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General Relativity, Supergravity and Tangent Groups

by

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An Abstract of the Thesis of

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The Lorentz group SO(1,3) is a local symmetry of space-time and is used to define a local orthonormal frame. This symmetry could be used to define General Relativity that allows spinors using the Cartan formulation. Some years ago, it was shown by Chamseddine and Mukhanov, contrary to statements made by Weinberg, that the local symmetry could be enlarged to the groups SO(1,4) or SO(2,3), and that the gravitational action is equivalent to the Einstein-Hilbert action. A Lagrangian for Supergravity could be constructed by considering the Poincare extension of the Lorentz group, as was shown by Chamseddine and West in 1976. A recent paper by Chamseddine and Mukhanov have shown that gravity and gauge theories could be unified in one geometric construction, using a large Lorentz tangent group SO(1,13), provided that a metricity condition is imposed on vielbein. We will discuss this unification in Topological field theories in 3-dimension. This implies that the Chern-Simons Theory exists in 3-dimensions for both gauge theories and gravity with the same quantized coupling constants.

Contents

Introduction

General relativity has been so far the best theory to describe gravity and spacetime for spacetime tells matter how to move, and matter tells spacetime how to curve. General relativity treats the gravitational field as a classical dynamical field, represented by (pseudo-) Riemannian metric of spacetime. However a quantum field theory of gravity would require a quantization of the gravitational field. There have been several attempts to construct semi-classical modifications of the Einstein field equations where gravity is treated classically and everything else is treated quantum mechanically; as well as, quantum field theories in curved spacetime. A quantum theory of gravity is needed to reconcile general relativity with the principles of quantum mechanics, but applying the usual recipe of a quantum field theory to gravity via the bosonic force carrier, the graviton, gives a non renormalizable theory that cant be used to make useful physical predictions. There have been several approaches towards a theory of quantum gravity such as string theory, supergravity, non-commutative geometry and loop quantum gravity. The aim of quantum gravity is only to describe the behavior of the gravitational field quantum mechanically; it is not aiming to unify all the fundamental interactions into a single mathematical framework. But a theory of quantum gravity could lead the way to several branches in high energy physics that aim for the unification of all of the fundamental forces of nature which is also called The Theory of Everything.

Speaking about the standard model of particle physics, the matter fields which are the fermions are classified according to how they interact and described by representations of their corresponding symmetry group. There are six quarks and six leptons coming in three flavors, having color and charge. The gauge bosons are defined as force carriers that mediate the fundamental interactions. We have three fundamental interactions in the standard model, the strong nuclear, weak nuclear and the electromagnetic interactions. For the electromagnetic interactions we have U(1) symmetry, SU(2) for the weak interactions and SU(3) for the strong interactions, thus the gauge group for the standard model is U(1)xSU(2)xSU(3). Although the standard model agrees perfectly with experiment, it leaves the physicist unsatisfied because it does not address some fundamental questions. For example, the different coupling constants for the electromagnetic, weak and strong interactions, the amount of parameters in the model, the quantization of the charge, and the hierarchy problem. We know that the values of the renormalized coupling constants depend on the renormalization scale M, so a theory unifying those three fundamental interactions requires finding a renormalization scale at which the three coupling constants satisfy the group property which already embed the electroweak and QCD in a grand unified group. In other words, the three coupling constants unify at the unification scale where the new gauge fields are associated with the larger gauge group. This is called a Grand Unified Theory at high energy at which all forces are merged into one single force. As an example, we have GUTs using the symmetry groups SU(5), SO(8) ,SO(10) and E8.

Moreover, we can have supersymmetric extensions to the Standard Model. Supersymmetry is a symmetry between the fermions and bosons, it predicts supersymmetric bosonic partners for the fermions and supersymmetric fermionic partners for the bosons. Since supersymmetry involves new fields, there will be new contributions to the evolution of the coupling constants where these changes are due to the superpartners. As a result the three couplings meet exactly at the same point (unification scale). If we take supersymmetry as a local symmetry

group, this will give us a theory that includes gravity as well and is called Supergravity. This was one of the attempts to unify the four forces of nature by combining the principles of supersymmetry and general relativity.
The quest for the theory of everything has been a motivation for mathematicians and even lead the way for new mathematical fields for all of theoretical physics is based on consistent mathematical frameworks. Group theory is a main ingredient in the standard model, grand unified theories, and supersymmetry. Non-Euclidean geometry is what general relativity is based on. Much more than that, most theorists dream of an abstract mathematical description of the universe with elegance and beauty. Topological field theories turned out to be useful in physics as well. The Chern-Simons theory has been used in mathematics to calculate knot invariants and three manifold invariants such as the Jones polynomials, while in physics to describe the topological order in fractional quantum Hall effect states. Moreover, topology is applied to condensed matter and used in topological insulators and topological phases of matter. Chern-Simons theory turned out to me most useful in three dimensional manifolds, this can give insights to the physics in four dimensions and is of special interest experimentally and theoretically. We are interest here in investigating the unification of gauge interaction and gravity in a Chern-Simons theory in three dimensions.

The Lorentz group SO(1,3) is a local symmetry of space-time and is used to define a local orthonormal frame. General Relativity could be constructed by taking the local symmetry of the Lorentz group, the Cartan formalism of General Relativity. The local symmetry could be enlarged to the SO(1,4) or SO(2,3) or to the Pioncare extension, as well as Graded Lie Algebras in order to include Supergravity as well. In a recent paper by Chamseddine and Mukhanov have shown that gravity and gauge theories could be unified in one geometric construction of a larger group SO(1,13) provided that a metricity condition is imposed on vielbien. In 3-dimensions the Chern-Simons action using such group with a specific gauge can be split into two parts, the gravitational Chern-Simons part and the Yang-Mills Chern-Simons part, which will have several applications.

We will begin by the work of Chamseddine and Mukhanov [**1**] which introduces the large group that unifies gravity and gauge theories and see how we can split the gravitational and gauge indices by impposing a metricity condition, then construct the action. In the second chapter we'll introduce the Chern-Simons Thoery for both Gravity and Yang-Mills then discuss their relation to topological invariances [**2**][**3**][**4**]. In the third chapter we'll try to unify the gauge and gravitational interactions in the Chern-Simons theory. Starting from the larger group we reduce to 3-dimensions then split the action into two parts resulting with the same quantized coupling constants for gravitational and gauge interactions in the case of compact groups.

In the last chapter we will discuss Super Algebras, specifically the Super Poincare algebra. This provides a symmetry between the bosonic and fermionic sectors of the Standard Model of Particle Physics. Then the Supergravity Lagrangian will be constructed as a Gauge theory of Supersymmetry as proposed by Chamseddine and West. At last, we'll talk about how our Unified Chern-Simons action can be extended to Supersymmetry.

Chapter 1

The Unification using Tangent Groups in a Cartan Formalism

A recent paper by Chamseddine and Mukhanov [**1**] have managed to unify gravity and gauge interactions by considering a higher dimensional Lorentz group as the symmetry of the tangent space in a Cartan formalism. We will consider a general case for any SO(N) symmetry group and elaborate how the spin connection of the gauged Lorentz group is responsible for both gravity and gauge fields.

In General Relativity the Lorentz group is taken as a local symmetry of the tangent manifold, the dimension of the tangent space is usually considered equal to the dimension of the curved manifold. The Lorentz symmetry is just a manifestation of the equivalence principle for spaces with zero torsion condition. The Cartan formalism of General Relativity considers the local Lorentz transformations in tangent space so that General Relativity is formulated as a gauge theory where the gauge fields are the spin-connections. If the dimensions of the tangent space is taken to be equal to that of space time then the gauge fields, the spin connections, include the same amount of information about the dynamics of the gravitational field as the affine connection only! However, the dimension of the tangent space must not necessarily be taken equal to that of space time.

Here we will consider the case of a higher dimensional tangent Lorentz group.

In which, the gauge transformations are realized as a subgroup of the tangent Lorentz group and the spinors describing matter are unified in the fundamental representation of this higher dimensional Lorentz group. Normally the group SO(10) is considered to be the group that unifies all matter in the Standard Model of particle physics and the SO(1,3) being the Lorentz group, thus the group SO(1,13) would be a good model.

As in [**1**], we consider a 4-dimensional manifold, where at every point of this manifold there is a real N-dimensional tangent space. This tangent space is spanned by linearly independent vectors v_A , A runs from 1 to N $(A = 1, 2, ..N)$, where $N \geq 4$. We also define the coordinate basis vectors spanning the 4-dimensional subspace by $e_\alpha = \partial/\partial x^\alpha$, where $\alpha = 1, .., 4$.
We define the scalar product in the tangent space as follows, where η_{AB} $(-, +, .., +)$ is the Minkowski matrix.

$$v_A.v_B = \eta_{AB} \tag{1.1}$$

and the scalar product of the coordinate basis vectors e_α in the tangent space induces the metric in the 4-dimensional manifold, which is defined by

$$e_\alpha.e_\beta = g_{\alpha\beta}(x^\gamma) \tag{1.2}$$

We define the vielbiens e^A_α as the coefficients of expansion of e_α in v_A-basis.

$$e_\alpha = e^A_\alpha v_A \tag{1.3}$$

by substituting (1.3) in (1.2) we can express the metric $g_{\alpha\beta}$ in terms of the velbiens.

$$g_{\alpha\beta} = e_{\alpha}^{A} e_{\beta}^{B} \eta_{AB} = e_{\alpha}^{A} e_{A\beta}$$

where we raise and lower tangent space indices using η_{AB}.

