AMERICAN UNIVERSITY OF BEIRUT

BOOLEAN-LIKE SEPARATION PROPERTIES IN RINGS

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ABSTRACT

The object of this thesis is to study separation properties of families of prime ideals in rings, similar to properties of some families of prime ideals in Boolean algebras. As such properties in Boolean algebras are connected with the ordering on the algebra, an ordering must be imposed on a ring before separation properties can be studied. Because of the nature of this study, all concepts defined in rings are analogues of similar concepts in Boolean algebras. Also all results found in rings reduce to known results in Boolean algebras.

As sections I to III of Ch. I are introductory to the study, a more detailed statement of the object of this thesis is placed as section IV of Ch. I.

CONVENTIONS AND NOTATIONS

$\subset$	indicates proper inclusion.
$\subseteq$	indicates ordinary inclusion.
s.t.	means such that
ring	means commutative ring with unity.
Z	stands for the set of all integers.
N	stands for the set of all positive integers.
Rf.	stands for reference.
x in P	stands for $x \in P$, since the letter $\in$ is not available.
x not in P	stands for $x \notin P$, for the same reason.

CHAPTER I

INTRODUCTION

I. Definitions and Facts Concerning Boolean Algebras

Boolean Algebra

Preceding the definition of a Boolean algebra, some concepts must be defined.

1. Definition: A partially ordered set is a set S with a binary relation $\leqslant$, satisfying the following conditions:

(i) For all x in S, $x \leqslant x$.

(ii) For all x, y, z in S, if $x \leqslant y$ and $y \leqslant z$ then $x \leqslant z$.

(iii) For all x, y in S, if $x \leqslant y$ and $y \leqslant x$ then $x = y$.

2. Definition: A lattice L is a partially ordered set in which every two elements have a greatest lower bound and a least upper bound. For all a, b in L, the greatest lower bound of a and b is denoted by ab and the least upper bound by $a + b$.

3. Definition: A lattice L is distributive iff for all x,y,z in L

$$x(y+z) = xy+xz \quad \text{and} \quad x+yz = (x+y)(x+z).$$

4. Notation: If a lattice L has a smallest element, that element will be denoted by 0. If it has a largest element, that element will be denoted by 1.

5. Definition: A lattice L with 0 and 1 is called complemented iff for all x in L, there exist $\bar{x}$ in L such

that $x + \bar{x} = 1$ and $x\bar{x} = 0$. ($\bar{x}$ is called the complement of x).

6. Definition: A Boolean algebra is a distributive complemented lattice.

However a Boolean algebra has another definition which can be proved equivalent to this one. The other definition is the following:

A Boolean algebra is an algebraic system with two selected elements, 0 and 1, and with two binary operations, $+$ and $\cdot$, and one unary operation, $\bar{\ }$, satisfying the following identities:

$$x + y = y + x \qquad xy = yx$$

$$x + (y + z) = (x + y) + z \qquad x(yz) = xy)z$$

$$x + x = x \qquad xx = x$$

$$x + xy = x \qquad x(x+y) = x$$

$$x(y + z) = xy + xz \qquad x + yz = (x+y)(x+z)$$

$$x + 0 = x \qquad x1 = x$$

$$x0 = 0 \qquad x + 1 = 1$$

$$x + \bar{x} = 1 \qquad x\bar{x} = 0.$$

7. Remark: In a Boolean algebra, complementation is unique.

Prime Ideals in a Boolean Algebra

8. Definition: An ideal I of a Boolean algebra B is a nonempty subset of B such that,

for all x, y in I, $x + y$ in I

and for all b in B, x in I, bx in I.

9. <u>Definition</u>: An ideal P of a Boolean algebra B is a prime ideal iff $P \neq B$ and

for all x,y in B such that xy in P, x in P or y in P.

10. <u>Definition</u>: An ideal M of a Boolean algebra B is a maximal ideal iff $M \neq B$, and if I is any ideal of B such that $I \supseteq M$, then $I = B$ or $I = M$.

11. <u>Theorem</u>: Let I be an ideal of a Boolean algebra B. The following conditions are equivalent:

(i) I is a prime ideal

(ii) I is a maximal ideal

(iii) For all x in B, x in I or $\bar{x}$ in I, but not both.

<u>Stone Families of Prime Ideals of a Boolean Algebra</u>

12. <u>Definition:</u> A family F of prime (hence maximal) ideals of a Boolean algebra B is said to be a Stone family iff $\bigcap_{I \in F} I = (0)$.

13. <u>Theorem</u>: Let F be a family of prime ideals of a Boolean algebra B. The following conditions are equivalent:

(i) F is a Stone family

(ii) For all x in B, $x \neq 0$, there exist I in F s.t. x not in I.

(iii) For all x,y in B s.t. $x \nleq y$, there exist I in F s.t. y in I and x not in I.

14. <u>Remark</u>: Let F be a family of prime ideals of a Boolean algebra B. The following conditions are equivalent:

(i) For all x in B s.t. $x \neq 0$, there exist I in F s.t. x not in I

(ii) For all x in B s.t. $x \neq 1$, there exist I in F s.t. x in I.

Proof:

Let F be a family of prime ideals of B with property (i). Let x in B s.t. $x \neq 1$. Then $\bar{x} \neq 0$, for $\bar{x} = 0$ implies $x = 1$, contradiction (I, 7), so there exist I in F s.t. $\bar{x}$ not in I, hence x in I, since I is prime (I, 11) $\therefore$ F has property (ii) and (i) implies (ii).

Similarly, let F be a family of prime ideals of B with property (ii). Let x in B s.t. $x \neq 0$, then $\bar{x} \neq 1$, so there exist I in F s.t. $\bar{\bar{x}}$ in I, hence x not in I and F has property (i), or (ii) implies (i).

15. Remark: From the above remark, and (I, 13), it follows that a family F of prime ideals of a Boolean algebra B is a Stone family iff

Fro all x in B, $x \neq 1$, there exist P in F s.t. $x \in P$,

i.e. $\{x \in B: x \neq 1\} \subseteq \bigcup_{P \in F} P$

In fact, since for any P in F, 0 in P, then 1 not in P for all P in F, or 1 not in $\bigcup_{P \in F} P$, hence,

$$\{x \in B: x \neq 1\} = \bigcup_{P \in F} P$$

iff F is a Stone family of prime ideals of B.

16. Remark: The family of all prime ideals of a Boolean algebra is a Stone family. In any finite Boolean algebra B, the family of all prime ideals is the only Stone family. However every infinite Boolean algebra has Stone families which do not contain all the

prime ideals of the algebra [Rf. 1].

Boolean Algebras and their Boolean Rings

17. Definition: A ring R is a Boolean ring iff for all x in R, $xx = x$.

18. Remarks:

(i) Every Boolean ring is commutative

(ii) If R is a Boolean ring, then for all x in R, $x + x = 0$.

The following theorem establishes a one to one correspendence between Boolean rings with unity and Boolean algebras.

19. Theorem: Let R be a Boolean ring with unity. Then R can be made into a Boolean algebra by defining $x \leq y$ iff $xy = x$. By this definition, Boolean product is ring product and the Boolean sum of two elements x and y is $x + y + xy$. The complement of an element x is $1 + x$. The zero and unit element of the Boolean algebra are the zero and unit element of the Boolean ring, respectively.

Let B be a Boolean algebra. Then B can be made into a Boolean ring with unity by defining ring product as Boolean product and ring sum of x and y as $x\bar{y} + \bar{x}y$. [Rf. 1].

20. Remarks:

(i) A subset I of a Boolean algebra B is an ideal in B iff it is an ideal in the corresponding Boolean ring.

(ii) An ideal I of a Boolean algebra B is prime (maximal) iff I is prime (maximal) in the corresponding Boolean ring.

(iii) It follows from (i) and (ii) that in a Boolean ring an ideal is prime iff it is maximal.

(iv) A family F of prime ideals of a Boolean algebra B is a Stone family iff the family of corresponding prime ideals in the Boolean ring of B has the property that the intersection of all its elements is 0.

II. Definitions and Facts Concerning Commutative Rings with Unity

1. **Definition:** An ideal P of a ring R is a prime ideal of R iff $P \neq R$ and for all x,y in R s.t. xy in P, x in P or y in P.

2. **Definition**: For any finite number of ideals $A_1, A_2, \ldots, A_n$ of a ring R, the product of these ideals, $\prod_{i=1}^{n} A_i$, is the set of all finite sums of products $a_1 a_2 \ldots a_n$ where a_i in A_i for all $i = 1, \ldots, n$.

3. **Remarks**:

(i) P is a prime ideal of R, iff for all ideals B, C of R, $BC \subseteq P$ implies $B \subseteq P$ or $C \subseteq P$.

(ii) Using mathematical induction, it follows from (i), that if P is a prime ideal of R, and $A_1, A_2, \ldots, A_n$ are ideals of R s.t. $\prod_{i=1}^{n} A_i \subseteq P$, then $A_i \subseteq P$ for some i, $1 \leq i \leq n$.

(iii) For any finite number of ideals of R, $A_1, \ldots, A_n$, the product of these ideals is contained in their intersection.

4. **Definition**: An ideal M of a ring R is a maximal ideal of R iff $M \neq R$ and for all ideals I of R s.t. $I \supseteq M$, $I = M$ or $I = R$.

5. **Remark**: In a ring R, every maximal ideal is prime, but not every prime ideal is maximal.

6. <u>Definition</u>: Let A be an ideal of a ring R. A prime ideal P of R is a minimal prime divisor of A iff $P \supseteq A$ and if P' is a prime ideal s.t. $P \supseteq P' \supseteq A$, then $P' = P$.

7. <u>Theorem</u>: If an ideal A of a ring R is contained in a prime ideal P of R, P contains a minimal prime divisor P' of A.

Proof:

Consider $S = \{Q : Q \text{ prime ideal of } R, A \subseteq Q \subseteq P\}$. Partial order S by the opposite of set inclusion. Let C be a chain in S, consisting of

$$P_1 \supseteq P_2 \supseteq \cdots \supseteq \bigcap P_i .$$

It can easily be shown that $\bigcap P_i$ is a prime ideal containing A, hence every chain of S is bounded above, and by Zorn's Lemma, S has a maximal element P' with respect to the partial ordering,

i.e. $P' \supseteq A$ and if Q in S s.t. $P' \supseteq Q \supseteq A$,

$Q = P'$. Therefore P' is a minimal prime divisor of A contained in P.

8. <u>Definition</u>: A prime ideal P of R is a minimal prime ideal iff P is a minimal prime divisor of (0), i.e. iff P is a prime ideal of R which contains no smaller non-zero prime ideal.

9. <u>Definition</u>: An element x in a ring R is nilpotent iff $x^n = 0$ for some n in N.

10. <u>Definition</u>: Let I be an ideal of a ring R. Then x in R is nilpotent mod I iff x^n in I for some n in N.

11. <u>Definition</u>: Let I be an ideal of a ring R. The radical of I (denoted Rad I) is the intersection of all prime

ideals of R containing I. Hence for any ideal I of R, Rad I is an ideal.

