ON THE Z = 0 CONDITION FOR DISTINGUISHING ELEMENTARY AND COMPOSITE PARTICLES

by

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ABSTRACT

The suggestion that a composite particle is characterized by the vanishing of its wave function renormalization constant is tested within the framework of non-relativistic potential scattering, for two specific models: the Hulthen potential and the square well potential.

First, normalization constants are calculated in terms of Z, the wave function renormalization constant, and then compared with the value of the normalization constant of the composite particle wave function. The particle considered is the deuteron. The two expressions agree in the limit as Z goes to zero

In the second part we consider the scattering length a_s and the effective range r_e for low energy neutron-proton scattering. These two parameters are expressed in terms of Z; then Z is allowed to approach zero, and the values $a_s(Z \to 0)$ and $r_e(Z \to 0)$ are obtained. These values are compared with experimental results, as well as with the model-independent approximations of S. Weinberg. The corrections to his results are found to give good agreement with experiment.

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I. INTRODUCTION

It has been suggested by various authors (see bibliography) that in field theory if one lets the wave function renormalization constant Z (defined in section II) approach zero, the particle becomes composite and the field no longer describes an elementary particle.* We shall test the validity of this conjecture in the simple case of the deuteron, which is well known to be a composite particle.

This will be done in two ways. The first method is general and is not limited to the deuteron, but can be applied to any system of two interacting particles: one considers the normalization constant A of the wave function describing the composite particle ;

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*Since the concept of the wave function renormalization constant arises in field theory, it seems appropriate to give here a short definition of it in field-theoretical terms, although this definition is not the one to be used in our calculations.

If the free fields [satisfy] $[\psi_{free}(x), \psi^{+}_{free}(y)]_{x_o=y_o} = \delta(\vec{x} - \vec{y})$, then the interacting fields will obey the relation

$$[\psi(x), \psi^{+}(y)]_{x_o=y_o} = \frac{1}{Z}\,\delta(\vec{x} - \vec{y}).$$

(See, for example, S. Gasiorowicz, "Elementary Particle Physics," John Wiley and Sons, Inc., New York, 1966, p. 112.)

by calculating A in terms of Z and then letting Z approach zero, one compares its value with the one obtained from normalizing the wave function. This is done in sections III and IV.*

The second method, applied in section V to the square well potential, consists in expressing the scattering length a_s and effective range r_e in terms of Z for low energy neutron-proton scattering. Then Z is made to approach zero and the results are compared with the experimental values and also with those of S. Weinberg.[1] The expressions obtained by Weinberg for a_s and r_e are model-independent approximations, and we see that by working in a specific model, one gets corrections to his expressions which bring the values of a_s and r_e closer to experiment. Furthermore, from the positive sign of the correction to r_e, one gains more confidence in the conclusion drawn by S. Weinberg; viz., that $Z = 0$ does indeed characterize a composite particle, and that low energy scattering data are capable of allowing one to determine the elementarity or

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*The problem of the normalization constant is treated more specifically in Ref. 8 of the References.

otherwise of a particle.

II. THE RENORMALIZATION CONSTANT

Let H represent the total Hamiltonian of the system we are considering; viz. a deuteron at rest. Divide this total Hamiltonian into a free-particle part H_o and an interaction part V:

$$H = H_o + V$$

The continuum eigenstates of H_o, which we shall call $|\alpha\rangle$, satisfy the equation

$$H_o |\alpha\rangle = E(\alpha) |\alpha\rangle ,$$

where $E(\alpha)$ denotes the energy eigenvalue of this free-particle state. The $|\alpha\rangle$'s are delta-normalized

$$\langle\beta|\alpha\rangle = \delta(\beta - \alpha).$$

In addition to these continuum states, H_o may also have discrete "bare-elementary particle" states $|n\rangle$ with energy E_n.

$$H_o|n\rangle = E_n|n\rangle .$$

The $|n\rangle$'s are orthogonal to the continuum states, and if they are normalized,

$$\langle\alpha|n\rangle = 0$$

$$\langle m|n\rangle = \delta_{mn}.$$

The eigenstates of H_o form a complete set. We can write the completeness relation as

$$1 = \sum_n |n\rangle\langle n| + \int |\alpha\rangle\langle\alpha| d\alpha, \qquad (1)$$

where $\sum_n$ and $\int d\alpha$ stand for all necessary summations and integrations. (For example, one must sum over all angular momenta of $|\alpha\rangle$.)

The deuteron wave-function $|d\rangle$ is a normalized eigenstate of the total Hamiltonian.

$$H|d\rangle = (H_o + V)|d\rangle = -B\,|d\rangle \qquad (2)$$

$$\langle d|d\rangle = 1, \qquad (2')$$

where B is the deuteron binding energy, the experimental value of which is about 2.2 Mev.

Multiplying equation (1) from the left by $\langle d|$ and from the right by $|d\rangle$, and using the normalization of $|d\rangle$, we get

$$\langle d|d\rangle = 1 = \sum_n \langle d|n\rangle\langle n|d\rangle + \int\langle d|\alpha\rangle\langle\alpha|d\rangle\, d\alpha$$

$$1 = \sum_n |\langle n|d\rangle|^2 + \int |\langle\alpha|d\rangle|^2\, d\alpha.$$

We define Z, the wave function renormalization constant, as the probability that the deuteron is an elementary particle:

$$Z = \sum_n |\langle n|d\rangle|^2.$$

(Conservation of probability demands that $0 \leq Z \leq 1$.)

Thus we obtain

$$1 = Z + \int |\langle \alpha|d\rangle|^2 \, d\alpha. \tag{3}$$

Multiplying equation (2) from the left by $\langle \alpha|$, we obtain

$$-B\langle \alpha|d\rangle = \langle \alpha|H_o + V|d\rangle = \langle \alpha|H_o|d\rangle + \langle \alpha|V|d\rangle$$

$$= E_\alpha \langle \alpha|d\rangle + \langle \alpha|V|d\rangle$$

or

$$\langle \alpha|d\rangle (E_\alpha + B) = - \langle \alpha|V|d\rangle$$

$$|\langle \alpha|d\rangle|^2 = \frac{|\langle \alpha|V|d\rangle|^2}{(E_\alpha + B)^2} .$$

Substitution of this result in equation (3) yields

$$1 - Z = \int \frac{|\langle \alpha|V|d\rangle|^2}{(E_\alpha + B)^2} \, d\alpha. \tag{4}$$

This is equation (19) of Weinberg.[1] The calculation of Z clearly depends on the form of the interaction assumed. At this point, to make a model-independent calculation, Weinberg makes the assumption that $\langle \alpha|V|d\rangle$ is constant (independent of the energy or momentum of $|\alpha\rangle$) and the integration becomes trivial. His results will be quoted later and compared with the present work.

Equation (4) is the exact equation for Z which we shall use in our later calculations.

III. APPLICATION TO THE HULTHÉN POTENTIAL

We shall always work in the center-of-mass system, and use units in which $\hbar = 1$, $2m = 1$, where m is the neutron-proton reduced mass.

