the methods which the pupil was actually using.

Several inappropriate habits were actually exhibited by the pupils in adding. It is the duty of the teacher to discover the habits of work which each pupil has formed and to give help where some of these round-about habits do appear. The following lists of inappropriate habits have been prepared by Buswell and John (1).

"1. Made errors in combinations.
2. Counted on fingers.
3. Added carried number last.
4. Forgot to add carried number.
5. Retracted work partly done.
6. Added carried number irregularly.
7. Wrote number to carried.
8. Carried wrong number.
9. Proceeded irregularly in column.
10. Grouped two or more numbers.
11. Used wrong number operations.
13. Lost place in column.
14. Depended on visualization.
15. Disregarded column position.
16. Omitted one or more digits.
17. Made errors in reading numbers.
18. Dropped back one or more decades.
19. Derived unknown combinations from familiar one.
20. Disregarded one column.
21. Made error in writing answer.
22. Skipped one or more decades.
23. Carried when there was nothing to carry.
24. Used scratched paper.
25. Added in pairs, giving last sum as answer.
26. Added same digit in two columns.
27. Wrote carried number in answer.
28. Added same number twice.
29. Began with left column.
30. Confused columns.
31. Added carried number twice.
32. Subtracted carried number.
33. Added imaginary column."

Subtraction

1. Errors in combinations.
2. Did not allow for having borrowed.
3. Counting.
4. Errors due to zero in minuend.
5. Said example backward.
6. Subtracted minuend from subtrahend.
7. Failed to borrow, gave zero as answer.

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(1) Ref. G. T. Buswell and L. John, "Diagnostic Studies in Arithmetic" pp. 136-139.

8. Added instead of subtracting.
9. Error in reading.
10. Used same digit in two columns.
11. Derived unknown from known combinations.
12. Omitted column.
13. Deducted from minuend when borrowing was not necessary.
14. Split numbers.
15. Used trial and errors addition.
16. Ignored addigit.
17. Deducted 2 from the minuend and subtrahend digits being same.
18. Used minuend or subtrahend as remainder.
19. Reversed digits in remainder.
20. Confused process with division or multiplication.
21. Skipped one or more decades.
22. Increased minuend digit after borrowing.
23. Based subtraction on multiplication combinations.
24. Errors in writing answer.
25. Began at left column.
26. Deducted all borrowed numbers from left-hand digit."

Multiplication

"1. Errors in multiplication combinations.
2. Errors in adding the carried number.
3. Wrote rows of zeros.
4. Carried a wrong number.
5. Errors in addition.
6. Used multiplicand as a multiplier.
7. Forgot to carry.
8. Errors in single zero combinations, zero as multiplier.
9. Errors due to zero in multiplier.
10. Used wrong process.
11. Counted to carry.
12. Omitted digit in multiplier.
13. Wrote carried number.
14. Omitted digit in multiplicand.
15. Errors due to zero in multiplicand.
16. Counted to get multiplication combinations.
17. Error in position of partail product.
18. Error in single zero combinations, zero as multiplicand.
19. Confused products when multiplier had two or more digits.
20. Repeated part of table.
21. Multiplied by adding.
22. Did not multiply a digit in multiplicand.
23. Derived unknown combination from another.
24. Errors in reading.
25. Omitted digit in product.
26. Errors in writing product.
27. Errors in carrying into a zero.
28. Illegible figures.
29. Forgot to add partial products.
30. Split multiplier.
31. Wrote wrong digit of product.
32. Multiplied by same digit twice.
33. Reversed digits in product.
34. Wrote tables.
35. Used multiplicand or multiplier as product.
36. Multiplied carried number.

37. Used digit in product twice.
38. Added carried number twice.
39. Carried when there was nothing to carry.
40. Began at left.
41. Multiplied partial products".

