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A NEW SIMULATION MODEL FOR THE HASEGAWA-MIMA PLASMA EQUATION

by

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AN ABSTRACT OF THE THESIS OF

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A nonlinear evolution of the drift-wave instability are investigated by means of numerical simulations based on a model equation derived from a two-fluid approximation that reduces to the Hasegawa-Mima equation. Although it was originally derived by Akira Hasegawa and Kunioki Mima in [2], it can be extended [5][4] and put as:

$$(\Delta - I)u_t + \{u, \Delta u\} + ku_y = 0 \tag{1}$$

We intent to first formulate the Hasegawa-Mima equation as a coupled system and then perform a new numerical simulation with the adequate boundary conditions and initial conditions. Experiments will be done to study the Modon steadiness solution for the nonlinear Hasegawa-Mima equation and to test the Periodic Boundary Conditions using finite element method.

Contents

Chapter 1

Introduction

1.1 Hasegawa-Mima (HM) Model for magnetized Plasma Physics

Confinement of magnetic plasma physics holds great potential in offering a clean source of energy in the future, thus diminishing the dependency on fossil fuels as a source of energy. This is a strong indicator of the importance and the strategic position of this type of research for the future.

Throughout this thesis, we focus on the numerical tools to simulate plasma turbulence using Hasegawa-Mima (HM) equation. We consider a new mathematical setting [19] to establish a numerical simulation for the two-dimensional (HM) model which is our main contribution in this thesis.

The (HM) was introduced by authors Hasegawa and Mima in 1977 [2] and can stated as follows:

Let $\Omega = (0, L) \times (0, L)$ an open bounded square domain with boundary $\Gamma = \partial\Omega$.

Seek $u : \overline{\Omega} \times [0, T] \longrightarrow \mathbb{R}$ such that:

$$(HM) \begin{cases} (\Delta - I)u_t + \{u, \Delta u\} + ku_y = 0, & \text{on } \Omega \times \text{[0,T]} \\ u(x, y, 0) = u_0(x, y), & \text{on } \Omega \\ \text{u and } \nabla\text{u satisfies periodic boundary conditions} \end{cases} \tag{1.1}$$

where:

- $u(x, y, t)$ describes the electrostatic fluctuations of the potential
- Δ is the Laplacian operator
- n_0 is the background particle density that depends only on the x-direction
- $k = \partial_x ln \dfrac{n_0}{w_{ci}}$
- w_{ci} is the ion cyclotron frequency that depends only on the initial magnetic field

Periodic boundary are imposed, in general, for modeling the physical interaction of a wave (in our case, plasma drift waves) with the boundary of its medium and can be expressed as follows:

$$(PBC) \begin{cases} v(0, y) = v(L, y) \text{ and } v_x(0, y) = v_x(L, y) \quad \forall y \in (0, L) \\ v(x, 0) = v(x, L) \text{ and } v_y(x, 0) = v_y(y, L) \quad \forall x \in (0, L) \end{cases} \tag{1.2}$$

1.2 Review of the Literature

1. **Existence and Uniqueness of Solutions**

   There are some existing results to the (HM) in the case where $\Omega = \mathbb{R}^2$, but there has been no clear results regarding the existence and uniqueness of the solution to the two-dimensional Hasegawa-Mima equation with periodic boundary conditions. Nevertheless, Lionel Paumond [9] worked on the perturbed Hasegawa-Mima equation and also formulated it as a semilinear abstract Cauchy problem and uses fractional powers of the perturbing operator. It proves local existence and uniqueness for $u_0 \in H^4(\mathbb{R}^2)$,whose global existence still remains open. It also proves global existence of a weak solution for $u_0 \in H^2(\mathbb{R}^2)$, whose uniqueness still remains open.

In fact, H.Karakazian, in his Master's thesis [6], showed that for a smooth enough given initial condition $u_0 \in H^4_P(\Omega)$, the problem (HM) with (PBC) above has a unique local $C^{\infty,2}$ solution on $(0, T^*) \times \Omega$ where $T^* > 0$ is a temporal value depending only on u_0.

2. **Numerical Simulations**

A computer model was designed in F.Hariri's thesis [5] for solving the two-dimensional Hasegawa- Mima equation based on a finite difference (FD) approach with the integration in time being carried out with a Euler explicit scheme that constraints the time-step size which limit the size of the time interval. Hence, such method is not well suited for periodic boundary conditions, has a major difficulty in discretizing the Poisson-bracket term, and not applicable for computations on long time intervals.

1.3 Outline of this thesis

There are five Chapters in this thesis.

In Chapter 2, we present a new mathematical model consisting of a coupled Poisson

Wave equation and summarize the theoretical results obtained by H.Karkazian-N.Nassif in their recent work [19].

In Chapter 3, based on the method used in Chapter 2 to obtain existence of solutions, we set a finite element approach using $\mathbb{P}_1$ elements to approximate solutions of the coupled Elliptic-Hyperbolic system.

In Chapter 4, we present results of our First numerical simulations.

Concluding in Chapter 5 in general remarks.

Chapter 2

Formulation of the Hasegawa-Mima Equation as a Coupled System of Partial Differential Equations

2.1 The Formulation

Given a time $T > 0$, we consider the Hasegawa-Mima problem on a square domain $\Omega = (0, L) \times (0; L)$ with boundary $\Gamma = \partial\Omega$: Seek $u : \overline{\Omega} \times [0, T] \to \mathbb{R}$ such that:

$$(HM) \begin{cases} (\Delta - I)u_t + \{u, \Delta u\} + ku_y = 0, & \text{on } \Omega \times \text{(0,T]} \\ u(x, y, 0) = u_0(x, y), & \text{on } \Omega \\ \text{u and } \nabla u \text{ satisfies periodic boundary condition} & \end{cases} \quad (2.1)$$

where $\{u,v\} := u_x v_y - u_y v_x$ is the Poisson bracket.

Since handling the non-linearity of the Poisson bracket is both theoretically and computationally expensive, we formulate (HM) as a coupled system of linear equations as follows.

Introduce the variable $w = -\Delta u + u$, then the Hasegawa-Mima equation becomes:

$$\begin{aligned}\partial_t w &= \{u, u-w\} + ku_y \\ &= \{u,u\} + \{u,-w\} + ku_y\end{aligned}$$

As $\{u,u\} = u_x u_y - u_y u_x = 0$, then one has

$$\begin{aligned}\partial_t w &= \{w,u\} + ku_y \\ &= w_x u_y - w_y u_x + ku_y\end{aligned}$$

Now define the divergence free vector field $\overrightarrow{V}(u) = \left(\begin{smallmatrix}-u_y \\ u_x\end{smallmatrix}\right)$, then

$$\partial_t w + \overrightarrow{V}(u). \bigtriangledown w = ku_y$$

At this point, we formulate the (HM) problem as the following Elliptic-Hyperbolic coupled system problem where one Seeks $\{u,w\} : \overline{\Omega} \times [0,T] \to \mathbb{R}^2$ such that

$$(HM)\begin{cases} u - \Delta u = w, & \text{on } \Omega \times (0,\mathrm{T}] \\ \partial_t w + \overrightarrow{V}(u). \bigtriangledown w = ku_y, & \text{on } \Omega \times (0,\mathrm{T}] \\ \text{u and } \nabla \text{u satisfy the periodic boundary condition} & \\ u(x,y,0) = u_0(x,y), & \text{on } \overline{\Omega} \end{cases} \tag{2.2}$$

Before putting (HM) in varational form, we introduce Periodic Sobolev space.

2.2 The periodic Sobolev spaces $H^1_p(\Omega)$ and $H^2_p(\Omega)$

In this section, we review the construction of a two-dimensional Periodic Sobolev space $H^1_p(\Omega)$ and study its properties[6]. We start by defining the Sobolev space $H^1(\Omega)$.

Definition 2.1. We Define $C_0^\infty = \{$f $\in C^\infty$ /f is compactly supported in $\Omega\}$.