Let us first calculate the free-particle wavefunctions $|\alpha\rangle$, obeying the Schroedinger equation

$$H_o|\alpha\rangle = E_\alpha|\alpha\rangle ,$$

which can be written

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \frac{\ell(\ell+1)}{r^2}R + ER = 0,$$

where $|\alpha\rangle = |K, \ell, m\rangle = R_{k\ell}(r)\; Y_m^\ell(\theta,\varphi)$,

where Y_m^ℓ are the spherical harmonics and $K^2 = E > 0$. The solution is given in terms of Bessel functions.

$$|\alpha\rangle = \sqrt{\frac{2}{\pi}}\, K\, j_\ell(kr)\; Y_m^\ell(\theta, \varphi). \qquad (5)$$

The normalization is

$$\langle\alpha|\beta\rangle = \langle K, \ell, m|K', \ell', m'\rangle = \delta_{\ell\ell'}\,\delta_{mm'}\,\delta(K - K')$$
$$= \delta(\alpha - \beta).$$

The integration over α in equation (1) must be substituted by

$$\int d\alpha \longrightarrow \sum_\ell \sum_m \int_0^\infty dK.$$

We now proceed to calculate the deuteron wavefunction $|d\rangle$, obeying the Schroëdinger equation

$$(H_o + V)|d\rangle = -B|d\rangle \quad ,$$

where V, if we assume a Hulthén potential[10], is given by

$$V = -V_o \frac{e^{-\frac{r}{a}}}{1 - e^{-\frac{r}{a}}} = \frac{V_o}{1 - e^{\frac{r}{a}}} . \qquad V_o > 0$$

For simplicity, we shall use a = 1, so that energy will be measured in units of $\frac{\hbar^2}{2ma^2}$. Thus

$$V = \frac{V_o}{1 - e^r} \tag{6}$$

With the assumption that the deuteron is in the S-state ($\ell = 0$), and with the definition $|d\rangle = \frac{U(r)}{r}$, the Schroëdinger equation for any energy $E = K^2$ becomes

$$\left[\frac{d^2}{dr^2} + E - \frac{V_o}{1-e^r}\right] U(r) = 0.$$

Substituting $U(r) = e^{iKr} f(r)$, we get

$$f''(r) + 2iKf'(r) - \frac{V_o}{1-e^r} f(r) = 0. \tag{7}$$

Let $z = e^{-r}$. $f(r) = F(z) = F(e^{-r})$.

Then $f'(r) = - z \frac{dF}{dz}$

$$f''(r) = z^2 \frac{d^2F}{dz^2} + z \frac{dF}{dz} .$$

By substituting in equation (7) and simplyfying, we obtain

$$z(1 - z) \frac{d^2F}{dz^2} + \left\{ 1 - 2iK - z(1 - 2iK) \right\} \frac{dF}{dz} + V_o F(z) = 0.$$

One recognizes this as the differential equation for the hypergeometric function

$$z(1-z) \frac{d^2F}{dz^2} + [c - z(1 + a + b)] \frac{dF}{dz} - abF = 0$$

with the constants a, b, c, given by

$$a = -iK \left(1 - \sqrt{1 - \frac{V_o}{K^2}}\right)$$

$$b = -iK\left(1+ \sqrt{1 - \frac{V_o}{K^2}}\right)$$

$$c = 1 - 2iK$$

The first solution of the hypergeometric equation is $F(a,b;c;z)$, while the second independent solution is given by

$$z^{1-c} F(a + 1 - c, b + 1 - c; 2 - c; z).$$

But $a + 1 - c = iK\left(1 + \sqrt{1 - \frac{V_o}{K^2}}\right) = -b$

$b + 1 - c = iK \left(1 - \sqrt{1 - \frac{V_o}{K^2}}\right) = - a$

$2 - c = 1 + 2iK$

$1 - c = 2iK$

$z^{1-c} = e^{-2iKr}$

Since the general solution is a linear combination of these two solutions, we have

$$f(r) = F(z) = C_1 F(a,b;c;e^{-r}) + C_2 \quad {}^{-2iKr} F(-b,-a;2-c,e^{-r})$$

or

$$U(r) = C_1 e^{iKr} F(a,b;c;e^{-r}) + C_2 e^{-iKr} F(-b,-a;2-c;e^{-r}).$$

We are looking at a bound state of the system, $K^2 = -B$; therefore $K = i\sqrt{B}$. The wave function should vanish for large distances, a condition satisfied by the first term, but not by the second, since $e^{-iKr} = e^{\sqrt{B}r}$ and $F(-b,-a,2-c;e^{-r}) \longrightarrow 1$ for large r, so that the second term increases exponentially. We conclude that $C_2 = 0$, and therefore, for the Hulthén potential, the bound state (negative energy) wave function is given by

$$|d\rangle = \frac{U(r)}{r} = A \frac{e^{-\sqrt{E}r}}{r} F(a,b;c;e^{-r}),$$

where E is the binding energy and a,b,c are given by

$$a = \sqrt{E} \left(1 - \sqrt{1 + \frac{V_o}{E}}\right)$$

$$b = \sqrt{E} \left(1 + \sqrt{1 + \frac{V_o}{E}}\right)$$

$$c = 1 + 2\sqrt{E}.$$

Since $|d\rangle$ must be finite everywhere including the origin, we get the condition $U(0) = 0$, or

$$F(a,b;c;1) = 0.$$

In appendix A we show that this condition, in the case of the deuteron, leads to the result

$$V_o = 1 + 2\sqrt{B}.$$

Thus a,b, and c simplify to

$$a = \sqrt{B}\ (1 - \sqrt{1 + \frac{V_o}{B}}\ ) = \sqrt{B}\ (1 - \sqrt{1 + \frac{1+2\sqrt{B}}{B}})$$

$$= \sqrt{B}\ (1 - \frac{1}{\sqrt{B}}\sqrt{1 + 2\sqrt{B} + B}) = \sqrt{B}\ (1 - \frac{1+\sqrt{B}}{\sqrt{B}} = \sqrt{B}\ (1 - \frac{1}{\sqrt{B}} - 1)$$

$$= -1$$

$$b = \sqrt{B}\ (1 + \frac{1}{\sqrt{B}} + 1) = 1 + 2\sqrt{B}$$

$$c = 1 + 2\sqrt{B} = b.$$

Substituting these values into the solution of the Schroedinger equation, we get

$$U(r) = Ae^{-\sqrt{B}r}\ F\ (-1,b;b;e^{-r}).$$

But $F(-n,b;b;z\ ) = (1 - z)^n$.

Therefore $F(-1,\ b;\ b;\ e^{-r}) = 1 - e^{-r}$ and the deuteron wave-function is given by

$$|d\rangle = A\,\frac{e^{-\sqrt{B}r}}{r}\,(1 - e^{-r}), \qquad (8)$$

where A is the normalization constant.

