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The ℓ^2 Decoupling Conjecture

by

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An Abstract of the Thesis of

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Suppose f_1,f_2, ...,f_N are functions in a Lebesgue space $L^p(\mu)$.
Then

$$\|f_1 + \cdots + f_N\|^p_{L^p(\mu)} = \int |f_1 + \cdots + f_N|^p \; d\mu \leq N^{\frac{p}{q}} \sum_{l=1}^{N} \|f_l\|^p_{L^p(\mu)}$$

so that we get

$$\|f_1 + \cdots + f_N\|_{L^p(\mu)} \leq N^{\frac{1}{q}} \left(\sum_{l=1}^{N} \|f_l\|^p_{L^p(\mu)} \right)^{\frac{1}{p}}.$$

In the special case of $p = 2$, this inequality is reduced to

$$\|f_1 + \cdots + f_N\|_{L^2(\mu)} \leq \left(\sum_{l=1}^{N} \|f_l\|^2_{L^2(\mu)} \right)^{\frac{1}{2}}$$

provided the f_l are mutually orthogonal.

One of the guiding principles of harmonic analysis (more precisely, restriction theory) states that if the functions $f_1, \dots, f_N$ are Fourier transforms of measures of the form $g_1 d\sigma, \dots, g_N d\sigma$ supported on a paraboloid S of surface measure $d\sigma$, then $f_1, \dots, f_N$ are almost orthogonal provided the functions $g_1, \dots, g_N$ are appropriately separated. One particular manifestation of this principle is the l^2 decoupling conjecture which asserts that to every $\varepsilon > 0$, there is a constant C_ε such that

$$\|f_1 + \cdots + f_N\|_{L^p(\mu)} \leq C_\varepsilon N^\varepsilon \left(\sum_{l=1}^{N} \|f_l\|^2_{L^p(\mu)} \right)^{\frac{1}{2}}$$

provided $g_1, \ldots, g_N$ are appropriately separated, when $2 \leq p \leq \frac{2n+2}{n-1}$.

This conjecture has recently been solved in [4] using multilinear theory. The purpose of our thesis is to present the solution to this important conjecture following the exposition in [1] and [3].

Contents

Chapter 1

Notations

We will denote the truncated paraboloid in $\mathbb{R}^n$ by

$$\mathbb{P}^{n-1} := \{(\xi_1, \xi_2, ..., \xi_{n-1}, \xi_1^2 + \xi_2^2 + ... + \xi_{n-1}^2) : 0 \leq \xi_i \leq 1\}$$

We will denote cubes in the frequency space $[0,1]^{n-1}$ with the letters Q, q and cubes in the physical space $\mathbb{R}^n$ with the letters B, Δ.

We will associate with each cube $B = B(c_B, R)$ in $\mathbb{R}^n$ the weight

$$w_B(x) = \frac{1}{(1 + \frac{|x-c_B|}{R})^E}$$

We will always assume that all cubes have side lengths in the set $2^{\mathbb{Z}} := \{2^k : k \in \mathbb{Z}\}$ We will denote by σ the lift of the Lebesgue measure from $[0,1]^{n-1}$ to the paraboloid $\mathbb{P}^{n-1}$.

We define the extension operator $E_Q^{(n)} = E_Q$ as follows: for a function $g : Q \to \mathbb{C}$, $E_Q g$ is defined as the Fourier transform of the measure $g d\sigma$, that is $E_Q g := \widehat{g d\sigma}$. More explicitly,

$$E_Q g(x) = \int_Q g(\xi_1, ..., \xi_{n-1}) e^{2\pi i (\xi_1 x_1 + ... + \xi_{n-1} x_{n-1} + (\xi_1^2 + ... + \xi_{n-1}^2) x_n)} d\xi,$$

where $\xi = (\xi_1, ..., \xi_{n-1})$ and $x = (x_1, ..., x_n)$.

For positive smooth functions η defined on $\mathbb{R}^n$ we will denote by $\eta_B(x)$ the function $\eta(\frac{x-c}{R})$ where B is the cube $B(c, R)$ of cetner c and side length $\ell(B) = R$.

For positive measurable functions $\nu : \mathbb{R}^n \to [0, \infty)$ which we shall call weights, and for a function $f : \mathbb{R}^n \to \mathbb{C}$, we define $\|f\|_{L^p(\nu)} := \big(\int_{\mathbb{R}^n} |f|^p \nu d\lambda\big)^{\frac{1}{p}}$ where λ is the Lebesgue measure on $\mathbb{R}^n$.

We also define the norm $\|F\|_{L^p_\#(w_B)} := \big(\frac{1}{|B|}\int |F|^p w_B d\lambda\big)^{\frac{1}{p}}$

We will denote by $|A|$ the Lebesgue measure of A if A has positive Lebesgue measure and the cardinality of A if A is finite, where the power E can be chosen large enough (usually $E \geq 100n$) to satisfy integrability conditions.

We will use the notation $\lesssim$ to mean "less than or equal to a constant times" where most of the constants will depend only on the Lebesgue index p and the dimension n.

Chapter 2

The Restriction Problem and the Kakeya Conjecture

The restriction problem was first posed explicitly by Elias Stein and now it has many equivalent formulations. In the study of restriction theory, one possible approach to problems would be to take linear combinations of waves of the form $e^{i\phi(\xi).x}f(\xi)$ and look at operators of the form

$$Ef(x) := \int_{B^{n-1}} e^{i\phi(\xi).x}f(\xi)d\xi$$

where $\phi(\xi)$ is called the frequency of the wave. One possible choice for this frequency would be the parametrization of the paraboloid:

$$\phi(\xi) = (\xi_1, ..., \xi_{n-1}, |\xi|^2).$$

A typical question that can be asked would be: for what values of p does the following hold:

$$\|Ef\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^\infty(\mathbb{R}^n)}.$$

A famously unsolved conjecture in the study of restriction theory is what is called Linear Restriction, as stated in [2] and it goes as follows:

Conjecture 1 (Linear Restriction). *Let U be a compact neighborhood of the origin in $\mathbb{R}^n$. Let S be an $(n-1)-$ dimensional submanifold of $\mathbb{R}^n$ that has everywhere non-vanishing gaussian curvature. If $q > \frac{2n}{n-1}$ and $p' \leq \frac{n-1}{n+1}q$, then there exists a finite positive constant C that depends only on n and ϕ such that*

$$\|Eg\|_{L^q(\mathbb{R}^n)} \leq C\|g\|_{L^p(U)}$$

for all $g \in L^p(U)$.

This conjecture which remains unsolved in the general case implies what is known as the Linear Kakeya conjecture which can be stated in many forms. One possible form goes as follows: The Hausdorff dimension of a borel set in $\mathbb{R}^n$ that contains a unit line segment in every direction is n. A stronger version which is more quantitative says:

Conjecture 2 (Linear Kakeya)**.** *let $0 < \delta << 1$. Define a $\delta-$tube to be a rectangular box T of dimensions $\delta \times \delta \times ... \times \delta \times 1$ (and hence of volume $\sim \delta^{n-1}$). Let $\mathbb{T}$ be an arbitrary collection of such $\delta-$tubes whose orientations form a $\delta-$separated set of points on S^{n-1}. Then*

$$\|\sum_{T\in\mathbb{T}} \chi_T\|_{L^q(\mathbb{R}^n)} \leq C\delta^{\frac{n-1}{q}}(\#\mathbb{T})^{\frac{1}{q}}.$$

Although both of these conjectures remain unsolved in their general forms, their respective multilinear versions have been formulated and proven successfully as we shall show shortly. It is also the case that the Multilinear Restriction and Multilinear Kakeya are equivalent to each other. However, the multilinear versions do not imply the linear versions. This however isn't the case for decoupling as we shall discuss later.

We shall be using a version of Multilinear Kakeya as proved by Bennett, Carbery, and Tao in [2] and by Guth in [3].

Lemma 1 (Multilinear Kakeya)**.** *Suppose that $\ell_{j,a}$ are lines in $\mathbb{R}^n$, where $j = 1, 2, ..., n$, and $a = 1, 2, ..., N_j$. Let $T_{j,a}$ be the characteristic function of the 1-neighborhood of $\ell_{j,a}$. Suppose each of these lines $\ell_{j,a}$ makes an angle of at most $(10n)^{-1}$ with the $x_j - axis$. Let Q_R be a cube of side length R. Then for any $\epsilon > 0$ and any $R \geq 1$, the following inequality holds:*

$$\int_{Q_R} \prod_{j=1}^{n} \left(\sum_{a=1}^{N_j} T_{j,a} \right)^{\frac{1}{n-1}} \leq C_\epsilon R^\epsilon \prod_{j=1}^{n} N_j^{\frac{1}{n-1}}.$$

We can deduce from this another version of multilinear Kakeya also proved by Guth in [3] which will be useful for us in proving mutilinear restriction. We state it as a corollary:

Corollary 1. *Let $\ell_{j,a}, T_{j,a}, Q_R$ be as in the previous lemma.*
Let $w_{j,a} \geq 0$ be numbers. Let

$$f_j := \sum_a w_{j,a} T_{j,a}$$

Then for any $\epsilon > 0$ and any $R \geq 1$ the following inequality holds:

$$\int_{Q_R} \prod_{j=1}^{n} f_j^{\frac{1}{n-1}} \leq C_\epsilon R^\epsilon \prod_{j=1}^{n} \Big(\sum_a w_{j,a} \Big)^{\frac{1}{n-1}}$$

We now state some results that will help us in the proofs that follow.

Suppose μ is a non-negative rapidly decaying function on $\mathbb{R}^n$, and $A \subset \mathbb{R}^n$ is a compact convex set. Then there is an affine function $h : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$B(c_n) \subset h(A) \subset B(C_n). \tag{2.1}$$

The affine function has the form $h(x) = Lx + v$, where L is a linear map and $v \in \mathbb{R}^n$. We define the cutoff function μ_A by

$$\mu_A(x) = |\mathrm{Det}\, L|\, \mu(h(x)).$$

Suppose $K \subset \mathbb{R}^n$ is a compact convex set with center of mass w_K. The dual convex body K^* is defined by

$$K^* = \{x \in \mathbb{R}^n : |x \cdot (w_K - w)| \leq 1\ \forall\, w \in K\}.$$

Proposition 1 (The locally constant property)**.** *Suppose μ is a non-negative rapidly decaying function on $\mathbb{R}^n$, and c_n, C_n are the constants in* (2.1)*. Then there is a non-negative rapidly decaying function η on $\mathbb{R}^n$ such that*

$$\sup_{b+K^*} f * \mu_{K^*} \leq \inf_{b+K^*} f * \eta_{K^*}$$

for all compact convex sets $K \subset \mathbb{R}^n$, $\forall\, f \in \mathcal{O}_+$, and $\forall\, b \in \mathbb{R}^n$.