A function f $\in L^2\Omega$ is said to have a weak derivative $D_w f \in L^2\Omega$ iff $\forall\Psi \in C_0^\infty(\Omega)$, $< f, \Psi' > =$

$- < D_w f, \Psi >$.

$L^2(\Omega) = \{f/ \int_\Omega dx < \infty\}$

Definition 2.2. The Sobolev spaces $H^1(\Omega)$ and $H^2(\Omega)$ are defined as follows:

$$H^1(\Omega) = \{f \in L^2(\Omega)/D_w^1 f \in L^2(\Omega)\}$$

$$H^2(\Omega) = \{f \in L^2(\Omega)/D_w^\alpha f \in L^2(\Omega), \alpha = \{1, 2\}\}$$

We define v to satisfy the periodic boundary conditions PBC^k of order k if and only if

For $k \geq 0$:

$$(PBC^k) \begin{cases} (\text{PBC}_x^k)\text{: } \partial_x^k v(0, y) = \partial_x^k v(L, y) \quad \forall y \in (0, L) \\ \\ (\text{PBC}_y^k)\text{: } \partial_y^k v(x, 0) = \partial_y^k v(x, L) \quad \forall x \in (0, L) \end{cases} \tag{2.3}$$

where ∂_x^k and ∂_y^k denote the differential operators $\dfrac{\partial^k}{\partial x}$ and $\dfrac{\partial^k}{\partial y}$, respectively, for short.

Definition 2.3. Define the Periodic Sobolev space of order 1 and 2 as

$$H_P^1(\Omega) = \{v \in H^1(\Omega) : v \text{ satisfies } \mathbf{PBC^0}\}$$

$$H_P^2(\Omega) = \{v \in H^2(\Omega) : v \text{ satisfies } \mathbf{PBC^0} \text{ and } \mathbf{PBC^1}\}$$

Properties of $H_P^1(\Omega)$:

- For $v \in H_P^1(\Omega)$ its trace Tr(v) on Γ is well-defined as the extension of

  $$Tr_0 : C^\infty \longrightarrow L^2(\Gamma)$$

  given that C_0^∞ is dense in $H^1(\Omega)$.

- Furthermore, $Tr : H^1(\Omega) \longrightarrow L^2(\Gamma)$ is continuous, i.e it satisfies :

  $$\|Tr(v)\|_{L^2(\Gamma)} \leq C\|v\|_{H^1(\Omega)}$$

2.3 Variational Formulation of (HM) equation as a coupled system

We intend to derive weak formulations of (2.2) that are well suited for deriving weak solutions to (HM) equation and also to obtain simple simulation models.

2.3.1 Variational Formulation of the Poisson Elliptic equation

We start with the Variational formulations of the Poisson Elliptic equation with periodic boundary condition.

This is now a well-known procedure based on Green's Formula.

Using a set of test functions this formulation generalizes the one dimensional integration by parts. To illustrate this, let us start with the formulation of a one-dimensional Poisson's equation:

$$\begin{cases} -u'' + u = w & u \in \Omega \times (0,T] \\ u(0) = u(L) \ \text{ and } \ u'(0) = u'(L) \end{cases} \tag{2.4}$$

Let $v \in C^1_P(0,L) = \{v \in C^1 \ / \ \ v(0) = v(L)\}$, then

$$-\int_0^L u''(x)v(x)dx + \int_0^L u(x)v(x)dx = \int_0^L w(x)v(x)dx$$

$$-[u'(x)v(x)]_0^L + \int_0^L u'(x)v(x)dx + \int_0^L u(x)v'(x)dx = \int_0^L w(x)v(x)dx$$

$$-u'(L)v(L) + u'(0)v(0) + \int_0^L u'(x)v'(x)dx + \int_0^L u(x)v(x)dx = \int_0^L w(x)v(x)dx$$

But since $u'(0) = u'(L)$ and by choosing $v(x)$ such that $v(0) = v(L) \Rightarrow -u'(L)v(L) + u'(0)v(0) = 0$

Define now:

$$a(u,v) = \int_0^L u'(x)v'(x)dx + \int_0^L u(x)v(x)dx$$

$$f(v) = \int_0^L w(x)v(x)dx$$

Then if $u(x) \in C^1_P(0,L)$ solves (2.4) then it is also solution to:

$$a(u,v) = f(v) \qquad \forall v \in C^1_P(0,L) \tag{2.5}$$

Obviously, (2.5) can be generalized if we replace $C^1(0,L)$ by $H^1(0,L)$, and $C^1_P(0,L)$ by $H^1_P(0,L)$.

The above can be extended to the two-dimensional Periodic Poisson's equation in (HM) with $\Omega = (0,L) \times (0,L)$.

It uses Green's formula (Divergence form) which states the following in two dimensions:

$\forall \vec{V} \in (C^1(\Omega) \cap C(\overline{\Omega}))^2$ and $\forall v \in C^1(\Omega) \cap C(\overline{\Omega})$

$$\int_\Omega (div(\vec{V}))v dxdy = \int_\Gamma v\nu.\vec{V} dxdy - \int_\Omega \nabla v \vec{V} dxdy$$

When applied to:

$$\begin{cases} -\Delta u + u = w & u \in \Omega \times (0,T] \\ \text{u satisfies } (PBC^0) \text{ and } (PBC^1) \end{cases} \tag{2.6}$$

Let $v \in C^1_P(\Omega) = \{v \in C^1(\Omega) \ / \ \ v \in (PBC^0)\}$, then

$$-\int_\Omega \Delta u.v dxdy + \int_\Omega uv d\mu = \int_\Omega wv dxdy$$

Green's Formula (2D) with $\Delta u = div(\nabla u)$ and Γ is the boundary of Ω

$$\int_\Omega (div(\nabla u))v dxdy = \int_\Gamma v\nu.\nabla u dxdy - \int_\Omega \nabla v \nabla u dxdy$$

$$-\int_\Gamma v\nu.\nabla u dxdy + \int_\Omega \nabla v \nabla u dxdy + \int_\Omega uv dxdy = \int_\Omega wv dxdy$$

But since $\nabla u(x,y)$ satisfies $(PBC)^1$ since $u \in C^1_P(\Omega)$ and by choosing $v(x,y) \in C^1_P(\Omega)$ $\Rightarrow \int_\Gamma v\nu.\nabla u = 0$

Define now:

$$a(u,v) = \int_\Omega \nabla v \nabla u dxdy + \int_\Omega uvdxdy =< \nabla v, \nabla u > + < u, v > \tag{2.7}$$

$$f(v) = \int_\Omega wvdxdy =< w, v > \tag{2.8}$$

Then if $u(x) \in C^1_P(\Omega)$ solves 2.6 then it is also solution to:

$$a(u,v) = f(v) \quad \forall v \in C^1_P(\Omega) \tag{2.9}$$

Obviously, (2.9) can be generalized if we replace $C^1(\Omega)$ by $H^1(\Omega)$, and $C^1_P(\Omega)$ by $H^1_P(\Omega)$.

$a : H^1_P(\Omega) \times H^1_P(\Omega) \longrightarrow \mathbb{R}$ defined in 2.7 is a bilinear form that is coercive and bicontinuous. In addition, $f : H^1_P(\Omega) \longrightarrow \mathbb{R}$ from 2.8 is a continuous linear operator. Hence a and f satisfy Lax-Milgram conditions that prove the existence of a unique solution u satisfying (2.9).

Proposition 2.4. *For $w \in L^2(\Omega)$, $u \in H^1_P(\Omega) \cap H^2(\Omega)$, with*

$\|u\|_{H^2(\Omega)} \leq C\|w\|_{L^2(\Omega)}$ *and $T : w \longrightarrow u$ is a compact map.*

Note that, the variational formulation does not necessitate that the solution $u(x, y)$ that belongs to the Test space satisfies (PBC^1)

2.3.2 The Variational Formulation of the Hyperbolic part of the Coupled System

The Variational formulations of the two dimensional Hyperbolic part is carried out in the same way as for the Poisson's equation with $w(x, y, t)$ satisfying:

$$\begin{cases} \partial_t w + \overrightarrow{V}(u).\nabla w = k u_y \\ \\ w(x,y,0) = w_0(x,y) = -\Delta u_0 + u_0 \end{cases} \tag{2.10}$$

For the initial condition, the least regularity that is required is $u_0 \in H^2(\Omega)$, $w_0 \in H^1(\Omega)$ which is sufficient for numerical simulation.