We now have all the necessary expressions to calculate the matrix element $\langle \alpha|V|d \rangle$. Using equations (5), (6) and (8), we get

$$\langle \alpha|V|d\rangle = \sqrt{\frac{2}{\pi}}\; AV_0 K \sum_{\ell,m} \int Y_m^{*\ell}(\theta, \varphi) j_\ell (Kr) \frac{e^{-\sqrt{B}r}(1-e^{-r})}{r(1-e^{r})} r^2 dr\, d\Omega$$

$$= \sqrt{\frac{2}{\pi}}\; AV_0 K \sqrt{4\pi} \int_0^\infty j(Kr) \frac{e^{-Br}}{r} (-e^{-r})\, r^2\, dr \; .$$

Since

$$\int Y_m^{x\ell}(\theta, \varphi)\, d\Omega = \sqrt{4\pi} \int Y_m^{*\ell}(\theta, \varphi) Y_0^0\, d\Omega = \sqrt{4\pi}\, \delta_{\ell o}\, \delta_{mo}$$

and $\dfrac{1 - e^{-r}}{1 - e^{r}} = -e^{-r}$

and since $j_0(Kr) = \dfrac{\sin (Kr)}{Kr}$, the matrix element reduces to a simple integration

$$\langle \alpha|V|d\rangle = -\sqrt{\frac{2}{\pi}} \sqrt{4\pi}\; AV_0 K \int_0^\infty \frac{\sin Kr}{K}\, e^{-(1+\sqrt{B})r}\, dr$$

$$= -\sqrt{8}\; A V_0\, I_m \int_0^\infty e^{-(1+\sqrt{B})r}\, e^{iKr}\, dr$$

$$= -\sqrt{8}\; AV_0\, I_m \frac{1}{1+\sqrt{B} - iK} = \frac{-\sqrt{8}\; A V_0 K}{K^2 + (1+\sqrt{B})^2} \; .$$

Defining C $= (1 + \sqrt{B})^2$, we have

$$|\langle \alpha | V | d \rangle|^2 = 8A^2 V_o^2 \left[\frac{K^2}{(C + K^2)^2}\right] .$$

Substituting this expression into equation (4), we get

$$1 - Z = 8A^2 V_o^2 \int_0^\infty \frac{K^2 dK}{(C + K^2)^2 (B + K^2)^2} . \qquad (9)$$

In appendix B it is shown that

$$\int_0^\infty \frac{K^2 \, dK}{(C + K^2) (B + K^2)^2} = \frac{\pi}{4 \sqrt{BC} (C - B)^3}$$

$$= \frac{\pi}{4 \sqrt{B}(1 + \sqrt{B})(1 + 2\sqrt{B})^3} \qquad (10)$$

Substituting this value of the integral in equation (9), we get the required expression of A in terms of Z:

$$1 - Z = 8A^2 V_o^2 \frac{\pi}{4 \sqrt{B}(1 + \sqrt{B})(1 + 2\sqrt{B})^3}$$

or, since $V_o = 1 + 2\sqrt{B}$,

$$A^2 = \frac{\sqrt{B}}{2\pi} (1 + \sqrt{B})(1 + 2\sqrt{B})(1 - Z).$$

We see then that, as $Z \longrightarrow 0$, this expression is identical with the value of A^2 obtained from the normalization of $|d\rangle$; viz.,

$$A^2 = \frac{\sqrt{B}}{2\pi}(1 + \sqrt{B})(1 + 2\sqrt{B}), \tag{11}$$

which is derived in appendix C.

Thus, we have established the result that the limit of $Z \longrightarrow 0$ correctly reproduces the bound state normalization.

IV. APPLICATION TO THE SQUARE WELL POTENTIAL

In this part we suppose that the neutron-proton interaction can be approximated by a square well potential of depth $V_o > 0$ and range a.

$$V(r) = -V_o \qquad r \leq a$$

$$V(r) = 0 \qquad r > a$$

Assuming again that the deuteron is an S-state, the Schroedinger equation for the wave function $|d\rangle = \frac{U(r)}{r}$ can be written as

$$\left[\frac{d^2}{dr^2} - B - V(r)\right] U(r) = 0,$$

where B is the deuteron binding energy.

For $r \leq a$, this equation becomes

$$\left[\frac{d^2}{dr^2} - B + V_o\right]U(r) = \frac{d^2U}{dr^2} + K^2 U = 0, \text{ where}$$

$$K^2 = (V_o - B) > 0. \qquad (12)$$

The solution of this equation satisfying the condition $U(0) = 0$ is $U(r) = A_1 \sin Kr$, where A_1 is the normalization constant.

For $r > a$, the Schroedinger equation becomes

$$\left[\frac{d^2}{dr^2} - B\right] U(r) = 0,$$

whose solution, vanishing at infinity, is $U(r) = A_2 e^{-\sqrt{B}\, r}$.

The continuity of $U(r)$ at $r = a$ yields

$$A_1 \sin Ka = A_2 e^{-\sqrt{B}\, a},$$

while the continuity of $U'(r)$ at $r = a$ yields

$$A_1 K \cos Ka = -A_2 \sqrt{B}\, e^{-\sqrt{B}\, a}.$$

Combining these two conditions, we get

$$\tan Ka = -\frac{K}{\sqrt{B}} \qquad (13)$$

One uses this relation to calculate the binding energy in terms of V_o and a. However, we shall use this equation for another purpose, as will be seen in later calculations.

Proceeding in the same way as we did for the Hulthén potential, we now find the matrix element $\langle a|V|d\rangle$.

$$\langle a|V|d\rangle = \int_{\Omega} \int_{r=0}^{\infty} \langle a|V|d\rangle\, r^2\, dr\, d\Omega = \int_{\Omega} \int_{r=0}^{r=a} \langle a\, |V|d\rangle\, r^2\, dr\, d\Omega ,$$

since the potential vanishes for $r > a$.

Writing $|\alpha\rangle = \sqrt{\frac{2}{\pi}}\, P\, j_{\ell}\,(Pr)\, Y^{\ell}_{m}(\theta, \varphi)$ (we change from K to P in order to avoid confusion with K defined in equation (12)), we have

$$\langle \alpha|V|d\rangle = \sqrt{\frac{2}{\pi}}\, PA_1 \int_0^a j_{\ell}\,(Pr)\,(-V_o)\,\frac{\sin Kr}{r}\, r^2\, dr \int Y^{*\ell}_{m}(\theta, \varphi)\, d\,?.$$

$$\langle \alpha|V|d\rangle = \sqrt{\frac{2}{\pi}}\,\sqrt{4\pi}\; A_1\, V_o\, P \int_0^a j_o\,(Pr)\,\frac{\sin Kr}{r}\, r^2\, dr$$

$$= -2\sqrt{2}\; A_1\, V_o \int_0^a \sin Pr \sin Kr\, dr$$

$$= -\sqrt{2}\, A_1\, V_o \int_0^a [\cos (P-K)r - \cos (P+K)r]\, dr$$

$$= \sqrt{2}\, A_1\, V_o\, \frac{\sin (P+K)a}{P+K} - \sqrt{2}\, A_1\, V_o\, \frac{\sin (P-K)a}{P-K}$$

$$= \frac{\sqrt{2}\, A_1\, V_o}{P^2 - K^2} \Big\{ P[\sin (P+K)a - \sin (P-K)a]$$

$$-K\, [\sin (P+K)a + \sin (P-K)a] \Big\}$$

$$= \frac{\sqrt{2}\, A_1\, V_o}{P^2 - K^2} \Big\{ 2P \sin aK \cos aP - 2K \cos aK \sin aP \Big\}.$$

