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# LOCAL HOLOMORPHIC EXTENSION OF CR FUNCTIONS


by
MARIAM ABBAS AL AKHDAR


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MARIAM ABBAS AL AKHDAR

Approved by:

Dr. Della Sala Giuseppe, Assistant Professor — Advisor
Mathematics

Dr. Abi Khuzam Farouk, Professor — Member of Committee
Mathematics

Dr. Bertrand Florian, Assistant Professor — Member of Committee
Mathematics

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# AN ABSTRACT OF THE THESIS OF

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This thesis involves the extension behavior of holomorphic functions in several variables, and the corresponding extension problem for CR functions defined on a real hypersurface of a complex manifold. However, the extension behavior of a holomorphic function f defined on a domain $U$ turns out to depend on the complex geometry of the boundary $\partial U$ of $U$: thus the problem can be often formulated in terms of the trace of $f$ on $\partial U$ rather than $f$ itself. Then what follows is the notion of CR functions: complex-valued functions defined on real submanifolds of $\mathbb{C}^n$. Specifically we will discuss some important extension results for CR functions, with particular emphasis on H.Lewy's classical result about the extension from strictly pseudoconvex hypersurfaces.

# Contents

# Chapter 1

# Introduction

The main subject of the present thesis is the extension behavior of holomorphic functions in several variables, and correspondingly the extension behavior of CR functions defined on a real hypersurface of a complex manifold. This is a very important questions in the field of Several Complex Variables, and is still at the center of current, active research.

Some of the fundamental differences between Complex Analysis in several variables and in one variable were discovered in the beginning of the 20th century. Poincaré was the first to observe that an analogue of the Riemann mapping theorem does not hold in $\mathbb{C}^n$, by showing that the ball and bi-disc are not biholomorphically equivalent. Another striking difference is Hartogs' phenomenon: holomorphic functions defined outside of a compact set $K$ of $\mathbb{C}^n$ extend holomorphically through $K$. Together, these properties show that the study of domains of existence of holomorphic functions in $\mathbb{C}^n$ is much more complicated than in $\mathbb{C}$. It was this study which ultimately led to the notions of domain of holomorphy and of pseudoconvexity.

The research on holomorphic functions has since uncovered various other extension phenomena, such as Bochner's tube theorem and Bogolyubov's edge-of-the-wedge theorem, some of which have relevant applications in Physics. As a general thread, the extension behavior of a holomorphic function $f$ defined on a domain $U$ turns out to depend on the complex geometry of the boundary $\partial U$ of $U$: the problem can be thus often formulated in terms of the trace on $\partial U$ rather than $f$ itself.

This fact motivated the introduction of CR functions: complex-valued functions defined on real submanifolds of $\mathbb{C}^n$, solving a system of PDE's which generalizes the usual Cauchy-Riemann equation. These functions have become an important topic in their own right, lying at the intersection between Complex Analysis and PDE. In the thesis we will review some of the most significant extension results for CR functions: mostly, we will focus on H.Lewy's classical result about the extension from strictly pseudoconvex hypersurfaces.

The organization of the thesis is as follows. In the first section we review some of the main notions of complex analysis in several variables: the definition of holomorphy, Cauchy theorem, and the Cauchy integral formula. In the second section we discuss some of the main differences between one complex variable and several complex variables: in particular we introduce the notion of the domain of holomorphy and discuss Hartogs theorem. In the third section we give a very brief overview of CR geometry by introducing real hypersurfaces in $\mathbb{C}^n$, the Levi form, and defining CR functions means of suitable vector fields. Then in the fourth section we go over the holomorphic extension of CR functions. Specifically we discuss the fundamental result proved by Hans Lewy[4] and we provide a more or less detailed proof of this result in the case of the *Lewy hypersurface* $M = \{\Im w = |z|^2\} \subset \mathbb{C}^2$. first we considered a compact domain where the CR function $f$ will First we extend the CR function $f$ on suitable compact one dimensional slices of $M$ to a function $F$ which is holomorphic w.r.t $z$ and then we show that $F$ is a holomorphic function in $\{\Im w > |z|^2\}$ and $F \restriction_M = f$. In the fifth section we expose subsequent developments on the extension theorem 5.1 both for the local version and for the global one. Then we ended our discussion by referring to the most far reaching generalizations of the one-sided local extension results of CR functions ,obtained by Treprean [8] and Tumanov [9].

# Chapter 2

# Complex analysis in several variables

Let $\mathbb{C}^n$ denote the complex Euclidean space. We denote by $z = (z_1, z_2, ..., z_n)$ the coordinates of $\mathbb{C}^n$. Let $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$ denote the coordinates in $\mathbb{R}^n$. We identify $\mathbb{C}^n$ with $\mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$ by letting $z = x + iy$. Just as in one complex variable we write $\bar{z} = x - iy$. We call $z$ the holomorphic coordinates and $\bar{z}$ the antiholomorphic coordinates.
We will use the Euclidean inner product on $\mathbb{C}^n$

$$\langle z, w \rangle = z_1\overline{w}_1 + \cdots + z_n\overline{w}_n$$

and the corresponding Euclidean norm on $\mathbb{C}^n$

$$\|z\| = \sqrt{\langle z, z \rangle} = \sqrt{(\ |z_1|^2 + \cdots + |z_n|^2)}\ .$$

In the following we will outline the basic notions of the function theory of several complex variables.

## 2.1 Basic Definitions

In this section we will give some alternative definitions of holomorphic function in several complex variables which extend the ones that are given in one complex variable. As in the case of one complex variable these definitions turn out to be equivalent.

**Definition 2.1** *Let $U \subset \mathbb{C}^n$ be an open set, and let $f : U \longrightarrow \mathbb{C}$ be a function of class $C^1$. Suppose that $f = u + iv$ satisfies the Cauchy-Riemann equations*

$$\begin{cases} \frac{\partial u}{\partial x_j} = \frac{\partial v}{\partial y_j} \\ \frac{\partial u}{\partial y_j} = -\frac{\partial v}{\partial x_j} \ \textit{for } j = 1, 2, ..., n. \end{cases}$$

*We then say that f is holomorphic.*

The following is another way to define a holomorphic function.