However, a stronger regularity condition may be a necessary when demonstrating existence of solutions to the coupled system. Following [19] we consider the following formulations for the hyperbolic system:

Given $u \in H^1_P(\Omega) \cap H^2(\Omega)$ one seeks $w : [0,T] \longrightarrow H^1_P(\Omega)$ such that: $\partial_t w : [0,T] \longrightarrow L^2(\Omega)$

$$\begin{cases} < \partial_t w, v > + < \overrightarrow{V}(u).\nabla w, v > = < k u_y, v > & \forall v \in L^2(\Omega) \\ \\ w_0(x,y) = -\Delta u_0 + u_0 & w_0 \in L^2(\Omega) \end{cases} \tag{2.11}$$

This is obtained simply by multiplying the equation in 2.11 by $v \in L^2(\Omega)$ and integrating over Ω.

For our computational model, we use an integral form of 2.11, obtained by integration from 0 to Δt.

Given $u \in H^1(\Omega)$, and $w_0 \in L^2(\Omega)$, one seeks $w : [0,T] \longrightarrow H^1_P(\Omega)$ such that:

$$\begin{cases} < w(\Delta t), v > + \int_0^{\Delta t} < \overrightarrow{V}(u).\nabla w, v > ds = \int_0^{\Delta t} < k u_y, v > ds + < w_o, v > \quad \forall v \in L^2(\Omega) \end{cases} \tag{2.12}$$

Or more generally from t to $t + \Delta t$.

Given $w(t) \in H^1(\Omega)$, one seeks $w : [t, t+\Delta t] \longrightarrow H^1_P(\Omega)$ such that:

$$\begin{cases} < w(t+\Delta t), v > + \int_t^{t+\Delta t} < \overrightarrow{V}(u).\nabla w, v > ds = \int_t^{t+\Delta t} < k u_y, v > ds + < w(t), v > \quad \forall v \in L^2(\Omega) \end{cases}$$

(2.13)

However, in view of the results obtained in [19], we may consider a formulation where we take $u_0 \in H^3(\Omega) \cap H^1_P(\Omega)$ and $u \in H^1_P(\Omega)$ and seek: $w : [0, T] \longrightarrow H^1_P(\Omega)$ such that:

$$\begin{cases} < w(t+\Delta t), v > - \int_t^{t+\Delta t} < \overrightarrow{V}(u).\nabla v, w > ds = \int_t^{t+\Delta t} < ku_y, v > ds + < w(t), v > \quad \forall v \in H^1_P(\Omega) \end{cases} \tag{2.14}$$

This is obtained by performing Green's Formula on the term $< \overrightarrow{V}(u).\nabla v, w > ds$ with $\overrightarrow{V}(u) = \left(\begin{smallmatrix} -u_y \\ u_x \end{smallmatrix} \right)$:

$$\begin{aligned} \int_\Omega \overrightarrow{V}(u).\nabla w v dxdy &= \int_\Gamma wv\overrightarrow{V}(u).\mu ds - \int_\Omega \overrightarrow{V}(u).\nabla v w ds \\ &= \int_{\Gamma_1} wv(-1\ ,\ 0) \left(\begin{smallmatrix} -u_y \\ u_x \end{smallmatrix} \right) ds + \int_{\Gamma_2} wv(1\ ,\ 0) \left(\begin{smallmatrix} -u_y \\ u_x \end{smallmatrix} \right) ds \\ &+ \int_{\Gamma_3} wv \left(\begin{smallmatrix} -u_y \\ u_x \end{smallmatrix} \right) (0\ ,\ -1) ds + \int_{\Gamma_4} wv(0,1) \left(\begin{smallmatrix} -u_y \\ u_x \end{smallmatrix} \right) ds - \int_\Omega \overrightarrow{V}(u).\nabla v w ds \\ &= \int_{\Gamma_1} wvu_y ds + \int_{\Gamma_2} wv(-u_y) ds \\ &+ \int_{\Gamma_3} wv(-u_x) ds + \int_{\Gamma_4} wvu_x ds - \int_\Omega \overrightarrow{V}(u).\nabla v w ds \end{aligned}$$

With Γ_1 and Γ_2 are the two opposite vertical sides of the square $(0, L) \times (0.L)$ and by the periodicity of u:

$\Longrightarrow \int_{\Gamma_1} wvu_y ds + \int_{\Gamma_2} wv(-u_y) ds = 0$

Moreover, Γ_3 and Γ_4 are the two opposite horizontal sides of the square $(0, L) \times (0.L)$ and by the periodicity of u:

$\Longrightarrow \int_{\Gamma_3} wv(-u_x) ds + \int_{\Gamma_4} wvu_x ds = 0$

Remark: Formulation 2.13 has the weakest form and will be used for computations.

2.3.3 Varitional formulation used in this thesis

From this section onward, the { , } will represent the couple of unknowns instead of the poisson Bracket.

We seek the pair $\{u,w\} \in L^2(0,T,H^2(\Omega)\cap H^1_P(\Omega))\times H^1_P(0,T,L^2(\Omega))$ where we use the notation $u(t) = u(x,y,t)$ and $w(t) = w(x,y,t)$ such that:

$$
\begin{cases}
a(u(t),v) =< w(t),v > & \forall v \in H^1_P(\Omega), \quad \forall t \in (0,T] \\
< w(\Delta t),v >= -\int_0^{\Delta t}(< \overrightarrow{V}(u).\nabla w(t),v > ds + < ku_y, v >)ds + < w_o, v > & \forall v \in L^2,\ 0 \le t \le T \\
u(0) = u_0 & u_0 \in H^2(\Omega) \\
w(0) = w_0 & w_0 \in L^2(\Omega)
\end{cases}
\tag{2.15}
$$

Or more generally given $\{u(t),w(t)\}$ we obtain $\{u(s),w(s)\}$ $\forall s \in (t,t+\Delta t]$ by solving:

$$
\begin{cases}
a(u(s),v) =< w(s),v > & \forall s \in (t,t+\Delta t] \\
< w(t+\Delta t),v >= -\int_t^{t+\Delta t}(< \overrightarrow{V}(u).\nabla w(t),v > + < ku_y, v >)ds + < w(t), v > & \forall v \in L^2(\Omega) \\
 & \forall s \in (t,t+\Delta t]
\end{cases}
\tag{2.16}
$$

Remark:

Note that (2.16) may be replaced by seeking $w(t) \in H^1_P(\Omega)$.

$$
\begin{cases}
a(u(s),v) = < w(s),v > & \forall s \in (t,t+\Delta t] \\
< w(t+\Delta t),v >= -\int_t^{t+\Delta t}(<\overrightarrow{V}(u).\nabla w(t),v> + < ku_y,v >)ds + < w(t),v > & \forall v \in H_P^1(\Omega) \\
& \forall s \in (t,t+\Delta t]
\end{cases}
\tag{2.17}
$$

Both 2.16 and 2.17 lead to the same finite element discrete system.

2.4 Recent theoretical results of Karakazian and Nassif[19]

In their research, Karakazian and Nassif consider first a Hilbert basis $\{\phi_i\}_{i=1}^{\infty} \in H_P^1(\Omega)$ consisting of the eigenfunctions of the periodic Laplacian operator with non-decreasing eigenvalues, set $E_N = span\{\phi_1, \cdots, \phi_N\} \subset H_P^1(\Omega)$, and formulate a Petrov-Galerkin approximation of the coupled system as a fixed point problem on $C(0,T;E_N) \times C(0,T;E_N)$. They then obtain a sequence $\{u_N, w_N\}$ of approximate solutions to the coupled system which are uniformly bounded in $L^\infty(0,T;H_P^1(\Omega) \cap H^2(\Omega)) \times L^\infty(0,T;H_P^1(\Omega))$. Finally, they extract a weakly convergent subsequence to construct a weak solution $\{u,w\} \in L^2(0,T;H_P^2(\Omega)) \times L^2(0,T;L^2(\Omega))$. Below is their main theorem whose corollary validates our FEM scheme.