In order to determine the rules of parallel transport for the coordinate basis vectors and the vielbiens in the nearby tangent space, we use the affine and spin-connections that are respectively defined by

$$\nabla_{e_\beta} e_\alpha = \nabla_\beta e_\alpha = \Gamma^{\nu}_{\alpha\beta} e_\nu \tag{1.4}$$

$$\nabla_\beta v_A = -w^{B}_{\beta A} v_B \tag{1.5}$$

∇_β is the directional derivative along the coordinate basis vector e_β. Using the important fact that η_{AB} and $g_{\alpha\beta}$ are sets of scalar functions, we deduce the metricity condition $\nabla_\beta \eta_{AB} = 0$ and $\nabla_\gamma g_{\alpha\beta} = \partial g_{\alpha\beta}/\partial x^\gamma = \partial_\gamma g_{\alpha\beta}$.

A consequence of the metricity condition would be that the spin-connection should be antisymmetric in tangent indices $w_{\alpha AB} = -w_{\alpha BA}$, this is easily seen by

$$\nabla_\alpha \eta_{AB} = (\nabla_\alpha v_A).v_B + v_A.(\nabla_\alpha v_B)$$

$$= -w^{C}_{\alpha A} v_C.v_B - v_A.w^{C}_{\alpha B} v_c = -w_{\alpha AB} - w_{\alpha BA} = 0$$

and if we apply ∇_β to $e_{A\alpha}$ that is defined as $e_{A\alpha} = (v_A.e_\alpha)$,we get

$$\partial_\beta e_{A\alpha} = (\nabla_\beta v_A.e_\alpha + v_A.(\nabla_\beta e_\alpha),$$

$$\partial_\beta e_{A\alpha} = -w^{B}_{\beta A} e_{B\alpha} + \Gamma^{\nu}_{\alpha\beta} e_{A\nu} \tag{1.6}$$

where the space-time here is assumed to be torsion free, that is the affine connection is symmetric with respect to the coordinate basis indices $\Gamma^{\nu}_{\alpha\beta} = \Gamma^{\nu}_{\beta\alpha}$.
We also note that in the absence of torsion, the well known Christoffel connections are expressed by the following, and are determined from eq.(1.6) irrespective of the dimension of the tangent space

$$\Gamma^{\gamma}_{\alpha\beta} = \frac{1}{2} g^{\gamma\sigma}(g_{\alpha\sigma,\beta} + g_{\sigma\beta,\alpha} - g_{\alpha\beta,\sigma}), \tag{1.7}$$

where $g^{\gamma\sigma}$ is the inverse of $g_{\sigma\beta}$, $g^{\alpha\sigma} g_{\sigma\beta} = \delta^{\alpha}_{\beta}$.
We will also need to use the vielbien satisfying the metricity condition

$$e^{\alpha}_{A} = g^{\alpha\gamma} e_{A\gamma} \tag{1.8}$$

$$\partial_{\beta} e^{\alpha}_{A} = -w^{B}_{\beta A} e^{\alpha}_{B} - \Gamma^{\alpha}_{\beta\nu} e^{\nu}_{A}. \tag{1.9}$$

where $e^{\alpha}_{A} e^{B}_{\beta} \neq \delta^{A}_{B}$. However, only if the dimension of the tangent space is equal to that of the manifold then e^{α}_{A} is inverse to e^{B}_{β}.

$$e^{\alpha}_{A} e^{A}_{\beta} = g^{\alpha\gamma} e_{A\gamma} e^{A}_{\beta} = g^{\alpha\gamma} g_{\gamma\beta} = \delta^{\alpha}_{\beta}$$

It is important to mention that in the vielbien formalism, e^{α}_{A} is a fundamental quantity and is invariant under the group of local Lorentz transformations $\Lambda^{B}_{A}(x)$.

In order to show how the larger dimension Lorentz group unifies gauge theories and gravity, we will split the tangent space basis v_A into a basis of $(e_{\alpha}, n_{\hat{\jmath}})$. Let us introduce $N-4$ orthonormal vectors $n_{\hat{\jmath}}$ orthogonal to the subspace spanned by e_{α}, that is,

$$n_{\hat{\jmath}}.e_{\alpha} = 0$$

$$n_{\hat{J}}.n_{\hat{I}} = \delta_{\hat{J}\hat{I}}$$

where $\hat{J}, \hat{I} = 5, 6, ..., N.$ and the vectors $n_{\hat{J}}$, e_α form a complete basis in tangent space. Taking into account $e_{A\alpha} = (v_A.e_\alpha)$ we have

$$e_{A\gamma} = (v_A.e_\gamma) = v_A^\alpha g_{\alpha\gamma} \tag{1.10}$$

and hence, $v_A^\alpha = g^{\alpha\gamma} e_{A\gamma} = e_A^\alpha$, that is, the coefficients v_A^α in (1.10) coincide with soldering form e_A^α. From this one gets

$$\eta_{AB} = v_A.v_B = v_A^\alpha v_B^\beta g_{\alpha\beta} + n_A^{\tilde{J}} n_{\tilde{J}B} = e_A^\alpha e_{\alpha B} + n_A^{\hat{J}} n_{\hat{J}B}, \tag{1.11}$$

or after raising the tangent space index B

$$e_A^\alpha e_\alpha^B = \delta_A^B - n_A^{\hat{J}} n_{\hat{J}}^B \equiv P_B^A$$

(1.12)

where P_B^A is defined as a projection operator: $P_C^A P_B^C = P_B^A$.

It is convenient consider the expansion

$$n_{\hat{J}} = l_{\hat{J}}^B v_B \tag{1.13}$$

Substituting this expression into $n_{\hat{J}A} = n_{\hat{J}}.v_A$ we obtain $n_{\hat{J}A} = l_{\hat{J}}^B \eta_{BA}$ and hence $l_{\hat{J}}^B = n_{\hat{J}}^B$; therefore

$$n_{\hat{J}} = n_{\hat{J}}^B v_B = n_{\hat{J}}^B \left(e_B^\alpha e_\alpha + n_B^{\hat{I}} n_{\hat{I}} \right) \tag{1.14}$$

Therefore the components $n^{\hat{J}}_A$ satisfy the following relations

$$n^A_{\hat{J}} e^\alpha_A = 0$$

$$n^A_{\hat{J}} n^{\hat{I}}_A = \delta^{\hat{I}}_{\hat{J}}$$

Before we get to the curvature of the spin connection, we will introduce the Dirac action. Consider a spinor ψ which transforms in tangent space under the transformation generated by the representation of the Lie algebra according to

$$\psi \to \exp(\frac{1}{4}\lambda^{AB}\Gamma_{AB})\psi \tag{1.15}$$

where $\Gamma_{AB} = \frac{1}{2}(\Gamma_A\Gamma_B - \Gamma_B\Gamma_A)$ are generators of the Lie algebra in the spinor representation and Γ_A are N Dirac matrices satisfying

$$\left\{\Gamma^A, \Gamma^B\right\} = 2\eta^{AB}$$

$$\Gamma = \Gamma^0\Gamma^A\Gamma^0.$$

The Lorentz and Gauge invariant gauge Dirac action is

$$\int d^4x\sqrt{g}\,\overline{\psi} i\Gamma^C e^\alpha_C D_\alpha\psi \tag{1.16}$$

where

$$D_\alpha \equiv \partial_\alpha + \frac{1}{4}\omega^{AB}_\alpha\Gamma_{AB} \tag{1.17}$$

and the hermiticity of the Dirac action is guaranteed by the metricity condition.

Taking the commutator of Dirac operators allows us to define the spin-connection curvature $R^{AB}_{\alpha\beta}$.

$$[D_\alpha, D_\beta] = \frac{1}{4} R_{\alpha\beta}{}^{AB} \Gamma_{AB} \tag{1.18}$$

where

$$R_{\alpha\beta}{}^{AB}(\omega) = \partial_\alpha \omega_\beta{}^{AB} - \partial_\beta \omega_\alpha{}^{AB} + \omega_\alpha{}^{AC} \omega_{\beta C}{}^{B} - \omega_\beta{}^{AC} \omega_{\alpha C}{}^{B}. \tag{1.19}$$

and under lorentz transformation the spin curvature transforms as $(R_{\mu\nu})_A{}^B \to (\Lambda R \Lambda^{-1})_A{}^B$.

If we use the identity

$$\partial_\beta \partial_\alpha e_{A\gamma} - \partial_\alpha \partial_\beta e_{A\gamma} = 0. \tag{1.20}$$

Substituting here $\partial_\beta e_{A\alpha} = -w^B_{\beta A} e_{B\alpha} + \Gamma^\nu_{\alpha\beta} e_{A\nu}$ and using this metricity condition one more time to express ∂e which appears after taking the derivative, we immediately arrive at the following relation

$$R_{\alpha\beta}{}^{AB}(\omega)\, e_{B\gamma} = R^\rho{}_{\gamma\alpha\beta}(\Gamma)\, e^A_\rho \tag{1.21}$$

where

$$R^\rho{}_{\gamma\alpha\beta}(\Gamma) = \partial_\alpha \Gamma^\rho_{\beta\gamma} - \partial_\beta \Gamma^\rho_{\alpha\gamma} + \Gamma^\rho_{\alpha\kappa} \Gamma^\kappa_{\beta\gamma} - \Gamma^\rho_{\beta\kappa} \Gamma^\kappa_{\alpha\gamma} \tag{1.22}$$

is the Riemann curvature. Thus we have a relation between the spin-connection curvature and the affine connection curvature.

