REDISTRIBUTION OF BENDING MOMENTS

IN

UNDERREINFORCED COLUMNS

THESIS

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Kevork B. Parseghian

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APPROVED:

Hanna M. Makhlouf
Supervising Professor - H. Makhlouf

Zuhir Alami
Z. Alami

R. Iliya
R. Iliya

J. Nasser

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ABSTRACT

The case of an external reinforced concrete column subjected to both bending and axial compression is considered. Usually such a column is limited in size due to architectural requirements, and consequently cannot be designed to resist the bending and axial stresses except with a rather high percentage of reinforcement. For this reason, and for the sake of simplicity in analysis and design, such columns are designed as concentrically loaded.

This investigation deals with the overall safety of such under reinforced columns and how do they adjust to the applied bending which is a consequence of monolithic construction.

It is found out that, the relative stiffness of the column decreases greatly upon cracking, thus rendering a more flexible column. This is substantiated by a design example where it is found out that the column moment predicted by conventional analysis is about twice the value of the moment that the column actually resists.

In the frame analysis account is made of the effect of cracking upon the members' stiffnesses, carry-over factors, fixed and moments and the like. Due to the fact that the cracking pattern is progressive with increased loading it was necessary to divide the load into three stages. The cracks occuring under one stage of loading, influence the redistribution of moments of that stage as well as the distribution of moments of the following stage of loading.

Specific conclusions are drawn out at the end of this work. In general, it is found out that such underreinforced columns are capable of resisting the axial and bending stresses but with a lesser factor of safety than specified by recognized codes of practice. It is also concluded that the conventional method of frame analysis, treating the concrete elements as uncracked sections, gives erroneous column moments that are highly on the safe side.

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LIST OF SYMBOLS

a = depth of equivalent rectangular stress block

a_b = depth of equivalent rectangular stress block for balanced conditions

A_g = gross area of tied column

A_s = area of tension reinforcement

A'_s = area of compression reinforcement

b = width of a rectangular flexural member

c = distance from extreme compression fiber to neutral axis

c_b = distance from extreme compression fiber to neutral axis for balanced conditions

C = carry-over factor

C_c = resultant compressive stress in concrete

$\bar{C}_c$ = C_c/ϕ

C_s = total compressive force in steel

$\bar{C}_s$ = C_s/ϕ

d = distance from extreme compression fiber to centroid of tension reinforcement

d' = distance from extreme compression fiber to centroid of compression reinforcement

e = eccentricity of the load on a column, measured from the gravity axis

E_c = modulus of elasticity of concrete

E_s = modulus of elasticity of steel

ε'_s = unit strain of steel in compression

ε_y = unit strain of steel at yielding

f'_c = compressive cylinder strength of concrete at 28 days

f_s = calculated tensile stress in steel when less than the yield strength, f_y

f'_s = compressive stress in steel

f''_s = effective compressive stress in steel considering the concrete displaced by compression reinforcement

f_t = maximum tensile stress in concrete

f_y = yield strength of steel

ϕ = capacity reduction factor

h = total depth of flexural members

I = moment of inertia

k = distribution factor

K = stiffness factor

L = unsupported height or span

m = n - 1

m' = 2n - 1

M = span moment in beams

$\overline{M}$ = M/ϕ

M_c = cracking moment

n = modular ratio

p = ratio of area of tension reinforcement to effective area of concrete (As/bd)

p' = ratio of area of compression reinforcement to effective area of concrete (A'_s/bd)

P = vertical load on column

$\overline{P}$ = P/ϕ

P_y = column load that would cause yielding of tension reinforcement

R_A = reaction at A

R_{B_1} = reaction at left of B

R_{B_2} = reaction at right of B

T = total tensile force in reinforcement

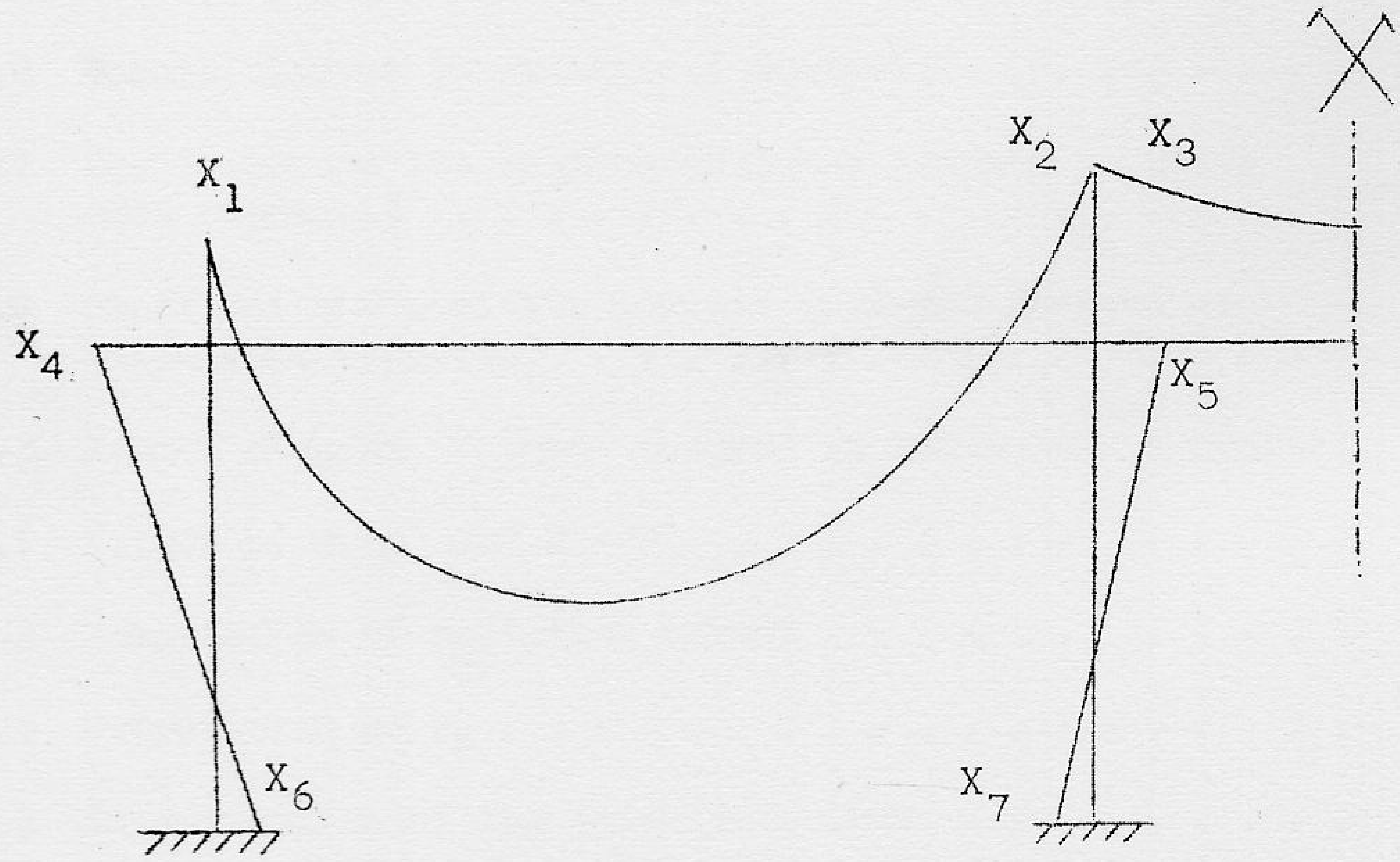
$\bar{T}$ = T/ϕ

w = uniformly distributed load per unit length

X = see key sketch below

Y_1 = distance from extreme compression fiber to neutral axis of transformed uncracked section

Y_2 = distance from extreme compression fiber to neutral axis of transformed cracked section



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INTRODUCTION

Though suggested by its name, "The Inernational Concrete Code" is not internationally adopted. On the other hand, there is no Lebanese code of practice for reinforced concrete. Depending upon his culture and education, the Lebanese engineer is forced to use one of the two codes widely in use in this country, namely, the French code and the American Concrete Institute Standard Building Code, hereafter referred to as ACI-Code. The rapid developments in the technique of reinforced concrete design, backed by extensive tests, have introduced great changes between the new and old codes. This fast pace of technological change, in recent years, have also resulted in wide differences between the two codes mentioned above. The 1963 ACI-Code authorises the design by ultimate strength method and the French code BA '60 specifies high working stresses, thus implicity recognizing ultimate strength behavior.

The new trend in reinforced concrete design puts more emphasis on ultimate strength method than working stress method, and as stated by Prof. Phil M. Ferguson, a member of ACI Committee 318, the working stress method of design is expected to continue only as a method in the appendix of the ACI-Code of 1970*.

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*"Ultimate Strength Design" by Phil M. Ferguson, Proc. Conf. on Reinf. Conc. & Found., Faculty of Engineering and Architecture, American University of Beirut, May 1968.

Years ago, in designing reinforced concrete columns, little importance was given to end moments. Except in extreme cases, columns were generally designed as axially loaded. To make this practice safe the factors of safety used were kept extremely high. In ultimate strength theory, however, the proportioning of sections is based on their actual strength, as confirmed by reliable tests, with much lower factors of safety. Due to this fact, the design of reinforced concrete columns by ultimate strength method gives considerably more economical results than working stress method.

In spite of the above mentioned advantages, the structural engineer often encounters difficulty when designing comparatively small, last floor columns supporting large span beams. It is a matter of common observation that, due to frame action, such columns are subject to high bending moments.

The design of these columns according to ACI-Code provisions, for instance, necessitates the use of a high percentage of reinforcement which is both expensive and not practical for execution. This difficulty suggests the possibility of designing such columns as axially loaded and overreinforcing the beams to carry their load as being simply supported over the columns. Such an approach has the advantage of simplifying the analysis and design procedure of the above mentioned columns. In addition to this, it helps the structural engineer in meeting architectural requirements.

The structural analysis of such a column, subjected to eccentric loading while being designed only for axial loading, is the main objective of this thesis. In the coming pages the theoretical background of the problem will be stated, and then followed by detailed analysis of a representative example. Conclusions will be drawn out from both the theoretical background and the solved example.

It is necessary to mention here that this work is not intended to be a complete one. The subject is rather extensive and involves many factors that may form subjects of separate studies.

It is expected that in the near future, many of the unknowns of today will be exposed to the light of knowledge, and the engineer of the future will have to deal with a minimum number of unknowns and assumptions.

CHAPTER ONE

THEORY

1.1 General

Most of the structures encountered in practice are statically indeterminate. A thorough inspection of as many structures as possible, in search for a simple beam or an axially loaded column, will reveal the fact that they are very rare. They are so rare that, they do not seem to exist outside the testing laboratories.

The characteristics of statically indeterminate reinforced concrete frames is the continuity between the units forming the frame by means of rigid joints. It is a matter of common observation that, when there is practically no deformation in the joint itself, moments, shears, and thrusts are transmitted from one member to others meeting at the same joint.

There is a mutual action of members in such a frame. The columns assist the beams when the latters are subjected to vertical loads and the beams assist the columns when the columns are subjected to horizontal loads.

Until about 1932, the analysis of rigid-joint structures was a laborious and time consuming mathematical work. But when the moment-distribution method was presented by Prof. Hardy Cross[(1)*], the

* The superscripts refer to the number of the reference listed under Bibliography.

analysis of statically indeterminate structures was reduced to a simple application of the four elementary operations of arithmetic.

1.2 Consistency of analysis and design

All present methods of indeterminate structural analysis are based on the elastic theory. Their consistency with the elastic-stress method of design may not be very much objectionable. The case is not the same, however, with the ultimate-strength design. It is known that concrete does not respond elastically to loads of more than about one-half the ultimate, and the ultimate-strength method of design takes into account the inelastic behavior of concrete. Therefore, the presently accepted procedure by which elastic analysis is coupled with inelastic design of sections is not consistant.

If a reinforced-concrete structure is loaded beyond its elastic limit, as a result of yielding of reinforcing steel, plastic hinges will develop at points of maximum bending moment. Depending upon the type of structure one or more plastic hinges would form before the structure collapses. In statically determinate structures the formation of one plastic hinge is enough to render the structure a mechanism and cause failure. In the case of indeterminate structures, stability may still be maintained in spite of the formation of several hinges at different cross sections. Upon the formation of plastic hinges the elastic curve shows an abrupt change in slope and certain inelastic readjustments of moments take place in the structure enabling it to carry a higher load than forseen.

It follows that plastic hinges may be intentionally formed at desired sections of a structure thus allowing a full inelastic redistribution of moments, and a consequent increase in the load carrying capacity. This would necessitate an efficient control on rotations at the plastic hinges. Also formation of objectionable amount of cracks and excessive deflections should be prevented. Methods for accomplishing these are not yet fully developed but current investigations in this respect might bring them soon within the reach of engineers.

1.3 Values of E_c

The modulus of elasticity of concrete was previously assumed to vary with the ultimate compressive strength as follows $E_c = 1000\ f'_c$. This formula is reasonably accurate for ordinary concrete. However, there are many factors that affect the value of E_c. The main difficulties come from the inherent properties of concrete. It is plastic, weak in tension, continuously changes properties with time and is very much affected by atmospheric conditions. In addition to these, concrete cannot be practically produced so as to give standard properties as steel. The kind, shape and the size of the aggregates, proportioning of the mix, the temperature and humidity at mixing, the method of depositing into forms, the kind of forms and curing have a big effect on the value of E_c. Even if it is assumed that the production of concrete can be rigorously controlled so as to give a material of standard qualities, there still remain the effects of age, creep under sustained loads, creep due to load repetition, and shrinkage. Another addition to these, is the effect of the increased use of light weight aggregate. However, this is overcome by the development of a new expression for E_c which includes the density of the concrete. This expression, $E_c = 33\ w^{1.5}\sqrt{f'_c}$, appears in the 1963 ACI-Code and is suitable as a design value for both normal weight and light weight aggregate concrete.

1.4 Crack occurrence in reinforced-concrete

Cracking of concrete was not an unknown occurrence in reinforced-concrete structures, but until lately appropriate attention was never given to it. With the introduction of the ultimate-strength method of design, this subject gained more weight because its effect in the bahavior of reinforced-concrete structures cannot be overlooked.

The concrete is a brittle material and when subjected to tension will crack without showing any notable elongation. Experiments show that the elongation of concrete in rupture is hardly equal to one-tenth of the elongation of steel at working stress level[2].

When a reinforced-concrete structure is subjected to flexure due to a small load that produces tension stresses in the concrete smaller than the modulus of rupture, the entire section of concrete is effective in resisting the stresses. These are compression and tension stresses on either side of the neutral axis.

As the load increases and the tensile strength of concrete is reached at any section, tension cracks develop and start to propagate quickly upward to or close to the level of the neutral axis. The neutral axis in turn may shift upwards with progressive cracking[3].

In a bent member of reinforced-concrete, cracks are always initiated in the area surrounding the reinforcing steel. Right after the first crack is developed, the whole tensile force is resisted by the reinforcing bars only. With the elongation of these bars the tension stresses in concrete which were reduced to zero at the face of the crack, start building up again until they reach the tensile strength of concrete at a neighbouring section thus producing another crack there. At present there are no methods that enable the engineer to predict the shape, the width and the pattern of tension cracks, but it can easily be observed that under service load a considerable portion of a beam will be cracked. The effect of cracks on different values such as moments of inertia, carry-over factors, stiffness factors and fixed end moments will be discussed in the articles that follow.

1.5 Values of the moment of inertia

Various values of moments of inertia, I, are used in practice. Some use the moment of inertia of the full transformed section, others neglect the reinforcement and use $A_c h^2/12$, where A_c is the total area of concrete and h is the full depth.

In practice the value of I is assumed to be constant along the member. As discussed in the previous article, reinforced-concrete is an inelastic material, subject in normal service to the discontinuities of cracks. Therefore if an exact analysis is desired, a constant value of I along any member cannot be used. For such purposes the most satisfactory treatment seems to be the use of different values of I along the member. This can be accomplished as outlined below:

1) Before the occurrence of cracks:

By using two different values of I of the uncracked transformed sections one for the portions subjected to positive moments and the other for the portions subjected to negative moments.

2) After the occurrence of cracks:

By using appropriate values of I as necessary of uncracked and cracked transformed sections for the portions subjected to positive and negative moments respectively.

The moment of inertia of cracked transformed sections may be computed by considering the area of concrete in compression plus "n" and "2n" times the area of tension and compression steel respectively.

Tests show that when a beam carries one half of its total service load, the cracks that have developed propagate up to the vicinity of the neutral axis[4]. Therefore the above procedure sounds to be good enough. On the other hand, there is still the problem of the sections between cracks where the full cross-section of concrete is active. A good answer would be the use of the moment of inertia of an equivalent uniform section all over the length, but as long as the pattern of cracking is not determined to a certain degree of accuracy a proper assumption for such a section is not possible.

1.6 Stiffness and carry-over factors

A cracked member, though prismatic in shape, cannot be treated like a prismatic member. Due to the fact that I is not a constant along the member, the simple expressions for stiffness, $K = I/L$, and carry-over factor, $C = 1/2$, are not applicable after the occurrence of cracks. New expressions for these quantities must be developed by taking into account the variations of I. These expressions are developed at a later section by making use of the area under the appropriate M/EI diagrams. It will be shown that the effect of crack occurrence on these quantities is appreciable.

1.7 Fixed End Moments

Like stiffness and carry-over factors the fixed end moments also are greatly affected by the occurrence of cracks. They are no more the same as the ones for prismatic members and are also computed by making use of the area under the corresponding M/EI diagram.

1.8 Readjustment of moments due to crack occurrence

In a statically indeterminate structure, any moment applied to a rigid joint is distributed to all members meeting at the same joint in proportion to their relative stiffnesses. The stiffer the member is the more the moment absorbed will be. It may be concluded, therefore, that the distribution of moments at any joint is dependent upon changes in the stiffnesses of members. The stiffness in turn is very much affected by the variation of the moment of inertia.

When applying the method of moment distribution to reinforced concrete members, it is generally accepted to consider the moment of inertia of the uncracked sections and neglect the reinforcement. This is good enough for the early stages of loading where the whole section of concrete is active. After the development of cracks at different sections, the moment of inertia of the member is reduced. Consequently its stiffness is no more the same as the previous one. As for the reinforcement, it cannot be any more neglected as its contribution to the value of I at cracked sections is predominant. It follows that overreinforced cracked members have considerably higher values of I than underreinforced cracked members. The case would not be so if the effects of cracks were not taken into consideration. Upon the formation of cracks the original conditions of design change and new conditions of stability come into existance. The structure adjusts itself to these new conditions by allowing a redistribution of moments. This results in completely different moments than the ones originally predicted.

The above mentioned difference becomes more striking when a joint is formed by only two members. If it is assumed that such a joint is formed by an underreinforced column and an overreinforced girder, upon cracking the large reduction in the column stiffness compared to the one of the girder causes a sharp cut on the column moment. The result is a much lower column moment than foreseen.

This partly explains the structural stability of presumably failing structures. The example worked out in the appendix, clearly illustrates this phenomenon.

CHAPTER TWO

DESIGN EXAMPLE

2.1 General

An illustrative example is a good tool for a better understanding of the problem on hand. With this in mind, a representative example is worked out in the appendix. It consists of the detailed analysis and investigation of a last floor reinforced concrete frame. The frame is supposed to be that of an office building with two end spans of 7.00 meters and a central span of 2.50 meters. The floor consists of 10 cm. wide ribs, 24 cm. thick hollow concrete blocks and a 6 cm. thick concrete slab. Therefore the total thickness of the ribbed floor is 30 cm. The beams are 1.00 meter wide and are concealed within the thickness of the slab to allow flexibility in partitioning the floor area. For architectural reasons the size of columns are limited to 25 cm x 25 cm. The frames are 5.00 meters apart and are tied by the ribs of the slab. The roof is supposed to be accessible and therefore a live load of 250 kg/m^2 is used throughout the analysis. To obtain the maximum moment on the exterior column, the central span is not loaded with the live load.

During this analysis the load factors are not applied. They are applied to the moments and reactions whenever needed for design purposes. The structural analysis, and the effect of cracking on the redistribution of moments in the frame, are described in the following articles.

2.2 Analysis by the conventional method

The first step consists of the analysis of the frame by the conventional method suggested in the ACI-Code. The following assumptions are maintained throughout the analysis:

a) The columns are assumed to be fixed at their far ends.

b) The structure has lateral stability and therefore no lateral displacement is allowed.

c) Wind loads are either negligible or accounted for by shear walls that relieve the frame under study from any lateral loading.

In computing the values of I for the relative flexural stiffnesses of beams and columns, the reinforcement is neglected as suggested in art. 905 c of the 1963 ACI-Code. As the members are of constant cross section, their stiffnesses are computed by the application of the simple formula $K = I/L$. To take advantage of the symmetry in the structure and its loading, one half of the stiffness of the central span is used. The carry-over factors are 1/2, as for prismatic members, and the fixed end moments are computed by the conventional way.

The moment distribution method is used in the analysis with the following sign convention:

a) A moment applied to a joint is positive when it tends to turn the joint clockwise.

b) A moment applied to a joint is negative when it tends to turn the joint counter clockwise.

The analysis for dead and live loads are conducted separately. The final moments and reactions are tabulated in the appendix on page 33.

2.3 Beam analysis assuming columns as simple supports

As mentioned previously, this is achieved by considering the beams as simply supported over knife edge supports, thus reducing the frame to a three span continuous beam. As the cross-section of the beam is constant all over the length, the values of I are assumed to be equal. The stiffness factor for the end span is reduced by 25 % and the fixed end moment at the interior support of the end span is increased by 50 % to provide for the free end support. Here also the symmetry in the structure is taken into account the same way as explained in the previous article.

As in the first step, different analysis for dead and live loads are conducted and a table of final moments and reactions are added, as shown on page 37.

The beams are designed for these moments using the ultimate strength method. Due to the difficulty of bending of high tensile reinforcement in shallow members, no bent ups are used in the beams. Cut-off points of reinforcing bars are computed according to the requirements of the

ACI-Code. The columns are designed for axial load only and the reinforcements used selected according to the Code minimums.

An investigation is carried out to see if the columns designed for axial load only would carry the moments and loads obtained from the conventional method of analysis outlined in article 2.2. Details of the computations are included in the appendix on page 47.

The investigation shows that the exterior columns, as well as the interior columns, would not be adequate under the above mentioned loads and moments. This suggests the main problem which is probed in this thesis. Many such frames have already been designed and built by assuming that the columns are knife edge supports and concentrically loaded.

The above mentioned results indicate that such columns should fail under the action of bending moments not allowed for in the design. The immediate conclusion may be that either the conventional method of analysis is wrong, thus over-estimating the bending on the column, or that the column sections are stronger than predicted, thus resisting such heavy bending.

In article 1.8, it was stated that crack occurrence reduces the stiffnesses of members and that the distribution of moments at any joint is dependent upon changes in the stiffnesses of members. It follows that without considering the cracks developed in the frame, which cause a redistribution of moments, the real rotations and moments cannot be properly predicted. This can be achieved by analysing the frame under successive increments of loading as described in the following article.

2.4 The analysis of the frame under successive increments of loading

A complete analysis of the frame under successive increments of loading, adding up to the full expected load, is carried out. Each analysis is based upon the relative stiffnesses of the members obtained from the cracked and uncracked transformed sections resulting from the previous increment of loading.

The three stages of loading are considered as follows:

1) Analysis right after removal of formwork.
2) Analysis with the total dead load acting.
3) Analysis with the total dead and live load acting.

Before going into the description of these stages, expressions for some design values must be developed. These are:

a) Values of I for transformed uncracked sections.
b) Values of I for transformed cracked sections.
c) Carry-over factors for non prismatic members.
d) Stiffness factors for nonprismatic members.
e) Fixed end moments for nonprismatic members.

Although expressions for the last three can be found in any structural book, a short review of these values will be useful.

a) Values of I for transformed uncracked sections

The position of the neutral axis is determined by taking the moments of the transformed areas about the upper fiber a-a in fig. 2.1a. The above procedure yields the expression.

$$Y_1 = \frac{h^2 + 2d\ (pmd + p'm'd')}{2\left[h + d\ (pm + p'm')\right]} \tag{1}$$

Then

$$I = \frac{bY_1^3}{3} + \frac{b(h - Y_1)^3}{3} + As.m(d - Y_1)^2 + A's\ m'(Y_1 - d')^2 \tag{2}$$

Where

Y_1 = Distance to neutral axis from the extreme upper fiber of uncracked section.

b = Width of section.

h = Total depth of section.

d = Effective depth of section.

d' = Distance from center line of compressive reinforcement to the extreme upper fiber.

m = n - 1

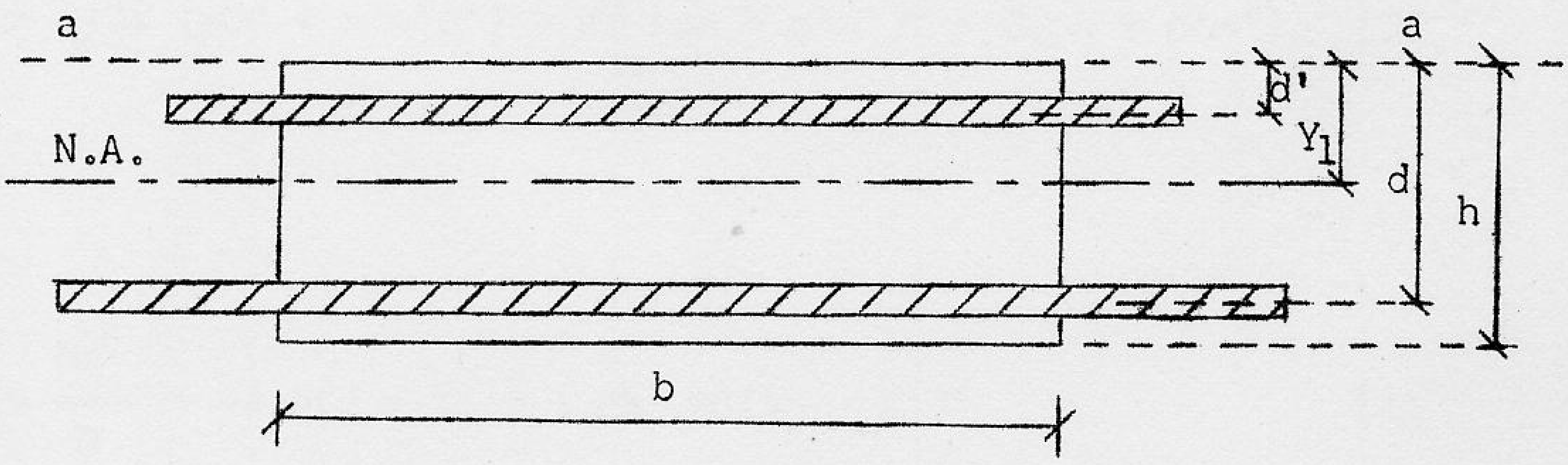
m' = 2n - 1

p = Percentage of tensile reinforcement.