Theorem 2.5. *Given* $0 < T < (C_E|k|+1)^{-1}$ *and* $u_0 \in H^2(\Omega)$*, there exist*

$$
u \in L^2(0,T;(\Omega) \cap H^2(\Omega)) \text{ and } u_t \in L^2(0,T;H_p^1(\Omega) \cap H^2(\Omega)) \tag{2.18}
$$

such that

$$\begin{cases} u_t = P(I-\Delta)^{-1}uQu - Q(I-\Delta)^{-1}uPu + k(1-\Delta)^{-1}u_y & \text{in } L^2(0,T;H^2_p(\Omega))^* \\ u(0) = u_0, & \text{on } \Omega \end{cases} \tag{2.19}$$

Where: $P = \partial_x^2 - \partial_y^2$

$Q = \partial_{xy}$

Remark. Statement (2.18) implies that $u \in C(0,T;L^2(\Omega))$.

Corollary 2.6. *Further if* $u \in L^2(0,T;H^3_P(\Omega))$, *then*

$$\begin{cases} (I-\Delta)u_t = \{u, \Delta u\} + ku_y & \text{in } L^2(0,T;H^2_p(\Omega))^* \\ u(0) = u_0, & \text{on } \Omega \end{cases} \tag{2.20}$$

or equivalently $\{u,w\} \in L^2(0,T;H^3_P(\Omega)) \times L^2(0,T;H^1_P(\Omega))$ *is a solution to the coupled system satisfying*

$$\begin{cases} \int_0^T < -\Delta u + u, v > \; dt = \int_0^T < w, v > \; dt & \text{a.e. on } \Omega \\ \int_0^T < w_t, v > \; dt + \int_0^T < \vec{V}(u)\cdot\nabla w, v > \; dt = \int_0^T < ku_y, v > dt & \forall v \in L^2(0,T;H^2_P(\Omega)) \\ u(0) = u_0, & \text{on } \Omega \end{cases} \tag{2.21}$$

Chapter 3

Finite Element Discretization of the coupled Hasegawa-Mima system

Our approach is to solve the coupled Hasegawa-Mima equation using for each of the PDE's a Finite Element approach. The main reason for choosing this approach is to handle the Periodic Boundary Conditions that are suitable to a variational formulation in which we force the approximate solution to satisfy (PBC^0) and to avoid (PBC^1) in contrary to was done in [5] where the Finite difference method lead to major difficulties handling (PBC^1). Our starting point is the variational formulation derived in Chapter 2.

Find $\{u,w\} \in L^2(0,T,H^2(\Omega)\cap H^1_P(\Omega)) \times H^1_P(0,T,L^2(\Omega))$ such that:

$$\begin{cases} a(u(s),v) =< w(s),v > & \forall s \in (t,t+\Delta t] \\ < w(t+\Delta t),v > & \\ = -\int_t^{t+\Delta t}(< \overrightarrow{V}(u).\nabla w(t),v > + < ku_y,v >)ds + < w(t),v > & \forall v \in L^2(\Omega)\ \forall s \in (t,t+\Delta t] \end{cases} \tag{3.1}$$

In [19], the above formulation was used to obtain existence of solutions proved in Chapter 2 Section 2.3.1 using Petrov-Galerkin approximations on spaces generated by the spectral eigenfunctions that satisfy:

$$-\Delta\Phi_n + \Phi_n = \lambda_n \Phi_n$$

with $\Phi_n \in (PBC^1)$, $n = (n_1,n_2)$ and $\Phi_n(x,y) = \varphi_{n_1}(x)\varphi_{n_2}(y)$ with $\varphi_{n_1}(x)$ and $\varphi_{n_2}(y)$ being trigonometric functions.

In our case, Finite Element subspaces will be used, although spectral methods based on $\{\Phi_n\}$ should be also investigated in the future.

3.1 $\mathbb{P}_1$ - Finite Elements on Triangles

Conformal meshing is obtained using the built-in `MATLAB` function `Delaunay`.

The set of triangles E $=\{T_i | 1 \le i \le M\}$ that is obtained is such that:

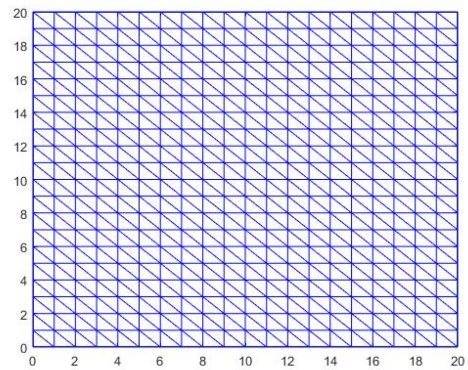


Figure 3.1: A conformal finite element meshing of the rectangle

$$T_i \cap T_j = \begin{cases} \text{Triangle itself when i=j.} \\ \text{vertex} \\ \text{one side} \\ \phi \end{cases} \tag{3.2}$$

In addition, $\Omega = \cup_{i=1}^{M} T_i$, See figure (3.1)

3.1.1 Definition of $S^1(E)$

Definition 3.1. The space of $\mathbb{P}_1-$finite elements on E is defined as follows:

$$S_1(E) = \{\Phi \in C(\Omega) | \Phi|_T \in \mathbb{P}_1\}$$

Where $\mathbb{P}_1$ is the set of polynomials of degree 1, in x and y.

Specifically, let $x = (x, y) \in$T, then for $\Phi \in S_1(E)$:

$$\Phi(x) = b_T + {a_T}^1 x + {a_T}^2 y$$

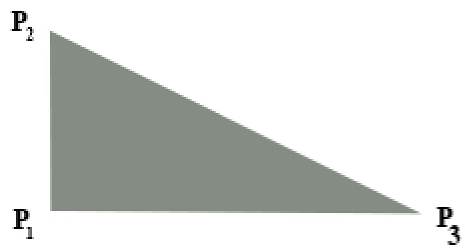


Figure 3.2: Right angle triangle in E

On every triangle, Φ is defined on the basis of 3 parameters $\{\Phi(P_1), \Phi(P_2), \Phi(P_3)\}$, then

$$\Phi(x, y) = \Phi(P_1)\psi_{1,T}(x, y) + \Phi(P_2)\psi_{2,T}(x, y) + \Phi(P_3)\psi_{3,T}(x, y)$$

where

$\psi_{i,T}(x, y) = \alpha_i x + \beta_i y + \gamma_i$, with $\psi_{i,T}(P_j) = \delta_{ij}$ and δ_{ij} being the kronecker delta.

Since all triangles in E are right angles (see figure (3.2)), then $\psi_{i,T}$ can be written as:

$$\psi_{i,T}(x, y) = 1 - c_i(x_- x_{P_i}) - d_i(y - y_{P_i})$$

Definition 3.2. Let $\aleph$ be the set of nodes associated with the conformal triangulation E. For every P $\in \aleph$, let $E_P = \{T \in E | P \textit{ is a vertex of } T\}$ be the set of triangles from E having P as a vertex. Then, $S_1(E) = span\{\psi_P(x,y)/P \in \aleph\}$, where $\psi_P(x,y)$ is the basis function associated with node P.

One notes that $Supp(\psi_P) = E_P$, i.e

$$\begin{cases} \psi_P(x,y) = 0 & (x,y) \notin E_P \\ \psi_P(x,y) \neq 0 & (x,y) \in E_P \end{cases}$$

Hence for $\Phi \in S_1(E)$, and $\aleph$ being the set of nodes of E. One writes

$$\Phi(x,y) = \sum_{P \in \aleph} \Phi(P)\psi_P(x,y)$$

And therefore to any $\Phi \in S^1(E)$ one associate uniquely

$$V = \begin{pmatrix} \Phi(P_1) \\ \vdots \\ \Phi(P_{N^2}) \end{pmatrix} \in \mathbb{R}^{N^2}$$

This implies that finding a function $\Phi \in S^1$ reduces to finding its values at the nodes of E. Thus given a triangulation $E = \{T_i | 1 \leq i \leq M\}$ that covers Ω in a conformal way, implementing the finite element method requires two basic data structures: `Nodes` and `Elements`.

