ON THE MIQUEL GEOMETRY OF THE TRIANGLE

By

Faruk Abi-Khuzam

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Approved:

Peter Yff
Advisor

David Singmaster
Member of Committee

Member of Committee

Member of Committee

Date of Thesis Presentation:

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NOTATION AND CONVENTIONS

$A_1A_2A_3$	triangle of reference
α_i	the angle at A_i
a_i	the side opposite A_i, or its length
R	the circumradius of $A_1A_2A_3$
O	the circumcenter of $A_1A_2A_3$
H	the orthocenter of $A_1A_2A_3$
K	the Lemoine point of $A_1A_2A_3$
G	the centroid of $A_1A_2A_3$.
$\sim$	equivalent to
β_i	the angle at X_i of the Miquel triangle $X_1X_2X_3$.
k,p.N	Reference book whose number is k in the entries of bibliography; page N.

ABSTRACT

By an inscribed triangle is meant one whose vertices are on the respective sides of the triangle of reference. The Miquel point of the inscribed triangle is the concurrence point of the three circles, each passing through a vertex of the triangle of reference and two adjacent vertices of the inscribed triangle. Since a triangle can be inscribed in six different ways, it has in general six Miquel points.

The special case of inscribed triangles similar to the triangle of reference is treated in Chapter II and the six points are found to lie on a circle.

Chapter III deals with the general case, and it is found that there are in general twelve points whose Miquel triangle has a given shape. These twelve points lie six by six on two circles inverse with respect to the circumcircle.

Chapter IV introduces the idea of a circumscribed triangle and some properties of the Miquel points of the triangle of reference with respect to the circumscribed triangle are investigated.

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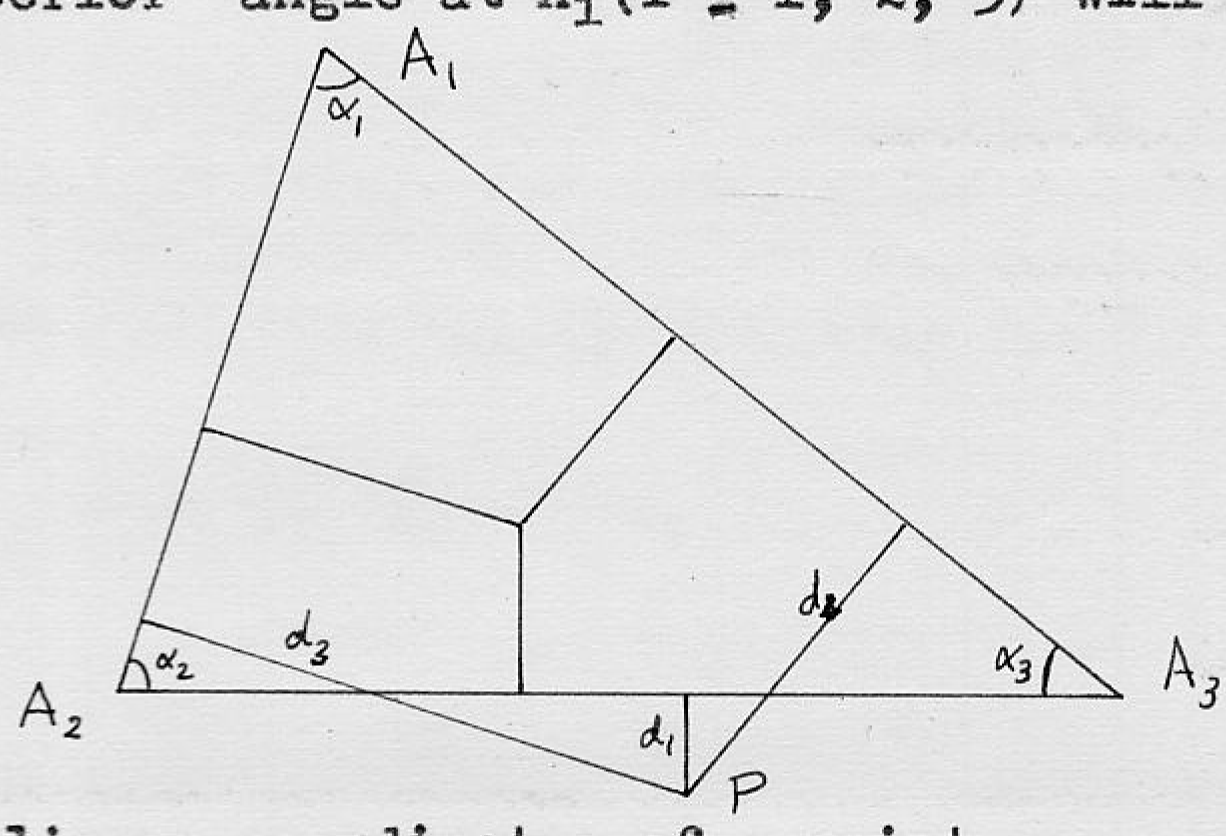
CHAPTER I

INTRODUCTION

1. Trilinear Coordinates

Let P be a point in the plane of a triangle with finite vertices $A_1 A_2 A_3$. The trilinear coordinates of P are the triple (d_1, d_2, d_3) or (kd_1, kd_2, kd_3) $k \neq 0$; d_1, d_2 and d_3 being the distances of point P from the three sides $A_2 A_3$, $A_3 A_1$ and $A_1 A_2$ respectively. Thus two points coincide if, and only if, their respective trilinear coordinates are proportional.

The sides $A_2 A_3$, $A_3 A_1$, $A_1 A_2$ (possibly extended) will be denoted by a_1, a_2, a_3 respectively. In the appropriate context, these letters will also be used for the lengths of the sides. The interior angle at $A_i (i = 1, 2, 3)$ will be designated by α_i.



The trilinear coordinates of a point are never all zero. In general for a point $P(x_1, x_2, x_3)$; x_i is negative if P and A_i are on opposite sides of a_i. For example in Fig. 1 $d_1 < 0$ since A_1 and P are separated by the side a_1. The distances of any point P satisfy:

$$a_1d_1 + a_2d_2 + a_3d_3 = 2\Delta$$

where Δ is the area of the triangle of reference.

This formula is valid for any position of P, provided that the correct signs for the distances are chosen. However not more than one coordinate of a point need be negative; if two or more coordinates are negative, multiplication by -1 leaves at most one negative coordinate.

In this paper two kinds of abbreviation will be employed: for example, $(uv) = (u_iv_i)$ means (u_1v_1, u_2v_2, u_3v_3), while $(u_2v_3) = (u_2v_3, u_3v_1, u_1v_2)$. Similarly $\sum uv = u_1v_1 + u_2v_2 + u_3v_3$, and $\sum u_2 v_3 = u_2v_3 + u_3v_1 + u_1v_2$.

2. Equations of lines and curves

In trilinear coordinates an equation of the form

$$c_1x_1 + c_2x_2 + c_3x_3 = \sum_{i=1}^{3} c_ix_i = 0$$

represents a straight line. For example the equation of the straight line through (b_1, b_2, b_3) and (c_1, c_2, c_3) is:

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0.$$

The equation of $A_2 A_3$ is : $x_1 = 0$.

The equation of the line at infinity is:

$$a_1x_1 + a_2x_2 + a_3x_3 = 0.$$

A <u>conic</u> is the locus of a point $P(x_1, x_2, x_3)$

whose coordinates satisfy the equation:

$$\sum_{i.j=1}^{3} a_{ij}\, x_i x_j = 0$$

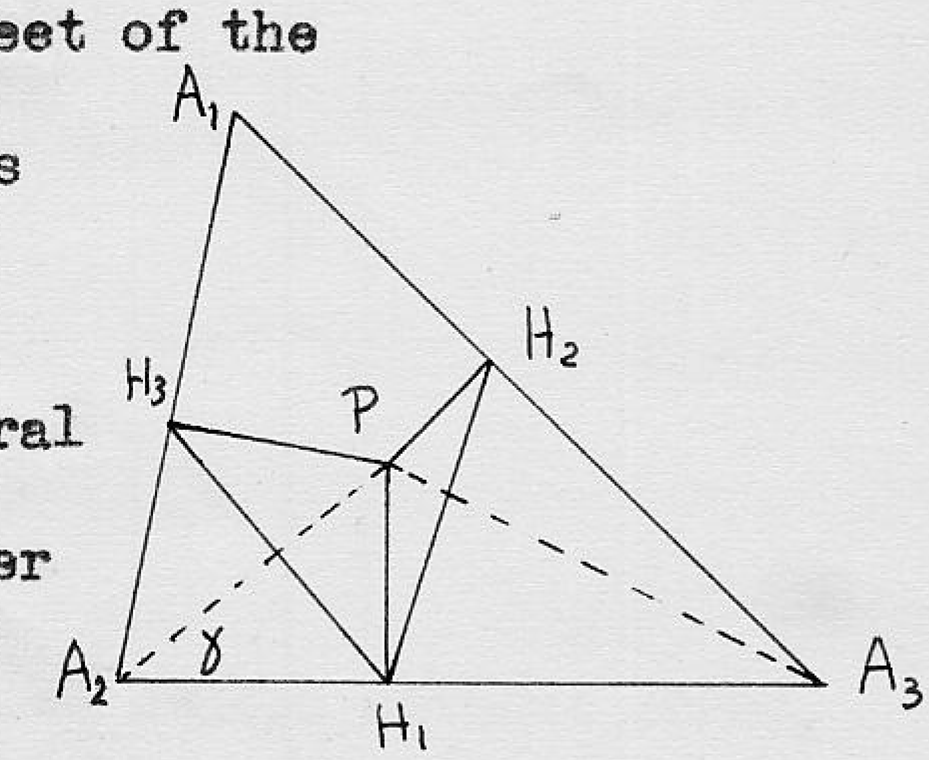
where a_{ij} is identified with a_{ji} when this is expanded it becomes:

$$a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{23}x_2x_3 + 2a_{31}x_3x_1 = 0.$$

We end this discussion by a theorem which relates the trilinear coordinates of a point and the actual distances of that point from the sides of the triangle of reference.

<u>Theorem:</u> The actual distances of (x_1, x_2, x_3) from the sides of the triangle of reference are $\left(\frac{2\Delta x_1}{\sum ax}, \frac{2\Delta x_2}{\sum ax}, \frac{2\Delta x_3}{\sum ax}\right)$.

<u>Proof</u>: Let H_1, H_2 and H_3 be the feet of the perpendiculars from P to the three sides respectively.



$PH_3A_2H_1$ being a cyclic quadrilateral we have $\angle PH_3H_1$ and $\angle PA_2H_1 = \gamma$ are either equal or supplementary, and $\angle PH_1H_3$ and $\angle PA_2H_3$ are either equal or supplementary. In triangle PH_3H_1 we have by the law of sines:

$$\frac{x_3}{x_1} = \frac{PH_3}{PH_1} = \frac{\sin \angle PH_1H_3}{\sin \angle PH_3H_1} = \frac{\sin(\alpha_2 - \gamma)}{\sin \gamma} = \sin \alpha_2 \cot \gamma - \cos \alpha_2$$

$$= \sin \alpha_2 \cdot \frac{A_2H_1}{PH_1} - \cos \alpha_2 \,;$$

simplifying we get $A_2H_1 = \left(\frac{x_3 + x_1 \cos \alpha_2}{x_1 \sin \alpha_2}\right) \cdot PH_1$.

a similar argument in triangle PH_1H_2 gives:

$$H_1A_3 = \left(\frac{x_2 + x_1\cos\alpha_3}{x_1\sin\alpha_3}\right) PH_1 ,$$

$$a_1 = A_2H_1 + H_1A_3 = \frac{PH_1}{x_1}\left(\frac{x_3 + x_1\cos\alpha_2}{\sin\alpha_2} + \frac{x_2 + x_1\cos\alpha_3}{\sin\alpha_3}\right) ,$$

$$a_1 = \frac{PH_1}{x_1}\left(\frac{x_3\sin\alpha_3 + x_2\sin\alpha_2 + x_1\sin\alpha_1}{\sin\alpha_2\ \sin\alpha_3}\right) ;$$

therefore

$$PH_1 = \frac{a_1x_1 . \sin\alpha_2\ \sin\alpha_3}{x_3\sin\alpha_3 + x_2\sin_2 \quad x_1\sin\alpha_1} ,$$

and by the law of sines again:

$$PH_1 = \frac{(a_1a_2\sin\alpha_3)x_1}{a_1x_1 + a_2x_2 + a_3x_3} ,$$

$$PH_1 = \frac{2\Delta . x_1}{\sum ax} .$$

Similarly

$$PH_2 = \frac{2\Delta x_2}{\sum ax} \quad \text{and} \quad PH_3 = \frac{2\Delta x_3}{\sum ax} .$$

The following theorems are basic and will be stated without proof.

<u>Theorem (Miquel)</u>: If a point is marked on each side of a triangle, and through each vertex of the triangle and the marked points on the adjacent sides a circle is drawn, these three circles meet at a point.

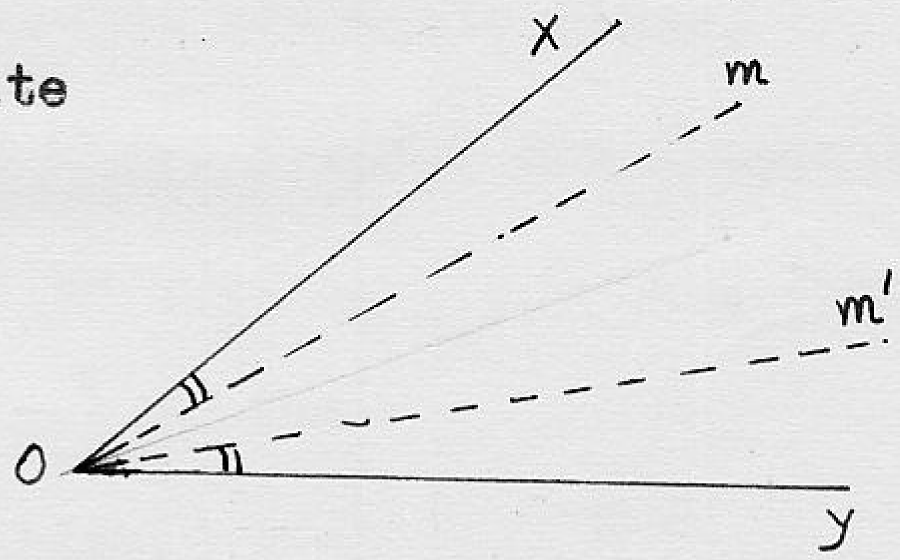
<u>Definition</u>: The point of concurrency whose existence is asserted by the Miquel theorem, is called the Miquel point for the triad of points that were marked, with respect to the given triangle.

<u>Theorem</u>: The lines from the Miquel point to the marked points make equal angles with the respective sides.

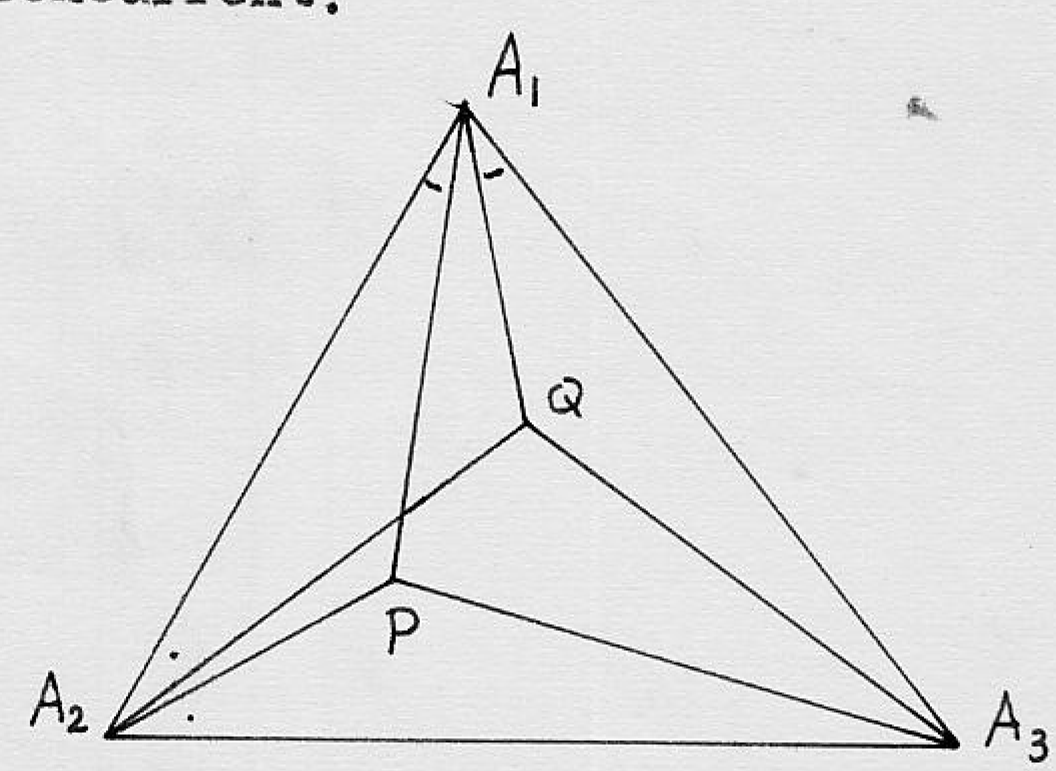
<u>Theorem</u>: If P is any fixed point in the plane of triangle $A_1 A_2 A_3$, it is possible to determine in an infinite number of ways a Miquel triangle for P.

<u>Theorem</u>: All the Miquel triangles of a given point P are directly similar, and P is the center of similitude or self-homologous point in every case.

<u>Definition</u>: Let XOY be an angle. Two lines m and m' from O are said to be isogonal conjugate lines if they are symmetric with respect to the bisector of angle XOY.



<u>Theorem:</u> If three lines from the vertices of a triangle are concurrent, their isogonals are also concurrent.



<u>Proof</u>: Let A_1P and A_1Q,

A_2P and A_2Q,

A_3P and A_3Q

be isogonal respectively; then:

$$\frac{\sin\angle A_2A_1P}{\sin\angle A_3A_1P} = \frac{\sin\angle QA_1A_3}{\sin\angle QA_1A_2},$$

$$\frac{\sin\angle A_1A_3P}{\sin\angle A_2A_3P} = \frac{\sin\angle QA_3A_2}{\sin\angle QA_3A_1},$$

$$\frac{\sin\angle A_3A_2P}{\text{Sin}\angle A_1A_2P} = \frac{\sin\angle QA_2A_1}{\sin\angle QA_2A_3}.$$

multiplying:

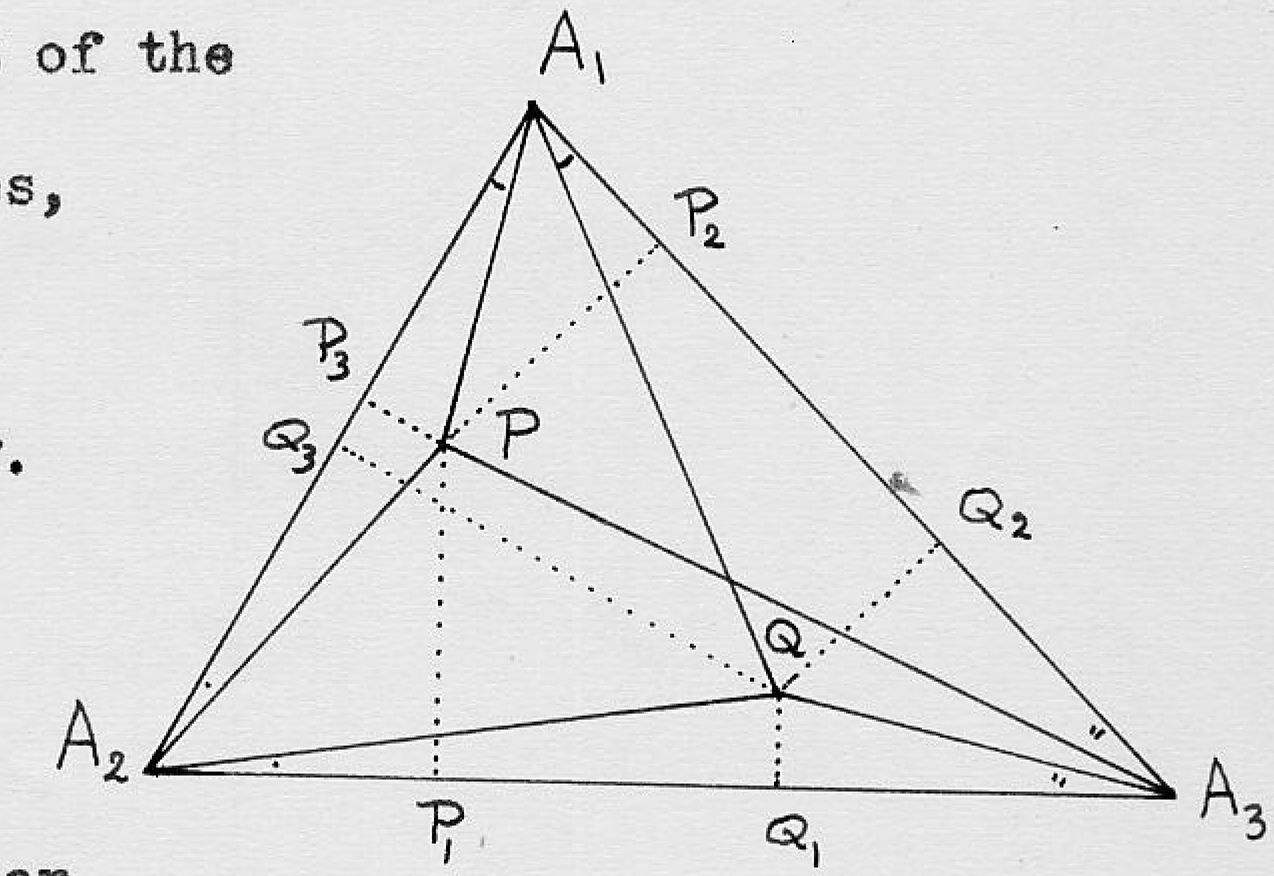
$$\frac{\sin\angle A_2A_1P.\sin\angle A_1A_3P.\sin\angle A_3A_2P}{\sin\angle A_3A_1P.\sin\angle A_2A_3P.\sin\angle A_1A_2P} = \frac{\sin\angle QA_1A_3.\sin\angle QA_3A_2.\sin\angle QA_2A_3}{\sin\angle QA_1A_2.\sin\angle QA_3A_1.\sin\angle QA_2A_3}.$$

If P is a concurrence point, the left hand side is -1 by the theorem of Ceva which implies that the right hand side equals -1 or that Q is a concurrence point by the converse of the theorem of Ceva.