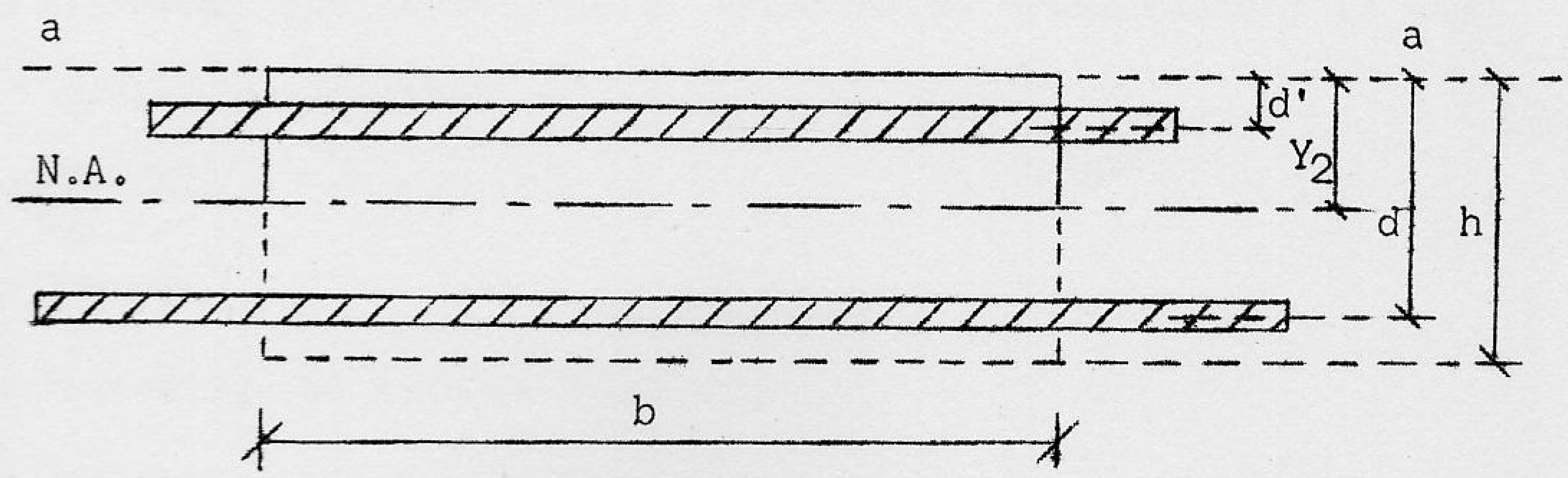
p' = Percentage of compressive reinforcement.

b) Values of I for transformed cracked section

As previously stated, it is assumed that the cracks have already propagated up to the neutral axis. By taking moments of the transformed



(a)



(b)

Fig. 2.1 (a) Transformed area of uncracked section

(b) Transformed area of cracked section

areas about the upper fiber a-a in fig. 2.1b, the following relation for the position of the neutral axis of the transformed cracked section is obtained.

$$Y_2 = \sqrt{2d\,(pnd + p'm'd') + d^2\,(pn + p'm')^2} - d(pn + p'm') \qquad (3)$$

Then

$$I = \frac{bY_2^3}{3} + As.n(d - Y_2)^2 + A's\,.m'(Y_2 - d')^2 \qquad (4)$$

Where

Y_2 = Distance to neutral axis from the extreme upper fiber of cracked section.

b = Width of section.

d = Effective depth of section.

d' = Distance from centerline of compressive reinforcement to the extreme upper fiber.

$n = \frac{Es}{Ec}$

m' = 2n - 1

p = Percentage of tensile reinforcement.

p' = Percentage of compressive reinforcement.

c) Carry-over factors for nonprismatic member

The carry-over factor may be defined as the ratio of the moment induced at a fixed end of a member to the moment producing rotation at the other end.

This definition can be expressed by the formula

$$C_{AB} = \frac{M_{BA}}{M_{AB}} \qquad (5)$$

If M_{AB} is equal to unity then

$$C_{AB} = M_{BA} \qquad (6)$$

This leads to the following simpler definition:

"The carry-over factor is the end moment induced at the fixed end of a member when the opposite end is rotated by an end moment of unity".[5]

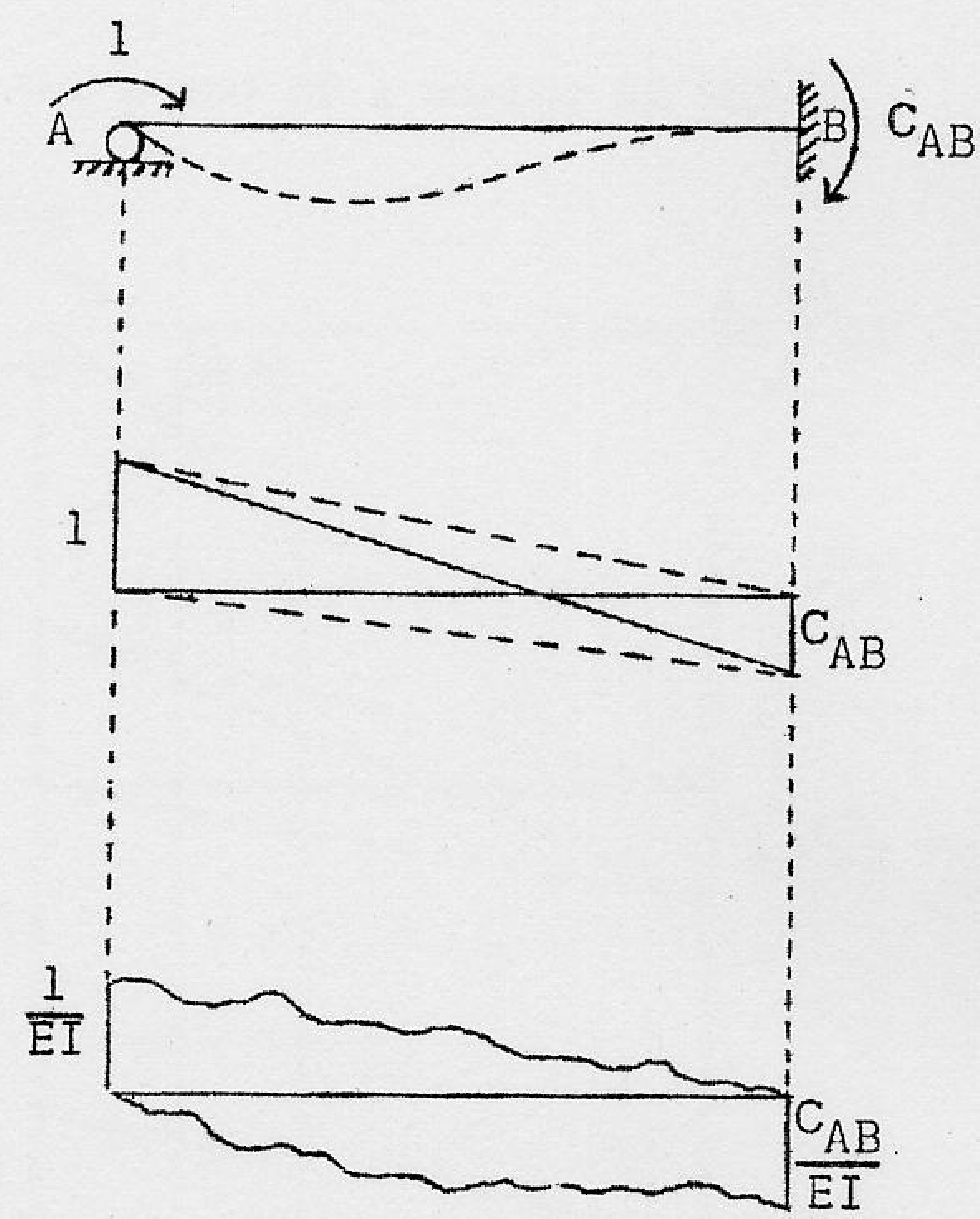


Fig. 2.2 Carry-over factors

To compute the carry-over factors of any member AB, a moment of unity is applied at A. According to the above definition the moment induced at B is equal to C_{AB}. By drawing the moment diagram and transforming it to the M/EI diagram, the second moment-area theorem can be applied. As the deflection of point A on the elastic curve from the tangent at B is equal to zero, the static moment of the composite M/EI diagram about an axis through point A must be equal to zero. From the resulting relation C_{AB} is readily computed.

To obtain C_{BA} the procedure is reversed. Different carry-over factors for different members are computed in the example that is worked out in the appendix.

d) Stiffness factors for non prismatic members

The stiffness factor K_{AB} may be defined as the end moment required to rotate the tangent at the A end of member AB through a unit angle when the B end is fixed.

By applying a moment equal to K_{AB} at the A end of a member AB which in turn will induce a moment equal to $K_{AB} \cdot C_{AB}$ at the B end, and following the same procedure as for computation of C_{AB}, the stiffness factors can be evaluated.

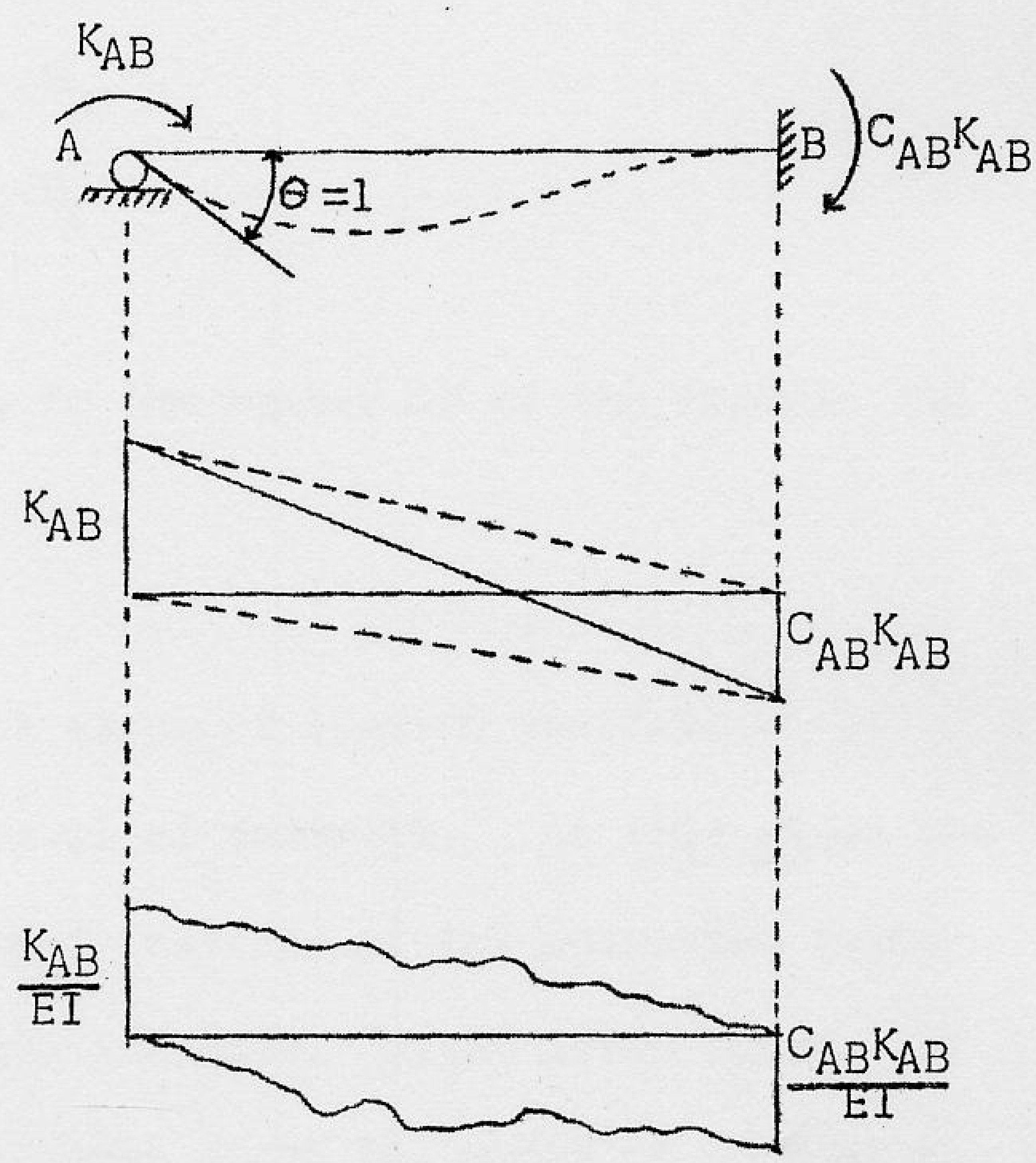


Fig. 2.3 Stiffness factors

As the angle of rotation at A must be equal to unity the area under the M/EI diagram is equal to one. This relation gives an expression containing the values of C_{AB} and K_{AB}. As C_{AB} has previously been computed, the determination of K_{AB} involves no difficulty.

Like carry-over factors various values of stiffness factors for various members are computed in the above mentioned example.

e) Fixed end moments for non prismatic members

The fixed end moments of non prismatic members are computed by applying the principle of superposition. Any member AB acted upon by a loading is temporarily assumed to be a simply supported beam and the end rotations θ_A and θ_B are computed by applying the conjugate beam or any other convenient method. Then one of the ends is assumed to be locked and the other end is rotated back to zero slope. By locking this end at zero slope the supposedly locked end is also released and rotated back to horizontal position. The resulting end moments are:

$$\begin{aligned} F.E.M_{AB} &= K_{AB}\cdot\theta_A - C_{BA}\cdot K_{BA}\cdot\theta_B \\ F.E.M_{BA} &= -K_{BA}\cdot\theta_B + C_{AB}\cdot K_{AB}\cdot\theta_A \end{aligned} \qquad (7)$$

This procedure is applied twice to the member AB of the example and the prepared tables are self explanatory.

As previously outlined the first stage of loading consists of an analysis of the frame right after removal of formwork. At this stage the sections are uncracked and the moments of inertia of the uncracked transformed sections are used throughout the analysis. Right after removal of the formwork the load applied to the frame is of the order of 80 % of the total dead load.

The reactions due to the total dead load were previously computed by the conventional method of frame analysis. The points of contraflexture are determined by using the 80 % of the total dead load and the reactions mentioned above. The points of contraflexure and the various sections of the frame having different values of I are as shown in fig. E.1 page 53

The bottom reinforcement in fig. E.1 is intentionally shown in two layers to indicate the cut off points.

The carry-over factor computed by the method previously outlined came out to be almost the same as for prismatic members. This shows that as long as the sections are uncracked, the contribution of the reinforcement to the values of I is not considerable. Consequently the analysis is continued by using the values of I of the transformed uncracked sections with carry-over factors equal to one half.

The second stage of loading considers the total dead load acting upon the frame. At this stage the sections are cracked and as the loading of the previous stage was more than one half of the total service load, the cracks are assumed to have already propagated up to the neutral axis.

The maximum tensile stress of concrete varies between 10 to 15 precent of f'_c. Occasionally it may still attain higher values[6], however an average value of 12 % is chosen for use in this investigation. Therefore the maximum tensile strength of concrete becomes

$$f_t = \frac{12}{100} \times 176^{*} = 21.12 \text{ kg/cm}^2$$

By using this value and the values of moments and reactions obtained from the loading of previous stage, the cracked and uncracked portions of the whole frame are determined. These sections and sections of different values of I are shown in fig. F.3 page 68.

After computing the values of carry-over factors, stiffness factors and F E M as previously outlined, the distribution of moments is carried out and the results tabulated in the appendix on page 75.

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* $f'_c = 176$ kg/cm², see pg. 38.

It can clearly be observed that readjustment of moments due to the occurrence of cracks as discussed in detail in art. 1.8 has already taken place at the end of this stage.

The column moment and the reaction R_A acting on the column were 4640 kgm and 11214 kg respectively as calculated by the conventional method of analysis. At the end of this stage, however, they are 2590 kgm and 10854 kg respectively. The vertical load on the column is practically unchanged but the moment is decreased by a considerable amount of 44.18 %. Consequently the eccentricity also is decreased and the load carrying capacity of the column increased.

At the last stage of loading the frame is loaded with the full expected service load. The cracked and uncracked sections are computed from the values of moments obtained at the end of the second stage of loading. Right before the last stage of loading the frame is as shown in fig. G.1 on page 79.

The carry-over factors, the stiffness factors and the F E M are computed in the same way as for the previous stage. The results obtained from moment distribution are tabulated in the appendix on page 85.

The moments and the reaction R_A obtained from the conventional method of analysis were 6290 kgm and 15145 kg respectively. At the end of the last stage of loading they are 3420 kgm and 14559 kg respectively. Again the vertical load is almost unchanged but a further slight decrease in the value of column moment is observed. The total reduction in the column moment at the end of the last stage of loading is 45.63 %. This is a supporting evidence to the discussion in art. 1.8.

2.5 The actual factor of safety against primary failure*

In the preceeding article, it is shown that, at the end of the third stage of loading, the column moment is much lower than the one found by the conventional method of analysis. It is clear that the supposedly failing column is not failing. It can sustain further loading and consequently accommodate some more rotation.

To determine the actual factor of safety, the load P_y that would cause primary failure of the column must be computed.

It is worth noting here that almost all the reduction on the column moment has taken place between the first and second stages of loading, in other words, right after the development of cracks. In spite of additional cracks between the second and third stages of loading, the relative stiffness factors are almost unchanged and consequently the extra reduction in the column moment is only 1.45 %. It follows that, once the members are cracked the additional cracking due to further loading does not introduce considerable changes in the distribution of moments. Therefore P_y can be computed with sufficient accuracy by assuming that the column load and moment will increase proportionally with further loading, and that the eccentricity will remain constant until the yeilding of tension reinforcement.

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*Yielding of tension reinforcement.

The computation for P_y is carried out in the appendix on page 86. It is computed by successive trials. The value of C_c is found by computing the area under the stress-strain curve as shown in fig. H.3 on page 88, instead of using the conventional compressive block. The eccentricity is found by using the value of column moment at the bottom of the beam.

Although the column is found to be safe, it does not have a sufficient factor of safety as specified in the ACI-Code. The actual factor of safety is 1.48, compared to a code value of 2.25*. However, it has an additional factor of safety against rotation which enables the column to rotate and release further bending. This is due to the fact that the concrete strain at primary failure is .001718 compared to an ultimate strain of at least .003.

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* $$\frac{3600 \times 1.50 + 1250 \times 1.80}{3600 + 1250} \times \frac{1}{0.70} = 2.25$$

CHAPTER THREE

SUMMARY AND CONCLUSIONS

1. Columns subjected to loads and bending moments but designed for axial load only are found to be inadequate to carry the load and bending moment obtained from conventional structural analysis (Allowed by the ACI-Code).

2. Cracking of underreinforced columns causes a sharp drop in the column stiffness and consequently in the column bending moment.

3. Almost all the reduction in the column moments takes place right after the development of cracks. The extra reduction due to additional cracks caused by further loading is negligible.

4. An exact analysis of the frame, taking into account the crack occurrence and the subsequent reduction in the members' stiffnesses have resulted in a reduction of the column moment, in this particular example, from 6290 kgm to 3420 kgm; a reduction of 45.63 %.

5. In the example, the exterior column which would have failed had the moment been as predicted by the conventional method of analysis, turned out to be safe against yielding of tension steel by a factor of 1.48. In addition it has some additional factor of safety against further rotation to release further bending and transmit it back to the beam.

6. Although the column is safe, it is not sufficiently safe according to

the ACI-Building Code which specifies a factor of safety equal to $\frac{\text{Load factor}}{\phi}$, namely 2.25 in the case of the example chosen.

It is in order to note, at this stage, that the above mentioned results and conclusions are only indicative in nature and not conclusive as they may seem at first glance. Most of them were based on a one particular example which, at its best, can not represent all possible cases met in practice. Moreover, the structural analysis was based on certain assumptions, that are relatively valid but not necessarily exact. These assumptions were stated and justified above in the appropriate places.

For these reasons combined it is felt necessary to devise an experimental program, for further study, which verifies the effect of these assumptions upon the above mentioned findings.

APPENDIX

The complete analysis and the design of the frame, referred to in the text, are carried out in this section. The main points of the problem are already discussed in Chapter Two. The computations, without going into the detail of simple mathematical operations follow.

A

ANALYSIS BY THE CONVENTIONAL METHOD

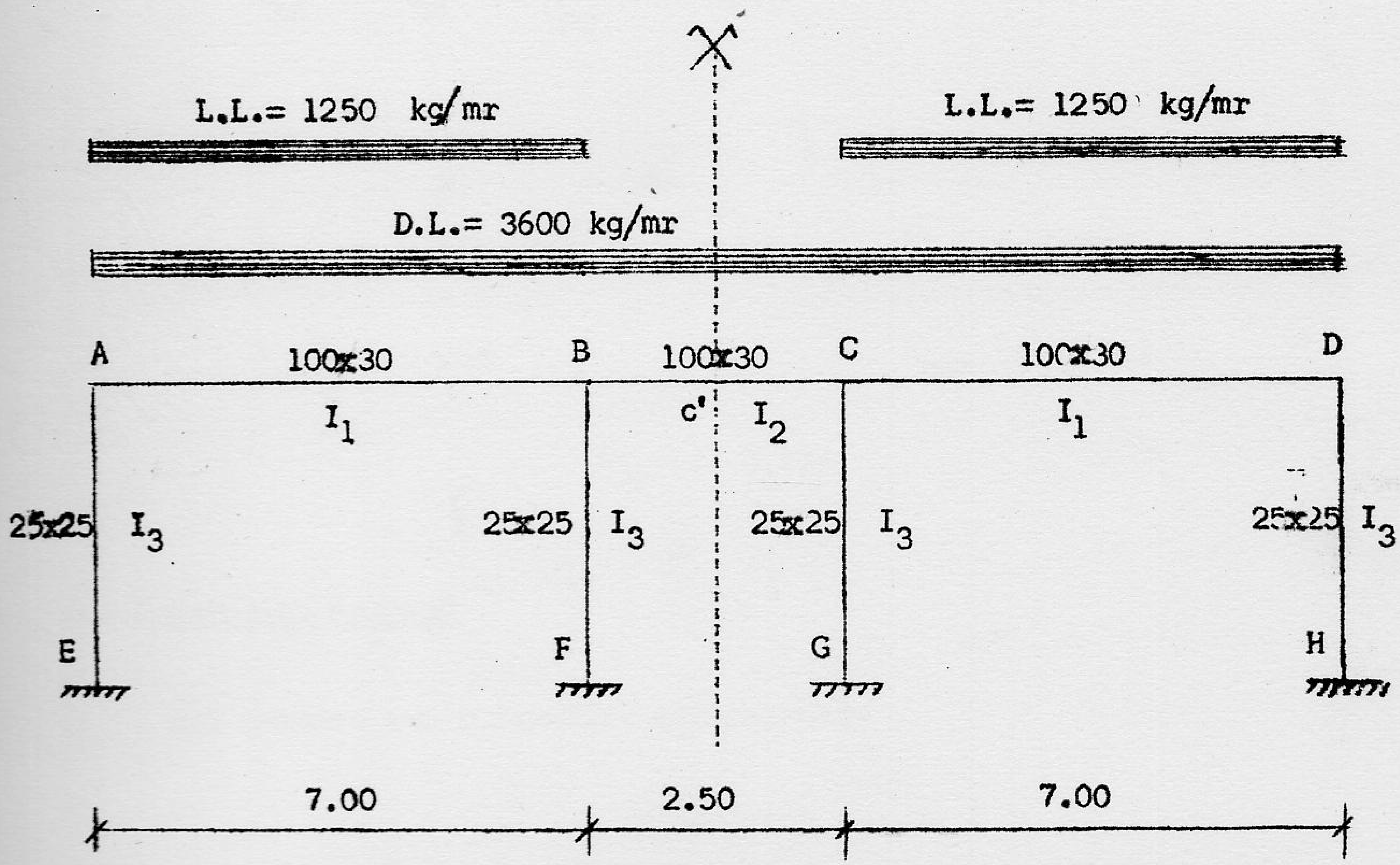


Fig. A.1 The frame and its loading

A.1 Moments of Inertia

$$I_1 = I_2 = \frac{1}{12} \times 10 \times 3^3 = 22.5 \text{ dm}^4.$$

$$I_3 = \frac{1}{12} \times 2.5^4 = 3.26 \text{ dm}^4.$$

A.2 Stiffness Factors

$$K_{AB} = K_{BA} = \frac{22.5}{70} = 0.321$$

$$K_{BC'} = \frac{1}{2} \times \frac{22.5}{25} = 0.450$$

$$K_{AE} = K_{EA} = \frac{3.26}{30} = 0.109$$

A.3 Distribution Factors

$k_{AB} = 0.746$ $\qquad$ $k_{BA} = 0.365$

$k_{AE} = 0.254$ $\qquad$ $k_{BC'} = 0.511$

$k_{BF} = 0.124$

A.4 Carry-over Factors

All members are prismatic, consequently

$$C = 0.5$$

A.5 Fixed End Moments

a) For dead load

$$FEM_{AB} = FEM_{BA} = \frac{1}{12} \times 3600 \times \overline{7.00}^2 = 14700 \text{ Kgm}$$

$$FEM_{BC} = FEM_{CB} = \frac{1}{12} \times 3600 \times \overline{2.50}^2 = 1875 \text{ Kgm}$$

b) For live load

$$FEM_{AB} = FEM_{BA} = \frac{1}{12} \times 1250 \times \overline{7.00}^2 = 5100 \text{ Kgm}$$

$$FEM_{BC} = FEM_{CB} = 0$$

A.6 Moment Distribution for Dead Load

A: Col.	A: Beam		B: Beam	B: Col.	B: Beam
0.254	0.746		0.365	0.124	0.511
	+14.70		-14.70		+ 1.88
- 3.73	-10.97	→	- 5.48		
	+ 3.34	←	+ 6.68	+2.27	+ 9.35
- 0.85	- 2.49	→	- 1.24		
	+ 0.22	←	+ 0.45	+0.16	+ 0.63
- 0.06	- 0.16	→	- 0.08		
			+ 0.03	+0.01	+ 0.04
- 4.64	+ 4.64		-14.34	+2.44	+11.90