These auxiliary structure are needed to generate discrete systems used to obtain the pair $\{u_N, w_N\}$ of finite element approximations to the coupled Elliptic-Hyperbolic system (3.1).

Theorem 3.3 (The Basic approximation result for finite-elements subspaces)**.** *[20]* $\forall v \in H^2(\Omega) \exists v_N \in S_1(E)$ *such that*

$$\|v - v_N\|_{H^1(\Omega)} \leq \frac{C}{N} \|v\|_{H^2(\Omega)} \tag{3.3}$$

3.1.2 Definition of $S^1_P(E)$

Definition 3.4. Define the $S_{1,P}$ as follows:

$$S_{1,P} = \{v \in S_1(E)/v \in (PBC) \text{ on } \Gamma\}$$

To find a basis for $S_{1,P}(E)$, we proceed as follows:

For each I or J belonging to the the boundary $\Gamma = \partial\Omega$ with $y(I) = 1$ or $x(J) = 1$ (I and $J \in PERIODIC_1$), corresponding nodes K and L $\in PERIODIC_2$ respectively such that $y(K) = 0$ or $x(L) = 0$.

Therefore our work is reduced into solving $N^2 - (2N - 1) = N_1$ equations instead of a N^2 equations, where N^2 is the total number of nodes.

$$u_N = \sum_{I \in INTERIOR} U_I \psi_I + \sum_{I \in PERIODIC_1} U_I \psi_I + \sum_{I' \in PERIODIC_2} U_{I'} \psi_{I'}$$

$$u_N = \sum_{I \in INTERIOR} U_I \psi_I + \sum_{I \in PERIODIC_1 \cup PERIODIC_2} U_I (\psi_I + \psi_{I'})$$

$$u_N = \sum_{I \in INTERIOR} U_I \psi_I + \sum_{I \in PERIODIC_1} U_I (\tilde{\psi_I})$$

Define: $\{\varphi_I, \;\; 1 \le I \le N_1\} = \{\psi_I, I \text{ interior node}\} \cup \{\tilde{\psi_I}, \text{ I } \in PERIODIC_1\}$

and $S_{N,P} = span\{\varphi_I, \;\; 1 \le I \le N_1\}$

Hence the approximation property:

$$\forall v \in H^1_P(\Omega), \exists v_N \in S_{N,P} : lim_{N \to \infty} \|v - v_N\|_H = 0 \tag{3.4}$$

3.2 Finite Element Discretization of the equations in the coupled system

The Poisson's part is discretized by seeking

$$u_N(s) \in S_P^1(\Omega) \qquad a(u_N(s), v) =< w(s), v >, \qquad \forall s \in (0, T], \ \ \forall v \in S_{1,P}(\Omega)$$

with $u_N(s) = \sum_{j=1}^{N} U_j(s)\varphi_j \in S_{1,P}(E)$

For the Hyperbolic part Starting with the variational formulation written in integral form (2.16):

Given $w(t) \in S_{N,P}$ such that: $< w(t + \Delta t), v >= -\int_t^{t+\Delta t}(< \overrightarrow{V}(u).\nabla w(t), v > + < ku_y, v >)ds+ < w(t), v >$

$\forall v \in L^2(\Omega)$ and $\forall s \in (t, t + \Delta t]$

One defines then the Finite element discrete system for such equation, where as given $w_N(t)$ one seeks $\forall s \in (t, t + \Delta t] : w_N(s) = \sum_{i=1}^{N_1} w_i(s)\varphi_i$

$$< w_N(t + \Delta t), v >=< w_N(t), v > - \int_t^{t+\Delta t}(V(u_N(s))\nabla w_N - ku_{N,y})vds, \quad \forall v \in S_{N,P}$$

3.3 Finite Element method for the coupled (HM) system

The method is formulated through a Petrov-Galerkin procedure on (3.1).

Given $\{u_N(t), w_N(t)\}$, we seek $\{u_N, w_N\} : (t, t + \Delta t] \longrightarrow S_{N,P} \times S_{N,P}$

$$(HM_d) \begin{cases} a(u_N(s), v) =< w_N(s), v > & \forall s \in [t, t + \Delta t], \forall v \in S_{N,P} \\ < w_N(t + \Delta t), v >=< w_N(t), v > - \int_t^{t+\Delta t}(V(u_N(s))\nabla w_N - ku_{N,y})vds & \forall v \in S_{N,P} \end{cases} \tag{3.5}$$

This system can be written in matrix form as:

$$\begin{cases} AU(s) = MW(s) & t \leq s \leq t+\Delta t \\ MW(t+\Delta t) = MW(t) - \int_t^{t+\Delta t} P(U(s))W(s)ds + \int_t^{t+\Delta t} RU(s) & t \leq s \leq t+\Delta t \end{cases} \tag{3.6}$$

Where we have to generate these matrices with A,M, $P(U(\Omega))$ being finite-element sparse matrices $N_1 \times N_1$ defined as follows:

$A = \{< \varphi_i, \varphi_j > + < \nabla\varphi_i, \nabla\varphi_j > /i, j = 1..N_1\}$ is the stiffness matrix

$M = \{< \varphi_i, \varphi_j > /i, j = 1..N_1\}$ is the mass matrix (symmetric positive definite).

$P(u_N) = \{< \overrightarrow{V}(u).\nabla\varphi_i, \varphi_j > /i, j = N_1\}$ is the skew-symmetric matrix. $R = \{< k\varphi_{iy}, \varphi_j > /i, j = 1..N_1\}$

With A, M and R constant matrices over the time, while $P(u_N)$ changes with time.

The coupled system obtained in (3.6) (highly implicit and non linear) is used to advance the pair {U(s),W(s)} by computing successively $\{U(\Delta t), W(\Delta t)\}, \{U(2\Delta t), W(2\Delta t)\}...\{U(m\Delta t), W(m\Delta t)\}$.

3.4 Predictor-Corrector Scheme

We advance the solution in time from $\{U(t), W(t)\}$ to $\{U(t+\Delta t), W(t+\Delta t)\}$ at time $(t+\Delta t)$ by letting $\{U(t+\Delta t), W(t+\Delta t)\} = \{Y, Z\}$. Specifically, assume $\{U, W\} = \{U(t), W(t)\}$ is given, we obtain $\{Y, Z\} = \{U(t+\Delta t), W(t+\Delta t)\}$ by replacing $\int_t^{t+\Delta t}$ using and a second order Trapezoidal rule and then targeting to solve the implicit system:

Given $\{U, W\}$, find $\{Y, Z\}$

$$\begin{cases} AY = MZ \\ MZ = MW - \frac{\Delta t}{2}[P(Y)Z + P(U)W] + R\frac{\Delta t}{2}(U+Y) \end{cases} \tag{3.7}$$

(3.7) is a Crank-Nicolson type implicit system that is solved by predictor-corrector scheme.