Definition: Two points P and Q are isogonal conjugate points if their corresponding cevians are isogonal conjugate lines.

Theorem: If $P(x_1, x_2, x_3)$ and $Q(y_1, y_2, y_3)$ are isogonal conjugate points then y_i may be chosen so that $x_iy_i = 1$ $(i = 1,2,3)$.

Proof: Let P_1, P_2, P_3 be the feet of the perpendiculars from P to the three sides, and Q_1, Q_2, Q_3 of those dropped from Q. The triangles PA_1P_3 and QA_1Q_2 are similar. Therefore



$$\frac{PP_3}{QQ_2} = \frac{PA_1}{QA_1}.$$

Also triangles PA_1P_2 and QA_1Q_3 are similar.

Therefore

$$\frac{PP_2}{QQ_3} = \frac{PA_1}{QA_1}.$$

Therefore

$$\frac{PP_3}{QQ_2} = \frac{PP_2}{QQ_3} \quad \text{or} \quad PP_3.QQ_3 = PP_2.QQ_2.$$

If we let $PP_1 = d_1$, $PP_2 = d_2$, $PP_3 = d_3$

and

$$QQ_1 = d'_1,\ QQ_2 = d'_2,\ QQ_3 = d'_3$$

then we have

$$d_3.d'_3 = d_2.d'_2, d_1.d'_1 = d_2.d'_2,$$

so that

$$d_1.d'_1 = d_2.d'_2 = d_3.d'_3 = k \ (k \neq 0)$$

but $\quad d_i = k_1 x_i \quad$ and $\quad d'_i = k_2 y_i \ (k_2.k_1 \neq 0)$

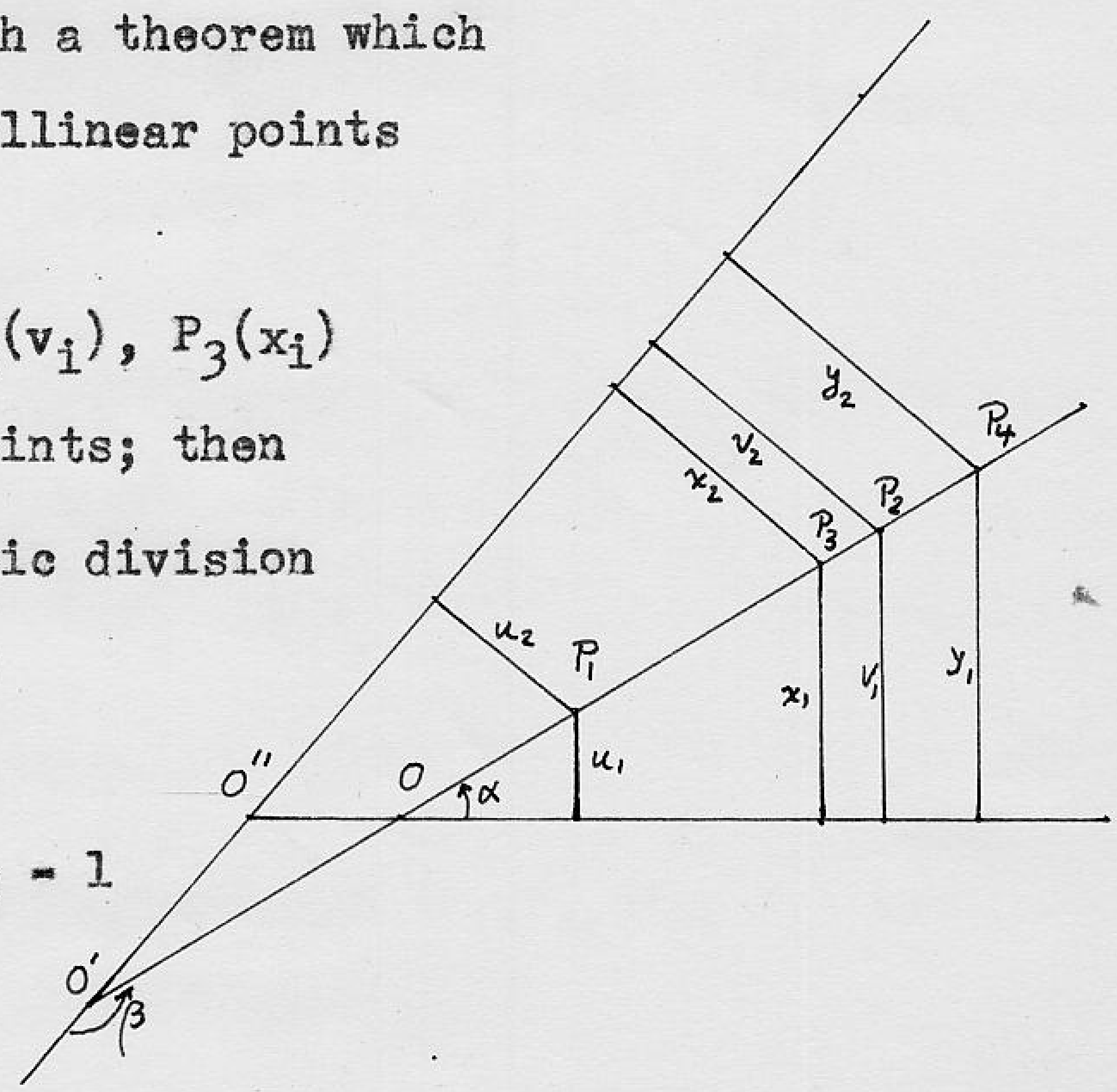
therefore

$$x_1.y_1 = x_2 y_2 = x_3 y_3 = \frac{k}{k_1 k_2} = K.$$

Thus if P is (x_1, x_2, x_3) then Q is

$$\left(\frac{K}{x_1}, \frac{K}{x_2}, \frac{K}{x_3}\right) \sim \left(\frac{1}{x_1}, \frac{1}{x_2}, \frac{1}{x_3}\right).$$

We end this chapter with a theorem which enables us to tell when four collinear points form a harmonic division.



__Theorem__: Let $P_1(u_i)$, $P_2(v_i)$, $P_3(x_i)$ and $P_4(y_i)$ be four collinear points; then P_1, P_2, P_3 and P_4 form a harmonic division if and only if the cross-ratio:

$$\mathcal{CR} = \frac{\frac{u_1}{u_2} - \frac{x_1}{x_2}}{\frac{u_1}{u_2} - \frac{y_1}{y_2}} \div \frac{\frac{v_1}{v_2} - \frac{x_1}{x_2}}{\frac{v_1}{v_2} - \frac{y_1}{y_2}} = -1$$

or any other cyclic permutation of the subscripts modulo 3.

__Proof__:

$u_1 = \overline{OP_1} \sin\alpha$, $v_1 = \overline{OP_2} \sin\alpha$, $x_1 = \overline{OP_3} \sin\alpha$ and $y_1 = \overline{OP_4} \sin\alpha$

$u_2 = \overline{O'P_1} \sin\beta$, $v_2 = \overline{O'P_2} \sin\beta$, $x_2 = \overline{O'P_3} \sin\beta$ and $y_2 = \overline{O'P_4} \sin\beta$.

Thus

$$\frac{u_1}{u_2} = \frac{\overline{OP_1}\sin\alpha}{\overline{O'P_1}\sin\beta} = \frac{\overline{OP_1}}{\overline{O'O} + \overline{OP_1}} \cdot \frac{\sin\alpha}{\sin\beta}$$

let $\overline{O'O} = m$ and $\frac{\sin\alpha}{\sin\beta} = k$ then:

$$\frac{u_1}{u_2} = \frac{\overline{OP_1}}{\overline{OP_1} + m} \cdot k .$$

Similarly

$$\frac{v_1}{v_2} = \frac{\overline{OP_2}}{\overline{OP_2} + m} \cdot k, \quad \frac{x_1}{x_2} = \frac{\overline{OP_3}}{\overline{OP_3} + m} \cdot k \text{ and } \frac{y_1}{y_2} = \frac{\overline{OP_4}}{\overline{OP_4} + m} \cdot k .$$

Then

$$\frac{\dfrac{u_1}{u_2} - \dfrac{x_1}{x_2}}{\dfrac{u_1}{u_2} - \dfrac{y_1}{y_2}} = \frac{(\overline{OP_4} + m)(\overline{OP_1} - OP_3)}{(\overline{OP_3} + m)(\overline{OP_1} - \overline{OP_4})} = \frac{(\overline{OP_4} + m)}{(\overline{OP_3} + m)} \cdot \frac{\overline{P_1P_3}}{\overline{P_1P_4}}$$

and

$$\frac{\dfrac{v_1}{v_2} - \dfrac{x_1}{x_2}}{\dfrac{v_1}{v_2} - \dfrac{y_1}{y_2}} = \frac{(\overline{OP_4} + m)(\overline{OP_2} - \overline{OP_3})}{(\overline{OP_3} + m)(\overline{OP_2} - \overline{OP_4})} = \frac{(\overline{OP_4} + m)}{(\overline{OP_3} + m)} \cdot \frac{\overline{P_2P_3}}{\overline{P_2P_4}}$$

$$\mathscr{CR} = \frac{\dfrac{u_1}{u_2} - \dfrac{x_1}{x_2}}{\dfrac{u_1}{u_2} - \dfrac{y_1}{y_2}} \div \frac{\dfrac{v_1}{v_2} - \dfrac{x_1}{x_2}}{\dfrac{v_1}{v_2} - \dfrac{y_1}{y_2}} = \frac{\overline{P_1P_3}}{\overline{P_1P_4}} \div \frac{\overline{P_2P_3}}{\overline{P_2P_4}} .$$

Now if $\mathscr{CR} = -1$ then $\dfrac{\overline{P_1P_3}}{\overline{P_1P_4}} = -\dfrac{\overline{P_2P_3}}{\overline{P_2P_4}}$ and (P_1, P_2, P_3, P_4) is a harmonic division.

Conversely if (P_1, P_2, P_3, P_4) is a harmonic division, then

$$\frac{\overline{P_1P_3}}{\overline{P_1P_4}} = -\frac{\overline{P_2P_3}}{\overline{P_2P_4}} \quad \text{and } \mathscr{CR} = -1.$$

CHAPTER II

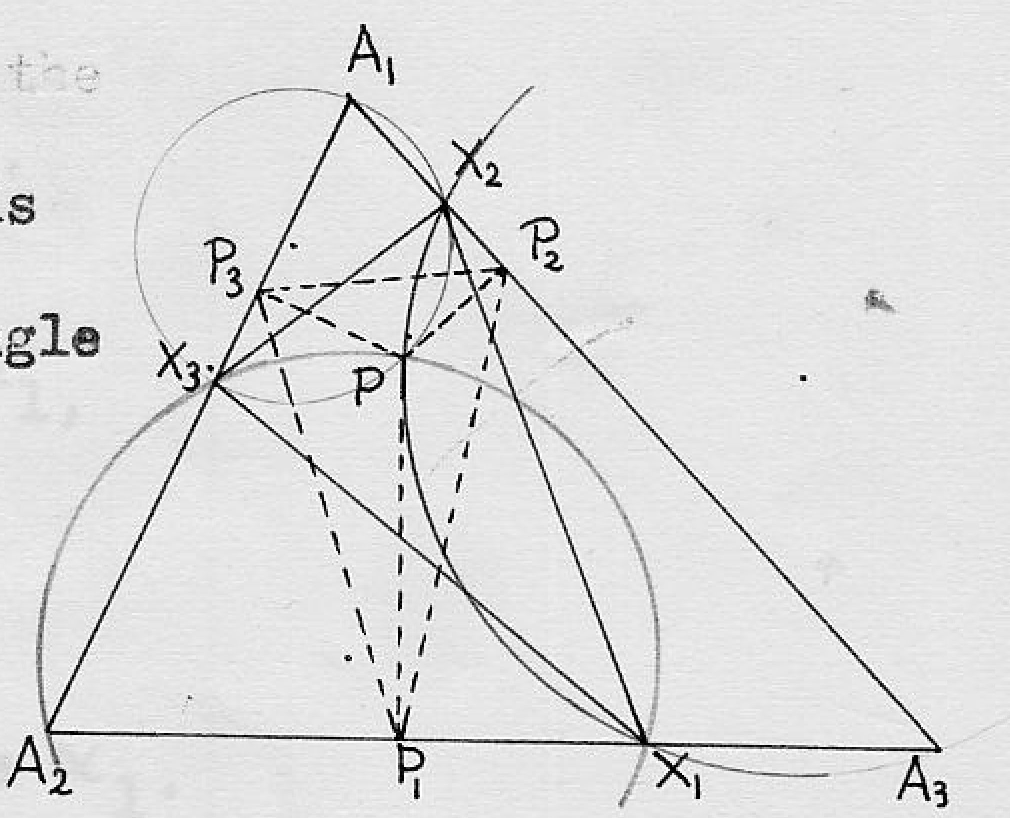
INSCRIBED TRIANGLES SIMILAR TO THE TRIANGLE OF REFERENCE

1. Introduction

If one vertex of a triangle lies on A_2A_3 or its extension, another vertex on A_3A_1 or its extension and the third vertex on A_1A_2 or its extension then we shall call the triangle an inscribed triangle.

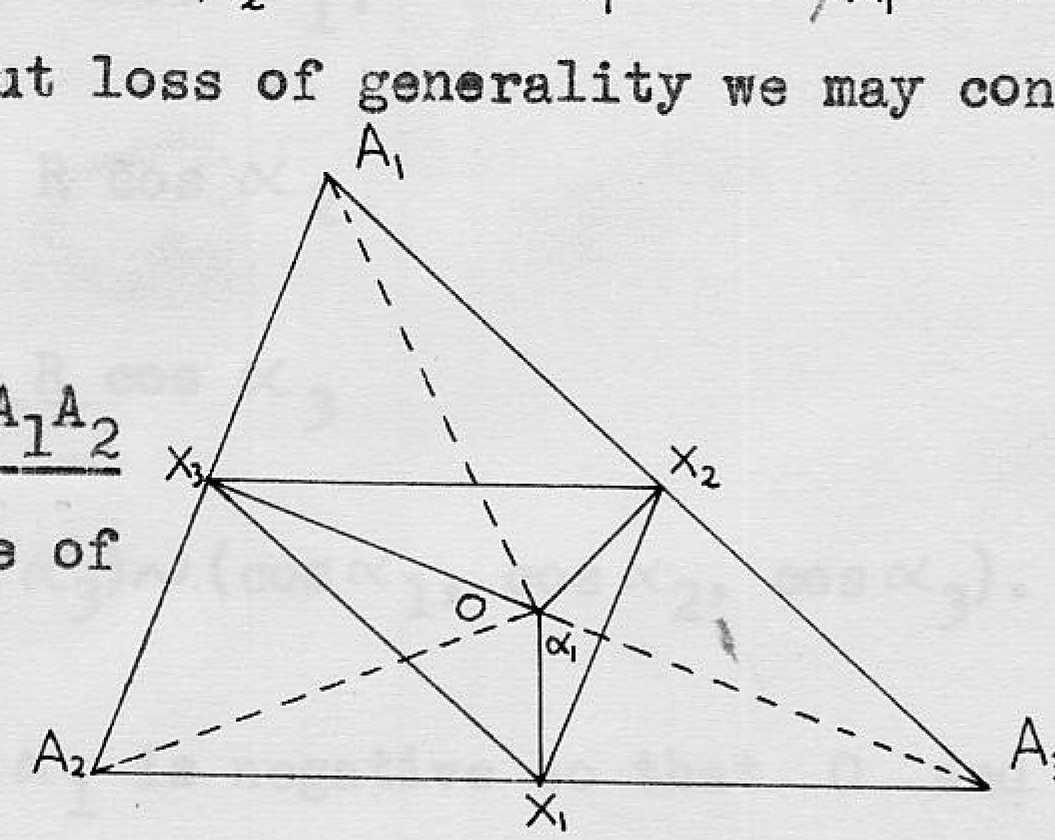
Let $X_1X_2X_3$ be an inscribed triangle similar to $A_1A_2A_3$. $X_1X_2X_3$ can be inscribed in six different ways.

In this Chapter we shall determine the Miquel points of $X_1X_2X_3$, and investigate the properties of these points. We first note that if P is the Miquel pt. of $X_1X_2X_3$, then the pedal triangle of P, whose vertices are the feet of the perpendiculars from P to the sides of $A_1A_2A_3$, is similar to $X_1X_2X_3$ and also has P as its Miquel point, so that without loss of generality we may consider $X_1X_2X_3$ as the pedal triangle of P.



2. X_1 on A_2A_3, X_2 on A_1A_3 and X_3 on A_1A_2

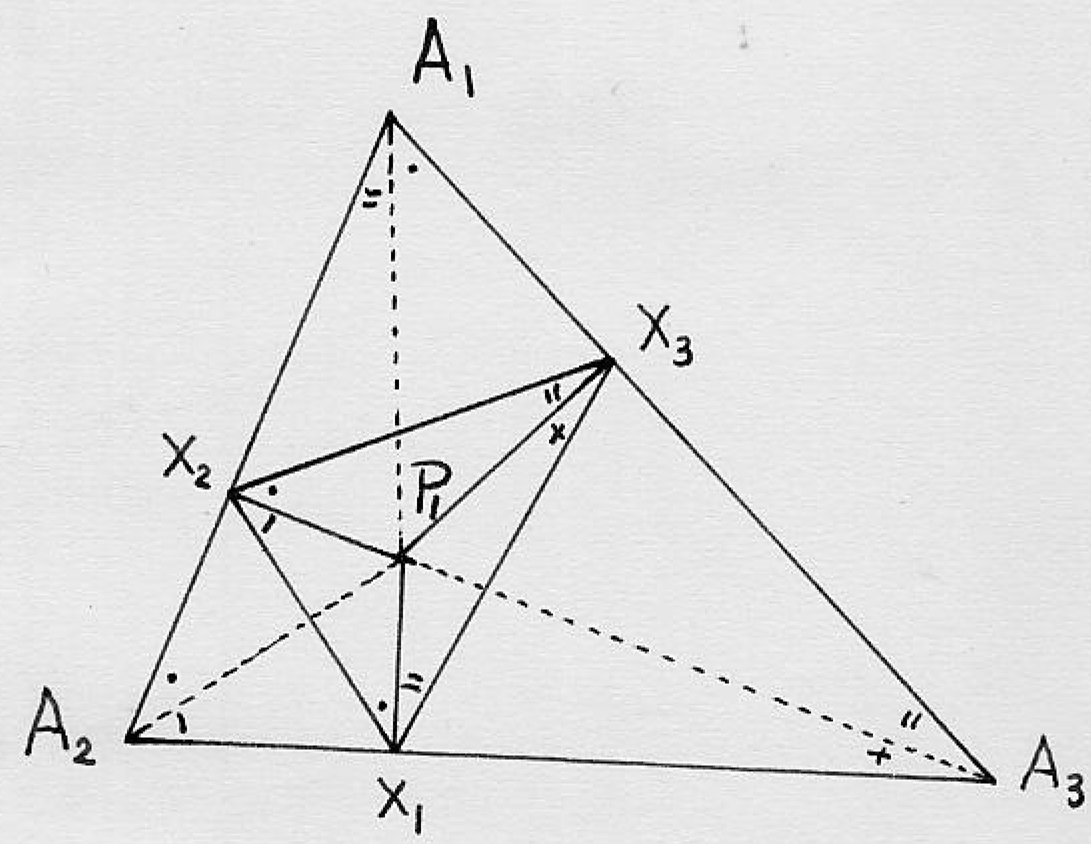
X_2X_3 is a chord of the circle of diameter A_1O; and in that circle the angle inscribed in arc X_2X_3 is α_1.