12. <u>Remarks</u>:

(i) Rad (0) is the intersection of all prime ideals containing 0, hence the intersection of all prime ideals of R.

(ii) For any ideal I of R,

$$\text{Rad } I = \{x : x \text{ nilpotent mod } I\}.$$

Hence $\text{Rad}(0) = \{x : x \text{ nilpotent mod } (0)\}$

$$= \{x : x \text{ nilpotent in } R\}.$$ [Rf.3, p.5].

For any ring R, Rad (0) is called the radical of the ring and denoted by Rad R. Hence, for any ring R, Rad R is the intersection of all prime ideals of R.

13. <u>Theorem</u>: The radical of any ideal I of R is the intersection of all minimal prime divisors of I. [Rf. 2, p. 104].

14. <u>Definition</u>: An element x of a ring R is quasi-regular (q.r.) iff there exist x' in R s.t. $x + x' + xx' = 0$.

15. <u>Definition</u>: Let I be an ideal of a ring R, then x in R is quasi-regular mod I (q.r. mod I) iff there exist x' in R s.t. $x + x' + xx'$ in I.

16. <u>Definition</u>: Let I be an ideal of a ring R. The J-radical of I (denoted J-Rad I) is the intersection of all maximal ideals of R containing I. Hence, for any ideal I, J-Rad I is an ideal.

17. <u>Remark</u>: J-Rad(0) is the intersection of all maximal ideals of R containing (0), hence the intersection of all maximal ideals of R.

18. <u>Definition</u>: The Jacobson radical of a ring R (denoted by J(R))

is the intersection of all maximal ideals of R.

Hence from (17), J(R) = J-Rad (0).

19. <u>Remark</u>: For any ring R,

$$\text{J-Rad}\ (0) = J(R) = \{x \in R : rx \ \text{ q.r.} \ \forall\, r \in R\}.$$

20. <u>Proposition</u>: For any ideal I of a ring R,

$$\text{J-Rad}\ I = \{x \in R : rx \ \text{ q.r. mod } I\}.$$

<u>Proof</u>:

Consider the quotient ring $R/_I$. There is a one to one correspondence between the ideal of R which contain I and the ideals of $R/_I$, such that, under the canonical homomorphism $p:R \to R/_I$, for any ideal A of R, containing I, A corresponds to P(A) [Rf. 4, p. 35]. In fact, an ideal A in R, containing I is maximal iff $p(A) = A'$ is maximal in $R/_I$, for let A be a maximal ideal in R s.t. $A \supseteq I$, and suppose there exist an ideal N in $R/_I$ s.t. $p(A) \subset N \subset R/_I$. This implies that $p^{-1}(p(A)) \subset p^{-1}(N) \subset R$ since p is an inclusion-preserving map, hence $A \subset p^{-1}(N) \subset R$. But $p^{-1}(N)$ is an ideal and A is maximal, contradiction.

Similarly, if B is a maximal ideal in $R/_I$, $p^{-1}(B)$ is an ideal in R, containing I. Let N be an ideal of R s.t. $p^{-1}(B) \subset N \subset R$. Then $B \subset p(N) \subset R/_I$, and p(N) is an ideal, contradiction, since B is maximal.

Therefore, there is a one to one correspondence between all maximal ideals of R, containing I and all maximal ideals of $R/_I$.

$$\text{J-Rad}\ I = \bigcap_{\substack{M \supseteq I \\ M \text{ max} \\ \text{in } R}} M$$

By the above mentioned correspondence of maximal ideals in R and $R/_I$,

$$\text{J-Rad } I \longleftrightarrow J(R/_I)$$

but

$$J(R/_I) = \{x + I : rx + I \text{ is q.r. } \forall\, r \in R\} \quad \text{(II,19)}$$

and it can be trivially shown that for all x in R, rx + I is q.r. in $R/_I$ for all r in R iff rx is q.r. in R for all r in R hence I + x in $J(R/_I)$ iff x is q.r. mod I or

$$\{x : rx \text{ q.r. mod } I\ \forall\, r \in R\} \longleftrightarrow J(R/_I).$$

Since the family of all ideals of $R/_I$ is in one to one correspondence with the family of all ideals of R, containing I, then

$$\text{J-Rad } I = \{x : rx \ \text{ q.r. mod } I\ \forall\, r \in R\}.$$

21. <u>Remark</u>: For any ring R, Rad R $\subseteq$ J(R).

22. <u>Definition</u>: A subset S of a ring R is a multiplicatively closed subset iff for all x,y in S, xy in S.

23. <u>Definition</u>: A binary relation $\prec$ defined on a ring R is a quasi-ordering on R iff

(i) For all x in R, $x \prec x$

(ii) For all x,y,z in R, $x \prec y$ and $y \prec z$ implies $x \prec z$.

24. <u>Remark</u>: Any partial ordering on a ring R is a quasi-ordering on R, but not conversely.

III. <u>Simple Separation Properties Common to Rings and Boolean Algebras</u>

1. <u>Remarks</u>: (i) Let B be a Boolean algebra. Let x be any

nonunit in B(i.e. $x \neq 1$), then there exist a prime (hence maximal) ideal of B containing x. This follows from the fact that the family of all prime ideals of a Boolean algebra is a Stone Family.

(ii) Let R be a ring. Let x be a nonunit in R, then there exist a maximal ideal of R containing x.

Proof:

Let x be a nonunit in R, then $(x) \subset R$ for $(x) = R$ implies $1 = rx$ for some r in R and x unit, contradiction. $(x) \subset R$, so there exist a maximal ideal M of R s.t. $(x) \subseteq M$ hence $1 \cdot x = x$ in M.

2. <u>Remarks:</u> i) Let B be a Boolean algebra, then for all x in R s.t. $x \neq 0$, there exist a prime ideal P of R s.t. x not in P. This follows from the fact that the family of all prime ideals of B is a Stone family. In fact, 2(i) is equivalent to 1(i)(I,14).

(ii) Let R be a ring, then for all x in R s.t. x not in Rad R, there exist a prime ideal P of R s.t. x not in P. This follows from the definition of Rad R as the intersection of all prime ideals of R. However, it will be shown later that 2(ii) and 1(ii) are not equivalent properties.

3. <u>Lemma</u>: Let P be an ideal of R s.t. for all ideals A,I of R, $A \supset P$ and $I \supset P$ implies $AI \not\subseteq P$, then P is a prime ideal of R.

Proof:

Let B, C be 2 ideals of R s.t. $B \not\subseteq P$ and $C \not\subseteq P$. Consider $(P,B) = \{p + b : p \text{ in } P \text{ and } b \text{ in } B\}$ and $(P,C) = \{p + c : p \text{ in } P, c \text{ in } C\}$. (P,B) and (P,C) are ideals.

$$(P,B) \supset P \qquad \text{and} \qquad (P,C) \supset P$$

for $(P,B) = P$ implies $B \subseteq P$ contradiction, then, by hypothesis $(P,B).(P,C) \nsubseteq P$. This implies $BC \nsubseteq P$, for suppose $BC \subseteq P$, then for all

$$x \text{ in } (P,B).(P,C),\ x = (p + b)(p^1 + c)$$

where p, p^1 in P, b in B, and c in C. bc in $BC \subseteq P$ so $x = pp^1 + bp^1 + cp + bc$ is the sum of elements of P, hence x in P and $(P,B).(P,C) \subseteq P$ contradiction, hence $BC \nsubseteq P$. So if B,C ideals of R s.t. $B \nsubseteq P$, $C \nsubseteq P$, then $BC \nsubseteq P$. Therefore P is prime (II,3).

4. Theorem: [Rf. 3, p. 4] Let S be a multiplicatively closed subset of a ring R. If an ideal A of R does not meet S, then there exist an ideal P, maximal with respect to the property that $P \supseteq A$ and P does not meet S. Such a P is necessarily prime.

Proof: Let $F = \{I : I \text{ ideal of } R,\ I \supseteq A,\ I \cap S = \emptyset\}$.

Partial order F by set inclusion. F is an inductive set, hence Zorn's lemma applied on F, implies the existence of a maximal element P of F.

To show P is prime, let B, C be two ideals of R s.t. $P \subset B$ and $P \subset C$. then B,C not in F hence $B \cap S \neq \emptyset$, and $C \cap S \neq \emptyset$, so $BC \cap S \neq \emptyset$, hence BC not in F and $BC \nsubseteq P$.

$\therefore$ P is prime (lemma 3).

5. Corollary: By taking $S = \{1\}$, it follows from theorem 4, that every proper ideal is contained in a maximal ideal.

6. Proposition: Let B be a Boolean algebra. Let S be a subset of B s.t. x,y in S implies xy in S. If an ideal A of B does not meet S, then there exist an ideal P of B, maximal with respect to the property that $P \supseteq A$ and $P \cap S = \emptyset$. P is

necessarily prime.

Proof:

Consider the corresponding Boolean ring R of B. Since ring and algebra products are the same, S is a multiplicatively closed subset of R. A is an ideal in B, hence A is an ideal in R and $A \cap S = \phi$, so by (III, 4) there exist an ideal P of the ring R, maximal with respect to property of not meeting S and containing A, and P is prime in R, hence P is prime in B. (I, 20).

Further, since the ideals in the algebra and the ring are the same, P has the desired maximality property in the algebra.

IV. Object of the Thesis

It is apparent from the above (III), that there are some simple separation properties common to prime ideals in Boolean algebras, and commutative rings with unity. However, prime ideals of a Boolean algebra seem to separate elements in more ways than a ring; for example, in a Boolean algebra, if $x \nleq y$, there exist a prime ideal P in the algebra s.t. y in P but x not in P. If some kind of ordering is defined on a ring, a similar property of prime ideals of the ring could be investigated.

Furthermore, in a Boolean algebra, families of prime ideals are defined, namely Stone families, such that the ideals of one family do not meet at more elements than necessary, hence meet at 0. The prime ideals of a Stone family separate elements according to the partial ordering in the algebra, and the separation of a Stone family is as strong as that of the family of all prime ideals of the algebra.

The object of this thesis is to define a quasi-ordering on a ring, then to study families of prime ideals of a ring, similar to Stone families in Boolean algebras, and investigate how strongly do such families (in a ring) separate elements according to the quasi-ordering defined. One would expect that families of prime ideals in a ring, which resemble Stone families in a Boolean algebra, should be defined in such a way that the ideals of one family meet at no more elements than necessary, namely at Rad R.

As in a ring, unlike the Boolean algebra, not all prime ideals are maximal, the investigation of separation properties, mentioned above, will be done, first using prime ideals and second using maximal ideals only.

CHAPTER II

A QUASI-ORDERING DEFINED ON A RING

I. Ordering Imposed on a Ring

1. Definition: Let R be a ring, then for all a,b in R, $a \prec b$ iff $a^n = cb$ for some n in N and c in R.

2. Remark: If this relation $\prec$ is defined, in the same way, on a Boolean algebra B, i.e. by replacing ring product by algebra product, then, for any two elements a,b in B, $a \prec b$ iff $a \leqslant b$.