Division

"1. Errors in division combinations.
2. Errors in subtraction.
3. Errors in multiplication.
4. Used remainder larger than divisor.
5. Found quotient by trial multiplication.
6. Neglected to use remainder within example.
7. Omitted digit in dividend.
8. Used wrong operation.
9. Omitted zero resulting from another digit.
10. Counted to get quotient.
11. Repeated part of multiplication table.
12. Used short division form for long division.
13. Wrote remainders within example.
14. Omitted final remainder.
15. Omitted zero resulting from zero dividend.
16. Used long division form for short division.
17. Counted in subtracting.
18. Used too large a product.
19. Said example backward.
20. Used remainder without new dividend.
21. Derived unknown combination from known one.
22. Grouped too many digits in dividend.
23. Had right answer by used wrong one.
24. Error in reading.
25. Used dividend or divisor as quotient.
26. Reversed dividend and divisor.
27. Found quotient by adding.
28. Used digits of divisor separately.
29. Wrote all remainders at end of example.
30. Misinterpreted table.
31. Used digit in dividend twice.
32. Used second digit of divisor to find quotient.
33. Began dividing at unit's digit of dividend.
34. Split dividend.
35. Used endings to find quotient. (long division).
36. Added remainder to quotient.
37. Added zeros to dividend when quotient was not a whole number.
38. Added remainder to next digit of dividend.
39. Wrote rows of zeros.
40. Illegible figures.
41. Dropped zero from divisor and not from dividend".

DIAGNOSTIC CHART FOR FUNDAMENTALS OF ARITHMETIC

Commonwealth Fund Arithmetic Investigation
(Experimental Edition)
Prepared by G. T. Buswell & Lenore John

Name ______________ School ______________________ Date ___________

Age _________ Grade _____ I.Q. _____ Arith. Age _____ on _________Test

Teacher's preliminary diagnosis ______________________________________

ADDITION: (Place a check before each habit observed in the pupil's work)

__ a1 Errors in combinations
__ a2 Counting
__ a3 Split numbers
__ a4 Added carried number last
__ a5 Forgot to add carried number
__ a6 Irregular procedure in column
__ a7 Carried wrong number
__ a8 Grouped two or more numbers
__ a9 Dropped back one or more tens
__ a10 Added carried number irregularly
__ a11 Omitted one or more digits
__ a12 Disregarded column position
__ a13 Wrote number to be carried
__ a14 Depended on visualization
__ a15 Added same digit in two columns.
__ a16 Skipped one or more details
__ a17. Retraced work after partly done.
__ a18 Used wrong fundamental operation.
__ a19 Derived unknown combination from familiar one
__ a20 Error in writing answer
__ a21 Carrying when there was nothing to carry
__ a22 Disregarded one column
__ a23 Errors in reading numbers (1)

(Write observation notes on pupil's work in space opposite examples)

---

(1)
5 6
2 3

(2)
2 8
9 4

(3)
12 13
2 5

(4)
19 17
2 9

(5)
6 + 2 =
3 + 4 =

(6)
52 40
13 39

(7)
78 46
71 92

(8)
3 8
5 7
8 9
2 7

---

(1) There are five habits and fifteen examples on the original chart not listed here.
This chart is adapted from G.T.Buswell, & L.John, "Diagnostic Studies in Arithmetic", page 108.

Specific Skills

Addition

There are eight such skills in addition to be mastered by the pupil.

1. 100 combinations with the ten digits, including the zeros.
2. 225 combinations in the higher decade:
    a. 135 of which consist of sums that do not require bridging the tens, such as 43 + 4
    b. 90 combinations require bridging the tens, such as 43 + 8.
3. The child must learn to read an example when it is:
    a. In equation form, as 4 + 2 = ?
    b. In column for, as $\begin{array}{r}4\\ \underline{2}\end{array}$
    c. In the form of words, as 4 plus 2.
4. He must also be able to read the problem when the figures are:
    a. in words, as four plus two, or four and two.
    b. in the form of pictures, as in a picture of four sticks and two sticks.
5. In adding problems that do not require carrying he must be able to:
    a. Add one digit to another.
    b. Two digits to another.
    c. Two or more digits to two or more digits:
        (1) no zero digits.
        (2) some digits are zeros.
6. He must also do each of these with carrying.
7. He must learn how:
    a. to carry when one column consists of zeros.
    b. to put down properly the sum of the last column to the left when it is greater than nine.
8. He must learn:
    a. to proceed from right to left.
    b. to keep his place in the column.
    c. to add unseen sums to seen numbers.
    d. to ignore empty spaces
    e. to ignore zeros.
    f. to record only the units of each column at its foot except in the last column. (1)