Taking $e^\alpha_A e^A_\beta = g^{\alpha\gamma} e_{A\gamma} e^A_\beta = g^{\alpha\gamma} g_{\gamma\beta} = \delta^\alpha_\beta$ into account and irrespective of the dimensions of the tangent space, we can express the 4d Riemann curvature in terms of $R_{\alpha\beta}{}^{AB}(\omega)$ as

$$R^{\sigma}{}_{\gamma\alpha\beta}(\Gamma) = e^{\sigma}_{A} R_{\alpha\beta}{}^{AB}(\omega)\, e_{B\gamma} \tag{1.23}$$

By using the pervious derived property $e^{\alpha}_{A} e^{B}_{\alpha} = \delta^{B}_{A} - n^{\hat{I}}_{A} n^{B}_{\hat{I}}$, we can express $R_{\alpha\beta}{}^{AB}(\omega)$ in terms of $R^{\sigma}{}_{\gamma\alpha\beta}(\Gamma)$

$$R_{\alpha\beta}{}^{AB}(\omega) = R_{\alpha\beta}{}^{AC}(\omega)\, n^{\hat{I}}_{C} n^{B}_{\hat{I}} + R^{\rho}{}_{\gamma\alpha\beta}(\Gamma)\, e^{A}_{\rho} e^{B\gamma}. \tag{1.24}$$

Calculations in this paper shows that the first term on the right hand side of this equation $R_{\alpha\beta}{}^{AC}(\omega)\, n^{\hat{I}}_{C} n^{B}_{\hat{I}}$ can be entirely expressed in terms of the spin-connections defining the parallel transport of vectors $n_{\hat{J}}$. This tangent subspace is orthogonal to that spanned by the 4 coordinate basis vectors e_{α}.
It is convenient to define the following

$$\nabla_{\alpha} n_{\hat{J}} = -A_{\alpha\hat{J}}{}^{\hat{I}} n_{\hat{I}} + B_{\alpha\hat{J}}{}^{\beta} e_{\beta} \tag{1.25}$$

where indices $\hat{J}$ and $\hat{I}$ run over values $5, 6, ..., N$. These indices are also raised and lowered with the Minkowski metric $\eta_{\hat{I}\hat{J}}$. In [**1**], Chamseddine and Mukhanov have shown that $B_{\alpha\hat{J}}{}^{\beta} = 0$ and derived the metricity conditions for $A_{\alpha\hat{J}}{}^{\hat{I}}$.
Defining another covaraint derivative allows us to compute the curvature $R^{C}_{\alpha\beta A}(w)$ as a function of the gauge fields A and a term proportional to the affine curvature.

$$D_{\alpha}(\omega)\, n^{\hat{I}}_{A} \equiv \partial_{\alpha} n^{\hat{I}}_{A} + \omega_{\alpha A}{}^{C} n^{\hat{I}}_{C}, \tag{1.26}$$

and consider the commutator that defines the curvature in terms of the spin connection

$$[D_\alpha(\omega), D_\beta(\omega)]\, n_A^{\hat{I}} = R_{\alpha\beta A}{}^{C}(\omega)\, n_C^{\hat{I}} \tag{1.27}$$

and using the following derived equation from that paper

$$D_\alpha(\omega)\, n_A^{\hat{I}} = n_A^{\hat{J}} A_{\alpha\hat{J}}{}^{\hat{I}} \tag{1.28}$$

we get

$$[D_\alpha(\omega), D_\beta(\omega)]\, n_A^{\hat{I}} = D_\alpha(\omega)\left(n_A^{\hat{J}} A_{\beta\hat{J}}{}^{\hat{I}}\right) - (\alpha \leftrightarrow \beta) = n_A^{\hat{J}} F_{\alpha\beta\hat{J}}{}^{\hat{I}}(A), \tag{1.29}$$

where

$$F_{\alpha\beta}{}^{\hat{I}\hat{J}}(A) = \partial_\alpha A_\beta^{\hat{I}\hat{J}} - \partial_\beta A_\alpha^{\hat{I}\hat{J}} + A_\alpha^{\hat{I}\hat{L}} A_{\beta\hat{L}}{}^{\hat{J}} - A_\beta^{\hat{I}\hat{L}} A_{\alpha\hat{L}}{}^{\hat{J}}. \tag{1.30}$$

$F_{\alpha\beta}{}^{\hat{I}\hat{J}}(A)$ here describes the curvature in terms of the gauge fields of the subspace spanned by the $n_{\hat{I}}$ coordinate vectors (not the affine curvature). to conclude that

$$R_{\alpha\beta A}{}^{C}(\omega)\, n_C^{\hat{I}} = n_A^{\hat{J}} F_{\alpha\beta\hat{J}}{}^{\hat{I}}(A) \tag{1.31}$$

Now, using this result in (1.23) we finally obtain

$$R_{\alpha\beta}{}^{AB}(\omega) = F_{\alpha\beta}{}^{\hat{J}\hat{I}}(A)\, n_{\hat{J}}^A n_{\hat{I}}^B + R^\rho{}_{\gamma\alpha\beta}(\Gamma)\, e_\rho^A e^{B\gamma}. \tag{1.32}$$

Contracting the tangent space index in $R_{\alpha\beta}{}^{AB}$ with e_A^σ always removes the F term in (1.31).

We are interested in computing the curvature invariants out of $R_{\alpha\beta}{}^{AB}(\omega)$ and e_A^γ. In order to get the Lagrangian for the theory we construct

$$R_{\alpha\beta}{}^{AB}(\omega)\, e_A^\alpha e_B^\beta = R(\Gamma) \tag{1.33}$$

which is a scalar invariant linear order in curvature, where $R(\Gamma)$ is the usual scalar curvature of 4d manifold which gives the Einstein action.

However, the second order curvature invariants are obtained by contracting $R_{\alpha\beta}{}^{AB} R_{\gamma\delta}{}^{CD}$ with $e_A e_B e_C e_D$ in all possible combibations of indices $\alpha\beta\gamma\delta$, that are

$$R^2(\Gamma)\,,\;\; R_{\alpha\beta}(\Gamma)\, R^{\alpha\beta}(\Gamma)\,,\;\; R_{\alpha\beta\gamma\delta}(\Gamma)\, R^{\alpha\beta\gamma\delta}(\Gamma)$$

We still have to generate the kinetic terms for $A_\beta^{\hat{I}\hat{J}}$ that is achieved only by contracting the tangent space indices with themselves. This shows that the Yang-Mills kinetic terms are a part of the curvature square term

$$g^{\alpha\gamma} g^{\beta\delta} R_{\alpha\beta}{}^{AB}(\omega)\, R_{\gamma\delta AB}(\omega)$$

$$= g^{\alpha\gamma} g^{\beta\delta} \left(F_{\alpha\beta}{}^{\hat{I}\hat{J}}(A)\, F_{\gamma\delta\hat{I}\hat{J}}(A) \right) + R_{\alpha\beta\gamma\delta}(\Gamma)\, R^{\alpha\beta\gamma\delta}(\Gamma)$$

Then the most general action up to quadratic order can be be expressed as follows as in the referred paper

$$\begin{aligned} I = &\int d^4x \sqrt{-g} \left[\frac{1}{16\pi G} R_{\alpha\beta}{}^{AB}(\omega)\, e_A^\alpha e_B^\beta - \frac{1}{4} g^{\alpha\gamma} g^{\beta\delta} R_{\alpha\beta}{}^{AB}(\omega)\, R_{\gamma\delta AB}(\omega) \right. \\ &\left. + R_{\alpha\beta}{}^{AB} R_{\gamma\delta}{}^{CD} \left(a e_A^\alpha e_B^\beta e_C^\gamma e_D^\delta + b e_A^\alpha e_C^\beta e_B^\gamma e_D^\delta + c e_C^\alpha e_D^\beta e_A^\gamma e_B^\delta \right) \right] \end{aligned} \tag{1.34}$$

$$\begin{aligned} = &\int d^4x \sqrt{-g} \left[\frac{1}{16\pi G} R(\Gamma) + a R^2(\Gamma) - b R_{\alpha\beta}(\Gamma)\, R^{\alpha\beta}(\Gamma) \right. \\ &\left. + \left(c - \frac{1}{4} \right) R_{\alpha\beta\gamma\delta}(\Gamma)\, R^{\alpha\beta\gamma\delta}(\Gamma) - \frac{1}{4} g^{\alpha\gamma} g^{\beta\delta} F_{\alpha\beta}{}^{\hat{I}\hat{J}}(A)\, F_{\gamma\delta\hat{I}\hat{J}}(A) \right] \end{aligned} \tag{1.35}$$

where a, b, and c are dimensionless constants. To avoid the ghost in the graviton propagator, a possible gauge choice will do well.