**Definition 2.2** *Let $U \subset \mathbb{C}^n$ be a domain. A function $f : U \longrightarrow \mathbb{C}$ is called $\mathbb{C}$-differentiable at $z^0 \in U$ if there exist $l_1, \cdots, l_n \in \mathbb{C}$ such that*
$|f(z) - f(z^0) - \sum_{j=1}^n l_j(z_j - \ z_j^0\ )| = O_2(|z - z^0|).$

If a function $f$ is $\mathbb{C}$-differentiable on $z^0$, then, for every $j = 1, \cdots, n$, we have

$$l_j = \frac{\partial f}{\partial z_j}(z^0),$$

where

$$\frac{\partial f}{\partial z_j}(z^0) = \frac{1}{2}(\ \frac{\partial f}{\partial x_j} - i\frac{\partial f}{\partial y_j})\ (z^0) = \lim_{h \to 0} \frac{f(z_1^0, \ldots, z_j^0 + h, \ldots, z_n^0) - f(z^0)}{h}.$$

$\partial f / \partial z_j(z^0)$ is called the partial derivative of $f$ at $z^0$.

**Definition 2.3** *For $r = (r_1, \ldots, r_n)$ where $r_j > 0$ and $a \in \mathbb{C}^n$, define a polydisc*

$$P(a, r_1, \ldots, r_n) = \{z \in \mathbb{C}^n : |z_j - a_j| < r_j\}.$$

*We call $a$ the center and $r$ the polyradius or simply the radius of the polydisc $P$. If $r_1 = r_2 = \cdots = r > 0$ then we write*

$$P(a, r) = \{z \in \mathbb{C}^n : |z_j - a_j| < r\}.$$

*If $D$ denotes the unit polydisc in one complex variable, then the polydisc can be seen as a product of $D$ :*

$$D^n = D \times D \times \cdots \times D = P(0, 1) = \{z \in \mathbb{C}^n : |z_j| < 1\}.$$

As in one variable we define the Wirtinger operators

$$\frac{\partial}{\partial z_j} = \frac{1}{2}(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j})$$

$$\frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}).$$

The expression of the chain rule in terms of these operators will be useful later; the proof can be achieved by a straightforward computation.

**Lemma 2.1** *(Chain rule for complex functions) Let $f = f(z)$ and let $g = g(z)$ be two complex valued functions which are differentiable in the real sense, $h(z) = g(f(z))$. Then we have:*

$$\frac{\partial h}{\partial z} = \frac{\partial g}{\partial z} \cdot \frac{\partial f}{\partial z} + \frac{\partial g}{\partial \overline{z}} \cdot \frac{\partial \overline{f}}{\partial z}$$

$$\frac{\partial h}{\partial \overline{z}} = \frac{\partial g}{\partial z} \cdot \frac{\partial f}{\partial \overline{z}} + \frac{\partial g}{\partial \overline{z}} \cdot \frac{\partial \overline{f}}{\partial \overline{z}}$$

Using these operators the Cauchy Riemann condition in Definition 2.1 can be written in a more compact way as

$$\frac{\partial f}{\partial \bar{z}_j} = 0 \text{ for } j = 1, \cdots, n.$$

Indeed,

$$\frac{\partial f}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial f}{\partial x_j} + i\frac{\partial f}{\partial y_j}\right) = \frac{1}{2}\left(\frac{\partial u}{\partial x_j} + i\frac{\partial v}{\partial x_j} + i\frac{\partial u}{\partial y_j} - \frac{\partial v}{\partial y_j}\right) = \frac{1}{2}\left(\frac{\partial u}{\partial x_j} - \frac{\partial v}{\partial y_j} + i\left(\frac{\partial v}{\partial x_j} + \frac{\partial u}{\partial y_j}\right)\right) = 0$$

$$\Leftrightarrow \begin{cases} \frac{\partial u}{\partial x_j} - \frac{\partial v}{\partial y_j} = 0 \\ \frac{\partial v}{\partial x_j} + \frac{\partial u}{\partial y_j} = 0 \end{cases} \Leftrightarrow \begin{cases} \frac{\partial u}{\partial x_j} = \frac{\partial v}{\partial y_j} \\ \frac{\partial u}{\partial y_j} = -\frac{\partial v}{\partial x_j} \end{cases}.$$

**Remark 2.1** *A function $f$ is $\mathbb{C}$-differentiable at $z^0$ if and only if it is $\mathbb{R}$-differentiable at $z^0$ and the Cauchy-Riemann conditions are fulfilled.*

The proofs of this fact is identical to the case $n = 1$.

## 2.2 Cauchy Formula

In this subsection we will highlight how some basic theorems in one complex variable generalize to several complex variables. We will start by the Cauchy theorem which is an important statement about line integrals for holomorphic functions in the complex plane. Essentially, it says that if two different paths connect the same two points, and a function is holomorphic everywhere in between the two paths, then the two path integrals of the function will be the same. Beside its importance in practical applications this theorem is a cornerstone in the development of complex analysis in one variable. Similarly, its generalization to several variables provides the foundation for the proof of many of the essential features of holomorphic functions.