$A_1A_2A_3$, which is the case; so the given coordinates are true for all shapes of $A_1A_2A_3$.

3. <u>X_1 on A_2A_3, X_2 on A_1A_2 and X_3 on A_1A_3</u>

It can be shown, using the cyclic quadrilaterals $A_1X_2P_1X_3$, $A_2X_2P_1X_1$ and $A_3X_1P_1X_3$, that:



$$\angle A_1P_1A_2 = \angle A_1P_1A_3$$

and

$$\angle A_2P_1A_3 = 2\alpha_1.$$

In this case:

$$X_2X_3 = A_1P_1 \sin\alpha_1,$$

$$X_1X_2 = A_2P_1 \sin\alpha_2,$$

$$X_3X_1 = A_3P_1 \sin\alpha_3,$$

so that $A_1P_1 \sin\alpha_1 : A_3P_1 \sin\alpha_3 : A_2P_1 \sin\alpha_2 = a_1 : a_2 : a_3$.

In the similar triangles $P_1X_3A_1$ and $P_1X_2A_2$ we have:

$$\frac{P_1X_3}{P_1X_2} = \frac{A_1P_1}{A_2P_1} = \frac{a_1 \sin\alpha_2}{a_3 \sin\alpha_1} = \frac{a_2 \cdot \sin\alpha_2}{a_3 \cdot \sin\alpha_2} = \frac{a_2}{a_3};$$

in triangle $A_2P_1A_3$:

$$A_2P_1 \cdot A_3P_1 \sin 2\alpha_1 = P_1X_1 \cdot A_2A_3,$$

or

$$A_2P_1 \cdot A_3P_1 \sin 2\alpha_1 = P_1X_1 \cdot a_1;$$

and in triangle $A_1P_1A_2$:

$$A_2P_1 \cdot A_1P_1 \sin\angle A_1P_1A_2 = P_1X_2 \cdot A_1A_2 = P_1X_2 \cdot a_3 .$$

Dividing:

$$\frac{A_2P_1 . A_3P_1 \sin 2\alpha_1}{A_2P_1 . A_1P_1 \sin\left\{\frac{2\pi - 2\alpha_1}{2}\right\}} = \frac{P_1X_1 . a_1}{P_1X_2 . a_3} ,$$

$$\frac{A_3P_1 \sin 2\alpha_1}{A_1P_1 \sin\alpha_1} = \frac{P_1X_1}{P_1X_2} . \frac{a_1}{a_3} .$$

Therefore

$$\frac{P_1X_1}{P_1X_2} = \frac{a_3}{a_1} . \frac{\sin 2\alpha_1}{\sin\alpha_1} . \frac{A_3P_1}{A_1P_1} = \frac{2a_3\cos\alpha_1}{a_1} . \frac{a_2 . \sin\alpha_1}{\sin\alpha_3 . a_1}$$

$$= \frac{2a_3\cos\alpha_1}{a_1} . \frac{a_2 . \sin\alpha_3}{\sin\alpha_3 . a_3} .$$

$$\frac{P_1X_1}{P_1X_2} = \frac{2a_2\cos\alpha_1}{a_1} .$$

Hence $P_1(P_1X_1, P_1X_3, P_1X_2) \sim \left(\frac{P_1X_1}{P_1X_2} , \frac{P_1X_3}{P_1X_2} , 1\right) = \left(\frac{2a_2\cos\alpha_1}{a_1} , \frac{a_2}{a_3} ,1\right)$

$$\sim P_1(2a_3a_2\cos\alpha_1, a_1a_2, a_1a_3).$$

4. <u>X_1 on A_1A_2, X_2 on A_3A_1 and X_3 on A_2A_3</u>

We obtain this case from case 3 by cyclic permutation of subscripts and coordinates.

If in this case P_2 is the Miquel point of $X_1X_2X_3$,

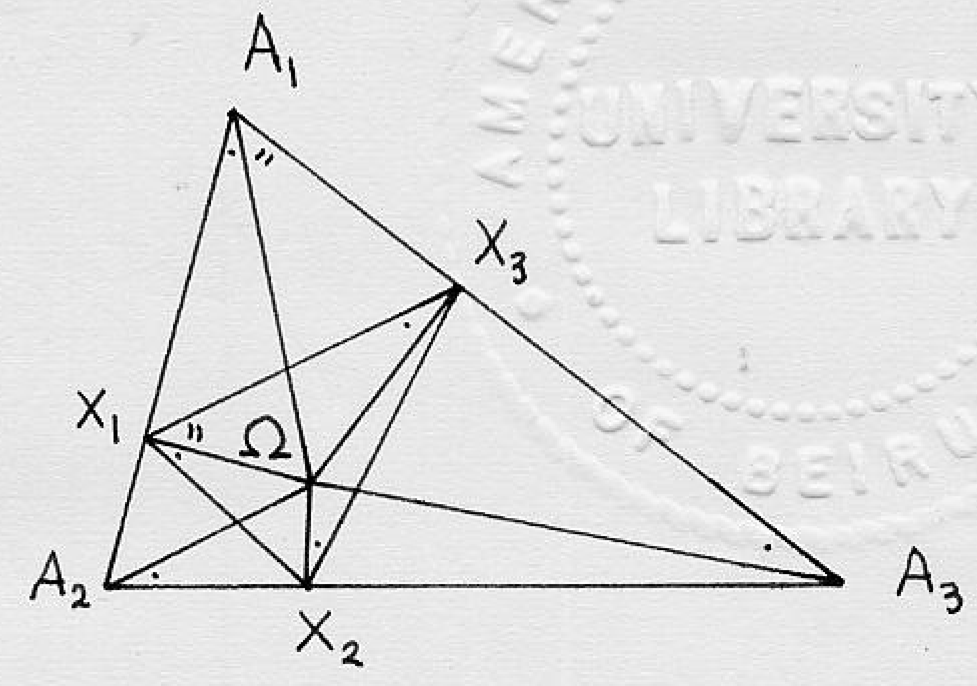
$$P_2\left(1, \frac{2a_3\cos\alpha_2}{a_2} , \frac{a_3}{a_1}\right) \sim P_2(a_1a_2, 2a_3a_1\cos\alpha_2, a_2a_3).$$

5. <u>X_1 on A_3A_1, X_2 on A_2A_3 and X_3 on A_1A_2</u>

The same argument applies here and

$$P_3(a_3a_1, a_2a_3, 2a_1a_2\cos\alpha_3).$$

6. X_1 on A_1A_2, X_2 on A_2A_3 and X_3 on A_3A_1



Let Ω be the Miquel point of $X_1X_2X_3$. The cyclic quadrilateral $A_1X_1\Omega X_3$ gives

$$\angle \Omega A_1X_3 = \angle \Omega X_1X_3 ;$$

and since

$$\angle X_2X_1X_3 = \angle A_2A_1A_3,$$

we have

$$\angle \Omega A_1A_2 = \angle \Omega X_1X_2 ;$$

but from the cyclic quadrilateral $A_2X_2\Omega X_1$ we have:

$$\angle \Omega X_1X_2 = \angle \Omega A_2X_2 .$$

Therefore

$$\angle \Omega A_1A_2 = \angle \Omega A_2A_3 .$$

Similarly it can be shown that $\angle \Omega A_2A_3 = \angle \Omega A_3A_1$.

Let us denote the common value of these angles by ω so that:

$$\angle \Omega A_1A_2 = \angle \Omega A_2A_3 = \angle \Omega A_3A_1 = \omega.$$

The similar triangles ΩA_1X_1, ΩA_2X_2 and ΩA_3X_3 give

$$\Omega X_1 : \Omega X_2 : \Omega X_3 = \Omega A_1 : \Omega A_2 : \Omega A_3 ,$$

but

$$\Omega A_1 : \Omega A_2 : \Omega A_3 = \frac{X_3X_1}{\sin \alpha_1} : \frac{X_1X_2}{\sin \alpha_2} : \frac{X_2X_3}{\sin \alpha_3} ,$$

and

$$X_2X_3 : X_3X_1 : X_1X_2 = a_1 : a_2 : a_3 .$$

Therefore

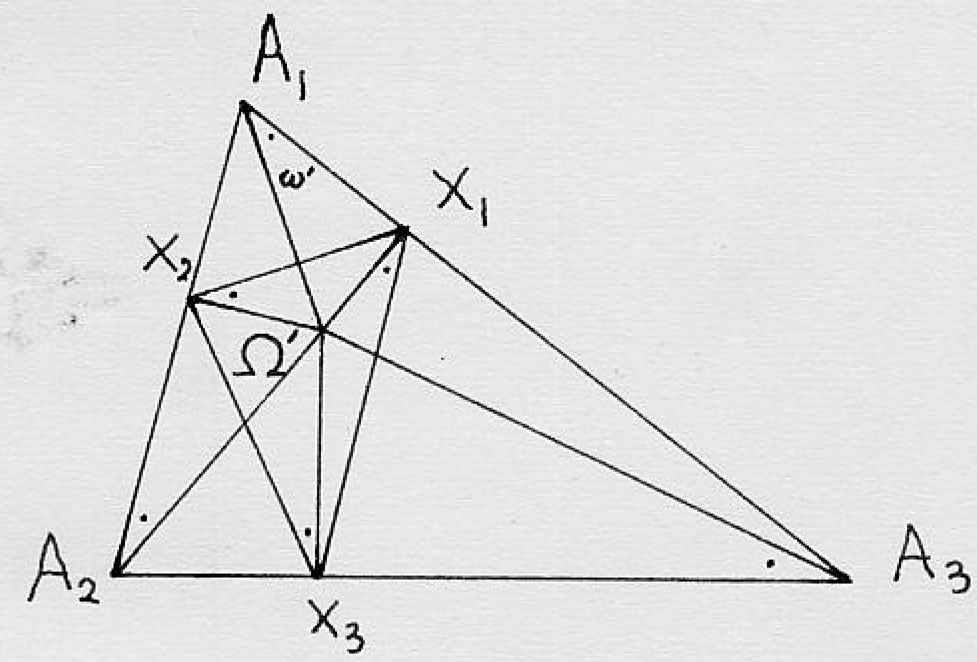
$$\Omega A_1 : \Omega A_2 : \Omega A_3 = \frac{a_2}{\sin \alpha_1} : \frac{a_3}{\sin \alpha_2} : \frac{a_1}{\sin \alpha_3} = \Omega X_1 : \Omega X_2 : \Omega X_3 .$$

Hence

$$\Omega(\Omega X_2, \Omega X_3, \Omega X_1) = \left(\frac{Ka_3}{\sin\alpha_2}, \frac{Ka_1}{\sin\alpha_3}, \frac{Ka_2}{\sin\alpha_1}\right) (K \neq 0)$$

$$\sim \Omega\left(\frac{a_3}{\sin\alpha_2}, \frac{a_1}{\sin\alpha_3}, \frac{a_2}{\sin\alpha_1}\right) \sim \Omega\left(\frac{a_3}{a_2}, \frac{a_1}{a_3}, \frac{a_2}{a_1}\right).$$

7. X_1 on A_1A_3, X_2 on A_1A_2 and X_3 on A_2A_3



Let Ω' be the Miquel point of $X_1X_2X_3$ then $\angle\Omega' A_2A_1 = \angle\Omega' A_3A_2 = \angle\Omega' A_1A_3 = \omega'$. The similar triangles $\Omega' X_1A_1$, $\Omega' X_2A_2$ and $\Omega' X_3A_3$ give

$$\Omega' X_1 : \Omega' X_2 : \Omega' X_3 = \Omega' A_1 : \Omega' A_2 : \Omega' A_3,$$

but

$$\Omega' A_1 : \Omega' A_2 : \Omega' A_3 = \frac{X_1X_2}{\sin\alpha_1} : \frac{X_2X_3}{\sin\alpha_2} : \frac{X_3X_1}{\sin\alpha_3},$$

and always

$$X_2X_3 : X_3X_1 : X_1X_2 = a_1 : a_2 : a_3$$

so that

$$\Omega' X_1 : \Omega' X_2 : \Omega' X_3 = \frac{a_3}{\sin\alpha_1} : \frac{a_1}{\sin\alpha_2} : \frac{a_2}{\sin\alpha_3}.$$

Hence

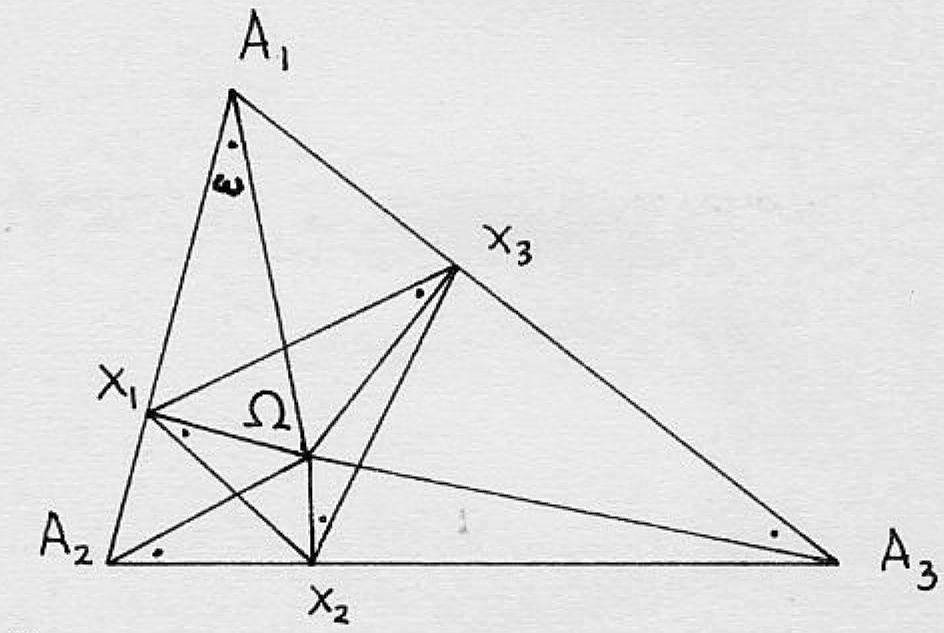
$$\Omega'(\Omega' X_3, \Omega' X_1, \Omega' X_2) = \Omega'\left(\frac{Ka_2}{\sin\alpha_3}, \frac{Ka_3}{\sin\alpha_1}, \frac{Ka_1}{\sin\alpha_2}\right) (K \neq 0)$$

$$\sim \Omega'\left(\frac{a_2}{a_3}, \frac{a_3}{a_1}, \frac{a_1}{a_2}\right).$$

It is clear that Ω' and Ω are isogonal conjugate points, so that $\omega = \omega'$.

<u>Definition</u> Ω, Ω' are called the Brocard points of $A_1A_2A_3$.

<u>Theorem:</u>



$$\sin\omega = \frac{2\Delta}{\sqrt{a_2^2a_3^2 + a_3^2a_1^2 + a_1^2a_2^2}}$$

$$\cot\omega = \frac{a_1^2 + a_2^2 + a_3^2}{4\Delta}$$

$$\cot\omega = \cot\alpha_1 + \cot\alpha_2 + \cot\alpha_3.$$

<u>Proof</u>: (a) $\angle A_1 \Omega A_2 = \pi - \alpha_2$ in triangle $A_1 \Omega A_2$.

The area is given by:

$$\frac{1}{2} \cdot \Omega X_1 \cdot A_1A_2 = \frac{1}{2} \Omega A_1 \cdot \Omega A_2 \sin \angle A_1 \Omega A_2$$

$$\frac{\Omega X_1 \cdot A_1A_2}{\Omega X_1 \cdot \Omega X_2} = \frac{\Omega A_1}{\Omega X_1} \cdot \frac{\Omega A_2}{\Omega X_2} \sin\alpha_2 = \frac{\sin\alpha_2}{\sin^2\omega}.$$

Therefore

$$\sin^2\omega = \frac{\sin\alpha_2}{a_3} \cdot \Omega X_2 .$$

Using the theorem for the actual distances of a point, substitute for ΩX_2:

$$\sin^2\omega = \frac{\sin\alpha_2}{a_3} \cdot \frac{2\Delta\left(\frac{a_3}{a_2}\right)}{\sum_{i=1}^{3} a_i u_i},$$

where u_i are the coordinates of Ω.

$$\sin^2\omega = \frac{\sin\alpha_2}{a_3} \cdot \frac{2\Delta a_3}{a_2\left(\frac{a_1^2a_3^2 + a_1^2a_2^2 + a_2^2a_3^2}{a_1a_2a_3}\right)}$$

$$\sin^2\omega = \frac{4\Delta^2}{(a_2^2a_3^2 + a_3^2a_1^2 + a_1^2a_2^2)}.$$

(b) The law of cosines gives in triangle $A_1 \Omega A_2$

$$\overline{\Omega A_2}^2 = \overline{\Omega A_1}^2 + \overline{A_1A_2}^2 - 2\,\overline{\Omega A_1}.\overline{A_1A_2}\cos\angle \Omega A_1A_2 ,$$

and the law of sines gives:

$$\frac{\Omega A_2}{\sin\omega} = \frac{A_1A_2}{\sin\alpha_2} = \frac{a_3}{\sin\alpha_2},$$

and from triangle $A_1 \Omega A_3$ we have:

$$\frac{\Omega A_1}{\sin\omega} = \frac{A_1A_3}{\sin\alpha_1} = \frac{a_2}{\sin\alpha_1};$$

substituting and simplifying we get:

$$\frac{a_3^2}{\sin^2\alpha_2} - \frac{a_2^2}{\sin^2\alpha_3} + \frac{2a_2a_3\cot\omega}{\sin\alpha_1} = \frac{a_3^2}{\sin^2\omega} .$$

Using the result of part (a) of the theorem:

$$\frac{4R^2a_3^2}{a_2^2} - \frac{4R^2a_2^2}{a_1^2} + \frac{4Ra_2a_3}{a_1}\cot\omega = \frac{4R^2.a_3^2.\sum au}{a_1a_2a_3},$$

$$\frac{a_2a_3}{a_1}\cot\omega = R\left[\frac{a_1a_2a_3^2\left(\frac{a_1a_3}{a_2} + \frac{a_1a_2}{a_3} + \frac{a_2a_3}{a_1}\right) + a_2^4a_3 - a_1^2a_3^3}{a_1^2a_2^2a_3}\right],$$

$$\frac{a_2a_3}{a_1}\cot\omega = \frac{R(a_1^2 + a_2^2 + a_3^2)}{a_1^2} .$$

Finally

$$\cot\omega = \frac{(a_1^2 + a_2^2 + a_3^2)}{\frac{a_1a_2a_3}{R}} = \frac{a_1^2 + a_2^2 + a_3^2}{4\Delta} .$$

(c) $$\cot\alpha_1 = \frac{a_2^2 + a_3^2 - a_1^2}{4\Delta},\quad \cot\alpha_2 = \frac{a_3^2 + a_1^2 - a_2^2}{4\Delta},\quad \cot\alpha_3 = \frac{a_1^2 + a_2^2 - a_3^2}{4\Delta}$$

addition and the result of part (b) give:

$$\cot\omega = \cot\alpha_1 + \cot\alpha_2 + \cot\alpha_3.$$

<u>Theorem</u>: The six points O, Ω, Ω', P_1, P_2 and P_3 lie on a circle.

<u>Proof</u>: We find^equation of the conic through Ω, Ω, P_1, P_2, P_3, check that it passes through O and that it is a circle.