A B C'

E F

A.7 Moment Distribution for Live Load

A

Col.	Beam
0.254	0.746
	+ 5.10
- 1.30	- 3.80 →
	+ 1.28
- 0.33	- 0.95 →
	+ 0.08
- 0.02	- 0.06 →
- 1.65	+ 1.65

B

Beam	Col.	Beam
0.365	0.124	0.511
- 5.10		
- 1.90		
← + 2.56	+ 0.86	+ 3.58
- 0.47		
← + 0.17	+ 0.08	+ 0.24
- 0.03		
+ 0.01		+ 0.02
- 4.76	+ 0.92	+ 3.84

E F c'

A.8 Reactions and Positive Moments

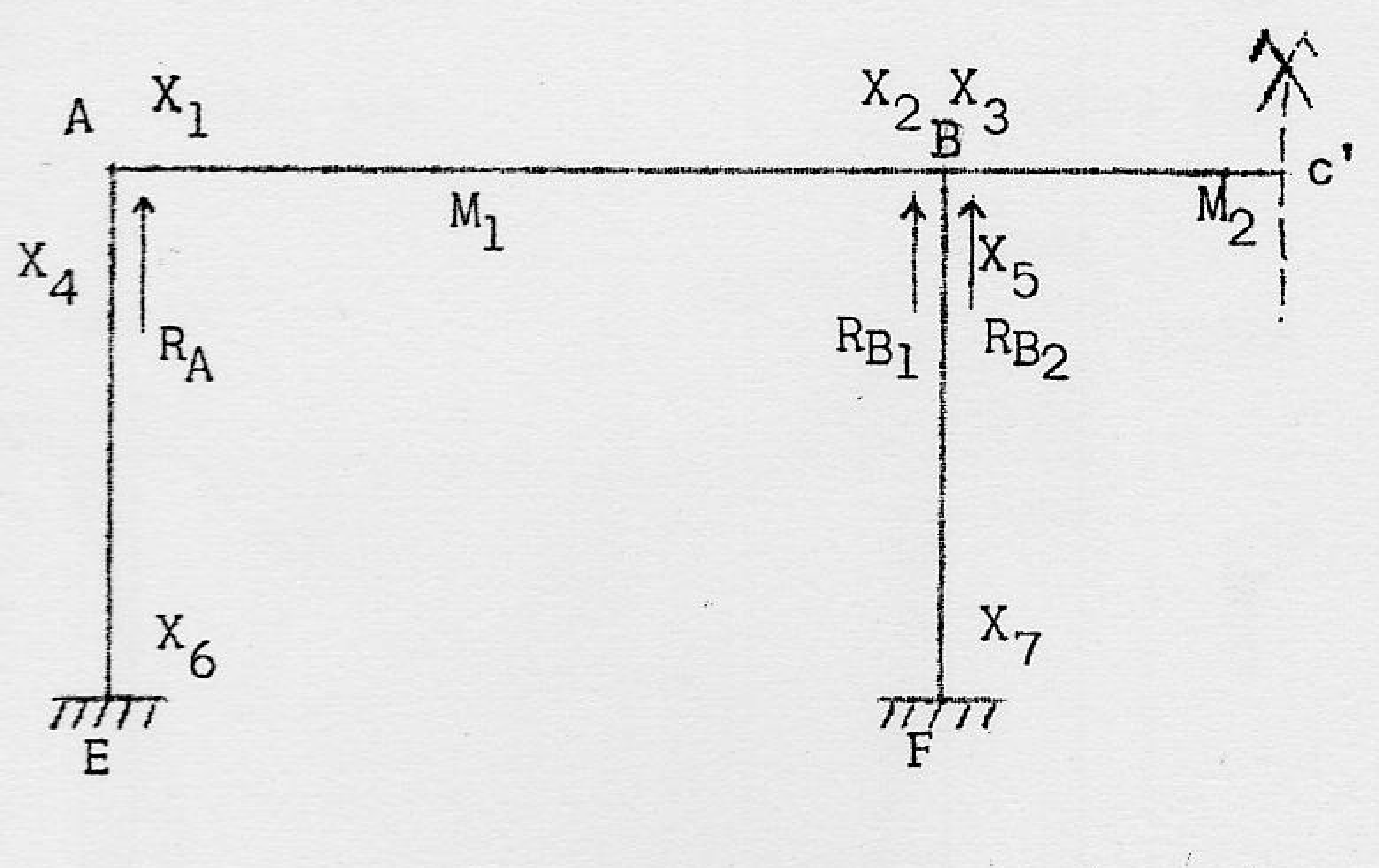


Fig. A.2 Reactions and positive moments
(Conventional analysis)

		For Dead Load	For Live Load
R_A	=	11214 Kg	3931 Kg
R_{B1}	=	13986 Kg	4819 Kg
R_{B2}	=	4500 Kg	0
M_1	=	12826 Kgm	4531 Kgm
M_2	=	-9088 Kgm	-3840 Kgm

A.9 Final Moments

		Total Moments	Ultimate Total Moments
X_1	=	-6290 Kgm	-9930 Kgm
X_2	=	-19100 Kgm	-30078 Kgm
X_3	=	-15740 Kgm	-24762 Kgm
X_4	=	-6290 Kgm	-9930 Kgm
X_5	=	-3360 Kgm	-5316 Kgm
X_6	=	+3145 Kgm	+4965 Kgm
X_7	=	+1680 Kgm	+2658 Kgm
M_1	=	+17357 Kgm	+27395 Kgm
M_2	=	-12928 Kgm	-20544 Kgm

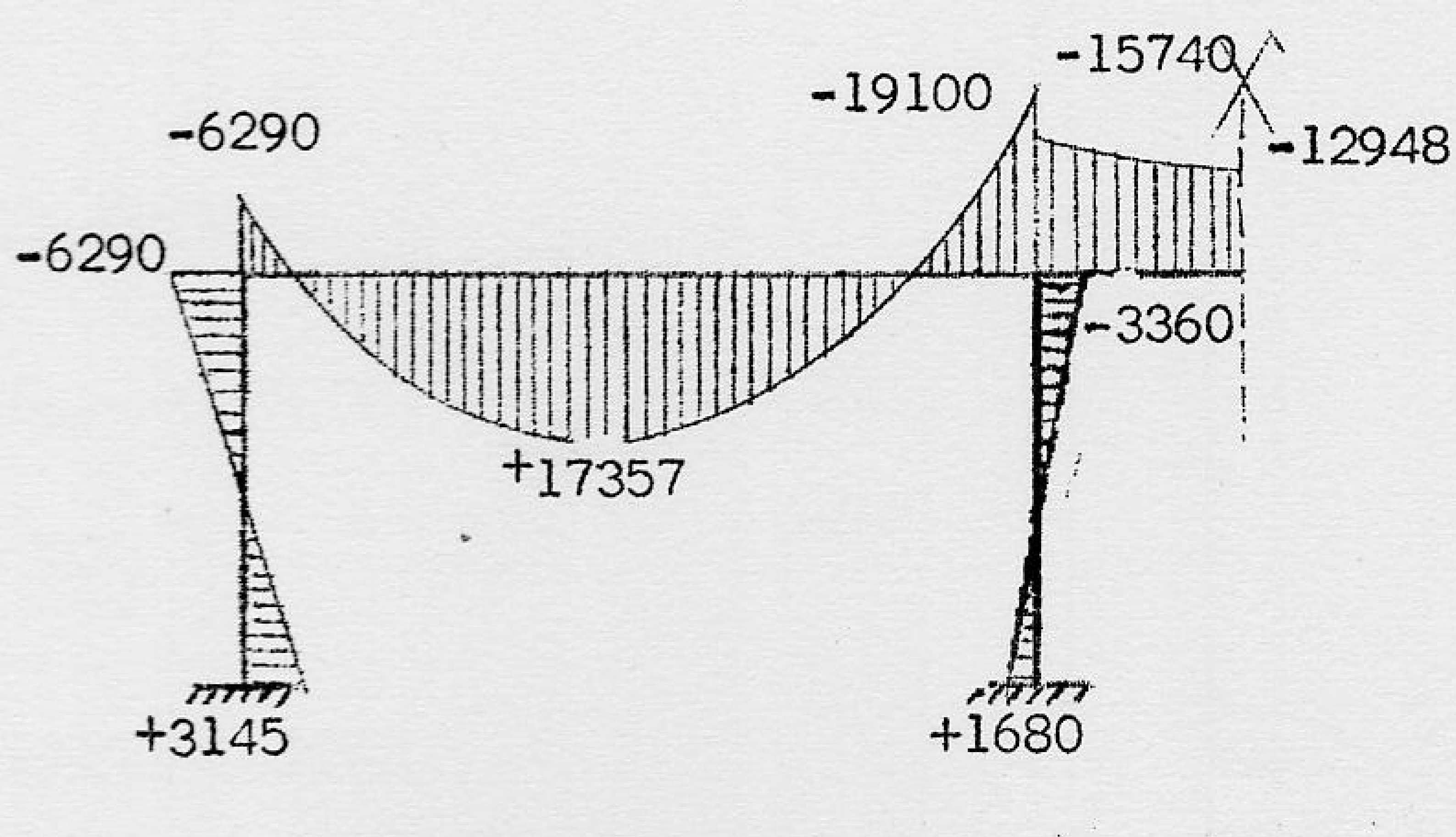


Fig. A.3 Moment diagram (Conventional analysis)

B

BEAM ANALYSIS ASSUMING COLUMNS AS SIMPLE SUPPORTS

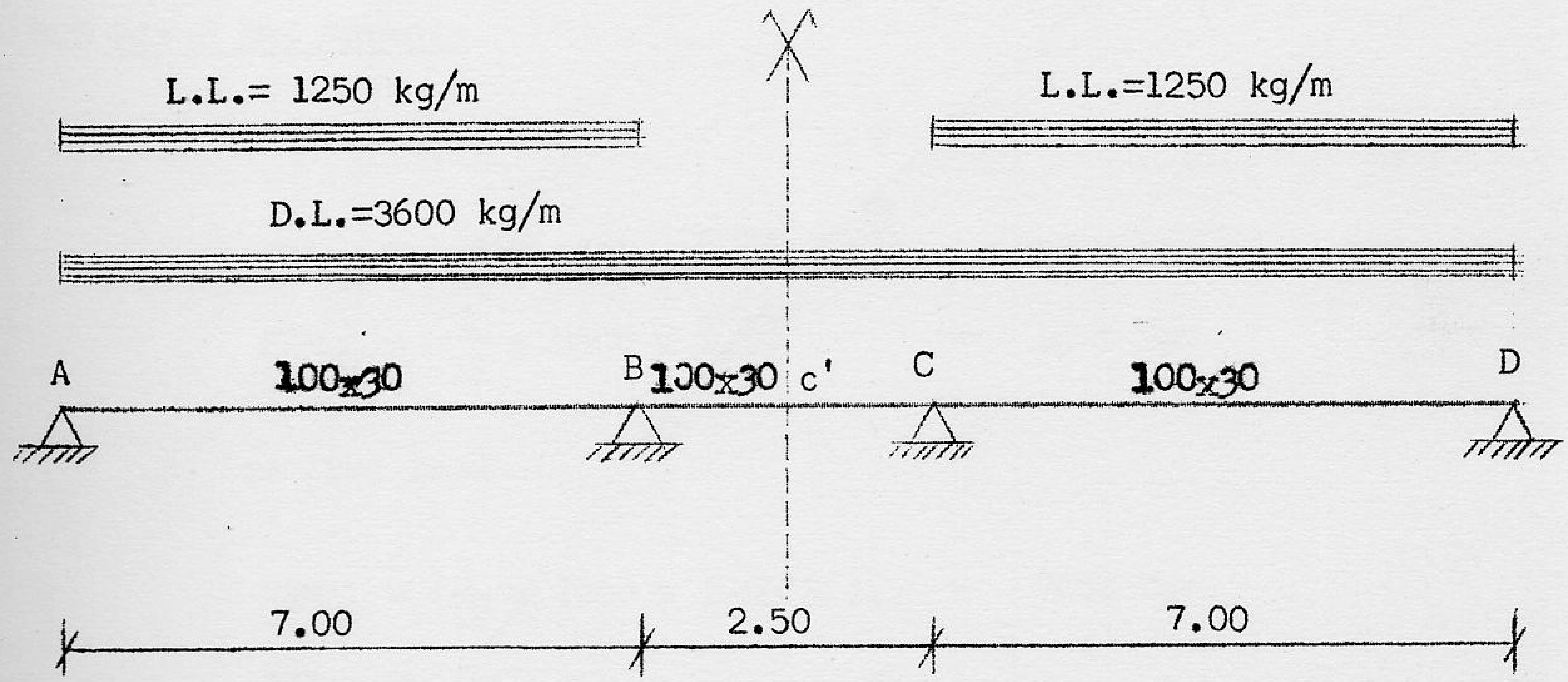


Fig. B.1 The frame and its loading
(Columns assumed as knife edge supports)

B.1 Moments of Inertia

All beams have the same cross-section, therfore the moments of inertia can be taken as unity.

B.2 Stiffness Factors

$$K_{BA} = 0.75 \times \frac{1}{7.00} = 0.107$$

$$K_{BC'} = \frac{1}{2} \times \frac{1}{2.50} = 0.200$$

B.3 Distribution Factors

$K_{BA} = 0.349$ $\qquad$ $K_{BC'} = 0.651$

B.4 Carry-over Factors

$C = 0.5$ for all members

B.5 Fixed End Moments

a) For dead load

$$FEM_{BA} = \left(\frac{1}{12} \times 3600 \times \overline{7.00}^2\right) 1.5 = 22050 \text{ Kgm}$$

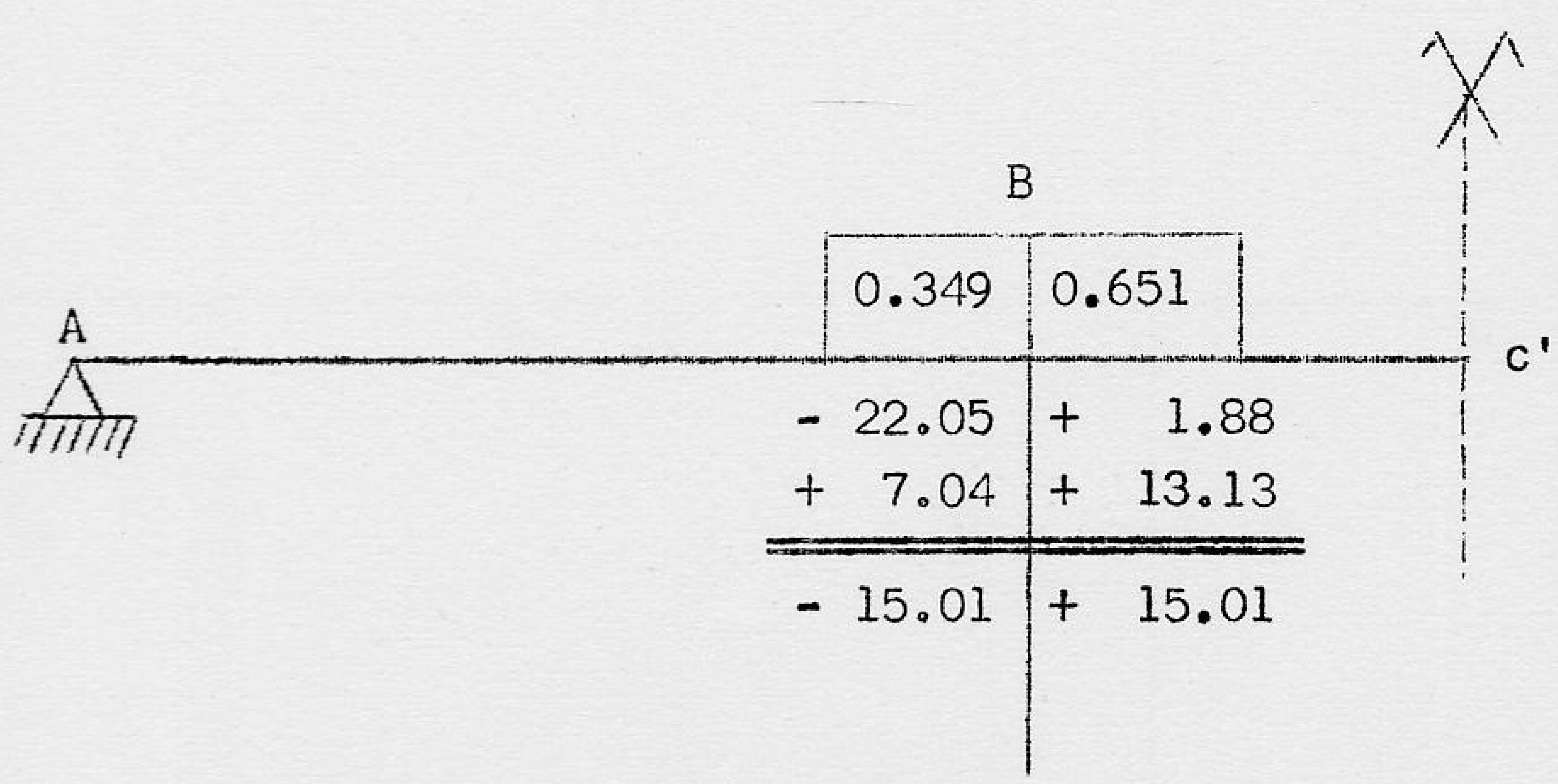
$$FEM_{BC} = \frac{1}{12} \times 3600 \times \overline{2.50}^2 = 1875 \text{ Kgm}$$

b) For live load

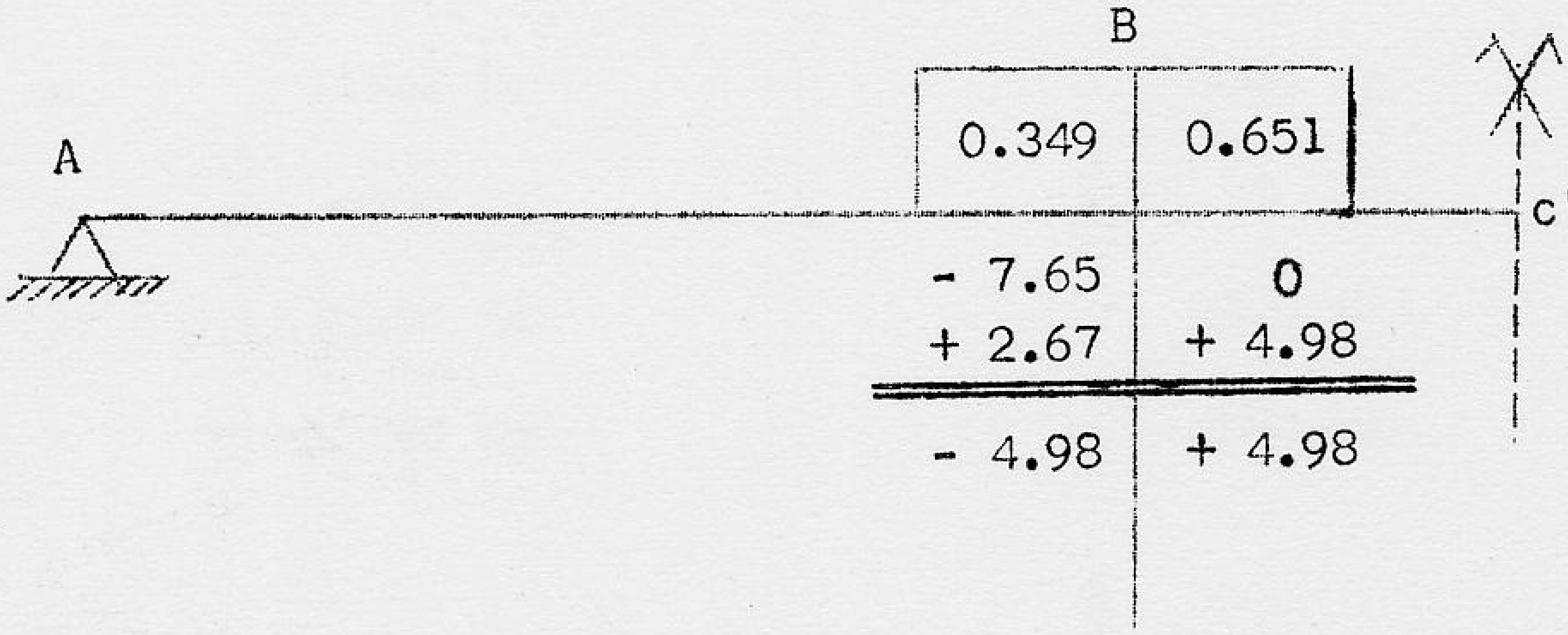
$$FEM_{BA} = \left(\frac{1}{12} \times 1250 \times 7.00^2\right) 1.5 = 7650 \text{ Kgm}$$

$$FEM_{BC} = 0$$

B.6 Moment Distribution for Dead Load



B.7 Moment Distribution for Live Load



B.8 Reactions and Positive Moments

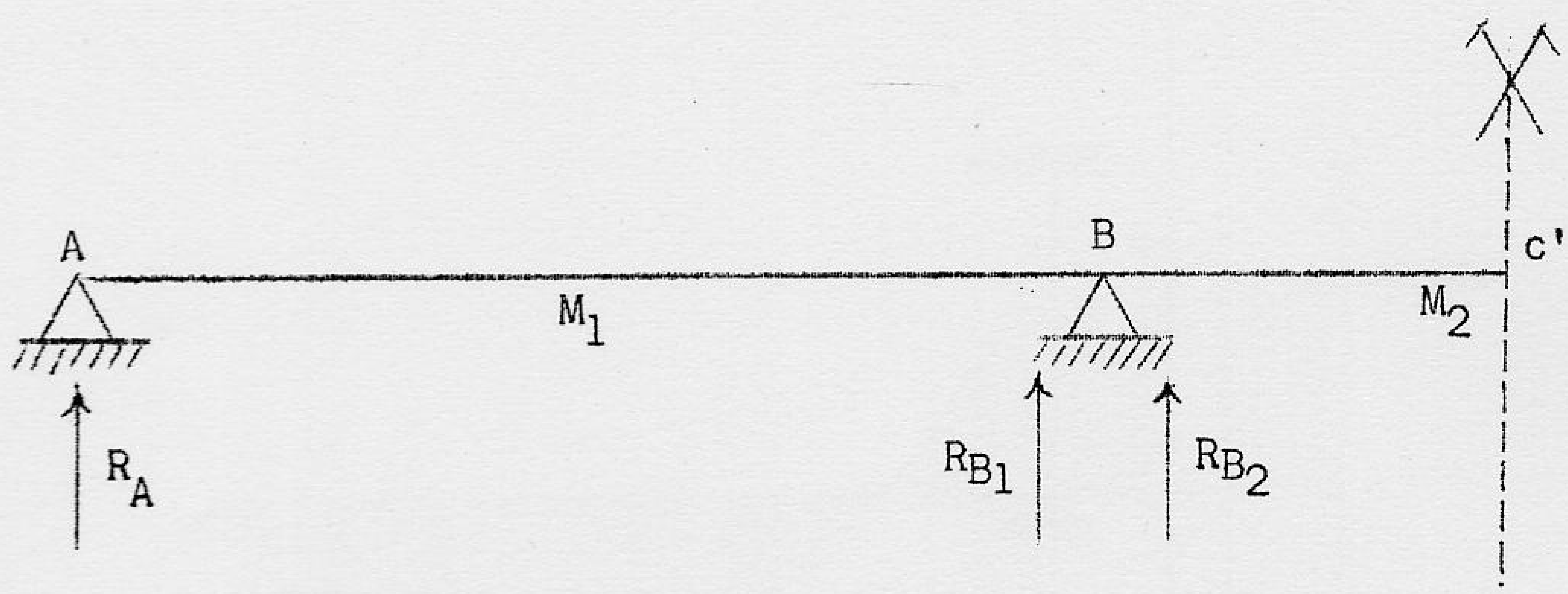


Fig. B.2 Reactions and positive moments
(Columns assumed as knife edge supports)