Proof. There is an affine function $h : \mathbb{R}^n \to \mathbb{R}^n$ (which depends on K) such that $B(c_n) \subset h(K^*) \subset B(C_n)$. Write $h = L + v$ as above. For $x \in b + K^*$, we have

$$\begin{aligned} f * \mu_{K^*}(x) &= \int f(y)\mu_{K^*}(x-y)dy \\ &= |\mathrm{Det}\, L| \int f(y)\mu\big(L(x-y)+v\big)dy \\ &= |\mathrm{Det}\, L| \int f(y)\mu(Lx - Ly + v)dy. \end{aligned}$$

Applying the change of variables $u = Ly$, this becomes

$$\begin{aligned} f * \mu_{K^*}(x) &= \int f(L^{-1}u)\mu(Lx - u + v)du \\ &= \int f(L^{-1}u)\mu(Lx - Lb + Lb - u + v)du \\ &= \int f(L^{-1}u)\mu\big(L(x-b) + v + Lb - u\big)du. \end{aligned}$$

Therefore,

$$f * \mu_{K^*}(x) = \int f(L^{-1}u)\mu\big(h(x-b) + Lb - u\big)du \leq \int f(L^{-1}u)\psi(Lb-u)du,$$

where $\psi : \mathbb{R}^n \to [0, \infty)$ is defined by

$$\psi(z) = \sup_{|z'-z| \le C_n} \mu(z'),$$

and where we used the observation

$$x \in b + K^* \iff x - b \in K^* \implies h(x-b) \in B(C_n).$$

Thus

$$\sup_{b+K^*} f * \mu_{K^*} \le \int f(L^{-1}u)\psi(Lb-u)du.$$

On the other hand, for $x \in b + K^*$, we also have

$$\begin{aligned}
\int f(L^{-1}u)\psi(Lb-u)du &= \int f(L^{-1}u)\psi(Lx - Lx + Lb - u)du \\
&= \int f(L^{-1}u)\psi\big(Lx - L(x-b) - u\big)du \\
&= \int f(L^{-1}u)\psi\big(Lx + v - L(x-b) - v - u\big)du \\
&= \int f(L^{-1}u)\psi\big(Lx + v - h(x-b) - u\big)du.
\end{aligned}$$

Applying the change of variables $u = Ly$, this becomes

$$\begin{aligned}
\int f(L^{-1}u)\psi(Lb-u)du &= |\mathrm{Det}\, L| \int f(y)\psi\big(Lx + v - h(x-b) - Ly\big)dy \\
&= |\mathrm{Det}\, L| \int f(y)\psi\big(L(x-y) + v - h(x-b)\big)dy \\
&= |\mathrm{Det}\, L| \int f(y)\psi\big(h(x-y) - h(x-b)\big)dy \\
&\le \int f(y)|\mathrm{Det}\, L|\eta(h(x-y))dy \\
&= \int f(y)\eta_{K^*}(x-y)dy \;=\; f * \eta_{K^*}(x),
\end{aligned}$$

where

$$\eta(z) = \sup_{|z'-z| \le C_n} \psi(z').$$

Thus

$$\sup_{b+K^*} f * \mu_{K^*} \le \int f(L^{-1}u)\psi(Lb-u)du \le \inf_{b+K^*} f * \eta_{K^*},$$

as desired. □

Proposition 2 (B. Shayya)**.** *[5]*

Suppose
θ is a box in $\mathbb{R}^n$ of center ξ_0 and dimensions $R^{-1/2} \times \cdots \times R^{-1/2} \times R^{-1}$,
θ^* is the box dual to θ (i.e., θ^* is centered at the origin and has dimensions $R^{1/2} \times \cdots \times R^{1/2} \times R$),
η is a non-negative rapidly decaying function on $\mathbb{R}^n$,
ϕ is a Schwartz function on $\mathbb{R}^n$ which is ~ 1 on $B(0,1)$ and whose Fourier transform is supported in $B(0,1)$,
$\{\delta_t\}_{t>0}$ is a dilation group on $\mathbb{R}^n$ given by $\delta_t y = (ty_1, \ldots, ty_{n-1}, t^2 y_n)$, and
ψ is the Schwartz function on $\mathbb{R}^n$ defined by $\psi(\xi) = \phi\big(\delta_{\sqrt{R}}(\xi - \xi_0)\big)$.
Then:
(i) *There is a rapidly decaying function μ on $\mathbb{R}^n$ such that $\eta_{\sqrt{R}} * |\widehat{\psi}|(x) \leq \mu_{\theta^*}(x)$ for all $x \in \mathbb{R}^n$.*
(ii) *There is a rapidly decaying function μ^+ on $\mathbb{R}^n$ such that: to every function $f \in L^1(\theta)$ there is a function $h \in L^1(\theta)$ such that $|h| \sim |f|$ on θ and*

$$\sup_{b+\theta^*} |\widehat{f}| * \eta_{\sqrt{R}} \leq \inf_{b+\theta^*} |\widehat{h}| * \mu^+_{\theta^*}$$

for all $b \in \mathbb{R}^n$.

Proof. (i) For $x \in \mathbb{R}^n$, we have

$$\widehat{\psi}(x) = \frac{1}{(\sqrt{R})^{n-1}} \frac{1}{R} e^{-2\pi i x \cdot \xi_0} \widehat{\phi}\big(\delta_{1/\sqrt{R}}\, x\big)$$

and

$$\begin{aligned} \eta_{\sqrt{R}} * |\widehat{\psi}|(x) &= \frac{1}{(\sqrt{R})^n} \frac{1}{(\sqrt{R})^{n-1}} \frac{1}{R} \int \big|\widehat{\phi}\big(\delta_{1/\sqrt{R}}(x-y)\big)\big| \eta(y/\sqrt{R}) dy \\ &= \frac{1}{R^{n+(1/2)}} \Big(\int_{B(0,\sqrt{R})} \big|\widehat{\phi}\big(\delta_{1/\sqrt{R}}(x-y)\big)\big| \eta(y/\sqrt{R}) dy \\ &\quad + \sum_{j=1}^{\infty} \int_{B(0,2^j\sqrt{R}) \setminus B(0,2^{j-1}\sqrt{R})} \big|\widehat{\phi}\big(\delta_{1/\sqrt{R}}(x-y)\big)\big| \eta(y/\sqrt{R}) dy \Big) \\ &\leq \frac{1}{R^{n+(1/2)}} \sum_{j=0}^{\infty} (\sup_{A_j} \eta) \int_{B(0,2^j\sqrt{R})} \big|\widehat{\phi}\big(\delta_{1/\sqrt{R}}(x-y)\big)\big| dy, \end{aligned}$$

where $A_0 = B(0,1)$ and $A_j = B(0,2^j) \setminus B(0,2^{j-1})$ for $j \geq 1$.

Since $\widehat{\phi}$ is supported in $B(0,1)$, $\widehat{\phi}\big(\delta_{1/\sqrt{R}}(x-y)\big)$, as a function of x (with fixed y), is supported in $y + \theta^*$. Since $y \in B(0, 2^j\sqrt{R})$, it follows that

$$x \in \text{ box of center 0 and dimensions } 2^j\sqrt{R} \times \cdots \times 2^j\sqrt{R} \times 2^j R,$$

and so

$$\delta_{1/\sqrt{R}}\, x \in \text{ box of center 0 and dimensions } 2^j \times \cdots \times 2^j, \text{ i.e. } \delta_{1/\sqrt{R}}\, x \in B(0, 2^j).$$

Therefore,

$$\begin{aligned} \eta_{\sqrt{R}} * |\widehat{\psi}|(x) &\leq \frac{1}{R^{n+(1/2)}} \sum_{j=0}^{\infty} (\sup_{A_j} \eta) \|\widehat{\phi}\|_{L^\infty} |B(0, 2^j \sqrt{R})| \chi_{B(0,2^j)}(\delta_{1/\sqrt{R}}\, x) \\ &= \frac{|B(0,1)|}{R^{(n+1)/2}} \|\widehat{\phi}\|_{L^\infty} \sum_{j=0}^{\infty} 2^{nj} (\sup_{A_j} \eta) \chi_{B(0,2^j)}(\delta_{1/\sqrt{R}}\, x). \end{aligned}$$

Letting

$$\mu = |B(0,1)| \|\widehat{\phi}\|_{L^\infty} \sum_{j=0}^{\infty} 2^{nj} (\sup_{A_j} \eta) \chi_{B(0,2^j)},$$

we get a rapidly decaying function on $\mathbb{R}^n$ with

$$\eta_{\sqrt{R}} * |\widehat{\psi}|(x) \leq \mu_{\theta^*}(x) \qquad \forall\, x \in \mathbb{R}^n.$$

(ii) Given f, we let $h = f/\psi$ to get $|f| = |\psi|\,|h| \sim |h|$ on θ, $\widehat{f} = \widehat{h} * \widehat{\psi}$, and

$$|\widehat{f}| * \eta_{\sqrt{R}} \leq |\widehat{h}| * |\widehat{\psi}| * \eta_{\sqrt{R}} \leq |\widehat{h}| * \mu_{\theta^*}$$

(by part (i)). Now the locally constant property provides us with a rapidly decaying function μ^+ on $\mathbb{R}^n$ such that

$$\sup_{b+\theta^*} |\widehat{f}| * \eta_{\sqrt{R}} \leq \sup_{b+\theta^*} |\widehat{h}| * \mu_{\theta^*} \leq \inf_{b+\theta^*} |\widehat{h}| * \mu^+_{\theta^*}$$

for all $b \in \mathbb{R}^n$. □

We now prove some important lemmas that will be used to prove multilinear restriction.

Lemma 2. *Let S_j smooth compact transverse hypersurfaces in $\mathbb{R}^n$*
$f_j = \widehat{g_j}$ *and* $\operatorname{supp} g_j \subset N_{1/R} S_j$
$\psi \in C_0^\infty(\mathbb{R}^n)$
$\mu_{B_r}(x) = \frac{1}{r^n} \left| \widehat{\psi}\left(\frac{x - x_0}{r}\right) \right|$ *if* $B_r = B(x_0, r)$
$p_0 = (2n)/(n-1)$.
Then

$$\mathrm{Avg}_{B_{\sqrt{R}} \subset B_R} \prod_{j=1}^{n} \|f_j\, \mu_{B_{\sqrt{R}}}\|_{L^2}^{p_0/n} \lesssim R^\epsilon \prod_{j=1}^{n} \left(\frac{1}{R^n} \int |f_j|^2 dx \right)^{p_0/(2n)}.$$

Proof. We have

$$\int |f_j|^2 \mu_{B_{\sqrt{R}}}^2 dx = \frac{1}{R^n}\int \Big| f_j(x)\widehat{\psi}(\frac{x-x_0}{\sqrt{R}})\Big|^2 dx = \frac{1}{R^n}\int \Big| f_j(x+x_0)\widehat{\psi}(\frac{x}{\sqrt{R}})\Big|^2 dx.$$

Since $f_j(x+x_0)$ is the Fourier transform of $e^{-2\pi i x_0\cdot\xi} g_j(\xi)$, it follows that

$$\int |f_j(x)|^2 \mu_{B_{\sqrt{R}}}(x)^2 dx = \frac{1}{R^n}\int \Big| \big(e^{-2\pi i x_0\cdot\xi} g_j\big) * \psi_{\sqrt{R^{-1}}}(\xi)\Big|^2 d\xi.$$