Proof:

Let a,b in B s.t. $a \prec b$, then $a^n = cb$ for some c in B and n in N. But $a^n = a = cb \leqslant b$, hence $a \leqslant b$.

Let a,b in B s.t. $a \leqslant b$, then $a = ab$ or $a^1 = ab$ hence $a \prec b$.

3. Proposition: If u is a unit in a ring R, then for all x in R,

(i) $x \prec u$

(ii) $u \prec x$ iff x is a unit in R.

Proof:

(i) Let u be a unit in R, then there exist v in R s.t. $uv = 1$. For all x in R, $x = x^1 = x.1 = x(uv) = (xv)u$, hence $x \prec u$.

(ii) Let u be a unit in R, and let x in R s.t. $u \prec x$, then there exist n in N and r in R s.t. $u^n = rx$, but u unit, so there is v in R s.t. $uv = 1$. Hence $(uv)^n = u^n v^n = 1$. But $u^n = rx$ implies $u^n v^n = rxv^n = 1$. Therefore x is a unit.

Let x be a unit. u in R so by (i) $u \prec x$.

4. <u>Proposition</u>: If y is a nonunit in R, then $x \prec y$ iff x in $\mathrm{Rad}(y)$.

Proof:

Let y be a nonunit in R. Let x in R s.t. $x \prec y$, then $x^n = cy$ for some n in N, and c in R.

Let P be any prime ideal of R containing (y), then $x^n = cy$ in P, hence x in P. Therefore, x in P for all prime ideals P of R, containing (y), and x in their intersection i.e. x in $\mathrm{Rad}(y)$.

Let $x \nprec y$, then $x^n \neq cy$ for all n in N and c in R, hence x^n not in (y) for all n in N. Consider $S = \{x^n : n \text{ in } N\}$. Clearly, S is a multiplicatively closed subset (or system) in R, and $S \cap (y) = \emptyset$, therefore, by (Ch. 1, III, 4) there exist a prime ideal P of R s.t. $(y) \subseteq P$ and $P \cap S = \emptyset$. Since x in S, x not in P, hence x not in $\mathrm{Rad}(y)$.

5. <u>Remark</u>: For any z in Z, $\mathrm{Rad}(z) = (\prod_{p/z} p)$ where the p's are the prime factors of z.

Proof:

Let z be in Z and x in Z s.t. x in $\mathrm{Rad}(z)$, then $x^n = cz$ for some n in N and c in Z. For any prime factor p of z, p is a prime factor of x, so if $p_1, p_2, \ldots, p_t$ are all the prime factors of z, $p_1, p_2, \ldots, p_t$ are prime factors of x and x in $(\prod_{p/z} p)$.

Let x in $(\prod_{p/z} p)$. Let the unique factorization of z into the product of primes be $z = p_1^{e_1} p_2^{e_2} \ldots p_t^{e_t}$, where $e_1, \ldots, e_t$ in N.

Now $x = rp_1p_2 \cdots p_t$ for some r in R, so x raised to the summation of the e_i's is some multiple of z, hence x in $Rad(z)$.

6. Remark: In a ring R, $x \prec 0$ iff x is nilpotent, for let $x \prec 0$, then $x^n = c.0$ for some n in N and c in R. $x^n = c.0 = 0$ hence x nilpotent. Let x be nilpotent, $x^n = 0 = 1.0$ for some n in N hence $x \prec 0$.

7. Remark: The definition of a prime ideal P in a ring R, given in the introduction, requires that $P \subsetneq R$. However, relaxing this definition, to consider R itself as a prime ideal of R, has the advantage that for any unit u in R, $Rad(u) = R = \{x : x \prec u\}$ (Ch. 2, I, 3) and hence propositions 3 and 4 can be combined into the following assertion.

For all r in R, $x \prec r$ iff x in $Rad(r)$.

Order Properties of the relation

In this section, the nature of the order imposed on a ring R, by the relation $\prec$, will be studied. Some reference will be made, to the extent that the relation $\prec$ on a ring resembles the partial ordering $\leq$ on a Boolean algebra.

8. Proposition: The relation $\prec$, defined on a ring R is a quasiordering on R.

Proof:

This involves showing reflexivity and transitivity.

Let x in R, $x^1 = 1.x$ hence $x \prec x$.

Let x,y,z in R s.t. $x \prec y$ and $y \prec z$. Then there exist m,n in N and c,d in R s.t. $x^m = cy$ and $y^n = dz$, then $(x^m)^n = (cy)^n$ or $x^{mn} = c^n y^n = c^n(dz) = (c^n d)z$. Therefore $x \prec z$.

9. <u>Remark</u>: In general, the relation $\prec$ defined on a ring R is only a quasi-ordering and not a partial ordering on R i.e. $x \prec y$ and $y \prec x$ do not imply that $x = y$. As a counterexample, consider 6, 12 in Z. $6^2 = 36 = 3.12$ hence $6 \prec 12$. Also $12^1 = 2.6$ so $12 \prec 6$ yet $6 \nmid 12$.

However, if a new relation $\sim$ is defined on R by $a \sim b$ iff $a \prec b$ and $b \prec a$, then $\sim$ is an equivalence relation, and $\prec$ induces a partial ordering on the ring of equivalence classes of R, under $\sim$. The relation $\sim$ will be discussed in more detail, in section II of this chapter.

10. <u>Proposition</u>: Let R be a ring, and x,y in R s.t. $x \prec y$, then for all z in R, $xz \prec yz$.

Proof:

Let $x \prec y$, then $x^n = cy$ for some n in N and c in R. $x^n z^n = cyz^n$ or $(xz)^n = (cz^{n-1})(yz)$, therefore $xz \prec yz$.

11. <u>Corollary</u>: Let R be a ring, and x,y,a,b in R s.t. $x \prec a$ and $y \prec b$, then $xy \prec ab$.

Proof:

By the above proposition, $x \prec a$ implies $xy \prec ay$ and $y \prec b$ implies $ay \prec ab$. Hence, by trasitivity, $xy \prec ab$,.

In any lattice L, for any two elements a,b in L there exist a greatest lower bound (denoted by ab) and a least upper bound (denoted by $a + b$). The property of ab in L is that $ab \leqslant a$ and $ab \leqslant b$, and further, if u is any element of L s.t. $u \leqslant a$ and $u \leqslant b$, then $u \leqslant ab$. The property of $a + b$ is that $a \leqslant a + b$ and $b \leqslant a + b$, and further if v is any element of L s.t. $b \leqslant v$ and $a \leqslant v$, then $a + b \leqslant v$.

The following will show, that with respect to the quasi-ordering $\prec$, defined on a ring R, any two elements of R will have a greatest lower bound, but no least upper bound, in the sense of that in a lattice.

12. <u>Proposition</u>: For all a,b in any ring R, the following holds:

(i) $ab \prec a$ and $ab \prec b$.

(ii) For all r in R s.t. $r \prec a$ and $r \prec b$, $r \prec ab$.

Proof:

(i) $(ab)^1 = (b)a$ hence $ab \prec a$. Similarly $ab \prec b$.

(ii) Let r in R s.t. $r \prec a$ and $r \prec b$, then there exist m,n in N and c,d in R s.t. $r^m = ca$ and $r^n = db$. Hence $r^{m+n} = r^m.r^n = ca.db = (cd)(ab)$. Therefore $r \prec ab$.

13. <u>Proposition</u>: For all a,b in a ring R, if r in R s.t. $a \prec r$ and $b \prec r$, then $a + b \prec r$ and $a + b + ab \prec r$.

Proof:

If r is a unit, then $a + b \prec r$ and $a + b + ab \prec r$ (Ch. 2,I,3).

If r is not a unit, then $a \prec r$ and $b \prec r$ imply a,b in Rad(r) (Ch. 2, I, 4) hence $a + b$ in Rad(r) and $a + b + ab$ in Rad(r). Therefore $a + b \prec r$ and $a + b + ab \prec r$.

<u>Note</u>: In the above proposition, the second part $(a + b + ab \prec r)$ was included, since a Boolean ring is transformed into a Boolean algebra by defining the algebra sum of two elements x and y as $x + y + xy$ (Ch. 1, I, 19). So one might expect that in a ring, $a + b + ab$, if not $a + b$, will be the least upper bound of a and b, relative to the relation $\prec$. However, as will be seen from the following, the desired analogy does not hold.

14. Remark: In general, for a,b in R, $a \not\prec a + b$, and $a \not\prec a + b + ab$. As counterexamples, consider 2 and 3 in Z. $2 + 3 = 5$ but $2^n \neq 5z$ for all n in N and z in Z (since 5 does not divide 2). Hence $2 \not\prec 2 + 3$. Also $2 + 3 + 2.3 = 11$, and $2^n \neq 11z$ for all n in N and z in Z (since 11 does not divide 2). Hence $2 \not\prec 2 + 3 + 2.3$.

II. Equivalence Relation Derived from $\prec$, Defined on a Ring

1. Definition: Let a,b in R, then $a \sim b$ iff $a \prec b$ and $b \prec a$.

2. Remark: It was proved (Ch. 2, I, 2), that in a Boolean algebra, the relation $\prec$, defined on the algebra is equivalent to the partial ordering $\leq$ of the algebra. Hence the relation $\sim$ defined on a Boolean algebra, is the equality relation on the algebra. However, since it was proved, that in general, the relation $\prec$, defined on a ring is not a partial ordering (Ch. 2, I, 9) then in a ring $\sim$ is not necessarily the equality relation.

3. Proposition: The relation $\sim$, defined on a ring R, is an equivalence relation on R.

Proof:

Let x in R, then $x \prec x$ hence $x \sim x$.

Let x,y in R s.t. $x \sim y$ then $x \prec y$ and $y \prec x$ hence $y \sim x$.

Let x,y,z in R s.t. $x \sim y$ and $y \sim z$, then $x \prec y$, $y \prec x$, $y \prec z$ and $z \prec y$, so by transitivity of $\prec$, it follows that $x \prec z$ and $z \prec x$ or $x \sim z$. Therefore $\sim$ is an equivalence relation on R.

4. Proposition: If y is a unit in R, then for all x in R, $x \sim y$ iff x is a unit.

Proof:

$x \sim y$ so $x \prec y$ and $y \prec x$. But y unit hence by (Ch. 2, I, 3)

x is a unit.

Now, let x be a unit in R, then by (Ch. 2, I, 3), $y \prec x$ and $x \prec y$, hence $x \sim y$.

5. <u>Proposition</u>: Let x and y be nonunits in a ring R, then $x \sim y$ iff $\mathrm{Rad}(x) = \mathrm{Rad}(y)$.

Proof:

Let $x \sim y$, then $x \prec y$ and $y \prec x$, so by (Ch. 2, I, 4) $x \prec y$ implies x in $\mathrm{Rad}(y)$. Hence x is in every prime ideal of R containing y and
$$\bigcap_{\substack{P \text{ prime} \\ \text{ideal} \\ P \supseteq (x)}} P \subseteq \bigcap_{\substack{P \text{ prime} \\ \text{ideal} \\ P \supseteq (y)}} P$$

i.e. $\mathrm{Rad}(x) \subseteq \mathrm{Rad}(y)$.