		For Dead Load	For Live Load
R_A	=	10456 Kg	3664 Kg
R_{B_1}	=	14744 Kg	5086 Kg
R_{B_2}	=	4500 Kg	0
M_1	=	15184 Kgm	5370 Kgm
M_2	=	-12198 Kgm	-4980 Kgm

B.9 Final Moments

		Total Moments	Ultimate Total Moments
X_1	=	0	0
X_2	=	-19990 Kgm	-31479 Kgm
M_1	=	+20554 Kgm	+32442 Kgm
M_2	=	-17178 Kgm	-27261 Kgm

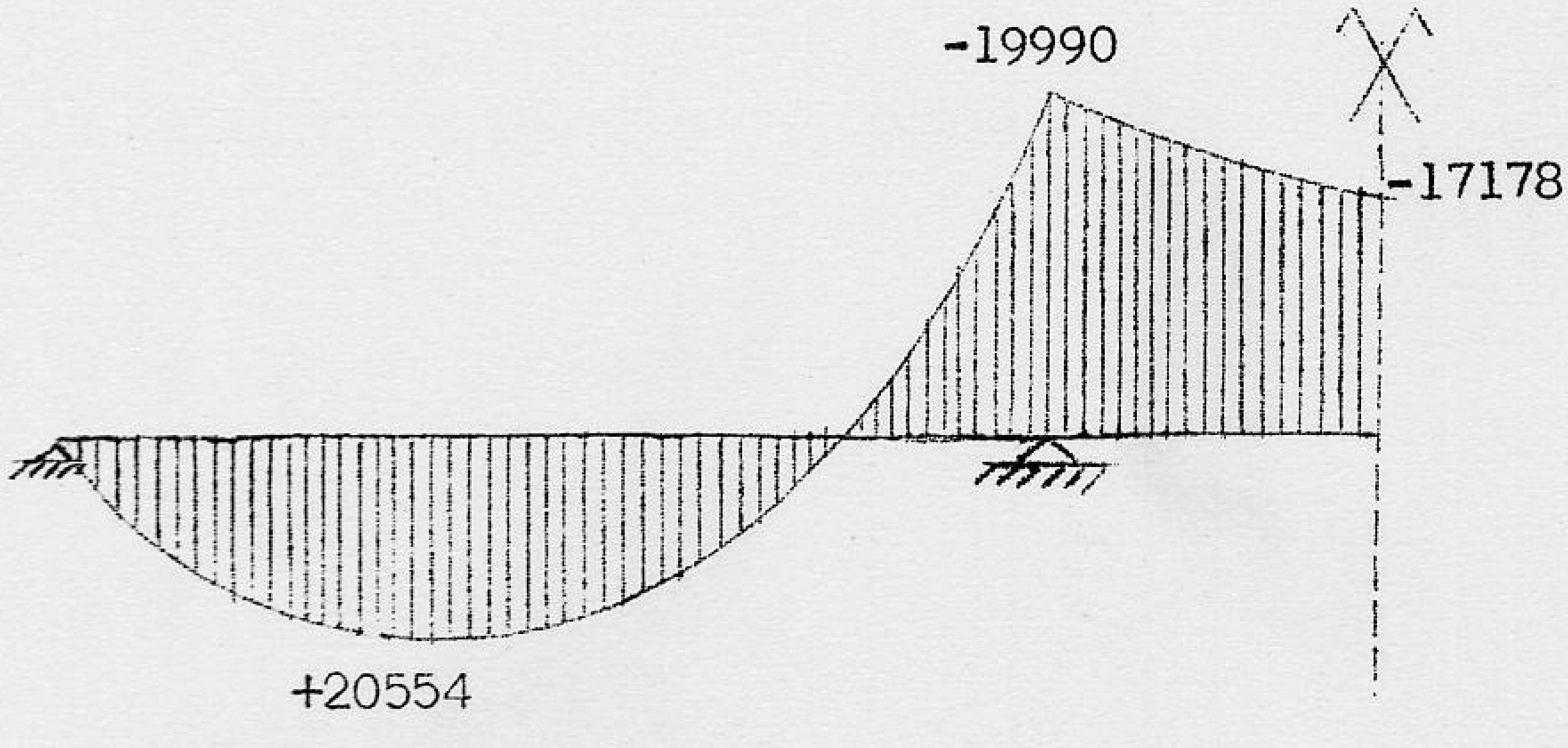


Fig. B.3 Moment diagram (Columns assumed as knife edge supports)

C

THE DESIGN OF THE MEMBERS FOR THE BENDING MOMENTS AND REACTIONS OBTAINED IN B.6, B.7, B.8 AND B.9

C.1 Limiting Values for Rectangular Beams

$$f_y = 4000 \text{ Kg/cm}^2$$

$$f'_c = 176 \text{ Kg/cm}^2$$

$$E_s = 2039 \times 10^3 \text{ Kg/cm}^2$$

$$n = 10$$

$$c_b = \frac{6117}{6117 + 4000} \cdot d \overset{*}{=} 0.608\ d$$

$$a_b = 0.85\ c_b = 0.85 \times 0.608\ d = 0.518\ d$$

$$a_{max} = 0.75 \times 0.518\ d = 0.389\ d$$

$$\bar{C}_{max} = 0.85\ f'_c\ (0.389\ d)\ b = 0.329\ f'_c bd$$

$$Jd = d - \frac{a}{2} = d - \frac{0.389}{2} d = 0.805\ d$$

$$\bar{C} = \bar{T} = As.f_y = pbdf_y$$

$$0.329\ f'_c bd = pbdf_y$$

$$\therefore\ p = 0.329 \times \frac{176}{4000} = 0.01445$$

$$\bar{M}_{max} = \bar{T}.jd = 0.01445\ bd \times 4000 \times 0.805\ d = 46.10\ bd^2$$

---

*Metric equivalent of $c_b = d(87000)/(87000 + f_y)$, ACI-Code pg. 140.

$$M_{u_{max}} = \phi \; \bar{M}_{max} = 0.9 \times 46.10 \; bd^2 = 41.40 \; bd^2$$

R_u	$\bar{R}$	100p	a/d
41.40	46.10	1.445	0.389

C.2 Design of the Beams

a) Design for M_1

$$M_u = 32442 \text{ Kgm}$$

$$\bar{M} = \frac{32442}{0.9} = 36200 \text{ Kgm}$$

$$\bar{R} = \frac{36200 \times 100}{100 \times 25.5^2} = 55.6$$

$$55.6 > 46.10$$

∴ Compr. Reinf. needed

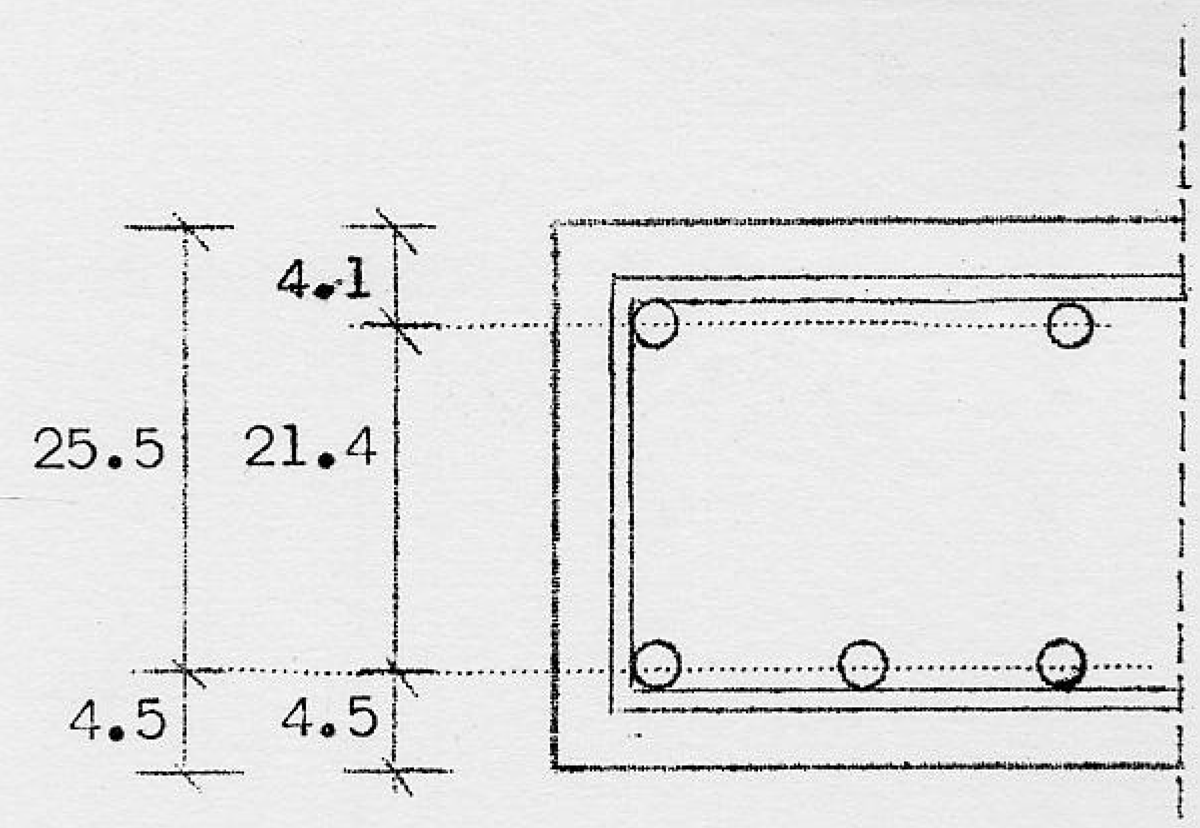


Fig. C.1 Values of d, d' and d - d'

$$\bar{M}_1 = 46.10 \times 100 \times \overline{25.5}^2$$

$$= 29900 \text{ Kgm}$$

$$A_{s1} = \frac{1.445}{100} \times 100 \times 25.5 = 37.00 \text{ cm}^2$$

$$\bar{M}_2 = 36200 - 29900 = 6300 \text{ Kgm}$$

$$\bar{C}_2 = \bar{T}_2 = \frac{\bar{M}_2}{d - d'} = \frac{6300 \times 100}{21.4} = 29500 \text{ Kg}$$

Considering the concrete displaced by A'_s

$$f''_s = 4000 - 0.85 \times 176 = 3850.40 \text{ Kg/cm}^2$$

$$A'_s = \frac{29500}{3850.40} = 7.70 \text{ cm}^2$$

using 6 Ф 16 mm $\qquad A'_s = 12.06 \text{ cm}^2$

Check if compr. steel yields or not

$$\varepsilon_y = \frac{4000}{2039 \times 10^3} = 0.00196$$

$$a = 0.389 \times 25.5 = 9.93 \text{ cm}$$

$$c = \frac{9.93}{0.85} = 11.70 \text{ cm}$$

$$\varepsilon'_s = 0.003 \left(\frac{11.70 - 4.10}{11.70}\right) = 0.00195 \sim 0.00196 \quad \text{O.K.}$$

$$A'_s f'_s = A_{s_2} \cdot f_y$$

$$A_{s_2} = \frac{7.70 \times 3850.40}{4000} = 7.45 \text{ cm}^2$$

$$\therefore A_s = 37.00 + 7.45 = 44.45 \text{ cm}^2$$ Use 10 Ф 25 mm

b) Design for X_2

The value of X_2 (31479 kgm) is very close to the one of M_1 (32442 kgm). No separate design for X_2 is needed. Use the same reinforcement as for M_1.

c) Design for M_2

The moment diagram in fig. B.3 page 37 shows that, like X_2,

M_2 also is negative and its value is not very much different from the value of X_2. It follows that, the reinforcement provided for X_2 can also be used for M_2. Besides, due to the shortness of the centerspan, the negative reinforcement extended from the supports, will either meet at the middle of the centerspan or come too close to each other. Taking these two points into consideration the reinforcement provided for X_2 will be continued in the centerspan and 4 Φ 16 mm bars will be used as bottom reinforcement.

C.3 Cut-off Points

One third of the positive reinforcement, in this case 4 Φ 25 mm bars, will be extended into the support. The rest of the bars will be stopped at a distance equal to 12 bar diameter, beyond the points at which they are no more needed. In other words, the positive reinforcement will be stopped at a distance equal to 0.30 m* beyond the points at which the moment is equal to the resisting moment of the section with only 4 Φ 25 mm bars at the bottom and 6 Φ 16 mm bars at the top. The above mentioned resisting moment and the cut-off points for positive reinforcement are computed as follows:

$$A'_s = A_{s_2} = 12.06 \text{ cm}^2$$

$$A_{s_1} = 19.69 - 12.06 = 7.63 \text{ cm}^2$$

$$a = \frac{A_{s_1} f_y}{0.85 f'_c b} = \frac{7.63 \times 4000}{0.85 \times 176 \times 100}$$

$$= 2.05 \text{ cm}$$

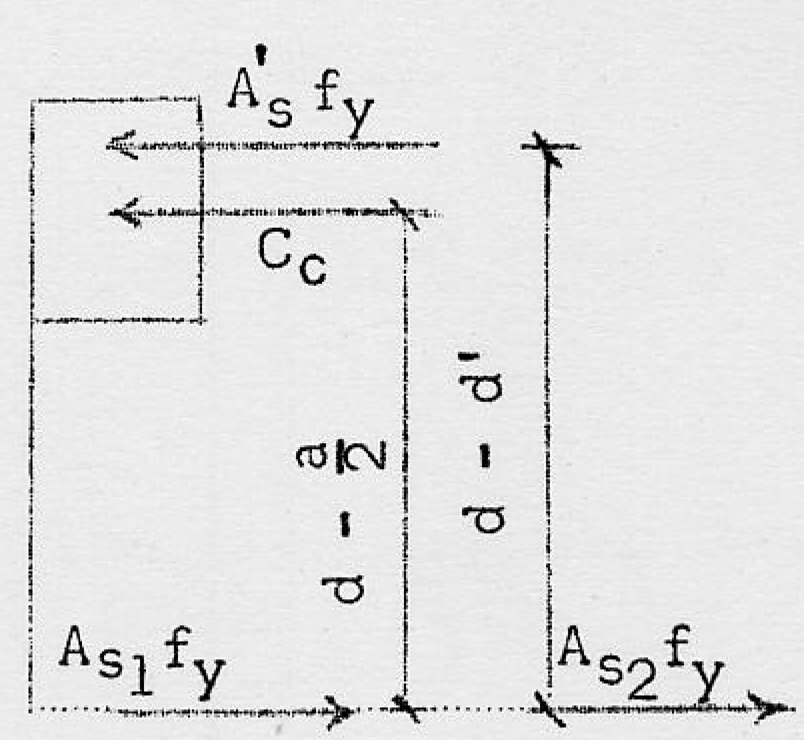


Fig. C.2 Computation of resisting moment

*0.30 = 12 x 0.025 = 12 x bar diameter.

$$c = \frac{a}{0.85} = \frac{2.05}{0.85} = 2.41 \text{ cm}$$

$$c < d'$$

therefore the compressive steel does not yeild; proceed with general analysis.

By trial and error f'_s is found to be 1247 kg/cm^2

$$f''_s = 1247.00 - 0.85 \times 176 = 1097.40 \text{ kg/cm}^2$$

$$A'_s . f''_s = A_{s2} . f_y$$

$$\therefore A_{s2} = \frac{A'_s . f''_s}{f_y} = \frac{12.06 \times 1097.40}{4000} = 3.3086 \text{ cm}^2$$

$$A_{s1} = 19.69 - 2.3086 = 16.3814 \text{ cm}^2$$

$$a = \frac{A_{s1} f_y}{0.85 f'_c b} = \frac{16.3814 \times 4000}{0.85 \times 176 \times 100} = 4.38 \text{ cm}$$

$$c = \frac{4.38}{0.85} = 5.15 \text{ cm}$$

$$\varepsilon = 0.003 \left(\frac{5.15 - 4.10}{5.15}\right) = 0.00061165$$

$$f'_s = 2039 \times 10^3 \times 0.00061165 = 1247.15 \sim 1247 \quad \text{O.K.}$$

$$\therefore \bar{M}_2 = A'_s . f''_s (d - d') = 12.06 \times 1097.40 \left(\frac{25.50 - 4.10}{100}\right)$$

$$= 2832.21 \text{ kgm}$$

$$\bar{M}_1 = A_{s1} f_y (d - \frac{a}{2}) = 16.38 \times 4000 (25.50 - \frac{4.38}{2}) \frac{1}{100}$$

$$= 15272.71 \text{ kgm}$$

$$\bar{M} = M_1 + M_2 = 15272.71 + 2832.21 = 18104.92 \text{ kgm}$$

$$M_u = 18104.92 \times 0.9 = 16294.42 \text{ kgm}$$

If x_1 is the distance from the centerline of the exterior support to the point at which the moment is equal to 18104.92 kgm, then

$$R_A \cdot x_1 - \frac{wx_1^2}{2} = 18104.92$$

Where

$$R_A = 10456 \times 1.50 + 3664 \times 1.80 = 22279.20 \text{ kg}$$

$$= 3600 \times 1.50 + 1250 \times 1.80 = 7650 \text{ kg/m}$$

Therefore

$$22279.20 \times x_1 - \frac{7650 \times x_1^2}{2} = 18104.92 \text{ kgm}$$

Or

$$x_1^2 - 5.82\, x_1 + 4.74 = 0 \qquad \therefore \quad x_1 = \begin{cases} 4.84 \text{ m} \\ 0.98 \text{ m} \end{cases}$$

Therefore the first cut off point will be at

$$0.98 - \frac{0.25}{2} - 0.30 = 0.555 \text{ m}$$ from the face of the exterior support.

This point is too close to the support. Obviously it is preferable to extend all bars right into the exterior support. The second cut off point will be at

$$7.00 - 4.84 - 0.30 = 1.86 \text{ m} \quad \text{say } 1.80 \text{ m}$$ from the center-line of the int. support.

According to the ACI-Building Code Art. 918 e at least one-third of the total reinforcement provided for negative moment at the support should be extended beyond the point of contra flexure a distance not less than 1/16 of the clear span, or the effective depth, whichever is greater. To avoid unnecessary complications when computing the stiffness factors of the beam by considering its different cracked and uncracked transformed sections, all bars provided for negative moment will be extended to the point specified above.

Let x_2 be the distance between the point of contra flexure and centerline of exterior support. Then,

$$R_A \cdot x_2 - \frac{w\, x_2^2}{2} = 0$$

$$22279.20 \times x_2 - \frac{7650 \times x_2^2}{2} = 0$$

$$x_2^2 - 5.82\, x_2 = 0$$

$$\therefore \quad x_2 = \begin{cases} 5.82 \\ 0 \end{cases}$$

Therefore the cut off point for the negative reinforcement will be at

$$7.00 - 5.82 + \frac{(7.00 - 0.25)^*}{16} = 1.602 \text{ m}$$ from the center-line of int. support.

The cut off points for positive and negative reinforcements are very close to each other. For simplicity both negative and positive reinforcements will be stopped at a distance of 1.80 m from the centerline of the interior support.

---

* 1/16 of the clear span.

C.4 Design of the Columns (For axial load only)

a) Exterior Column

$R_A = 10456 \times 1.5 + 3664 \times 1.8 = 22279.20$ kg

$r = \sqrt{\frac{I}{A}} = \sqrt{\frac{3.26 \times 10^4}{25 \times 25}} = 7.22$ cm

$\frac{h}{r} = \frac{300 - 15}{7.22} = 39.47 < 60$ $\therefore$ no strength reduction is necessary.

$\phi = 0.70$

$\bar{P} = \frac{22279.20}{0.70} = 31827.43$ kg

$\bar{P} = 0.85 f'_c (A_g - A_{st}) + f_y . A_{st}$

$= 0.85 \times 176 (25 \times 25 - A_{st}) + 4000 A_{st}$

$= 93500 + 3850.40 A_{st}$

The section is ample, therefore use minimum reinforcement.

$p_{min} = 0.01$

$\therefore A_{st_{min}} = 0.01 \times 25 \times 25 = 6.25$ cm^2

4 Φ 14 mm would be enough but use 4 Φ 16 mm according ACI-Building Code art. 913 a.

Use 6 mm ties s = 25 cm*

* 25 cm = Least dimension of column.

b) Interior Column

R_{B_1} = 14744 x 1.5 + 5086 x 1.8 = 31270.80 kg

R_{B_2} = 4500 x 1.50 + 0 x 1.80 = 6750.00 kg

$\therefore$ P_u = 31270.80 + 6750.00 = 38020.80 kg

$\frac{h}{r}$ = 39.47

$\frac{h}{r} < 60$ $\therefore$ no strength reduction is necessary

$\emptyset$ = 0.70

$\bar{P} = \frac{38020.80}{0.70} = 54315.43$ kg

$\bar{P} = 0.85\ f'_c\ (A_g - A_{st}) + f_y.A_{st}$

$= 0.85 \times 176\ (25 \times 25 - A_{st}) + 4000\ A_{st}$

$= 93500 + 3850.40\ A_{st}$

The section is ample. Use min. reinforcement.

Use 4 Ⓣ 16 mm $\emptyset$ 6 mm ties s = 25 cm*

---

* 25 cm = Lease dimension of column.

D

INVESTIGATION OF THE COLUMNS FOR THE MOMENTS AND REACTIONS OBTAINED FROM THE CONVENTIONAL ANALYSIS

D.1 Investigation of the exterior column

$R_A = 11214 \times 1.5 + 3931 \times 1.8 = 23896.80$ kg

$\therefore P_u = 23896.80$ kg

$M_u = 9930$ kgm

$\frac{h}{r}$ 60 as before. No strength reduction.

$e = \frac{M_u}{P_u} = \frac{9930 \times 100}{23896.80} = 41.55$ cm

$c_b = \frac{6117}{6117 \times 4000} \times 21.1 = 12.76$ cm

$a_b = 12.76 \times 0.85 = 10.85$ cm

$\bar{C}_c = 0.85 \times 176 \times 10.85 \times 25 =$ 40579.00 kg

$\varepsilon_y = \frac{4000}{2039 \times 10^3} = 0.00196$

$\varepsilon_s' = 0.003 \left(\frac{12.76 - 3.90}{12.76}\right) =$ 0.00208

$\varepsilon_s' > \varepsilon_y$ O.K.