<u>The line $\Omega\Omega'$ is given by</u>:

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ \frac{a_3}{a_2} & \frac{a_1}{a_3} & \frac{a_2}{a_1} \\ \frac{a_2}{a_3} & \frac{a_3}{a_1} & \frac{a_1}{a_2} \end{vmatrix} = 0,$$

i.e., $$a_2a_3(a_1^4 - a_2^2a_3^2)x_1 + a_1a_3(a_2^4 - a_1^2a_3^2)x_2 + a_1a_2(a_3^4 - a_1^2a_2^2)x_3 = 0.$$

<u>The line P_1, P_2</u> :

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ 2a_2a_3\cos\alpha_1 & a_1a_2 & a_1a_3 \\ a_1a_2 & 2a_1a_3\cos\alpha_2 & a_2a_3 \end{vmatrix} = 0,$$

i.e., $$a_1a_3(2a_2^2 - a_1^2 - a_3^2)x_1 + a_2a_3(2a_1^2 - a_2^2 - a_3^2)x_2$$

$$+ (a_3^4 - a_1^4 - a_2^4 + a_1^2\,a_2^2)x_3 = 0.$$

Multiplication gives a degenerate conic through Ω, Ω', P_1, P_2:

$$\left\{a_2a_3(a_1^4-a_2^2a_3^2)x_1+a_1a_3(a_2^4-a_1^2a_3^2)x_2+a_1a_2(a_3^4-a_1^2a_2^2)x_3\right\}\left\{a_1a_3(2a_2^2-a_1^2-a_3^2)x_1\right.$$
$$\left.+a_2a_3(2a_1^2-a_2^2-a_3^2)x_2+(a_3^4-a_1^4-a_2^4+a_1^2a_2^2)x_3\right\}=0.$$

Similarly the line ΩP_1 is given by:

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ \frac{a_3}{a_2} & \frac{a_1}{a_3} & \frac{a_2}{a_1} \\ 2a_2a_3\cos\alpha_1 & a_1a_2 & a_1a_3 \end{vmatrix} = 0,$$

i.e.,
$$a_1a_2a_3x_1 - a_3(a_2^2+a_3^2)x_2 + a_1^2a_2x_3 = 0,$$

and the line $\Omega' P_2$ is given by:

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ \frac{a_2}{a_3} & \frac{a_3}{a_1} & \frac{a_1}{a_2} \\ a_1a_2 & 2a_1a_3\cos\alpha_2 & a_2a_3 \end{vmatrix} = 0,$$

i.e.,
$$a_3(a_3^2+a_1^2)x_1 - a_1a_2a_3x_2 - a_1a_2^2x_3 = 0.$$

Multiplication gives another degenerate conic through Ω, Ω', P_1, P_2:

$$\left\{a_1a_2a_3x_1 - a_3(a_3^2+a_2^2)x_2 + a_1^2a_2x_3\right\}\left\{a_3(a_3^2+a_1^2)x_1 - a_1a_2a_3x_2 - a_1a_2^2x_3\right\} = 0.$$

Now the equation of the family of conics passing through Ω, Ω', P_1, P_2 is given by:

$$\left\{a_2a_3(a_1^4-a_2^2a_3^2)x_1 + a_1a_3(a_2^4-a_1^2a_3^2)x_2 + a_1a_2(a_3^4-a_1^2a_2^2)x_3\right\}\left\{a_1a_3(2a_2^2-a_1^2-a_3^2)x_1\right.$$
$$\left. + a_2a_3(2a_1^2-a_2^2-a_3^2)x_2 + (a_3^4-a_1^4-a_2^4+a_1^2a_2^2)x_3\right\} + \lambda\left\{a_1a_2a_3x_1 - a_3(a_3^2+a_2^2)x_2 + a_1^2a_2x_3\right\} \times$$
$$\left\{a_3(a_3^2+a_1^2)x_1 - a_1a_2a_3x_2 - a_1a_2^2x_3\right\} = 0$$

where λ is a constant.

To determine λ we substitute the coordinates of P_3 and solve; this gives:

$$= - a_1^4 - a_2^4 - a_3^4 + a_1^2a_2^2 + a_2^2a_3^2 + a_3^2a_1^2 .$$

<u>Equation of the conic through $\Omega, \Omega', P_1, P_2, P_3$:</u>

Substitution of the value of λ obtained and simplification gives:

$$a_1a_2a_3(x_1^2+x_2^2+x_3^2) - a_3^3x_1x_2 - a_2^3x_1x_3 - a_1^3x_2x_3 = 0.$$

An easy computation shows that $O(\cos\alpha_1, \cos\alpha_2, \cos\alpha_3)$ and $K(a_1, a_2, a_3)$ lie on this conic .

Now the equation of the line at infinity is:

$$a_1x_1 + a_2x_2 + a_3x_3 = 0$$

$$x_1 = \frac{-(a_2x_2 + a_3x_3)}{a_1} .$$

Substituting in the equation of the conic we get:

$$a_1a_2a_3\left\{\frac{a_2^2x_2^2 + a_3^2x_3^2 + 2a_2a_3x_2x_3}{a_1^2} + x_2^2 + x_3^2\right\} + \frac{(a_2x_2+a_3x_3)}{a_1}\cdot(a_3^3x_2 + a_2^3x_3) - a_1^3x_2x_3 = 0,$$

which reduces to

$$a_2a_3x_2^2 + (a_2^2 + a_3^2 - a_1^2)x_2x_3 + a_2a_3x_3^2 = 0.$$

or

$$a_2a_3x_2^2 + (2a_2a_3\cos\alpha_1)x_2x_3 + a_2a_3x_3^2 = 0$$

i.e. ,

$$x_2^2 + 2\cos\alpha_1 x_2x_3 + x_3^2 = 0$$

$$\frac{x_2}{x_3} = -\cos\alpha_1 \pm \sqrt{\cos^2\alpha_1 - 1} = -\cos\alpha_1 \pm i\cdot\sin\alpha_1$$

and

$$\frac{x_1}{x_3} = -\cos\alpha_2 \mp i\sin\alpha_2 .$$

Finally

$$(x_1, x_2, x_3) \sim (-\cos\alpha_2 \mp i\cdot\sin\alpha_2, -\cos\alpha_1 \pm i\cdot\sin\alpha_1, 1)$$

which shows that the line at infinity $\sum ax = 0$, cuts this conic in the two circular points at infinity i.e. the conic is a circle. See appendix (It is called the Brocard circle).

The center of the Brocard circle

Let y_1, y_2, y_3 be the coordinates of the center. Then since the center is the pole of the line at infinity, its coordinates are the solutions of the following set of equations:

$$a_{11}y_1 + a_{12}y_2 + a_{13}y_3 \sim a_1$$

$$a_{21}y_1 + a_{22}y_2 + a_{23}y_3 \sim a_2$$

$$a_{31}y_1 + a_{32}y_2 + a_{33}y_3 \sim a_3$$

$$y \sim \begin{vmatrix} a_1 & a_{12} & a_{13} \\ a_2 & a_{22} & a_{23} \\ a_3 & a_{32} & a_{33} \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & -a^3_{3/2} & -a^3_{2/2} \\ a_2 & a_1a_2a_3 & -a^3_{1/2} \\ a_3 & -a^3_{1/2} & a_1a_2a_3 \end{vmatrix}$$

$$= \frac{a_1a_2a_3}{2}(a_1^2 + a_2^2 + a_3^2)(a_2a_3 + a_1^2 \cos\alpha_1).$$

Similarly

$$y_2 \sim \frac{a_1a_2a_3}{2}(a_1^2 + a_2^2 + a_3^2)(a_3a_1 + a_2^2 \cos\alpha_2),$$

and

$$y_3 \sim \frac{a_1a_2a_3}{2}(a_1^2 + a_2^2 + a_3^2)(a_1a_2 + a_3^2 \cos\alpha_3),$$

so the center is

$$(a_2a_3 + a_1^2\cos\alpha_1,\ a_3a_1 + a_2^2\cos\alpha_2,\ a_1a_2 + a_3^2\cos\alpha_3).$$

$$a_2a_3 + a_1^2 \cos\alpha_1 = 4R^2(\sin\alpha_2\sin\alpha_3 + \sin^2\alpha_1 \cos\alpha_1) =$$

$$= 4R^2(\sin\alpha_2 \sin\alpha_1\cos\alpha_2 + \sin^2\alpha_2\cos\alpha_1 + \sin^2\alpha_1\cos\alpha_1)$$

$$= 4R^2(\cos\alpha_1(\sin^2\alpha_1 + \sin^2\alpha_2) + \sin\alpha_1 \sin\alpha_2\cos\alpha_2)$$

$$= 4R^2(\cos\alpha_1\sin^2\alpha_3 + 2\sin\alpha_1\sin\alpha_2\cos\alpha_1\cos\alpha_3$$
$$+ \sin\alpha_1\sin\alpha_2\cos\alpha_2)$$

$$= 4R^2\{\cos\alpha_1(\sin^2\alpha_3 + \sin\alpha_1\sin\alpha_2\cos\alpha_3)$$
$$+ \sin\alpha_1\sin\alpha_2(\cos\alpha_1\cos\alpha_3 + \cos\alpha_2)\}$$

$$= 4R^2\{\cos\alpha_1(1 + \cos\alpha_1\cos\alpha_2\cos\alpha_3) + \sin^2\alpha_1\sin\alpha_2\sin\alpha_3\}$$

$$= 4R^2 \sin\alpha_1\sin\alpha_2\sin\alpha_3(\cos\alpha_1 \cot\omega + \sin\alpha_1)$$

$$= 4R^2 \sin\alpha_1\sin\alpha_2\sin\alpha_3 \cdot \frac{\cos(\alpha_1 - \omega)}{\sin\omega}.$$

Therefore the center is $(\cos(\alpha_1 - \omega), \cos(\alpha_2 - \omega), \cos(\alpha_3 - \omega))$.

Finally, since the determinant

$$\begin{vmatrix} \cos(\alpha_1 - \omega) & \cos(\alpha_2 - \omega) & \cos(\alpha_3 - \omega) \\ \cos\alpha_1 & \cos\alpha_2 & \cos\alpha_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = 0,$$

the center lies on OK, i.e., the Brocard circle has OK as diameter.

CHAPTER III

INSCRIBED TRIANGLES OF A GIVEN SHAPE

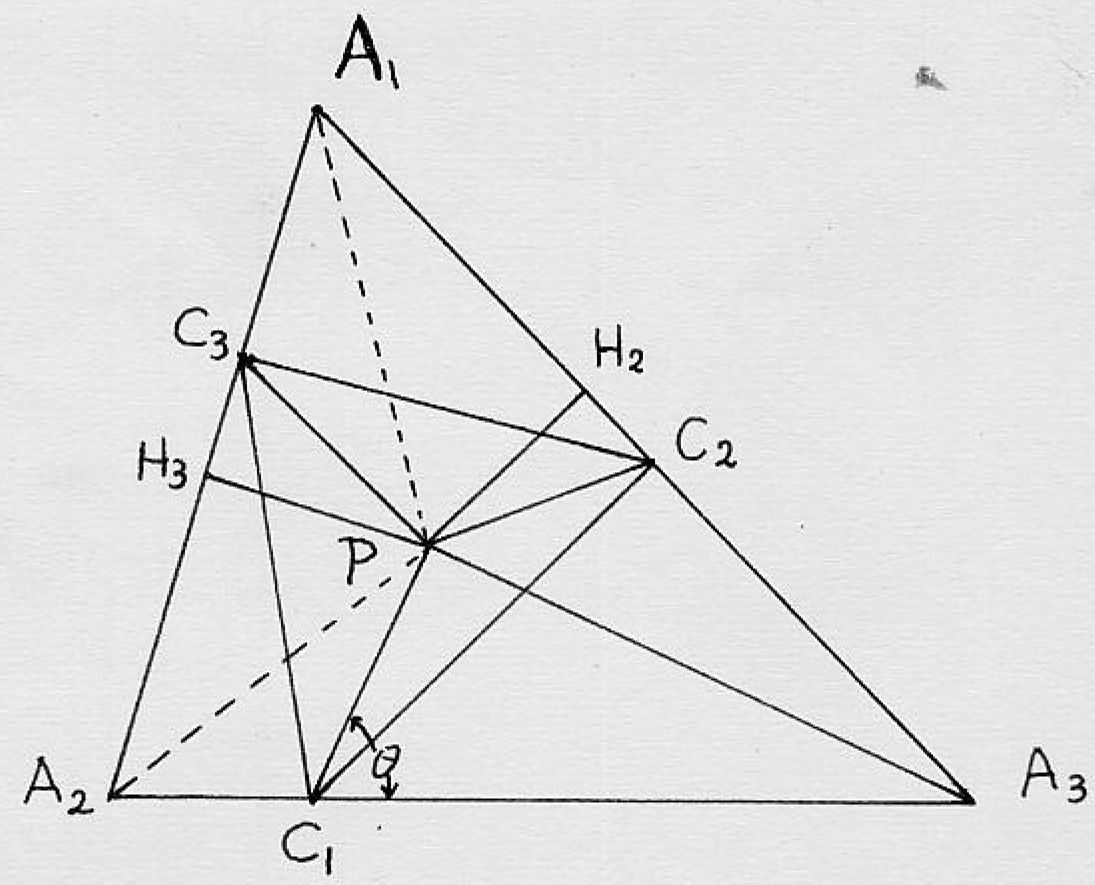
<u>Theorem</u>: Let $P(u_1, u_2, u_3)$ be a point in the plane of the triangle of reference. If C_1, C_2 and C_3 are the three points on A_2A_3, A_3A_1 and A_1A_2 respectively, with

$$\angle PC_1A_3 = \angle PC_2A_1 = \angle PC_3A_2 = \theta,$$

then $C_1C_2C_3$ is a Miquel triangle of P and

$$C_2C_3 = \frac{A_1P.\sin\alpha_1}{\sin\theta} = \frac{2\Delta\sqrt{u_2^2+u_3^2+2u_2u_3\cos\alpha_1}}{\sin\theta(a_1u_1+a_2u_2+a_3u_3)}, \text{ etc.}$$

<u>Proof</u>: Let d be the diameter of the circle $A_1C_2PC_3$. In this circle C_2C_3 and A_1P are chords with α_1 the angle inscribed in arc C_2C_3 and θ the angle inscribed in arc A_1P.



Therefore

$$C_2C_3 = d\sin\alpha_1,$$

$$A_1P = d\sin\theta,$$

i.e.,

$$C_2C_3 = \frac{A_1P\sin\alpha_1}{\sin\theta}, \text{ etc.}$$

Let H_2 and H_3 be the feet of the perpendiculars from P to A_3A_1 and A_1A_2 respectively; then in the circle $PH_2A_1H_3$, A_1P is a diameter and H_2H_3 is a chord subtending an angle α_1; therefore

$$H_2H_3 = A_1P \sin \alpha_1$$

and

$$C_2C_3 = H_2H_3/\sin \theta \quad .$$

The theorem for the actual distances of a point gives:

$$PH_2 = \frac{2\Delta u_2}{\sum au} \quad \text{and} \quad PH_3 = \frac{2\Delta u_3}{\sum au}$$

and the cosine law in triangle PH_2H_3 gives

$$\overline{H_2H_3}^2 = \overline{PH_2}^2 + \overline{PH_3}^2 - 2.\overline{PH_2}.\overline{PH_3} \cos \angle H_2PH_3 ,$$

$$\times\ \overline{H_2H_3}^2 = \left(\frac{2\Delta u_2}{\sum au}\right)^2 + \left(\frac{2\Delta u_3}{\sum au}\right)^2 - 2.\left(\frac{2\Delta u_2}{\sum au}\right)\left(\frac{2\Delta u_3}{\sum au}\right)\cos(\pi - \alpha_1) ,$$

$$\overline{H_2H_3}^2 = \frac{4\Delta^2}{(\sum au)^2}.(u_2^2 + u_3^2 + 2u_2u_3\cos\alpha_1).$$

Therefore

$$C_2C_3 = \frac{2\Delta\sqrt{u_2^2 + u_3^2 + 2u_2u_3\cos\alpha_1}}{\sin\theta(\sum au)} = \frac{2\Delta\sqrt{u_2^2 + u_3^2 + 2u_2u_3\cos\alpha_1}}{\sin\theta(a_1u_1 + a_2u_2 + a_3u_3)} .$$

Also

$$C_3C_1 = \frac{2\Delta\sqrt{u_3^2 + u_1^2 + 2u_3u_1\cos\alpha_2}}{\sin\theta(a_1u_1 + a_2u_2 + a_3u_3)}$$

and

$$C_1C_2 = \frac{2\Delta\sqrt{u_1^2 + u_2^2 + 2u_1u_2\cos\alpha_3}}{\sin\theta(a_1u_1 + a_2u_2 + a_3u_3)}$$

<u>Theorem</u>: Conversely, suppose that a Miquel triangle has its sides in the ratios $v_1 : v_2 : v_3$, then it is possible to solve for $u_1 : u_2 : u_3$. It is also possible to solve for (u_1, u_2, u_3) in terms of the angles β_i of the Miquel triangle:

<u>Proof</u>: The following equation is immediate from the previous theorem:

$$v_1:v_2:v_3 = \sqrt{u_2^2+u_3^2+2u_2u_3\cos\alpha_1} : \sqrt{u_3^2+u_1^2+2u_3u_1\cos\alpha_2} : \sqrt{u_1^2+u_2^2+2u_1u_2\cos\alpha_3},$$

and if we let $c = u_1^2+u_2^2+2u_1u_2\cos\alpha_3$,

then:

$$\frac{v_1^2}{v_3^2} = \frac{u_2^2+u_3^2+2u_2u_3\cos\alpha_1}{c} \quad\text{and}\quad \frac{v_2^2}{v_3^2} = \frac{u_3^2+u_1^2+2u_3u_1\cos\alpha_2}{c},$$

or

$$u_2^2+u_3^2+2u_2u_3\cos\alpha_1 - \frac{cv_1^2}{v_3^2} = 0 \quad\text{and}\quad u_3^2+u_1^2+2u_3u_1\cos\alpha_2 - \frac{cv_2^2}{v_3^2} = 0,$$

$$(u_3+u_2\cos\alpha_1)^2 = \frac{cv_1^2}{v_3^2} - u_2^2\sin^2\alpha_1 \quad\text{and}\quad (u_3+u_1\cos\alpha_2)^2 = \frac{cv_2^2}{v_3^2} - u_1^2\sin^2\alpha_2,$$

which give:

$$u_3+u_2\cos\alpha_1 = \pm\sqrt{\frac{cv_1^2-v_3^2u_2^2\sin^2\alpha_1}{v_3^2}} \quad \text{........} \quad (1)$$

$$u_3+u_1\cos\alpha_2 = \pm\sqrt{\frac{cv_2^2-v_3^2u_1^2\sin^2\alpha_2}{v_3^2}} \quad \text{.......} \quad (2)$$

Eliminating u_3,

$$u_2\cos\alpha_1-u_1\cos\alpha_2 = \pm\sqrt{\frac{cv_1^2-v_3^2u_2^2\sin^2\alpha_1}{v_3^2}} \pm \sqrt{\frac{cv_2^2-v_3^2u_1^2\sin^2\alpha_2}{v_3^2}},$$

squaring both sides:

$$u_2^2\cos^2\alpha_1+u_1^2\cos^2\alpha_2 - 2u_1u_2\cos\alpha_1\cos\alpha_2 = \frac{cv_1^2-v_3^2u_2^2\sin^2\alpha_1+cv_2^2-v_3^2u_1^2\sin^2\alpha_2}{v_3^2}$$

$$\pm 2\sqrt{\frac{cv_1^2-v_3^2u_2^2\sin^2\alpha_1}{v_3^2}}\sqrt{\frac{cv_2^2-v_3^2u_1^2\sin^2\alpha_2}{v_3^2}},$$

or

$$\frac{c(v_1^2+v_2^2-v_3^2)+2v_3^2u_1u_2\sin\alpha_1\sin\alpha_2}{v_3^2} = \pm 2\sqrt{\frac{cv_1^2-v_3^2u_2^2\sin^2\alpha_1}{v_3^2}}\sqrt{\frac{cv_2^2-v_3^2u_1^2\sin^2\alpha_2}{v_3^2}};$$

squaring again:

$$\frac{c^2(v_1^2+v_2^2-v_3^2)^2+4cv_3^2(v_1^2+v_2^2-v_3^2)u_1u_2\sin\alpha_1\sin\alpha_2+4v_3^4u_1^2u_2^2\sin^2\alpha_1\sin^2\alpha_2}{v_3^4}$$

$$=\frac{4c^2v_1^2v_2^2-4cv_1^2v_3^2u_1^2\sin^2\alpha_2-4cv_2^2v_3^2u_2^2\sin^2\alpha_1+4v_3^4u_1^2u_2^2\sin^2\alpha_1\sin^2\alpha_2}{v_3^4}$$

or $$c\Big[u_1^2\Big\{(v_1^2+v_2^2-v_3^2)^2-4v_1^2v_2^2+4v_1^2v_3^2\sin^2\alpha_2\Big\}+u_2^2\Big\{(v_1^2+v_2^2-v_3^2)^2-4v_1^2v_2^2+4v_2^2v_3^2\sin^2\alpha_1\Big\}$$
$$+2u_1u_2\Big\{(v_1^2+v_2^2-v_3^2)^2\cos\alpha_3+2v_3^2(v_1^2+v_2^2-v_3^2)\sin\alpha_1\sin\alpha_2-4v_1^2v_2^2\cos\alpha_3\Big\}\Big]=0.$$