Similarly, it can be shown that $\mathrm{Rad}(y) \subseteq \mathrm{Rad}(x)$, therefore $\mathrm{Rad}(x) = \mathrm{Rad}(y)$.

Conversely, let $\mathrm{Rad}(x) = \mathrm{Rad}(y)$ then x in $\mathrm{Rad}(x) \subseteq \mathrm{Rad}(y)$, hence $x \prec y$. Similarly $y \prec x$, therefore $x \sim y$.

6. <u>Definition</u>: Let $R/\sim$ be the set of equivalence classes of a ring R with respect to $\sim$. Then for $[a]$, $[b]$ in $R/\sim$, $[a] \prec [b]$ iff $a \prec b$ in R.

7. <u>Proposition</u>: Let R be a ring, then the relation $\prec$ defined on $R/\sim$ is a well-defined partial ordering on $R/\sim$.

Proof:

Let $[a'] = [a]$ and $[b'] = [b]$ and suppose that $[a] \prec [b]$, then $a^1 \sim a$, $b^1 \sim b$ and $a \prec b$ in R. Hence $a' \prec a$ and $b \prec b'$, but the relation $\prec$, defined on R is transitive, hence $a' \prec a$, $a \prec b$ and $b \prec b'$ imply $a' \prec b'$. Therefore $[a'] \prec [b']$ and the relation $\prec$, on $R/\sim$ is well-defined.

Let $[a]$ in $R/\sim$, then $a \prec a$ in R hence $[a] \prec [a]$.

Let $[a]$, $[b]$, $[c]$ in $R/\sim$ s.t. $[a] \prec [b]$ and $[b] \prec [c]$, then $a \prec b$ and $b \prec c$ in R, hence $a \prec c$ and $[a] \prec [c]$. Also, let $[a]$, $[b]$ in $R/\sim$ s.t. $[a] \prec [b]$ and $[b] \prec [a]$, then $a \prec b$ and $b \prec a$ in R, hence $a \sim b$ and $[a] = [b]$. Therefore $\prec$ is a partial ordering on $R/\sim$.

III. Imbedding of a Ring into a Boolean Algebra

A family F of subsets of a set S is called a field of sets iff F is closed under finite set-theoretic union and intersection, and under set-theoretic complementation. Thus, every field of sets is a Boolean algebra.

Conversely, every Boolean algebra B is isomorphic to a field of sets. The latter is obtained by considering the set S of all prime ideals of B and the mapping $f: B \longrightarrow 2^S$, given by

$$f(x) = \{P \text{ in } S : x \text{ not in } P\}.$$

f is a Boolean-algebraic isomorphism, that is, it is 1 - 1, onto, and preserves the 3 Boolean-algebraic operations [Rf. 1, pp. 17, 18].

Attempting to get a similar representation for a ring R, let S be the family of all prime ideals of R, and consider the mapping $f: R \longrightarrow 2^S$, given by $f(x) = \{P \text{ in } S : x \text{ not in } P\}$.

In general f, so defined, is not even a 1 - 1 map, for let x, y in R, s.t. $x \neq y$, yet $x \sim y$. Then $x \prec y$ and $y \prec x$, hence for any P prime ideal of R, x in P iff y in P therefore x not in P iff y not in P and $f(x) = f(y)$. Therefore f is not one to one.

However, consider $R/\sim$, the set of equivalence classes of a ring R, with respect to $\sim$, and consider the mapping

$$g : R/\sim \longrightarrow 2^S$$

given by

$$g[x] = \{P \text{ in } S : x \text{ not in } P\}.$$

Then g is a 1 - 1 map which preserves products. To show this, let $[x], [y]$ in $R/\sim$ s.t. $g[x] = g[y]$. Then for all P prime ideal of R, x not in P iff y not in P hence for all P prime ideal of R, x in P iff y in P and $x \sim y$, which means $[x] = [y]$. Therefore g is 1 - 1. Now, let $[x], [y]$ in $R/\sim$)

$$\begin{aligned} g([x]\cdot[y]) = g[xy] &= \{P \text{ in } S : xy \text{ not in } P\} \\ &= \{P \text{ in } S ; x \text{ not in } P \text{ and } y \text{ not in } P\} \\ &= \{P \text{ in } S : x \text{ not in } P\} \cap \{P \text{ in } S : y \text{ not in } P\} \\ &= g[x] \cap g[y] \end{aligned}$$

If intersection in 2^S is considered as taking products, then we can say that the map g is a product-preserving map.

Further, the intersection of two elements of 2^S corresponds to their greatest lower bound with respect to set-theoretic inclusion. Similarly, since $R/\sim$ is partially ordered by $<$, one might expect that the product $[ab]$ of any two elements $[a], [b]$ in $R/\sim$ correspond to their greatest lower bound with respect to $<$, and this is actually the case. This follows trivially from the fact that in R, ab is the greatest lower bound of a and b with respect to $<$ (Ch. 2, I, 12) and from the definition of $<$ on $R/\sim$.

Also, in 2^S, the union of two elements corresponds to their least upper bound with respect to set-theoretic inclusion. However $[x + y]$ in $R/\sim$ does not correspond to the least upper bound of $[x]$ and $[y]$, with respect to $\prec$, (Ch. 2, I, 14). Unfortunately, we could not find a least upper bound, for any two elements of $R/\sim$; that is, we could not find an element $[z]$ in $R/\sim$ s.t. $g[z] = g[x] \cup g[y]$.

So, in conclusion, we can say that, in general, $R/\sim$ is not lattice, but a partially ordered set where only greatest lower bounds exist, that the mapping g is a 1 - 1 product-preserving map, and that $R/\sim$ can be represented by a family of subsets of 2^S which is closed under finite set-theoretic intersection. Also, $R/\sim$ is imbedded (by g) in 2^S, which is a Boolean algebra.

However, if the discussion is restricted to principal ideal domains, much more can be said.

1. <u>Remarks</u>:

(i) In any principal ideal domain (P.I.D.), the concepts of prime element, factorization, greatest common factor and least common multiple carry through as in the integers. Every two elements of a P.I.D. have a greatest common factor (G.C.D.) and a least common multiple (L.C.M.)

(ii) Any P.I.D. is a unique factorization domain, that is, every element has a unique factorization into primes.

2. <u>Theorem:</u> Let D be a principal ideal domain. Then $D/\sim$ is a distributive lattice with 0 and 1, and the imbedding $g: D/\sim \longrightarrow 2^S$, where S is the set of all prime ideals of D, and

$g(x) = \{P$ in $S : x$ not in $P\}$ is a 1 - 1 lattice homomorphism, that, is, g is a 1 - 1 map which preserves the 2 lattice operations.

Proof:

(g.l.b. will denote greatest lower bound and l.u.b. least upper bound).

In $D/\sim$, $[a] = [b]$ iff the prime ideals which contain a are the prime ideals which contain b. Since D is a P.I.D., then for every prime ideal P of D, $P = (p)$ for some prime element p in D. Hence in $D/\sim$, $[a] = [b]$ iff a and b have the same prime factors and so $[a] = [b] = [\prod_{p/a} p]$

Since D is a ring, it was shown above that in $D/\sim$, $[ab] = $ g.l.b. $([a], [b])$ with respect to the partial ordering $\prec$. For all $[a], [b]$ in $D/\sim$, define $[a] \vee [b] = [\prod_{p/a \text{ and } p/b} p]$. Then $[a] \vee [b]$ is the l.u.b. of $[a]$ and $[b]$, with respect to $\prec$, for $a = \prod_{p/a \text{ and } p/b} p.x$ for some x in D. Hence $a \prec \prod_{p/a \text{ and } p/b} p$ a a b

Similarly, $[b] \prec [a] \vee [b]$. Now, let $[u]$ in $D/\sim$ s.t. $[a] \prec [u]$ and $[b] \prec [u]$. Then $a \prec u$ and $b \prec u$ in D, so $a^m = xu$ and $b^n = yu$ for some m,n in N and x,y in D. Hence the prime factors of u are common prime factors of a and b, so if $u = p_1^{e_1} p_2^{e_2} \dots p_n^{e_n}$, and since $\prod_{p/a \,\&\, p/b} p = r.\, p_1 p_2 \dots p_n$ for some r in D, then, there exist k in N s.t. $\left(\prod_{p/a \,\&\, p/b} p\right)^k = r'u$ for some r' in D. Therefore $\prod p \prec u$ and $[a] \vee [b] \prec [u]$. So for any two elements $[a], [b]$ in $D/\sim$, $[a] \vee [b]$ is their l.u.b. with respect to $\prec$. Therefore $D/\sim$ is a lattice.

Further, consider $[1] = [\prod_{p/1} P]$. For any $[a]$ in $D/\sim$, $a \prec 1$ in D, hence $[a] \prec [1]$, and $[1]$ is the largest element of the lattice $D/\sim$. Also, for any $[a]$ in $D/\sim$, $0^1 = 0.a$ in D, hence $0 \prec a$ in D and $[0] \prec [a]$. Therefore $[0]$ is the smallest element of $D/\sim$.

Since, for any two elements $[a]$, $[b]$ in $D/\sim$, $[a] \cdot [b] = [ab] = [\prod_{p/ab} p] = [\prod_{p/a \text{ or } p/b} p]$ and $[a] \vee [b] = [\prod_{p/a \text{ and } p/b} p]$, then it follows from the properties of "and" and "or", that $D/\sim$ is a distributive lattice.

Now consider the mapping $g: D/\sim \longrightarrow 2^S$ where S is the set of all prime ideals of D, and $g[x] = \{P \text{ in } S : x \text{ not in } P\}$. Then we know g is 1 - 1 and preserves products. Let $[a]$, $[b]$ in $D/\sim$,

$$g([a] \vee [b]) = g[\prod_{p/a \text{ and } p/b} p]$$

that is,

$$g([a] \vee [b]) = \{P \text{ in } S : P = (p),\ p \nmid a \text{ or } p \nmid b\}$$

$$= \{P \text{ in } S : P = (p),\ p \nmid a\} \cup \{P \text{ in } S : P = (p), p \nmid b\}$$

$$= g[a] \cup g[b].$$

Hence g preserves sums. Therefore g is a 1 - 1 lattice homomorphism. However it is not onto 2^S because any non-empty finite subset of S corresponds to no element of $D/\sim$.

CHAPTER III

RADICAL FAMILIES OF PRIME IDEALS OF A RING

In this chapter, families of prime ideals of a ring, analogous to Stone families in Boolean algebras, are investigated. The separation properties of such families with respect to the quasiordering $\prec$, as well as other properties of these families are studied.

I. Radical Families in General

The following proposition will show that the family of all prime ideals of a ring has similar properties as the family of all prime ideals of a Boolean algebra.

1. Proposition: Let R be a ring, then the following holds in R:

(i) For all x in R s.t. $x \not\prec 0$, there exist a prime ideal P of R s.t. x not in P.