We write $g_j = \sum_\theta g_{j,\theta}$, and we recall that each θ is a $\sqrt{R^{-1}}\times\cdots\times\sqrt{R^{-1}}\times R^{-1}$ box. Since $\psi_{\sqrt{R^{-1}}}$ is supported in a ball of center 0 and radius $\sim\sqrt{R^{-1}}$, it follows that each function

$$\big(e^{-2\pi i x_0\cdot\xi} g_{j,\theta}\big) * \psi_{\sqrt{R^{-1}}}$$

is supported in a ball of the same center as θ and of radius $\sim\sqrt{R^{-1}}$. So

$$\Big\|\sum_\theta \big(e^{-2\pi i x_0\cdot\xi} g_{j,\theta}\big) * \psi_{\sqrt{R^{-1}}}\Big\|_{L^2}^2 \lesssim \sum_\theta \Big\|\big(e^{-2\pi i x_0\cdot\xi} g_{j,\theta}\big) * \psi_{\sqrt{R^{-1}}}\Big\|_{L^2}^2$$

(where we used the fact that if $\theta\neq\theta'$, then the distance between the center of θ and the center of θ' is $\sim\sqrt{R^{-1}}$), and so

$$\int |f_j(x)|^2 \mu_{B_{\sqrt{R}}}(x)^2 dx \lesssim \frac{1}{R^n}\sum_\theta\int \Big| \big(e^{-2\pi i x_0\cdot\xi} g_{j,\theta}\big) * \psi_{\sqrt{R^{-1}}}(\xi)\Big|^2 d\xi.$$

Denoting the Fourier transform of $e^{-2\pi i x_0\cdot\xi} g_{j,\theta}$ by $f'_{j,\theta}$, we obtain

$$\begin{aligned}\int |f_j(x)|^2 \mu_{B_{\sqrt{R}}}(x)^2 dx &\lesssim \sum_\theta \int |f'_{j,\theta}(x)|^2 \frac{1}{R^n}\Big|\widehat{\psi}(\frac{x}{\sqrt{R}})\Big|^2 dx \\ &\lesssim \sum_\theta \oint_{Q_{\sqrt{R}}} |f'_{j,\theta}| * \eta_{Q_{\sqrt{R}}}(x)^2 dx \\ &\leq \sum_\theta \sup_{Q_{\sqrt{R}}} \big(|f'_{j,\theta}| * \eta_{Q_{\sqrt{R}}}\big)^2,\end{aligned}$$

where η is a rapidly decaying function on $\mathbb{R}^n$ which depends only on ψ. So

$$\begin{aligned}\prod_{j=1}^n \|f_j\,\mu_{B_{\sqrt{R}}}\|_{L^2}^{p_0/n} &\lesssim \prod_{j=1}^n \Big(\sum_\theta \sup_{Q_{\sqrt{R}}} \big(|f'_{j,\theta}| * \eta_{Q_{\sqrt{R}}}\big)^2\Big)^{\frac{p_0}{2n}} \\ &= \left(\prod_{j=1}^n \Big(\sum_\theta \sup_{Q_{\sqrt{R}}} \big(|f'_{j,\theta}| * \eta_{Q_{\sqrt{R}}}\big)^2\Big)^{\frac{1}{n-1}}\right)^{\frac{p_0(n-1)}{2n}} \\ &\lesssim \left(\oint_{Q_{\sqrt{R}}} \prod_{j=1}^n \Big(\sum_\theta \sup_{Q_{\sqrt{R}}} \big(|f'_{j,\theta}| * \eta_{Q_{\sqrt{R}}}\big)^2\Big)^{\frac{1}{n-1}} dx\right)^{\frac{p_0(n-1)}{2n}} \\ &\leq \oint_{Q_{\sqrt{R}}} \prod_{j=1}^n \Big(\sum_\theta |f'_{j,\theta}| * \eta^+_{Q_{\sqrt{R}}}(x)^2\Big)^{\frac{1}{n-1}} dx.\end{aligned}$$

We recall that $f'_{j,\theta}$ is the Fourier transform of $e^{-2\pi i x_0\cdot\xi} g_{j,\theta}$. Denoting the Fourier transform of $g_{j,\theta}$ by $f_{j,\theta}$, we see that

$$\begin{aligned}\prod_{j=1}^n \|f_j\,\mu_{B_{\sqrt{R}}}\|_{L^2}^{p_0/n} &\lesssim \oint_{Q_{\sqrt{R}}} \prod_{j=1}^n \Big(\sum_\theta |f_{j,\theta}| * \eta^+_{Q_{\sqrt{R}}}(x+x_0)^2\Big)^{\frac{1}{n-1}} dx\\ &= \oint_{x_0+Q_{\sqrt{R}}} \prod_{j=1}^n \Big(\sum_\theta |f_{j,\theta}| * \eta^+_{Q_{\sqrt{R}}}(x)^2\Big)^{\frac{1}{n-1}} dx.\end{aligned}$$

So

$$R^{\frac{n}{2}} \mathrm{Avg}_{B_{\sqrt{R}}\subset B_R} \prod_{j=1}^n \|f_j\,\mu_{B_{\sqrt{R}}}\|_{L^2}^{\frac{p_0}{n}} \lesssim R^{-\frac{n}{2}} \int_{B_R} \prod_{j=1}^n \Big(\sum_\theta |f_{j,\theta}| * \eta^+_{Q_{\sqrt{R}}}(x)^2\Big)^{\frac{1}{n-1}} dx,$$

and so

$$\mathrm{Avg}_{B_{\sqrt{R}}\subset B_R} \prod_{j=1}^n \|f_j\,\mu_{B_{\sqrt{R}}}\|_{L^2}^{\frac{p_0}{n}} \lesssim \oint_{B_R} \prod_{j=1}^n \Big(\sum_\theta |f_{j,\theta}| * \eta^+_{Q_{\sqrt{R}}}(x)^2\Big)^{\frac{1}{n-1}} dx.$$

For each θ, we let $\mathbb{T}(\theta)$ be a finitely overlapping family of $\sqrt{R}\times\cdots\times\sqrt{R}\times R$ tubes pointing in the direction of θ and covering B_R. Part (ii) of Proposition 2 provides us with functions $h_{j,\theta}\in L^1(\theta)$ and a rapidly decaying function μ^+ on $\mathbb{R}^n$ such that

$$\sup_T |\widehat{g_{j,\theta}}| * \eta^+_{Q_{\sqrt{R}}} = \sup_T |f_{j,\theta}| * \eta^+_{Q_{\sqrt{R}}} \le \inf_T |\widehat{h_{j,\theta}}| * \mu^+_{\theta^*}$$

for all $T\in\mathbb{T}(\theta)$. Now, we define the functions g_j^+ by

$$g_j^+ = \sum_{\theta\subset N_{1/R}S_j} \sum_{T\in\mathbb{T}(\theta)} \big(\inf_T |\widehat{h_{j,\theta}}| * \mu^+_{\theta^*}\big)^2 \chi_T,$$

and we observe that

$$\sum_{\theta\subset N_{1/R}S_j} \big(|f_{j,\theta}| * \eta^+_{Q_{\sqrt{R}}}\big)^2 \le g_j^+ \le \sum_{\theta\subset N_{1/R}S_j} \big(|\widehat{h_{j,\theta}}| * \mu^+_{\theta^*}\big)^2$$

on B_R, so that

$$\prod_{j=1}^n \Big(\sum_{\theta\subset N_{1/R}S_j} |f_{j,\theta}| * \eta^+_{Q_{\sqrt{R}}}(x)^2\Big)^{\frac{1}{n-1}} \le \prod_{j=1}^n g_j^+(x)^{1/(n-1)}$$

for all $x\in B_R$. Thus

$$\mathrm{Avg}_{B_{\sqrt{R}}\subset B_R} \prod_{j=1}^n \|f_j\,\mu_{B_{\sqrt{R}}}\|_{L^2}^{p_0/n} \lesssim \oint_{Q_R} \prod_{j=1}^n g_j^+(x)^{1/(n-1)} dx.$$

It is now time to appeal to the multilinear Kakeya inequality

$$\fint_{Q_R} \prod_{j=1}^{n} g_j^+(x)^{1/(n-1)} dx \lesssim R^\epsilon \prod_{j=1}^{n} \Big(\fint_{B_R} g_j^+(x) dx \Big)^{\frac{1}{n-1}}$$

to get

$$\mathrm{Avg}_{B_{\sqrt{R}} \subset B_R} \prod_{j=1}^{n} \| f_j \, \mu_{B_{\sqrt{R}}} \|_{L^2}^{p_0/n} \lesssim R^\epsilon \prod_{j=1}^{n} \Big(\fint_{B_R} \sum_{\theta \subset N_{1/R} S_j} |\widehat{h_{j,\theta}}| * \mu_{\theta^*}^+(x)^2 dx \Big)^{\frac{1}{n-1}}.$$

The right-hand side of this inequality can be written as

$$R^\epsilon \prod_{j=1}^{n} \Big(R^{-n} \sum_{\theta \subset N_{1/R} S_j} \big\| |\widehat{h_{j,\theta}}| * \mu_{\theta^*}^+ \big\|_{L^2}^2 \Big)^{\frac{1}{n-1}}.$$

By Hausdorff-Young and Plancherel,

$$\big\| |\widehat{h_{j,\theta}}| * \mu_{\theta^*}^+ \big\|_{L^2}^2 \le \| \widehat{h_{j,\theta}} \|_{L^2}^2 \| \mu_{\theta^*}^+ \|_{L^1}^2 \lesssim \| h_{j,\theta} \|_{L^2}^2 \sim \| g_{j,\theta} \|_{L^2}^2,$$

so

$$\sum_{\theta \subset N_{1/R} S_j} \big\| |\widehat{h_{j,\theta}}| * \mu_{\theta^*}^+ \big\|_{L^2}^2 \lesssim \sum_{\theta \subset N_{1/R} S_j} \| g_{j,\theta} \|_{L^2}^2 = \| g_j \|_{L^2}^2 = \| f_j \|_{L^2}^2,$$

and so

$$\mathrm{Avg}_{B_{\sqrt{R}} \subset B_R} \prod_{j=1}^{n} \| f_j \, \mu_{B_{\sqrt{R}}} \|_{L^2}^{p_0/n} \lesssim R^\epsilon \prod_{j=1}^{n} \Big(R^{-n} \| f_j \|_{L^2}^2 \Big)^{\frac{1}{n-1}},$$

as desired. □

Theorem 1 (Multilinear restriction). *Suppose f_j, g_j, S_j, and p_0 are as in Lemma 2. Then*

$$\Big\| \prod_{j=1}^{n} |f_j|^{1/n} \Big\|_{L^p_{\mathrm{avg}}(B_R)} \lesssim R^\epsilon \prod_{j=1}^{n} \Big(\frac{1}{R^n} \int |f_j|^2 dx \Big)^{1/(2n)}$$

for $1 \le p \le p_0$.