We have done all of this detailed calculation as presented in the paper[**1**] to emphasizes on how gauge interactions and gravity have been unified. The results showed that the $SO(1, N-1)$ invariants split into $SO(1,3)$ and $SO(N-4)$ invariants, To work in a special gauge, we first split the constraint of

$$\partial_\beta e_A^\alpha = -w_{\beta A}^B e_B^\alpha - \Gamma_{\beta\nu}^\alpha e_A^\nu$$

into $A = a = 1, ..., 4$ and $A = \hat{I} = 5, ...N$:

$$0 = \partial_\mu e_a^\nu + \omega_{\mu a}{}^b e_b^\nu + \omega_{\mu a}{}^{\hat{I}} e_{\hat{I}}^\nu + \Gamma_{\mu\rho}^\nu e_a^\rho \tag{1.36}$$

$$0 = \partial_\mu e_{\hat{I}}^\nu + \omega_{\mu\hat{I}}{}^a e_a^\nu + \omega_{\mu\hat{I}}{}^{\hat{J}} e_{\hat{J}}^\nu + \Gamma_{\mu\rho}^\nu e_{\hat{J}}^\rho \tag{1.37}$$

And under $SO(1, N-1)$ transformation, the vielbiens transform according to $e_A^\mu \to \tilde{e}_A^\mu = \Lambda_{AB} e^{\mu B}$. If we take the gauge subspace only we get $e_{\hat{I}}^\mu \to \tilde{e}_{\hat{I}}^\mu = \Lambda_{\hat{I}a} e^{\mu a} + \Lambda_{\hat{I}\hat{J}} e^{\mu\hat{J}}$. Thus, it is possible to use the gauge invariance and the freedom in the choice of gauge parameters $\Lambda_{\hat{I}a}$ to set $e_{\hat{I}}^\mu$ to zero

$$e_{\hat{I}}^\mu = 0. \tag{1.38}$$

The action is invariant under transformations of the group $SO(1, N-1)$. It is also invariant under transformations deduced from equation (1.37) leading to the transfromations under the subgroup $SO(1,3) \times SO(N-4)$ paramatrized by Γ_{ab} and $\Gamma_{\hat{I}\hat{J}}$.

This gauge choice implies that $\omega_{\mu\hat{I}}{}^a = 0$.

In this special gauge $\omega_{\mu\hat{I}}{}^{\hat{J}} = A_{\mu\hat{I}}{}^{\hat{J}}$ and

$$R_{\mu\nu}{}^{a\hat{I}} = 0, \tag{1.39}$$

and clearly the nonvanishing components of the curvature $R_{\mu\nu}{}^{ab}$ and $R_{\mu\nu}{}^{\hat{I}\hat{J}}$ are responsible for the gravity and gauge fields respectively. Thus achieving our goal of unification of gravity and gauge interactions in a Cartan formalism!

More clearly stated, the gauge groups can be considered as subgroup of the Lorentz group of a higher dimensional tangent space. This means that the connections $A_{\alpha}^{\hat{I}\hat{J}}$ transform under $SO\,(N-4)$ rotations in a subspace orthogonal to the space spanned by coordinate tangent vectors e_{α} . Thus the gauge fields are already unified with gravity within $SO\,(1, N-1)$ Lorentz group.

Chapter 2

Chern-Simons Action and Topology

After elaborating how Chamseddine and Mukhanov managed to unify gauge interaction with gravity in one geometric construction by enlarging the group and dimensions of the tangent space in a vielbien formalism in chapter 1, our next aim is to use this large group in Chern-Simons theory in 3-dimensions. Chern-Simons Theory is a topological field theory; thus in this chapter we will explain what we mean by a topological field theory and present the gravitational Chern-Simons action and the Yang-Mills Chern-Simons action independently before unifying them in the next chapter.

Topological Field Theories are metric independent theories that define invariants of the manifold M which they are defined on. A topological invariant or topological property is a property of the topological space which is invariant under Homeomorphism. By Homeomprphism we mean a continuous function between topological spaces that has a continuous inverse function.

For example, a compact (topological) gauge group is a topological group whose topology is compact. Compactness generalizes the notion of a subset of Euclidean space being closed and bounded. Compact groups are also a natural generalization of finite groups with the discrete topology.

We note that a quantum field theory defined on a manifold M without any prior choice of metric on M is said to be generally covariant, but a quantum field theory in which all observables are topological invariants can naturally be seen as a generally covariant quantum field theory.

In general, we call the topological field theory whose action is explicitly independent of the metric a Schwarz type topological field theory

$$\frac{\delta S}{\delta g_{\mu\nu}} = 0 \tag{2.1}$$

examples of such are Chern-Simons theories and BF theories. We also have the Witten type topological field theories, such as the Donaldson-Witten theory that was formulated by Witten in 1988 and is used to compute Donaldson invariants.

We are interested in Chern-Simons theory for our problem. There have been a lot of interest in this field of study recently, especially after the 2016 Nobel Prize for Topological Phases of Matter. Chern-Simons theory has many physical application like describing the topological order in fractional quantum Hall effect states. While in mathematics, it has been used to calculate Knot invariants and three-manifold invariants such as the Jones polunomials.

Since our theory is topological, indicating that we can't define a metric to construct the action. We will have to use differential forms !

If $x^1 ... x^n$ are local coordinates then a differential k-form is

$$w = w_{i_1 ... i_k} dx^{i_1} \wedge ... \wedge dx^{i_k}$$

the differential forms also have the antisymmetric property

$$dx^i \wedge dx^j = -dx^j \wedge dx^i$$

More generally stated for vectors α, β of degrees p, q respectively , the wedge product has the following property

$$\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha$$

Given a manifold M and a Lie Algebra valued 1-form A over it, we can define the family Chern-Simons of p-forms as follows.

In 1-dim, the Chern-Simon 1-from is the trace $Tr(A)$. In 3-dim, the Chern-Simons 3-form is

$$Tr[F \wedge A - \frac{1}{3} A \wedge A \wedge A]$$

where the curvature F is a 2-form is defined by $F = dA + A \wedge A$.

The Chern-Simon action with a compact and simple gauge group G = SU(N) on a generic 3-manifold as defined by Witten [**4**] is

$$S = \frac{k}{4\pi} \int Tr(A \wedge dA + \frac{2}{3} A \wedge A \wedge A), \tag{2.2}$$

$$= \frac{k}{8\pi} \int \epsilon^{ijk} Tr(A_i(\partial_j A_k - \partial_k A_j) + \frac{2}{3} A_i [A_j, A_k])$$

Here k is the coupling constant and A is a G-gauge connection on the trivial bundle on M.

He showed that the partition function

$$Z(M) = \int DAe^{iS}, \tag{2.3}$$

is a topological invariant of the manifold M [**4**]. For more complicated topological spaces when L is a collection of oriented and non-intersecting knots C_i, we have the partition function

$$Z(M, L) = \int DAexp(iS) \prod_{i=1}^{r} W_{R_i}(C_i) \tag{2.4}$$

with

$$W_R(C) = Tr_R Pexp \int_C A \tag{2.5}$$

representing a certain type of Knot invariant called Jones Polynomial.
Topological spaces with knots and links are not of our interest in this problem.

We will now talk about the Chern-Simons term in a Yang-Mills or a Non-Abelian gauge theory. The action is still the same as in eq.(2.2) where A is the group's gauge connection on the trivial bundle on M. The group could be chosen as SU(2) for weak interactions, SU(3) for strong interaction, $U(1)\times SU(2)\times SU(3)$ for the standard model, SO(10) for the unification case , or generally any SU(N) gauge group.

In paper [**2**] , Deser , Jackiw and Templeton have added a topological mass term to the lagrangian of a Non-Abelian gauge theory. Similarly, as explained above, the Lagrangian with an additional topological mass term is written as

$$L_G = \frac{1}{2g^2} trF^{\mu\nu}F_{\mu\nu} - \frac{\mu}{2g^2}\epsilon^{\mu\nu\alpha}tr(F_{\mu\nu}A_\alpha - \frac{2}{3}A_\mu A_\nu A_\alpha) \tag{2.6}$$

the notation used here was as in [**2**]

$$A_\mu = gT^a A^a_\mu$$

$$F_{\mu\nu} = gT^a F^a_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

$[T^a, T^b] = f^{abc}T^c$ where T^a are the representation matrices of the group, and g is a coupling constant.

As for the case of the gravitational Chern-Simons action, the same 3-from action of eq.(2.2) applies but using the spin-connection $\omega^A_{\mu B}$ as defined in chapter 1 instead of the gauge field A. The gravitational Chern-Simons action is thus given by

$$I = \int [\omega^B_A(e)d\omega^A_B(e) + \frac{2}{3}\omega^B_A(e)\omega^C_B(e)\omega^A_C(e)] \tag{2.7}$$

e^A is the dreibein $e = e^A_\mu dx^\mu$.