β_i being the angles of the Miquel triangle:

$$v_1 : v_2 : v_3 = \sin\beta_1 : \sin\beta_2 : \sin\beta_3;$$

and the cosine law gives:

$$v_1^2+v_2^2-v_3^2 = 2v_1v_2\cos\beta_3.$$

Thus the equation becomes:

$$c\Big[4v_1^2u_1^2(\sin^2\beta_3\sin^2\alpha_2-\sin^2\beta_2\sin^2\beta_3)+4v_2^2u_2^2(\sin^2\beta_3\sin^2\alpha_1-\sin^2\beta_1\sin^2\beta_3)$$
$$+8v_1v_2u_1u_2(\sin\beta_1\sin\beta_2\cos^2\beta_3\cos\alpha_3+\sin^2\beta_3\cos\beta_3\sin\alpha_1\sin\alpha_2$$
$$-\sin\beta_1\sin\beta_2\cos\alpha_3)\Big]=0$$

$$4c.\sin^2\beta_3\Big[v_1^2u_1^2(\sin^2\alpha_2-\sin^2\beta_2)+v_2^2u_2^2(\sin^2\alpha_1-\sin^2\beta_1)$$
$$2v_1v_2u_1u_2(\sin\alpha_1\sin\alpha_2\cos\beta_3-\sin\beta_1\sin\beta_2\cos\alpha_3)\Big]=0.$$

Now $c = u_1^2 + u_2^2 + 2u_1u_2\cos\alpha_3 \neq 0$ for any real value of $\frac{u_1}{u_2}$;
and if the triangle is not degenerate, $\pi \neq \beta_i \neq 0$ and $\sin\beta_i \neq 0$
so that:

$$v_1^2(\sin^2\alpha_2-\sin^2\beta_2)u_1^2+v_2^2(\sin^2\alpha_1-\sin^2\beta_1)u_2^2$$

$$+2v_1v_2(\sin\alpha_1\sin\alpha_2\cos\beta_3-\sin\beta_1\sin\beta_2\cos\alpha_3)u_1u_2 = 0.$$

The discriminant is given by:

$$v_1^2v_2^2(\sin\alpha_1\sin\alpha_2\cos\beta_3-\sin\beta_1\sin\beta_2\cos\alpha_3)^2-v_1^2v_2^2(\sin^2\alpha_2-\sin^2\beta_2)(\sin^2\alpha_1-\sin^2\beta_1)$$

$$= v_1^2v_2^2(\cos\alpha_2\sin\alpha_1\sin\beta_2\cos\beta_1-\sin\alpha_2\cos\alpha_1\cos\beta_2\sin\beta_1)^2$$

Therefore

$$\frac{u_1}{u_2} = \frac{-v_1v_2(\sin\alpha_1\sin\alpha_2\cos\beta_3-\sin\beta_1\sin\beta_2\cos\alpha_3)\pm v_1v_2(\cos\alpha_2\sin\alpha_1\sin\beta_2\cos\beta_1-\sin\alpha_2\cos\alpha_1\cos\beta_2\sin\beta_1)}{v_1^2(\sin^2\alpha_2-\sin^2\beta_2)}$$

$$\frac{u_1}{u_2} = \frac{v_2}{v_1}\cdot\frac{(\sin\beta_1\sin\beta_2\cos\alpha_3-\sin\alpha_1\sin\alpha_2\cos\beta_3)\pm(\cos\alpha_2\sin\alpha_1\sin\beta_2\cos\beta_1-\sin\alpha_2\cos\alpha_1\cos\beta_2\sin\beta_1)}{(\sin^2\alpha_2-\sin^2\beta_2)}$$

$$\frac{u_1}{u_2} = \frac{v_2}{v_1}\cdot\frac{(\sin\alpha_1\sin\alpha_2\cos\beta_1\cos\beta_2\pm\cos\alpha_2\sin\alpha_1\sin\beta_2\cos\beta_1)-(\sin\beta_1\sin\beta_2\cos\alpha_1\cos\alpha_2\pm\sin\alpha_2\cos\alpha_1\cos\beta_2\sin\beta_1)}{\sin(\alpha_2+\beta_2)\sin(\alpha_2-\beta_2)}$$

$$\frac{u_1}{u_2} = \frac{v_2}{v_1}\cdot\frac{\sin(\alpha_1\pm\beta_1)}{\sin(\alpha_2\pm\beta_2)} = \frac{\sin\beta_2}{\sin\beta_1}\cdot\frac{\sin(\alpha_1\pm\beta_1)}{\sin(\alpha_2\pm\beta_2)} = \frac{\dfrac{\sin(\alpha_1\pm\beta_1)}{\sin\beta_1}}{\dfrac{\sin(\alpha_2\pm\beta_2)}{\sin\beta_2}}$$

Thus

$$\frac{u_2}{u_3} = \frac{\dfrac{\sin(\alpha_2 \pm \beta_2)}{\sin \beta_2}}{\dfrac{\sin(\alpha_3 \pm \beta_3)}{\sin \beta_3}} .$$

This may be used to calculate $\frac{u_1}{u_3}$ and $\frac{u_2}{u_1}$.

Finally

$$u_1 : u_2 : u_3 = \frac{\sin(\alpha_1 \pm \beta_1)}{\sin \beta_1} : \frac{\sin(\alpha_2 \pm \beta_2)}{\sin \beta_2} : \frac{\sin(\alpha_3 \pm \beta_3)}{\sin \beta_3} .$$

Hence

$$(u_1, u_2, u_3) \sim \left(\frac{\sin(\alpha_1 \pm \beta_1)}{\sin \beta_1}, \frac{\sin(\alpha_2 \pm \beta_2)}{\sin \beta_2}, \frac{\sin(\alpha_3 \pm \beta_3)}{\sin \beta_3}\right)$$

$$\sim (a_1(\cot \alpha_1 \pm \cot \beta_1), a_2(\cot \alpha_2 \pm \cot \beta_2), a_3(\cot \alpha_3 \pm \cot \beta_3)) .$$

We note that this is in fact a pair of points which correspond to one of the six positions in which the given triangle can be inscribed, and we shall call them associated points.

We now investigate the properties of the <u>twelve</u> Miquel points obtained for a given shape of the Miquel triangle.

<u>Theorem</u>: Every two associated points are inverse with respect to the circumcircle of the triangle of reference.

<u>Proof</u>: Two things must be shown:

(i) The two associated points $P(a(\cot \alpha + \cot \beta))$ and $P'(a(\cot \alpha - \cot \beta))$ and the circumcenter O are collinear.

(ii) The line PP' meets the circumcircle in two points M and N, harmonic conjugates with respect to P and P'.

(i) Since the following determinant:

$$\begin{vmatrix} \cos\alpha_1 & \cos\alpha_2 & \cos\alpha_3 \\ a_1(\cot\alpha_1+\cot\beta_1) & a_2(\cot\alpha_2+\cot\beta_2) & a_3(\cot\alpha_3+\cot\beta_3) \\ a_1(\cot\alpha_1-\cot\beta_1) & a_2(\cot\alpha_2-\cot\beta_2) & a_3(\cot\alpha_3-\cot\beta_3) \end{vmatrix}$$

$$= \begin{vmatrix} \cos\alpha_1 & \cos\alpha_2 & \cos\alpha_3 \\ 2a_1\cot\alpha_1 & 2a_2\cot\alpha_2 & 2a_3\cot\alpha_3 \\ a_1(\cot\alpha_1-\cot\beta_1) & a_2(\cot\alpha_2-\cot\beta_2) & a_3(\cot\alpha_3-\cot\beta_3) \end{vmatrix}$$

(by adding the third row to the second)

$$= 4R \begin{vmatrix} \cos\alpha_1 & \cos\alpha_2 & \cos\alpha_3 \\ \cos\alpha_1 & \cos\alpha_2 & \cos\alpha_3 \\ a_1(\cot\alpha_1-\cot\beta_1) & a_2(\cot\alpha_2-\cot\beta_2) & a_3(\cot\alpha_3-\cot\beta_3) \end{vmatrix}$$

(using the law of sines)

$= 0.$

Then O, P and P' are collinear and (i) follows.

Now for the proof of (ii) we note the following:

$P(u_1, u_2, u_3)$, $P'(v_1, v_2, v_3)$, $M(x_1, x_2, x_3)$ and $N(y_1, y_2, y_3)$

will form a harmonic division iff:

$$\frac{\frac{u_1}{u_2} - \frac{x_1}{x_2}}{\frac{u_1}{u_2} - \frac{y_1}{y_2}} = - \frac{\frac{v_1}{v_2} - \frac{x_1}{x_2}}{\frac{v_1}{v_2} - \frac{y_1}{y_2}}$$

or

$$\left(\frac{x_1}{x_2} + \frac{y_1}{y_2}\right) \cdot \left(\frac{u_1}{u_2} + \frac{v_1}{v_2}\right) = 2 \cdot \left(\frac{x_1}{x_2} \cdot \frac{y_1}{y_2} + \frac{u_1}{u_2} \cdot \frac{v_1}{v_2}\right).$$

Thus we only need to find the sum $\frac{x_1}{x_2} + \frac{y_1}{y_2}$ and the product $\frac{x_1}{x_2} \cdot \frac{y_1}{y_2}$ without actually calculating (x_i) and (y_i), and then to check the second formula; this is now done:

The equation of PP':

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ a_1(\cot\alpha_1+\cot\beta_1) & a_2(\cot\alpha_2+\cot\beta_2) & a_3(\cot\alpha_3+\cot\beta_3) \\ a_1(\cot\alpha_1-\cot\beta_1) & a_2(\cot\alpha_2-\cot\beta_2) & a_3(\cot\alpha_3-\cot\beta_3) \end{vmatrix} = 0,$$

i.e., $a_2a_3(\cot\alpha_3\cot\beta_2-\cot\alpha_2 \cdot \cot\beta_3)x_1+a_3a_1(\cot\alpha_1\cot\beta_3-\cot\alpha_3\cot\beta_1)x_2$

$$+a_1a_2(\cot\alpha_2\cot\beta_1-\cot\alpha_1\cot\beta_2)x_3 = 0.$$

The equation of the circumcircle is:

$$a_1x_2x_3 + a_2x_3x_1 + a_3x_1x_2 = 0$$

or

$$a_3x_1x_2 + (a_1x_2 + a_2x_1)x_3 = 0.$$

Eliminating x_3 from both equations we get:

$$a_2^2(\cot\alpha_3\cot\beta_2-\cot\alpha_2\cot\beta_3)x_1^2$$

$$+a_1a_2(\cot\alpha_1\cot\beta_2-\cot\alpha_2\cot\beta_1+\cot\alpha_3\cot\beta_2-\cot\alpha_2\cot\beta_3$$

$$+\cot\alpha_1\cot\beta_3-\cot\alpha_3\cot\beta_1)x_1x_2$$

$$+a_1^2(\cot\alpha_1\cot\beta_3 - \cot\alpha_3\cot\beta_1)x_2^2 = 0.$$

$\frac{x_1}{x_2}$ and $\frac{y_1}{y_2}$ are the two roots of this equation.

Therefore

$$\frac{x_1}{x_2}+\frac{y_1}{y_2}=\frac{a_1a_2(\cot\alpha_2\cot\beta_1-\cot\alpha_1\cot\beta_2+\cot\alpha_2\cot\beta_3-\cot\alpha_3\cot\beta_2+\cot\alpha_3\cot\beta_1-\cot\alpha_1\cot\beta_3)}{a_2^2(\cot\alpha_3\cot\beta_2-\cot\alpha_2\cdot\cot\beta_3)}$$

and

$$\frac{x_1}{x_2}\cdot\frac{y_1}{y_2}=\frac{a_1^2(\cot\alpha_1\cot\beta_3-\cot\alpha_3\cot\beta_1)}{a_2^2(\cot\alpha_3\cot\beta_2-\cot\alpha_2\cot\beta_3)};$$

on the other hand:

$$\frac{u_1}{u_2}+\frac{v_1}{v_2}=\frac{a_1(\cot\alpha_1+\cot\beta_1)}{a_2(\cot\alpha_2+\cot\beta_2)}+\frac{a_1(\cot\alpha_1-\cot\beta_1)}{a_2(\cot\alpha_2-\cot\beta_2)}$$

$$=\frac{a_1}{a_2}\cdot\frac{2(\cot\alpha_1\cot\alpha_2-\cot\beta_1\cot\beta_2)}{(\cot^2\alpha_2-\cot^2\beta_2)}$$

and

$$\frac{u_1}{u_2}\cdot\frac{v_1}{v_2}=\frac{a_1^2}{a_2^2}\cdot\frac{\cot^2\alpha_1-\cot^2\beta_1}{\cot^2\alpha_2-\cot^2\beta_2}.$$

Using the relations $\sum\cot\alpha=\cot\omega$ and $\sum\cot\beta=\cot\omega'$ - where ω and ω' are the Brocard angles of the triangle of reference and the Miquel triangle respectively - and simplifying.

$$\frac{x_1}{x_2}+\frac{y_1}{y_2}=\frac{a_1}{a_2^2}\cdot\frac{\cot^2\beta_1\csc^2\alpha_2-\cot^2\alpha_1\csc^2\beta_2+\cot^2\alpha_2-\cot^2\beta_2}{\cot\beta_1\cot\beta_2\csc^2\alpha_2-\cot\alpha_1\cot\alpha_2\csc^2\beta_2+\cot^2\beta_2-\cot^2\alpha_2}$$

and

$$\frac{x_1}{x_2}\cdot\frac{y_1}{y_2}=\frac{a_1^2}{a_2^2}\cdot\frac{\cot\alpha_1\cot\alpha_2\csc^2\beta_1-\cot\beta_1\cot\beta_2\csc^2\alpha_1+\cot^2\alpha_1-\cot^2\beta_1}{\cot\beta_1\cot\beta_2\csc^2\alpha_2-\cot\alpha_1\cot\alpha_2\csc^2\beta_2+\cot^2\beta_2-\cot^2\alpha_2}$$

Then

$$\left\{\frac{u_1}{u_2}+\frac{v_1}{v_2}\right\}\left\{\frac{x_1}{x_2}+\frac{y_1}{y_2}\right\}=\frac{2a_1^2}{a_2^2}\cdot\frac{(\cot\alpha_1\cot\alpha_2-\cot\beta_1\cot\beta_2)(\cot^2\beta_1\csc^2\alpha_2-\cot^2\alpha_1\csc^2\beta_2+\cot^2\alpha_2-\cot^2\beta_2)}{(\cot^2\alpha_2-\cot^2\beta_2)(\cot\beta_1\cot\beta_2\csc^2\alpha_2-\cot\alpha_1\cot\alpha_2\csc^2\beta_2+\cot^2\beta_2-\cot^2\alpha_2)}$$

and

$$2\left[\frac{u_1}{u_2}\cdot\frac{v_1}{v_2}+\frac{x_1}{x_2}\cdot\frac{y_1}{y_2}\right]$$

$$= 2\left[\frac{a_1^2}{a_2^2}\cdot\frac{(\cot^2\alpha_1-\cot^2\beta_1)}{(\cot^2\alpha_2-\cot^2\beta_2)}+\frac{a_1^2(\cot\alpha_1\cot\alpha_2\csc^2\beta_1-\cot\beta_1\cot\beta_2\csc^2\alpha_1+\cot^2\alpha_1-\cot^2\beta_1)}{a_2^2(\cot\beta_1\cot\beta_2\csc^2\alpha_2-\cot\alpha_1\cot\alpha_2\csc^2\beta_2+\cot^2\beta_2-\cot^2\alpha_2)}\right]$$

$$=\frac{2a_1^2}{a_2^2}\left[\frac{\begin{array}{r}\cot\alpha_1\cot\alpha_2(\cot^2\beta_1\csc^2\beta_2-\cot^2\alpha_1\csc^2\beta_2+\cot^2\alpha_2\csc^2\beta_1-\cot^2\beta_2\csc^2\beta_1)-\cot\beta_1\cot\beta_2\cdot\\ (\cot^2\beta_1\csc^2\alpha_2-(\cot^2\alpha_1\csc^2\alpha_2+\cot^2\alpha_2\csc^2\alpha_1-\cot^2\beta_2\csc^2\alpha_1)\end{array}}{(\cot^2\alpha_2-\cot^2\beta_2)(\cot\beta_1\cot\beta_2\csc^2\alpha_2-\cot\alpha_1\cot\alpha_2\csc^2\beta_2+\cot^2\beta_2-\cot^2\alpha_2)}\right]$$

and the equalities:

$$\cot^2\beta_1\csc^2\beta_2-\cot^2\alpha_1\csc^2\beta_2+\cot^2\alpha_2\csc^2\beta_1-\cot^2\beta_2\csc^2\beta_1$$

$$=\cot^2\beta_1\csc^2\alpha_2-\cot^2\alpha_1\csc^2\alpha_2+\cot^2\alpha_2\csc^2\alpha_1-\cot^2\beta_2\csc^2\alpha_1$$

$$=\cot^2\beta_1\csc^2\alpha_2-\cot^2\alpha_1\csc^2\beta_2+\cot^2\alpha_2-\cot^2\beta_2$$

give

$$\left[\frac{u_1}{u_2}+\frac{v_1}{v_2}\right]\left[\frac{x_1}{x_2}+\frac{y_1}{y_2}\right]=2\left[\frac{u_1}{u_2}\cdot\frac{v_1}{v_2}+\frac{x_1}{x_2}\cdot\frac{y_1}{y_2}\right]$$

hence (ii).

<u>Theorem</u>: The twelve Miquel points of a given triangle, lie six by six on two circles. The two circles are inverse with respect to the circumcircle, and thus are coaxal with the circumcircle.

<u>Proof</u>: Consider the six points:

$Q_1(a_1(\cot\alpha_1+\cot\beta_1),\ a_2(\cot\alpha_2+\cot\beta_2),\ a_3(\cot\alpha_3+\cot\beta_3))$

$Q_2(a_1(\cot\alpha_1+\cot\beta_1),\ a_2(\cot\alpha_2+\cot\beta_3),\ a_3(\cot\alpha_3+\cot\beta_2))$

$Q_3(a_1(\cot\alpha_1+\cot\beta_2),\ a_2(\cot\alpha_2+\cot\beta_1),\ a_3(\cot\alpha_3+\cot\beta_3))$

$$Q_4(a_1(\cot\alpha_1+\cot\beta_2),\ a_2(\cot\alpha_2+\cot\beta_3),\ a_3(\cot\alpha_3+\cot\beta_1))$$

$$Q_5(a_1(\cot\alpha_1+\cot\beta_3),\ a_2(\cot\alpha_2+\cot\beta_2),\ a_3(\cot\alpha_3+\cot\beta_1))$$

$$Q_6(a_1(\cot\alpha_1+\cot\beta_3),\ a_2(\cot\alpha_2+\cot\beta_1),\ a_3(\cot\alpha_3+\cot\beta_2)).$$

The line Q_1Q_2 is given by

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ a_1(\cot\alpha_1+\cot\beta_1) & a_2(\cot\alpha_2+\cot\beta_2) & a_3(\cot\alpha_3+\cot\beta_3) \\ a_1(\cot\alpha_1+\cot\beta_1) & a_2(\cot\alpha_2+\cot\beta_3) & a_3(\cot\alpha_3+\cot\beta_2) \end{vmatrix} = 0,$$

i.e. $$a_2a_3\left(\frac{\cot\omega+\cot\omega'}{\cot\alpha_1+\cot\beta_1}-1\right)x_1 - a_3a_1x_2 - a_1a_2x_3 = 0.$$

The line Q_3Q_4 is:

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ a_1(\cot\alpha_1+\cot\beta_2) & a_2(\cot\alpha_2+\cot\beta_1) & a_3(\cot\alpha_3+\cot\beta_3) \\ a_1(\cot\alpha_1+\cot\beta_2) & a_2(\cot\alpha_2+\cot\beta_3) & a_3(\cot\alpha_3+\cot\beta_1) \end{vmatrix} = 0,$$

i.e., $$a_2a_3\left(\frac{\cot\omega+\cot\omega'}{\cot\alpha_1+\cot\beta_2}-1\right)x_1 - a_3a_1x_2 - a_1a_2x_3 = 0.$$

Multiplication gives a degenerate conic through Q_1, Q_2, Q_3, Q_4:

$$\left\{a_2a_3\left(\frac{\cot\omega+\cot\omega'}{\cot\alpha_1+\cot\beta_1}-1\right)x_1-a_3a_1x_2-a_1a_2x_3\right\}\left\{a_2a_3\left(\frac{\cot\omega+\cot\omega'}{\cot\alpha_1+\cot\beta_2}-1\right)x_1-a_3a_1x_2-a_1a_2x_3\right\}=0.$$

Similarly, the lines Q_1Q_3 and Q_2Q_4 are given by

$$a_2a_3x_1 + a_3a_1x_2 - a_1a_2\left(\frac{\cot\omega+\cot\omega'}{\cot\alpha_3+\cot\beta_3}-1\right)x_3 = 0$$

and

$$a_2a_3x_1 - a_3a_1\left(\frac{\cot\omega + \cot\omega'}{\cot\alpha_2 + \cot\beta_3} - 1\right)x_2 + a_1a_2x_3 = 0,$$

respectively.