(ii) For all x in R s.t. $1 \not\prec x$, there exist a prime ideal P of R s.t. x in P.

(iii) For all x,y in R s.t. $x \not\prec y$, there exist a prime ideal P of R s.t. y in P, x not in P.

Proof:

(i) Let $x \not\prec 0$, then by (Ch. 2, I, 6) x not in Rad R, hence there exist a prime ideal P of R s.t. x not in P.

(ii) Let $1 \not\prec x$, then by (Ch. 2, I, 3) x is a nonunit in R, so (x) is a proper ideal in R, for suppose (x) = R, then 1 = xr for some r in R and $1 \prec x$, contradiction. Since (x) is a proper ideal, it is contained in a maximal, hence prime, ideal P of R. Hence $x = 1.x$ in P.

(iii) Let $x \not\prec y$, then by (Ch. 2, I, 4) x not in Rad(y), so there exist a prime ideal P of R s.t. $(y) \subseteq P$ and x not in P. But 1.y in P, hence y in P, x not in P.

2. <u>Definition</u>: A family F of prime ideals of a ring R is called proper iff F does not contain all the prime ideals of R.

3. <u>Definition</u>: Let F be a non-empty family of prime ideals of a ring R, then F is a Radical family iff $\bigcap_{P \text{ in } F} P = \text{Rad } R$.

4. <u>Remark</u>: It follows from the definition of Rad R, that the family of all prime ideals of a ring R is a Radical family. However there exist proper Radical families of prime ideals of R. As an example, consider the family F of prime ideals of Z, given by

$$F = \{(p) : p \text{ prime in } Z, p \neq 2\}$$

Since no non-zero element of Z has infinitely many prime factors, then the intersection of all elements of F is (0), and as Rad Z = (0), then F is a Radical family. Yet (2) is prime in Z and (2) is not in F.

5. <u>Proposition</u>: A family of prime ideals of a ring, containing a Radical family is itself a Radical family.

Proof:

Let F be a family of prime ideals of a ring R s.t. $F \supseteq S$ where S is a Radical family. Then Rad $R \subseteq P$ for all P in F, hence Rad $R \subseteq \bigcap_{P \text{ in } F} P$. Also $\bigcap_{P \text{ in } F} P \subseteq P$ for all P in S, so

$\bigcap_{P \text{ in } F} P \subseteq \bigcap_{P \text{ in } S} P$. But S is a Radical family, hence $\bigcap_{P \text{ in } S} P = \text{Rad } R$, so Rad R $\subseteq \bigcap_{P \text{ in } F} P \subseteq \bigcap_{P \text{ in } S} P \subseteq$ Rad R. Therefore $\bigcap_{P \text{ in } F} P = \text{Rad } R$, and F is a Radical family.

6. <u>Note</u>: One might expect, that with the above definition of Radical families, one might get a theorem for Radical families, similar to Stone's theorem on Stone families, (Ch. 1, I, 13), that is, one might expect to prove that,

For a family F of prime ideals of a ring R, the following are equivalent:

(i) F is a Radical family

(ii) For all x in R s.t. $x \nless 0$, there exist a prime ideal P in F s.t. x not in P.

(iii) For all x,y in R s.t. $x \nless y$, there exist a prime ideal P in F s.t. y in P, x not in P.

However this equivalence does not hold in general. As a counterexample, consider the family F of prime ideals of Z, given by $F = \{(p) : p \text{ prime in } Z \text{ and } p \neq 2\}$. Then by Remark 3, F is a Radical family. Consider 2, 3 in Z. 2 in (2) and 3 not in (2) hence 3 not in Rad(2), and $3 \nless 2$. Yet there exist no prime ideal of F which contains 2, hence there exist no P in F s.t. 2 in P and 3 not in P.

A weaker form of Stone's theorem does hold in a ring. It is the following:

7. <u>Theorem</u>: Let F be a family of prime ideals of a ring R, then the following conditions are equivalent:

(i) F is a Radical family.

(ii) For all x in R s.t. $x \nless 0$, there exist P in F s.t. x not in P.

(iii) For all x, y in R s.t. $x \nprec 0$ and $x\,y \prec 0$, there exist P in F s.t. y in P, x not in P.

Proof:

(i) implies (ii): Let F be a Radical family, then $\bigcap_{P \text{ in } F} P = \text{Rad } R$. Let x in R s.t. $x \nprec 0$, then x not in Rad R, hence there exist P in F s.t. x not in P.

(ii) implies (iii): Let F be a family of prime ideals of R with the property in (ii). Let x,y in R s.t. $x \nprec 0$ and $x\,y \prec 0$. $x \nprec 0$, hence there exist P in F s.t. x not in P, but $x\,y \prec 0$ so xy in Rad R and xy in every prime ideal of R, hence xy in P but x not in P, hence y in P.

(iii) implies (i): Let F be a family of prime ideals of R with the property in (iii). For any P in F, Rad $R \subseteq P$, hence Rad $R \subseteq \bigcap_{P \text{ in } F} P$. Suppose there exist x in $\bigcap_{P \text{ in } F} P$ s.t. x not in Rad R, then $x \nprec 0$ but $x.0 = 0 \prec 0$ so there exist P in F s.t. 0 in P, x not in P, contradiction. Therefore Rad $R = \bigcap_{P \text{ in } F} P$ and F is a Radical family.

8. <u>Corollary</u>: The family of all minimal prime ideals of a ring R is a Radical family.

Proof:

Let x in R s.t. $x \nprec 0$. Then there exist a prime ideal P of R s.t. x not in P. But 0 in P or the ideal $(0) \subseteq P$, hence by (Ch. 1, II, 7) there exist a minimal prime divisor P_o of 0, contained in P, i.e. $(0) \subseteq P_o \subseteq P$. But a minimal prime divisor of (0) is, by definition, a minimal prime ideal, so P_o is a minimal prime ideal, and x not in P, hence x not in P_o. Therefore, by the above theorem, the family of all minimal prime ideals of R is a

Radical family.

9. Definition: A family F of prime ideals of a ring R is said to have the strong separation property iff F has the property that, for all x,y in R s.t. $x \nmid y$, there exist P in F s.t. y in P, x not in P.

10. Remarks:

(i) The family of all prime ideals of a ring R is a Radical family with the strong separation property (this is a restatement of Remark 4 and Proposition 1). However there exist proper Radical families of prime ideals of R having the strong separation property. An example of such a family is given in item 16 of this section.

(ii) The family of all minimal prime divisors of all principal ideals of a ring R is a Radical family with the strong separation property

Proof of (ii):

Let F be the family of all minimal prime divisors of all principal ideals of a ring R. Then F contains the family S of all minimal prime ideals of R, since these are the minimal prime divisors of the principal ideal (0). By Corollary 8, S is a Radical family, hence F is a Radical family (Proposition 5). Let x,y in R s.t. $x \nmid y$. Then x not in Rad(y), so there exist a prime ideal Q in R s.t. $(y) \subseteq Q$, and x not in Q. Hence there exist P in F s.t. $(y) \subseteq P \subseteq Q$ (Ch. 1, II, 7). But x not in Q so x not in P, while $y = 1.y$ in P. Therefore F has the strong separation property.

11. Proposition: A family F of prime ideals of a ring R has the strong separation property iff, for all y in R

$$\bigcap_{\substack{y \text{ in } P \\ P \text{ in } F}} P = \text{Rad}\,(y)$$

Proof:

Let F be a family of prime ideals of R with the strong separation property. If y is a unit in R, then y is not contained in any prime ideal of R, so $\text{Rad}(y) = R$ (empty intersection is the universe). Also, y not in P for all P in F, so $\bigcap_{\substack{y \text{ in } P \\ P \text{ in } F}} P = R = \text{Rad}(y)$. If y is a nonunit in R, then for all P in F s.t. y in P, $(y) \subseteq P$ and hence $\text{Rad}\,(y) \subseteq P$. So $\text{Rad}(y) \subseteq \bigcap_{\substack{y \text{ in } P \\ P \text{ in } F}} P$.

Suppose there exist x in $\bigcap_{\substack{y \text{ in } P \\ P \text{ in } F}} P$ s.t. x not in $\text{Rad}\,(y)$. Then $x \not\prec y$, so there exist P in F s.t. y in P, x not in P, contradiction. Therefore $\text{Rad}\,(y) = \bigcap_{\substack{y \text{ in } P \\ P \text{ in } F}} P$. Conversely, let F be a family of prime ideals of R s.t. for all y in R, $\bigcap_{\substack{y \text{ in } P \\ P \text{ in } F}} P = \text{Rad}(y)$.

Let x,y in R s.t. $x \not\prec y$, then x not in $\text{Rad}\,(y)$, hence x not in $\bigcap_{\substack{y \text{ in } P \\ P \text{ in } F}} P$, so there exist P in F s.t. y in P, x not in P. Therefore F has the strong separation property.

12. <u>Proposition</u>: If a family F of prime ideals of a ring R has the strong separation property, then F is a Radical family.

Proof:

Let F have the strong separation property. Let x be in R s.t. $x \not\prec 0$, then there exist P in F s.t. 0 in P and x not in P, i.e. there exist P in F s.t. x not in P. Hence, by Theorem 7, F is a Radical family.

13. Proposition: If a family F of prime ideals of a ring R has the strong separation property, then for all y in R s.t. $1 \not\prec y$ (i.e. s.t. y nonunit), there exist P in F s.t. y in P.

Proof:

Let F have the strong separation property. Let y be in R s.t. $1 \not\prec y$, then there exist P in F s.t. y in P, 1 not in P, i.e. there exist P in F s.t. y in P.

14. Proposition: A sufficient condition for a family F of prime ideals of R to have the strong separation property is that, for any y in R and for any prime ideal P of R containing y, there exist Q in F s.t. y in $Q \subseteq P$.

Proof:

Let F be a family of prime ideals of R with the above property. Let x,y be in R s.t. $x \not\prec y$, then x not in Rad(y), so there exist a prime ideal P of R s.t. $(y) \subseteq P$ and x not in P. $y = 1.y$ in P, so there exist Q in F s.t. y in $Q \subseteq P$. But x not in P, hence x not in Q, and F has the strong separation property.

15. Corollary: Let F be a family of prime ideals of R s.t. for all y in R, and P prime ideal of R containing y, there exist Q in F s.t. y in $Q \subseteq P$, then F is a Radical family (follows from items 12 and 14).

16. Example: The following is an example of a proper Radical family of prime ideals of a ring which has the strong separation property.

Consider $Z[x]$, the ring of polynomials in one indeterminate over Z. $Z[x]$ is a unique factorization domain, that is every element a in $Z[x]$ has a unique factorization into primes.