Proof. We consider a sequence of scales. We pick a large integer M and we let $r = R^{2^{-M}}$. We will use Lemma 2 at scales $r, r^2, \ldots, r^{2^M} = R$. We abbreviate $r_a = r^{2^a}$.

We begin with a crude inequality at the small scale r. Applying Bernstein's inequality to each f_j on each $B_r \subset B_R$, we get

$$\int_{B_r} \prod_{j=1}^{n} |f_j|^{p/n} dx \le |B(0,1)| r^n \prod_{j=1}^{n} C_n^{1/n} r^{2p} \| f_j \|_{L^p(\psi_r)}^{p/n} = C_n r^n r^{2pn} \prod_{j=1}^{n} \| f_j \|_{L^p(\psi_r)}^{p/n},$$

where $|B(0,1)|$ was absorbed into the constant C_n, and where we used the fact that $r \geq 1$. Therefore,

$$\int_{B_R} \prod_{j=1}^{n} |f_j|^{p/n} dx \leq \frac{R^n}{r^n} C_n r^n r^{2pn} \mathrm{Avg}_{B_r \subset B_R} \prod_{j=1}^{n} \|f_j\|_{L^p(\psi_r)}^{p/n}.$$

Therefore,

$$\fint_{B_R} \prod_{j=1}^{n} |f_j|^{p/n} dx \leq C_n r^{2pn} \mathrm{Avg}_{B_r \subset B_R} \prod_{j=1}^{n} \|f_j\|_{L^p(\psi_r)}^{p/n}.$$

But

$$\mathrm{Avg}_{B_r \subset B_R} = \mathrm{Avg}_{B_{r_1} \subset B_R} \mathrm{Avg}_{B_r \subset B_{r_1}},$$

so the result follows by repeated (more precisely, M) applications of Lemma 2. □

Chapter 3

The ℓ^2 Decoupling Theorem

For $2 \le p \le \infty$ and $R \in \mathbb{R}$, we let $D_{n,p}(R)$ be the smallest constant such that the inequality:

$$\|Eg\|_{L^p(w_B)} \le D_{n,p}(R)\Big(\sum_{Q \in Part_{R^{-1/2}}([0,1]^{n-1})} \|Eg\|^2_{L^p(w_B)}\Big)^{1/2}$$

holds for every cube $B \subset \mathbb{R}^n$ with side length R and every $g : [0,1]^{n-1} \to \mathbb{C}$. We call $D_{n,p}(R)$ the decoupling constant. The following theorem was stated and proved by Bourgain and Demeter in [6].

Theorem 2 (ℓ^2 decoupling)**.** *Let $D_{n,p}(R)$ be as above. Then*

$$D_{n,p}(R) \lesssim R^{\epsilon}$$

for $2 \le p \le \frac{2(n+1)}{n-1}$. The implicit constant depends on ϵ, p, n but not on R.

Chapter 4

Key Lemmas and Orthogonality

Lemma 3. *Let $\mathcal{W}$ be the collection of all positive integrable functions on $\mathbb{R}^n$. Let $R > 0$ and fix E. Suppose $Oi : \mathcal{W} \to [0, \infty], i = 1, 2$ satisfy the following four properties:*
(P1) $O_1(\chi_B) \lesssim O_2(w_{B,E})$ for all cubes $B \subset \mathbb{R}^n$ with $\ell(B) = R$
(P2) $O_1(\alpha u + \beta v) \leq \alpha O_1(u) + \beta O_1(v)$, for every $u, v \in \mathcal{W}$ and $\alpha, \beta > 0$
(P3) $O_2(\alpha u + \beta v) \geq \alpha O_2(u) + \beta O_2(v)$, for every $u, v \in \mathcal{W}$ and $\alpha, \beta > 0$
(P4) If $u, v \in \mathcal{W}$ and $u \leq v$ then $O_i(u) \leq O_i(v)$.
Then

$$O_1(w_{B,E}) \lesssim O_2(w_{B,E})$$

for each cube B with side length R, where the implicit constant depends only on the implicit constant from the first property (P1) and is independent of R, B .

Proof. Let $\mathcal{B}$ be a finitely overlapping cover of $\mathbb{R}^n$ with cubes $B' = B'(c_{B'}, R)$. To prove this lemma, we need to prove three basic inequalities involving characteristic functions and weights. The inequalities are:

$$\chi_B \lesssim \sum_{B' \in \mathcal{B}} w_{B'} \lesssim w_B \tag{4.1}$$

$$w_B(x) \lesssim \sum_{B' \in \mathcal{B}} \chi_{B'}(x) w_B(c_{B'}) \tag{4.2}$$

$$\sum_{B' \in \mathcal{B}} w_{B'}(x) w_B(c_{B'}) \lesssim w_B(x) \tag{4.3}$$

First, we will show that if these inequalities hold, then the lemma is true:

$$O_1(w_B) \lesssim O_1\Big(\sum_{B' \in \mathcal{B}} \chi_{B'} w_B(c_{B'})\Big)$$

$$\lesssim \sum_{B' \in \mathcal{B}} w_B(c_{B'}) O_1(\chi_{B'}) \lesssim \sum_{B' \in \chi_{B'}} w_B(c_{B'}) O_2(w_{B'})$$

$$\lesssim \sum_{B' \in \chi_{B'}} O_2(w_B(c_{B'})w_{B'})$$

$$\lesssim O_2\Big(\sum_{B' \in \mathcal{B}} w_B(c_{B'})w_{B'}\Big) \lesssim O_2(w_B).$$

Now we prove the inequalities.
(1) is easy to check.

To prove (2), first notice that if we take two cubes in $\mathbb{R}^n$ having the same side length and such that they have at least one point in common, then their centers are at most at a distance of $R\sqrt{n}$ from each other (by taking the case where one diagonal coincides, as this gives the furthest position of the cetners).

$$\sum_{B' \in \mathcal{B}} \chi_{B'}(x) w_B(c_{B'}) = \sum_{B' \in \mathcal{B}} \chi_{B'}(x) \frac{1}{(1 + \frac{|c_B - c'_B|}{R})^E}$$

$$\geq \sum_{B' \in \mathcal{B}} \chi_{B'}(x) \frac{1}{(1 + \frac{2R\sqrt{n}}{2R})^E} \geq \frac{1}{(1+\sqrt{n})^E} \sum_{B' \in \mathcal{B}} \chi_{B'}(x)$$

$$\geq \frac{1}{(1+\sqrt{n})^E} w_B(x)$$

the last inequality is true since at least one of the B' has to contain x and the weight of a cube is always at most 1.

To prove (3), just notice that $w_B(c_{B'}) \leq 1$ and so we get

$$\sum_{B' \in \mathcal{B}} w_{B'}(x) w_B(c_{B'}) \leq \sum_{B' \in \mathcal{B}} w_{B'}(x) \lesssim w_B(x)$$

where in the last step we used inequality (1). □

Proposition 3 (L^2 decoupling or orthogonality)**.** *Let Q be a cube with $l(Q) \geq R^{-1}$. Then for each cube $B_R \subset \mathbb{R}^n$ with side length R we have:*

$$\|E_Q g\|_{L^2(w_{B_R})} \lesssim \Big(\sum_{q \in Part_{\frac{1}{R}}(Q)} \|E_q g\|^2_{L^2(w_{B_R})}\Big)^{1/2}$$

Proof. Pick a positive Schwartz function η whose square root has fourier support contained in a small neighborhood of the origin, and such that $\eta \geq 1$ on $B(0,1)$. We apply Lemma 3 by setting

$$O_1(v) = \|E_Q g\|^2_{L^2(v)}$$

and

$$O_2(v) = \sum_{q \in Part_{\frac{1}{R}}(Q)} \|E_Q g\|^2_{L^2(v)}.$$

All properties are straightforward and follow directly from the basic properties of the norms except (1). So we only prove (1):

$$\|E_Q g\|^2_{L^2(B')} \leq \|E_Q g\|^2_{L^2(\eta_{B'})} = \|\sqrt{\eta_{B'}} E_Q g\|^2_{L^2(\mathbb{R}^n)}.$$

Now, the functions $\sqrt{\eta_{B'}} E_q g$ are orthogonal and hence the result follows directly. □

Lemma 4 (Reverse Minkowski's Inequality). *Consider the measure space $(X, \mathcal{M}, \mu)$. Let $p \in (0,1]$ and f, g be two positive measurable functions. Then we have the following inequality:*

$$\left(\int (f+g)^p d\mu\right)^{\frac{1}{p}} \geq \left(\int f^p d\mu\right)^{\frac{1}{p}} + \left(\int g^p d\mu\right)^{\frac{1}{p}}$$

Proof. The function defined for $x > 0$ by $\phi(x) = x^p$ is concave for $p \in (0,1]$. This can be shown very easily since $\phi''(x) < 0$ for the given range of p. Hence ϕ satisfies the concavity inequality: $\phi((1-\alpha)x + \alpha y) \geq (1-\alpha)\phi(x) + \alpha\phi(y)$ for any $x, y > 0$ and for any $\alpha \in [0,1]$.

Define the functions F and G by

$$F = \frac{f}{(\int f^p d\mu)^{\frac{1}{p}}) + (\int g^p d\mu)^{\frac{1}{p}}}$$

and

$$G = \frac{g}{(\int f^p d\mu)^{\frac{1}{p}}) + (\int g^p d\mu)^{\frac{1}{p}}}$$

and notice that

$$\int F^p d\mu = \frac{1}{\left[(\int f^p d\mu)^{\frac{1}{p}}) + (\int g^p d\mu)^{\frac{1}{p}}\right]^p} \int f^p d\mu$$

and

$$\int G^p d\mu = \frac{1}{\left[(\int f^p d\mu)^{\frac{1}{p}}) + (\int g^p d\mu)^{\frac{1}{p}}\right]^p} \int g^p d\mu$$

so that we get

$$\left(\int F^p d\mu\right)^{\frac{1}{p}} + \left(\int G^p d\mu\right)^{\frac{1}{p}} = 1.$$

Now set $(\int F^p d\mu)^{\frac{1}{p}} = 1 - \alpha$ and $(\int G^p d\mu)^{\frac{1}{p}} = \alpha$.