Thus we obtain the equation of another degenerate conic through Q_1, Q_3, Q_2, Q_4:

$$\left\{a_2a_3x_1 + a_3a_1x_2 - a_1a_2\left(\frac{\cot\omega+\cot\omega'}{\cot\alpha_3+\cot\beta_3} - 1\right)x_3\right\}\left\{a_2a_3x_1 - a_3a_1\left(\frac{\cot\omega+\cot\omega'}{\cot\alpha_2+\cot\beta_3} - 1\right)x_2 + a_1a_2x_3\right\} = 0$$

The equation of the family of conics passing through Q_1, Q_2, Q_3 and Q_4 may be expressed as:

$$\left\{a_2a_3\left(\frac{\cot\omega+\cot\omega'}{\cot\alpha_1+\cot\beta_1} - 1\right)x_1 - a_3a_1x_2 - a_1a_2x_3\right\}\left\{a_2a_3\left(\frac{\cot\omega+\cot\omega'}{\cot\alpha_1+\cot\beta_2} - 1\right)x_1 - a_3a_1x_2 - a_1a_2x_3\right\}$$

$$+\lambda\left\{a_2a_3x_1 + a_3a_1x_2 - a_1a_2\left(\frac{\cot\omega+\cot\omega'}{\cot\alpha_3+\cot\beta_3} - 1\right)x_3\right\}\left\{a_2a_3x_1 - a_3a_1\left(\frac{\cot\omega+\cot\omega'}{\cot\alpha_2+\cot\beta_3} - 1\right)x_2 + a_1a_2x_3\right\} = 0$$

where λ is a constant.

To determine λ we substitute the coordinates of Q_5 and solves; this gives:

$$\lambda = -\frac{(\cot\alpha_3 + \cot\beta_3)(\cot\alpha_2 + \cot\beta_3)}{(\cot\alpha_1 + \cot\beta_1)(\cot\alpha_1 + \cot\beta_2)}.$$

Equations of the conic through Q_1, Q_2, Q_3, Q_4, Q_5

Substituting the value of λ obtained we get after simplification:

$$\frac{\cot\omega - \cot\alpha_1}{a_1^2}x_1^2 + \frac{\cot\omega - \cot\alpha_2}{a_2^2}x_2^2 + \frac{\cot\omega - \cot\alpha_3}{a_3^2}x_3^2$$

$$+ \frac{\cot\alpha_3 - \cot\omega'}{a_1a_2}x_1x_2 + \frac{\cot\alpha_2 - \cot\omega'}{a_3a_1}x_3x_1 + \frac{\cot\alpha_1 - \cot\omega'}{a_2a_3}x_2x_3 = 0$$

or

$$x_1^2 + x_2^2 + x_3^2 - \sum\frac{\sin(\alpha_1 - \omega')}{\sin\omega'}x_2x_3 = 0.$$

The symmetry of this equation shows that Q_6 lies on this conic.

From the line at infinity $\sum ax = 0$

we have

$$x_3 = \frac{-a_2x_2 - a_1x_1}{a_3},$$

which we substitute for x_3 in the equation of the conic, and cancelling the common factor $\dfrac{\sin\alpha_1 \sin\alpha_2 \sin\alpha_3 \sin(\omega+\omega')}{\sin^2\alpha_3 \sin\omega \sin\omega'}$ we get

$$x_1^2 + x_2^2 + 2\cos\alpha_3 x_1x_2 = 0,$$

which yields

$$\frac{x_1}{x_2} = -\cos\alpha_3 \pm \sqrt{\cos^2\alpha_3 - 1} = -\cos\alpha_3 \pm i.\sin\alpha_3.$$

This may be used to find $\frac{x_1}{x_3}$ or $\frac{x_3}{x_2}$. Finally we have

$$(x_1, x_2, x_3) \sim (-\cos\alpha_3 \pm i\cdot\sin\alpha_3, 1, -\cos\alpha_1 \mp i.\sin\alpha_1).$$

Therefore the conic cuts the line $\sum ax = 0$ in the two circular points and is therefore a circle. (See appendix).

If we consider the other six points

$$Q_1'(a_1(\cot\alpha_1 - \cot\beta_1), a_2(\cot\alpha_2 - \cot\beta_2), a_3(\cot\alpha_3 - \cot\beta_3)) \text{ etc.}$$

We also find that $Q_1', Q_2', Q_3', \dots, Q_6'$ all lie on a circle of equation:

$$x_1^2 + x_2^2 + x_3^2 + \sum \frac{\sin(\alpha_1+\omega')}{\sin\omega'} x_2x_3 = 0.$$

This proves the first part of the theorem.

Now consider the first circle of equation:

$$x_1^2 + x_2^2 + x_3^2 - \sum \frac{\sin(\alpha_1-\omega')}{\sin\omega'} x_2x_3 = 0$$

and let y_1, y_2 and y_3 be the coordinates of the center of this circle, then y_1, y_2 and y_3 are the solutions of the following set of equations:

$$y_1 - \frac{\sin(\alpha_3 - \omega')}{2.\sin\omega'}\, y_2 - \frac{\sin(\alpha_2 - \omega')}{2\sin\omega'}\, y_3 \sim a_1$$

$$- \frac{\sin(\alpha_3 - \omega')}{2\sin\omega'}\, y_1 + y_2 - \frac{\sin(\alpha_1 - \omega')}{2\sin\omega'} . y_3 \sim a_2$$

$$- \frac{\sin(\alpha_2 - \omega')}{2\sin\omega'} . y_1 - \frac{\sin(\alpha_1 - \omega')}{2\sin\omega'} . y_2 + y_3 \sim a_3$$

$$y_1 \sim \begin{vmatrix} a_1 & \frac{\sin(\omega' - \alpha_3)}{2\sin\omega'} & \frac{\sin(\omega' - \alpha_2)}{2\sin\omega'} \\ a_2 & 1 & \frac{\sin(\omega' - \alpha_1)}{2\sin\omega'} \\ a_3 & \frac{\sin(\omega' - \alpha_1)}{2\sin\omega'} & 1 \end{vmatrix}$$

$$\sim \cos(\alpha_1 - \omega').$$

Therefore the center $(y_1, y_2, y_3) \sim (\cos(\alpha_1 - \omega'), \cos(\alpha_2 - \omega'), \cos(\alpha_3 - \omega'))$. If now we consider the second circle of equation

$$x_1^2 + x_2^2 + x_3^2 + \sum \frac{\sin(\alpha_1 + \omega')}{\sin\omega'}\, x_2 x_3 = 0$$

its center is found to be $(y_1', y_2', y_3') \sim (\cos(\alpha_1 + \omega'), \cos(\alpha_2 + \omega'), \cos(\alpha_3 + \omega'))$.

To determine the relation of these centers to the circumcenter, we see that

$$\begin{vmatrix} \cos\alpha_1 & \cos\alpha_2 & \cos\alpha_3 \\ \cos(\alpha_1 - \omega') & \cos(\alpha_2 - \omega') & \cos(\alpha_3 - \omega') \\ \cos(\alpha_1 + \omega') & \cos(\alpha_2 + \omega') & \cos(\alpha_3 + \omega') \end{vmatrix}$$

$$= \begin{vmatrix} \cos\alpha_1 & \cos\alpha_2 & \cos\alpha_3 \\ 2\cos\alpha_1\cos\omega' & 2\cos\alpha_2\cos\omega' & 2\cos\alpha_3\cos\omega' \\ \cos(\alpha_1+\omega') & \cos(\alpha_2+\omega') & \cos(\alpha_3+\omega') \end{vmatrix}$$

$$= 2\cos\omega' \begin{vmatrix} \cos\alpha_1 & \cos\alpha_2 & \cos\alpha_3 \\ \cos\alpha_1 & \cos\alpha_2 & \cos\alpha_3 \\ \cos(\alpha_1+\omega') & \cos(\alpha_2+\omega') & \cos(\alpha_3+\omega') \end{vmatrix} = 0,$$

which means that the three points are collinear. Also we see that

$$\begin{vmatrix} \sin\alpha_1 & \sin\alpha_2 & \sin\alpha_3 \\ \cos(\alpha_1-\omega') & \cos(\alpha_2-\omega') & \cos(\alpha_3-\omega') \\ \cos(\alpha_1+\omega') & \cos(\alpha_2+\omega') & \cos(\alpha_3+\omega') \end{vmatrix}$$

$$= \begin{vmatrix} \sin\alpha_1 & \sin\alpha_2 & \sin\alpha_3 \\ 2\sin\alpha_1\sin\omega' & 2\sin\alpha_2\sin\omega' & 2\sin\alpha_3\sin\omega' \\ \cos(\alpha_1+\omega') & \cos(\alpha_2+\omega') & \cos(\alpha_3+\omega') \end{vmatrix} = 0,$$

and so the Lemoine point K is also on the line.

Thus the centers of the two circles are always on the line OK. In other words their line of centers is fixed and independent (of the shape) of the inscribed triangle. The inverse of the first circle with respect to the circumcenter, must contain the inverse of the six points Q_i ($i = 1, \ldots, 6$) that is must contain

Q_i' (i = 1, ..., 6) and therefore has to coincide with the second circle and conversely; so the two circles are inverse with respect to the circumcircle. Finally since the circle of inversion is always coaxal with pairs of inverse circles with respect to it, the two circles are coaxal with the circumcircle.

An important special case of this theorem occurs when the inscribed triangle is similar to the triangle of reference.

Theorem: If the inscribed triangle is similar to the triangle of reference, then it has eleven Miquel points which lie six on a circle called the Brocard circle and five on a straight line called the Lemoine axis.

Proof: Putting $\beta_i = \alpha_i$ (i = 1, 2, 3) gives us eleven points and Q_1 reduces to (0,0,0) which is meaningless. Replacing ω' by ω the equation of the first circle

$$x_1^2 + x_2^2 + x_3^2 - \sum \frac{\sin(\alpha_1 - \omega')}{\sin \omega'} x_2 x_3 = 0$$

reduces to

$$x_1^2 + x_2^2 + x_3^2 - \frac{\sin^3\alpha_3 . x_1x_2 + \sin^3\alpha_2 . x_3x_1 + \sin^3\alpha_1 . x_2x_3}{\sin\alpha_1 . \sin\alpha_2 . \sin\alpha_3} = 0$$

which is the equation of the Brocard circle as obtained in chapter two.

The center $(\cos(\alpha_1 - \omega'))$ becomes $(\cos(\alpha_1 - \omega))$ which is the center of the Brocard circle.

On the other hand replacing ω' by ω in the equation of the second circle

$$x_1^2 + x_2^2 + x_3^2 + \sum \frac{\sin(\alpha_1 + \omega')}{\sin \omega'} x_2 x_3 = 0$$

reduces to

$$\left\{\sin\alpha_2.\sin\alpha_3x_1+\sin\alpha_3.\sin\alpha_1x_2+\sin\alpha_1.\sin\alpha_2x_3\right\}\left\{\sin\alpha_1.x_1+\sin\alpha_2.x_2+\sin\alpha_3x_3\right\} = 0$$

or

$$\left\{\frac{x_1}{a_1}+\frac{x_2}{a_2}+\frac{x_3}{a_3}\right\}\left\{a_1x_1+a_2x_2+a_3x_3\right\} = 0$$

and since for every finite point $\sum ax \neq 0$ we have $\frac{x_1}{a_1}+\frac{x_2}{a_2}+\frac{x_3}{a_3} = 0$ as the equation of the straight line (the Lemoine axis) containing the five associated points.

<u>Corollary</u>: The Lemoine axis is perpendicular to the line OK at the point $(\cos(\alpha_1+\omega))$.

<u>Proof</u>: OK being the diameter of the Brocard circle, and 0 the center of inversion, the Lemoine axis being the inverse of the circle is perpendicular to OK.

<u>Theorem</u>: If the inscribed triangle is equilateral, the twelve points reduce to two points which separate the Brocard diameter harmonically, called the isodynamic points.

<u>Proof</u>: Put $\beta = \pi/3$. The isodynamic points are

$$(\sin(\alpha_1+\pi/3),\ \sin(\alpha_2+\pi/3),\ \sin(\alpha_3+\pi/3))$$

and

$$(\sin(\alpha_1-\pi/3),\ \sin(\alpha_2-\pi/3),\ \sin(\alpha_3-\pi/3))$$

or

$$\sim (a_1(\cot\alpha_1+\frac{1}{\sqrt{3}}),\ a_2(\cot\alpha_2+\frac{1}{\sqrt{3}}),\ a_3(\cot\alpha_3+\frac{1}{\sqrt{3}}))$$

and

$$(a_1(\cot\alpha_1-\frac{1}{\sqrt{3}}),\ a_2(\cot\alpha_2-\frac{1}{\sqrt{3}}),\ a_3(\cot\alpha_3-\frac{1}{\sqrt{3}})).$$

In fact any two points of the form $\sin(\alpha \pm \theta)$ have the property of separating OK harmonically. To prove this we first note that the determinants

$$\begin{vmatrix} \sin\alpha_1 & \sin\alpha_2 & \sin\alpha_3 \\ \sin(\alpha_1+\theta) & \sin(\alpha_2+\theta) & \sin(\alpha_3+\theta) \\ \sin(\alpha_1-\theta) & \sin(\alpha_2-\theta) & \sin(\alpha_3-\theta) \end{vmatrix}$$

and

$$\begin{vmatrix} \cos\alpha_1 & \cos\alpha_2 & \cos\alpha_3 \\ \sin(\alpha_1+\theta) & \sin(\alpha_2+\theta) & \sin(\alpha_3+\theta) \\ \sin(\alpha_1-\theta) & \sin(\alpha_2-\theta) & \sin(\alpha_3-\theta) \end{vmatrix}$$

are both zero for any θ so that the points $(\sin(\alpha_i \pm \theta))$ are always collinear with 0 and k.

The second part of the proof consists in evaluating the cross ratio:

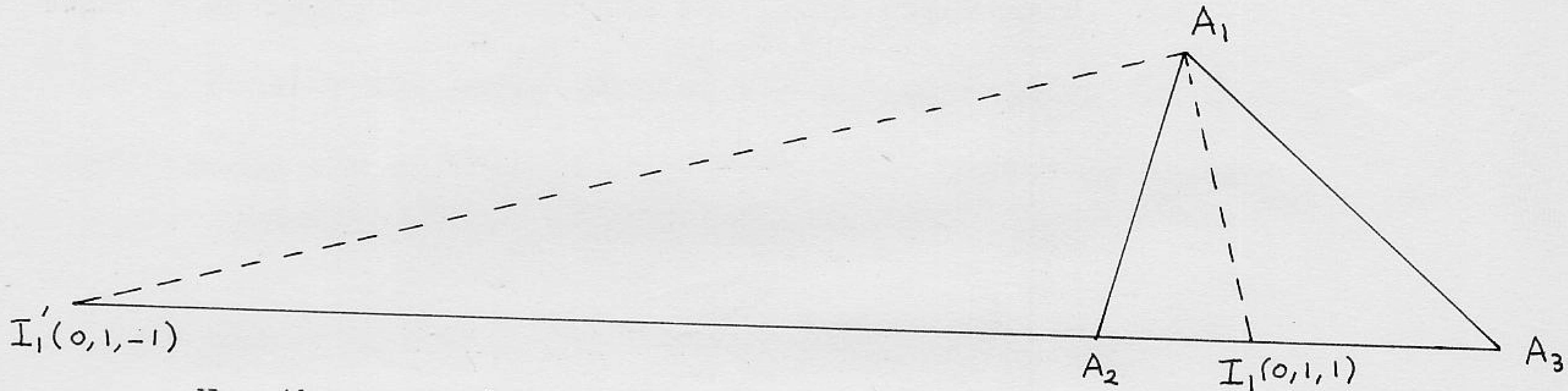
$$\frac{\frac{u_1}{u_2} - \frac{x_1}{x_2}}{\frac{u_1}{u_2} - \frac{y_1}{y_2}} \div \frac{\frac{v_1}{v_2} - \frac{x_1}{x_2}}{\frac{v_1}{v_2} - \frac{y_1}{y_2}}$$

$$= \frac{\frac{\sin(\alpha_1+\theta)}{\sin(\alpha_2+\theta)} - \frac{\sin\alpha_1}{\sin\alpha_2}}{\frac{\sin(\alpha_1+\theta)}{\sin(\alpha_2+\theta)} - \frac{\cos\alpha_1}{\cos\alpha_2}} \div \frac{\frac{\sin(\alpha_1-\theta)}{\sin(\alpha_2-\theta)} - \frac{\sin\alpha_1}{\sin\alpha_2}}{\frac{\sin(\alpha_1-\theta)}{\sin(\alpha_2-\theta)} - \frac{\cos\alpha_1}{\cos\alpha_2}}$$

$$= \frac{\frac{\sin\theta.\sin(\alpha_2-\alpha_1)}{\sin\alpha_2\sin(\alpha_2+\theta)}}{\frac{\cos\theta.\sin(\alpha_1-\alpha_2)}{\cos\alpha_2.\sin(\alpha_2+\theta)}} \div \frac{\frac{\sin\theta.\ \sin(\alpha_1-\alpha_2)}{\sin\alpha_2\sin(\alpha_2-\theta)}}{\frac{\cos\theta.\ \sin(\alpha_1-\alpha_2)}{\cos\ _2.\sin(\alpha_2-\theta)}} = -1.$$

<u>Theorem</u>: The isodynamic points are the two points of intersection of the three circles of Apollonius.

<u>Proof</u>: The first circle of Apollonius passes through the vertex $A_1(1, 0, 0)$ of the triangle of reference and has as diameter the line joining the foot of the interior bisector of $\angle A_3A_1A_2$, $I_1(0, 1, 1)$ and that of the exterior bisector of $\angle A_3A_1A_2$, $I_1'(0, 1, -1)$.



Now the general equation of a conic is:

$$a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{23}x_2x_3 + 2a_{31}x_3x_1 = 0.$$

Substitution of the coordinates of A_1, I_1 and I_1' gives

$$a_{11} = 0, \quad a_{22} = -a_{33}, \quad a_{23} = 0.$$

Substitution of the coordinates of the circular points at infinity gives:

$$\frac{a_{12}}{a_{22}} = \cos\alpha_3 \quad \text{and} \quad \frac{a_{31}}{a_{22}} = -\cos\alpha_2.$$

So that the equation of the first circle of Apollonius is:

$$x_2^2 - x_3^2 + (2\cos\alpha_3)x_1x_2 - (2\cos\alpha_2)x_3x_1 = 0.$$

The equations of two other circles of Apollonius follow at once:

$$- x_1^2 + x_3^2 - (2\cos\alpha_3)x_1x_2 + (2\cos\alpha_1)x_2x_3 = 0$$

$$x_1^2 - x_2^2 - (2\cos\alpha_1)x_2x_3 + (2\cos\alpha_2)x_3x_1 = 0$$

The fact that only two of these equations are independent shows that the three circles are concurrent.

Substitution of the coordinates $(\sin(\alpha + \pi/3))$ or $(\sin(\alpha - \pi/3))$ in any of the equations gives zero.

Finally the three circles are not coinciding for each contains one and only one vertex of $A_1A_2A_3$, hence the theorem.