Consider the family F of prime ideals of Z [x] given by,

$$F = \{(p) : p \text{ prime in } Z[x]\}.$$

Then F is a proper Radical family with the strong separation property. To show this, let a in Z [x], then $a = p_1^{e_1} p_2^{e_2} \cdots p_n^{e_n}$, where $p_1, \ldots, p_n$ are primes in Z [x] and $e_1, \ldots, e_n$ in N. Let P be any prime ideal of Z [x] containing a, then p_i in P for some $1 \leqslant i \leqslant n$, hence $(p_i) \subseteq P$. But a in $(p_i) \subseteq P$ and (p_i) in F, hence by Propositions 14 and 12, F is a Radical family with the strong separation property. Also, it can be shown that the ideal (x,2) is prime in Z [x] and (x,2) is not in F, hence F is proper.

17. <u>Proposition</u>: A family F of prime ideals of a ring R has the property that for all y in R s.t. $1 \nmid y$, there exist P in F s.t. y in P iff $\bigcup_{P \text{ in } F} P = M$, where M is the set of nonunits of R.

Proof:

Let F be a family of prime ideals of R. Then F has the property that for all y in R s.t. $1 \nmid y$ there exist P in F s.t. y in P iff every nonunit of R is in some element of F, iff

$$M = \{y : y \text{ nonunit in } R\} \subseteq \bigcup_{P \text{ in } F} P \text{ iff } M = \bigcup_{P \text{ in } F} P$$

(since for any family F of prime ideals of R, $\bigcup_{p \text{ in } F} P$ contains only nonunits, so $\bigcup_{P \text{ in } F} P \subseteq M$)

18. <u>Proposition</u>: Let F be a family of prime ideals of a ring R containing all the maximal ideals, of R. Then F has

the property that every nonunit of R is contained in some element of F.

Proof:

Let F contain all the maximal ideals of R. Let y be any nonunit in R, then y is contained in some maximal ideal M of R (Cha. 1, III, 1 (iii)), and M in F.

<u>II. Relationships among Properties of Families of Prime Ideals of a Ring</u>

In section I, three properties of some families F of prime ideals of a ring R have appeared:

<u>Property 1</u>: For all x in R s.t. $x \not\prec 0$, there exist P in F s.t. x not in P.

<u>Property 2:</u> For all x in R s.t. $1 \not\prec x$, there exist P in F s.t. x in P.

<u>Property 3</u>: (strong separation property) For all x,y in R s.t. $x \not\prec y$, there exist P in F s.t. y in P, x not in P.

In a Boolean algebra any Stone family has all three properties (with $\prec$ replaced by $\leqslant$ of the algebra), and if a family of prime ideals of a Boolean algebra has one of these three properties, it is a Stone family (Ch. 1, I, 13). However, in a ring, the case is different.

In a ring R, a family F of prime ideals of R is a Radical family iff it has Property 1, (Ch. 3, I, 7).

Property 2 is neither necessary nor sufficient for a family of prime ideals of a ring to be a Radical family, that is there exist a Radical family which does not have Property 2 (Counterexample 1) and there exist a family of prime ideals of a ring which has Property 2, but is not a Radical family (counterexample 2).

1. <u>Counterexample</u>: Consider Z, the ring of integers, and the family F of prime ideals of Z given by

$$F = \{(p) : p \text{ prime in } Z,\ p \neq 2\}.$$

By (Ch. 3, I, 4) F is a Radical family. However 2 is a nonunit in Z, so $1 \notin 2$, yet there exist no element of F which contains 2. Therefore, in general, Property 1 does not imply Property 2.

2. <u>Counterexample</u>: Consider Q, the ring of rationals, and its subring R, given by

$$R = \left\{\frac{a}{b} : (a,b) = 1 \text{ and } (b,3) = 1\right\}.$$

Then R is the ring of quotients of Z with respect to the set $S = \{s \text{ in } Z \text{ s.t. } (s,3) = 1\}$. The only prime ideals of R are the principal ideals (0) and (3) [Rf. 4, p.43]. Hence (3) is the only maximal ideal of R. Consider the family F of prime ideals of R, given by $F = \{(3)\}$. Then every nonunit in R is contained in (3), since every nonunit must be in some maximal ideal. Hence F has Property 2. Yet $\bigcap_{P \text{ in } F} P = (3) \neq (0)$ and Rad R $\subseteq$ Rad Q $=$ (0) so Rad R $=$ (0) hence F is not a Radical family. Therefore, in general, Property 2 does not imply Property 1.

3. <u>Proposition</u>: Let R be a ring s.t. for all $a \neq 0$, $a + 1$ or $a - 1$ is not a unit, then any family F of prime ideals of R having Property 2, has Property 1. Hence any family with Property 2 is a Radical family.

Proof:

Let R be such a ring. Let F be a family of prime ideals of R having Property 2. Let a be in R s.t. $a \nmid 0$. Then $a + 1$ or $a - 1$ is a nonunit, i.e. $1 \nmid a + 1$ or $1 \nmid a - 1$, so there exist P in F s.t. $a + 1$ in P or $a - 1$ in P, hence a not in P, for in either case, a in P, implies 1 in P, contradiction. Therefore F has Property 1.

Property 3 (the strong separation property) is sufficient for a family of prime ideals to be a Radical family (Ch. 3, I, 12). However Property 3 is not necessary for a family of prime ideals to be Radical, i.e. Property 1 does not imply Property 3, for suppose that Property 1 implies Property 3, and we know (Ch. 3, I, 13) that Property 3 implies Property 2, hence Property 1 implies Property 2, contradiction (counterexample 1). Therefore, in general, Property 1 does not imply Property 3, i.e. not every Radical family has the strong separation property.

There remains to study the relationship between Properties 2 and 3. By (Ch. 3, I, 13) Property 3 implies Property 2, for any family F of prime ideals of a ring R. However, in general, Property 2 does not imply Property 3, for suppose Property 2 implies Property 3, and we know (Ch. 3, I, 12) that Property 3 implies Property 1, hence Property 2 implies Property 1, contradiction (counterexample 2). In fact, even Properties 1 and 2 together do not imply Property 3. The following counterexample will prove this.

4. <u>Counterexample:</u> Consider the family F of prime ideals of $Z[x]$, given by

$$F = \{(x,p) : p \text{ prime } Z\} \cup \{(p(x)) : p(0) = 1,\ p(x) \text{ prime in } Z[x]\}.$$

The following will show that F has Properties 1 and 2, but does not have Property 3.

It can be easily shown that, for any prime p in Z, the ideal (x,p) is the set of all elements of $Z[x]$ whose constant term is divisible by p. To show that F is a Radical family (i.e. has Property 1), let a(x) be in $\bigcap_{P \text{ in } F} P$ s.t. $a(x) \neq 0$, then for any prime p(x) in $Z[x]$ s.t. $p(0) = 1$, p(x) divides a(x), hence a(x) has infinitely many factors, contradiction. Therefore $\bigcap_{P \text{ in } F} P = (0)$ $=$ Rad $Z[x]$ and F is a Radical family, i.e. has Property 1. To show F has Property 2, let a(x) in $Z[x]$ s.t. $1 \nmid a(x)$, then a(x) is a nonunit, i.e. $a(x) \neq \pm 1$. Let the constant term of a(x) be a_0. If $a_0 = \pm 1$, then a(x) has some prime factor p(x) s.t. $p(0) = 1$ hence a(x) in (p(x)) in F. If $a_0 \neq \pm 1$, then a_0 has some prime factor p in Z, so a(x) in (x,p) in F. Therefore, every element of $Z[x]$ is in some element of F, i.e. F has Property 2. However F does not have Property 3, for consider the prime element $x^2 + 2$ in $Z[x]$. $x^2 + 2$ in (x, 2) in F, and $x^2 + 2$ is contained in no other element of F, so $\bigcap_{\substack{x^2+2 \text{ in } P \\ P \text{ in } F}} P = (x,2)$. But Rad $(x^2 + 2) = (x^2 + 2)$ (since $(x^2 + 2)$ is a prime ideal) and $(x^2 + 2) \neq (x,2)$ for x in (x,2) and x not in $(x^2 + 2)$, hence $\text{Rad}(x^2 + 2) \neq \bigcap_{\substack{x^2+2 \text{ in } P \\ P \text{ in } F}} P$ and F doesn not have Property 3(Ch. 3, I, 11)

III. <u>Finite Radical Families</u>

1. <u>Proposition</u>: Let F be a finite Radical family of prime ideals of a ring R, then for any prime ideal P of R there exist Q in F s.t. $Q \subseteq P$.

Proof:

Let $F = \{Q_1, Q_2, \ldots, Q_n\}$, be a finite Radical family of prime ideals of a ring R. Let P be any prime ideal of R. Then Rad R $\subseteq$ P. But Rad R $= \bigcap_{i=1}^{n} Q_i \subseteq P$, and $\prod_{i=1}^{n} Q_i \subseteq \bigcap_{i=1}^{n} Q_i \subseteq P$, so $Q_i \subseteq P$ for some i between 1 and n. (Ch. 1, II, 3).

2. Corollary: Any finite Radical family of prime ideals of a ring R contains all the minimal prime ideals of R.

Proof:

Let F be a finite Radical family and let P be a minimal prime ideal of R. Then, by the above proposition, there exist Q in F s.t. $Q \subseteq P$. But P is minimal, hence Q = P and P in F.

3. Remark: From the above corollary and (Ch. 3, I, 8), it follows that a ring R has a finite Radical family iff R has a finite number of minimal prime ideals.

4. Proposition: Let F be a finite family of prime ideals of a ring R, with the property that for any nonunit y in R, there exist P in F s.t. y in P, then F contains all the maximal ideals of R, that is, a sufficient condition for a ring R to have finitely many maximal ideals is that R have a finite Radical family with the above-mentioned property.

Proof:

Let F be such a finite Radical family in a ring R, and denote the elements of F by $Q_1, Q_2, \ldots, Q_n$. Suppose there exist a maximal ideal M in R s.t. M not in F. Then $M \not\subseteq Q_1$, $M \not\subseteq Q_2$ ..., $M \not\subseteq Q_n$, hence there exist a in M s.t. a not in Q_i for all i = 1, ..., n [Rf. 4, p. 12]. But a in M, implies a nonunit, contradiction.

5. <u>Proposition</u>: Let F be a finite family of prime ideals of a ring R, with the strong separation property, then, for all y in R, and all P prime ideal of R s.t. y in P, there exist Q in F s.t. y in $Q \subseteq P$. (This is the converse of (Ch. 3, I, 14)).

Proof:

Let y be a nonunit in R, and P be any prime ideal of R containing y. y in P implies $\text{Rad}(y) \subseteq P$. But F has the strong separation property, so $\text{Rad}(y) = \bigcap_{\substack{y \text{ in } Q \\ Q \text{ in } F}} Q$. Since F is finite then the set of all elements of F containing y is finite, so let $Q_1, Q_2, \ldots, Q_n$ be the elements of F which contain y. Then $\bigcap_{i=1}^{n} Q_i \subseteq P$, but $\prod_{i=1}^{n} Q_i \subseteq \bigcap_{i=1}^{n} Q_i \subseteq P$, hence $Q_i \subseteq P$ for some i between 1 and n. (Ch. 1, II, 3).

6. <u>Corollary</u>: Let F be a finite family of prime ideals of a ring R, with the strong separation property, then F contains all the minimal prime divisors of all principal ideals of R.