$\therefore \bar{C}_s = 4.02\,(4000 - 0.85 \times 176) =$ 15478.61 kg

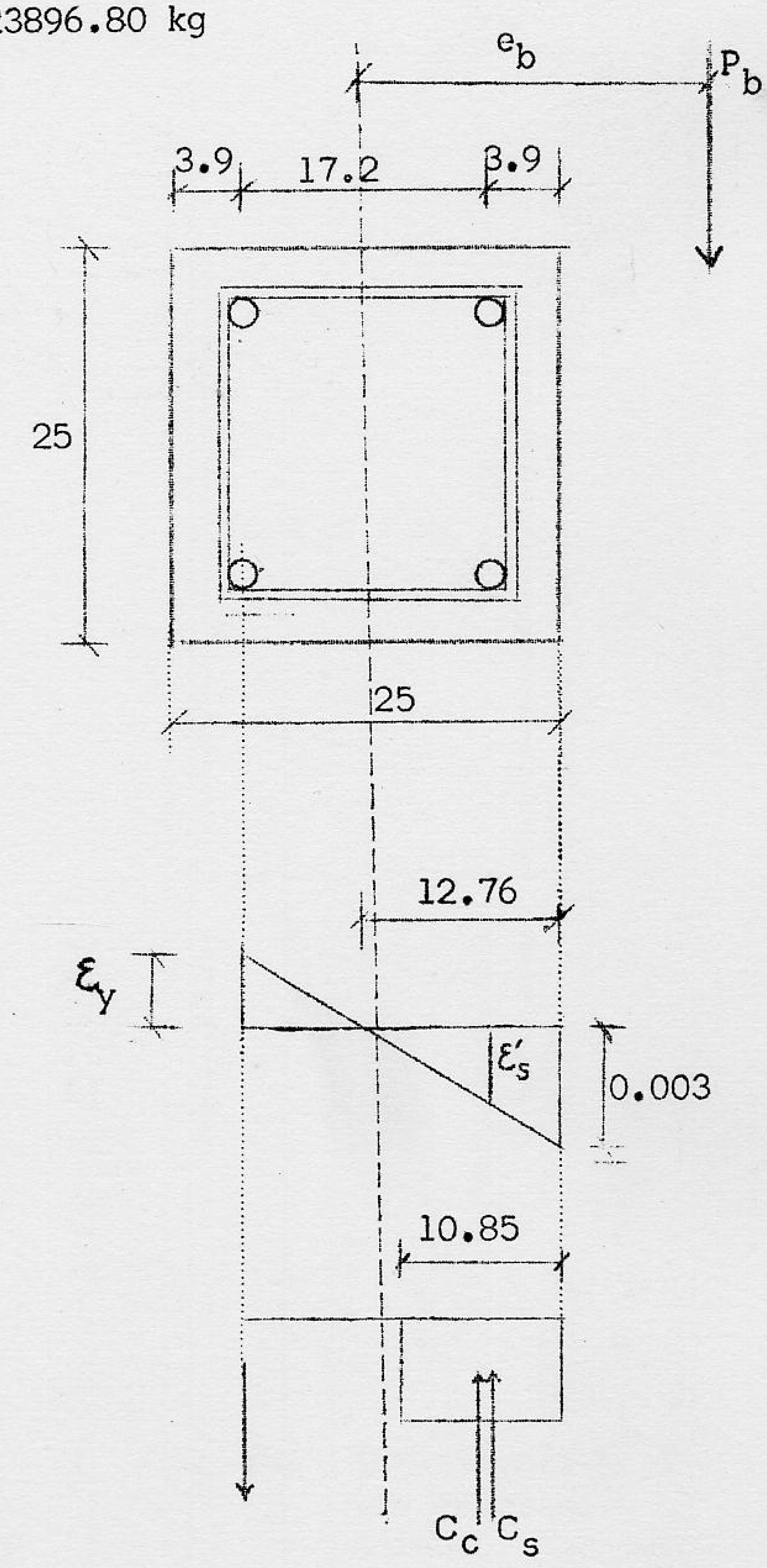


Fig. D.1 Computation of P_b

$\bar{T} = 4.02 \times 4000 = 16080.00$ kg

$\bar{P}_b = \bar{C}_c + \bar{C}_s - \bar{T}$

$\bar{P}_b = 40579.00 + 15478.61 - 16080.00 = 39977.61$ kg

$P_b = 39977.61 \times 0.70 = 27984.33$ kg

Σ M about centerline = 0

$$16080.00 \times \frac{17.2}{2} + 15478.61 \times \frac{17.2}{2} + 40579.00\left(12.50 - \frac{10.85}{2}\right) - \bar{P}_b e_b = 0$$

$\therefore \quad e_b = 13,97$ cm

$M_b = 27984.33 \times 0.1397$

$= 3909.41$ kgm

$P_u < P_b$

therefore tension governs

$$z = e - \frac{t}{2} + \frac{a}{2}$$

$$z = 41.55 - \frac{25}{2} + \frac{a}{2}$$

$$z = 29.05 + \frac{a}{2}$$

$$d - d' = 25.0 - 2 \times 2.5 - 2 \times 0.6 - 2 \times \frac{1.6}{2} = 17.2 \text{ cm}$$

Neglecting the displaced concrete

$$\bar{P}z = \bar{T}(d - d') = A_s f_y (d - d')$$

$$\therefore \bar{P} = \frac{A_s f_y (d - d')}{z}$$

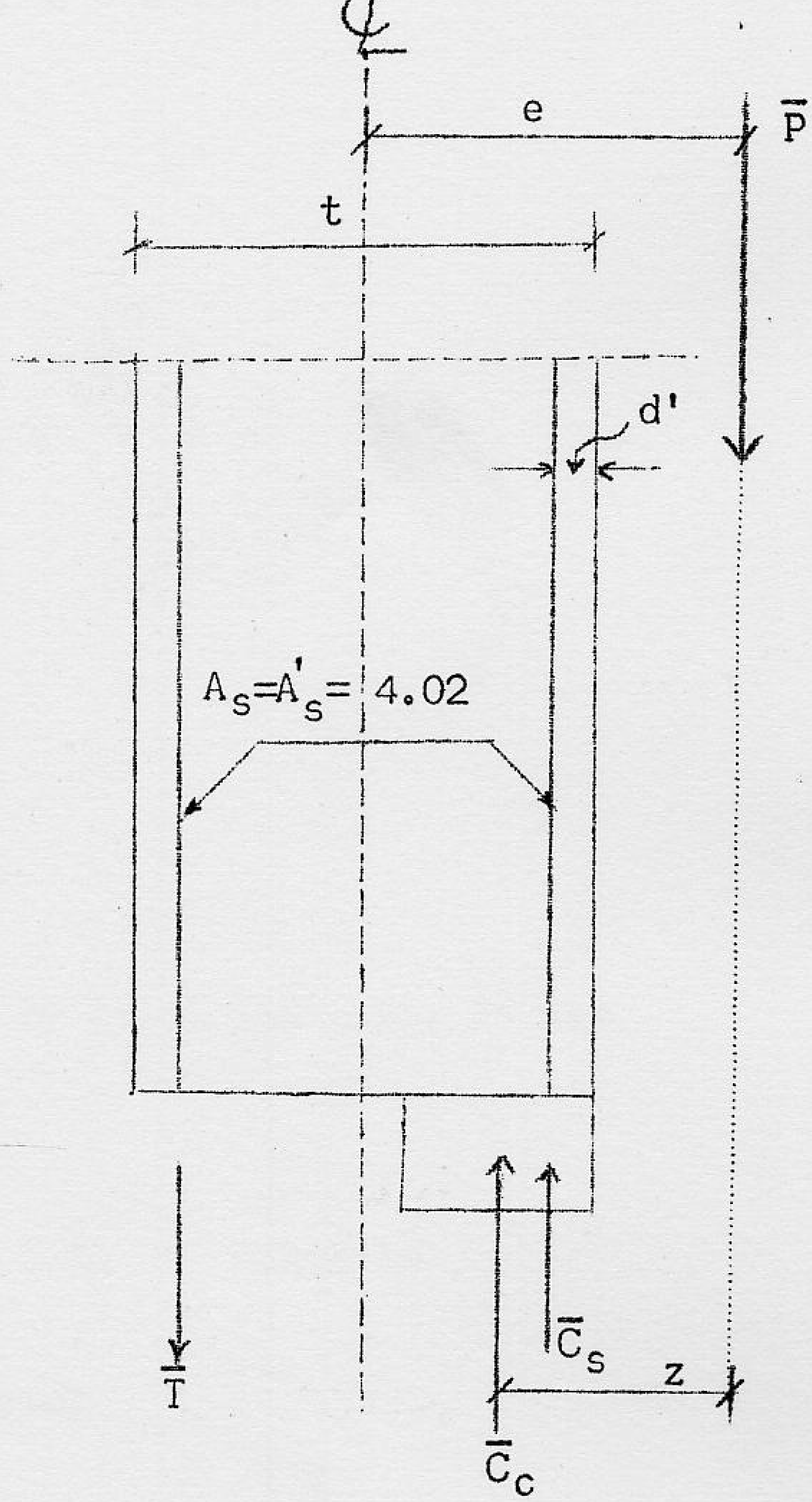


Fig. D.2 Investigation of the exterior column replacing the value of z

$$\bar{P} = \frac{4.02 \times 4000 \times 17.2}{29.05 + \frac{a}{2}} = \frac{553152.00}{58.10 + a}$$

Also .

$$\bar{P} = 0.85\, f'_c\, ba = 0.85 \times 176 \times 25 \times a = 3740\, a$$

Equating these two values of $\bar{P}$

$$3740\, a = \frac{553152.00}{58.10 + a}$$

$$3740\, a^2 + 217294.00\, a - 553152.00 = 0$$

$$a^2 + 58.10\, a - 147.90 = 0$$

$$\therefore\ a = 2.44 \text{ cm}$$

$$c = \frac{2.44}{0.85} = 2.87 < d'' \quad \text{N.G.}$$

So proceed with general analysis. By trial and error it was found that

$$c = 5.70 \text{ cm} \qquad \therefore\ a = 0.85 \times 5.70 = 4.845 \text{ cm}$$

$$\varepsilon'_s = 0.003 \left(\frac{5.70 - 3.90}{5.70}\right) = 0.000947$$

$$f'_s = 0.000947 \times 2039 \times 10^3 = 1930.93 \text{ kg/cm}^2$$

$$f''_s = 1930.93 - 0.85 \times 176 = 1781.33 \text{ kg/cm}^2$$

$$\therefore\ \bar{C}_s = 1781.33 \times 4.02 = 7160.95 \text{ kg}$$

$$\bar{C}_c = 0.85 \times 176 \times 25 \times 4.845 = 18120.30 \text{ kg}$$

$$\bar{T} = 4000 \times 4.02 = 16080.00 \text{ kg}$$

$$\therefore \quad \bar{P} = \bar{C}_c + \bar{C}_s - \bar{T} = 18120.30 + 7160.95 - 16080.00$$

$$= 9201.25 \text{ kg}$$

$$\Sigma M = 0$$

$$\therefore \quad (16080.00 + 7160.95)\frac{17.2}{2} + 18120.30\left(12.50 - \frac{4,845}{2}\right) -$$

$$9201.25 \times 41.55 = 0$$

$$382524.79 - 382311.94 \sim 0 \quad \text{O.K.}$$

$$\therefore \quad \bar{P} = 9201.25 \text{ kg}$$

$$\& \quad P_u = 9201.25 \times 0.7 = 6440.88 < 23896.80 \quad \text{N.G.}$$

D.2 <u>Investigation of the interior column</u>

$$R_{B_1} = 13986 \times 1.5 + 4819 \times 1.8 = 29653.20 \text{ kg}$$

$$R_{B_2} = 4500 \times 1.5 + 0 \times 1.8 = 6750.00 \text{ kg}$$

$$\therefore \quad P_u = 29653.20 + 6750.00 = 36403.20 \text{ kg}$$

$$\frac{h}{r} < 60 \quad \text{No strength reduction}$$

$$M_u = 5316 \text{ kgm}$$

$$e = \frac{M_u}{P_u} = \frac{5316 \times 100}{36403.20} = 14.60 \text{ cm}$$

$$P_u > P_b \quad \text{therefore compression governs}$$

$$P_u = \frac{P_o}{1 + \left[\frac{P_o}{P_b} - 1\right]\frac{e}{e_b}}$$

$$P_o = 0.85\, f'_c\,(A_g - A_{st}) + f_y\, A_{st}$$

$$= 0.85 \times 176\,(25 \times 25 - 8.04) + 4000 \times 8.04$$

$$= 124457.22 \text{ kg}$$

$$P_b = 27984.33 \text{ kg}$$

$$e_b = 13.97 \text{ cm}$$

$$\therefore\; P_u = \frac{124457.22}{1 + \left[\frac{124457.22}{27984.33} - 1\right]\frac{14.60}{13.97}} = 26938.79 \text{ kg} < 36403.20 \text{ N.G.}$$

E

INVESTIGATION OF THE FRAME RIGHT AFTER REMOVAL OF FORMWORK

E.1 Points of Contraflexure

Right after removal of formwork the load on the frame is 79.44 % of the total dead load. The sections are uncracked. To determine the sections of different moment of inertias along the beam, the points of contraflexure must be computed. They can be found by correcting the dead load reactions and moments obtained from the conventional method of analysis to the present loading as follows:

$$R_A = 11214 \times \frac{79.44}{100} = 8908.40 \text{ kg}$$

$$X_1 = -4640 \times \frac{79.44}{100} = -3686.02 \text{ kgm}$$

$$w = 3600 \times \frac{79.44}{100} = 2859.84 \text{ kg/m}$$

Let x be the distance to the points of contraflexure from the center of the left support, Then

$$R_A x - \frac{wx^2}{2} + X_1 = 0$$

$$8908.40\, x - \frac{2859.84\, x^2}{2} - 3686.02 = 0$$

$$x^2 - 6.23\, x + 2.58 = 0$$

$$\therefore\ x = \begin{cases} 5.784 \text{ m say } 5.80 \text{ m} \\ 0.446 \text{ m say } 0.45 \text{ m} \end{cases}$$

Right after the removal of formwork the different sections of the frame are as shown in fig. E.1.

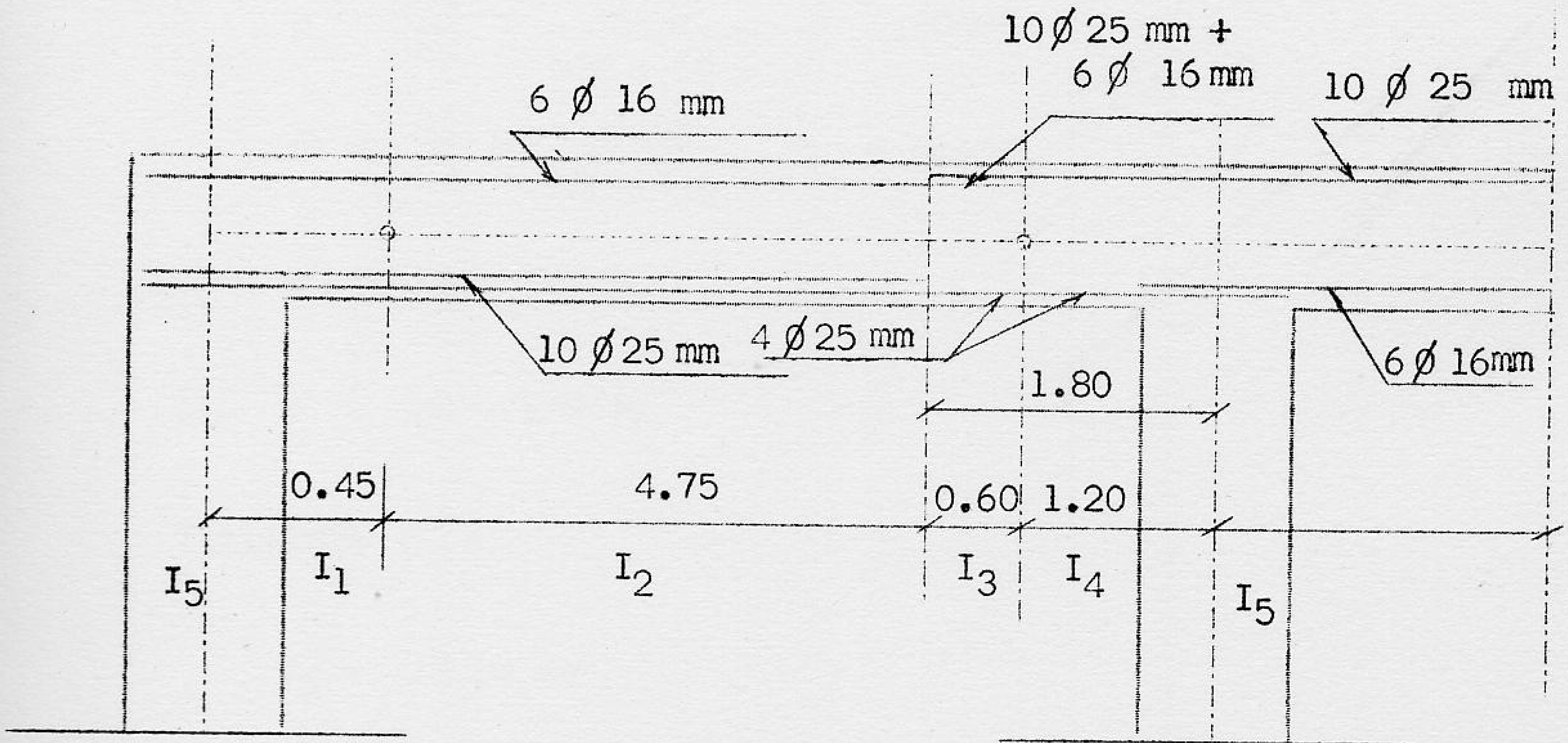


Fig. E.1 Different sections of the frame (1st stage of loading)

E.2 Moment of Inertias

I_1)

$$Y_1 = \frac{h^2 + 2d\,(pmd + p'm'd')}{2\,[h + d\,(pm + p'm')]}$$

$h = 30$ cm

$h^2 = 900 \text{ cm}^2$

$d = 30.0 - 2.5 - 0.8 - \frac{1.6}{2} = 25.9 \text{ cm}$

$d' = 2.5 + 0.8 + \frac{2.5}{2} = 4.55 \text{ cm}$ say 4.5 cm

$p = \frac{A_s}{bd} = \frac{12.06}{100 \times 25.9} = 0.00466$

$p' = \frac{A'_s}{bd} = \frac{49.22}{100 \times 25.9} = 0.0190$

$m = 10 - 1 = 9$

$m' = 20 - 1 = 19$

Therefore

$Y_1 = 12.86 \text{ cm}$

$I_1 = 396481.23 \text{ cm}^4$ or 39.65 dm^4

I_2)

$h = 30 \text{ cm}$

$h^2 = 900 \text{ cm}^2$

$d = 30.0 - 2.5 - 0.8 - \frac{2.5}{2} = 25.45 \text{ cm}$ say 25.5 cm

$d' = 2.5 + 0.8 + \frac{1.6}{2} = 4.1 \text{ cm}$

$p = \frac{A_s}{bd} = \frac{49.22}{100 \times 25.5} = 0.0193$

$p' = \frac{A'_s}{bd} = \frac{12.06}{100 \times 25.5} = 0.00473$

$m = 10 - 1 = 9$

$m' = 20 - 1 = 19$

Therefore

$$Y_1 = 15.59 \text{ cm}$$

$$I_2 = 299799.60 \text{ cm}^4 \quad \text{or} \quad 29.98 \text{ dm}^4$$

I_3)

$$h = 30 \text{ cm}$$

$$h^2 = 900 \text{ cm}^2$$

$$d = 30.0 - 2.5 - 0.8 - \frac{2.5}{2} = 25.45 \text{ cm} \quad \text{say} \quad 25.5 \text{ cm}$$

$$d_1' = 2.5 + 0.8 + \frac{2.5}{2} = 4.55 \quad \text{say} \quad 4.5 \text{ cm}$$

$$d_2' = 2.5 + 0.8 + 2.5 + 2.5 + \frac{1.6}{2} = 9.1 \text{ cm}$$

$$d' = \frac{49.22 \times 4.5 + 12.06 \times 9.1}{49.22 + 12.06} = 5.405 \text{ cm} \quad \text{say } 5.4 \text{ cm}$$

$$p = \frac{A_s}{bd} = \frac{19.69}{100 \times 25.5} = 0.00772$$

$$p' = \frac{A_s'}{bd} = \frac{49.22 + 12.06}{100 \times 25.5} = 0.024$$

$$m = 10 - 1 = 9$$

$$m' = 20 - 1 = 19$$

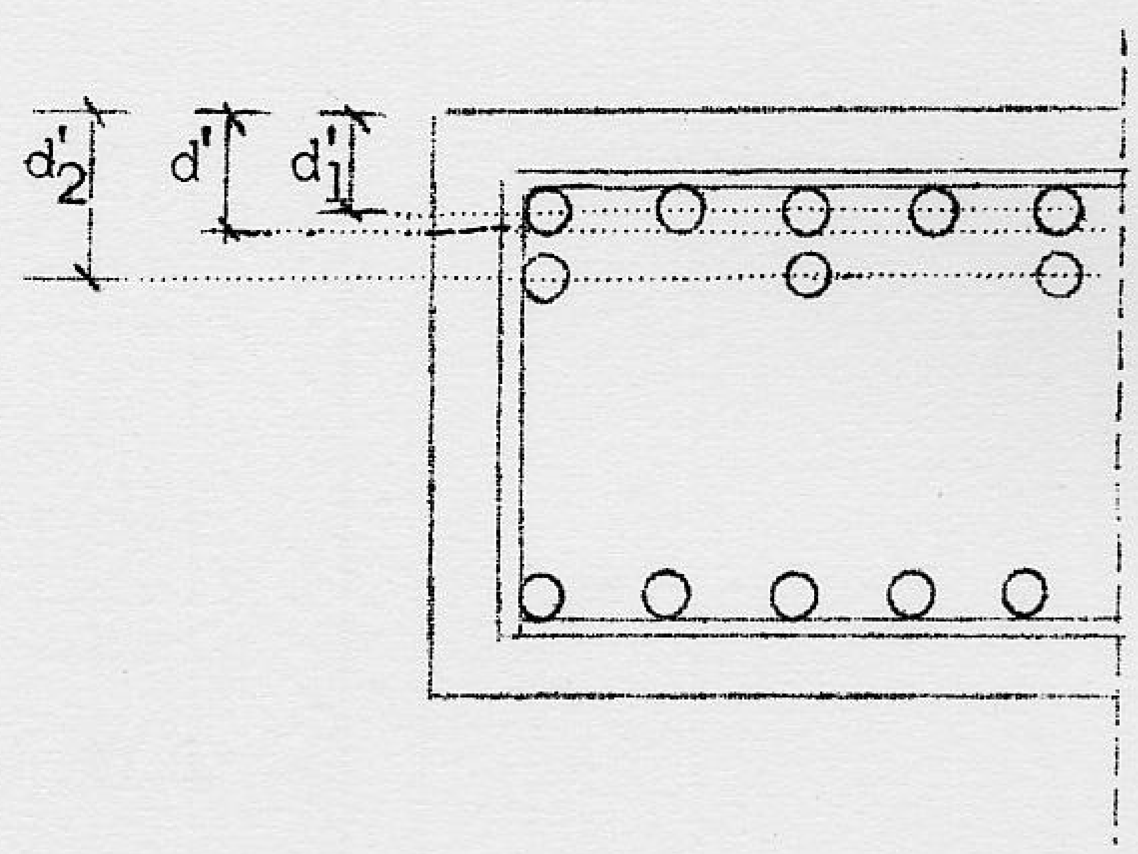


Fig. E.2 Center of gravity of compr. steel

Therefore

$$Y_1 = 12.86 \text{ cm}$$

$$I_3 = 331847.84 \text{ cm}^4 \quad \text{or} \quad 33.18 \text{ dm}^4$$

I_4)

$h = 30$ cm

$h^2 = 900$ cm^2

$d = 30.0 - 2.5 - 0.8 - \frac{2.5}{2} = 25.45$ cm say 25.5 cm

$d' = 2.5 + 0.8 + \frac{2.5}{2} = 4.55$ cm say 4.5 cm

$$p = \frac{A_s}{bd} = \frac{49.22}{100 \times 25.5} = 0.0193$$

$$p' = \frac{A'_s}{bd} = \frac{19.69}{100 \times 25.5} = 0.00772$$

$m = 10 - 1 = 9$

$m' = 20 - 1 = 19$

Therefore

$Y_1 = 15.18$ cm

$I_4 = 314947.51$ cm^4 or 31.49 dm^4

I_5)

$h = 25$ cm

$h^2 = 625$ cm^2

$d = 25.0 - 2.5 - 0.6 - \frac{1.6}{2} = 21.1$ cm

$d' = 2.5 + 0.6 + \frac{1.6}{2} = 3.9$ cm

$$p = p' = \frac{A_s}{bd} = 0.00762$$

$m = 10 - 1 = 9$

$m' = 20 - 1 = 19$

Therefore

$Y_1 = 12.02$ cm

$I_5 = 40715.06$ cm^4 or 4.07 dm^4

E.3 Carry-Over Factors

C_{AB})

As previously mentioned the carry-over factors are computed by making use of the area under the appropriate M/EI diagram. The M/EI diagram for C_{AB} is shown in fig. E.3 on page 58. The computations for C_{AB} follow.

$$A_1 = \frac{4.50}{2}\left(\frac{0.0252 + 0.0236}{E}\right) = \frac{0.110}{E}$$

$$y_1 = \frac{4.50}{3} \times \frac{0.0252 + 2 \times 0.0236}{0.0252 + 0.0236} = 2.23 \text{ dm}$$

$$A_2 = \frac{47.50}{2}\left(\frac{0.0312 + 0.0086}{E}\right) = \frac{0.945}{E}$$

$$y_2 = \frac{47.50}{3} \times \frac{0.0312 + 2 \times 0.0086}{0.0312 + 0.0086} = 19.25 \text{ dm}$$

$$A_3 = \frac{6.00}{2}\left(\frac{0.0078 + 0.0052}{E}\right) = \frac{0.039}{E}$$

$$y_3 = \frac{6.00}{3} \times \frac{0.0078 + 2 \times 0.0052}{0.0078 + 0.0052} = 2.80 \text{ dm}$$

$$A_4 = \frac{12.00}{2} \times \frac{0.0054}{E} = \frac{0.032}{E}$$

$$y_4 = \frac{12.00}{3} = 4.00 \text{ dm}$$

$$A_5 = \frac{12.00}{2} \times \frac{(0.0263 + 0.0318)\, C_{AB}}{E} = \frac{0.349\, C_{AB}}{E}$$