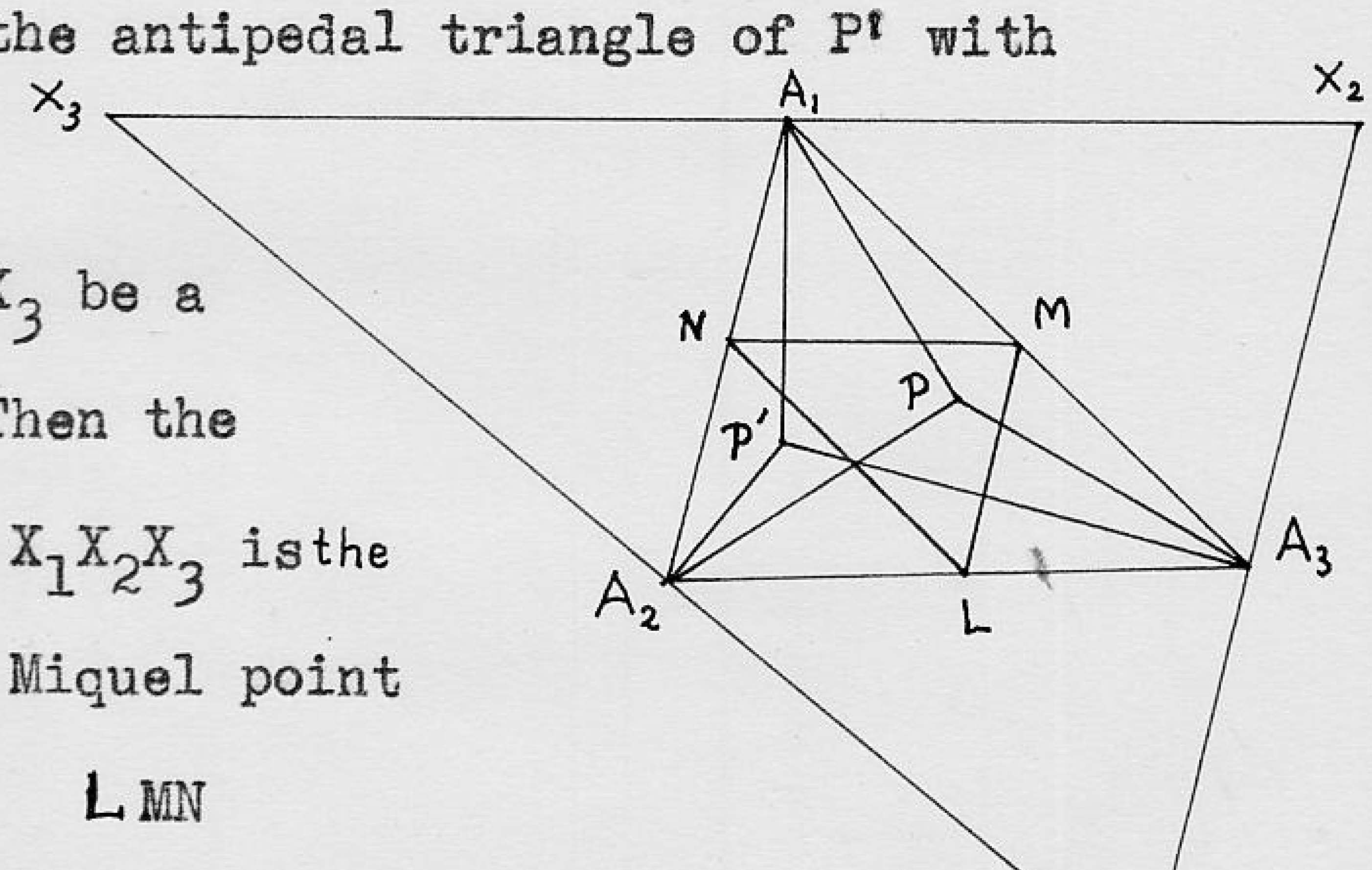
CHAPTER IV

TRIANGLES CIRCUMSCRIBED TO THE TRIANGLE OF REFERENCE

<u>Definition</u>: A triangle is circumscribed to the triangle of reference if the latter is inscribed in it.

<u>Definition</u>: If any point is connected with the vertices of a triangle, the lines through the vertices and perpendicular to these connectors are the sides of a triangle called the antipedal triangle of the point with regard to the given triangle.

Let $X_1X_2X_3$ be a triangle circumscribed to $A_1A_2A_3$. $X_1X_2X_3$ can be circumscribed in six different ways. In this chapter we shall determine the Miquel points of $A_1A_2A_3$ with respect to $X_1X_2X_3$, and investigate some of their properties. We first note that if P' is the Miquel point of $A_1A_2A_3$ with respect to $X_1X_2X_3$, then it is also the Miquel point of $A_1A_2A_3$ with respect to the antipedal triangle of P', which is similar to $X_1X_2X_3$. Therefore we may take $X_1X_2X_3$ to be the antipedal triangle of P' with respect to $A_1A_2A_3$.



<u>Theorem</u>: Let $X_1X_2X_3$ be a circumscribed triangle. Then the Miquel point of $A_1A_2A_3$ in $X_1X_2X_3$ is the isogonal conjugate of the Miquel point of the inscribed triangle LMN similar to $X_1X_2X_3$ and similarly placed.

Proof: Let P and P' be two isogonal conjugate points, with $X_1X_2X_3$ the antipedal triangle of P' and LMN the pedal triangle of P.

In the cyclic quadrilateral PMA_1N

$$\angle PA_1M = \angle PNM$$

and since

$$\angle PA_1M = \angle P'A_1N \quad \text{then } \angle PNM = \angle P'A_1N$$

then MN is perpendicular to A_1P'

or MN is parallel to X_2X_3.

Similarly LM is parallel to X_1X_2 and NL is parallel to X_3X_1 so that $X_1X_2X_3$ and LMN are similar and similarly placed.

This theorem enables us to determine the six Miquel points of $A_1A_2A_3$ in a circumscribed triangle and their six associated points by using the results of chapter three and a theorem of chapter one concerning the coordinates of isogonal conjugates.

Theorem: When the circumscribed triangle is similar to the triangle of reference, it has eleven Miquel points. Five of these lie on an ellipse called Steiner's ellipse and another four lie on a circle.

Proof: We first exhibit the points:

$$A_1(1, 0, 0)$$

$$A_2(0, 1, 0)$$

$$A_3(0, 0, 1)$$

$$\left(\frac{\sin\alpha_3}{\sin(\alpha_1-\alpha_3)}, \frac{\sin\alpha_1}{\sin(\alpha_2-\alpha_1)}, \frac{\sin\alpha_2}{\sin(\alpha_3-\alpha_2)}\right)$$

$$\left(\frac{\sin\alpha_2}{\sin(\alpha_1-\alpha_2)}, \frac{\sin\alpha_3}{\sin(\alpha_2-\alpha_3)}, \frac{\sin\alpha_1}{\sin(\alpha_3-\alpha_1)}\right).$$

These five points are the isogonal conjugates of those five points lying on the Lemoine axis and whose pedal triangles are similar to $A_1A_2A_3$. To find the equation of the conic through these we could start with the general equation of a conic $\sum_{i,j=1}^{3} a_{ij}X_iX_j = 0$ and substitute the coordinates of the five points to determine the coefficients a_{ij}. In this case simple computations give:

$$a_{11} = a_{22} = a_{33} = 0$$

$$a_{23} = \frac{1}{a_1}, \quad a_{31} = \frac{1}{a_2} \quad \text{and} \quad a_{12} = \frac{1}{a_3}.$$

Therefore the equation of the conic passing through these five points is:

$$\frac{x_2x_3}{a_1} + \frac{x_3x_1}{a_2} + \frac{x_1x_2}{a_3} = 0.$$

In order to find the type of this conic, we find its intersection with the line at infinity $\sum ax = 0$.

Simultaneous solution with the equation of the conic gives

$$-\frac{x_1x_2}{a_3} - \frac{a_1x_1^2}{a_2a_3} - \frac{a_2x_2^2}{a_1a_3} = 0$$

or

$$a_1^2x_1^2 + a_2^2x_2^2 + a_1a_2x_1x_2 = 0$$

$$\frac{x_1}{x_2} = \frac{-a_1a_2 \pm \sqrt{a_1^2a_2^2 - 4a_1^2a_2^2}}{2a_1^2} = \frac{-a_2 \pm a_2\sqrt{3}\cdot i}{2a_1} .$$

This shows that the line $\sum ax = 0$ cuts this conic in two imaginary points different from the circular points, it is therefore an ellipse.

The six other points are:

$$\Omega(\frac{a_3}{a_2}, \frac{a_1}{a_3}, \frac{a_2}{a_1}), \Omega'(\frac{a_2}{a_3}, \frac{a_3}{a_1}, \frac{a_1}{a_2})$$

$$P'_1(\frac{\sin\alpha_1}{2\cos\alpha_1}, \sin\alpha_3, \sin\alpha_2),$$

$$P'_2(\sin\alpha_3, \frac{\sin\alpha_2}{2\cos\alpha_2}, \sin\alpha_1),$$

$$P'_3(\sin\alpha_2, \sin\alpha_1, \frac{\sin\alpha_3}{2\cos\alpha_3})$$

$$H(\sec\alpha_1, \sec\alpha_2, \sec\alpha_3)$$

the orthocenter.

These six points are the isogonal conjugates of those six points on the Brocard circle and with pedal triangles similar to the triangle of reference.

From these six points, four are on a circle which also passes through $G(\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3})$ the centroid of $A_1 A_2 A_3$.

<u>The line HP'_1 is given by</u>

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ \frac{1}{\cos\alpha_1} & \frac{1}{\cos\alpha_2} & \frac{1}{\cos\alpha_3} \\ \frac{\sin\alpha_1}{2\cos\alpha_1} & \sin\alpha_3 & \sin\alpha_2 \end{vmatrix} = 0,$$

i.e., $(2\cos\alpha_1)x_1 - (\cos\alpha_2)x_2 - (\cos\alpha_3)x_3 = 0.$

<u>The line GP'_2 is</u>

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ \frac{1}{a_1} & \frac{1}{a_2} & \frac{1}{a_3} \\ \sin\alpha_3 & \frac{\sin\alpha_2}{2\cos\alpha_2} & \sin\alpha_1 \end{vmatrix} = 0,$$

i.e., $a_1x_1 - a_3x_3 = 0.$

Then

$$\left\{a_1x_1 - a_3x_3\right\}\left\{(2\cos\alpha_1)x_1-(\cos\alpha_2)x_2-(\cos\alpha_3)x_3\right\} = 0$$

is a degenerate conic through H, P'_1, G, P'_2.

Similarly, using the lines GP'_1 and HP'_2, we obtain a second degenerate conic

$$\left\{-a_2x_2 + a_3x_3\right\}\left\{(\cos\alpha_1)x_1-(2\cos\alpha_2)x_2 + (\cos\alpha_3)x_3\right\} = 0.$$

Then the family of conics through H, G, P'_1, P'_2 is given by

$$\left\{a_1x_1 - a_3x_3\right\}\left\{(2\cos\alpha_1)x_1 - (\cos\alpha_2)x_2-(\cos\alpha_3)x_3\right\}$$
$$+ \lambda\left\{-a_2x_2 + a_3x_3\right\}\left\{(\cos\alpha_1)x_1 - (2\cos\alpha_2)x_2 + (\cos\alpha_3)x_3\right\} = 0,$$

where $\lambda \neq 0$.

To determine λ for the conic through P'_3 substitute the coordinates of P'_3, which gives after simplification $\lambda = 1$, and the equation of the conic through P'_1, P'_2, P'_3, G, H is:

$$(\sin 2\alpha_1)x_1^2+(\sin 2\alpha_2)x_2^2+(\sin 2\alpha_3)x_3^2-(\sin\alpha_3)x_1x_2-(\sin\alpha_2)x_3x_1-(\sin\alpha_1)x_2x_3 = 0.$$

Calculations analogous to previous ones show that this conic is a circle.

<u>Theorem</u>: The circle through P_1', P_2', P_3', G and H has diameter GH.

<u>Proof</u>: Let y_1, y_2 and y_3 be the coordinates of the center of this circle.

$$y_1 \sim \begin{vmatrix} a_1 & \frac{-\sin\alpha_3}{2} & \frac{-\sin\alpha_2}{2} \\ a_2 & \sin 2\alpha_2 & \frac{-\sin\alpha_1}{2} \\ a_3 & \frac{-\sin\alpha_1}{2} & \sin 2\alpha_3 \end{vmatrix}$$

$$= 3R \sin\alpha_1 \sin\alpha_2 \sin\alpha_3(\cos\alpha_1 + 4\cos\alpha_2\cos\alpha_3)$$

similarly

$$y_2 \sim 3R \sin\alpha_1 \sin\alpha_2 \sin\alpha_3(\cos\alpha_2 + 4\cos\alpha_3\cos\alpha_1)$$

and

$$y_3 \sim 3R \sin\alpha_1 \sin\alpha_2 \sin\alpha_3(\cos\alpha_3 + 4\cos\alpha_1\cos\alpha_2).$$

Finally the center is:

$$(y_1, y_2, y_3) \sim (\cos\alpha_1 + 4\cos\alpha_2\cos\alpha_3, \cos\alpha_2 + 4\cos\alpha_3\cos\alpha_1, \cos\alpha_3 + 4\cos\alpha_1\cos\alpha_2).$$

Now consideration of the determinant

$$\begin{vmatrix} \cos\alpha_1 + 4\cos\alpha_2\cos\alpha_3 & \cos\alpha_2 + 4\cos\alpha_3\cos\alpha_1 & \cos\alpha_3 + 4\cos\alpha_1\cos\alpha_2 \\ \cos\alpha_1 & \cos\alpha_2 & \cos\alpha_3 \\ \frac{1}{\sin\alpha_1} & \frac{1}{\sin\alpha_2} & \frac{1}{\sin\alpha_3} \end{vmatrix}$$

$$= 4 \begin{vmatrix} \cos\alpha_2\cos\alpha_3 & \cos\alpha_3\cos\alpha_1 & \cos\alpha_1\cos\alpha_2 \\ \cos\alpha_1 & \cos\alpha_2 & \cos\alpha_3 \\ \frac{1}{\sin\alpha_1} & \frac{1}{\sin\alpha_2} & \frac{1}{\sin\alpha_3} \end{vmatrix}$$

$$= 4 \begin{vmatrix} \cos\alpha_2\cos\alpha_3 & \cos\alpha_3\cos\alpha_1 & \cos\alpha_1\cos\alpha_2 \\ \sin\alpha_2\sin\alpha_3 & \sin\alpha_3\sin\alpha_1 & \sin\alpha_1\sin\alpha_2 \\ \frac{1}{\sin\alpha_1} & \frac{1}{\sin\alpha_2} & \frac{1}{\sin\alpha_3} \end{vmatrix}$$

$$= K \begin{vmatrix} \cos\alpha_2\cos\alpha_3 & \cos\alpha_3\cos\alpha_1 & \cos\alpha_1\cos\alpha_2 \\ \sin\alpha_2\sin\alpha_3 & \sin\alpha_3\sin\alpha_1 & \sin\alpha_1\sin\alpha_2 \\ \frac{1}{\sin\alpha_1} & \frac{1}{\sin\alpha_2} & \frac{1}{\sin\alpha_3} \end{vmatrix} = 0$$

where $K = 4\sin\alpha_1\sin\alpha_2\sin\alpha_3$.

This shows that the points $G(\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}) \sim (\frac{1}{\sin\alpha_1}, \frac{1}{\sin\alpha_2}, \frac{1}{\sin\alpha_3})$, $H(\sec\alpha_1, \sec\alpha_2, \sec\alpha_3)$ and the center of this circle are collinear. G and H being on this circle, GH is a diameter.

We shall call this circle the GH circle.

<u>Theorem:</u> P'_1, P'_2 and P'_3 are the intersections of circle GH with the medians through A_1, A_2, and A_3 respectively, and the triangles $P'_1 P'_2 P'_3$ and $A_1 A_2 A_3$ are inperspective with G the center of perspective .

Proof: It is enough to show that P'_1 is on the median through A_1 etc.

P'_1 will be on A_1G if the following determinant:

$$\begin{vmatrix} \dfrac{\sin\alpha_1}{2\cos\alpha_1} & \sin\alpha_3 & \sin\alpha_2 \\ \dfrac{1}{\sin\alpha_1} & \dfrac{1}{\sin\alpha_2} & \dfrac{1}{\sin\alpha_3} \\ 1 & 0 & 0 \end{vmatrix} \text{ is zero .}$$

Evaluation of this determinant gives :

$$-\sin\alpha_3\left(\frac{-1}{\sin\alpha_3}\right) + \sin\alpha_2\left(\frac{-1}{\sin\alpha_2}\right) = 0.$$

Note: Since GH is a diameter, $\angle HP'_1G = \pi/2$ so that the preceding theorem can be stated as: The points P'_1, P'_2 and P'_3 are the three projections of the orthocenter H of $A_1A_2A_3$ upon the three medians.

Corollary: P_1, P_2, P_3 are the intersections of the Brocard circle with the symmedians (isogonal conjugates of the medians) through A_1, A_2 and A_3 respectively and the triangles $P_1P_2P_3$ and $A_1A_2A_3$ are in perspective with G the center of perspective.

Proof: It is enough to show that P_1 is on the symmedian through A_1 etc. and this follows immediately since the symmedian is the isogonal conjugate line of the median and since P_1 and P'_1 are isogonal conjugate points.

Note: The Brocard circle has OK as diameter, so $\angle OP_1K = \pi/2$

and this corollary becomes: P_1, P_2, and P_3 are the three orthogonal projections of the circumcenter O of $A_1A_2A_3$ upon the three symmedians.

Theorem: When the circumscribed triangle is equilateral, the twelve Miquel points reduce to two points; these are called the isogonic centers of triangle $A_1A_2A_3$ and they separate the line joining the Lemoine point K and the mid point of GH harmonically.

Proof: If the circumscribed triangle is equilateral the points are $\left(\frac{1}{\sin(\alpha_1+\pi/3)}\right)$ and $\left(\frac{1}{\sin(\alpha_1-\pi/3)}\right)$, because the isogonic centers are the isogonal conjugates of the isodynamic points.

$$\frac{\frac{u_1}{u_2}-\frac{x_1}{x_2}}{\frac{u_1}{u_2}-\frac{y_1}{y_2}} \div \frac{\frac{v_1}{v_2}-\frac{x_1}{x_2}}{\frac{v_1}{v_2}-\frac{y_1}{y_2}}$$

$$= \frac{\frac{\sin(\alpha_2+\pi/3)}{\sin(\alpha_1+\pi/3)}-\frac{\cos\alpha_1+4\cos\alpha_2\cos\alpha_3}{\cos\alpha_2+4\cos\alpha_3\cos\alpha_1}}{\frac{\sin(\alpha_2+\pi/3)}{\sin(\alpha_1+\pi/3)}-\frac{\sin\alpha_1}{\sin\alpha_2}} \div \frac{\frac{\sin(\alpha_2-\pi/3)}{\sin(\alpha_1-\pi/3)}-\frac{\cos\alpha_1+4\cos\alpha_2\cos\alpha_3}{\cos\alpha_2+4\cos\alpha_3\cos\alpha_1}}{\frac{\sin(\alpha_2-\pi/3)}{\sin(\alpha_1-\pi/3)}-\frac{\sin\alpha_1}{\sin\alpha_2}}$$

$$= \frac{\sin\alpha_2(3\cos\alpha_3\cos\pi/3-\sin\alpha_3\sin\pi/3)}{\sin(\alpha_3-\pi/3)(\cos\alpha_2+4\cos\alpha_3\cos\alpha_1)} \div \frac{\sin\alpha_2(\sin\alpha_3\sin\pi/3+3\cos\alpha_3\cos\pi/3)}{\sin(\alpha_3+\pi/3)(\cos\alpha_2+4\cos\alpha_1\cos\alpha_3)}$$

$$= \frac{-1+4\cos^2\alpha_3}{-4\cos^2\alpha_3+1} = -1 .$$

Of course the four points must be collinear; this results from the following considerations:

The equation of the line joining the isogonic centers is:

$$(\sin^2\alpha_1-\sin^2\pi/3)\sin(\alpha_2-\alpha_3)x_1+(\sin^2\alpha_2-\sin^2\pi/3)\sin(\alpha_3-\alpha_1)x_2$$
$$+(\sin^2\alpha_3-\sin^2\pi/3)\sin(\alpha_1-\alpha_2)x_3 = 0$$

and the equation of the line OK is:

$$\sin(\alpha_2-\alpha_3)x_1+\sin(\alpha_3-\alpha_1)x_2+\sin(\alpha_1-\alpha_2)x_3 = 0.$$

The intersection of these two lines is found as follows: [ ]

$$x_1 \sim \begin{vmatrix} \sin(\alpha_3-\alpha_1)(\sin^2\alpha_2-\sin^2\pi/3) & \sin(\alpha_1-\alpha_2)(\sin^2\alpha_3-\sin^2\pi/3) \\ \sin(\alpha_3-\alpha_1) & \sin(\alpha_1-\alpha_2) \end{vmatrix}$$

$$= \sin^2\pi/3\,\sin(\alpha_3-\alpha_1)\sin(\alpha_1-\alpha_2)\sin(\alpha_2-\alpha_3)\sin\alpha_1 .$$