Proof:

Let y be a nonunit in R, and let P be any minimal prime divisor of (y). Then, by the above propesition, there exist Q in F s.t. y in $Q \subseteq P$. So $(y) \subseteq Q \subseteq P$. But P is a minimal prime divisor of (y), so $Q = P$ and P in F.

7. <u>Remark</u>: For the above corollary, and (Ch. 3, I, 10) it follows that a ring R has a finite family with the strong separation property iff the number of minimal prime divisors of all principal ideals of R is finite.

8. Proposition: Let R be a ring which has a finite family with the strong separation property, then, every maximal ideal of R is a minimal prime divisor of some principal ideal of R.

Proof:

Since R has a finite Radical family with the strong separation property, then the family S of all minimal prime divisors of all principal ideals of R is a finite family (Remark 7). But S is a Radical family with the strong separation property (Ch. 3,I,10), hence S has the property that every nonunit of R is in some element of S. (Ch. 3, I, 3), so by Proposition 4, S contains all the maximal ideals of R.

9. Remarks: In a finite ring R, the following holds:

(i) A family F of prime ideals of R is a Radical family iff F contains all the minimal prime ideals of R.

Proof:

Necessity follows from Corollary 2. To prove sufficiency, let F be a family of prime ideals of R, containing the family S of all minimal prime ideals of R. S is a Radical family (Ch.3,I,8). Hence, by (Ch. 3, I, 5), F is a Radical family.

(ii) A family F of prime ideals of R has the property that every nonunit is in some element of F iff F contains all the maximal ideals of R.

Proof:

Necessity follows from Proposition 4. Sufficiency follows from (Ch. 3, I, 18).

(iii) A family F of prime ideals of R has the strong separation property iff F contains all minimal prime divisors of all principal ideals of R.

Proof:

Necessity follows from Corollary 6. To prove sufficiency, let F be a family of prime ideals of R, containing the family S of all minimal prime divisors of principal ideals of R. S has the strong separation property (Ch. 3, I, 10). Let x, y in R s.t. $x \nmid y$, then there exist P in S s.t. y in P, x not in P. But $S \subseteq F$, hence P in F, and F has the strong separation property.

IV. Radical families in Principal ideal Rings

We have shown (Ch. 3, I, 4) that there exist a proper Radical family in the ring of integers, which is a P.I.R. However, for the family described in (Ch. 3, I, 4), not every nonunit was contained in some element of the family. The following proposition will clarify the situation in general.

1. Propositions: Let R be a P.I.R., and let F be a family of prime ideals of R with the property that every nonunit of R is contained in some element of F, then F is the family of all prime ideals of R.

Proof:

Let P be any prime ideal of R, then P = (p) for some p, prime element of R. p is a nonunit (since it generates a prime ideal) so there exist Q in F s.t. $p \subseteq Q$. But Q is a prime ideal, hence Q = (q) for some prime element q in R. p in (q) implies p = rq for some r in R. Since distinct primes can not divide each other, $p = q$ and $P = (p)$ in F.

2. <u>Proposition</u>: Let R be a finite P.I.R. Then the only Radical family in R is the family of all prime ideals of R.

Proof:

Let F be a Radical family of prime ideals of a finite P.I.R. R. Then F is a finite Radical family, so F contains all the minimal prime ideals of R(Ch. 3, III, 2). But, in a P.I.R. every prime ideal is maximal, hence every prime ideal is a minimal prime ideal. Therefore F is the family of all prime ideals of R.

CHAPTER IV

FAMILIES OF MAXIMAL IDEALS

AND THEIR SEPARATION PROPERTIES

In this chapter, a quasi-ordering will be defined on a ring to play the same role for maximal ideals as the quasi-ordering $\prec$ (defined in Ch. 2) does for prime ideals. Also families of maximal ideals will be defined, analogous to Radical families of prime ideals, and the seperation properties of the former, relative to the new quasi-ordering defined, will be studied.

I. Quasi-ordering $\prec'$ Defined on a Ring

It was shown in the introduction (Ch. 1, II, 20) that for any y in R,

$$J - \text{Rad}(y) = \bigcap_{\substack{y \text{ in } M \\ M \text{ max. in } R}} M = \{x: rx \text{ q.r. mod}(y) \text{ for all } r \text{ in } R\}.$$

This motivates the following definition.

1. <u>Definition</u>: Let R be a ring, then $x \prec' y$ iff rx q.r. mod(y) for all r in R, i.e. $x \prec' y$ iff for all r in R there exist r' in R s.t. $rx + r' + rxr' = cy$ for some c in R (Ch. 1, II, 15).

2. <u>Remark</u>: If the relation $\prec'$ is defined on a Boolean ring, and then, by the usual transformation (Ch. 1, I, 19), on the corresponding Boolean algebra B, then in B, $x \prec' y$ iff $x \leq y$ (where $\leq$ is the partial ordering of the algebra).

Proof:

Let $+$ denote summation in the Boolean ring and $\oplus$ summation in the corresponding Boolean algebra B.

Let x,y in B s.t. $x \prec' y$, then for all r in B there exist r' in B s.t. $rx + r' + rxr' = cy$ for some c in B. But 1 in B so there exist r' in B s.t. $1 \cdot x + r' + 1.xr' = cy$ for some c in B, i.e. $x + r' + xr' = cy$ or $x \oplus r' = cy$, but $x \leq x \oplus r' = cy \leq y$, therefore $x \leq y$.

Let x,y in B s.t. $x \leq y$, then for any r in B, $rx \leq x \leq y$, hence $rx \oplus y = cy$ or $rx + y + rxy = y = 1.y$, therefore $x \prec' y$.

3. <u>Corollary</u>: In any Boolean algebra B, $x \prec y$ iff $x \prec' y$.

Proof:

In any Boolean algebra B, $x \prec y$ iff $x \leq y$ (Ch. 2, I, 2) and by the above remark, $x \prec' y$ iff $x \leq y$, hence $x \prec y$ iff $x \prec' y$.

4. <u>Proposition</u>: Let R be a ring, then $x \prec' y$ iff x in J-Rad(y).

Proof:

If y is a unit, then y is contained in no maximal ideal of R, hence $\text{J-Rad}(y) = \bigcap_{\substack{y \text{ in } M \\ M \text{ max. in } R}} M = R$. Let x in R, then x in J-Rad(y), so $x \prec' y$ implies x in J-Rad(y). Now let x in R = J-Rad(y) then for all r in R,

$$rx + 0 + rx.0 = (rxy^{-1})y$$

hence $x \prec' y$.

If y is a nonunit, then J-Rad(y) $= \{x\colon rx$ q.r. mod(y) for all r in R$\} = \{x\colon x \prec' y\}$ (Ch. 1, II, 20).

5. <u>Corollary</u>: Let R be a ring, and u be any unit in R, then for all x in R, $u \prec' x$ iff x is a unit.

Proof:

Let u be a unit in R. Let x in R s.t. $u \prec' x$, then u in J-Rad(x), hence $1 = uu^{-1}$ in J-Rad(x), J-Rad$(x) = R$ and x is a unit. Now, let x be a unit, then

$$\text{J-Rad}(x) = \bigcap_{\substack{x \text{ in } M \\ M \text{ max. in } R}} M = R$$

(since empty intersection is the universe), hence u in J-Rad x, and $u \prec' x$.

6. <u>Corollary</u>: Let R be a ring in which every prime ideal is maximal, then for all x,y in R, $x \prec' y$ iff $x \prec y$.

Proof:

Let R be such a ring, then for any y in R, J-Rad$(y) =$ Rad(y), so by Proposition 4, and (Ch. 2, I, 4), $x \prec' y$ iff $x \prec y$.

7. <u>Remark</u>: Let R be any ring, then for all x,y in R, $x \prec y$ implies $x \prec' y$, but the converse is not true in general.

Proof:

Let x,y in R s.t. $x \prec y$, then x in Rad(y). Now, for any maximal ideal M of R, M is prime hence Rad$(y) \subseteq M$, so

$$\text{Rad}(y) \subseteq \bigcap_{\substack{y \text{ in } M \\ M \text{ max. in } R}} M = \text{J-Rad}(y),$$

but x in Rad(y) hence x in J-Rad(y) and $x \prec' y$.

To show that the converse is not true, i.e. $x \prec' y$ does not imply $x \prec y$, consider the subring R of $Q[x]$ (Q is the ring of rationals) given by

$$R = \left\{ \frac{a}{b} : a,b \text{ in } Z[x],\ (a,b) = 1,\ b \text{ not in } (x,3) \right\}.$$

Then R is the quotient ring of $Z[x]$ with respect to the set $S = \{b \in Z[x] : b \text{ not in } (x,3)\}$. Also R has a unique maximal ideal $M = (x,3) = \{xc + 3d : c,d \text{ in } R\}$, and all the prime ideals of R are the extended prime ideals of $Z[x]$, which are contained in M (since M contains all the nonunits) [Rf. 4, p. 43]. Consider x, 3 in R. x, 3 in $(x, 3)$, so x is in every maximal ideal containing 3, i.e. x in J-Rad(3), and $x \prec' 3$. Yet 3 in $(3) \subseteq R$, (3) is a prime ideal, and x not in (3), hence x not in Rad(3), i.e. $x \not\prec 3$.

8. Remark: The relation $\prec'$, defined on a ring R is a quasi-ordering on R, yet in general, $\prec'$ is not a partial ordering on R. To show $\prec'$ is quasi-ordering, let x in R, then x in J-Rad(x), hence $x \prec' x$. Let x, y, z in R. s.t. $x \prec' y$ and $y \prec' z$, then x in J-Rad(y) and y in J-Rad(z), so y is in every maximal ideal containing z, and x is in every maximal ideal containing y, hence x is in every maximal ideal containing z, and x in J-Rad(z), then $x \prec' z$.

To show that, in general, $\prec'$ is not a partial ordering, consider 4, 2 in Z. (2) is the only maximal ideal containing each of 4 and 2, hence J-Rad(4) = J-Rad(2) = (2), so $4 \prec' 2$ and $2 \prec' 4$, yet $4 \neq 2$.

It was found in Ch. 2, (Ch. 2, I, 12 and 13) that, with respect to the quasi-ordering $\prec$ defined on a ring, any 2 elements have a greatest lower bound but no least upper bound in the sense of a lattice. A similar result holds with respect to the quasi-ordering $\prec'$.

9. Proposition: For any two elements a,b of a ring R, the following holds:

, (i) ab $\prec'$ a, ab $\prec'$ b, and for any u in R s.t. u $\prec'$ a and u $\prec'$ b, u $\prec'$ ab.

(ii) For any v in R s.t. a $\prec'$ v and b $\prec'$ v, a + b $\prec'$ v and a + b + ab $\prec'$ v.

Proof:

(i) Let a,b in R, then ab is in any maximal ideal containing a, hence ab in J-Rad(a) and ab $\prec'$ a. Similarly, ab $\prec'$ b. Now, let u in R s.t. u $\prec'$ a and u $\prec'$ b, then u in J-Rad(a) and u in J-Rad(b), so u is in any maximal ideal containing either a or b, hence u is in any maximal ideal containing ab. Therefore u in J-Rad(ab) and u $\prec'$ ab.