By the concavity of ϕ as defined above, we get the inequality

$$(F+G)^p = ((1-\alpha)\frac{F}{1-\alpha} + \alpha\frac{G}{\alpha})^p \geq (1-\alpha)\frac{F^p}{(1-\alpha)^p} + \alpha\frac{G^p}{\alpha^p}$$

and hence we get

$$\int (F+G)^p d\mu \geq \frac{1-\alpha}{(1-\alpha)^p}\int F^p d\mu + \frac{\alpha}{\alpha^p}\int G^p d\mu = 1$$

so that

$$\int \frac{f}{(\int f^p d\mu)^{\frac{1}{p}}) + (\int g^p d\mu)^{\frac{1}{p}}} + \frac{g}{(\int f^p d\mu)^{\frac{1}{p}}) + (\int g^p d\mu)^{\frac{1}{p}}})^p d\mu \geq 1$$

so that

$$\int (f+g)^p d\mu \geq ((\int f^p d\mu)^{\frac{1}{p}} + (\int g^p d\mu)^{\frac{1}{p}})^p$$

and therefore

$$(\int (f+g)^p d\mu)^{\frac{1}{p}} \geq (\int f^p d\mu)^{\frac{1}{p}} + (\int g^p d\mu)^{\frac{1}{p}}.$$

□

Lemma 5 (Reverse Hölder's Inequality)**.** *Consider cubes Q and B according to the convention mentioned in the beginning such that the side length of Q is $\ell(Q) = R^{-1}$ and the side length of B is $\ell(B) = R$. Let $q \geq p \geq 1$.*

Then $\|E_Q g\|_{L^q_{\#}(w_{B,E})} \lesssim \|E_Q g\|_{L^p_{\#}(w_{B,\frac{Ep}{q}})}$ where the constant is independent of R, Q, B and the function g.

Proof. Let η be a positive smooth function on $\mathbb{R}^n$ that satisfies $\chi_{B(0,1)} \leq \eta_{B(0,1)}$ and such that $supp(\widehat{\eta^{\frac{1}{p}}}) \subset B(0,1)$. The following inequality follows easily:

$$\|E_Q g\|_{L^q(B)} \leq \|E_Q g\|_{L^q(\eta_B^{p/q})} = \|\eta_B^{1/p} E_Q g\|_{L^q(\mathbb{R}^n)}.$$

Let ψ be a Shwartz function which is constant ($=1$) on the cube of center the origin and radius 10. An easy computation using the fact that the fourier support of $\eta_B^{1/p} E_Q g$ is contained in the cube $3Q$ (that is a cube having the same center as Q but thrice its side length) shows that:

$$(\eta_B^{1/p} E_Q g) * \widehat{\psi} = \eta_B^{1/p} E_Q g.$$

So for $\frac{1}{q} = \frac{1}{p} + \frac{1}{r} - 1 = \frac{1}{p} - \frac{1}{r'}$ apply Young's inequality to get:

$$\|\eta_B^{1/p} E_Q g\|_{L^q(\mathbb{R}^n)} \leq \|\eta_B^{1/p} E_Q g\|_{L^p(\mathbb{R}^n)} \|\hat{\psi_Q}\|_{L^r(\mathbb{R}^n)} \lesssim R^{-n/r'} \|E_Q g\|_{L^p(\eta_B)}.$$

Now we use the same notation and some of the inequalities mentioned in the proof of lemma 3 to continue in the following way:

$$\|E_Q g\|_{L^q(w_{B,E})} = \int |E_Q g|^q w_{B,E} \lesssim \sum_{B' \in \mathcal{B}} w_{B,E}(c_{B'}) \int_{B'} |E_Q g|^q$$

$$\lesssim R^{-\frac{nq}{r}} \sum_{B' \in \mathcal{B}} w_{B,E}(c_{B'}) \|E_Q g\|^q_{L^p(\eta_{B'})}$$

$$\lesssim R^{-\frac{nq}{r}} \Big(\sum_{B' \in \mathcal{B}} [w_{B,E}(c_{B'})]^{\frac{p}{q}} \|E_Q g\|^p_{L^p(\eta_{B'})} \Big)^{\frac{q}{p}}$$

$$\lesssim R^{-\frac{nq}{r}} \Big(\int |E_Q g|^p w_{B,\frac{Ep}{q}} \Big)^{\frac{q}{p}}$$

where all integrals are with respect to the Lebesgue measure.

□

Chapter 5

The Decoupling Norm

In this section $S \subset \mathbb{R}^n$ will denote a compact positively oriented curved C^3 hypersurface. (Our truncated paraboloid was one example of such a hypersurface). Now we define what we shall call the decoupling norm. Let $supp\hat{f} \subset N_\delta S$ and let Ω be a domain in $\mathbb{R}^n$. Then the decoupling norm is defined by Guth in [4] as:

$$\|f\|_{L^{p,\delta}_{avg}(\Omega)} := \Big(\sum_{\theta \in Part_{\delta^{1/2}}(N_\delta S)} \|f_\theta\|^2_{L^p_{avg}(\Omega)} \Big)^{1/2}$$

It can be shown very easily that this is an actual norm.

We now look at the situation where a domain is partitioned into smaller subsets and we try to estimate the decoupling norm over the whole set in terms of the decoupling norm over its partitioned subsets. For the usual L^p norms, we would get equality. For the decoupling norm we get the following:

Lemma 6. *Let $\Omega = \bigsqcup \Omega_j$, $p \geq 2, \delta \geq 0$, and f is such that $supp\hat{f} \subset N_\delta S$. Then we have:*

$$\sum_j \|f\|^p_{L^{p,\delta}(\Omega_j)} \leq \|f\|^p_{L^{p,\delta}(\Omega)}$$

Proof. Starting from the left-hand-side and replacing the decoupling norm by its explicit expression, we get

$$\sum_j \|f\|^p_{L^{p,\delta}(\Omega_j)} = \sum_j \Big(\sum_\theta \|f_\theta\|^2_{L^p(\Omega_j)} \Big)^{\frac{p}{2}} = \Big\| \sum_\theta \|f_\theta\|^2_{L^p(\Omega_j)} \Big\|^{\frac{p}{2}}_{\ell_j^{p/2}}$$

Now applying Minkowski inequality for the $\ell_j^{p/2}$ norm and using the fact that $\sum_j \|f\|^p_{L^p(\Omega_j)} = \|f\|^p_{L^p(\Omega)}$, we get

$$\Big\| \sum_\theta \|f_\theta\|^2_{L^p(\Omega_j)} \Big\|^{\frac{p}{2}}_{\ell_j^{p/2}} \leq \Big(\sum_\theta \big\| \|f_\theta\|^2_{L^p(\Omega_j)} \big\|_{\ell_j^{p/2}} \Big)^{\frac{p}{2}} = \Big(\sum_\theta \|f_\theta\|^2_{L^p(\Omega)} \Big)^{\frac{p}{2}} = \|f\|^p_{L^{p,\delta}(\Omega)}$$

□

The next question we ask is if we get a good decoupling over the subsets forming the partition, does this give us a decoupling for the whole set? And the answer is given by the proposition below:

Proposition 4 (Parallel Decoupling). *Let* Ω, Ω_j, p, f *be as in the above lemma. Moreover, suppose that the following inequality holds for every* j*:*

$$\|f\|_{L^p(\Omega_j)} \leq M\|f\|_{L^{p,\delta}(\Omega_j)}.$$

Then we get:

$$\|f\|_{L^p(\Omega)} \leq M\|f\|_{L^{p,\delta}(\Omega)}$$

Proof. This will follow as a corollary to the previous lemma. Again we expand

$$\|f\|^p_{L^p(\Omega)} = \sum_j \|f\|^p_{L^p(\Omega_j)} \leq \sum_j (M\|f\|_{L^{p,\delta(\Omega_j)}})^p = M^p \sum_j \|f\|^p_{L^{p,\delta}(\Omega_j)} \leq M^p\|f\|^p_{L^{p,\delta}(\Omega)},$$

where the last upper bound is given by the result of the previous lemma. Raising both sides to the power $\frac{1}{p}$ ends the proof.

□

In terms of our new notation, the decoupling constant $D_p(R)$ is the smallest constant such that the inequality:

$$\|f\|_{L^p_{avg}(B_R)} \leq D_p(R)\|f\|_{L^{p,1/R}_{avg}(B_R)}$$

holds for all f with $supp\hat{f} \subset N_{1/R}(S)$ and all cubes B_R. So the ℓ^2 decoupling theorem asserts that $D_p(R) \lesssim R^\varepsilon$ for every $\varepsilon > 0$ and $2 \leq p \leq \frac{2(n+1)}{n-1}$.

Chapter 6

Parabolic Rescaling

We start by proving that the decoupling constants are invariant under affine changse of coordinates.

Lemma 7. *Let $h : \mathbb{R}^n \to \mathbb{R}^n$ be an affine map. Let Ω be a domain in $\mathbb{R}^n$ partitioned into a disjoint union of subsets θ. We denote the decoupling constant over Ω given by its partition to the $\theta's$ by $D(\Omega = \bigsqcup \theta)$. Then we have:*

$$D(\Omega = \bigsqcup \theta) = D\big(h(\Omega) = \bigsqcup h(\theta)\big)$$

Proof. Let $\bar{\Omega} = h(\Omega)$ and $\bar{\theta} = h(\theta)$ Suppose g is a function with $supp\hat{g} \subset \bar{\Omega}$. Then $g = \sum_{\bar{\theta}} g_{\bar{\theta}}$. Applying the change of coordinates given by this affine transformation, we get a function f with fourier support in Ω such that $f_\Omega = \sum_\theta f_\theta$. Denote the Jacobian of this affine transformation by $|J|$. We notice that $\|g\|_{L^p(\bar{\Omega})} = |J|^{\frac{1}{p}} \|f\|_{L^p(\Omega)}$ and $\|g_{\bar{\theta}}\|_{L^p(\bar{\theta})} = |J|^{\frac{1}{p}} \|f\|_{\theta_{L^p(\theta)}}$ This finishes the proof. □

Theorem 3 (Parabolic Rescaling)**.** *Let $r \leq R$. Let $\tau \in Part_{r^{-1/2}}(N_{R^{-1}}S)$ and suppose f has Fourier support in τ. If $\theta \in Part_{R^{-1/2}}(N_{R^{-1}}S)$, then we have:*

$$\|f\|_{L^p_{avg}(B_R)} \lesssim D_{p,n}(R/r)\Big(\sum_{\theta \subset \tau} \|f_\theta\|^2_{L^p_{avg}(B_R)}\Big)^{1/2}$$

Proof. After an affine transformation which leaves the decoupling constant invariant, we can shift τ to the region:

$$\{(\xi_1, ..., \xi_{n-1}, \xi_n) : 0 < \xi_n < r^{-1}, 0 < |\xi_i| < r^{-1/2}; i = 1, 2, ..., n-1\}$$

Define the new coordinates $\bar{\xi_n} := r\xi_n$ and $\bar{\xi_i} := r^{1/2}\xi_i$ where $i = 1, 2, ..., n-1$. We will denote the image of any set A in the new coordinates by $\bar{A}$. So now we have $\bar{\theta}$ as a $(R/r)^{-1/2} \times ... \times (R/r)^{-1/2} \times (R/r)^{-1} - block$.
The corresponding change of coordinates that takes place in physical space is given by: $\bar{x_n} = r^{-1}x_n$ and $\bar{x}_i = r^{-1/2}x_i$ for $i = 1, 2, ..., n-1$.