Subtraction

The following list, prepared by Miss Morton, consists of the basic elements which may be considered to make up the general ability to subtraft whole numbers. (2)

1. The 100 basic subtraction combinations.
2. Three ideas in one's subtraction concept
    a. taking away idea : 15 - 7
    b. adding idea: what number added to 7 equals 15
    c. difference idea: 15 is how many more than 7.
3. The meaning of the following terms: minus, less, subtrahend, minuend, borrowing, difference and remainder.
4. The meaning of the subtraction sign.
5. That the complete minuend must always be larger than the complete subtrahend.

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(1) Ref. H. B. Reed, "Psychology of Elementary School Subjects" p. 112.
(2) Ref. L. J. Brueckner and B.O. Melby "Diagnostic and Real Teaching" p. 196.

6. That in writing the examples, units must be placed under units, tens, under tens, etc.
7. That one must begin at the right and work to the left.
8. That the order of units in the subtrahend must be subtracted from the same order in the minuend.
9. How to proceed when the first number to be subtracted in the minuend is larger than the corresponding number in the subtrahend.
10. That one must not borrow unless the number in the subtrahend is larger than the corresponding number in the minuend.
11. How to proceed when a number in the subtrahend is larger than the corresponding number in the minuend. i.e., "borrowing".
12. What is meant to place a 1 (one) in front of a number when necessary $\begin{array}{r}423\\ \underline{219}\\ 204\end{array}$ 1 added in terms of ten.
13. What it does to the next number in the minuend when a 1 has been placed before the following number.
14. Must be able to remember the new number made through borrowing: $\begin{array}{r}628\\ \underline{239}\\ 389\end{array}$ after subtracting 9 from 18, the child is dealing with 11, not 12.
15. How to proceed when the need for borrowing and no borrowing are met alternately in the example.
16. How to borrow when two or more successive digits in the subtrahend are larger than the corresponding digits in the minuend.
17. How to proceed when there are fewer figures in the subtrahend than in the minuend.
18. How to proceed when the last subtraction takes place with the subtrahend and minuend the same. $\begin{array}{r}649\\ \underline{625}\\ 025\end{array}$ the zero must not be put in the remainder.
19. Ability to handle a zero or a succession of zeros in the subtrahend.
20. Ability to handle a zero or a succession of zeros in the minuend.
21. How to check for correct answers.

The following set of examples suggests a suitable arrangement for the step by step introduction of these skills:-

$$\begin{array}{rrrrrrrrrrrrrr}8 & 75 & 14 & 47 & 27 & 128 & 95 & 60 & 75 & 962 & 708 & 845 & 45 & 435\\ \underline{6} & \underline{12} & \underline{6} & \underline{3} & \underline{24} & \underline{75} & \underline{28} & \underline{47} & \underline{74} & \underline{138} & \underline{495} & \underline{296} & \underline{8} & \underline{86}\end{array}$$

$$\begin{array}{rrrrrrr}600 & 5276 & 8764 & 9562 & 6001 & 8070 & 8100\\ \underline{497} & \underline{1427} & \underline{5278} & \underline{7698} & \underline{2746} & \underline{5042} & \underline{7263}\end{array}$$ (1)

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(1) Ref. L. J. Brueckner and E.O. Melby: "Diagnostic and Remedial Teaching", pp. 197-198.