This gravitational action can also be written in term of the Christoffel symbols as defined in eq.(1.7), where the action is

$$I' = \int \epsilon^{\mu\nu\rho}(\Gamma^\sigma_{\mu\kappa}\partial_\nu\Gamma^\kappa_{\rho\sigma} + \frac{2}{3}\Gamma^\sigma_{\mu\kappa}\Gamma^\kappa_{\nu\lambda}\Gamma^\lambda_{\rho\sigma}) \tag{2.8}$$

the relation derived in chapter 1 $\omega^A_{\mu B}(e) = \Gamma^\rho_{\mu\nu}e^A_\rho e^v_B - \partial_\mu e^A_\nu e^\nu_B$ allows us to find a relation between the action I in terms of the spin-connection and I' in terms of the Christofell symbols. This is straightforward and derived in the appendix of

[**3**].

$$I(e) = I'(g_{\mu\nu}) + \frac{i}{12\pi}\int_{M_3}(e^{-1}de)^B_A(e^{-1}de)^C_B(e^{-1}de)^A_C \qquad (2.9)$$

$$+\frac{i}{4\pi}\int_{\partial M_3}\epsilon^{ij}\partial_i e^a_\kappa \Gamma^\kappa_{j\lambda} e^\lambda_a d^2x$$

this relation shows that the two actions are the same up to boundary terms.

In the next chapter we will derive how we can unify the Yang-Mills and gravitational Chern-Simons actions by starting with the larger tangent group of Chamseddine and Mukhanov that is explained in details in the first chapter.

Chapter 3

Unification of the Gravitational and Gauge Chern-Simons Action

3.1 The Unification in two ways

We have discussed in the first chapter the unification of gauge interactions and gravity by considering a larger tangent group as done by Chamseddine and Mukhanov. In the second chapter, we introduced the Chern-Simons theory and what we mean by Topological field theories. We are now interested in unifying gravity and gauge interactions in a 3-dimensional Chern-Simons theory and examining it's consequences.
So, we will use a higher dimensional Lorentz tangent group as in chapter 1, which contains subgroups that allows it to split into a gravitational and a gauge interaction parts, in a Chern-Simons theory. We can work directly on a 3-dimensional manifold or start by a 4-dimensional manifold and take its limits to 3-dimensions[**13**].

We will start by the method of directly working in 3-D. The topological Chern-

Simon term, as explained in the second chapter, corresponding to a large group SO(N) is

$$I_{CS} = \frac{k}{4\pi}\int Tr(A \wedge dA + \frac{2}{3}A \wedge A \wedge A) \tag{3.1}$$

Where A is the 1-form connection of the group, $A = dx^{\mu}\frac{1}{4}A_{\mu}^{AB}\Gamma_{AB}$ and A,B,C range from 1 to N.

The Γ-matrices are a set of 2^D matrices resulting from the repeated multiplication of the γ-matrices of the D-dimensional Clifford algebra

$$\gamma_a\gamma_b + \gamma_b\gamma_a = 2\delta_{ab}I \tag{3.2}$$

where a,b range from 1 to D.

Using properties of Γ matrices in an arbitrary space-time dimension we find that the Chern-Simon term in the action can be written in the form

$$I_{CS} = \frac{k}{4\pi}\int (A^{AC}dA^{CA} + \frac{2}{3}A^{AB}A^{BC}A^{CA}) \tag{3.3}$$

Now, we choose the same gauge as in [**1**] as explained chapter 1 ,and split the indices A, B, C into a, b, c which go from 1 to 3 and $\hat{I}, \hat{J}, \hat{K}$ which ranges from 4 to N. We re-emphasize that this specific gauge is of vanishing mixed spin-connections $e^{\nu}_{\hat{J}} = 0$;

the action can directly be split into the two parts; thus becomes

$$I_{CS} = \frac{k}{4\pi}\int A^{AC}dA^{CA} + \frac{2}{3}A^{AB}A^{BC}A^{CA} =$$

$$\frac{k}{4\pi}\int(A^{ab}dA^{ba}+\frac{2}{3}A^{ab}A^{bc}A^{ca})+\frac{k}{4\pi}\int(A^{\hat{I}\hat{J}}dA^{\hat{J}\hat{I}}+\frac{2}{3}A^{\hat{I}\hat{J}}A^{\hat{J}\hat{K}}A^{\hat{K}\hat{I}}) \quad (3.4)$$

The first term in equation (3.3) corresponds to the gravitational Chern-Simon action and the second one to the gauge Chern-Simon action with both having the same coupling coefficient.
It is so important to note that by using the first fundamental property of Chern-Simons theories we get that the coupling coefficient is quantized for compact groups. The quantization is because the group G of continuous maps $M \to G$ is not connected. In fact, in the homotopy classification of groups, the action is not invariant under gauge transformations of non-zero "winding number".

Now, we will get the same result by starting in 4-dimensions and taking the limits to the Chern-Simons term in 3-dimensions.

Our starting point are objects called Chern-Pontryagin densities. We preferred to explain them here with the unification rather than in the second chapter to avoid repetition. On a 2n dimensional manifold, these are of the form:

$$P^{2n} \propto \epsilon^{\mu_1\mu_2...\mu_{2n}}Tr(F_{\mu_1\mu_2}....F_{\mu_{2n-1}\mu_{2n}}) \quad (3.5)$$

where F (field strength) is the curvature 2-form ($dA+A\wedge A$) of some G-connection (G is the gauge group). These are gauge-invariant, closed, and their integral over the manifold M (compact, no boundary) is an integer which is a topological invariant. (*note that these sorts of invariants are examples of characteristic classes.)

The Pontryagin density of any gauge theory is

$$P_4 = -\frac{1}{16\pi^2} Tr(^* F^{\mu\nu} F_{\mu\nu}) \tag{3.6}$$

Where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A_{\mu b} A_{\nu c}$$

$$^* F^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$$

and that for gravity is the (Hirzebruch-Pontryagin) given by

$$^* RR = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} R_{\mu\nu\rho\sigma} R^{\rho\sigma}_{\alpha\beta} \tag{3.7}$$

After we have explained what we mean by the Pontryagin densities it is time to make use of it in our case. We start from the results of chapter 1, where the procedure is applicable for every N. We start by the compact group SO(N) in 4-D space-time and consider the field strength corresponding to it

$$R^{AB}_{\alpha\beta}(w) = F^{\hat{J}\hat{I}}_{\alpha\beta} n^A_{\hat{J}} n^B_{\hat{I}} + R^\rho_{\gamma\alpha\beta}(\Gamma) e^A_\rho e^{B\gamma} \tag{3.8}$$

where A and B range from 1 to N, Greek indices go from 1 to 4 and indices with hat go from 5 to N. e_α and $n_{\hat{J}}$ form a complete basis for SO(N), the former correspond to SO(4) and the latter to SO(N-4). Then any vector v_A can be expanded as:

$$v_A = v^\alpha_A e_\alpha + n^{\hat{J}}_A n_{\hat{J}}$$

The spin connection curvature of the SO(N) group is the $R^{AB}_{\alpha\beta}(\omega)$

$$R^{AB}_{\alpha\beta}(\omega) = \partial_\alpha \omega^{AB}_\beta - \partial_\beta \omega^{AB}_\alpha + \omega^{AC}_\alpha \omega^B_{\beta C} - \omega^{AC}_\beta \omega^B_{\alpha C} \tag{3.9}$$

The field strength of the SO(N-4) group is $F_{\alpha\beta}^{\hat{J}\hat{I}}$

$$F_{\alpha\beta}^{\hat{I}\hat{J}}(A) = \partial_\alpha A_\beta^{\hat{I}\hat{J}} - \partial_\beta A_\alpha^{\hat{I}\hat{J}} + A_\alpha^{\hat{I}\hat{L}} A_{\beta\hat{L}}^{\hat{J}} - A_\beta^{\hat{I}\hat{L}} A_{\alpha\hat{L}}^{\hat{J}} \tag{3.10}$$

Showing on steps in details, the Pontryagin density for the large group SO(N) in 4D is

$$\frac{1}{2}\epsilon^{\mu\nu\alpha\beta} R_{\mu\nu}^{AB} R_{\alpha\beta AB} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}(F_{\mu\nu}^{\hat{J}\hat{I}} n_{\hat{J}}^A n_{\hat{I}}^B + R_{\gamma\mu\nu}^{\rho} e_\rho^A e^{B\gamma})(F_{\alpha\beta}^{\hat{K}\hat{L}} n_{\hat{J}A} n_{\hat{L}B} + R_{\sigma\alpha\beta}^{\delta} e_{\delta A} e_B^{\sigma}) \tag{3.11}$$

using the relation $n_{\hat{J}}^A e_A^\alpha = 0$, and making a gauge choice means that the mixed terms vanish and we are left with two Pontryagin densities corresponding to gauge theory and gravity.

$$\frac{1}{2}\epsilon^{\mu\nu\alpha\beta} R_{\mu\nu}^{AB} R_{\alpha\beta AB} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} F_{\mu\nu\hat{K}\hat{L}} F_{\alpha\beta}^{\hat{K}\hat{L}} + \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} R_{\mu\nu}^{AB} R_{\alpha\beta AB} \tag{3.12}$$

We deduce that in 4-D space-time, the Pontryagin density can be split as well. The first term in the right hand side of eq.(3.12) corresponds to gauge interactions, while the other corresponds to gravity!

Now we move to determining the limits from 4 to 3-dimensions. We know that the Chern-Pontryagin densities in 4-D are the exterior derivatives of the Chern-Simons entities in 3-D. We have $I_{CS} = \int \omega_3$ in 3-D. Then using Stokes theorem

$$\int_{M_4} P_4 = \int_{M_4} d\omega_3 = \int_{\partial M_4} \omega_3 \tag{3.13}$$

where

$$
\begin{aligned}
P_4 = Tr(F \wedge F) &= Tr((dA + A \wedge A)(dA + A \wedge A)) \\
&= Tr(d(AdA + A^3)) = Tr(d\omega_3),
\end{aligned}
\tag{3.14}
$$

and $Tr(A^4) = 0$.