Implicit-Predictor Corrector:

One define the sequence $\{Y^k, Z^k\}$ in the following way: starting $Y^0 = U, Z^0 = W$ in the following way:

$$\begin{cases} (M + \frac{\Delta t}{2}P(Y^k))Z^{k+1} = (M - \frac{\Delta t}{2}P(U))W + \frac{\Delta t}{2}RU + \frac{\Delta t}{2}RY^k \\ AY^{k+1} = MZ^{k+1} \end{cases} \tag{3.8}$$

Theorem 3.5. *$(M + \frac{\Delta t}{2}P(Y^k))$ is positive definite which implies that $(M + \frac{\Delta t}{2}P(Y^k))$ is invertible.*

Proof. $v^T(M + \frac{\Delta t}{2}P(Y^k))v = v^TMv + v^T\frac{\Delta t}{2}P(Y^k)v$ Note that M is symmetric positive definite hence $v^TMv \geq 0$ and P is skew symmetric, i.e $P^T = -P$ which can be proved easily then $v^T\frac{\Delta t}{2}P(Y^k)v = 0$ Hence $v^T(M + \frac{\Delta t}{2}P(Y^k))v \geq 0$ and so Positive definite. □

$P(Y^K)$ changes at each correction step. Thus meshing this correction steps is highly expensive as one has to perform an LU decomposition at each time. Instead, we replace $(M + \frac{\Delta t}{2}P(Y^k))$ by $(M + \frac{\Delta t}{2}P(U))$ thus performing one LU decomposition at the prediction and using it at each correction.

$$\begin{cases} (M + \frac{\Delta t}{2}P(U))Z^{k+1} = (M - \frac{\Delta t}{2}P(U))W + \frac{\Delta t}{2}RU + \frac{\Delta t}{2}RY^k \\ AY^{k+1} = MZ^{k+1} \end{cases} \tag{3.9}$$

Where our skew-symmetric matrix P(U) remains constant while the Prediction-Correction steps.

Note that, using this semi-implicit method we can choose $\Delta t = \Delta x$ avoiding the explicit methods [5] that forces to take small Δt steps.

Remark: Explicit methods are obtained by a first order rectangular rule.

$$\int_t^{t+\Delta t} P(u_N)W(s)ds \approx \Delta t P(u_N(t))W(t)$$

3.5 Implementation aspects

There are essentially related to generating the matrices $A, M, R, P(U)$ using the `MATLAB` sparse command. For that purpose one has to compute local coefficients of these matrices, in the table of triangles. In what follows we will illustrate the procedures. Given the data of table `Nodes` and `Elements` relative to the nodes, the local matrix and vector elements can be all computed.

$$a(\varphi_I, \varphi_J) = \sum_{T \in E} a_T(\varphi_I, \varphi_J)$$

$$F(\varphi_J) = \sum_{T \in E} F_T(\varphi_J)$$

	P_1	P_2	P_3
P_1	$a_T(\varphi_I, \varphi_I)$	$a_T(\varphi_I, \varphi_J)$	$a_T(\psi_I, \varphi_K)$
P_2	$a_T(\varphi_J, \varphi_I)$	$a_T(\varphi_J, \varphi_J)$	$a_T(\varphi_J, \varphi_K)$
P_3	$a_T(\varphi_K, \varphi_I)$	$a_T(\varphi_K, \varphi_J)$	$a_T(\varphi_K, \varphi_K)$

INDEX	P_1	P_2	P_3	9 components of Local Matrix	3 components of Local Vector
1	–	–	–	–	–
2	–	–	–	–	–
⋮	⋮	⋮	⋮	⋮	⋮
M	–	–	–	–	–

3.5.1 Poisson's Local matrices

The integration of $\int_{T_k} \varphi_I \varphi_J dxdy$ and $\int_{T_k} \nabla\varphi_I \nabla\varphi_J dxdy$ are done using the formula $\int_{T_k} f(x,y)dxdy \approx \frac{Area(T_k)}{3} \sum_{P\in\aleph(T)} f(P)$

which is based on linear interpolation of $\varphi_I\varphi_J$ and $\nabla\varphi_I\nabla\varphi_J$ at each element (triangle) vertices, say I, J and L respectively.

$$\int_{T_k} \varphi_I \varphi_J dxdy \approx \frac{1}{3} \text{ Area(T)} \sum_{P=I,J,L} \varphi_I(P)\varphi_J(P)$$

Notice that since φ_K is a $\mathbb{P}_1$-function then $\nabla\varphi_K$ is a constant. Hence:

$$\int_{T_k} \nabla\varphi_J \nabla\varphi_J dxdy \approx \text{Area(T) } \nabla\varphi_J \nabla\varphi_J$$

$\int_{T_k} \varphi_J w dxdy \approx \dfrac{1}{3} \text{ Area(T)} \sum_{P=I,J,L} \varphi_I(P)w(P)$.

3.5.2 Hyperbolic equation's Local matrices

To compute the local stiffness matrix M, one needs to compute

$$M_T(\varphi_I, \varphi_J) = \int_T \varphi_I \varphi_J d\mu$$

Next, to compute $P(u_N)_T = \{< \overrightarrow{V}(u).\nabla\varphi_i, \varphi_j > \backslash i,j = 1..N\}$ with $\overrightarrow{V}(u)$ the vector field $\left(\begin{smallmatrix} -u_x \\ u_y \end{smallmatrix}\right)$.

Then $P(u_N)_T(\varphi_I, \varphi_J) = -\int_T u_y \nabla\varphi_I \varphi_J d\mu + \int_T u_x \nabla\varphi_I \varphi_J d\mu$

With u_x and u_y are computed using finite difference on the matrix solution u.

We ran into difficulties dealing with such matrix depending on $u_N(x,y,t)$, since the coefficients of this matrix are varying at each time step resulting into slow computations.

Finally, to compute $R(u_N) =< ku_{N,y}, \varphi_j >$ with $k = \partial_x ln\dfrac{n_0}{w_{ci}}$ with $ln\dfrac{n_0}{w_{ci}}$ not depending on y and its derivative with respect to x is the constant k, then

$$a_{R,T}(\varphi_I, \varphi_J) = k \int_T (\varphi_I)_y \varphi_J d\mu$$

Note that, all the previous integrations are treated by the same rule used in (3.5.1).

Chapter 4

Numerical Simulations

In Chapter 2, we studied the Hasegawa-Mima equation as a coupled system. In Chapter 3, we discussed the finite element formulation of the problem and the resulting discrete scheme. We present in this chapter simulations done by solving this equation. We aim at conducting a numerical test based on turbulence theory.

4.1 Nonlinear Solution for the HM Equation

A modon is the simplest member of the family of solutions of the HM equation.

If the Hasegawa-Mima equation is to have any exact solutions, then they must either be stationary, or moving at a constant velocity. The reason for this is the constraint of the conservation of the density function. [5] Thus, we are lead to look for stationary solutions.

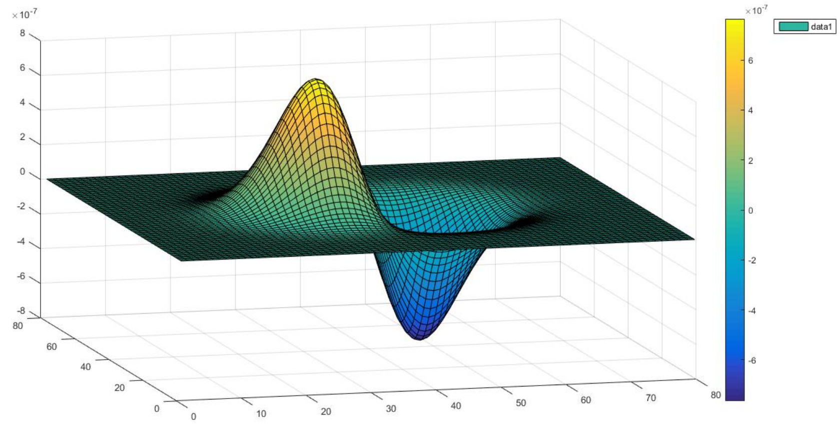


Figure 4.1: Initial potential profile for the modon dipole vortex solution with $a = 2$, $c = 10$ and $\gamma = 4 : 0914$.

4.1.1 Test for Stationarity and Conservation of Energy and Enstrophy

We aim at verifying that our Finite Element code preserves the steadiness of the solution as a function of time. Figure 4.1 a plot of the modon which will be used for this test as initial function. The continuity constant should be chosen carefully so as not to induce discontinuities in u. The code is run for 500 s with a time step $= \Delta x$. We use contour plots to show the time evolution of the simulated solution. It is striking that no modification is detected even after such a long period of time. We deduce that our code preserves the stationarity condition.