From equation (13) $K \cos aK = -\sqrt{B} \sin aK$. Therefore

$$\langle \alpha|V|d\rangle = \frac{2\sqrt{2}\, A_1\, V_o}{P^2 - K^2} \sin aK\, [P \cos aP + \sqrt{B} \sin aP]$$

and

$$|\langle \alpha|V|d\rangle|^2 = 8A_1^2 V_o^2 \sin^2 aK \frac{(P \cos aP + \sqrt{B} \sin aP)^2}{(P^2 - K^2)^2} .$$

We now substitute this expression in equation (4).

$$1 - Z = 8A_1^2 V_o^2 \sin^2 aK \int_o^\infty \frac{(P \cos aP + \sqrt{B} \sin aP)^2}{(P^2 - K^2)^2 (B + P^2)^2} dP, \qquad (14)$$

since $E_\alpha = P^2$ in our units.

It is shown in appendix D that the value of the above integral is

$$\int_o^\infty \frac{(P \cos aP + \sqrt{B} \sin aP)^2}{(P^2 - K^2)^2 (B + P^2)^2} dP = \frac{\pi (1 + a\sqrt{B})}{4K^2 \sqrt{B} (B + k^2} .$$

Let us only mention here that the integrand is finite for all P since, as P approaches K, the numerator $P \cos aP + \sqrt{B} \sin aP$ approaches $K \cos aK + \sqrt{B} \sin aK$, which is zero by condition (13). Also, by the same condition, we get

$$\tan^2 Ka = \frac{K^2}{B} \quad \text{or} \quad \sin^2 Ka = \frac{K^2}{B + K^2} .$$

Substituting these values into equation (14), we get

$$1 - Z = 8A_1^2 V_o^2 \frac{K^2}{B + K^2} \frac{\pi (1 + a\sqrt{B})}{4K^2 \sqrt{B} (B+K^2)} = 2\pi A_1^2 \frac{(1+a\sqrt{B})}{\sqrt{B}} \frac{V_o^2}{(B+K^2)^2}$$

$$= 2\pi A_1^2 \frac{(1 + a\sqrt{B})}{\sqrt{B}} \quad ,$$

since $V_o^2 = (B + K^2)^2$ from equation (12).

Thus the required expression is given as

$$A_1^2 = \frac{\sqrt{B}}{2\pi} \frac{(1 - Z)}{1 + a\sqrt{B}} \quad . \tag{15}$$

We see that as $Z \longrightarrow 0$, this expression reduces to the value of A_1^2 obtained from the normalization condition; viz.,

$$A_1^2 = \frac{\sqrt{B}}{2\pi\,(1 + a\sqrt{B})} \quad ,$$

which is derived in appendix E.

V. THE SCATTERING PARAMETERS IN TERMS OF Z

The scattering parameters we are interested in are a_s, the scattering length, and r_e, the effective range, which appear in the so-called shape-independent approximation which is useful for low energy scattering. The phase shift $\delta(K)$ (for S-waves) is related to these parameters through

$$K \operatorname{ctn} \delta(K) = -\frac{1}{a_s} + \frac{1}{2} r_e K^2 ,$$

where $E = K^2$ is the energy of the incident particle.

We want to determine whether low energy scattering data provide us with any information concerning the elementarity of a particle. For this purpose we have to express a_s and r_e in terms of Z and see if in the limit as $Z \longrightarrow 0$ we get agreement with experimental results. Furthermore, we want to compare our results with those of S. Weinberg, which are

$$(a_s)_w = \frac{2(1 - Z)}{\sqrt{B}\,(2 - Z)} \tag{16}$$

$$(r_e)_w = - \frac{Z}{\sqrt{B}\,(1 - Z)} . \tag{17}$$

(See equations (39) and (40) of reference 1 in references.)

For the square well potential which we considered in the previous section, the scattering length and the effective range are given by*

$$a_s = \frac{2}{2\sqrt{B} - aB} \tag{18}$$

$$r_e = a. \tag{19}$$

Solving equations (15) for a, we have

$$a = \frac{1 - Z}{2\pi A_1^2} - \frac{1}{\sqrt{B}} .$$

Substitution of this value in equation (18) gives

$$a_s = \frac{2}{2\sqrt{B} - \frac{B(1 - Z)}{2\pi A_1^2} + \sqrt{B}} = \frac{2}{3\sqrt{B} - \frac{(1 - Z)B}{2\pi A_1^2}} .$$

Substituting for the value of A_1^2 from the normalization condition equation (E1), we obtain

$$a_s = \frac{2}{3\sqrt{B} - \frac{(1 - Z)(1 + a\sqrt{B})B}{\sqrt{B}}} = \frac{2}{3\sqrt{B} - \sqrt{B} - aB + Z(1 + a\sqrt{B})\sqrt{B}}$$

$$a_s = \frac{2}{2\sqrt{B} - aB + Z(1 + a\sqrt{B})\sqrt{B}} . \tag{20}$$

Similarly

$$r_e = a = \frac{1 - Z}{2\pi A_1^2} - \frac{1}{\sqrt{B}} .$$

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*See, for example, D Park, Introd. to the Quantum Theory, McGraw-Hill, Inc. 1964, p. 287

Substituting from equation (E1), we obtain

$$r_e = \frac{(1 - Z)(1 + a\sqrt{B})}{\sqrt{B}} - \frac{1}{\sqrt{B}} = \frac{1 + a\sqrt{B}}{\sqrt{B}} - \frac{1}{\sqrt{B}} - Z\,\frac{(1+a\sqrt{B})}{\sqrt{B}}$$

$$= a - \frac{z(1 + a\sqrt{B})}{\sqrt{B}} \quad . \qquad (21)$$

Since, we want to compare our results with those of Weinberg, let us write equations (20) and (21) as

$$a_s = (a_s)_w + C_s \qquad (22)$$

$$r_e = (r_e)_w + C_e \; , \qquad (23)$$

where

$$C_s = \frac{2}{2\sqrt{B} - aB + Z\,(1 + a\sqrt{B})\sqrt{B}} - \frac{2(1 - Z)}{(2 - z)\sqrt{B}}$$

$$= \frac{2\sqrt{B} + 2Z\,(Z + Z\sqrt{B} - 2\sqrt{B})}{\sqrt{B}\,[4 - 2\sqrt{B} + Z\,(3\sqrt{B} - Z - Z\sqrt{B})]} \; ,$$

and

$$C_e = a - \frac{Z(1 + a\sqrt{B})}{\sqrt{B}} + \frac{Z}{(1 - z)\sqrt{B}}$$

$$= 1 + \frac{Z\,[1 - (1 - Z)(1 + \sqrt{B})]}{\sqrt{B}\,(1 - Z)} \; ,$$

where we have put a = 1, making $\frac{\hbar^2}{2ma^2}$ our unit of energy.