Based on this, we can easily translate the 4-D unification of the corresponding Chern-Simons terms $\int_{M_4} P_4 = \int_{M_4} P_4^{gauge} + \int_{M_4} P_4^{gravity}$ to 3-D

we see that ω_3 also splits into 2 parts, one corresponding to gravity and the other to gauge interactions

$$
\int_{M_3} \omega_3 = \int_{M_3} \omega_3^{gauge} + \int_{M_3} \omega_3^{gravity} \tag{3.15}
$$

where

$$
\begin{aligned}
\omega_3^{gauge} &= \epsilon^{ijk}(A_i^a \partial_j A_k^a + \frac{1}{3} f^{abc} A_i^a A_j^b A_k^c) \\
\omega_3^{gravity} &= \epsilon^{ijk}(R_{ijab}\omega_k^{ab} + \frac{2}{3}\omega_{ib}^c \omega_{jc}^a \omega_{ka}^b)
\end{aligned}
\tag{3.16}
$$

We note that eq.(3.16) can also be expressed as a 3-from of a topological Chern-simons theory, thus reaching the same results. After we have achieved the results in 2 different ways, we will discuss the quantization and other important consequences of this in the next section.

3.2 Consequences of Unification in Chern-Simons theory

The first consequence is that eq.(3.4) and eq.(3.12) represent topological invariant quantities as shown by Witten. However, Witten added the gravitational part to the gauge interaction part without showing how they can be related by enlarging the tangent group as a starting point. In order to prove that the partition function Z below represents a topological invariant quantity, Witten [**4**] worked the case of weak coupling limit (large k) leading to

$$Z = \int DAexp(\frac{ik}{4\pi}\int_M Tr(AdA + \frac{2}{3}A \wedge A \wedge A)) \tag{3.17}$$

which turned out to be a toplogical invariant quantity of the 3-D space-time manifold. We will continue with our case using Witten's technique and calculations. For the SO(6) group, we split the indices as in chapter 1 so that the partition function Z splits into two parts

$$\begin{aligned} Z = \int DAexp(iI_{CS}) = \int DAexp(iI_{gauge} + iI_{gravity}) = \\ \int Dwexp(iI_{gravity}) \int DBexp(iI_{gauge}) = Z_1.Z_2 \end{aligned} \tag{3.18}$$

the first term represents the SO(3) Gravity Chern-Simon term with ω as the gauge connection, and the second term represents the SO(3) gauge Chern-Simon term with B as the gauge field.

Now we consider the weak coupling limit of the gauge part. So Z_2 as named above is given by

$$Z_2 = \sum_\alpha \mu(B^\alpha) \tag{3.19}$$

Here $\mu(B^\alpha)$ is a function of flat connections for which the curvature vanishes as defined in Witten's paper. We expand the gauge field $B_i = B_i^{(\alpha)} + C_i$, the Chern-Simon gauge action term becomes

$$I_{CS}^{gauge} = kI(B^\alpha) + \frac{k}{4\pi}\int_M Tr(C \wedge DC) \tag{3.20}$$

D is the covariant derivative with respect to B^α.

In order to compute the Gaussian integral in eq(3.20), a gauge fixing is needed which can not be done without picking a metric. We choose such a metric to satisfy $D_i C^i = 0$ and from field theories we know that this will lead to ghosts in the action, so it becomes

$$S_{GF} = \int_M Tr(\phi D_i C^i + \bar{c} D_i D^i c) \tag{3.21}$$

where ϕ is a lagrangian multiplier enforcing the gauge condition $D_i C^i = 0$ and c,$\bar{c}$ are anticommuting ghosts.

Integrating out C, ϕ, c, $\bar{c}$, will lead to the following

$$exp(\frac{i\pi\eta(B^\alpha)}{2})T_\alpha \tag{3.22}$$

as Witten computed where $\eta(B^\alpha)$ is the " eta inavraint" and is defined by the following

$$\eta(B^\alpha) = \frac{1}{2}\lim_{s\to 0}\sum_i sign\lambda_i|\lambda_i|^{-s} \tag{3.23}$$

λ_i's are eigenvalues of operator L_i, ${}^*D_B + D_{B^*}$ is restricted on odd forms, and T_α is the torsion invariant of flat connection $B(\alpha)$.

Witten then used the Atiyah-Patodi-Singer index theorem so that the partition function can be written as

$$Z_2 = exp(i\frac{\pi}{2}\eta(0))\sum_{\alpha} e^{i(k+c_2(G)/2)I(B^{(\alpha)})}.T^{\alpha} \tag{3.24}$$

$\eta(0)$ is the eta invariant of the trivial gauge field and $c_2(G)$ is the Casimir operator of G. The function $\eta(0)$ is the only term that depends on the metric, which means that the partition function is not yet a topological invariant quantity. Witten suggested that by adding a counter term, the partition function will be regualrized and turned into a topological invariant quantity. This counterterm must be a multiple of the gravitational Chern-Simons term. Luckily, this term is already present in our case because we started from the large group and split the actions. Substituting the weak coupling limit of Z_2 from our case in eq.(3.25) we already get the gravitational term that Witten added. This leads to a topological invariant partition function and results similar to his

$$\begin{aligned} Z &= \int D\omega exp(iI_{grav}).Z_2 \\ &= \int D\omega exp(iI_{grav})exp(i\frac{\pi}{2}\eta(0))\sum_{\alpha} e^{i(k+c_2(G)/2)I(B^{(\alpha)})}.T^{\alpha} \\ &= \int D\omega exp(i(I_{grav}+\frac{\pi}{2}\eta(0)))\sum_{\alpha} e^{i(k+c_2(G)/2)I(B^{(\alpha)})}T^{\alpha} \end{aligned} \tag{3.25}$$

We conclude that the partition function of this unification of gauge interaction and gravity is already a topological invariant quantity without the need of adding any terms.

Another important consequence other than unifying gravity and gauge Chern-

Simons theories in 3-D space-time is the quantization of the coupling constants for compact groups cases. In fact, the coupling constant is quantized for compact groups but not for non-compact groups as shown in [**12**]. Actually, this is a first fundamental property of Chern-Simons theories which states that coupling coefficient is quantized for compact groups. This quantization is because the group G of continuous maps $M \to G$ is not connected, as mentioned before, and in the homotopy classification of groups the action is not invariant under gauge transformations of non-zero "winding number".

It should be noted that such Chern-Simon theories also admit local supersymmetric extensions to be discussed in the last chapter.

Chapter 4

Notes on Supergravity

Supergravity is a field theory that combines the principles of supersymmetry and general relativity. Supergravity is generally written in terms of Cartan connections as a gauge theory of supersymmetry. Supersymmetry is a spacetime symmetry relating particles with integer and half-integer spins by associating super-fermionic partners to each boson and super-bosonic partners to each fermion. Not only is supersymmetry elegant but also is of remarkable significance in theoretical physics. The supersymmetry algebras are the only extension of the Poincare algebra to graded lie algebras that are consistent with the symmetries of the S-matrix by Coleman Mandulas theorem. By considering equal fermionc and bosonic degrees of freedom, supersymmetry manages to cancel the contributions to the Higgs $mass^2$ quantum corrections and thus solves the hierarchy problem between the grand unified theories mass scale and the electroweak mass scale without requiring any fine tuning. Much more than that, supersymmetry has been used as a gauge theory to construct supergravity and serves as a corner stone for string theories. Therefore, tremendous efforts have been devoted to find supersymmetric particles throughout the decades. We'll begin with Supergravity now as we introduce the concept of Graded algebras and the supersymmetric algebra in the appendix.

As mentioned above, the theory of Supergravity can be obtained by taking the local gauge symmetry of the Supersymmetric algebra in the Cartan formalism of General Relativity. We will go through Supergravity briefly; then discuss how we can relate Supergravity to Chern-Simons theory to serve our objectives and see how these topics relate to each other.

In the first chapter, we have discussed how enlarging the tangent symmetry group in the tangent space to higher dimensions than the manifold allows us to unify gauge interactions with gravity. However, originally, the well-known Einstein-Cartan formalism for gravity was formulated by considering the Poincare gauge group alone as a local gauge symmetry.
Defining the covariant derivative to be

$$D_\mu = \partial_\mu + e^a_\mu P_a + \omega^{ab}_\mu M_{ab}$$

where M_{ab} are the generators of the Lorentz algebra $SO(1,3)$. To define the curvature we compute the commutator of the covariant derivatives

$$[D_\mu, D_\nu] = R^{ab}_{\mu\nu} M_{ab} + T^a_{\mu\nu} P_a$$

and

$$[M_{ab}, M_{cd}] = -\frac{1}{2}(\eta_{ac} M_{bd} - \eta_{ad} M_{bc} - \eta_{bc} M_{ad} + \eta_{bd} M_{ac})$$

we get

$$T^a_{\mu\nu} = \partial_\mu e^a_\nu - \partial_\nu e^a_\mu + \omega^a_{\mu c} e^c_\nu - \omega^a_{\nu c} e^c_\mu$$

$$R^{ab}_{\mu\nu} = \partial_\mu \omega^{ab}_\nu - \partial_\nu \omega^{ab}_\mu + \omega^{ac}_\mu \omega^b_{\nu c} - \omega^{ac}_\nu \omega^b_{\mu c}$$

$R_{\mu\nu}^{ab}$ is similar to the curvature tensor in General relativity. ω_{ν}^{ab} is called the spin connection which in the absence of matter is similar to the affine connection in gravity.