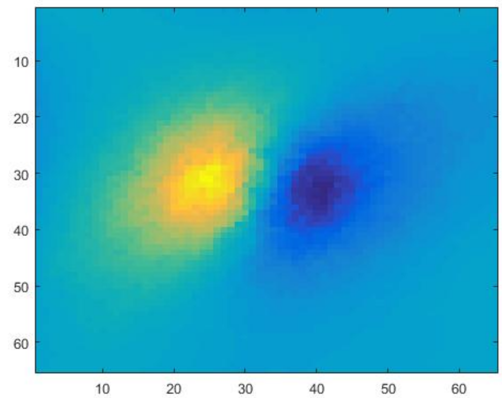
4.2 Test for the Periodic Boundary Conditions

An exponential initial profile for n_0 is used with gradient in the y-direction, we show its shape in figure 4.3, together with a First order derivative of a Gaussian (figure 4.1) as an initial profile to test its behavior after passing a boundary.

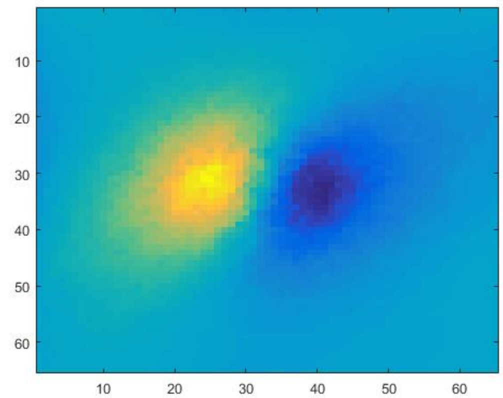
The gradient of $log(n_0)$ induces a convective motion perpendicular to its direction, that is, in the x-direction. Additionally, Φ_0 is rather close to the modon solution, hence, we expect this initial condition not only to travel in the direction perpendicular to the density gradient but also to have an evolution as it is not the exact solution for the non-linear HM equation. [5]

As it is clearly illustrated in Fig (4.5), the solution satisfy the periodic boundary conditions even when advancing in time and using a relatively big time step Δt.

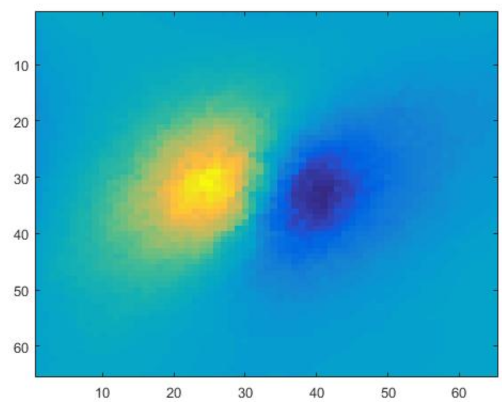
Hence the period boundary conditions work fine in our `Matlab` code.



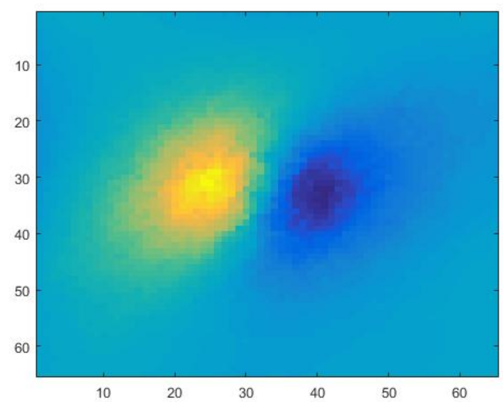
(a) 50s



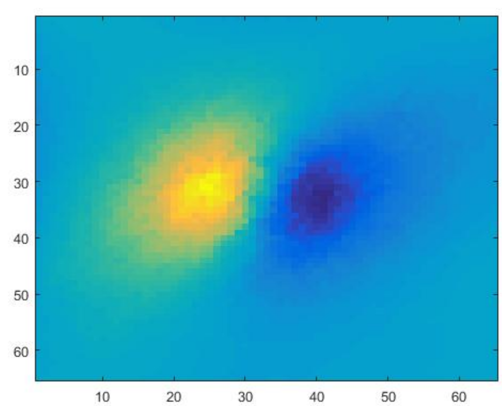
(b) 100s



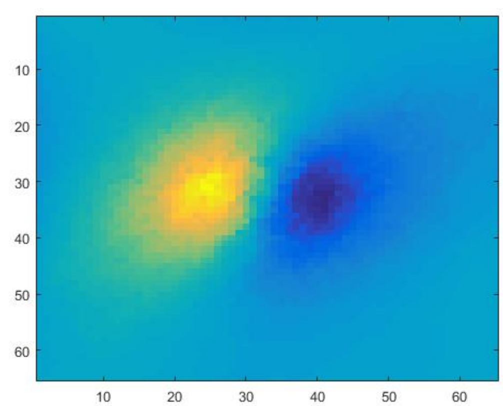
(c) 150s



(d) 200s



(e) 250s



(f) 300s

Figure 4.2: Plots of the electrostatic potential of the modon taken at different times t, with a time step $\Delta t = \Delta x = \frac{20}{64}$, on a 64 × 64 grid.

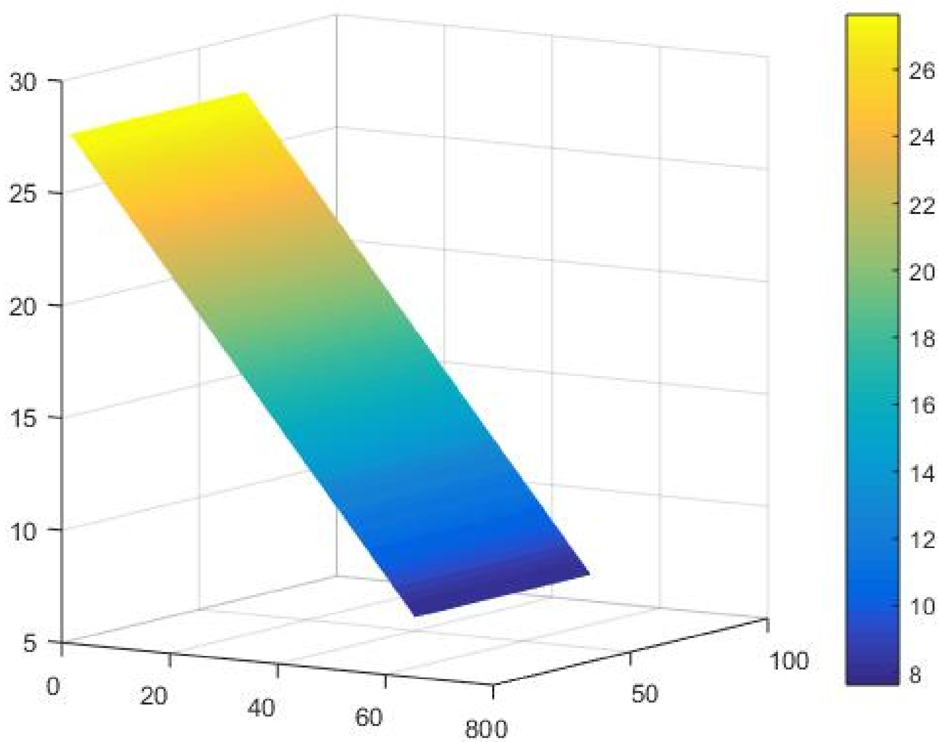
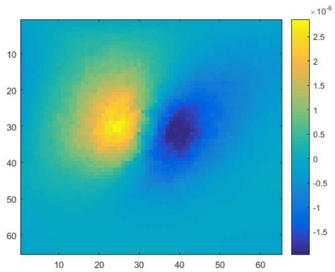
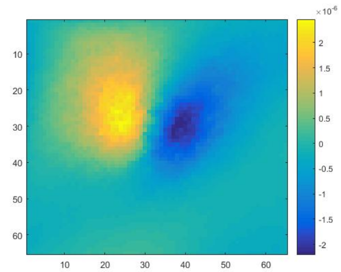