**Theorem 2.1** *(Cauchy theorem in one variable)*

*Let $D \subseteq \mathbb{C}$ domain, $K \subset D$ compact, and $f \in \mathcal{O}(D)$. Then*

$$\int_{\partial K} f(z)dz = 0.$$

**Theorem 2.2** *(Cauchy theorem in several variables)*
*Let $D \subseteq \mathbb{C}^n$ domain, $K = K_1 \times \cdots \times K_n \subset D$ compact, and $f \in \mathcal{O}(D)$. Then*

$$\int_{\partial K_1} \int_{\partial K_2} ... \int_{\partial K_n} f(z_1, z_2, ..., z_n) dz_n \cdots dz_2 dz_1 = 0.$$

Proof. Consider $z = (z_1, \cdots, z_n)$ where $z_j = x_j + iy_j$, $j = 1, \cdots, n$ and $f(z)$ to be $f(z) = u(z) + iv(z)$ ,then ,

$$\int_{\partial K_1} \int_{\partial K_2} \cdots \int_{\partial K_n} f(z_1, z_2, \cdots, z_n) dz_n \cdots dz_2 dz_1 = \int_{\partial K_1} \int_{\partial K_2} \cdots \int_{\partial K_n} (u(z)+iv(z))(dx_n+idy_n) dz_{n-1} \cdots dz_2 dz_1 =$$

$$\int_{\partial K_1} \int_{\partial K_2} \cdots \int_{\partial K_n} (u(z)dx_n + iv(z)dx_n + iu(z)dy_n - v(z)dy) dz_{n-1} \cdots dz_2 dz_1 =$$

$$\int_{\partial K_1} \int_{\partial K_2} \cdots \int_{\partial K_n} (u(z)dx_n - v(z)dy_n) \cdots dz_2 dz_1 + i \int_{\partial K_1} \int_{\partial K_2} \cdots \int_{\partial K_n} (v(z)dx_n + u(z)dy_n) dz_{n-1} \cdots dz_2 dz_1 =$$

$$\int_{\partial K_1} \int_{\partial K_2} \cdots \int_{K_n} (-\frac{\partial v(z)}{\partial x} - \frac{\partial u(z)}{\partial y}) d(x_n, y_n) \cdots dz_2 dz_1 + i \int_{\partial K_1} \int_{\partial K_2} \cdots \int_{K_n} (\frac{\partial u(z)}{\partial x} - \frac{\partial v(z)}{\partial y}) d(x_n, y_n) dz_{n-1} \cdots dz_2 dz_1 =$$

$$0 + i0 = 0$$

where in the forth line we have applied Green's theorem to $K_n$ and $d(x_n, y_n)$ is the area element in the $x_n y_n$ plane.

This is true since $f$ is holomorphic, then it satisfies the Cauchy-Riemann equations.□

As a consequence of Cauchy theorem one can derive the Cauchy integral formula which expresses the fact that a holomorphic function defined on a disk is completely determined by its values on the boundary of the disk. Using the cauchy formula one can show that holomorphic function has derivatives of all orders and is analytic (i.e. it can be represented by a power series).

We start by stating the Cauchy integral formula in one variable:

**Theorem 2.3** *(Cauchy integral formula in one variable)*
*Let $D$ be a disc in $\mathbb{C}$. Suppose $f : D \longrightarrow \mathbb{C}$ is a continous function holomorphic in $D$. Then for $z_0 \in D$*

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)}{z - z_0} dz.$$

The generalization to several variables can be given as follows:

**Theorem 2.4** *(Cauchy integral formula in several variables)*
*Let $P$ be a polydisc centered at $a \in \mathbb{C}^n$. Suppose $f : P \longrightarrow \mathbb{C}$ is a continuous function holomorphic in $P$. Write $\Gamma = \partial P_1 \times \partial P_2 \times \cdots \times \partial P_n$. Then for $z \in P$*

$$f(z) = \frac{1}{(2\pi i)^n} \int_{\Gamma} \frac{f(\zeta_1, \zeta_2, \cdots, \zeta_n)}{(\zeta_1 - z_1)(\zeta_2 - z_2) \cdots (\zeta_n - z_n)} d\zeta_n \cdots d\zeta_2 d\zeta_1$$

$$= \frac{1}{(2\pi i)^n} \int_{\partial P_1} \int_{\partial P_2} \cdots \int_{\partial P_n} \frac{f(\zeta_1, \zeta_2, \cdots, \zeta_n)}{(\zeta_1 - z_1)(\zeta_2 - z_2) \cdots (\zeta_n - z_n)} d\zeta_n \cdots d\zeta_2 d\zeta_1.$$

Proof. This can be obtained by an iterated application of the Cauchy formula in one variable

$$\begin{aligned}
&\int_{\partial P_1}\int_{\partial P_2}\cdots\int_{\partial P_n}\frac{f(\zeta_1,\cdots,\zeta_n)}{(\zeta_1-z_1)(\zeta_2-z_2)\cdots(\zeta_n-z_n)}d\zeta_n\cdots d\zeta_2 d\zeta_1\\
=&\int_{\partial P_1}\int_{\partial P_2}\cdots\int_{\partial P_{n-1}}\frac{1}{(\zeta_1-z_1)\cdots(\zeta_{n-1}-z_{n-1})}\int_{\partial P_n}\frac{f(\zeta_1,\cdots,\zeta_n)}{(\zeta_n-z_n)}d\zeta_n\cdots d\zeta_2 d\zeta_1\\
=&\int_{\partial P_1}\int_{\partial P_2}\cdots\int_{\partial P_{n-1}}\frac{2\pi i f(\zeta_1,\cdots,\zeta_{n-1},z_n)}{(\zeta_1-z_1)\cdots(\zeta_{n-1}-z_{n-1})}d\zeta_{n-1}\cdots d\zeta_2 d\zeta_1\\
=&\int_{\partial P_1}\int_{\partial P_2}\cdots\int_{\partial P_{n-2}}\frac{(2\pi i)^2 f(\zeta_1,\cdots,\zeta_{n-2},z_{n-1},z_n)}{(\zeta_1-z_1)\cdots(\zeta_{n-2}-z_{n-2})}d\zeta_{n-2}\cdots d\zeta_2 d\zeta_1\\
=&\cdots\\
=&(2\pi i)^n f(z_1,z_2,\cdots,z_n).\square
\end{aligned}$$

A similar formula holds for the derivatives of $f(z)$, and can be proved by differentiating the Cauchy formula inside the integral.