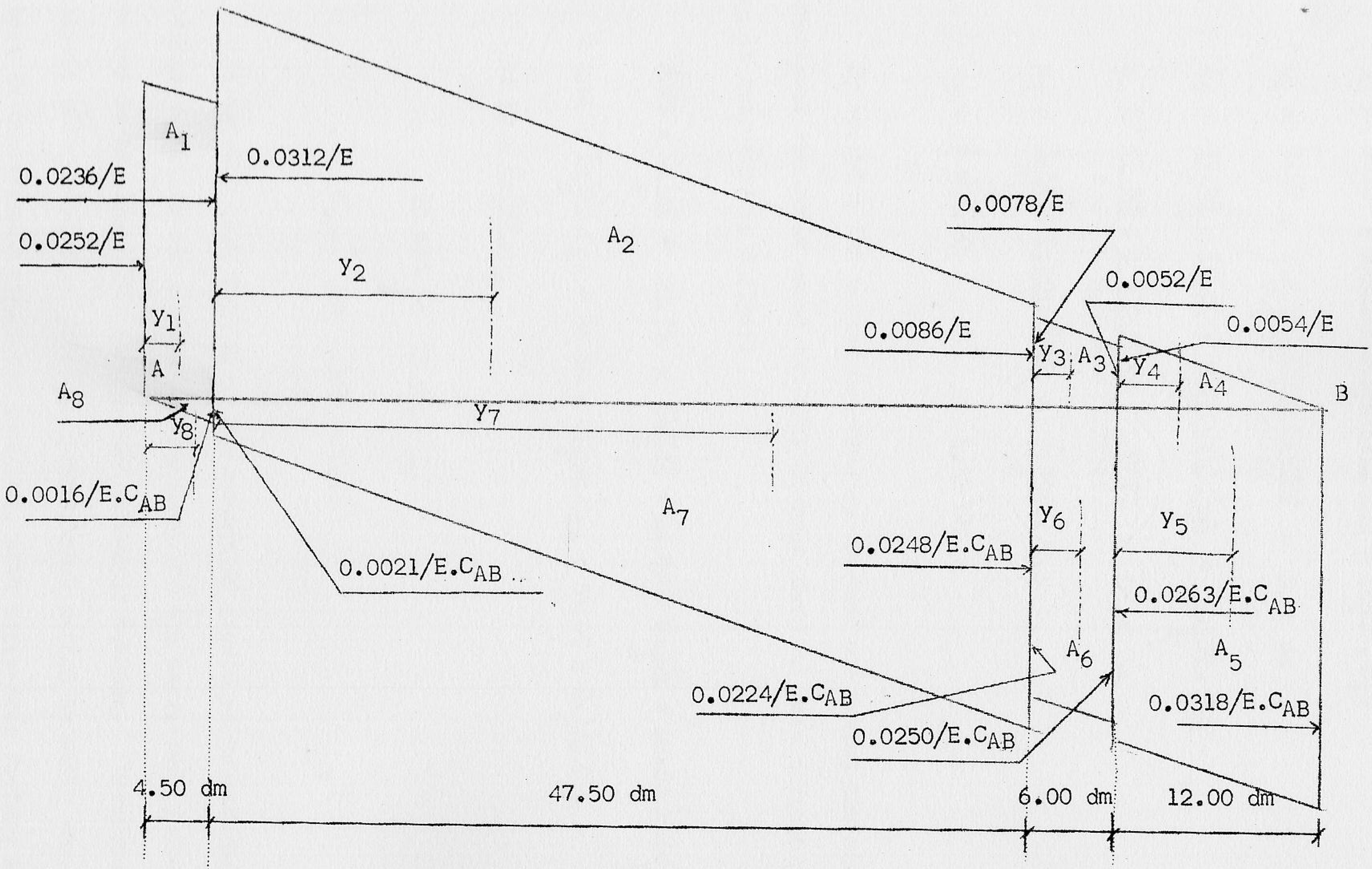


Fig. E.3 M/EI diagram for C_{AB}

$$y_5 = \frac{12.00}{3} \times \frac{2 \times 0.0318 + 0.0263}{0.0318 + 0.0263} = 6.19 \text{ dm}$$

$$A_6 = \frac{6.00}{2} \times \frac{(0.0224 + 0.0250)\, C_{AB}}{E} = \frac{0.142}{E} \cdot C_{AB}$$

$$y_6 = \frac{6.00}{3} \times \frac{2 \times 0.0250 + 0.0224}{0.0250 + 0.0224} = 3.05 \text{ dm}$$

$$A_7 = \frac{47.50}{2} \times \frac{(0.0248 + 0.0021)\, C_{AB}}{E} = \frac{0.639}{E} \cdot C_{AB}$$

$$y_7 = \frac{47.50}{3} \times \frac{2 \times 0.0248 + 0.0021}{0.0248 + 0.0021} = 30.40 \text{ dm}$$

$$A_8 = \frac{4.50}{2} \times \frac{0.0016\, C_{AB}}{E} = \frac{0.004}{E} \cdot C_{AB}$$

$$y_8 = \frac{2}{3} \times 4.50 = 3.00 \text{ dm}$$

By applying the second moment-area theorem and taking the static moments of the areas under the M/EI diagram about an axis through point A, C_{AB} is evaluated as follows:

$$2.23 \times \frac{0.110}{E} + 23.75 \times \frac{0.945}{E} + 54.80 \times \frac{0.039}{E} + 62.00 \times \frac{0.032}{E}$$

$$= 3.00 \times \frac{0.004}{E}.C_{AB} + 34.90 \times \frac{0.639}{E}.C_{AB} + 55.05 \times \frac{0.142}{E}.C_{AB} + 64.19 \times \frac{0.349}{E}.C_{AB}$$

$$26.810 = 52.532\, C_{AB}$$

$$\therefore C_{AB} = 0.510 \sim 1/2$$

It is seen that the use of the moment of ineria of transformed uncracked sections did not contribute much and the carry-over factor came out to be almost the same as for prismatic members. Therefore the analysis will be continued by the conventional method by using the I values of transformed uncracked sections, with carry-over factors equal to one half.

At this stage, the load is 79.44 % of the total dead load.

$$\therefore \quad w = 3600 \times \frac{79.44}{100} = 2859.84 \text{ kg/m} \quad \text{say} \quad 2860 \text{ kg/m}$$

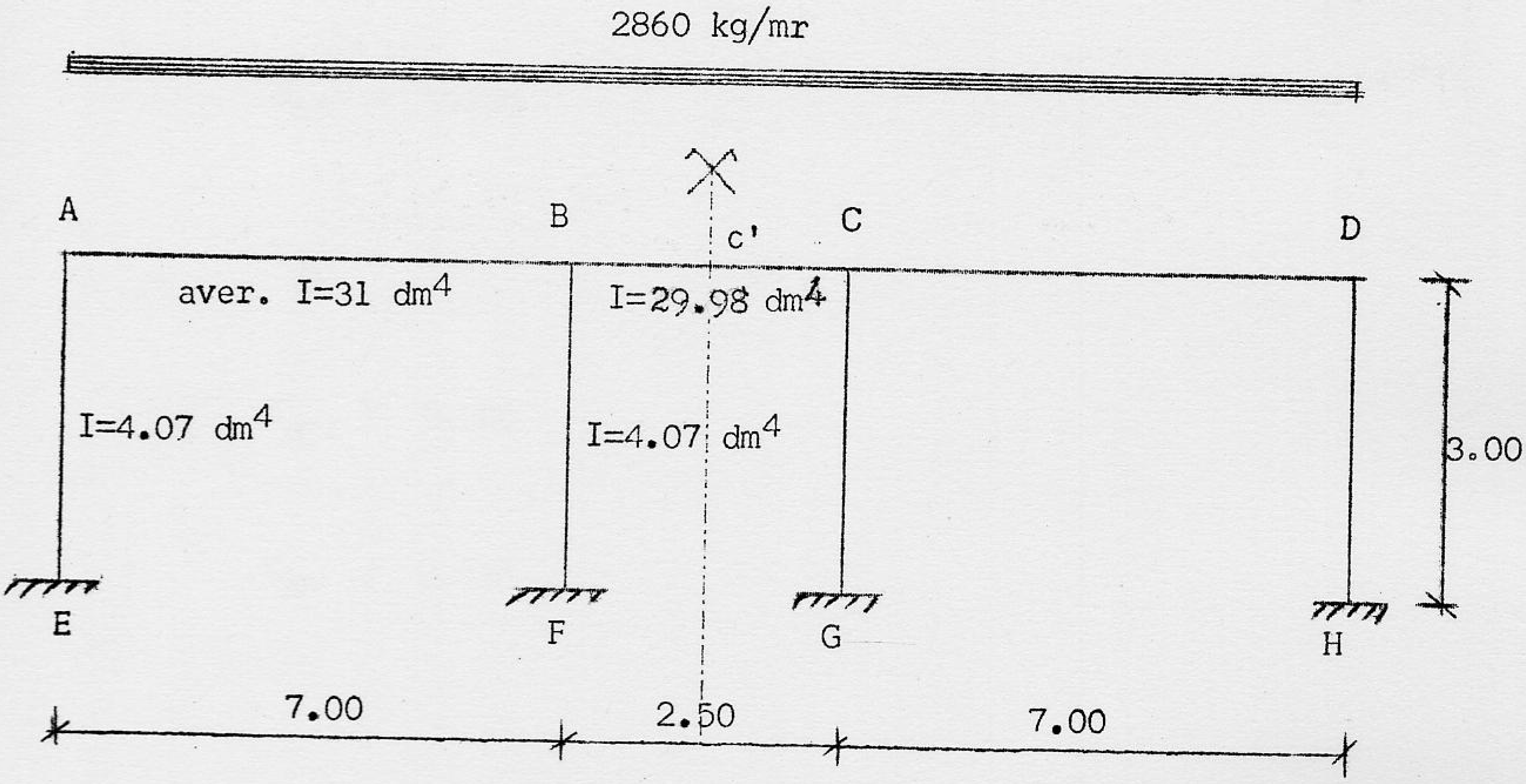


Fig. E.4 The frame and its loading
(1st stage of loading)

E.4 Stiffness Factors

$$K_{AB} = K_{BA} = \frac{31}{70} = 0.442$$

$$K'_{BC} = \frac{29.98}{2 \times 25} = 0.600$$

$$K_{AE} = K_{EA} = \frac{4.07}{30} = 0.136$$

$$K_{BF} = K_{FB} = \frac{4.07}{30} = 0.136$$

E.5 Distribution Factors

$$K_{AB} = 0.765$$

$$K_{AE} = 0.235$$

$$K_{BA} = 0.375$$

$$K_{BC'} = 0.509$$

$$K_{BF} = 0.116$$

E.6 Fixed End Moments

$$FEM_{AB} = FEM_{BA} = \frac{1}{12} \times 2860 \times \overline{7.00}^2 = 11678 \text{ kgm}$$

$$FEM_{BC} = \frac{1}{12} \times 2860 \times \overline{2.50}^2 = 1490 \text{ kgm}$$

E.7 Moment Distribution

A

Col.	Beam
0.235	0.765
	+11.68
- 2.75	- 8.93 →
	+ 2.74
- 0.65	- 2.09 →
	+ 0.20
- 0.05	- 0.15 →
- 3.45	+ 3.45

E

B

Beam	Col.	Beam
0.375	0.116	0.509
-11.68		+ 1.49
- 4.46		
← + 5.49	+ 1.70	+ 7.46
- 1.04		
← + 0.39	+ 0.12	+ 0.53
- 0.08		
+ 0.03	+ 0.01	+ 0.04
-11.35	+ 1.83	+ 9.52

F

E.8 Reactions and Positive Moments

R_A = 8881 kg M_1 = 10339 kgm

R_{B_1} = 11139 kg M_2 = -7286 kgm

R_{B_2} = 3575 kg

F

INVESTIGATION OF THE FRAME WITH THE DEAD LOAD COMPLETE

F.1 Points of Contraflexure

$R_A = 8881$ kg

$\omega = 2860$ kg/m

$X_1 = -3450$ kgm

If x is the distance between the center of the exterior support and the contraflexure points, then

$$8881\,x - \frac{2860\,x^2}{2} - 3450 = 0$$

$$x^2 - 6.21\,x + 2.41 = 0$$

$$\therefore\ x = \begin{cases} 5.794 \text{ m} & \text{say } 5.80 \text{ m} \\ 0.416 \text{ m} & \text{say } 0.45 \text{ m} \end{cases}$$

F.2 Determination of Cracked and Uncracked Sections

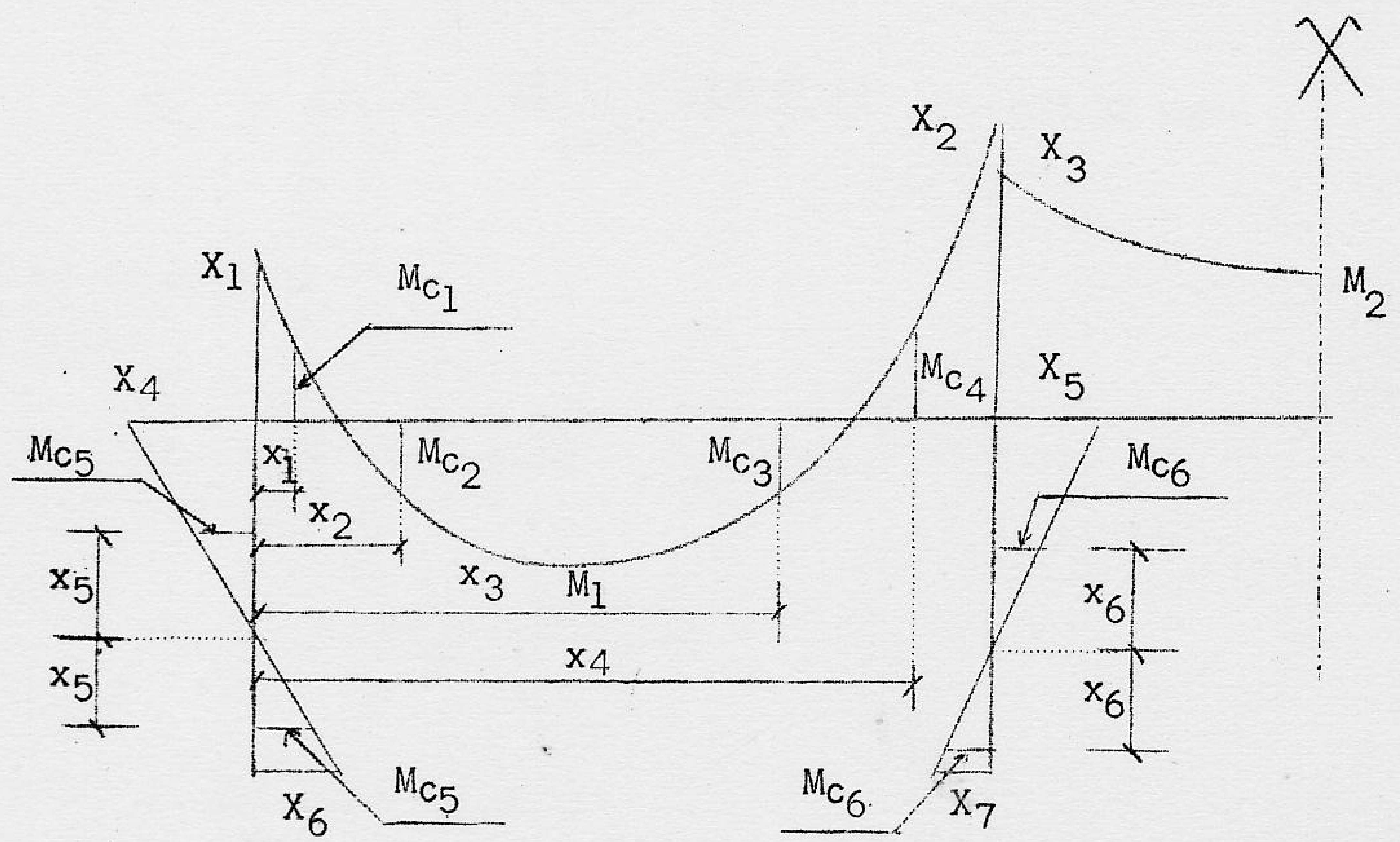


Fig. F.1 Location of cracking moments in beams and columns

The maximum tensile stress of concrete will be taken as 12 % of f'_c. Therefore

$$f_t = \frac{12}{100} \times 176 = 21.12 \text{ kg/cm}^2$$

If M_c is the moment that produces cracks, then

a) In the beam

$$M_{c1} = \frac{f_t \, I_1}{c_1} = \frac{21.12 \times 396481.23}{17.14 \times 100} = 4885.46 \text{ kgm}$$

$$M_{c_2} = \frac{f_t\, I_2}{c_2} = \frac{21.12 \times 299799.60}{14.41 \times 100} = 4394.01 \text{ kgm}$$

$$M_{c_3} = \frac{f_t\, I_3}{c_3} = \frac{21.12 \times 331847.84}{17.14 \times 100} = 4089.05 \text{ kgm}$$

$$M_{c_4} = \frac{f_t\, I_4}{c_4} = \frac{21.12 \times 314947.51}{14.82 \times 100} = 4488.32 \text{ kgm}$$

b) In columns

$$-f_t = \frac{P}{A} - \frac{M_{c_5} \times c_5}{I_5}$$

$$-21.12 = \frac{8881}{25^2} - \frac{M_{c_5} \times c_5}{I_5}$$

$$-\frac{M_{c_5} \times c_5}{I_5} = -\ 35.33$$

$$\therefore\ M_{c_5} = \frac{f_t\, I_5}{c_5} = \frac{35.33 \times 40715.06}{12.98 \times 100} = 1108.21 \text{ kgm}$$

$$-f_t = \frac{P}{A} - \frac{M_{c_6} \times c_5}{I_5}$$

$$-21.12 = \frac{11139 + 3575}{25^2} \quad \frac{M_{c_6} \times c_5}{I_5}$$

$$-\frac{M_{c_6} \times c_5}{I_5} = -\ 44.66$$

$$\therefore\ M_{c_6} = \frac{f_t\, I_5}{c_5} = \frac{44.66 \times 40715.06}{12.98 \times 100} = 1400.87 \text{ kgm}$$

Let x_1, x_2, x_3 and x_4 be the distance between the centerline of the exterior support and the sections on the beam where the moments are M_{c_1}, M_{c_2}, M_{c_3}, and M_{c_4} respectively. Then

$$R_A.x_1 - \frac{\omega x_1^2}{2} + X_1 = - 4885.46$$

$$8881\ x_1 - \frac{2860\ x_1^2}{2} - 3450 = - 4885.46$$

$$x_1^2 - 6.21\ x_1 - 1.01 = 0$$

$$\therefore\ x_1 = - 0.159\ m$$

So the beam is uncracked at the exterior support.

$$8881\ x_2 - \frac{2860\ x_2^2}{2} - 3450 = 4394.01$$

$$x_2^2 - 6.21\ x_2 + 5.48 = 0$$

$$\therefore\ x_2 = 1.06\ m \quad \text{say}\ 1.05\ m$$

$$8881\ x_3 - \frac{2860\ x_3^2}{2} - 3450 = 4089.05$$

$$x_3^2 - 6.21\ x_3 + 5.27 = 0$$

$$\therefore\ x_3 = 5.196\ m \quad \text{say}\ 5.20\ m$$

$$8881\ x_4 - \frac{2860\ x_4^2}{2} - 3450 = - 4488.32$$

$$x_4^2 - 6.21\ x_4 - 0.73 = 0$$

$$\therefore\ x_4 = 6.326\ m \quad \text{say}\ 6.30\ m$$

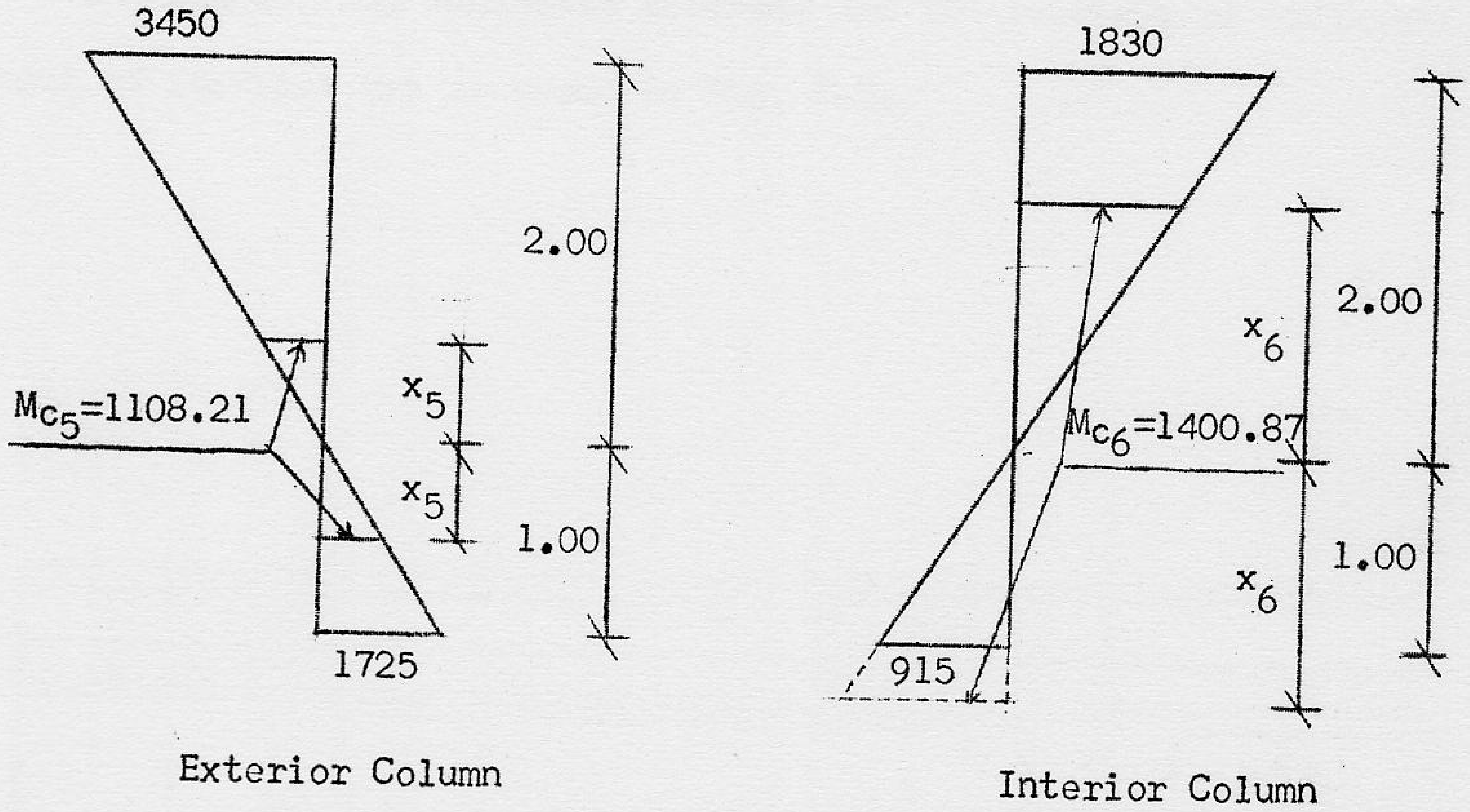


Fig. F.2 Computation for the location of cracking moments in columns

Let x_5 and x_6 be the distance between points of contraflexure and the sections on the columns where the moments are M_{c_5} and M_{c_6} respectively as shown in fig. F.2.

$$x_5 = \frac{100 \times 1108.21}{1725} = 0.64 \text{ m} \quad \text{say } 0.65 \text{ m}$$

$$x_6 = \frac{100 \times 1400.87}{915} = 1.53 \text{ m} \quad \text{say } 1.55 \text{ m}$$

The cracked and uncracked sections and the sections of different moment of inertias are as shown in fig. F.3.

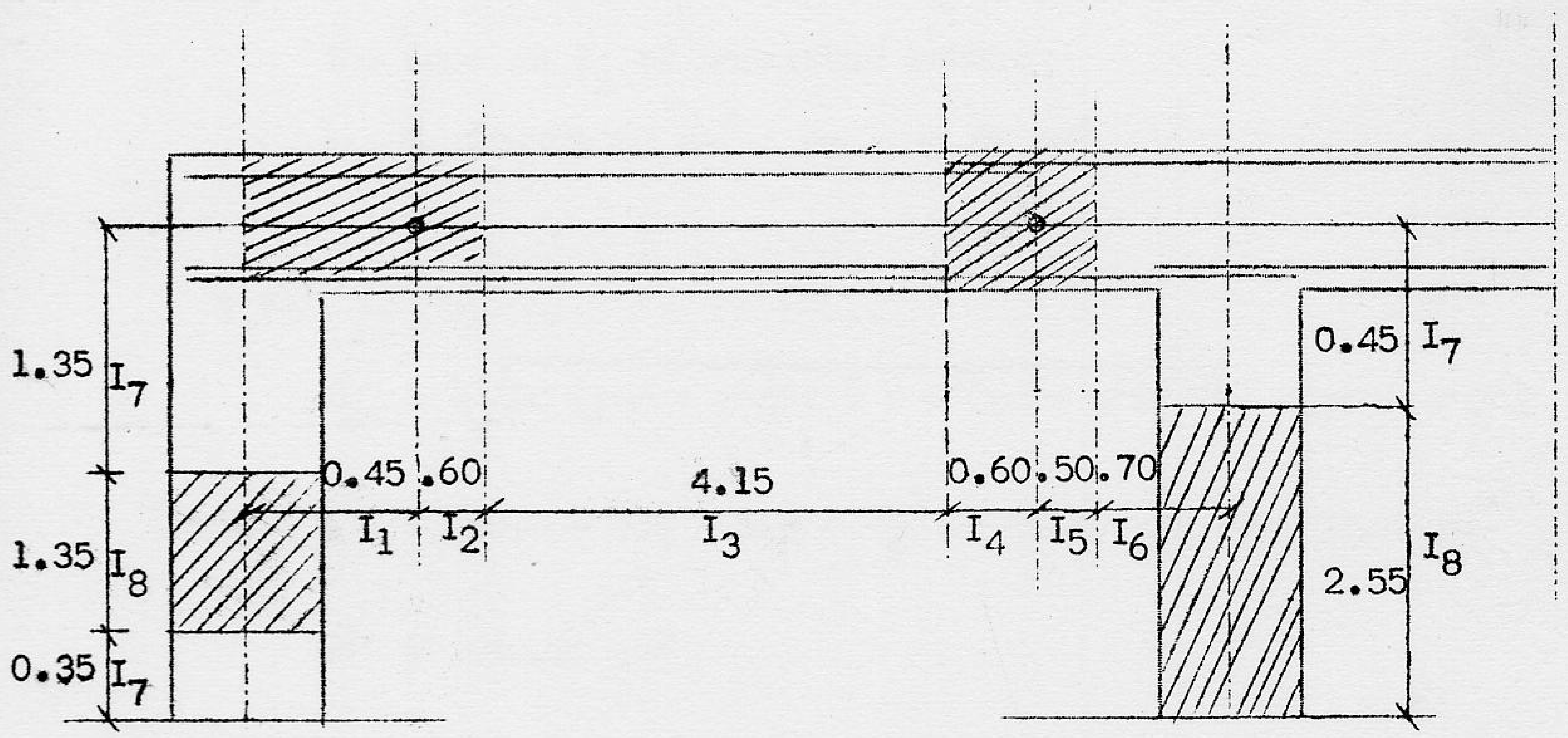


Fig. F.3 Cracked and uncracked sections of the frame (2nd stage of loading)

Notes: 1) The crosshatched sections are uncracked

2) The left limit of the interior uncracked section of the beam was 10 cm away from the cut off point of positive reinforcement. This limit is brought up to the said cut off point for reasons of simplicity.