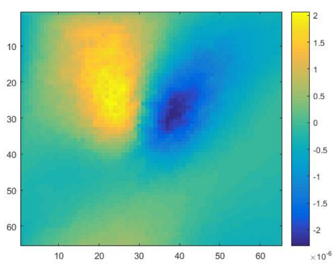
Figure 4.3: Plot of log(n0) where n0 is the initial density profile



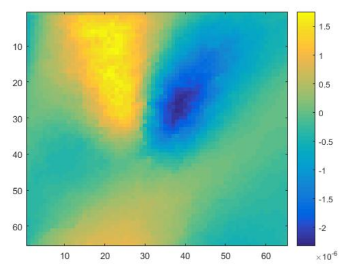
(a) 10s



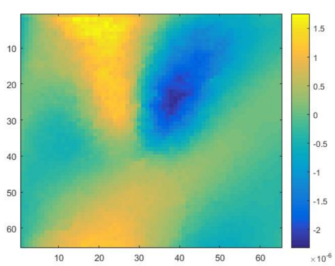
(b) 20s



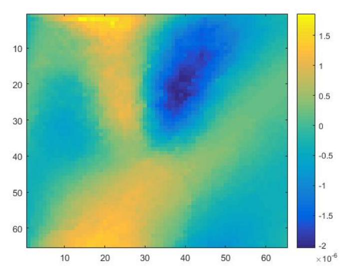
(c) 30s



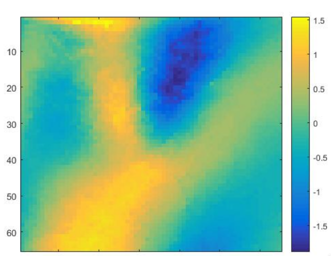
(d) 40s



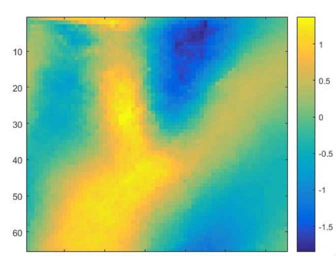
(e) 50s



(f) 60s

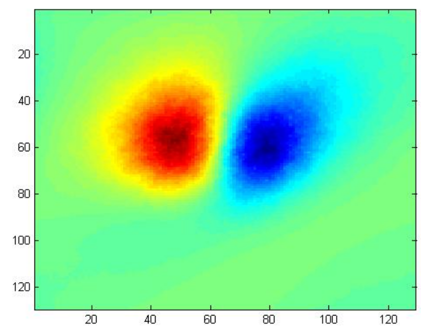


(g) 70s

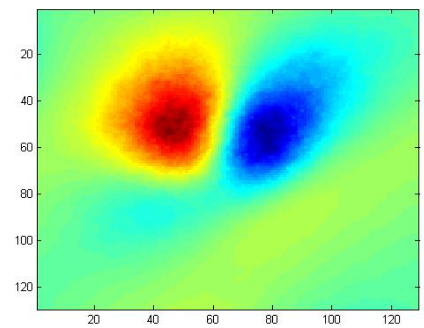


(h) 80s

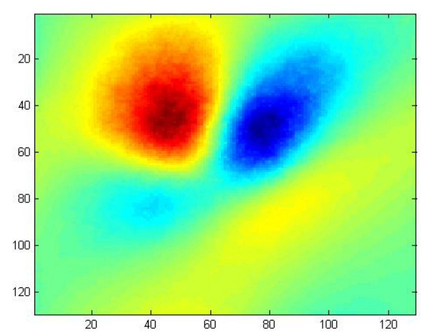
Figure 4.4: The time evolution of the spatial structure of the potential, with a time step $\Delta t = \Delta x = \dfrac{20}{64}s$ on a 64×64 grid.



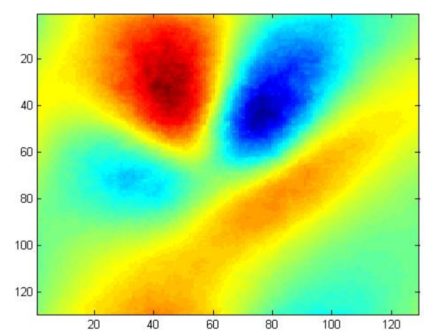
(a) 20s



(b) 40s



(c) 60s



(d) 80s

Figure 4.5: The time evolution of the spatial structure of the potential, with a time step $\Delta t = \Delta x = \dfrac{20}{64}s$ on a 64×64 grid.

Chapter 5

Conclusion

In this work, we present a new Simulation Model for the Hasegawa-Mima Plasma Equation. The derivations are given for the two-dimensional equation where we focus on discretizing the coupled Hasegawa-Mima system introduced in Chapter 2 using Finite Element method for spacial discretization and implicit Crank-Nicolson scheme for time discretization with predictor-corrector implementation discussed in Chapter 3.

First tests of the method have been conducted in Chapter 4. The code appear to be robust and flexible when it comes to the choice of Δt.

Such tests needs to be pursued in the future in view of validating this approach.

Bibliography

[1] ITER Physics Expert Groups on Confinement and Transport and Confinement Modeling and Database. Plasma confinement and transport , Nuclear Fusion, vol. 39 no. 12 pp. 2175 - 2249 (1999).

[2] A. Hasegawa and K. Mima Pseudo - three - dimensional turbulence in magnetized nonuniform plasma Physics of Fluids vol. 21 pp. 87 - 92 Jan (1978)

[3] A. Hasegawa and M. Wakatani , Plasma edge turbulence, Phys. Rev. Lett. vol. 50 pp. 682 - 686 Feb 1983

[4] B. Shivamoggi, , Charney-hasegawa-mima equation: A general class of exact solutions, Physics Letters A, vol . 138 pp. 37 - 42 Jun (1989).

[5] F. A. Hariri , Simulating bi-dimensional turbulence in fusion plasma with linear geometry, Master's thesis, American University of Beirut, (2010).

[6] H.K Karakazian, , Local Existence and Uniqueness of the Solution to the 2D Hasegawa-Mima Equation with Periodic Boundary Conditions, Master's thesis, American University of Beirut, (2016).

[7] M. Kono and E. Miyashita, , Modon formation in the nonlinear development of the collisional drift wave instability, Phys. Fluids, 31(2) (1988).

[8] Paul Terry and Wendell Horton , Stochasticity and the random phase ap- proximation for three electron drift waves, Phys. Fluids, 25(3) March

[9] L. Paumond , Some remarks on a hasegawa-mima-charney-obukhov equation, Physica D: Nonlinear Phenomena, vol. 195, p. 379390 Aug (2004)

[10] A. Pazy , Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer (1983)

[11] B. Guo and Y. Han , Existence and uniqueness of global solution of the hasegawa-mima equation, Journal of Mathematical Physics, Apr (2004).

[12] A. Arakawa, , Computational design for long-term numerical integration of the equations of fluid motion: two-dimensional incompressible flows. part I, J. Comput, Phys. 1:119, (1966)

[13] V. Naulin, K. H. Spatschek, S. Musher and L. Piterbarg, , Properties of a two-nonlinearity model for drift-wave turbulence, Phys. Plasmas, 2: 2640-52, (1995)

[14] Wendell Horton, , Statistical properties and correlation functions of drift waves, Phys. Fluids, 29(5):1491-1503, May (1986)

[15] H. Brezis, , Functional Analysis,Sobolev Spaces and Partial Differential Equations. Springer, Chapter 8 p. 201, May (2011)

[16] Kaw, Autar; Kalu, Egwu, , Numerical Methods with Applications, (2008)

[17] Tuncer Cebeci, , Convective Heat Transfer, Springer, (2002)

[18] Yinnian He and Weiwei Sun, , Stability and Convergence of the Crank-Nicolson/Adams-Bashforth Scheme for the Time-Dependent Navier-Stokes Equations, SIAM Journal on Numerical Analysis Vol. 45, No. 2 pp. 837-869 (2007)

[19] Karkazian-Nassif, , manuscript, Waves 2017 conference

[20] Society for Industrial and Applied Mathematics, , The Finite Element Method for Elliptic Problems, 2002