**Theorem 2.5** *Let $P$ be a polydisc centered at $a \in \mathbb{C}^n$. Suppose $f : P \longrightarrow \mathbb{C}$ is a continuous function holomorphic in $P$. Write $\Gamma = \partial P_1 \times \partial P_2 \times \cdots \times \partial P_n$. Then for $z \in P$ and $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$ we have*

$$\begin{aligned}
f^{(\alpha)}(z) &= \frac{\alpha!}{(2\pi i)^n}\int_\Gamma \frac{f(\zeta_1,\zeta_2,\cdots,\zeta_n)}{(\zeta_1-z_1)^{\alpha_1+1}(\zeta_2-z_2)^{\alpha_2+1}\cdots(\zeta_n-z_n)^{\alpha_n+1}}d\zeta_n\cdots d\zeta_2 d\zeta_1\\
&= \frac{\alpha!}{(2\pi i)^n}\int_{\partial P_1}\int_{\partial P_2}\cdots\int_{\partial P_n}\frac{f(\zeta_1,\zeta_2,\cdots,\zeta_n)}{(\zeta_1-z_1)^{\alpha_1+1}(\zeta_2-z_2)^{\alpha_2+1}\cdots(\zeta_n-z_n)^{\alpha_n+1}}d\zeta_n\cdots d\zeta_2 d\zeta_1
\end{aligned}$$

*where $\alpha! = \alpha_1!\alpha_2!\cdots\alpha_n!$ and $f^{(\alpha)} = \frac{\partial^{|\alpha|}f}{\partial z_1^{\alpha_1}\cdots\partial z_n^{\alpha_n}}$.*

The following result shows that a holomorphic function is analytic.

**Theorem 2.6** *Let $f$ be holomorphic in a domain $\Omega \subset \mathbb{C}^n$ and let $a \in \Omega$. Then, $f$ can be expanded in an absolutely convergent power series:*

$$f(z) = \sum_{\alpha\in\mathbb{N}^n} c_\alpha (z-a)^\alpha,$$

*in a neighborhood of $a$. The coefficients $c_\alpha$ are given by*

$$c_\alpha = \frac{f^{(\alpha)}(z)}{\alpha!}.$$

*The representation of $f$ as the sum of its Taylor series is valid in any polydisc centered at $a$ and contained in $\Omega$.*

The proof is similar to the case $n = 1$.
As a consequence of analyticity we have the following unique continuation result.

**Theorem 2.7** *(Identity theorem)*
*Given functions $f$ and $g$ holomorphic on a connected open set $D$, if $f = g$ on some non-empty open subset of $D$, then $f = g$ on $D$.*

Proof.The connectedness assumption on the domain $D$ is necessary. Under this assumption, since we are given that the set is not empty, we can topologically prove that the set $S \in D$ where $f$ and $g$ coincide is both open and closed.
The closedness is immediate from the continuity of $f$ and $g$: Let $S = \{z \in \mathbb{C} : f(z) = g(z)\}$ and $h(z) = f(z)-g(z)$ a continuous function. Since $S$ is the zero set of $h(z)$ a continuous function, $S$ is closed.
Now we want to prove that $S$ is open. Both $f$ and $g$ are holomorphic, then $h$ is holomorphic. So it is analytic, i.e. $\forall p \in D\ \exists U$ a neighborhood of $p$ such that $h\restriction_U =$ sum of the power series centered at $p$. Let $q \in \partial$(interior of

$S$), then $h(q) = 0 \implies$ all the derivatives of $h$ will be zero $\implies$ the Taylor series of $h$ at $q$ will be zero. Then, $q \in$ ( interior of $S$). Therefore, $S$ is open.

Then $f(z) = g(z)$ on the set $S \in D$ which is both open and closed.
But, as $D$ is connected, the only closed and open subset at once is $D$ itself.
Thus $f(z) = g(z)$ for $z \in D$.$\square$

# Chapter 3

# Domain of holomorphy and Hartogs theorem

One might be tempted to think of the analysis of several complex variables as being essentially one variable theory with the additional complications of multi-indices. However, this view turns out to be incorrect. So in this section we will discuss several aspects in which one complex variable and several complex variables are different.
One example is the Riemann mapping theorem, that is any two simply connected regions in $\mathbb{C}$ can be mapped conformally onto each other. The corresponding result is not true in $\mathbb{C}^n$: indeed, domains in $\mathbb{C}^n$ can behave differently to each other in several ways. One of the ways to distinguish domains in $\mathbb{C}^n$ is given by the extension properties of holomorphic functions. In order to discuss this topic in a precise way we first need a few definitions.

**Definition 3.1** *(Analytic completion) Let $D \subseteq \mathbb{C}^n$ be a domain and let $\Delta$ be a subdomain of $D$. A domain $\widehat{\Delta} \subseteq \mathbb{C}^n$ is called an $\mathcal{O}(D)$-analytic completion of $\Delta$,or ,simply ,an analytic completion of $\Delta$, if:*

*(i)$\Delta \subseteq \widehat{\Delta}$*
*(ii) for every $f \in \mathcal{O}(D)$, $f|_\Delta$ extends holomorphically on $\widehat{\Delta}$*

**Definition 3.2** *(Domain of holomorphy)*
*A domain $D \subseteq \mathbb{C}^n$ is said to be a domain of holomorphy if, for every $\Delta \subseteq D$, all analytic completions $\widehat{\Delta}$ of $\Delta$ are contained in $D$.*

**Definition 3.3** *(Domain of existence)*
*A domain $D \subseteq \mathbb{C}^n$ is said to be a domain of existance of $f \in \mathcal{O}(D)$ if $D$ is an $f$-domain of holomorphy.*

The Hartogs extension theorem is one of the crucial results of the theory. It highlights that, differently from what happens in $\mathbb{C}$, for $n \geq 2$, $\mathbb{C}^n$, contains domains $D$ with the property that every $f \in \mathcal{O}(D)$ extends holomorphically on a larger domain $\hat{D} \supset D$. This phenomenon, traditionally called Hartogs phenomenon, is at the core of the fundamental notion of domain of holomorphy.