We now take the limit of a_s and r_e in equations (22) and (23) as $Z \longrightarrow 0$.

$$\lim_{Z\to 0} a_s = \lim_{Z\to 0} C_s + \lim_{Z\to 0} (a_s)_w$$

$$= \lim_{Z\to 0} (a_s)_w + \frac{2\sqrt{B}}{\sqrt{B}\,(4 - 2\sqrt{B})}$$

$$= \lim_{Z\to 0} (a_s)_w + \frac{1}{2 - \sqrt{B}} .$$

$$\lim_{Z\to 0} r_e = \lim_{Z\to 0} (r_e)_w + \lim_{Z\to 0} C_e$$

$$= \lim_{Z\to 0} (r_e)_w + 1 .$$

For a numerical calculation, we use (see appendix A)

$B = 0.14$

$\sqrt{B} = 0.37$

$$\lim_{Z\to 0} a_s = \frac{1}{2 - 0.37} + \frac{1}{0.37} = 0.6 + 2.7 = 3.3$$

Or, since our unit of length is $a = 1.5$ F,

$$\lim_{Z\to 0} a_s \approx 5F,$$

$$\lim_{Z\to 0} r_e = 0 + 1 = 1$$

or

$$\lim_{Z \to 0} r_e = 1.5 \text{ F}.$$

In the table below we summarize our results and compare them with those of S. Weinberg and the experimental values.

	S. Weinberg	Calculated	Experimental
a_s	4.1 F	5.0 F	5.4 F
r_e	0	1.5 F	1.8 F

Besides getting results nearer to experiment, <u>the important thing in the above calculation is that the correction of r_e to S. Weinberg's model-independent calculation is positive.</u> As can be seen from equation (17), $(r_e)_W$ is negative for $Z \neq 0$ and goes to zero as Z approaches zero. But for the fact that the correction term is <u>positive</u>, his conclusion that $Z = 0$ characterizes a composite particle (such as the deuteron) would have been unwarranted.

VI. CONCLUSION

From our calculations in sections III and IV we have seen that if one expresses the normalization constant A of a particle in terms of the renormalization constant Z, and then lets Z approach zero, the result is the correct normalization constant for a composite particle. In other words if in equation (2') we change $\langle d|d\rangle = 1$ to $\langle d|d\rangle = A^2$, then equation (4) is replaced by

$$A^2 = Z + \int \frac{|\langle\alpha|V|d\rangle|^2}{(E_\alpha + B)^2}\, d_\alpha$$

and we have verified that

$$A^2_{composite} = \int \frac{|\langle\alpha|V|d\rangle|^2\, d_\alpha}{(E_\alpha + B)^2} .$$

The second conclusion is that, in some cases at least, one can use scattering experiments to decide whether a given particle is composite or not. Since for low energy scattering the shape-independent approximation is a good one, it is possible to express a_s and r_e in terms of Z, then let Z approach zero, and see if they agree with experimental values. We obtained very good agreement in the case of the square well potential (a similar calculation could be carried out for the Hulthén potential), and comparison with the results of S. Weinberg showed that the corrections were positive, thus supporting his claim that low energy scattering data can indeed provide information concerning the elementarity of a particle.

APPENDIX A

To prove that $V_o = 1 + 2\sqrt{B}$ for the Hulthén potential, we first need to express B in terms of our unit of energy, which is $\frac{\hbar^2}{2ma^2}$. So let us first calculate this unit.

Since m is the neutron-proton reduced mass, 2m is approximately equal to the mass of the proton (or neutron). We have already said that a stands for the range of the neutron-proton interaction, and is approximately 1.5 Fermis.

$$\hbar^2 \approx 1.0 \times 10^{-54} \text{ (erg-sec)}^2$$

$$2m \approx 1.7 \times 10^{-24} \text{ gm}$$

$$a^2 \approx 2.3 \times 10^{-26} \text{ cm}^2$$

$$\frac{\hbar^2}{2ma^2} \approx \frac{10^{-54}}{1.7 \times 10^{-24} \times 2.3 \times 10^{-26}} = 2.5 \times 10^{-5} \text{ ergs.}$$

Now, $B = 2.2$ Mev $= 3.6 \times 10^{-6}$ erg.

Therefore $B = \frac{3.6 \times 10^{-6}}{2.5 \times 10^{-5}} = 0.14$ units of $\frac{\hbar^2}{2ma^2}$ and $\sqrt{B} = 0.37$.

These values, $B = 0.14$ and $\sqrt{B} = 0.37$, are also used in section V.

We now show that the condition $F(a,b;c;1) = 0$ leads to

$V_o = 1 + 2\sqrt{B}$.

$$F(a,b;c;1) = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)} .$$

From the definitions of a, b, and c, we have $c-a-b = 1$, or $c = 1+a+b$. Therefore, since $\Gamma(1) = 1$,

$$F(a,b;c;1) = \frac{\Gamma(1+a+b)}{\Gamma(1+a)\,\Gamma(1+b)} = \prod_{n=1}^{\infty} \frac{(n+a)(n+b)}{n(n+a+b)} .$$

(For this relation, as well as other properties of the hypergeometric functions, see, for example, E.T. Whittaker and G.N. Watson, A Course of Modern Analysis, Cambridge, 1962.)

Again from the definition of a and b, $a + b = 2\sqrt{E}$, and $ab = -V_o$, so that

$$(n+a)(n+b) = n^2 + (a+b)n + ab = n^2 + 2\sqrt{E}\,n - V_o$$

and

$$\frac{(n+a)(n+b)}{n(n+a+b)} = \frac{n^2 + 2\sqrt{E}\,n - V_o}{n(n + 2\sqrt{E})} = 1 - \frac{V_o}{n(n + 2\sqrt{E})} .$$

Thus $$F(a,b;c;1) = \prod_{n=1}^{\infty} \left\{ 1 - \frac{V_o}{n(n+2\sqrt{E})} \right\} = 0.$$

This says that for at least one n, n = 1, 2, ... we must have

$$V_o = n(n + 2\sqrt{E_n}).$$

Furthermore, the n's are limited, since

$$\sqrt{E_n} = \frac{V_o - n^2}{2n} \geq 0, \text{ so that } n^2 \leq V_o.$$

One can easily see that the ground state must have the lowest n; viz., unity. Suppose $m < n$. Then

$$\sqrt{E_m} - \sqrt{E_n} = \frac{V_o - m^2}{2m} - \frac{V_o - n^2}{2n} = \frac{V_o (n - m) + mn (n - m)}{2mn}$$

$$\sqrt{E_m} - \sqrt{E_n} = \frac{(V_o + mn)(n - m)}{2mn} > 0.$$

Thus, if $m < n$, we have $E_m > E_n$. Therefore, for the deuteron in the ground state, n = 1. Let us show that no excited states are allowed, and that n = 1 is the only possible state of the deuteron in the Hulthén potential.