Define the function f as $f(x) = g(\bar{x})$ and so we get that since the fourier support of g is in τ then the fourier support of f would be in $\bar{\tau}$.
Easy computations show that B_R is in fact an ellipsoid E with minor axis of length R/r^{-1} and $n-1$ major axes of length $R/r^{-1/2}$.
Partition the ellipsoid E into balls of radius R/r. On each ball $B_{R/r}$, we have the estimate:

$$\|g\|_{L^p_{avg}(B_{R/r})} \leq D_p(R/r)\Big(\sum_{\theta} \|g_{\bar{\theta}}\|^2_{L^p_{avg}(B_{R/r})}\Big)^{1/2}.$$

By parallel decoupling, this gives us a decouping estimate on the entire ellipsoid as follows:

$$\|g\|_{L^p_{avg}(E)} \leq D_p(R/r)\Big(\sum_{\theta} \|g_{\bar{\theta}}\|^2_{L^p_{avg}(E)}\Big)^{1/2}.$$

The fact that all norms are averaged takes care of any Jacobian fators that might arise during the change back to our original coordinates and so we get the desired estimate for f. Note that if we replace balls by cubes and the ellipsoid by the corresponding box, the arguments will remain true.

$\square$

Proposition 5. *For any $R_1, R_2 \geq 1$, we have $D_{n,p}(R_1R_2) \lesssim D_{n,p}(R_1)D_{n,p}(R_2)$.*

Proof. We let $R = R_1 \times R_2$, f be the function with fourier support in $N_{1/R}S$, and $\tau \in Part_{R_1^{-1/2}}(N_{1/R}(S))$. We cover B_R with disjoint boxes of side length R_1 (this is possible since $R > R_1$). We thus get the inequality:

$$\|f\|_{L^p_{avg}(B_R)} \leq D_p(R_1)\Big(\sum_{\tau} \|f_\tau\|^2_{L^p_{avg}(B_R)}\Big)^{1/2}.$$

Let $\theta \in Part_{R^{-1/2}}(N_{1/R}S)$. Using parabolic rescaling, we can find an upper bound as follows:

$$\|f_\tau\|_{L^p_{avg}(B_R)} \lesssim D_p(R_2)\Big(\sum_{\theta\subset\tau} \|f_\theta\|^2_{L^2_{avg}}(B_R)\Big)^{1/2}.$$

Using the first estimate with this inequality we get:

$$\|f\|_{L^p_{avg}(B_R)} \leq D_p(R_1)D_p(R_2)\Big(\sum_{\theta} \|f_\theta\|^p_{L^p_{avg}(B_R)}\Big)^{1/2}.$$

$\square$

Chapter 7

Linear Decoupling and Multilinear Decoupling

We will denote the multilinear decoupling constant by $\tilde{D}$.

Theorem 4. *Suppose that in dimension $n-1$, $D_{n,p}(R) \lesssim R^\epsilon$ for every $\epsilon > 0$. Then, for any $\epsilon > 0$ we have:*

$$D_{n,p}(R) \lesssim R^\epsilon \tilde{D}_{n,p}(R)$$

and

$$\tilde{D}_{n,p}(R) \leq D_{n,p}(R)$$

for any n, p, R.

Proof. See Bourgain and Demeter's proof in [1]. □

So we can conclude that if decoupling holds in dimension $n-1$ then $\tilde{D}_{n,p}(R) \approx D_{n,p}(R) \approx R^\gamma$ for some $\gamma = \gamma(n,p)$.

Theorem 5 (Multilinear decoupling for $2 \leq p \leq 2n/(n-1)$)**.** *Suppose f_j, g_j, S_j, and p_0 are as in Lemma 2. Then*

$$\Big\| \prod_{j=1}^{n} |f_j|^{1/n} \Big\|_{L^p_{\text{avg}}(B_R)} \lesssim R^\epsilon \prod_{j=1}^{n} \|f_j\|^{1/n}_{L^{p,R^{-1}}(\eta_{B_R})}$$

for $1 \leq p \leq p_0$.

Proof. Let $\psi \in C_0^\infty(\mathbb{R}^n)$ be such that $\widehat{\psi} \geq 1$ on B_1. By multilinear restriction,

$$\begin{aligned} \Big\| \prod_{j=1}^{n} |f_j|^{1/n} \Big\|_{L^p_{\text{avg}}(B_R)} &\leq \Big\| \prod_{j=1}^{n} |f_j \widehat{\psi_{R^{-1}}}|^{1/n} \Big\|_{L^p_{\text{avg}}(B_R)} \\ &\lesssim R^\epsilon \prod_{j=1}^{n} \Big(\frac{1}{R^n} \int |f_j \widehat{\psi_{R^{-1}}}|^2 dx \Big)^{1/(2n)}. \end{aligned}$$

By Plancherel,

$$\frac{1}{R^n}\int |f_j\,\widehat{\psi_{R^{-1}}}|^2 dx = \frac{1}{R^n}\int |g_j * \psi_{R^{-1}}|^2 d\xi.$$

Writing $g_j = \sum_\theta g_{j,\theta}$, and observing that the functions $g_{j,\theta} * \psi_{R^{-1}}$ have essentially disjoint supports, we see that

$$\frac{1}{R^n}\int |g_j * \psi_{R^{-1}}|^2 d\xi \lesssim \frac{1}{R^n}\sum_\theta \int |g_{j,\theta} * \psi_{R^{-1}}|^2 d\xi.$$

Applying Plancherel again, we get

$$\frac{1}{R^n}\int |f_j\,\widehat{\psi_{R^{-1}}}|^2 dx \lesssim \frac{1}{R^n}\sum_\theta \int |f_{j,\theta}\,\widehat{\psi_{R^{-1}}}|^2 dx.$$

Defining $\eta_{B_R} = R^{-n}|\widehat{\psi_{R^{-1}}}|^2$, we get

$$\frac{1}{R^n}\int |f_j\,\widehat{\psi_{R^{-1}}}|^2 dx \lesssim \sum_\theta \int |f_{j,\theta}|^2 \eta_{B_R} dx = \sum_\theta \|f_{j,\theta}\|^2_{L^2(\eta_{B_R})}.$$

Thus

$$\begin{aligned} \Big\|\prod_{j=1}^n |f_j|^{1/n}\Big\|_{L^p_{\text{avg}}(B_R)} &\lesssim R^\epsilon \prod_{j=1}^n \Big(\sum_\theta \|f_{j,\theta}\|^2_{L^2(\eta_{B_R})}\Big)^{1/(2n)} \\ &\le R^\epsilon \prod_{j=1}^n \Big(\sum_\theta \|f_{j,\theta}\|^2_{L^p(\eta_{B_R})}\Big)^{1/(2n)} \\ &= R^\epsilon \prod_{j=1}^n \|f_j\|^{1/n}_{L^{p,R^{-1}}(\eta_{B_R})} \end{aligned}$$

provided $2 \le p \le p_0$. □

Lemma 8. *If* $\operatorname{supp} g_i \subset N_{1/R}S_i$, $f_i = \widehat{g_i}$, *and* S_i *are compact positively curved transverse hypersurfaces in* $\mathbb{R}^n$, *and if* $q \ge 2n/(n-1)$, *then*

$$\text{Avg}_{B_{\sqrt{R}}\subset B_R} \prod_{i=1}^n \|f_i\|^{\frac{q}{n}}_{L^2_{\text{avg}}(B_{\sqrt{R}})} \lesssim R^\epsilon \prod_{i=1}^n \|f_i\|^{\frac{q}{n}\alpha(q)}_{L^2_{\text{avg}}(B_R)} \prod_{i=1}^n \|f_i\|^{\frac{q}{n}(1-\alpha(q))}_{L^{q,R^{-1}}_{\text{avg}}(B_R)}.$$

If $q = 2n/(n-1)$, *then* $\alpha(q) = 1$. *If* $q = 2(n+1)/(n-1)$, *then* $\alpha(q) = 1/2$. *If* $q > 2(n+1)/(n-1)$, *then* $\alpha(q) < 1/2$.

Our goal is to show that $D_{n,q}(R) \lesssim R^\epsilon$ where $s = 2(n+1)/(n-1)$, for $2 \le q \le s$. We are going to use induction on the dimension, so we assume that $D_{n-1,q}(R) \lesssim R^\epsilon$ (if $y = 2(x+1)/(x-1)$, then $y' = -4/(x-1)^2$). We know that

$D_{n,q} \approx \widetilde{D}_{n,q} \approx R^{\gamma(n,q)}$. We just have to show that $\gamma(n,q) \le \epsilon$. We want to study $\widetilde{D}_{n,q}$, and so we bring the multilinear decoupling setup into the picture. We want to get an inequality of the following form:

$$\fint_{B_R} \prod_{i=1}^{n} |f_i|^{\frac{q}{n}} dx \lesssim R^{\epsilon} \prod_{i=1}^{n} \|f_i\|_{L^{q,R^{-1}}_{\text{avg}}(B_R)}^{\frac{q}{n}}.$$

To do this, we take a large integer M and we define a sequence of scales:$r, r^2, \ldots, r^{2^M} = R$ where $r = R^{2^{-M}}$. We will use Lemma 8 at scales $r, r^2, \ldots, r^{2^M} = R$. We write $r_a = r^{2^a}$.

By Bernstein's, we have the following inequality at the small scale r:

$$\fint_{B_R} \prod_{i=1}^{n} |f_i|^{\frac{q}{n}} dx \lesssim r^C \text{Avg}_{B_r \subset B_R} \prod_{i=1}^{n} \|f_i\|_{L^2_{\text{avg}}(B_r)}^{\frac{q}{n}}.$$

In this inequality we lost a factor of

$$r^C = R^{\frac{C}{2^M}}.$$

Next, we apply Lemma 8 on each ball of radius $r^2 = r_1$. We get

$$\text{Avg}_{B_r \subset B_R} \prod_{i=1}^{n} \|f_i\|_{L^2_{\text{avg}}(B_r)}^{\frac{q}{n}} \lesssim r_1^{\epsilon} \text{Avg}_{B_{r_1} \subset B_R} \prod_{i=1}^{n} \|f_i\|_{L^2_{\text{avg}}(B_{r_1})}^{\frac{q}{n}\alpha(q)} \prod_{i=1}^{n} \|f_i\|_{L^{q,r_1^{-1}}_{\text{avg}}(B_{r_1})}^{\frac{q}{n}(1-\alpha(q))}.$$

We use Hölder's inequality to separate the terms of the form $\|f_i\|_{L^{q,r_1^{-1}}_{\text{avg}}(B_{r_1})}$ from the L^2 terms, and then we will decouple them the rest of the way in terms of $D_{n,q}(R/r_1)$.