**Theorem 3.1** *(Hartogs theorem) Let $D \subseteq \mathbb{C}^n, n > 1$, be a domain, $K \subset D$ a compact subset such that $D \setminus K$ is connected. Then the restriction homomorphism*

$$\mathcal{O}(D) \xrightarrow{\text{res}} \mathcal{O}(D \setminus K)$$

*is an isomorphism.*

**Remark 3.1** *In the theorem the assumption $n > 1$ is essential: the restriction map $\theta$ need not to be surjective as the following example shows.*
*Consider the function $f : D\backslash K \longrightarrow \mathbb{C}$ given by $f(z) = \frac{1}{z}$ where $D = \{z \in \mathbb{C} : |z| < 1\}$ and $K = \{z \in \mathbb{C} : |z| \leq \frac{1}{2}\}$, then clearly $f$ can not extent holomorphicaly to all of $D$. In fact in a similar way one can show that the restriction map $\theta$ is* never *surjective.*

Denote the restriction map by $\theta : \mathcal{O}(D) \longrightarrow \mathcal{O}(D \setminus K)$.It is easy to show that $\theta$ is injective:
$Ker(\theta) = \{f \in \mathcal{O}(D) : \theta(f) = 0 \text{ i.e. } f \restriction_{D\setminus K} = 0\} = \{0\}$
Indeed, using the identity theorem 2.7, since $f = 0$ on a connected open subset $D \setminus K \subset D$ we have $f = 0$ on $D$, hence $Ker(\theta) = \{0\}$ and therefore $\theta$ is an injective map.
The proof that $\theta$ is surjective is much more involved and requires tools which go beyond the scope of our discussion. However, some particular cases can be treated as a direct application of Cauchy's integral formula 2.4. An especially notable instance is the following:

**Theorem 3.2** *(Hartogs figures for $n = 2$)*
*Let $a, b, c \in (0, 1), b < c$,*

$$D_1 = \{(z_1, z_2) \in \mathbb{C}^2 : a < |z_1| < 1, 0 < |z_2| < 1\},$$

$$D_2 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1, b < |z_2| < c\}$$

*and $D = D_1 \cup D_2$. Then every holomorphic function $f : D \longrightarrow \mathbb{C}$ extends holomorphically(and in a unique way) on the bidisc $P = P(0; 1)$.*

Proof. Let $r \in (a, 1)$,

$$P_r = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < r, |z_2| < 1\}$$

and let $\widetilde{f_r} : P_r \longrightarrow \mathbb{C}$ be defined by

$$\widetilde{f_r}(z_1, z_2) = \frac{1}{2\pi i} \int_{|\zeta_1|=r} \frac{f(\zeta_1, z_2)}{\zeta_1 - z_1} d\zeta_1.$$

$\widetilde{f_r}$ is continuous on $p_r$ and holomorphic with respect to each variable $z_1$ and $z_2$ (since $\frac{\partial \widetilde{f_r}}{\partial \bar{z_1}}(z_1, z_2) = \frac{\partial \widetilde{f_r}}{\partial \bar{z_2}}(z_1, z_2)$), therefore holomorphic. For every fixed $z_2$ with $z_2 \in (b, c), f(z_1, z_2)$ is holomorphic with respect to $z_1$ in the disc

$$\{z_1 \in \mathbb{C} : |z_1| < r\},$$

And

$$\widetilde{f_r}(z_1, z_2) = \frac{1}{2\pi i} \int_{|\zeta_1|=r} \frac{f(\zeta_1)}{\zeta_1 - z_1} d\zeta_1.$$

Therefore,$\widetilde{f_r}$ is the Cauchy representation of $f$. From the identity theorem $f$ and $\widetilde{f_r}$ coincide on $D_r = D \cap \{z_1 \in \mathbb{C} : |z_1| < r\}$. The function $\widetilde{f}$ defined as $\widetilde{f} = \widetilde{f_r}$ on $P_r$ and $\widetilde{f} = f$ on $D$ is the required holomorphic extension of $f$.$\square$

**Remark 3.2** *By using Theorem 3.2 (and its straightforward generalization to n variables) it is relatively easy to give a proof of theorem 3.1 in the case where $D = \mathbb{C}^n$. Indeed, one can always "fit" a compact set $K \subset \mathbb{C}^n$ inside a Hartog figure of the kind studied in theorem 3.2. However, the proof for a general domain $D$ is considered more difficult.*

# Chapter 4

# Hypersurfaces and CR functions

## 4.1 Hypersurfaces

A deeper study of the extension problem considered in the previous section shows that the behavior of holomorphic functions defined on a domain $D \subset \mathbb{C}^n$ is linked to the geometry of $\partial D$ which is typically a real hypersurface of $\mathbb{C}^n$. That is why the study of holomorphic functions in several Complex Variables leads naturally to the study of the Geometry of real hypersurfaces. We are going to review some of the main concepts involved in this study: to this aim we will need to give some definitions.

**Definition 4.1** *Let $M$ be a subset of $\mathbb{R}^n$ such that for every point $p \in M$ there exists a neighbourhood $U_p$ of $p$ in $\mathbb{R}^n$ and a continuously differentiable function $\rho : U \longrightarrow \mathbb{R}$ with $\nabla\rho \neq 0$ on $U$, such that*

$$M \cap U = \{x \in U | \rho(x) = 0\}.$$

*Then $M$ is called a real hypersurface of class $C^1$.*

**Example 4.1** *The unit sphere in $\mathbb{C}^2$ defined as $S = \{|z|^2 + |w|^2 = 1\}$, where $z = x + iy, w = u + iv$ is a real hypersurface of $\mathbb{C}^2 \cong \mathbb{R}^4$. We can write the equation of $S$ in terms of the real coordinates as $\rho = x^2+y^2+u^2+v^2-1$, since $\rho = 0 \implies x^2 + y^2 + u^2 + v^2 = 1 \implies |z|^2 + |w|^2 = 1$. Note that the gradient of $\rho$ is $\nabla\rho = (2x, 2y, 2u, 2v)$ hence it does not vanish on $S$.*

As it is well known if the hypersurface $M \subset \mathbb{R}^n$ is defined as $\{\rho = 0\}$, its convexity properties are encoded into the Hessian matrix of $\rho$ whose entries are the derivatives of $\rho$ of order two. However, the (usual) convexity is not an invariant property under holomorphic transformations and therefore it is not a natural notion in complex analysis. An alternative concept of convexity was introduced by E.E. Levi in [3]. As it turns out in order to define an invariant notion of convexity one needs to consider only the (1,1) part of the second order expansion of $\rho$. This is what motivates the following definition.