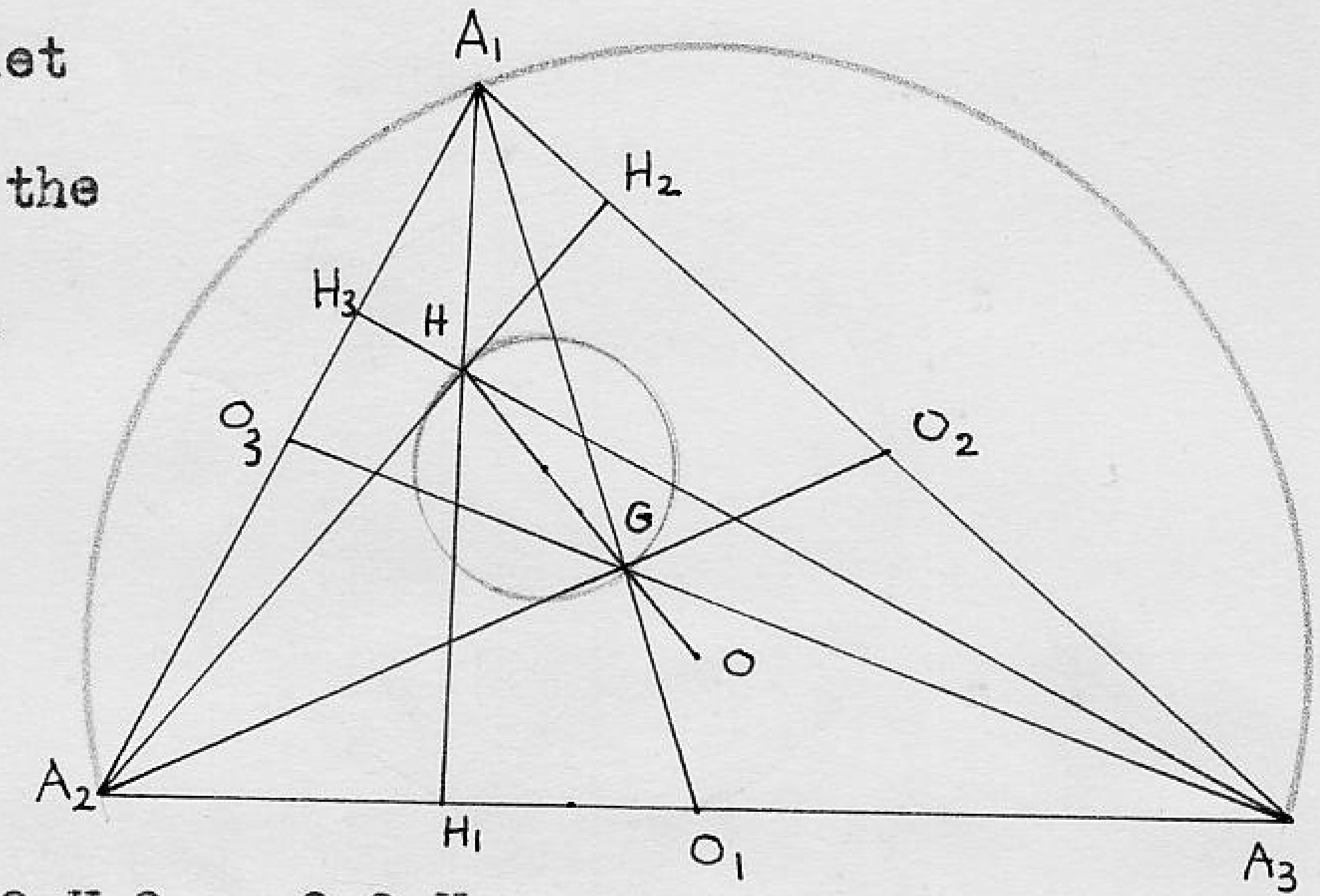
Similarly for X_2 and X_3, so that

$$(x_1, x_2, x_3) \sim (\sin\alpha_1, \sin\alpha_2, \sin\alpha_3) \sim (a_1, a_2, a_3).$$

Thus the Lemoine point K is collinear with the isogonic centers. Finally substitution of the coordinates of the mid point of GH in the equation of the line joining the isogonic centers, shows that it is on this line.

<u>Theorem</u>: There are eight Miquel triangles inscribed in the nine point circle. The Miquel points of four of these lie four by four on a circle and a straight line, and four of the Miquel points of the other triangles are on a circle.

<u>Proof</u>: Let O_1, O_2, and O_3 be the mid points of the sides, and let H_1, H_2, and H_3 be the feet of the altitudes. The following is a list of the Miquel triangles inscribed in the nine-point circle:



$$O_1O_2O_3,\quad H_1O_2O_3,\quad O_1H_2O_3,\quad O_1O_2H_3$$

and

$$H_1H_2H_3,\quad O_1H_2H_3,\quad H_1O_2H_3,\quad H_1H_2O_3\ .$$

It can be easily shown that the first four of these triangles are congruent to each other and similar to $A_1A_2A_3$. If β_1 represents the angle of the Miquel triangle whose vertex is on A_2A_3 etc. then we have for $O_1O_2O_3$: $\beta_1 = \alpha_1$, $\beta_2 = \alpha_2$, $\beta_3 = \alpha_3$

$H_1O_2O_3$ $\quad \beta_1 = \alpha_1$, $\beta_2 = \alpha_3$, $\beta_3 = \alpha_2$

$O_1H_2O_3$ $\quad \beta_1 = \;_3$, $\beta_2 = \alpha_2$, $\beta_3 = \alpha_1$

$O_1O_2H_3$ $\quad \beta_1 = \;_2$, $\beta_2 = \alpha_1$, $\beta_3 = \alpha_3$

and the corresponding Miquel points are:

$$O(\cos\alpha_1,\ \cos\alpha_2,\ \cos\alpha_3)$$

$$P_1(2\cot\alpha_1,\ \csc\alpha_3,\ \csc\alpha_2)$$

$$P_2(\csc\alpha_3,\ 2\cot\alpha_2,\ \csc\alpha_1)$$

$$P_3(\csc\alpha_2,\ \csc\alpha_1,\ 2\cot\alpha_3)$$

and their associated points

$(0, 0, 0)$ which is meaningless

$(0, a_2, - a_3)$

$(-a_1, 0, a_3)$

$(a_1, - a_2, 0)$.

The points O, P_1, P_2 and P_3 are on the Brocard circle as was proved in chapter two, and their associated points are on the Lemoine axis.

Now consider the other set of triangles. We have for

$H_1H_2H_3$ $\beta_1 = \pi - 2\alpha_1$, $\beta_2 = \pi - 2\alpha_2$, $\beta_3 = \pi - 2\alpha_3$

$O_1H_2H_3$ $\beta_1 = \pi - 2\alpha_1$, $\beta_2 = \alpha_1$, $\beta_3 = \alpha_1$

$H_1O_2H_3$ $\beta_1 = \alpha_2$, $\beta_2 = \pi - 2\alpha_2$, $\beta_3 = \alpha_2$

$H_1H_2O_3$ $\beta_1 = \alpha_3$, $\beta_2 = \alpha_3$, $\beta_3 = \pi - 2\alpha_3$

and the corresponding Miquel points are:

$H\left(\frac{1}{\cos\alpha_1}, \frac{1}{\cos\alpha_2}, \frac{1}{\cos\alpha_3}\right)$.

$P'_1\left(\frac{\sin\alpha_1}{2\cos\alpha_1}, \sin\alpha_3, \sin\alpha_2\right)$

$P'_2\left(\sin\alpha_3, \frac{\sin\alpha_2}{2\cos\alpha_2}, \sin\alpha_1\right)$

$P'_3\left(\sin\alpha_2, \sin\alpha_1, \frac{\sin\alpha_3}{2\cos\alpha_3}\right)$

H, P'_1, P'_2 and P'_3 are on the circle GH as was proved in the middle of this chapter.

The six points Q_1', ..., Q_6' do not enjoy all the properties of their isogonal conjugates Q_1, ..., Q_6. For example, Q_1, ..., Q_6 are always on a circle and their six associated points are also on a circle, but Q_1', ..., Q_6' are not on a circle, since in the special case of the circumscribed triangle similar to the triangle of reference, the Brocard points Ω and Ω' failed to lie on the circle GH.

In fact if one starts with the equation of a conic and replaces x by $\frac{1}{x}$ (to get the locus of the isogonal conjugate), one gets a fourth degree equation. Accordingly one can say that Q_1', ..., Q_6' are on a curve, of equation:

$$x_2^2x_3^2 + x_3^2x_1^2 + x_1^2x_2^2 - \sum \frac{\sin(\alpha_1 - \omega')}{\sin \omega'} x_1^2x_2x_3 = 0,$$

and their associated points are on another curve of equation:

$$x_2^2x_3^2 + x_3^2x_1^2 + x_1^2x_2^2 + \sum \frac{\sin(\alpha_1 + \omega')}{\sin \omega'} x_1^2x_2x_3 = 0.$$

It may be possible to sharpen this result. Another property of Q_1, ..., Q_6 and their associated points which remains to be checked for their isogonal conjugates is the following:

<u>Theorem</u>: The triangles $Q_1Q_4Q_6$ and $Q_2Q_3Q_5$ are always triply in perspective. The same is true if the associated points of Q_1, ..., Q_6 are considered. In each case the centers of perspective are the centers of the circles of Apollonius.

<u>Proof</u>: The line $Q_1\,Q_3$ is:

$$a_2a_3x_1 + a_3a_1x_2 - a_1a_2\left(\frac{\cot\omega + \cot\omega'}{\cot\alpha_3 + \cot\beta_3} - 1\right)x_3 = 0.$$

The line Q_4Q_5 is:

$$a_2a_3x_1 + a_3a_1x_2 - a_1a_2\left(\frac{\cot\omega + \cot\omega'}{\cot\alpha_3 + \cot\beta_1} - 1\right)x_3 = 0.$$

Q_1Q_3 and Q_4Q_5 intersect in the point (x_1, x_2, x_3) with:

$$x_1 \sim \begin{vmatrix} a_3a_1 & -a_1a_2\left(\dfrac{\cot\omega + \cot\omega'}{\cot\alpha_3 + \cot\beta_3} - 1\right) \\ \\ a_3a_1 & -a_1a_2\left(\dfrac{\cot\omega + \cot\omega'}{\cot\alpha_3 + \cot\beta_1} - 1\right) \end{vmatrix}$$

$$= \frac{a_1^2a_2a_3(\cot\omega + \cot\omega')(\cot\beta_1 - \cot\beta_3)}{(\cot\alpha_3 + \cot\beta_3)(\cot\alpha_3 + \cot\beta_1)} .$$

Similarly

$$x_2 \sim \frac{-a_1a_2^2a_3(\cot\omega + \cot\omega')(\cot\beta_1 - \cot\beta_3)}{(\cot\alpha_3 + \cot\beta_3)(\cot\alpha_3 + \cot\beta_1)} ,$$

and $x_3 \sim 0$.

Thus

$$(x_1, x_2, x_3) \sim (a_1^2a_2a_3, -a_1a_2^2a_3, 0) \sim (a_1, -a_2, 0).$$

This point is readily verified to be on the line $Q_2 Q_6$ of equation:

$$a_2a_3x_1 + a_3a_1x_2 - a_1a_2\left(\frac{\cot\omega + \cot\omega'}{\cot\alpha_3 + \cot\beta_2} - 1\right)x_3 = 0.$$

Similar calculations give the points $(-a_1, 0, a_3)$ and $(0, a_2, -a_3)$ as centers of perspective of the triangles $Q_1Q_4Q_6$ and $Q_2Q_3Q_5$. The three points $(a_1, -a_2, 0)$, $(-a_1, 0, a_3)$, $(0, a_2, -a_3)$ are the centers of the Apollonian circles, which are on the Lemoine axis.

The calculations for the associated points are analogous to these and the same result is obtained.

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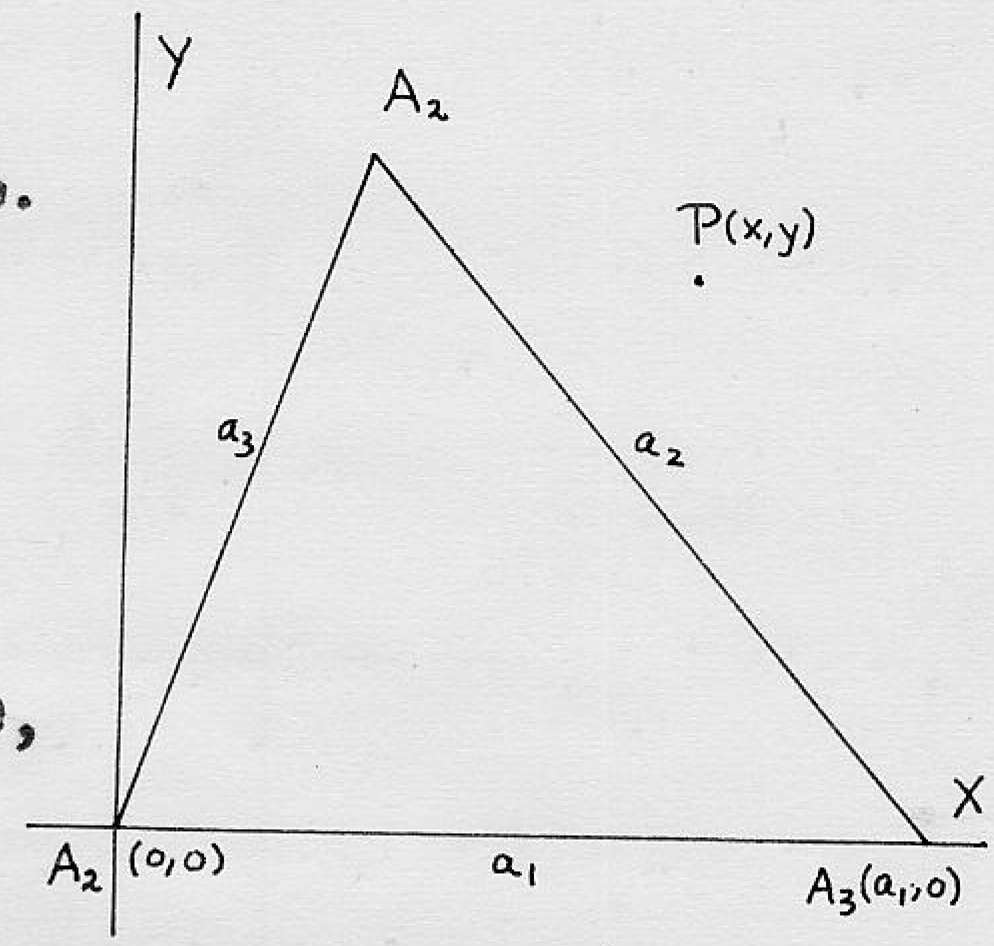
APPENDIX



Establish an ordinary Cartesian coordinate system as shown in the figure. The equations of the sides become the following:

$A_2A_3 : y = 0,$

$A_3A_1 : x \sin \alpha_3 + y \cos \alpha_3 - a_1 \sin \alpha_3 = 0,$

$A_1A_2 : x \sin \alpha_2 - y \cos \alpha_2 = 0.$

Consider any point $P(x,y)$ with trilinear coordinates (x_i). Let the distances from the sides of $A_1A_2A_3$ be given by ρx_1, ρx_2, ρx_3 respectively. Then, using the formula for the distance from a point to a line, with proper regard for sign, we obtain

$$\rho x_1 = y,$$

$$\rho x_2 = - x\sin \alpha_3 - y\cos \alpha_3 + a_1 \sin \alpha_3,$$

$$\rho x_3 = x\sin \alpha_2 - y\cos \alpha_2.$$

Setting $y = \rho x_1$ in the third equation, we have

$$\rho x_3 = x\sin \alpha_2 - \rho x_1 \cos \alpha_2, \text{ or } x = \rho(x_1 \cot \alpha_2 + x_3 \csc \alpha_2).$$

Then, substitution for x and y in the second equation gives

$$\rho x_2 = - \rho \sin \alpha_3 (x_1 \cot \alpha_2 + x_3 \csc \alpha_2) - \rho x_1 \cos \alpha_3 + a_1 \sin \alpha_3, \text{ or}$$

$$\rho = \frac{a_1 \sin \alpha_2 \sin \alpha_3}{\sum x \sin \alpha} .$$

From this it follows that

$$x = \frac{a_1 \sin\alpha_3(x_1\cos\alpha_2 + x_3)}{\Sigma x\sin\alpha} = \frac{a_1a_3(x_1\cos\alpha_2 + x_3)}{\Sigma ax} \text{ (by the law of sines),}$$

and

$$y = \frac{(a_1\sin\alpha_2\sin\alpha_3)x_1}{\Sigma x\sin\alpha} = \frac{(a_1a_2\sin\alpha_3)x_1}{\Sigma ax}.$$

Hence, for any finite point, $\Sigma ax \neq 0$.

Now take any circle $(x - h)^2 + (y - k)^2 = r^2$, and find its trilinear equation by means of the relations above:

$$\left[\frac{a_1a_3(x_1\cos\alpha_2 + x_3)}{\Sigma ax} - h\right]^2 + \left[\frac{(a_1a_2\sin\alpha_3)x_1}{\Sigma ax} - k\right]^2 = r^2.$$

Its intersection with the line $\Sigma ax = 0$ is found by multiplying through by Σax and then equating Σax to zero. Then

$$a_1a_3(x_1\cos\alpha_2+x_3)^2 + (a_1a_2x_1\sin\alpha_3)^2 = 0,$$

$$(x_1\cos\alpha_2+x_3)^2 + (x_1\sin\alpha_2)^2 = 0,$$

$$x_1^2 + 2x_1x_3\cos\alpha_2 + x_3^2 = 0,$$

$$\frac{x_1}{x_3} = -\cos\alpha_2 \pm \sqrt{\cos^2\alpha_2 - 1} = -\cos\alpha_2 \pm i.\sin\alpha_2.$$

This may be used to find $\frac{x_1}{x_2}$ or $\frac{x_2}{x_3}$. Finally, we have

$$(x_1, x_2, x_3) \sim (-\cos\alpha_2 \pm i.\sin\alpha_2, -\cos\alpha_1 \mp i.\sin\alpha_1, 1)$$

which shows that every circle in the plane is cut by the line $\Sigma ax = 0$ in two fixed points, which are called the circular points at infinity.

Let

$$a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{23}x_2x_3 + 2a_{31}x_3x_1 = 0$$

be the equation of a conic in homogoneous coordinates. If we assume that this conic passes through the two fixed circular points at infinity and substitute the coordinates of these points in the equation we get the following two conditions:

$$a_{11} + a_{22}\cos 2\alpha_3 + a_{33}\cos 2\alpha_2 - 2a_{12}\cos\alpha_3 + 2a_{23}\cos(\alpha_2 - \alpha_3)$$

$$- 2a_{31}\cos\alpha_2 = 0 \quad \ldots\ldots 1$$

$$-a_{22}\sin 2\alpha_3 + a_{33}\sin 2\alpha_2 + 2a_{12}\sin\alpha_3 + 2a_{23}\sin(\alpha_2 - \alpha_3)$$

$$- 2a_{31}\sin\alpha_2 = 0 \quad \ldots\ldots 2$$

on the other hand, writing the equation of the conic in cartisian coordinates gives the following quadratic terms:

$$\left\{a_{11} + a_{22}\cos^2\alpha_3 + a_{33}\cos^2\alpha_2 - 2a_{12}\cos\alpha_3 + 2a_{23}\cos\alpha_2\cos\alpha_3 - 2a_{31}\cos\alpha_2\right\} y^2$$

$$+ \left\{a_{22}\sin^2\alpha_3 + a_{33}\sin^2\alpha_2 - 2a_{23}\sin\alpha_2\sin\alpha_3\right\} x^2$$

$$+ \left\{a_{22}\sin 2\alpha_3 - a_{33}\sin 2\alpha_2 - 2a_{12}\sin\alpha_3 + 2a_{23}\sin(\alpha_3 - \alpha_2) + 2a_{31}\sin\alpha_2\right\} xy.$$

Condition (2) shows that the coefficient of xy is zero.

Also the difference between the coefficient of y^2 and that of x^2 is condition (1) which shows that the coefficients of x^2 and y^2 are equal.

Thus the equation of the conic expressed in cartesian coordinates becomes;

$$\left\{a_{22}\sin^2\alpha_3 + a_{33}\sin^2\alpha_2 - 2a_{23}\sin\alpha_2\sin\alpha_3\right\}\left\{x^2 + y^2\right\}$$

$$- 2a_1\sin\alpha_3\left\{a_{22}\sin\alpha_3 - a_{23}\sin\alpha_2\right\}x - 2a_1\sin\alpha_3\left\{a_{22}\cos\alpha_3 + a_{23}\cos\alpha_2 - a_{12}\sin\alpha_3\right\}y$$

$$+ a_{22}a_1^2\sin^2\alpha_3 = 0$$

which is a circle.

Thus every conic passing through the two circular points at infinity is a circle.

Combining the two results we have: A conic is a circle if and only if it passes through the two circular points at infinity.