Since B = 2.2 Mev is also the ground state energy, we can calculate

$$V_o = 1 + 2\sqrt{B} = 1 + 2(0.37) = 1.74.$$

For excited states, $n \geq 2$, we have

$$\sqrt{E}_n = \frac{V_o - n^2}{2n} = \frac{1.74 - n^2}{2n} < 0 \text{ for } n \geq 2.$$

We conclude that $n = 1$ is the only possible state and therefore

$$V_o = 1 + 2\sqrt{B}.$$

APPENDIX B

Here we shall outline the procedure to obtain the following result:

$$I = \int_o^\infty \frac{K^2\, dK}{[C + K^2]^2\, [B + K^2]^2} = \frac{\pi}{4\sqrt{B}\,(1 + \sqrt{B})(1 + 2\sqrt{B})^3} .$$

Let us divide the integrand into partial fractions.

$$\frac{K^2}{[C + K^2]^2\, [B + K^2]^2} = \frac{A_1}{(C + K^2)^2} + \frac{A_2}{C + K^2} + \frac{A_3}{(B + K^2)^2} + \frac{A_4}{B + K^2}$$

After some algebraic calculations, one obtains

$$A_1 = - \frac{C}{(C - B)^2} \tag{B1}$$

$$A_2 = \frac{1}{(C - B)^2} - \frac{2C}{(C - B)^3} \tag{B2}$$

$$A_3 = - \frac{B}{(C - B)^2} \tag{B3}$$

$$A_4 = \frac{1}{(C - B)^2} + \frac{2B}{(C - B)^3} . \tag{B4}$$

Define $I = I_{A_1} + I_{A_2} + I_{A_3} + I_{A_4}$, where

$$I_{A_1} = A_1 \int_o^\infty \frac{dK}{(C + K^2)^2}$$, and the others are defined

similarly. There occur two types of integrals, $\int_o^\infty \frac{dK}{(C + K^2)^2}$ and

$\int_o^\infty \frac{dK}{C + K^2}$, both of which can be evaluated by the substitution

$K = \sqrt{C}\tan\theta$, $dK = \sqrt{C}\sec^2\theta\, d\theta$.

$$\int_o^\infty \frac{dK}{C + K^2} = \sqrt{C}\int_o^{\frac{\pi}{2}} \frac{\sec^2\theta\, d\theta}{C(1+\tan^2\theta)} = \frac{1}{\sqrt{C}}\int_o^{\frac{\pi}{2}} d\theta = \frac{\pi}{2\sqrt{C}}$$

$$\int_o^\infty \frac{dK}{[C + K^2]^2} = \sqrt{C}\int_o^{\frac{\pi}{2}} \frac{\sec^2\theta\, d\theta}{C^2(1+\tan^2\theta)^2} = \frac{1}{C^{3/2}}\int_o^{\frac{\pi}{2}} \cos^2\theta\, d\theta$$

$$= \frac{1}{2C^{3/2}}\int_o^{\frac{\pi}{2}} (1 + \cos 2\theta)\, d\theta = \frac{\pi}{4\, C^{3/2}}$$

Substitution of the proper constants in each of the four integrals, yields

$$I = \pi\left\{ \frac{A_1}{4C^{3/2}} + \frac{A_2}{2\sqrt{C}} + \frac{A_3}{4B^{3/2}} + \frac{A_4}{2\sqrt{B}} \right\} .$$

Substituting the values of A_1, A_2, A_3, and A_4 from (B1-4) we get, after doing some algebra,

$$I = \frac{\pi\,(C\sqrt{C} - 3C\sqrt{B} + 3B\sqrt{C} - B\sqrt{B})}{4\sqrt{BC}\,(C - B)^3} = \frac{\pi\,(\sqrt{C} - \sqrt{B})^3}{4\sqrt{BC}\,(C - B)^3}$$

But $C = (1 + \sqrt{B})^2 = 1 + 2\sqrt{B} + B$, so that $\sqrt{C} - \sqrt{B} = 1$, $C - B = 1 + 2\sqrt{B}$. Therefore

$$I = \frac{\pi}{4\sqrt{B}\,(1 + \sqrt{B})(1 + 2\sqrt{B})^3}$$ which is equation (10).

APPENDIX C

Using the normalization condition, $\langle d|d\rangle = 1$, where, from equation (8)

$$|d\rangle = A\,\frac{e^{-\sqrt{B}r}}{r}\,(1 - e^{-r}).$$

We want to show that $A^2 = \frac{\sqrt{B}}{2\pi}(1+\sqrt{B})(1+2\sqrt{B})$.

$$1 = \langle d|d\rangle = A^2\int_0^\infty \frac{e^{-2\sqrt{B}r}}{r^2}(1-e^{-r})^2\, r^2\, dr \int d\Omega$$

$$= 4\pi A^2 \int_0^\infty e^{-2\sqrt{B}r}(1 - 2e^{-r} + e^{-2r})\, dr$$

$$= 4\pi A^2 \int_0^\infty [e^{-2\sqrt{B}r} - 2e^{-(1+2\sqrt{B})r} + e^{-2(1+\sqrt{B})r}]\, dr$$

Using $\int_0^\infty e^{-\alpha x}\, dx = \frac{1}{\alpha}$, we get

$$1 = 4\pi A^2\left\{\frac{1}{2\sqrt{B}} - \frac{2}{1+2\sqrt{B}} + \frac{1}{2(1+\sqrt{B})}\right\}$$

$$= 2\pi A^2\left\{\frac{1+3\sqrt{B}+2B-4\sqrt{B}-4B+\sqrt{B}+2B}{\sqrt{B}\,(1+\sqrt{B})\,(1+2\sqrt{B})}\right\}$$

$$= \frac{2\pi A^2}{\sqrt{B}\,(1+\sqrt{B})(1+2\sqrt{B})}.$$

Therefore

$A^2 = \frac{\sqrt{B}}{2\pi}(1+\sqrt{B})(1+2\sqrt{B})$ which is equation (11).

APPENDIX D

The evaluation of $\int_0^\infty \frac{(P \cos aP + \sqrt{B} \sin aP)^2 \, dP}{(P^2 - K^2)^2 \, (B + P^2)^2}$

turns out to be quite complicated if one uses real variables only, because when we break the integrand into partial fractions divergent terms occur, although we expect those terms to cancel in the end, since the whole integrand is finite even at $P = \pm K$, as mentioned earlier. The evaluation is somewhat simplified if one uses complex variables.

We must first find the contour on which the integrand vanishes for large imaginary P.

$$\frac{I}{2} = \int_0^\infty \frac{(P \cos aP + \sqrt{B} \sin aP)^2}{(P^2 - K^2)^2 \, (B + P^2)^2} \, dP, \text{ where}$$

$$I = \int_{-\infty}^\infty \frac{(P \cos aP + \sqrt{B} \sin aP)^2}{(P^2 - K^2)^2 \, (B + P^2)^2} \, dP \text{ , since the integrand}$$

is even in P.