**Definition 4.2** *Let $\rho : \mathbb{C}^n \longrightarrow \mathbb{R}$ be of class $C^2$ and fix $z^0 \in \mathbb{C}^n$. The Hermitian form*

$$L(\rho; z^0)(\zeta) = \sum_{\alpha,\beta=1}^{n} \frac{\partial^2\rho}{\partial z_\alpha \partial \bar{z_\beta}}(z^0)\zeta_\alpha\bar{\zeta_\beta}, \quad \zeta = (\zeta_1, ..., \zeta_n) \in \mathbb{C}^n,$$

*is called the Levi form of $\rho$ at $z^0$.*

For convenience we will now recall some facts about Hermitian forms on a complex vector space.

**Definition 4.3** *A Hermitian form on a vector space $V$ over $\mathbb{C}$ , is a function $\langle .,. \rangle : V \times V \longrightarrow \mathbb{C}$ satisfying the following properties:*

*1-* $\langle v, w\rangle = \overline{\langle w, v\rangle}$

*2-* $\langle \alpha v_1 + \beta v_2, w\rangle = \alpha\langle v_1, w\rangle + \beta\langle v_2, w\rangle \quad \forall \alpha, \beta \in \mathbb{C}$

*3-* $\langle v, \alpha w_1 + \beta w_2\rangle = \overline{\alpha}\langle v, w_1\rangle + \overline{\beta}\langle v, w_2\rangle \quad \forall \alpha, \beta \in \mathbb{C}$ *(note that* 3 *is a consequence of* 1 *and* 2*).*

*It is positive definite if it satisfies the following properties:*

*(i)* $\langle v, v\rangle \geq 0$

*(ii)* $\langle v, v\rangle = 0 \iff v = 0$

**Example 4.2** *In* $\mathbb{C}^2$ *the standard Hermitian product*

$$\langle v, w\rangle = z_1\overline{w}_1 + z_2\overline{w}_2,$$

*where* $v = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ *and* $w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$*, defines a positive definite Hermitian form.*

**Remark 4.1** *The Levi form of* $\rho$ *is by definition a Hermition form but needs not satisfy the positive definite property. If* $L(\rho; z)$ *is positive definite for any* $z \in \mathbb{C}^n$ *then* $\rho$ *is called* strictly plurisubharmonic.

In much the same way scalar products on $\mathbb{R}^n$ are represented by real symmetric matrices, Hermitian forms on $\mathbb{C}^n$ can be represented by matrices of the following kind:

**Definition 4.4** *A Hermitian matrix* $A$ *is a matrix of complex entries such that:*

$$A = \overline{A^t}$$

*It is well known that a Hermitian matrix satisfies the following properties:*

*1-Its diagonal entries are real (by definition).*

*2-Its eigenvalues are all real.*

*3- It is diagonalizable.*

**Remark 4.2** *The link between Hermitian forms and the corresponding matrices is given as follows:*

*For any Hermitian form* $L(v, w)$ *on* $\mathbb{C}^n$ *write*

$$L(v, w) = \sum_{i,j=1}^{n} a_{ij} z_i \overline{w}_j.$$

*Then the matrix* $A = (a_{ij})$ *is Hermitian and we say that* $A$ represents *the form* $L$*. Vice versa, given any Hermitian matrix* $A$ *the expression above defines a Hermitian form on* $\mathbb{C}^n$*.*

*The standard Hermitian product on* $\mathbb{C}^n$ *is positive definite and is defined by:*

$$L(u, v) = \langle Av, u\rangle = \sum_{i,j} a_{ij} z_i \overline{w}_j$$

*Indeed for any complex matrix* $A$*:*

$$\langle Av, u\rangle = \langle v, \overline{A^t}u\rangle$$

*But if* $A$ *is a Hermitian matrix then:*

$$\langle Av, u\rangle = \langle v, \overline{A^t}u\rangle = \langle v, Au\rangle$$

*Then,*

$$L(v, u) = \langle Av, u\rangle = \langle v, \overline{A^t}u\rangle = \langle v, Au\rangle = \overline{\langle Au, v\rangle} = \overline{L(u, v)}.$$

Let now $\rho : \mathbb{C}^n \longrightarrow \mathbb{R}$ be a function and fix $z^0 \in \mathbb{C}^n$ such that $\rho(z^0) = 0$. The order two Taylor expansion of $\rho$ about $z^0$ can be written in terms of the complex coordinates as

$$\rho(z) = \sum_{\alpha=1}^{n} \rho_{z_\alpha}(z^0) z_\alpha + \sum_{\alpha=1}^{n} \rho_{\bar{z}_\alpha}(z^0)\bar{z}_\alpha + \tag{4.1}$$

$$+\frac{1}{2}\sum_{\alpha,\beta=1}^{n} \rho_{z_\alpha z_\beta}(z^0) z_\alpha z_\beta + \frac{1}{2}\sum_{\alpha,\beta=1}^{n} \rho_{\bar{z}_\alpha \bar{z}_\beta}(z^0)\bar{z}_\alpha \bar{z}_\beta + \sum_{\alpha,\beta=1}^{n} \rho_{z_\alpha \bar{z}_\beta}(z^0) z_\alpha \bar{z}_\beta + O_3(z).$$