Now, as we take supersymmetry as a gauge theory we automatically take the symmetry to be local. This will give us the advantage of including gravity in the theory automatically!

We will begin by taking pure supergravity at which matter fields are still not included ! The covariant derivative now includes an extra term for supersymmetry

$$D_{\mu} = \partial_{\mu} + e_{\mu}^{a} P_{a} + \omega_{\mu}^{ab} M_{ab} + \psi_{\mu}^{\alpha} Q_{\alpha} \tag{4.1}$$

here ψ_{μ}^{α} is a spin $\frac{3}{2}$ vector contracted to the supersymmetry generator and is known as the gravitino field, while e_{μ}^{α} is the graviton of helicity 2. This forms the supergravity multiplet $(\frac{3}{2}, 2)$. The curvature $R_{\mu\nu}^{ab}$ remains the same while the torsion gets an extra term

$$T_{\mu\nu}^{a} = \partial_{\mu} e_{\nu}^{a} - \partial_{\nu} e_{\mu}^{a} + \omega_{\mu c}^{a} e_{\nu}^{c} - \omega_{\nu c}^{a} e_{\mu}^{c} + 2\bar{\psi}_{\mu} \gamma^{a} \psi_{v} \tag{4.2}$$

and we have

$$\psi_{\mu\nu}^{\alpha} = \partial_{\mu} \psi_{\nu}^{\alpha} - \partial_{\nu} \psi_{\mu}^{\alpha} + \frac{1}{4} \omega_{\mu}^{ab} (\gamma_{ab})_{\beta}^{\alpha} \psi_{\nu}^{\beta} - \frac{1}{4} \omega_{\nu}^{ab} (\gamma_{ab})_{\beta}^{\alpha} \psi_{\mu}^{\beta} \tag{4.3}$$

The field ω_{μ}^{ab} can be eliminated by solving some constraints and equations of motion. Moreover, constraints on these fields can be added to construct the desirable invariant actions.

The action can then be constructed out of invariant quantities built out of these fields! We will not elaborate in detailed calculations any further about this.

Our aims for introducing supersymmetry and supergravity were to elaborate how to construct a field theory out of a symmetry group, to show how gauging this symmetry group will lead to supergravity, and intentionally to show that Chern-Simons theories admit supersymmetric extensions as well!

In the previous chapter we have shown how gravitational and gauge Chern-Simons theories could be unified in 3-D space-time. We realized that the coupling constant is quantized for compact groups but not for non-compact cases. And that the partition function Z is directly a topological invariant quantity. Now, our concern is to see how Chern-Simon theories are related to supersymmetry; in fact, they admit local supersymmetric extensions. This is achieved by gauging the supergroups, by extending the space-time manifold to a supermanifold, or by both [**12**]. Therefore, we will consider the graded groups that are extensions of our SO(1,5) group as done in [**12**], for example, we could choose the supersymmetrization of the de-sitter group in six space-time dimensions

$$O(6,1) \oplus SU(2), (8,2) \tag{4.4}$$

where $O(6,1) \oplus SU(2)$ is the bosonic part, while the fermionic part live in the $(8,2)$ representation the 2 groups, and the group $SU(N,q)$ is the group of unitary quaternionic $N \times N$ matrices.

As an example consider the SU(4/N) group of graded connections

$$\phi^A_{\mu B} = \begin{pmatrix} M^\beta_{\mu\alpha} & \psi^i_{\mu\alpha} \\ -\bar{\psi}^\alpha_{\mu i} & A^i_{\mu j} \end{pmatrix}$$

the Chern-Simons theory for this connection will yield a supersymmetric topological theory. This and the other group will be considered in future works.

Conclusion

To sum it all up, we have used the large group SO(1,N-1) as a symmetry of the tangent space to unify gauge interactions and gravity as proposed by Chamseddine and Mukhanov. We have considered this group in Chern-Simons theory at 3-dimensions to find out that the quantized coupling constants is the same for gauge interactions and gravity. The quantization of the coupling constants emerges from the first fundamental property of Chern-Simons theory for compact groups. The resulting action is a topological invariant quantity as proved by Witten. However, in our case, we didn't need to add a gravitational term as Witten did because this term is already a consequence of our tangent symmetry group. In the end we discussed how all of the above can be extended to supersymmetry. By considering the SU(4/N) group of graded connections as an example, the Chern-Simons theory will yield a supersymmetric topological field theory.

In general relativity, usually the dimensions of the tangent space is taken to be equal to the dimensions of the manifold and the Lorentz group is taken to be the symmetry of the tangent space. However, by arguing that the dimensions of the tangent space could be taken different than that of the manifold or better stated that the symmetry group of that tangent space could be enlarged, we realize that a unification of gauge interaction and gravity emerges naturally. The chosen symmetry group of the tangent space provides description of the physical gauge interactions of the standard model as well as a description of gravity. A symmetry breaking mechanism is then used to produce the fermions' masses.
While, the advantage of the Chern-Simons theory in 3-dimensions is that the coupling constants are quantized in the case of compact symmetry groups. This

leads to the suspicious conclusion that the quantized coupling constants are the same for gauge interactions and gravity before symmetry breaking. Could this symmetry be truly a unifying symmetry of nature at a certain scale? And when it's broken, it is broken at exactly what scale? Much more than that, we get a topologically invariant quantity of 3-manifolds out of gravitational and gauge interaction contributions. Then what does this physically mean, how could it be useful, and how could be related to 4-dimensional physics? For example, topological field theories and Chern-Simons theories turned out to have several applications in condensed matter physics and even in quantum computing.

What we have presented is a mathematical realization aiming for a possible elegant mathematical description of nature. It is consistent within the mathematical framework and the principles of the addressed theories. But this does not imply that our proposed results must be of physical significance.

Appendices

Graded and Supersymmetric Algebras

We begin our discussion with the graded algebras [**14**]. A graded algebra is an algebra in which a grading exists. The Lorentz group provides an example of graded algebras because the rotation generators, denoted by L_0, and the boosts , denoted by L_1, together define a Z_2 grading

$$[L_i, L_j] \subset L_{i+j}$$

The super algebra is an algebra with a Z_2 grading of even and odd elements. The bracket of two odd elements is symmetric; however, the bracket of any other two generators is antisymmetric. The Jacobi identities (to be explained in a while) should be also satisfied. Since super algebras contains both even and odd generators, we are interested in them in order to construct a theory that unifies the even bosonic sector of particles with the odd fermionic sector.

We will consider only the super algebras whose structure constants are ordinary numbers. There are two main families of the simple super algebras, the orthosymplectic OSp(n/m) and the super-unitary SU(n/m). Super algebras contain also several semisimple super algebras, non-semisimple super algebras, and exceptional algebras such as E_6, E_7, E_8.

Consider algebras with even generators (E) and odd generators (O), this implies the following commutations relations with the respective structure constants

$$[E_i, E_j] = f_{ij}^k E_k$$

$$[E_i, O_\alpha] = g^{\beta}_{i\alpha} O_\beta$$

$$\{O_\alpha, O_\beta\} = h^{i}_{\alpha\beta} E_i$$

these should satisfy the Jacobi identity as mentioned above. The extended Jacobi identity which include anticommutators is

$$\{A, \{B, C]] \pm \{B, \{C, A]] \pm \{C, \{A, B]] = 0 \tag{5}$$

where A,B,C are the generators. The bracket structure $\{,]$ signifies signifies either commutator or anticommutator according to the even or odd character of the generators A,B,C. The odd elements determines the signs. The sign is positive if the odd elements are in cyclic permutation of the first term, and the sign is negative if the odd elements are not in in cyclic permutation of the first term.

Applying the Jacobi identities for the above commuatations relations, we get the usual Jacobi identities for ordinary Lie algebras for the even elements in addition to the following

$$[E_i, [E_j, O_\alpha]] - [E_j, [E_i, O_\alpha]] = [[E_i, E_j], O_\alpha] \tag{6}$$

$$[E_i, \{O_\alpha, O_\beta\}] = \{[E_i, O_\alpha], O_\beta\} + \{[E_i, O_\beta], O_\alpha\} \tag{7}$$

$$[O_\alpha, \{O_\beta, O_\gamma\}] = [\{O_\alpha, O_\beta\}, O_\gamma] + [\{O_\alpha, O_\gamma\}, O_\beta] \tag{8}$$

the identity of eq.(4.2) indicates that the O_α form a representation of the ordinary Lie algebra E_i if we consider the O_α as vectors on which E_i acts. Eq.(4.3) is equivenlent to eq.(4.2) if $g_{i\alpha\beta} = h^{*}_{\alpha\beta i}$. Eq.(4.4) forms a restriction on the possible representations O_α of the ordinary Lie algebra, thus not every ordinary Lie algebra can be extended to a super algebra. We will see in the next section that of all the graded algebras only the supersymmetry algebras generate symmetries of the S-matrix that are consistent with relativistic quantum field theory according to Coleman-Mandula theorem.