Substituting $\cos aP = \frac{e^{iaP} + e^{-iaP}}{2}$, $\sin aP = \frac{e^{iaP} - e^{-iaP}}{2i}$,

squaring the numerator, and collecting terms, we get

$$I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{(P - i\sqrt{B})(P + i\sqrt{B})\, dP}{(B + P^2)^2\,(P^2 - K^2)^2} + \frac{1}{4}\int_{-\infty}^{\infty} \frac{e^{2iaP}\,(P - i\sqrt{B})^2\, dP}{(B + P^2)^2\,(P^2 - K^2)^2}$$

$$+ \frac{1}{4}\int_{-\infty}^{\infty} \frac{e^{-2iaP}\,(P + i\sqrt{B})^2\, dP}{(B + P^2)^2\,(P^2 - K^2)^2} .$$

Changing P to -P in the last integral we see that it is identical with the second.

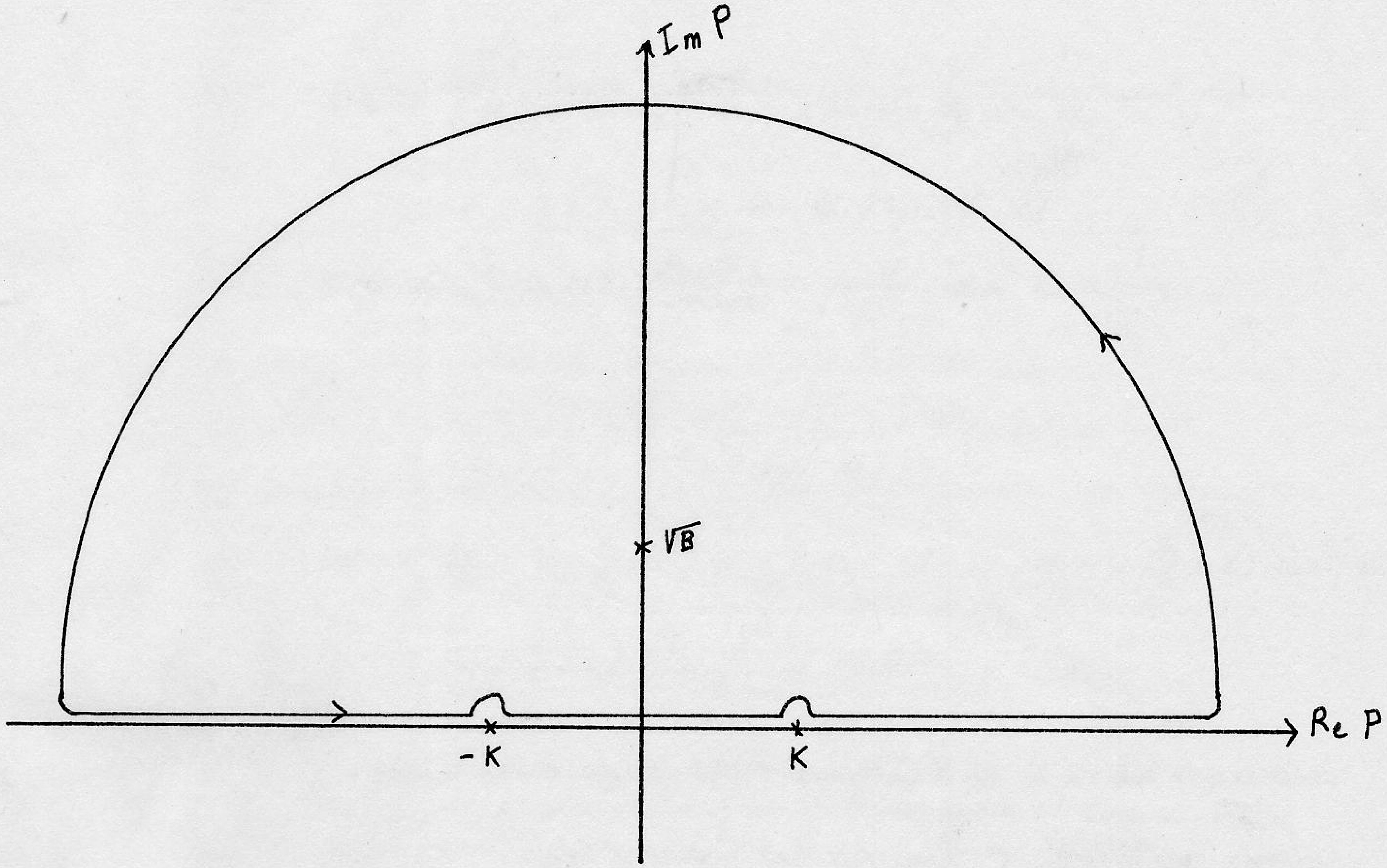
$$I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{(P - i\sqrt{B})(P + i\sqrt{B})\, dP}{(B + P^2)^2\,(P^2 - K^2)^2} + \frac{1}{2}\int_{-\infty}^{\infty} \frac{e^{2iaP}\,(P - i\sqrt{B})^2\, dP}{(B + P^2)^2(P^2 - K^2)^2}$$

$$I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{[P - i\sqrt{B}]\,[P + i\sqrt{B} + (P - i\sqrt{B})\, e^{2iaP}]\, dP}{(P - i\sqrt{B})^2\,(P + I\sqrt{B})^2\,(P - K)^2\,(P + K)^2}$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \frac{[P + i\sqrt{B} + (P - i\sqrt{B})\, e^{2aiP}]\, dP}{(P - i\sqrt{B})(P + i\sqrt{B})^2\,(P - K)^2\,(P + K)^2}$$

Now the integrand vanishes for large imaginary P. (That was the reason we expressed sin aP and cos aP in terms of e^{2aiP}.)

The figure below shows the contour of integration and the poles on the real and positive imaginary axes.



There is a simple pole at $P = i\sqrt{B}$, and two <u>simple</u> poles at $P = \pm K$, although $(P-K)^2$ appears in the denominator. To prove that the pole at $P = +K$ is simple, we have to show that the numerator vanishes at $P = K$.

$$N = K + i\sqrt{B} + (K - i\sqrt{B})e^{2iaK}$$

$$ImN = \sqrt{B} + K \sin 2aK - \sqrt{B} \cos 2aK$$

$$R_eN = K + K \cos 2aK + \sqrt{B} \sin 2aK$$

$$I_mN = \sqrt{B} + 2K \sin aK \cos aK - \sqrt{B} + 2\sqrt{B} \sin^2 aK$$

$$= 2 (K \sin aK \cos aK + \sqrt{B} \sin^2 ak)$$

$$\text{ReN} = K + K - 2K \sin^2 aK + 2\sqrt{B} \sin aK \cos aK$$

$$= 2K + 2(\sqrt{B} \sin aK \cos aK - K \sin^2 aK)$$

From condition (13), $\frac{\sin Ka}{\cos Ka} = -\frac{K}{\sqrt{B}}$, $\sin^2 Ka = \frac{K^2}{B+K^2}$.