If $\rho$ is a regular defining function for a real hypersurface $M = \{\rho = 0\}$, $z^0 \in M$, then the linear part of the expansion $\ell = \sum_{\alpha=1}^{n} \rho_{z_\alpha}(z^0) z_\alpha + \sum_{\alpha=1}^{n} \rho_{\bar{z}_\alpha}(z^0)\bar{z}_\alpha$ is not vanishing.
This fact allows to give the following definitions:

**Definition 4.5** *Let $M$ be a real hypersurface defined as $M = \{\rho = 0\}$, $z^0 \in M$.*
*The zero set of the linear part $\ell$ of the expansion of $\rho$ about $z^0$ is a real hyperplane of $\mathbb{C}^n$, called the tangent hyperplane of $M$ at $z^0$ and denoted as $T_{z^0}(M)$.*

**Definition 4.6** *Let $M, \rho$ be as in definition 4.5 and let $l : \mathbb{C}^n \longrightarrow \mathbb{C}$ be the complex linear function such that $\Re(l) = \ell$. (Note that $l = \frac{1}{2}\sum_{\alpha=1}^{n} \rho_{z_\alpha}(z^0) z_\alpha$)*
*Then the zero set of $l$ is a complex hyperplane of $\mathbb{C}^n$, called the complex tangent space of $M$ at $z^0$ and denoted by $H_{z^0}(M)$.*

If $D \subset \mathbb{C}^n$ is defined as $D = \{\rho < 0\}$ with $\rho : \mathbb{C}^n \longrightarrow \mathbb{R}$ a function of class $C^k$ such that $\nabla\rho \neq 0$ on $\partial D = \{\rho = 0\}$ then we say that $D$ is a *domain of class $C^k$ of $\mathbb{C}^n$*. In this case the boundary $\partial D$ is a real hypersurface of class $C^k$.
The notion of convexity introduced by E.E. Levi can now be defined as follows:

**Definition 4.7** *Let $D \subset \mathbb{C}^n, n > 1$ be a domain as before and let $z^0 \in \partial D$. We say that $D$ is Levi-convex at $z^0$ if the restriction of the Levi form of $\rho$ to $H_{z^0}(\partial D)$*

$$L(\rho; z^0)(\zeta) = \sum_{\alpha,\beta=1}^{n} \frac{\partial^2 \rho}{\partial z_\alpha \partial \bar{z_\beta}}(z^0)\zeta_\alpha \bar{\zeta_\beta}, \quad \zeta = (\zeta_1, ..., \zeta_n) \in H_{z^0}(\partial D),$$

*has non-negative eigenvalues. If the eigenvalues are strictly positive then we say that $D$ is strictly Levi convex at $z^0$.*

In other words, the notion of Levi convexity depends on the (1,1) part $\sum_{\alpha,\beta=1}^{n} \rho_{z_\alpha \bar{z}_\beta}(z^0) z_\alpha \bar{z}_\beta$ of the Taylor expansion (4.1). As it turns out this part is the (1,1) enjoining some invariant properties with respect to holomorphic changes of coordinates. In order to work with such coordinate changes it would be useful to recall the holomorphic version of the inverse mapping theorem.

**Theorem 4.1** *(Inverse mapping theorem for holomorphic functions)*
*Let $f : \mathbb{C}^n \to \mathbb{C}^n$ be a holomorphic function and let $z \in \mathbb{C}^n$ be such that $J_f(z)$ is invertible, where $J_f(z) = (\frac{\partial f_i}{\partial z_j})_{i,j=1,...,n}$ is the* complex Jacobin *of $f$ at $z$. Then there exists a local inverse of $f$ at $z$, that is, there are neighborhoods $U$ of $z$ and $V = f(U)$ of $f(z)$ such that $f : U \to V$ is bijective and the map $f^{-1} : V \to U$ giving the local inverse to $f$ is holomorphic.*

We also need the well known:

**Theorem 4.2** *(Implicit function theorem)*
*If $f_1, ..., f_n$ are differentiable functions on a neighborhood of the point $(x_0, y_0) = ({x_1}^0, ..., {x_n}^0, {y_1}^0, ..., {y_n}^0)$ in $\mathbb{R}^n \times \mathbb{R}^m$ , if $f_1(x_0, y_0) = f_2(x_0, y_0) = ... = f_n(x_0, y_0) = 0$, and and if the $n \times n$ matrix*

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}$$

*is invertible at* $(x_0, y_0)$*, then there is a neighborhood* $U$ *of the point* $y_0 = ({y_1}^0, ..., {y_n}^0)$ *in* $\mathbb{R}^m$*, there is a neighborhood* $V$ *of the point* $x_0 = ({x_1}^0, ..., {x_n}^0)$ *in* $\mathbb{R}^n$ *, and there is a unique mapping* $\varphi : U \longrightarrow V$ *such that* $\varphi(y_0) = x_0$ *and* $f_1(\varphi(y), y) = ... = f_1(\varphi(y), y)$ *for all* $y$ *in* $U$*. Furthermore,* $\varphi$ *is differentiable.*

The special role played by the Levi form in the Taylor expansion of the defining function $\rho$ can now be seen in the following lemma:

**Lemma 4.1** *let* $D \subset \mathbb{C}^n, n \geq 1$*, be a domain with differentiable boundary and* $z^0 \in \partial D$*. Then there exist a neighbourhood* $U$ *of* $z^0$ *and a biholomorphism* $f : U \longrightarrow f(U), z \mapsto w = f(z)$*, such that*
*(i)*$f(z^0) = 0$
*(ii)*$f(U \cap bD)$ *is defined by a differentiable function* $\widehat{\rho}$ *of the following form:*

$$\widehat{\rho}(w) = w_n + \overline{w}_n + L(\widehat{\rho}; 0)(w) + O_3(w),$$

*with* $|O_k(w)| < |w|^k$.