Recall that the multilinear Hölder inequality says that if $b_j > 0$ and $\sum b_j = 1$, then $\text{Avg} \prod_j A_j \le \prod_j (\text{Avg}\, A_j)^{b_j}$. We will apply the $(n+1)$-linear Hölder inequality with exponents $\alpha(q) + \frac{1-\alpha(q)}{n} + \cdots + \frac{1-\alpha(q)}{n} = 1$ to get

$$\begin{aligned}
&\text{Avg}_{B_r \subset B_R} \prod_{i=1}^{n} \|f_i\|_{L^2_{\text{avg}}(B_r)}^{\frac{q}{n}} \\
\lesssim\ & r_1^{\epsilon} \text{Avg}_{B_{r_1} \subset B_R} \Big(\prod_{i=1}^{n} \|f_i\|_{L^2_{\text{avg}}(B_{r_1})}^{\frac{q}{n}} \Big)^{\alpha(q)} \prod_{i=1}^{n} \Big(\|f_i\|_{L^{q,r_1^{-1}}_{\text{avg}}(B_{r_1})}^{q} \Big)^{\frac{1-\alpha(q)}{n}} \\
\le\ & r_1^{\epsilon} \Big(\text{Avg}_{B_{r_1} \subset B_R} \prod_{i=1}^{n} \|f_i\|_{L^2_{\text{avg}}(B_{r_1})}^{\frac{q}{n}} \Big)^{\alpha(q)} \\
& \times \prod_{i=1}^{n} \Big(\text{Avg}_{B_{r_1} \subset B_R} \|f_i\|_{L^{q,r_1^{-1}}_{\text{avg}}(B_{r_1})}^{q} \Big)^{\frac{1-\alpha(q)}{n}}.
\end{aligned}$$

The first factor is ready to apply Lemma 8 again at the next scale. The second factor is slightly decoupled, and now we explain how to decouple the rest of the way. Using Minkowski's inequality, we can bound

$$\mathrm{Avg}_{B_{r_1}\subset B_R}\|f_i\|^q_{L^{q,r_1^{-1}}_{\mathrm{avg}}(B_{r_1})} \leq \|f_i\|^q_{L^{q,r_1^{-1}}_{\mathrm{avg}}(B_R)}.$$

This expression involves decoupling f_i into contributions from caps of size $r_1^{-1/2}$. We want to decouple f_i into finer caps of size $R^{-1/2}$. To do so, we use parabolic scaling to decouple f_i further, bringing in a factor of $D_{n,q}(R/r_1)$:

$$\|f_i\|_{L^{q,r_1^{-1}}_{\mathrm{avg}}(B_R)} \leq D_{n,q}(R/r_1)\|f_i\|_{L^{q,R^{-1}}_{\mathrm{avg}}(B_R)} \lesssim \left(\frac{R}{r_1}\right)^{\gamma}\|f_i\|_{L^{q,R^{-1}}_{\mathrm{avg}}(B_R)}.$$

All together, we see that

$$\begin{aligned}
&\mathrm{Avg}_{B_r\subset B_R}\prod_{i=1}^n \|f_i\|^{\frac{q}{n}}_{L^2_{\mathrm{avg}}(B_r)} \\
\lesssim\ & r_1^{\epsilon}\Big(\mathrm{Avg}_{B_{r_1}\subset B_R}\prod_{i=1}^n \|f_i\|^{\frac{q}{n}}_{L^2_{\mathrm{avg}}(B_{r_1})}\Big)^{\alpha(q)}\prod_{i=1}^n\Big(\Big(\frac{R}{r_1}\Big)^{q\gamma}\|f_i\|^q_{L^{q,R^{-1}}_{\mathrm{avg}}(B_R)}\Big)^{\frac{1-\alpha(q)}{n}} \\
=\ & r_1^{\epsilon}\Big(\mathrm{Avg}_{B_{r_1}\subset B_R}\prod_{i=1}^n \|f_i\|^{\frac{q}{n}}_{L^2_{\mathrm{avg}}(B_{r_1})}\Big)^{\alpha(q)} \\
& \times\Big(\frac{R}{r_1}\Big)^{q\gamma(1-\alpha(q))}\Big(\prod_{i=1}^n \|f_i\|^q_{L^{q,R^{-1}}_{\mathrm{avg}}(B_R)}\Big)^{\frac{1-\alpha(q)}{n}}.
\end{aligned}$$

Putting together the whole argument so far, we have proven that

$$\begin{aligned}
\fint_{B_R}\prod_{i=1}^n |f_i|^{\frac{q}{n}}dx \quad \lesssim\quad & r^C r_1^{\epsilon}\Big(\mathrm{Avg}_{B_{r_1}\subset B_R}\prod_{i=1}^n \|f_i\|^{\frac{q}{n}}_{L^2_{\mathrm{avg}}(B_{r_1})}\Big)^{\alpha(q)} \\
& \times\Big(\frac{R}{r_1}\Big)^{q\gamma(1-\alpha(q))}\Big(\prod_{i=1}^n \|f_i\|^q_{L^{q,R^{-1}}_{\mathrm{avg}}(B_R)}\Big)^{\frac{1-\alpha(q)}{n}}.
\end{aligned}$$

Now we can iterate this computation. Repeating the computation one more time, we get

$$\mathrm{Avg}_{B_{r_1}\subset B_R}\prod_{i=1}^n \|f_i\|^{\frac{q}{n}}_{L^2_{\mathrm{avg}}(B_{r_1})} \lesssim r_2^{\epsilon}\,\mathrm{Avg}_{B_{r_2}\subset B_R}\prod_{i=1}^n \|f_i\|^{\frac{q}{n}\alpha(q)}_{L^2_{\mathrm{avg}}(B_{r_2})}\prod_{i=1}^n \|f_i\|^{\frac{q}{n}(1-\alpha(q))}_{L^{q,r_2^{-1}}_{\mathrm{avg}}(B_{r_2})},$$

where we have used Main lemma Lemma 8. The right-hand side of this inequality is

$$= r_2^{\epsilon}\,\mathrm{Avg}_{B_{r_2}\subset B_R}\Big(\prod_{i=1}^n \|f_i\|^{\frac{q}{n}}_{L^2_{\mathrm{avg}}(B_{r_2})}\Big)^{\alpha(q)}\prod_{i=1}^n\Big(\|f_i\|^q_{L^{q,r_2^{-1}}_{\mathrm{avg}}(B_{r_2})}\Big)^{\frac{1-\alpha(q)}{n}},$$

which, by Hölder, is

$$\le r_2^{\epsilon}\Big(\mathrm{Avg}_{B_{r_2}\subset B_R}\prod_{i=1}^{n}\|f_i\|_{L^2_{\mathrm{avg}}(B_{r_2})}^{\frac{q}{n}}\Big)^{\alpha(q)}\prod_{i=1}^{n}\Big(\mathrm{Avg}_{B_{r_2}\subset B_R}\|f_i\|^q_{L^{q,r_2^{-1}}_{\mathrm{avg}}(B_{r_2})}\Big)^{\frac{1-\alpha(q)}{n}},$$

which, by Minkowski, is

$$\le r_2^{\epsilon}\Big(\mathrm{Avg}_{B_{r_2}\subset B_R}\prod_{i=1}^{n}\|f_i\|_{L^2_{\mathrm{avg}}(B_{r_2})}^{\frac{q}{n}}\Big)^{\alpha(q)}\prod_{i=1}^{n}\Big(\|f_i\|^q_{L^{q,r_2^{-1}}_{\mathrm{avg}}(B_R)}\Big)^{\frac{1-\alpha(q)}{n}}.$$

Using parabolic scaling, we obtain

$$\begin{aligned}
&\mathrm{Avg}_{B_{r_1}\subset B_R}\prod_{i=1}^{n}\|f_i\|_{L^2_{\mathrm{avg}}(B_{r_1})}^{\frac{q}{n}}\\
&\lesssim\ r_2^{\epsilon}\Big(\mathrm{Avg}_{B_{r_2}\subset B_R}\prod_{i=1}^{n}\|f_i\|_{L^2_{\mathrm{avg}}(B_{r_2})}^{\frac{q}{n}}\Big)^{\alpha(q)}\prod_{i=1}^{n}\Big(\Big(\frac{R}{r_2}\Big)^{q\gamma}\|f_i\|^q_{L^{q,R^{-1}}_{\mathrm{avg}}(B_R)}\Big)^{\frac{1-\alpha(q)}{n}}\\
&=\ r_2^{\epsilon}\Big(\mathrm{Avg}_{B_{r_2}\subset B_R}\prod_{i=1}^{n}\|f_i\|_{L^2_{\mathrm{avg}}(B_{r_2})}^{\frac{q}{n}}\Big)^{\alpha(q)}\\
&\quad\times\Big(\frac{R}{r_2}\Big)^{q\gamma(1-\alpha(q))}\Big(\prod_{i=1}^{n}\|f_i\|^q_{L^{q,R^{-1}}_{\mathrm{avg}}(B_R)}\Big)^{\frac{1-\alpha(q)}{n}}.
\end{aligned}$$

Putting together the whole argument so far, we have proven that

$$\begin{aligned}
&\fint_{B_R}\prod_{i=1}^{n}|f_i|^{\frac{q}{n}}dx\\
&\lesssim\ r^C r_1^{\epsilon} r_2^{\epsilon\alpha(q)}\Big(\mathrm{Avg}_{B_{r_2}\subset B_R}\prod_{i=1}^{n}\|f_i\|_{L^2_{\mathrm{avg}}(B_{r_2})}^{\frac{q}{n}}\Big)^{\alpha(q)^2}\Big(\frac{R}{r_2}\Big)^{q\gamma(1-\alpha(q))\alpha(q)}\\
&\quad\times\Big(\frac{R}{r_1}\Big)^{q\gamma(1-\alpha(q))}\Big(\prod_{i=1}^{n}\|f_i\|^{\frac{q}{n}}_{L^{q,R^{-1}}_{\mathrm{avg}}(B_R)}\Big)^{(1-\alpha(q))(1+\alpha(q))}\\
&\le\ r^C(c\,r_1^{\epsilon})(c\,r_2^{\epsilon})^{\alpha(q)}\dots(c\,r_{M-1}^{\epsilon})^{\alpha(q)^{M-2}}\\
&\quad\times\Big(\mathrm{Avg}_{B_{r_{M-1}}\subset B_R}\prod_{i=1}^{n}\|f_i\|_{L^2_{\mathrm{avg}}(B_{r_{M-1}})}^{\frac{q}{n}}\Big)^{\alpha(q)^{M-1}}\\
&\quad\times\Big(\frac{c\,R}{r_{M-1}}\Big)^{q\gamma(1-\alpha(q))\alpha(q)^{M-2}}\dots\Big(\frac{c\,R}{r_2}\Big)^{q\gamma(1-\alpha(q))\alpha(q)}\Big(\frac{c\,R}{r_1}\Big)^{q\gamma(1-\alpha(q))}\\
&\quad\times\Big(\prod_{i=1}^{n}\|f_i\|^{\frac{q}{n}}_{L^{q,R^{-1}}_{\mathrm{avg}}(B_R)}\Big)^{(1-\alpha(q))\big(1+\alpha(q)+\dots+\alpha(q)^{M-2}\big)},
\end{aligned}$$

where c is a constant which is greater than the constant coming from Lemma 8 and greater than the constant coming from parabolic scaling.