Therefore

$$I_m N = 2(-\sqrt{B} \sin^2 aK + \sqrt{B} \sin^2 aK) = 0$$

$$\text{ReN} = 2K + 2(-\frac{B}{K} \sin^2 aK - K \sin^2 aK) = 2K - 2(\frac{B}{K} + K) \sin^2 aK$$

$$= 2K - \frac{2}{K}(B + K^2) \frac{K^2}{B + K^2} = 0.$$

A similar calculation, after changing K to -K in the expression for N, yields N = 0. We conclude that the poles at $P = \pm K$ are simple. We can thus evaluate the integral using the theory of residues.*

$$I = 2\pi i \text{ Res } [f(P), P = i\sqrt{B}] + \pi i \Sigma \text{ Res } [f(P), P = \pm K],$$

where $$f(P) = \frac{1}{2} \frac{[P + i\sqrt{B} + (P - i\sqrt{B}) e^{2aiP}]}{(P - i\sqrt{B})(P + i\sqrt{B})^2 (P - K)^2 (P + K)^2} .$$

$$\text{Res } [f(P), P = i\sqrt{B}] = \frac{1}{2} \frac{2i\sqrt{B}}{(2i\sqrt{B})^2 (-B - K^2)^2} = \frac{1}{4i\sqrt{B}(B + K^2)^2}$$

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*See, for example, E.T. Whittaker and G.N. Watson, A Course on Modern Analysis, Cambridge, 1962, p. 117.

$$\text{Res}\,[f(P),\ P=K] = \lim_{P\to K} \frac{1}{2} \frac{[P + i\sqrt{B} + (P - i\sqrt{B})\ e^{2aiP}]}{(P - i\sqrt{B})(P + i\sqrt{B})^2\ (P + K)^2\ (P - K)}$$

$$= \frac{1}{2(K + i\sqrt{B})^2(K - i\sqrt{B})(2K)^2} \lim_{P\to K} \frac{P + i\sqrt{B} + (P - i\sqrt{B})e^{2aiP}}{P - K}$$

$$\text{Res}\,[f(P),P=K] = \frac{[1 + (1 + 2aiK + 2a\sqrt{B})\ e^{2aiK}]}{2(K + i\sqrt{B})^2\ (K - i\sqrt{B})\ (2K)^2}\ ,$$

where l'Hopital's rule was used, since both numerator and denominator vanish at $P = K$.

Similarly,

$$\text{Res}\,[f(P),P=-K] = \frac{-[1 + (1 - 2aiK + 2a\sqrt{B})e^{-2aiK}]}{2(K - i\sqrt{B})^2\ (K + i\sqrt{B})(2K)^2}\ .$$

Adding these two residues, after a few pages of algebra where repeated use of condition (13) must be made, we get

$$\Sigma\ \text{Res}\,[f(P),P = \pm K] = -\frac{i}{2K^2(B+K^2)^3}\,[2aK^2B + B\sqrt{B} + aB^2 + K^2\sqrt{B} + aK^4].$$

Therefore, we have

$$I = 2\pi i\ \text{Res}\,[f(P),P = i\sqrt{B}] + \pi i\ \Sigma\ \text{Res}\,[f(P),P = \pm K]$$

$$= \frac{\pi}{2\sqrt{B}\ (B+K^2)^2} + \frac{\pi\,[2aK^2B + B\sqrt{B} + aB^2 + K^2\sqrt{B} + aK^4]}{2K^2\ (B + K^2)^3}$$

$$= \frac{\pi\,[K^4 + aK^4\sqrt{B} + 2BK^2 + 2aK^2\,B\sqrt{B} + B^2 + aB^2\sqrt{B}\,]}{2K^2\sqrt{B}\ (B + K^2)^3}$$

$$= \frac{\pi\,[1 + a\sqrt{B}][B + K^2]^2}{2K^2\sqrt{B}\,(B + K^2)^3} = \frac{\pi\,(1 + a\sqrt{B})}{2K^2\sqrt{B}\,(B + K^2)}$$

Thus we have the required result that

$$\int_0^\infty \frac{(P\cos aP + \sqrt{B}\sin aP)^2\,dP}{(P^2 - K^2)^2\,(B + P^2)^2} = \frac{I}{2} = \frac{\pi\,(1 + a\sqrt{B})}{4K^2\sqrt{B}\,(B + K^2)}\ .$$

APPENDIX E

From the normalization of $|d$,

$$1 = \langle d|d \rangle = \int_o^a \langle d|d \rangle r^2 \, dr \int d\Omega + \int_a^\infty \langle d|d \rangle r^2 \, dr \int d\Omega ,$$

where $|d\rangle = A_1 \dfrac{\sin Kr}{r} \qquad r \leq \alpha .$

$$|d\rangle = A_2 \frac{e^{-\sqrt{B}r}}{r} \qquad r > a.$$

$$1 = 4\pi A_1^2 \int_o^a \sin^2 Kr \, dr + 4\pi A_2^2 \int_a^\infty e^{-2\sqrt{B}r} \, dr$$

$$= 2\pi A_1^2 \int_o^a (1 - \cos 2 Kr) dr - \frac{4\pi A_2^2}{2\sqrt{B}} [e^{-2\sqrt{B}r}]_a^\infty$$

$$= \frac{2\pi A_2^2}{\sqrt{B}} e^{-2\sqrt{B}a} + 2\pi A_1^2 (a - \frac{\sin 2Ka}{2K})$$

From the continuity of $|d\rangle$ at $r = a$,

$$A_1 \sin Ka = A_2 e^{-\sqrt{B}a} .$$

Therefore

$$1 = 2\pi A_1^2 \cdot (a - \frac{\sin 2Ka}{2K}) + \frac{2\pi}{\sqrt{B}} A_1^2 \sin^2 Ka$$

$$= 2\pi A_1^2 \left\{ a + \sin^2 Ka \left(\frac{1}{\sqrt{B}} - \frac{\text{ctn } Ka}{K} \right) \right\} .$$

From equation (13), $\text{ctn } Ka = -\dfrac{\sqrt{B}}{K}$, $\sin^2 Ka = \dfrac{K^2}{B + K^2}$.

Therefore

$$1 = 2\pi A_1^2 \left\{ a + \frac{K^2}{B + K^2} \left(\frac{1}{\sqrt{B}} + \frac{\sqrt{B}}{K^2} \right) \right\} = 2\pi A_1^2 \left\{ a + \frac{K^2}{B+K^2} \left(\frac{K^2 + B}{K^2 \sqrt{B}} \right) \right\}$$

$$1 = 2\pi A_1^2 \left\{ a + \frac{1}{\sqrt{B}} \right\} = \frac{2\pi A_1^2}{\sqrt{B}} \quad (1 + a\sqrt{B}$$

$$A_1^2 = \frac{\sqrt{B}}{2\pi \, (1 + a\sqrt{B})} \qquad \text{(E1)}$$

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