F.3 Moments of Inertia

I_1)

The same as I_1 of uncracked section

$I_1 = 396481.23 \ cm^4$ or $39.65 \ dm^4$

I_2)

The same as I_2 of uncracked section

$I_2 = 299799.60 \ cm^4$ or $29.98 \ dm^4$

I_3)

$$Y_2 = \sqrt{2d\,(pnd + p'm'd') + d^2(pn + p'm')^2} - d(pn + p'm')$$

$$d = 30.0 - 2.5 - 0.8 - \frac{2.5}{2} = 25.45 \ cm \quad \text{say} \quad 25.5 \ cm$$

$$d^2 = 650.25 \ cm^2$$

$$d' = 2.5 + 0.8 + \frac{1.6}{2} = 4.1 \ cm$$

$$p = \frac{A_s}{bd} = \frac{49.22}{100 \times 25.5} = 0.0193$$

$$p' = \frac{A'_s}{bd} = \frac{12.06}{100 \times 25.5} = 0.00473$$

$$n = 10$$

$$m' = 20 - 1 = 19$$

Therefore

$$Y_2 = 10.72$$

$$I_3 = 158626.39 \ cm^4 \quad \text{or} \quad 15.86 \ dm^4$$

I_4)

The same as I_3 of uncracked section

$I_4 = 331847.84 \text{ cm}^4$ or 33.18 dm^4

I_5)

The same as I_4 of uncracked section

$I_5 = 314947.51 \text{ cm}^4$ or 31.49 dm^4

I_6)

$d = 30.0 - 2.5 - 0.8 - \frac{2.5}{2} = 25.45 \text{ cm}$ say 25.5 cm

$d^2 = 650.25 \text{ cm}^2$

$d' = 2.5 + 0.8 + \frac{2.5}{2} = 4.55 \text{ cm}$ say 4.5 cm

$p = \frac{A_s}{bd} = \frac{49.22}{100 \times 25.5} = 0.0193$

$p' = \frac{A'_s}{bd} = \frac{19.69}{100 \times 25.5} = 0.00772$

$n = 10$

$m' = 20 - 1 = 19$

Therefore

$Y_2 = 10.30 \text{ cm}$

$I_6 = 162727.18 \text{ cm}^4$ or 16.27 dm^4

I_7)

$d = 25.0 - 2.5 - 0.6 - \frac{1.6}{2} = 21.1 \text{ cm}$

$d^2 = 445.21 \text{ cm}^2$

$d' = 2.5 + 0.6 + \frac{1.6}{2} = 3.9 \text{ cm}$

$p = p' = \frac{A_s}{b_d} = \frac{4.02}{25 \times 21.1} = 0.00762$

$n = 10$

$m' = 20 - 1 = 19$

Therefore

$Y_2 = 5.98 \text{ cm}$

and $I_7 = 11302.81 \text{ cm}^4$ or 1.13 dm^4

I_8)

The same as I_5 of uncracked section

$I_8 = 40715.06 \text{ cm}^4$ or 4.07 dm^4

F.4 Carry-Over Factors

Applying the same principle as in art. E.3 the carry-over factors are computed and listed below

$C_{AB} = 0.543$ $C_{AE} = 0.538$

$C_{BA} = 0.581$ $C_{BF} = 0.573$

$C_{BC} = 0.500$*

* The member BC is cracked all through its length. It has no contraflexure points and its reinforcements are extended into the supports. As the moment of inertia of its transformed cracked sections are constant eveywhere, it can be treated as a prismatic member.

F.5 Stiffness Factors

With the advantage of the areas under the M/EI diagram already found while computing the carry-over factors, the stiffness factors are obtained.

$K_{AB} = 1.248\,E$ $\qquad$ $K_{AE} = 0.158\,E$

$K_{BA} = 1.218\,E$ $\qquad$ $K_{BF} = 0.243\,E$

$K_{BC'} = 1.269\,E^{*}$

F.6 Distribution Factors

$K_{AB} = 0.888$ $\qquad$ $K_{BA} = 0.446$

$K_{AE} = 0.112$ $\qquad$ $K_{BC'} = 0.465$

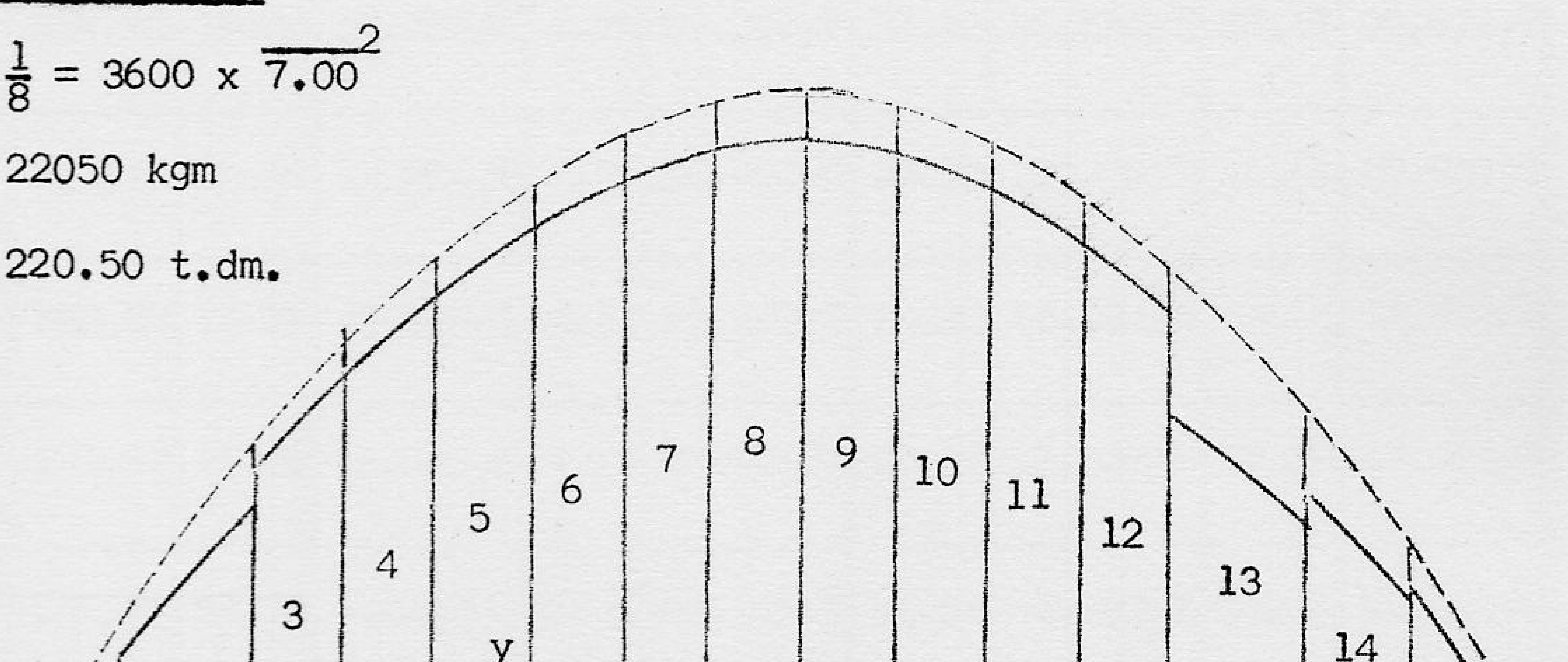
$K_{BF} = 0.089$

F.7 Fixed End Moments

$M = \frac{1}{8} = 3600 \times \overline{7.00}^{2}$

$= 22050$ kgm

or 220.50 t.dm.

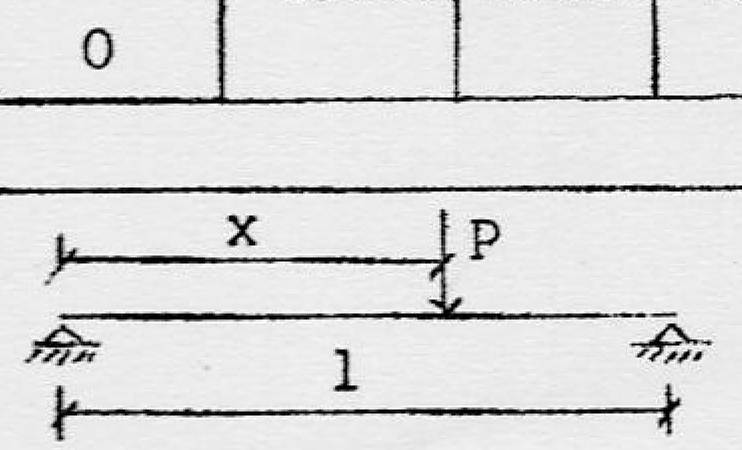


$^{*}K_{BC'} = \frac{1}{2} \times 4E\frac{I_3}{L} = \frac{1}{2} \times 4 \times E \times \frac{15.86}{25} = 1.269\,E$

Section	y t.dm	Aver.y ⇒ Y t.dm	I dm^4	Y/EI t/dm^3	b dm	$A = \frac{Y \times b}{EI}$ t/dm^2	l - x dm	$R_a = A \times \frac{l-x}{l}$ * t/dm^2	x dm	$R_b = A \times \frac{x}{l}$ t/dm^2
1	0 53.06	26.53	39.65	0.669/E	4.50	3.011/E	67.00	2.882/E	3.00	0.129/E
2	53.06 112.46	82.76	29.98	2.761/E	6.00	16.566/E	62.34	14.752/E	7.66	1.814/E
3	112.46 145.96	129.21	15.86	8.147/E	4.15	33.810/E	57.34	27.694/E	12.66	6.116/E
4	145.96 173.26	159.61	15.86	10.064/E	4.15	41.766/E	53.21	31.748/E	16.79	10.018/E
5	173.26 194.36	183.81	15.86	11.590/E	4.15	48.099/E	49.08	33.724/E	20.92	14.375/E
6	194.36 209.27	201.82	15.86	12.725/E	4.15	52.809/E	44.95	33.911/E	25.05	18.898/E
7	209.27 217.97	213.62	15.86	13.469/E	4.15	55.896/E	40.81	32.587/E	29.19	23.309/E
8	217.97 220.47	219.22	15.86	13.822/E	4.15	57.361/E	36.67	30.049/E	33.33	27.312/E
9	220.47 216.77	218.62	15.86	13.784/E	4.15	57.204/E	32.53	26.583/E	37.47	30.621/E
10	216.77 206.88	211.83	15.86	13.356/E	4.15	55.427/E	28.39	22.476/E	41.61	32.951/E
11	206.88 190.78	198.83	15.86	12.537/E	4.15	52.029/E	24.25	18.024/E	45.75	34.005/E
12	190.78 168.48	179.63	15.86	11.326/E	4.15	47.003/E	20.12	13.509/E	49.88	33.494/E
13	168.48 125.28	146.88	33.18	4.427/E	6.00	22.135/E	15.14	4.787/E	54.86	17.348/E
14	125.28 79.38	102.33	31.49	3.250/E	5.00	16.250/E	9.68	2.247/E	60.32	14.003/E
15	79.38 0	39.69	16.27	2.439/E	7.00	17.073/E	4.67	1.139/E	65.33	15.934/E

$\Sigma R_a = 296.112/E$ $\Sigma R_b = 280.327/E$

*



$$\therefore \quad \theta_a = \frac{296.112}{E}$$

$$\theta_b = \frac{280.327}{E}$$

$$FEM_{AB} = K_{AB}\,\theta_a - C_{BA} \cdot K_{BA}\,\theta_b$$

$$= 1.248\,E \times \frac{296.112}{E} - 0.581 \times 1.218\,E \times \frac{280.327}{E}$$

$$= 171.172 \text{ t.dm} \quad \text{or} \quad 17.12 \text{ tm}$$

$$FEM_{BA} = -K_{BA} \cdot \theta_b + C_{AB} \cdot K_{AB}\,\theta_a$$

$$= -1.218\,E \times \frac{280.327}{E} + 0.543 \times 1.248\,E \times \frac{296.112}{E}$$

$$= -140.774 \text{ t.dm} \quad \text{or} \quad 14.08 \text{ tm}$$

F.8 Moment Distribution

C_{AE} 0.538 ↓ C_{AB} 0.543 → C_{BA} 0.581 ← C_{BF} 0.573 ↓

A: Col.	A: Beam			B: Beam	B: Col.	B: Beam
0.112	0.888			0.446	0.089	0.465
	+17.12			-14.08		+ 1.88
- 1.92	-15.20	→		- 8.25		
	+ 5.30		←	+ 9.12	+ 1.82	+ 9.51
- 0.59	- 4.71	→		- 2.56		
	+ 0.66		←	+ 1.14	+ 0.23	+ 1.19
- 0.07	- 0.59	→		- 0.32		
	+ 0.08		←	+ 0.14	+ 0.03	+ 0.15
- 0.01	- 0.07			-14.81	+ 2.08	+12.73
- 2.59	+ 2.59					
	- 1.39 (E)					+ 1.19 (F)

F.9 Reactions and Positive Moments

R_A = 10854 kg

R_{B_1} = 14346 kg

R_{B_2} = 4500 kg

M_1 = 13772 kgm

M_2 = -9917 kgm

G

INVESTIGATION OF THE FRAME WITH THE TOTAL LOAD ACTING

G.1 Points of Contraflexure

$R_A = 10854$ kg

$\omega = 3600$ kg/m

$X_1 = -2590$ kgm

$$10854\,x - \frac{3600\,x^2}{2} - 2590 = 0$$

$$x^2 - 6.03\,x + 1.44 = 0$$

$$\therefore\ x = \begin{cases} 5.781 \text{ m} & \text{say } 5.80 \text{ m} \\ 0.249 \text{ m} & \text{say } 0.25 \text{ m} \end{cases}$$

G.2 Determination of Cracked and Uncracked Sections

a) In the beam

$$M_{c_1} = \frac{f_t\, I_1}{c_1} = \frac{21.12 \times 396481.23}{17.14 \times 100} = 4885.46 \text{ kgm}$$

$$M_{c_2} = \frac{f_t\, I_2}{c_2} = \frac{21.12 \times 299799.60}{14.41 \times 100} = 4394.01 \text{ kgm}$$

$$M_{c_3} = \frac{f_t\, I_4}{c_4} = \frac{21.12 \times 331847.84}{17.14 \times 100} = 4089.05 \text{ kgm}$$

$$M_{c_4} = \frac{f_t\, I_5}{c_5} = \frac{21.12 \times 314947.51}{14.82 \times 100} = 4488.32 \text{ kgm}$$

b) In columns

$$-f_t = \frac{P}{A} - \frac{M_{c_5} \times c_8}{I_8}$$

$$-21.12 = \frac{10854}{25^2} \quad \frac{M_{c_5} \times c_8}{I_8}$$

$$-\frac{M_{c_5} \times c_8}{I_8} = -38.49$$

$$\therefore M_{c_5} = \frac{38.49 \times 40715.06}{12.98 \times 100} = 1207.34 \text{ kgm}$$

$$-f_t = \frac{P}{A} \quad \frac{M_{c_6} \times c_8}{I_8}$$

$$-21.12 = \frac{14346 + 4500}{25^2} - \frac{M_{c_6} \times c_8}{I_8}$$

$$-\frac{M_{c_6} \times c_8}{I_8} = -51.27$$

$$\therefore M_{c_6} = \frac{51.27 \times 40715.06}{12.98 \times 100} = 1608.21 \text{ kgm}$$

Let x_1, x_2, x_3 and x_4 be the distance between the centerline of the exterior support and the sections on the beam where the moments are M_{c_1}, M_{c_2}, M_{c_3} and M_{c_4} respectively. Then

$$R_A x_1 - \frac{\omega x_1^2}{2} + X_1 = -4885.46$$

$$10854 x_1 - \frac{3600 . x_1^2}{2} - 2590 = -4885.46$$

$$x_1^2 - 6.03 x_1 - 1.27 = 0$$

$$\therefore x_1 = -0.204 \text{ m}$$

So the beam is still uncracked at the exterior support.

$$10854 x_2 - \frac{3600 x_2^2}{2} - 2590 = 4394.01$$

$$x_2^2 - 6.03 x_2 + 3.88 = 0$$

$$\therefore x_2 = 0.732 \quad \text{say} \quad 0.75 \text{ m}$$

$$10854\ x_3 \quad - \quad \frac{3600\ x_3^2}{2} \quad - \quad 2590 \quad = \quad 4089.05$$

$$x_3^2 \quad - \quad 6.03\ x_3 \quad + \quad 3.71 \quad = \quad 0$$

$$\therefore \quad x_3 \quad = \quad 5.334\ \text{m} \quad \text{say} \quad 5.30\ \text{m}$$

Assume x_3 = 5.20 m to have it on the cut off point of positive reinforcement.

$$10854\ x_4 \quad - \quad \frac{3600\ x_4}{2} \quad - \quad 2590 \quad = \quad -4488.32$$

$$x_4^2 \quad - \quad 6.03\ x_4 \quad - \quad 1.05 \quad = \quad 0$$

$$\therefore \quad x_4 \quad = \quad 6.199\ \text{m} \quad \text{say} \quad 6.20\ \text{m}$$

The moments on the columns are less than the previous stages. Obviously the cracked and uncracked sections of the columns will be the same as the ones of previous stage of loading. The cracked and uncracked sections of the frame and the sections of different moments of inertia are as shown in fig. G.1.

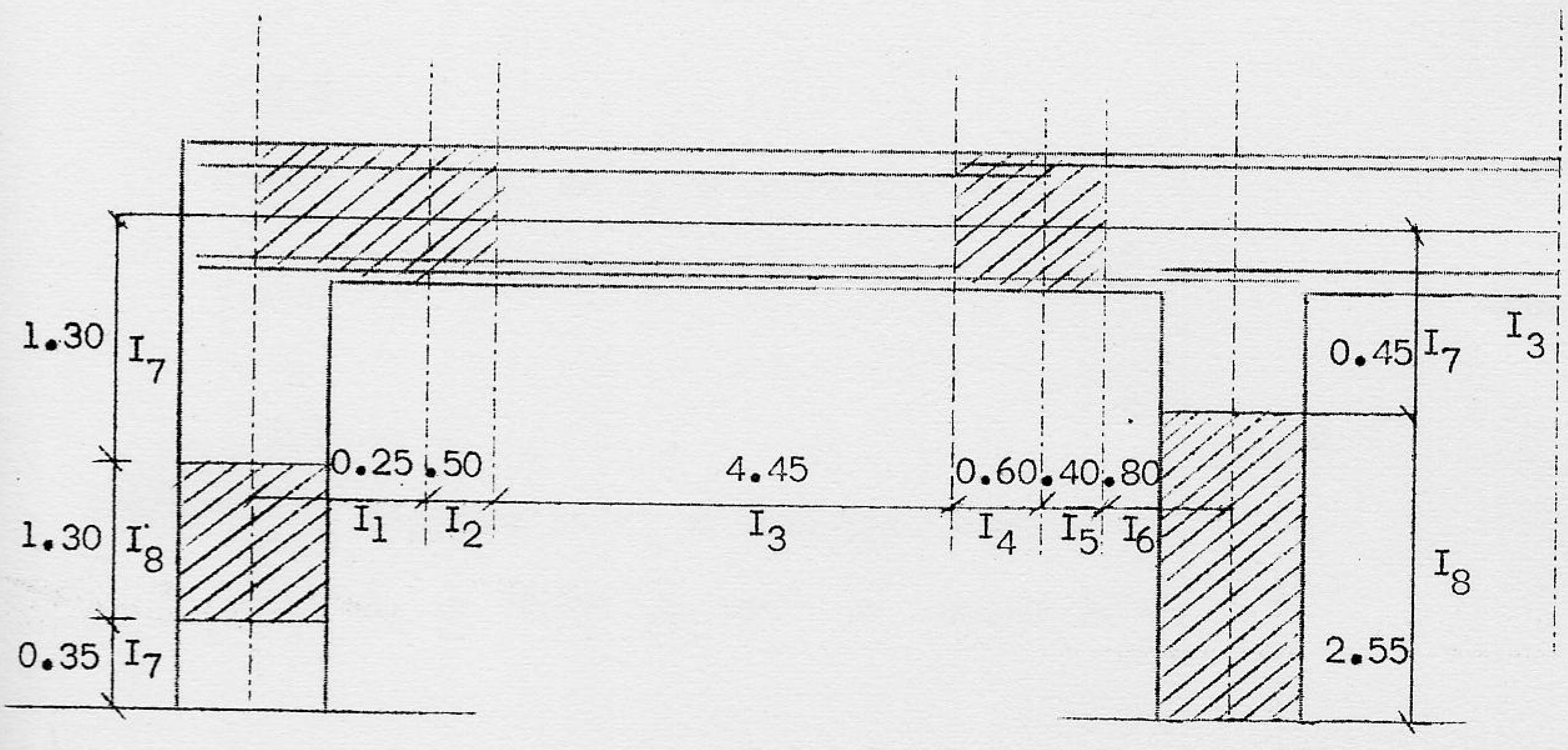


Fig. G.1 Cracked and uncracked sections of the frame (3rd stage of loading)

G.3 Moments of Inertia

I_1)

The same as I_1 of uncracked section

$I_1 = 396481.23 \text{ cm}^4$ or 39.65 dm^4

I_2)

The same as I_2 of uncracked section

$I_2 = 299799.60\ cm^4$ or $29.98\ dm^4$

I_3)

The same as I_3 of cracked section

$I_3 = 158626.39\ cm^4$ or $15.86\ dm^4$

I_4)

The same as I_3 of uncracked section

$I_4 = 331847.84\ cm^4$ or $33.18\ dm^4$

I_5)

The same as I_4 of uncracked section

$I_5 = 314947.51\ cm^4$ or $31.49\ dm^4$

I_6)

The same as I_6 of cracked section

$I_6 = 162727.18\ cm^4$ or $16.27\ dm^4$

I_7)

The same as I_7 of cracked section

$I_7 = 11302.81\ cm^4$ or $1.13\ dm^4$

I_8)

The same as I_5 of uncracked section

$I_8 = 40715.06\ cm^4$ or $4.07\ dm^4$

G.4 Carry-Over Factors

Following to the same procedure as in art. E.3 the carry-over factors are computed.

$C_{AB} = 0.541$ $C_{AE} = 0.538$

$C_{BA} = 0.546$ $C_{BF} = 0.573$

$C_{BC} = 0.500$

G.5 Stiffness Factors

The stiffness factors are computed by making use of the areas under the M/EI diagram already found while computing the carry-over factors.