(ii) Let v in R s.t. a $\prec'$ v and b $\prec'$ v, then a,b in J-Rad(v) which is an ideal, hence a + b in J-Rad(v). Also ab in J-Rad(v) so a + b + ab in J-Rad(v). Therefore a + b $\prec'$ v and a + b + ab $\prec'$ v.

The following is a counterexample to show that for any 2 elements a,b in a ring a $\prec'$ a + b does not always hold, and hence the sum of two elements is not their least upper bound with respect to $\prec'$.

10. Counterexample: Consider 2, 3 in Z. (5) is the only maximal ideal in which 2 + 3 = 5 is contained, so J-Rad(5) = (5), and 2 not in J-Rad(5) so 2 $\not\prec'$ 2 + 3. Similarly, J-Rad(2 + 3 + 2.3) = J-Rad(11) = (11) and 2 not in (11), hence 2 $\not\prec'$ 2 + 3 + 2.3.

11. Remark: Since in any P.I.D. every prime ideal is maximal, the theorem (Ch. 2, III, 2) can be proved using $\prec'$ instead of $\prec$.

II. Families of Maximal Ideals

The following proposition will show that, the family of all maximal ideals of a ring, has separation properties similar to the families of all prime ideals of a ring or a Boolean Algebra.

1. Proposition: Let R be a ring, then the following holds in R:

(i) For all x in R s.t. $x \not\prec' 0$, there exist a maximal ideal M of R s.t. x not in M.

(ii) For all x in R s.t. $1 \not\prec' x$, there exist a maximal ideal M of R s.t. x in M

(iii) For all x,y in R s.t. $x \not\prec' y$, there exist a maximal ideal M of R s.t. y in M, x not in M.

Proof:

(i) Let $x \not\prec' 0$, then x not in J-Rad(0), but

$$\text{J-Rad}(0) = J(R) = \bigcap_{\substack{M \text{ max.} \\ \text{in } R}} M,$$ so x not in J(R), hence x not in M for some maximal ideal M of R.

(ii) Let $1 \not\prec' x$, then x nonunit (Ch. 4, I, 5), so there exist a maximal ideal M of R s.t. x in M (Ch. 1, III, 1).

(iii) Let $x \not\prec' y$, then x not in J-Rad(y) (Ch. 4, I, 4), so there exist a maximal ideal M of R s.t. $(y) \subseteq M$, and x not in M, i.e. s.t. y = 1.y in M, x not in M.

2. Definition: Let R be a ring. A family F of maximal ideals of R is a J-family iff $\bigcap_{M \text{ in } F} M = J(R)$.

Using this definition, one might expect to get a theorem for families of maximal ideals of a ring similar to the theorem (Ch. 3, I, 7) obtained for families of prime ideals of a ring. In fact, such a theorem can be asserted.

3. <u>Theorem</u>: Let F be a family of maximal ideals of a ring R, then the following are equivalent:

(i) F is a J-family

(ii) For all x in R s.t. $x \not\prec' 0$, there exist M in F s.t. x not in M.

(iii) For all x, y in R s.t. $x \not\prec' 0$, $xy \prec 0$, there exist M in F s.t. y in M, x not in M.

Proof:

Replacing $\prec$ by $\prec'$, prime by maximal, and Rad(y) by J-Rad(y), the proof of this theorem carries through exactly like the proof of the analogous theorem for prime ideals (Ch. 3, I, 7).

4. <u>Remark</u>: The family of all maximal ideals of any ring is a J-family. This follows, either from the definition of $J(R)$ (Ch. 1, II, 18) or from Proposition 1 and the above theorem. However, there exist proper J-families of maximal ideals. As an example, consider the family F of maximal ideals of Z, given by $F = \{(p) : p \text{ prime in } Z, p \neq 2\}$, then $\bigcap_{P \text{ in } F} P = (0)$, since no non-zero element of Z has infinitely many prime factors. Also $J(Z) = \bigcap_{M \text{ max. in } Z} M = \bigcap_{p \text{ prime in } Z} (p) = (0)$, hence F is a J-family and the maximal ideal (2) is not in F, so F is proper.

5. <u>Remarks:</u>

(i) Let R be a ring s.t. $J(R) = \text{Rad } R$, then any J-family of maximal ideals is a Radical family

(ii) Let R be a ring in which every prime ideal is maximal, then a family of prime ideals is Radical iff it is a J-family.

6. <u>Definition</u>: A family F of maximal ideals of a ring R is said to have the strong separation property relative to $\prec'$, iff for all x,y in R s.t. $x \not\prec' y$, there exist M in F s.t. y in M, x not in M (For the sake of brevity, this property will be referred to in this and further sections as the strong separation property).

7. <u>Remark</u>: By Proposition 1, the family of all maximal ideals of any ring has the strong separation property.

8. <u>Remark</u>: If a family F of maximal ideals of a ring has the strong separation property, then F is a J-family. (This is the analogue of the result in (Ch. 3, I, 12) for families of prime ideals.

Proof:

By replacing $\prec$ by $\prec'$, the proof carries through exactly as that of (Ch. 3, I, 12).

9. <u>Remark</u>: If a family F of maximal ideals of a ring R has the strong separation property, then for all x in R s.t. $1 \not\prec' x$ there exist M in F s.t. x in M. (This is the analogue of (Ch. 3, I, 13) concerning families of prime ideals).

Proof:

By replacing $\prec$ by $\prec'$, and prime by maximal, the proof carries through exactly as for (Ch. 3, I, 13).

10. <u>Proposition:</u> A family F of maximal ideals of a ring R has the strong separation property iff, for all y in R,

$\bigcap_{\substack{y \text{ in } M \\ M \text{ in } F}} M = \text{J-Rad}(y)$. (Again, this is the analogue of (Ch. 3, I, 11) concerning families of prime ideals).

Proof:

By replacing Rad(y) by J-Rad(y), prime by maximal, and $\prec$ by $\prec'$, the proof carries through exactly as that of (Ch. 3, I, 11).

So, we conclude from the above, that families of maximal ideals of a ring behave with respect to the quasi-ordering $\prec'$, very similarly to the way families of prime ideals behave with respect to the quasi-ordering $\prec$. This is no surprise since this was intended in the very definition of $\prec'$, of J-families, and of the different separation properties.

III. Relationships among Properties of Families of Maximal Ideals

This section aims to investigate any existing implications among properties of families of maximal ideals, as has section II of Ch. 3 done for families of prime ideals.

In section II of this chapter, three properties of families F of maximal ideals have appeared:

Property 1: For all x in R s.t. $x \not\prec' 0$, there exist M in F s.t. x not in M.

Property 2: For all x in R s.t. $1 \not\prec' x$, there exist M in F s.t. x in M.

Property 3(or the strong separation property): For all x,y in R s.t. $x \not\prec' y$, there exist M in F s.t. y in M, x not in M.

A family F of maximal ideals of a ring R has Property 1 iff F is a J-family (Ch. 4, II, 3).

Property 2 is the property that every nonunit of R is in some element of F. Property 2 is not necessary for a family of maximal ideals to be a J-family i.e. Property 1 does not imply Property 2. The following counterexample will show this.

1. <u>Counterexample</u>: Consider the family F of prime ideals of Z, given by $F = \{(p) : p \text{ prime in } Z, p \neq 2\}$. F is a J-family (Ch. 4, II, 4), and 2 is a nonunit in Z, yet 2 is not contained in any element of F.

However, Property 2 is sufficient for a family F of maximal ideals of a ring R to be a J-family. The follwoing theorem will assert this.

2. <u>Theorem</u>: Let F be a family of maximal ideals of a ring R having Property 2, then F is a J-family (so has Property 1).

Proof:

Let F have Property 2, then every nonunit of R is in some element of F. $J(R) \subseteq M$ for all M in F, hence $J(R) \subseteq \bigcap_{M \text{ in } F} M$. Suppose there exist a in $\bigcap_{M \text{ in } F} M$ s.t. a not in J(R). Then $a \neq 0$, so there exist a maximal ideal M_0 of R s.t. a not in M_0 (Ch. 4,II, 1). Consider the ideal $(a, M_0) = \{ra + m : r \text{ in } R, m \text{ in } M_0\}$. $M_0 \subseteq (a, M_0) \subseteq R$. If $M_0 = (a, M_0)$, then a in M_0, contradiction, hence $M_0 \subset (a, M_0)$, and by the maximality of M_0, $(a, M_0) = R$. 1 is in R, hence there exist r in R and m_0 in M_0 s.t. $ra + m_0 = 1$, then $ra = 1 - m_0$. But a in M for all M in F so $ra = 1 - m_0$ in M for all M in F. Now m_0 in M_0, so m_0 nonunit, and there exist M in F s.t. m_0 in M. But $1 - m_0$ in M, so 1 in M, contradiction. Therefore $J(R) = \bigcap_{M \text{ in } F} M$ and F is a J-family.

Property 3, or the strong separation property, is not necessary for a family of maximal ideals of a ring to be a J-family, that is Property 1 does not imply Property 3. To prove this, suppose Property 1 implies Property 3. By (Ch. 4, II, 9) Property 3 implies Property 2, hence Property 1 implies Property 2, contradiction to counterexample 1. However Property 3 is a sufficient condition for a family of maximal ideals to be a J-family (Ch. 4,II, 8).

There remains to study the relationship between Properties 2 and 3. By (Ch. 4, II, 9), Property 3 implies Property 2. However we do not know whether, in general, Property 2 implies Property 3 or not.

So, in conclusion, the only other existing implications among properties of families of maximal ideals are,

Property 2 implies Property 1

and

Property 3 implies Property 2.

IV. Some Special Cases of Families of Maximal Ideals

1. Proposition: Let F be a finite J-family of maximal ideals of a ring R, then F is the family of all maximal ideals of R.

Proof:

Let M be any maximal ideal of R, then $J(R) \subseteq M$. But F is a finite J-family, hence $J(R) = \bigcap_{i=1}^{n} M_i$, where $M_1, M_2, \ldots, M_n$ are the elements of F. So $\bigcap_{i=1}^{n} M_i \subseteq M$, but M is maximal, hence prime, so $M_i \subseteq M$ for some i between 1 and n (Ch. 1, II, 3).

Therefore $M_i = M$ and M in F.

2. Corollary: In a ring R, in which the number of maximal ideals is finite, the only J-family is the family of all maximal ideals of R.

3. Proposition: Let F be a J-family of maximal ideals of a principal ideal ring R, with the property that every nonunit is in some element of F, then F is the family of all maximal ideals of R.

Proof:

Let F be such a family. Since R is a P.I.R., an ideal in R is prime iff it is maximal, hence F is a Radical family (Ch. 4, II, 5). Also F has the property that every nonunit of R is in some element of F, so by (Ch. 3, IV, 1), F is the family of all prime, hence maximal ideals of R.

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