THE MOST COMMON FAULTS

The following lists contain the chief difficulties which students meet in the simple processes. It is, without doubt, for the welfare of the pupils to have the teacher keep in mind these difficulties, and to determine how each of them should be satisfactorily met.

Addition

"1. Errors in combinations.
2. Counting.
3. Carrying:
    a- added carried number last
    b- added carried number irregularly
    c- added carried number twice
    d- forgot to add carried number
    e- subtracted carried number
    f- carried wrong number
    g- carried when nothing to carry
    h- wrote number to be carried.
4. Faulty procedure:
    a- retraced work partly done.
    b- irregular procedure.
    c- grouped numbers
    d- split number
    e- omitted dgits
    f- added in pairs
    g- added same digit in two columns
    h- began with left column.
5. Lapses and other miscellaneous faults:
    a- used wrong operation
    b- errors in reading numbers
    c- dropped back one or more tens
    d- derived unknown from know n
    e- skipped decades
    f- confused columns
    g- added imaginary numbers". (1)

Subtraction

"1. Errors in combinations.
2. Borrowing:
    a- did not allow for having borrowed
    b- errors due to zero in minuend
    c- subtracted minuend from subtrahend
    d- failed to borrow; gave zero as answer
    e- deducted two from minuend after borrowing
    f- deducted in minuend when no borrowing was necessary.
    g- increased minuend digit after borrowing
    h- deducted all borrowed numbers from left hand digit.
3. Counting.
4. Faulty procedure:
    a- said example backward
    b- added instead of subtracted
    c- used same digit in two columns
    d- omitted a column
    e- split numbers
    f- ignored a digit

---

(1) Ref. L. J. Brueckner and E. O. Melby: "Diagnostic and Remedial Teaching", p. 203.

  - g- used minuend or subtrahend as remainder
  - h- began at left column.
5. Lapses, etc.
  - a- derived unknown from known
  - b- error in reading
  - c- error due to numbers in minuend and subtrahend being the same.
  - d- reversed digits in remainder
  - e- confused process with division or multiplication
  - f- skipped one or more decades
  - g- based subtraction on multiplication combinations
  - h- error in writing answers". (1)

Multiplication

"1. Errors of multiplication combinations.
2. Errors in adding carried numbers.
3. Errors in misplaced zeros.
4. Errors in addition.
5. Carried a wrong number
6. Used multiplicand as multiplier
7. Forget to carry.
8. Error in zero combinations, zero as multiplier.
9. Used wrong process". (2)

Division

"1. Errors in division combinations.
2. Errors in subtraction:
  - a- subtraction combinations
  - b- borrowing.
3. Errors in multiplication:
  - a- multiplication combinations
  - b- carrying wrong number when multiplying
  - c- adding carried number
4. Ignorance of value of a cipher, zero difficulties in connection with dividend, divisor and quotient.
5. Used remainder larger than divisor.
6. Forgetting to put integers in the quotient.
7. Used the trial-and-error method of finding quotient.
8. Used wrong operation.
9. Omitted digit in dividend.
10. The assumption that the first integer of the divisor may be used always as a trial diviser.
11. Short division.
12. Long division with 2 or more digit divisors". (3)

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(1) Adapted from Buswell and John, p. 137.
(2) "National Society for the Study of Education", 29th Year Book, p.300.
(3) Department of Superintendance: "Fourth Year Book", p. 212.

CHART SHOWING ANALYSIS OF STEPS IN THE PROCESS OF PROBLEM SOLVING

Steps in Problem Solving	Factors Underlying Problem Solving	Types of Drill Provided
Comprehension	Vocabulary Ability to read numerals Ability to read rapidly Ability to comprehend a. Follow directions b. Make generalizations c. Select potent elements d. Discard irrelevant elements e. Determine problem setting as a unit f. Determine the outcome of the problem g. Grasp significance of problem cues	Vocabulary drill Comprehension drills a. Directions exercise b. Completion exercise c. Multiple choice exercises
Analysis and Organization	Selection of potent factors Selection of processes involved Determining what the problem calls for Determining what is given in the problem Determining process relationships	What is called for Process analysis What is given Problem relationships
Recognition	Choice of procedure Determining problem conditions Determining purpose of the problem Determining relevant elements in problem	Process analysis What is given What is called for
Solution	Selection of processes Organization of processes in order Knowledge of combinations Problem relationships	Process analysis Problem relationships Problem scales
Verification	Probable form of answer Probable magnitude of answer	Probable answer

This chart is adapted from the Second Yearbook of "The National Council of Teachers of Mathematics", page 66.