$K_{AB} = 1.142\,E$ $K_{AE} = 0.158\,E$

$K_{BA} = 1.143\,E$ $K_{BF} = 0.243\,E$

$K_{BC'} = 1.269\,E$

G.6 Distribution Factors

$K_{AB} = 0.878$ $K_{BA} = 0.431$

$K_{AE} = 0.122$ $K_{BC'} = 0.478$

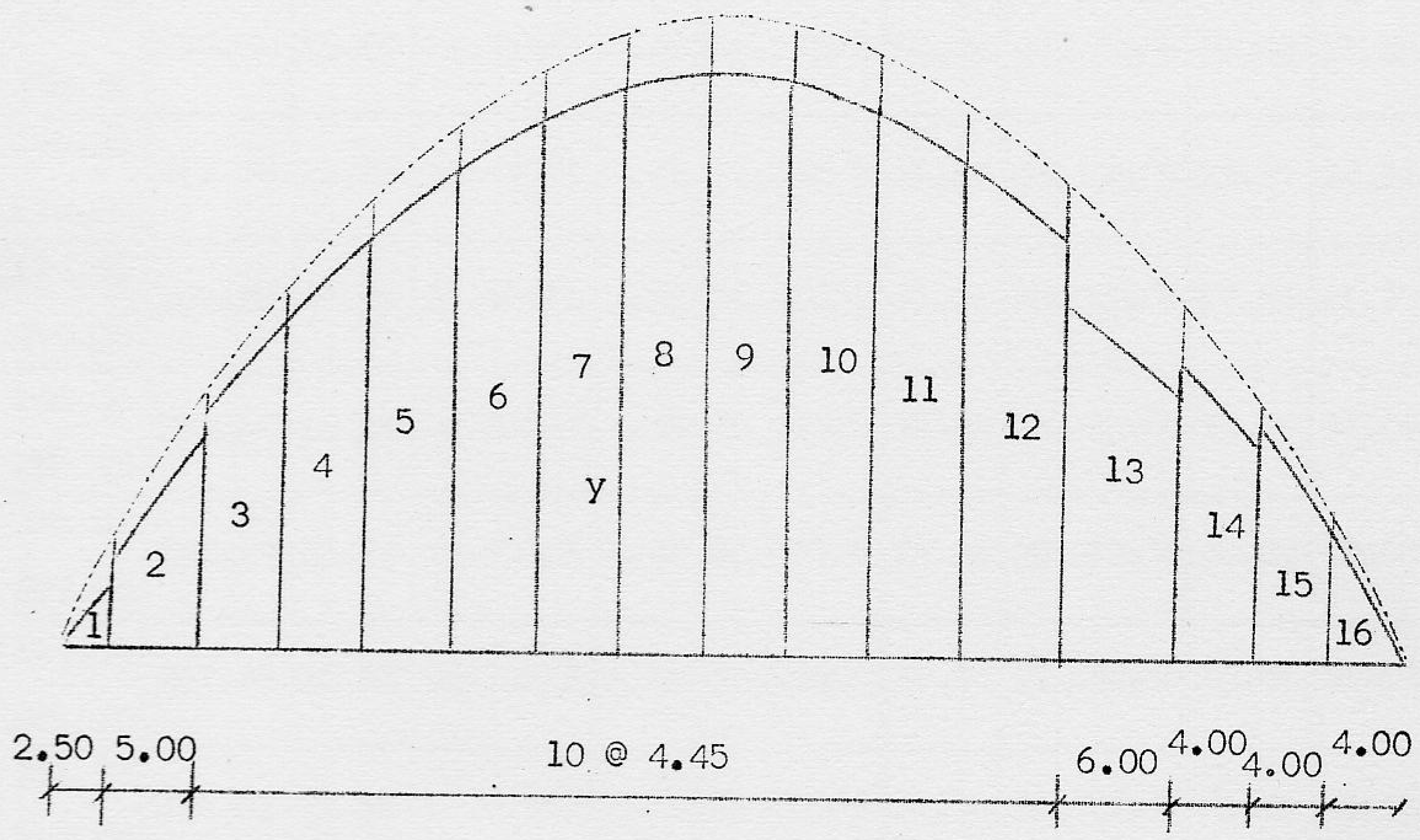
$K_{BF} = 0.091$

G.7 Fixed End Moments

$$M = \frac{1}{8} \times 4850 \times \overline{7.00}^2$$

$$= 29706.25 \text{ kgm}$$

$$= 297.06 \text{ t.dm}$$



Section	y t.dm	Aver.y =Y t.dm	I dm^4	Y/EI t/dm^3	b dm	A=Yxb/EI t/dm^2	l -x dm	$R_a=Ax\frac{l-x}{l}$ t/dm^2	x dm	$R_b=Ax\frac{x}{l}$ t/dm^2
1	0 40.92	20.46	39.65	0.516/E	4.50	2.322/E	68.33	2.267/E	1.67	0.055/E
2	40.92 113.67	77.30	29.98	2.578/E	3.00	7.734/E	64.60	7.137/E	5.40	0.597/E
3	113.67 168.22	140.95	15.86	8.887/E	4.45	39.547/E	60.13	33.971/E	9.87	5.576/E
4	168.22 213.17	190.70	15.86	12.024/E	4.45	53.507/E	55.74	42.607/E	14.26	10.900/E
5	213.17 248.51	230.84	15.86	14.555/E	4.45	64.770/E	51.31	47.476/E	18.69	17.294/E
6	248.51 274.25	261.38	15.86	16.480/E	4.45	73.336/E	46.89	49.125/E	23.11	24.211/E
7	274.25 290.38	282.32	15.86	17.801/E	4.45	79.214/E	42.45	48.038/E	27.55	31.176/E
8	290.38 296.91	293.64	15.86	18.515/E	4.45	82.392/E	38.02	44.751/E	31.98	37.641/E
9	296.91 293.83	295.37	15.86	18.624/E	4.45	82.877/E	33.68	39.876/E	36.32	43.001/E
10	293.83 281.15	287.49	15.86	18.127/E	4.45	80.665/E	29.14	22.053/E	40.86	58.612/E
11	281.15 258.87	270.01	15.86	17.025/E	4.45	75.761/E	24.70	26.733/E	45.30	49.028/E
12	258.87 226.98	242.93	15.86	15.317/E	4.45	68.161/E	20.27	19.743/E	49.73	48.418/E
13	226.98 168.78	197.88	33.18	5.964/E	6.00	35.784/E	15.15	7.745/E	54.85	28.039/E
14	168.78 120.28	144.53	31.49	4.590/E	4.00	18.360/E	10.11	2.652/E	59.89	15.708/E
15	120.28 64.02	92.15	16.27	5.664/E	4.00	22.656/E	6.21	2.010/E	63.79	20.646/E
16	64.02 0	32.01	16.27	1.967/E	4.00	7.868/E	2.67	0.301/E	67.33	7.567/E

$\sum R_a = 396.485/E$ $\sum R_b = 398.469/E$

$$\theta_a = \frac{396.485}{E}$$

$$\theta_b = \frac{398.469}{E}$$

$$FEM_{AB} = K_{AB}\,\theta_a - C_{BA} \cdot K_{BA} \cdot \theta_b$$

$$= 1.142\,E \times \frac{396.485}{E} - 0.546 \times 1.143\,E \times \frac{398.469}{E}$$

$$= 204.110 \text{ t.dm} \quad \text{or} \quad 20.41 \text{ tm}$$

$$FEM_{BA} = -K_{BA}\,\theta_b + C_{AB} \cdot K_{AB} \cdot \theta_a$$

$$= -1.143\,E \times \frac{398.469}{E} + 0.541 \times 1.142\,E \times \frac{396.485}{E}$$

$$= -210.493 \text{ t.dm} \quad \text{or} \quad 21.05 \text{ tm}$$

G.8 Moment Distribution

C_{AE} 0.538 ↓ C_{AB} 0.541 → C_{BA} 0.546 ← C_{BF} 0.573 ↓

Joint A:

Col.	Beam	
0.122	0.878	
	+20.41	
- 2.49	-17.92	→
	+ 6.79	
- 0.83	- 5.96	→
	+ 0.76	
- 0.09	- 0.67	→
	+ 0.09	
- 0.01	- 0.08	
- 3.42	+ 3.42	

Joint B:

	Beam	Col.	Beam
	0.431	0.091	0.478
	-21.05		+ 1.88
	- 9.69		
←	+12.44	+ 2.62	+13.80
	- 3.22		
←	+ 1.39	+ 0.29	+ 1.54
	- 0.36		
←	+ 0.16	+ 0.03	+ 0.17
	-20.33	+ 2.94	+17.39

Column base E: - 1.84

Column base F: + 1.68

E F

G.9 Reactions and Positive Moments

R_A = 14559 kg	M_1 = 18432 kgm
R_{B_1} = 19391 kg	M_2 = -15577 kgm
R_{B_2} = 4500 kg	

H

THE ACTUAL FACTOR OF SAFETY

AGAINST PRIMARY FAILURE

Value of X_1 at the bottom of the beam

$$x = \frac{3420}{3420 + 1840} \times 3.00$$

$$= 1.95 \text{ m}$$

$$X_1' = \frac{1.80}{1.95} \times 3420$$

$$= 3157 \text{ kgm}$$

$$P_u = 14559 \text{ kg}$$

$$M_u = 3157 \text{ kgm}$$

$$e = \frac{3157 \times 100}{14559} = 21.68 \text{ cm}$$

The value of "P_y" which would cause primary failure is found by trial and error; by assuming the strain in the concrete, computing c and C_c, and then checking the stability of the section. The final choice is shown in fig. H.2.

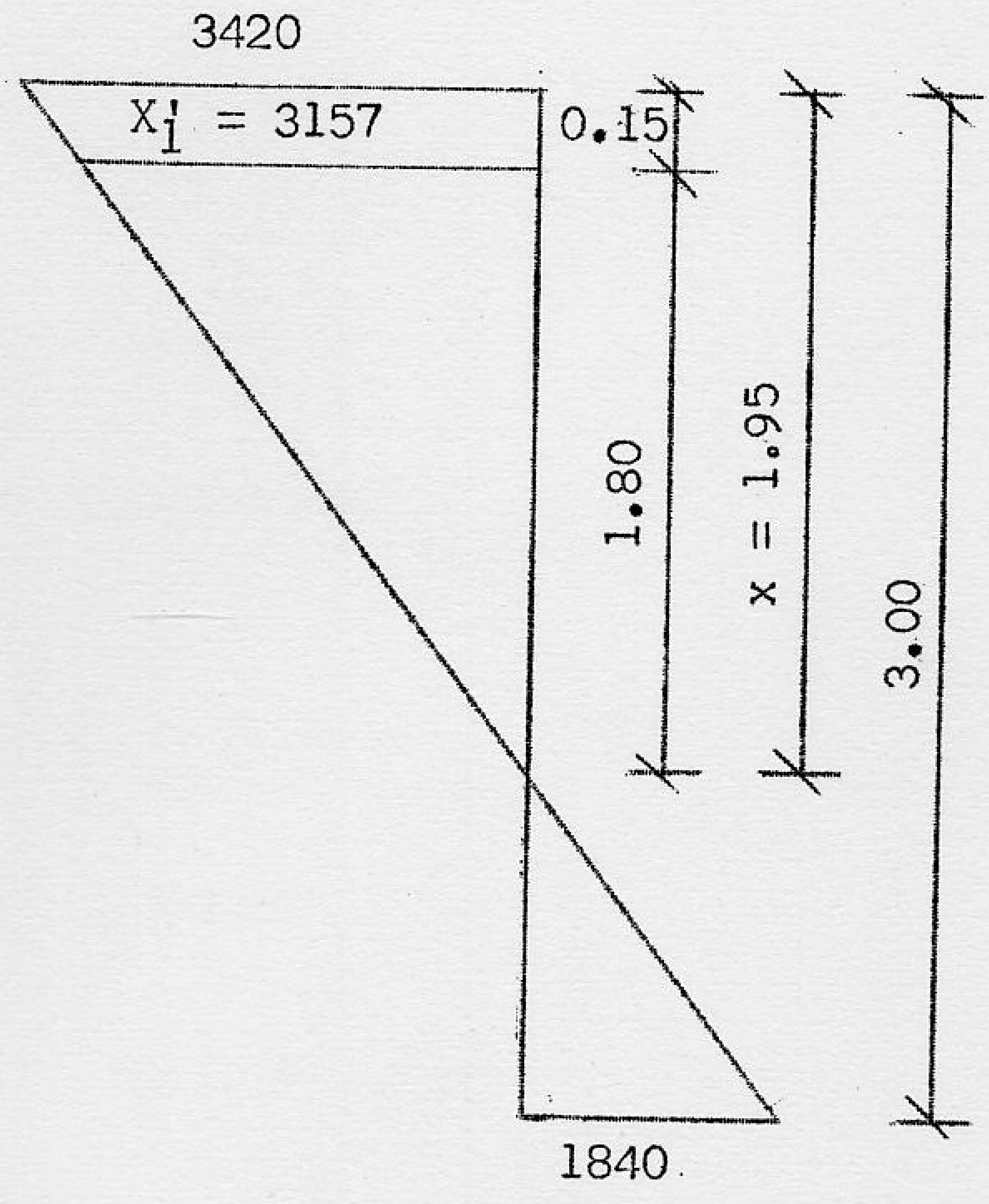


Fig. H.1 Value of the column moment at the bottom of the beam

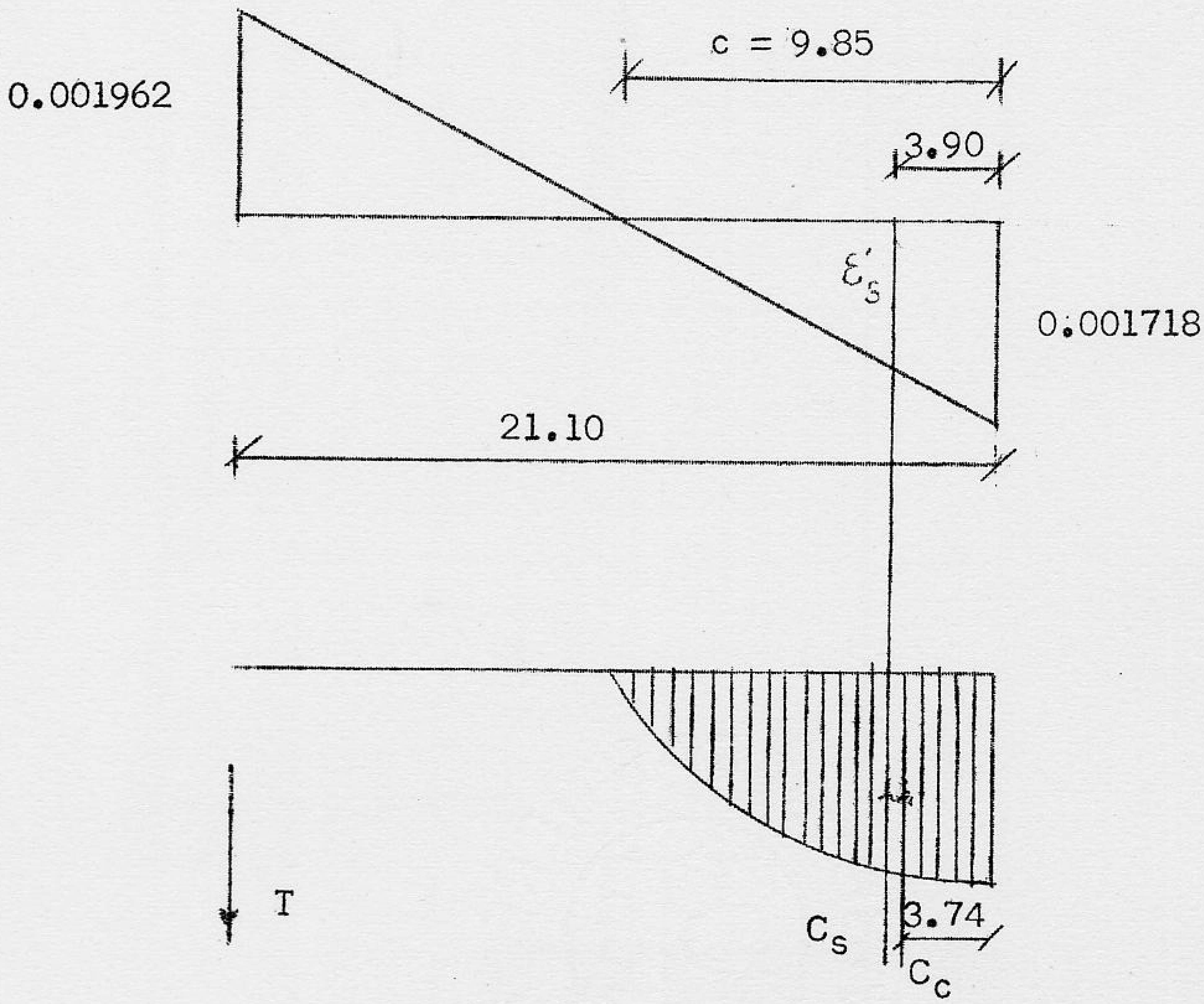


Fig. H.2 Compressive stress distribution

$$c = \frac{0.001718}{0.001962 + 0.001718} \times 21.10 = 9.85 \text{ cm}$$

$$\varepsilon_s' = \frac{9.85 - 3.90}{9.85} \times 0.001718 = 0.001038$$

$$f_s' = 0.001038 \times 2039 \times 10^3 = 2116.48 \text{ kg/cm}^2$$

$$f_s'' = 2116.48 - 151.00 = 1965.48 \text{ kg/cm}^2$$

$$\therefore C_s = 1965.48 \times 4.02 = 7901.23 \text{ kg}$$

$$T = 4000.00 \times 4.02 = 16080.00 \text{ kg}$$

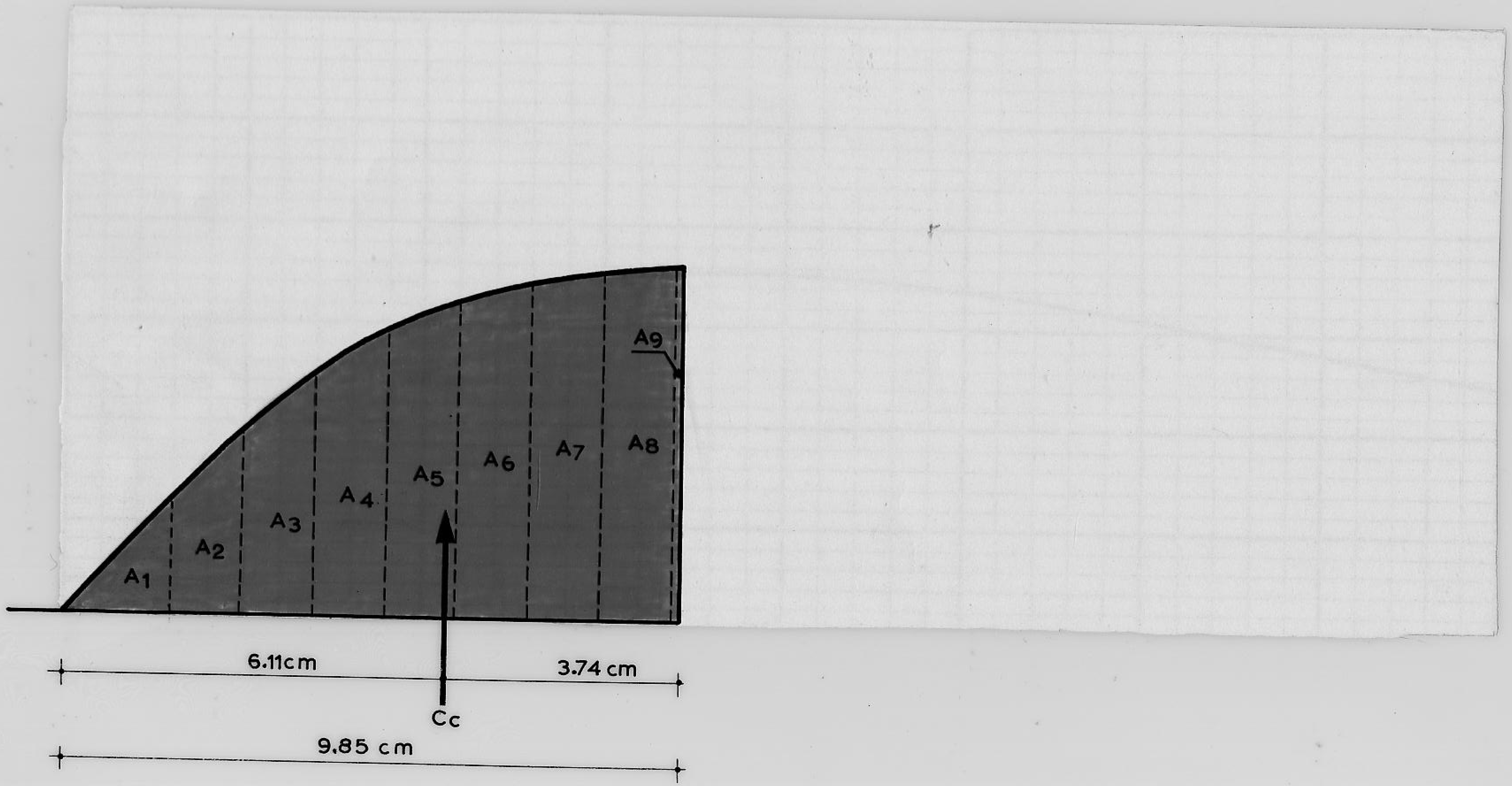
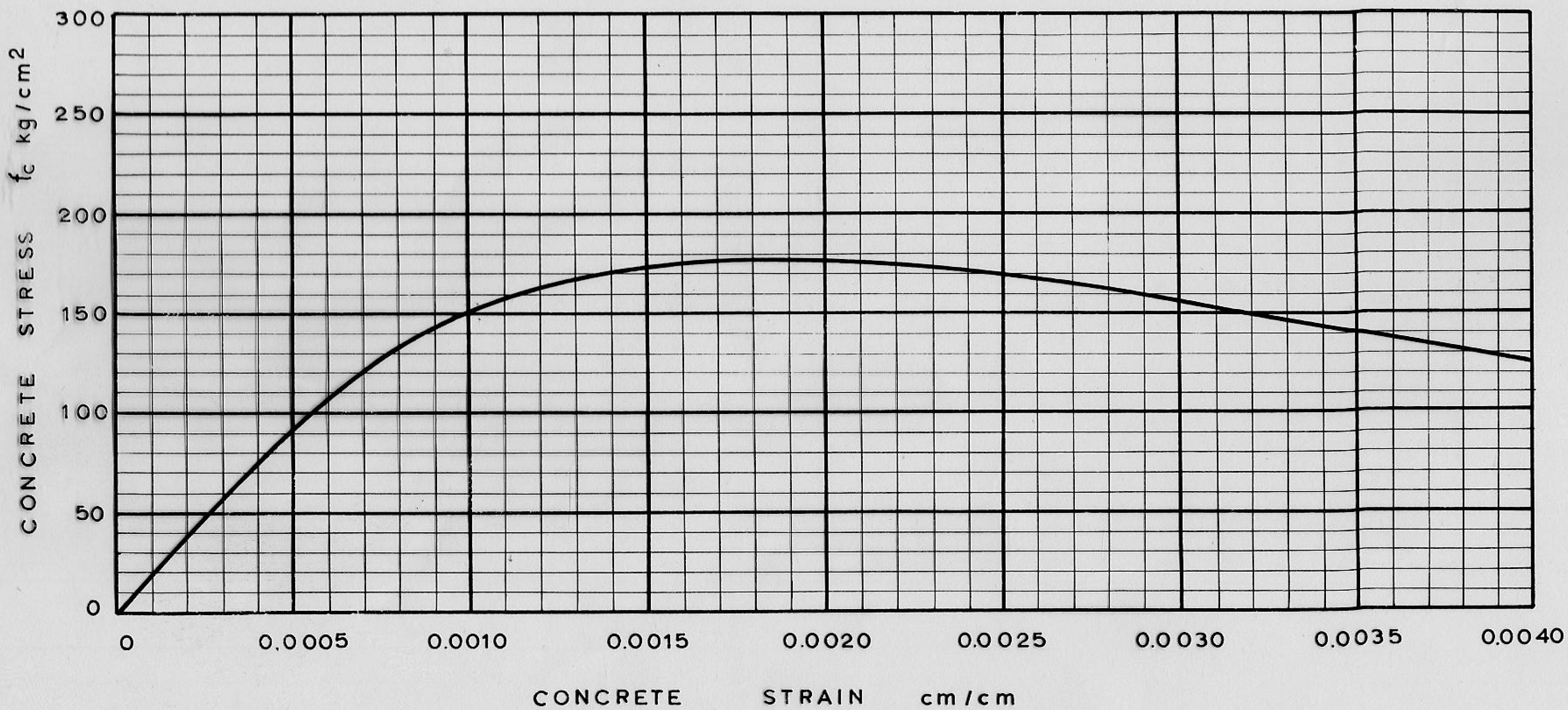


Fig. H. 3

DISTRIBUTION OF COMPRESSION STRESSES



STRESS STRAIN CURVE FOR CONCRETE [3]

$f'_c = 176\ kg/cm^2$

The value of C_c is found by computing the area under the stress-strain curve shown in fig. H.3 as follows:

$$A_1 = 0.5733^* \times 3 \times \frac{58}{2} = 49.88$$

$$A_2 = 0.5733 \times 2 \times 75 = 86.00$$

$$A_3 = 0.5733 \times 2 \times 108 = 123.83$$

$$A_4 = 0.5733 \times 2 \times 132 = 151.35$$

$$A_5 = 0.5733 \times 2 \times 150 = 171.99$$

$$A_6 = 0.5733 \times 2 \times 164 = 188.04$$

$$A_7 = 0.5733 \times 2 \times 172 = 197.22$$

$$A_8 = 0.5733 \times 2 \times 175 = 200.66$$

$$A_9 = 0.5733 \times 0.18 \times 176 = \underline{18.16}$$

$$1187.13$$

$$\therefore C_c = 1187.13 \times 25 = 29678.25 \text{ kg}$$

The point of action of C_c is at the center of gravity of the area under consideration and is computed as shown below.

$$49.88\ (0.5733 \times 15.18) = 434.09$$

$$86.00\ (0.5733 \times 13.18) = 649.82$$

$$123.83\ (0.5733 \times 11.18) = 793.69$$

$$151.35\ (0.5733 \times 9.18) = 796.54$$

$$171.99\ (0.5733 \times 7.18) = 707.96$$

$$188.04\ (0.5733 \times 5.18) = 558.42$$

$$197.22\ (0.5733 \times 3.18) = 359.55$$

$$200.66\ (0.5733 \times 1.18) = 135.75$$

$$18.16\ (0.5733 \times \frac{0.18}{2}) = \underline{0.94}$$

$$\Sigma M = 4436.76$$

---

* Each strip is 0.5733 cm wide.

REDISTRIBUTION OF BENDING

IN

UNDERREINFORCED CO

$\therefore$ point of action of $C_c = \frac{4436.76}{1187.13} = 2.74$ cm from the uppermost fiber.

$$\Sigma F_y = 0$$

$$C_c + C_s - T - P_y = 0$$

$\therefore$ $P_y = C_c + C_s - T = 29678.25 + 7901.23 - 16080.00 = 21499.48$ kg

$\Sigma M = 0$ taking moments about the centerline of the column.

$$(16080.00 + 7901.23)\ \frac{17.20}{2} + 29678.25\ (12.50 - 3.74) - 21499.48 \times 21.68 = 0$$

$$466020.05 - 466108.73 \sim 0 \quad \text{O.K.}$$

Therefore

$$P_y = 21499.48 \text{ kg}$$

$\therefore$ Actual load factor $= \frac{21499.48}{14559.00} = 1.48$

COMPARISON OF MOMENTS AND REACTIONS AT VARIOUS STAGES OF LOADING

Moments and Reactions	Successive increments of Loading			Conventional Analysis
	1st Stage	2nd Stage	3rd Stage	
X_1	- 3450	- 2590	- 3420	- 6290
X_2	- 11350	- 14810	- 20330	- 19100
X_3	- 9520	- 12730	- 17390	- 15740
X_4	- 3450	- 2590	- 3420	- 6290
X_5	- 1830	- 2080	- 2940	- 3360
X_6	+ 1725	+ 1390	+ 1840	+ 3145
X_7	+ 915	+ 1190	+ 1680	+ 1680
M_1	+ 10339	+ 13772	+ 18432	+ 17357
M_2	- 7286	- 9917	- 15577	- 12928
R_A	8881	10854	14559	15145
R_{B_1}	11139	14346	19391	18805
R_{B_2}	3575	4500	4500	4500

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