An important example of simple super algebras is the orthosymplectic super algebra OSp(N/M). Its bosonic part contains the ordinary Lie algebras SO(N) and Sp(M). It can be defined as the linear transformations that leave the bilinear real form F invariant

$$F = x^i y^j \delta_{ij} + \theta^\alpha \xi^\beta C_{\alpha\beta}$$

where θ and ξ are anticommuating objects such as the Grassmann numbers, C denotes an antisymmetry real metric, and i,j run from 1 to N while α, β run from 1 to M.

The term $\theta^\alpha \xi^\beta C_{\alpha\beta}$ is invariant under Sp(M).

We can use the diagonal form $F = x^2$ to define SO(N). And require that θ^α being anticommuting objects for Sp(M), then F can be considered as

$$F = x^i x^j \delta_{ij} + \theta^\alpha \theta^\beta C_{\alpha\beta} \equiv x^2 + \bar{\theta}\theta$$

Thus the orthosymplectic algebra OSp(N/M) consists of all real transformations that leaves F invariant. The diagonal parts are for the even SO(N) and Sp(M) generators while the odd generators of OSp(N/M) are represented by $(M + N) \times (M + N)$ matrices with anticommuting entries.

The set of generators of SO(N), Sp(M) and the $M \times N$ odd generators form a closed algebraic system under commutators only, but defining the set of $(M +$

$N) \times (M+N)$ matrices with ordinary numbers only allows us to form a closed algebraic system under commutators and anticommutators.
We note that if we take N=1 and M=4 we get OSp(1/4) the super de-sitter algebra. It is also called super de-sitter algebra because Sp(4) is locally isomorphic to O(3,2) that is the de-sitter algebra. OSp(1/4) can be contracted down to the super Poincare algebra using the so-called Wigner-Inonu contraction.

Another important example of super algebras are the super-unitary SU(N/M) algebras. They contain as bosonic part the $SU(N) \times SU(M) \times U(1)$ algebras. In a representation of $(N+M) \times (N+M)$ matrices , the representation SU(N) algebras lies in the first $N \times N$ submatrice and that of SU(M) algebras lie in the last $M \times M$ submatrices, while that of U(1) lies along the diagonal of $(N+M) \times (N+M)$.
The super-unitary algebras can be defined as the transformations that leave the real F form invariant, where F is

$$F = (z^i)^* z^j \delta_{ij} + (\theta^\alpha)^* \theta^\beta g_{\alpha\beta}$$

where $g_{\alpha\beta} = \pm\delta_{\alpha\beta}$, i runs from 1 to N, and α runs from 1 to M.
In the case of super-unitary algebras we do not need an antisymmetric metric $C_{\alpha\beta}$ because one can always diagonalize the θ-metric where M, N can be even or odd. So we can use the two diagonal metrices δ_{ij} and $g_{\alpha\beta}$. We note that θ are taken to be anticommuting objects. Because the transformations $x \to \theta$ contain anticommuting entries, then all of these transformations form form a closed algebra under commutation and anticommutation relations.

After providing an idea of what the super algebras are, in the next section we consider the super Poincare algebras.

Supersymmetry

Bosons correspond to even generators of Lie algebras. Fermions correspond to odd generators. What are the supersymmetric bosons and fermions described by the Super-Poincare algebra?

1. The Supersymmetry Algebra

The Super Poincare algebra or the so-called Supersymmetry algebra is

$$\{Q_\alpha^A, \bar{Q}_{\dot{\beta}B}\} = 2\sigma^m_{\alpha\dot{\beta}} P_m \delta^A_B$$

$$\{Q_\alpha^A, Q_\beta^B\} = \}\bar{Q}_{\dot{\alpha}A}, \bar{Q}_{\dot{\beta}B}\} = 0$$

$$[Q_\alpha^A, M_{mn}] = (\sigma_{mn})_\alpha^\beta Q_\beta^A$$

$$[P_m, Q_\alpha^A] = [P_m, \bar{Q}_{\dot{\alpha}A}] = 0$$

$$[P_m, P_n] = 0$$

where $\alpha, \beta, \dot{\alpha}, \dot{\beta}, ... = 1, 2$ denotes two-component Weyl spinors, and $m, n = 1, 2, 3, 4$ identify Lorentz four-vectors. A, B refer to an internal space, they run from 1 to $N \geq 1$. For N=1 we call it a supersymmetry algebra, for $N \geq 1$ we call it an extended supersymmetry.

The supersymmetry algebras (along with their extensions to include central charges) are the only graded Lie algebras of symmetries of the S-matrix consistent with relativistic quantum field theory. The proof is based on the Coleman-Mandula theorem which assumes that:
The S-matrix is based on a local, relativistic quantum field theory in 4-dim space-time.

There are only a finite number of particles associated with one-particle states of given mass.

There is an energy gap between the vacuum and the one particle states. Assuming as well,

The operators Q acts in Hilbert space with positive definite metric

Both Q and its hermitian conjugate belong to the algebra.

Decomposing the generators Q into a sum of irreducible representations under the homogeneous Lorentz group L :

$$Q = \Sigma Q_{\alpha_1 \ldots \alpha_a, \dot{\alpha}_1 \ldots \dot{\alpha}_b}$$

Since it is symmetric w.r.t. dotted and undotted indices it belongs to the irreducible spin $\frac{1}{2}(a+b)$ representation.
(a+b) is odd since the $Q's$ anticommute.
$\{Q_{\underbrace{1\ldots1}_{a},\underbrace{\dot{1}\ldots\dot{1}}_{b}}, \bar{Q}_{\underbrace{\dot{1}..\dot{1}}_{a},\underbrace{1\ldots1}_{b}}\}$ thus closes into an even (commuting) element of the algebra with spin-(a+b) . From Coleman-Mandula theorem, to be consistent with Lorentz symmetry, this element is either zero or a component of P_m(the energy-momentum operator) for $a+b=1$.

This anticommutator is a positive definite operator in Hilbert space with a positive metric, so for $\{Q_{\alpha_1 \ldots \alpha_a, \dot{\alpha}_1 \ldots \dot{\alpha}_b}, Q_{\alpha_1 \ldots \alpha_a, \dot{\alpha}_1 \ldots \dot{\alpha}_b}\}$ to vanish we should have

$Q_{\underbrace{1...1}_{\rm a}\underbrace{\dot{i}...\dot{i}}_{\rm b}} = 0$ for $(a+b) \geq 1$.[**6**][**7**] So the anticommuting part of the superalgebra is composed only of spin-$\frac{1}{2}$ operators Q^A_α and $\bar{Q}_{\dot{\alpha}B}$. We can choose it to be $\{Q^A_\alpha, \bar{Q}_{\dot{\beta}B}\} = 2\sigma^m_{\alpha\dot{\beta}} P_m \delta^A_B$.

2. Component Fields and Superfields

We are concerned now in representing the superalgebra in terms of fields that are not restricted to mass shell conditions. To do so, we will take advantage of the well-known mathematical tool of introducing Grassmann numbers (anticommuting parameters) $\xi_\alpha, \bar{\xi}_{\dot{\alpha}}$.

$$\{\xi^\alpha, \xi^\beta\} = \{\xi^\alpha, Q_\beta\} = ... = [P_m, \xi^\alpha] = 0.$$

From this we have

$$[\xi Q, \bar{\xi}\bar{Q}] = \xi Q \bar{\xi}\bar{Q} - \bar{\xi}\bar{Q}\xi Q = -\xi Q \bar{Q}\bar{\xi} - \xi \bar{Q} Q \bar{\xi} = -\xi\{Q, \bar{Q}\}\bar{\xi}$$

$$= -2\xi\sigma^m P_m \bar{\xi} = 2\xi\sigma^m \bar{\xi} P_m$$

Therefore we have the following commutators

$$[\xi Q, \bar{\xi}\bar{Q}] = 2\xi\sigma^m\bar{\xi}P_m$$

$$[\xi Q, \xi Q] = [\bar{\xi}\bar{Q}, \bar{\xi}\bar{Q}] = 0$$

$$[P^m, \xi Q] = [P^m, \bar{\xi}\bar{Q}] = 0$$

We define a set of fields $(A, \psi, ...)$ and an infinitesimal transformation $\delta_\xi = \xi Q + \bar{\xi}\bar{Q}$ acting on the fields such that:

$$\delta_\xi A = (\xi Q + \bar{\xi}\bar{Q}) \times A$$

$$\delta_\xi \psi = (\xi Q + \bar{\xi}\bar{Q}) \times \psi$$

using the commutators we just derived we can straight forwardly show that

$$(\delta_\eta \delta_\xi - \delta_\xi \delta_\eta)A = -2i(\eta\sigma^m\bar{\xi} - \xi\sigma^m\bar{\eta})\partial_m A \tag{9}$$

where both η and ξ are grassmann parameters.

We deduce that that the supersymmetry transformation acts by transforming the field of dimension L in a field of dimension $L + \frac{1}{2}$ or $L - \frac{1}{2}$. we can say that the supersymmetry transformation maps tensor fields into spinor fields, and scalar fields into spinor fields,..etc. Thus creating a symmetry between the fermionic and bosonic fields as desired.

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