GLOSSARY

ability. The capacity or power to produce.
abstract numbers. Those numbers whose unit has no name, as, 20, 4, 6.
accuracy. Ratio between number of exercises correctly done and number of exercises attempted.
achievement. The accomplishment or production of the child in his school work.
addends. The numbers to be added.
age norms. Tables of values representing typical or average performance on standardized tests for pupils in different age groups.
ambiguity. The quality of a test item which makes possible more than one logical interpretation of its intent or meaning.
analysis. Reduction or taking apart of a total performance in the process of identifying specific skills.
aptitude. Ability in a certain field or area of performance.
aptitude test. A test of specific intelligence, i.e., intelligence as it operates in a certain field or area of performance, which may be used for prognostic purposes.
average. A generic term covering the measures of central tendency, but commonly used to designate the arithmetic mean.
basic skills. Tool skills, such as those of reading, language, and arithmetic, essential to study of the content subjects.
classification. The process of assigning a pupil to the grade or unit of a school for which his abilities and training best fit him.
clues. Characteristics of test items which frequently aid the pupil in determining the correct answers.
coefficient of correlation (r). A measure of relationship which ranges in value from + 1.0 through zero to - 1.0; refers here mainly to Pearson product-moment coefficient.
completion exercise. A type of test exercise to which the pupil responds by filling the blanks of a paragraph with the words, numbers, phrase etc., which he believes will correctly complete the meaning.
concrete number. One whose unit is named, as, 3 feet.
correction. An adjustment used in computing the arithmetic mean, standard deviation, and correlation coefficient by the short method from a frequency distribution or correlation chart.
correction for attenuation. A correction for the effects of those chance or accidental errors in the two tests which lower the reliability coefficients of both tests and thus affect the correlation between them.
corrective teaching. Steps taken to remedy observed defects or difficulties in pupil learning.
correlation. The degree of relationship existing between two or more sets of measures.
correlation coefficient. See coefficient of correlation.
criterion. A standard by which a test or other product is judged or evaluated.
deviation. The amount by which a score or other measure differs from the central tendency of the group of scores in which it is included.
diagnosis. Exact identification and location of specific strengths or weaknesses in performance.
diagnostic test. A test used to locate the nature, and if possible the causes, of disability in performance.
difference. The number which added to the smaller produces the greater.
difficulty. The characteristic in a test item which results in a large percentage of incorrect responses.
digit. One of the symbols, 1, 2, 3, 0. 9, etc.
dividend. (a) The number to be divided.

divisor. The number that shows into how many parts the dividend is to be divided, or one of the parts.

drill. Repetition designed to improve skill or to make learnings permanent.

evaluate. To test, measure, and appraise the "whole" child by the use of tests and a wide variety of non-test techniques and devices.

examination. See test.

exercise. A unit of a test governed by a specific set of directions.

figures. (a) Symbols used to represent numbers. (b) Diagrams to represent geometrical forms.

general achievement tests. Educational tests covering several fields of subject matter and ordinarily adapted for use in several grades.

general intelligence test. A test of general mental ability.

grade. The administrative division of the school which indicates the level of advancement of the pupil.

grade norms. Tables of values representing typical or average performance on standardized tests for pupils in different grades.

group test. A test which can be administered to a number of pupils at the same time.

individual differences. The observed or measured variation of individual in ability, progress, achievement, etc.

individual test. A test which can be administered to only one pupil at a time.