Applying Main lemma Lemma 8 one more time, we arrive at

$$
\begin{aligned}
&\fint_{B_R}\prod_{i=1}^{n}|f_i|^{\frac{q}{n}}dx\\
\leq\ & r^C\prod_{l=1}^{M-1}(c\,r_l^{\epsilon})^{\alpha(q)^{l-1}}\Big(c\,r_m^{\epsilon}\prod_{i=1}^{n}\|f_i\|^{\frac{q}{n}\alpha(q)}_{L^2_{\mathrm{avg}}(B_R)}\prod_{i=1}^{n}\|f_i\|^{\frac{q}{n}(1-\alpha(q))}_{L^{q,R^{-1}}_{\mathrm{avg}}(B_R)}\Big)^{\alpha(q)^{M-1}}\\
&\times\prod_{l=1}^{M-1}\Big(\frac{c\,R}{r_l}\Big)^{q\gamma(1-\alpha(q))\alpha(q)^{l-1}}\Big(\prod_{i=1}^{n}\|f_i\|^{\frac{q}{n}}_{L^{q,R^{-1}}_{\mathrm{avg}}(B_R)}\Big)^{(1-\alpha(q))\big(1+\alpha(q)+\cdots+\alpha(q)^{M-2}\big)}\\
=\ & r^C C'\Big(\prod_{l=1}^{M}r_l^{\alpha(q)^{l-1}}\Big)^{\epsilon}\left(\prod_{l=1}^{M-1}\Big(\frac{R}{r_l}\Big)^{\alpha(q)^{l-1}}\right)^{q\gamma(1-\alpha(q))}\\
&\times\Big(\prod_{i=1}^{n}\|f_i\|^{\frac{q}{n}}_{L^{q,R^{-1}}_{\mathrm{avg}}(B_R)}\Big)^{(1-\alpha(q))\big(1+\alpha(q)+\cdots+\alpha(q)^{M-2}\big)+\alpha(q)^{M-1}},
\end{aligned}
$$

where we have used the fact that

$$\|f_i\|_{L^2_{\mathrm{avg}}(B_R)}=\|f_i\|_{L^{2,R^{-1}}_{\mathrm{avg}}(B_R)}\leq\|f_i\|_{L^{q,R^{-1}}_{\mathrm{avg}}(B_R)},$$

and where

$$
\begin{aligned}
C' &= \Big(\prod_{l=1}^{M}c^{\alpha(q)^{l-1}}\Big)\Big(\prod_{l=1}^{M-1}c^{\alpha(q)^{l-1}}\Big)^{q\gamma(1-\alpha(q))}\\
&= c^{\sum_{l=1}^{M}\alpha(q)^{l-1}}\Big(c^{\sum_{l=1}^{M-1}\alpha(q)^{l-1}}\Big)^{q\gamma(1-\alpha(q))}\\
&= c^{\frac{1-\alpha(q)^M}{1-\alpha(q)}}c^{\frac{1-\alpha(q)^{M-1}}{1-\alpha(q)}q\gamma(1-\alpha(q))}\\
&\leq c^{\frac{1}{1-\alpha(q)}+q\gamma}.
\end{aligned}
$$

Now

$$
\begin{aligned}
&(1-\alpha(q))\big(1+\alpha(q)+\cdots+\alpha(q)^{M-2}\big)+\alpha(q)^{M-1}\\
&\quad=\ 1+\alpha(q)+\cdots+\alpha(q)^{M-2}-\alpha(q)-\alpha(q)^2-\cdots-\alpha(q)^{M-1}\ =\ 1,
\end{aligned}
$$

and $r_l=r^{2^l}=R^{2^l/2^M}$, so

$$
\begin{aligned}
&\fint_{B_R}\prod_{i=1}^{n}|f_i|^{\frac{q}{n}}dx\\
\leq\ & r^C c^{\frac{1}{1-\alpha(q)}+q\gamma}\Big(\prod_{l=1}^{M}R^{(2\alpha(q))^{l-1}}\Big)^{\frac{2}{2^M}\epsilon}\left(\prod_{l=1}^{M-1}R^{\big(1-\frac{2^l}{2^M}\big)\alpha(q)^{l-1}}\right)^{q\gamma(1-\alpha(q))}\\
&\times\prod_{i=1}^{n}\|f_i\|^{\frac{q}{n}}_{L^{q,R^{-1}}_{\mathrm{avg}}(B_R)},
\end{aligned}
$$

and so

$$\begin{aligned} R^{q\gamma} &\approx \widetilde{D}_{n,q}(R) \\ &\leq r^C c^{\frac{1}{1-\alpha(q)}+q\gamma}\left(R^{\sum_{l=1}^{M}(2\alpha(q))^{l-1}}\right)^{\frac{2}{2^M}\epsilon}\left(R^{\sum_{l=1}^{M-1}\left(1-\frac{2^l}{2^M}\right)\alpha(q)^{l-1}}\right)^{q\gamma(1-\alpha(q))}. \end{aligned}$$

Also,

$$\sum_{l=1}^{M}(2\alpha(q))^{l-1} = \begin{cases} \frac{1-(2\alpha(q))^M}{1-2\alpha(q)} & \text{if } \frac{1}{2} < \alpha(q) < 1, \\ M & \text{if } \alpha(q) = \frac{1}{2} \end{cases}$$

and

$$\begin{aligned} \sum_{l=1}^{M-1}\left(1-\frac{2^l}{2^M}\right)\alpha(q)^{l-1} &= \sum_{l=1}^{M-1}\alpha(q)^{l-1} - \frac{2}{2^M}\sum_{l=1}^{M-1}\left(2\alpha(q)\right)^{l-1} \\ &= \begin{cases} \frac{1-\alpha(q)^{M-1}}{1-\alpha(q)} - \frac{2}{2^M}\frac{1-(2\alpha(q))^{M-1}}{1-2\alpha(q)} & \text{if } \frac{1}{2} < \alpha(q) < 1, \\ \frac{1-\alpha(q)^{M-1}}{1-\alpha(q)} - \frac{2}{2^M}(M-1) & \text{if } \alpha(q) = \frac{1}{2}. \end{cases} \end{aligned}$$

<u>Case 1:</u> $\alpha(q) = \frac{1}{2}$. Then

$$R^{q\gamma} \lesssim R^{\frac{C}{2^M}} c^{2+q\gamma} R^{\frac{2M}{2^M}\epsilon} R^{\left(1-\alpha(q)^{M-1}\right)q\gamma - \frac{2(M-1)}{2^M}q\gamma\frac{1}{2}}$$

so that

$$R^{\frac{q\gamma}{2^{M-1}} + \frac{M-1}{2^M}q\gamma} \lesssim R^{\frac{C}{2^M}} c^{2+q\gamma} R^{\frac{2M}{2^M}\epsilon},$$

i.e.

$$R^{\frac{M+1}{2^M}q\gamma - \frac{C}{2^M} - \frac{2M}{2^M}\epsilon} \lesssim c^{2+q\gamma}.$$

Letting $R \to \infty$, we get

$$\frac{M+1}{2^M}q\gamma \leq \frac{C}{2^M} + \frac{2M}{2^M}\epsilon,$$

i.e.

$$(M+1)q\gamma \leq C + 2M\epsilon,$$

i.e.

$$\frac{M+1}{M}q\gamma \leq \frac{C}{M} + 2\epsilon.$$

Letting $M \to \infty$, we get $q\gamma \leq 2\epsilon$, so that $\gamma \leq \frac{2\epsilon}{q} \leq \epsilon$.

<u>Case 2:</u> $\frac{1}{2} < \alpha(q) < 1$. Then

$$\begin{aligned} &R^{q\gamma} \\ &\lesssim R^{\frac{C}{2^M}} c^{\frac{1}{1-\alpha(q)}+q\gamma} R^{\frac{1-(2\alpha(q))^M}{1-2\alpha(q)}\frac{2}{2^M}\epsilon} R^{\left(1-\alpha(q)^{M-1}\right)q\gamma - \frac{2}{2^M}\frac{1-(2\alpha(q))^{M-1}}{1-2\alpha(q)}q\gamma(1-\alpha(q))} \end{aligned}$$

so that

$$R^{\alpha(q)^{M-1}q\gamma+\frac{2}{2^M}\frac{1-(2\alpha(q))^{M-1}}{1-2\alpha(q)}q\gamma(1-\alpha(q))} \lesssim R^{\frac{C}{2^M}} c^{\frac{1}{1-\alpha(q)}+q\gamma} R^{\frac{1-(2\alpha(q))^M}{1-2\alpha(q)}\frac{2}{2^M}\epsilon}.$$

Letting $R \to \infty$, we get

$$\alpha(q)^{M-1}q\gamma + \frac{2}{2^M}\frac{1-(2\alpha(q))^{M-1}}{1-2\alpha(q)}q\gamma(1-\alpha(q)) \leq \frac{C}{2^M} + \frac{1-(2\alpha(q))^M}{1-2\alpha(q)}\frac{2}{2^M}\epsilon,$$

i.e.

$$\begin{aligned} & q\gamma + \frac{1}{(2\alpha(q))^{M-1}}\frac{(2\alpha(q))^{M-1}-1}{2\alpha(q)-1}q\gamma(1-\alpha(q)) \\ \leq \quad & \frac{C}{2(2\alpha(q))^{M-1}} + \frac{(2\alpha(q))^M-1}{2\alpha(q)-1}\frac{\epsilon}{(2\alpha(q))^{M-1}}. \end{aligned}$$

Letting $M \to \infty$, we get

$$q\gamma + \frac{1-\alpha(q)}{2\alpha(q)-1}q\gamma \leq \frac{2\alpha(q)}{2\alpha(q)-1}\epsilon,$$

i.e.

$$\big(2\alpha(q)-1\big)q\gamma + \big(1-\alpha(q)\big)q\gamma \leq 2\alpha(q)\epsilon,$$

i.e.

$$\alpha(q)q\gamma \leq 2\alpha(q)\epsilon,$$

i.e.

$$\gamma \leq \frac{2\epsilon}{q} \leq \epsilon.$$

What happens when $\alpha(q) = 1$? Well, $C' = C^M$,

$$\sum_{l=1}^{M}(2\alpha(q))^{l-1} = \frac{1-2^M}{1-2} = 2^M - 1,$$

and

$$\begin{aligned} \sum_{l=1}^{M-1}\Big(1-\frac{2^l}{2^M}\Big)\alpha(q))^{l-1} &= M-1-\frac{1}{2^M}\sum_{l=1}^{M-1}2^l \\ &= M-1-\frac{1}{2^M}2\frac{1-2^M}{1-2} \\ &= M-1-\frac{2^M-1}{2^{M-1}}, \end{aligned}$$

so that

$$R^{q\gamma} \lesssim R^{\frac{C}{2^M}} c^M R^{(2^M-1)\frac{2}{2^M}\epsilon} R^{\left(M-1-\frac{2^M-1}{2^{M-1}}\right)q\gamma(1-1)},$$

i.e.

$$R^{q\gamma} \lesssim R^{\frac{C}{2^M}} c^M R^{\frac{2^M-1}{2^M}(2\epsilon)}.$$

Letting $R \to \infty$, we get

$$q\gamma \leq \frac{C}{2^M} + \frac{2^M - 1}{2^M}(2\epsilon).$$

Letting $M \to \infty$, we get

$$q\gamma \leq 2\epsilon.$$

This gives

$$\gamma \leq \frac{2\epsilon}{q} \leq \epsilon,$$

as desired.

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