informal objective test. A teacher-made objective test.

integer. A whole number.

integral number. A whole number, as 4, 1.

intelligence test. A test which measures ability to learn or to profit from experience.

interview. A personal conference technique frequently used in diagnosis and in the evaluation of attitudes.

inventory test. A test used as a preliminary check on the degree of mastery existing prior to instruction.

mark. The teacher's numerical or letter evaluation of pupil achievement in a course or area of performance.

matching exercise. A type of test exercise to which the pupil responds by attempting to pair the related items in two or more columns of related facts.

measure. To test by means of standardized and teacher-made instruments mainly in the fields of achievement and intelligence. Also a test score or other numerical rating.

minuend. The number from which another is taken.

multiple-choice-item. A type of test item to which the pupil responds by attempting to select the correct response from the several alternatives given.

multiplicand. The number to be multiplied.

multiplier. The number which shows how many times the multiplicand is taken.

new-type examination. See informal objective test.

norms. The median or average performances on standardized tests of pupils of different ages or grade placement, as determined by the testing of large numbers of pupils.

number. (a) A collection of units of the same kind.

objective test. A test for which the scoring procedure eliminates subjective opinion and judgment.

objectivity. The characteristic of a test which eliminates subjective opinion or judgment in the process of scoring it; an important criterion of a good examination.

oral examination. A test administered and answered orally.

performance. The accomplishment, achievement, or behavior of the pupil.

power test. A test which measures the difficulty of the task the pupil is just able to perform in terms of how far he can go through a test in which the items consistently increase in difficulty.

preventive teaching. Steps taken at the time of initial instruction to guard against the later appearance of defects or difficulties in pupil learning.

problem. A question to be solved.

product. The result of multiplication.

prognostic test. A test used to predict future success in specific subjects or fields.

progress record. A device similar to a profile chart on which pupil progress from year to year can be shown graphically for certain achievement tests.

quotient. The result of division.

range (R). The distance from the lowest to the highest score in a series of scores.

reliability. The degree to which a test measures what it does measure; consistency of measurement; a major criterion of a good examination.

reliability coefficient. The correlation coefficient obtained between scores made by the same pupils on two equivalent forms of a test.

remainder. Same as difference.

remedial. Having as a purpose the correction of observed difficulties and weaknesses in performance.

scale. An instrument used by the scorer in evaluating pupil performance or by the test-maker in constructing a test. Also the continuum from the lowest to the highest score in a frequency distribution.

scaled test. A test in which the items are arranged in an order of increasing difficulty.

score. A quantitative description of performance.

standardization. The process of constructing a test and establishing norms for it.

standardized test. A test for which the exercises have been carefully selected and evaluated and which is accompanied by norms.

subjectivity. The degree to which measurement results are influenced by personal opinions or judgment.

subtrahend. The number taken from the minuend.

sum. The result of addition.

survey test. A test which measures general achievement in certain subjects or fields.

T-scale. A method devised by McCall for scaling total scores, or groups of items, on a test.

tabulation. The process of grouping and classifying data for purposes of condensation and ease of interpretation. Also the distribution into which data are classified.

teacher-made tests. Tests constructed by the teacher, such as the essay and informal objective tests.

technique. A procedure or method.

test. In the general sense, any instrument used in the measurement of any educational or mental ability; in a specific sense, an instrument used by the pupil and ordinarily involving the use of paper and pencil. Also to measure by the use of tests.

test item. The smallest unit of a test; almost synonymous with test exercise.

tool subjects. Fields in which achievement consists mainly in the acquisition of skills and techniques useful in further learning, as reading, arithmetic, and spelling.

true-false item. A type of alternate-response item to which the pupil responds by indicating whether a statement is true or false.

validity. The degree to which a test measures what it purports to measure; the major criterion of a good examination.

Remark: The above material is adapted from the glossaries of Greene, Jorgensen, Gerberich, "Measurement and Evaluation in the Elementary School", and Robinson's "The Arithmetic Help".

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