Proof. Suppose that $n > 1$. The statement does not depend on translations, therefore we may assume that $z^0 = 0$. Let $\rho$ be a differentiable function which defines $bD$ at $z^0$. Using the derivatives $\rho_{z_\alpha}, \rho_{\bar{z}_\alpha}, \rho_{z_\alpha z_\beta}, \rho_{z_\alpha \bar{z}_\beta}, \rho_{\bar{z}_\alpha z_\beta}, \rho_{\bar{z}_\alpha \bar{z}_\beta}$, we write the Taylor expansion of $\rho$ as in (4.1) in the following form

$$\rho(z) = \sum_{\alpha=1}^{n} \rho_{z_\alpha}(0) z_\alpha + \sum_{\alpha=1}^{n} \rho_{\bar{z}_\alpha}(0) \bar{z}_\alpha +$$
$$+ \frac{1}{2} \sum_{\alpha,\beta=1}^{n} \rho_{z_\alpha z_\beta}(0) z_\alpha z_\beta + \frac{1}{2} \sum_{\alpha,\beta=1}^{n} \rho_{\bar{z}_\alpha \bar{z}_\beta}(0) \bar{z}_\alpha \bar{z}_\beta + \sum_{\alpha,\beta=1}^{n} \rho_{z_\alpha \bar{z}_\beta}(0) \rho_{z_\alpha \bar{z}_\beta} + O_3(z).$$

Since $d\rho(0) \neq 0$, there exist $\alpha \in \{1, ..., n\}$ say $\alpha = n$, such that $\rho_{z_\alpha} \neq 0$.
Consider the local biholomorphism $f$ given by

$$\begin{cases} w_1(z) = z_1 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ w_{n-1}(z) = z_{n-1} \\ w_n(z) = \sum_{\alpha=1}^{n} \rho_{z_\alpha}(0) z_\alpha + \frac{1}{2} \sum_{\alpha,\beta=1}^{n} \rho_{z_\alpha z_\beta}(0) z_\alpha z_\beta; \end{cases}$$

Let $g = f^{-1}$ be given by $g = (g_1, ..., g_n)$ and $\widehat{\rho} = \rho \circ g$.
Applying the Taylor formula to each component $g_\alpha$ of $g$, we obtain $g_\alpha(w) = z_\alpha(w) = \sum_{\gamma=1}^{n} \frac{\partial g_\alpha}{\partial w_\gamma}(0) w_\beta + O_2(w), 1 \leq \alpha \geq n$,
thus

$$L(\rho; 0)(z) = \sum_{\alpha,\beta=1}^{n} \rho_{z_\alpha \bar{z}_\beta (0) z_\alpha \bar{z}_\beta}$$
$$= \sum_{\alpha,\beta=1}^{n} \left[ \rho_{z_\alpha} \bar{z}_\beta(0) \left( \sum_{\gamma=1}^{n} \frac{\partial g_\alpha}{\partial w_\gamma}(0) w_\gamma + O_2(w) \right) \left( \sum_{\sigma=1}^{n} \frac{\partial g_\alpha}{\partial w_\sigma}(0) \bar{w}_\sigma + O_2(w) \right) \right]$$

Then by substitution we finally get

$$\rho(g(w)) = w_n + \bar{w}_n + \sum_{\alpha,\beta,\gamma,\sigma=1}^{n} \rho_{z_\alpha} \bar{z}_\beta(0) \frac{\partial g_\alpha}{\partial w_\gamma}(0) \overline{\frac{\partial g_\beta}{\partial w_\sigma}}(0) w_\gamma \bar{w}_\sigma + O_3(w) = w_n + \bar{w}_n + L(\widetilde{\rho}; 0)(w) + O_3(w). \quad \square$$

We want to illustrate lemma 4.1 by showing how the computations are carried out in the case of the unit sphere $S$ of $\mathbb{C}^2$ at $p = (0, 1)$ (cf. 4.1).

Recall that $S = \{\rho = 0\}$, where $\rho = x^2+y^2+u^2+v^2-1$ and $\nabla\rho = (2x, 2y, 2u, 2v)$. Since $\nabla\rho(p) = (0,0,2,0) \neq 0$, we can apply the implicit function theorem. Hence in a neighborhood of $p \in S$ we can express $S$ as a graph of a function: $S = \{u = \sqrt{1-x^2-y^2-v^2}\}$ for $(x^2+y^2+v^2) < 1$.

Putting $u(x,y,v) = \sqrt{1-x^2-y^2-v^2}$, we can find the Taylor expansion of $u$ by recalling the binomial expansion:

$$(1+\zeta)^\alpha = 1 + \alpha\zeta + \frac{\alpha(\alpha-1)}{2}\zeta^2 + O(\zeta^3)$$

valid for $\alpha \in \mathbb{R}$. Setting $\zeta = x^2+y^2+v^2$
We get

$$u = (1-\zeta)^{\frac{1}{2}} = 1 + \frac{1}{2}(-\zeta)^2 + \frac{1}{2}\frac{-1}{2}(-\zeta)^2 + O(\zeta^3) = 1 - \frac{1}{2}(x^2+y^2+v^2) - \frac{1}{4}(x^2+y^2+v^2)^2 + O(\zeta^3)$$

$$= 1 - \frac{1}{2}(x^2+y^2+v^2) + O(4)$$

Then

$$\rho = u - 1 + \frac{1}{2}(x^2+y^2+v^2) + O(4) = \frac{w}{2} + \frac{\bar{w}}{2} - \frac{1}{2}\left(|z|^2 + \left(\frac{w}{2i} - \frac{\bar{w}}{2i}\right)^2\right) + O(4)$$

$$= \frac{w}{2} + \frac{\bar{w}}{2} - \frac{|z|^2}{2} - \frac{1}{2}\left(-\frac{w^2}{4} - \frac{\bar{w}^2}{4} + \frac{w\bar{w}}{2}\right) + O(4) = \frac{w}{2} + \frac{\bar{w}}{2} + -\frac{w^2}{8} + \frac{\bar{w}^2}{8} - \frac{z\bar{z}}{2} - \frac{w\bar{w}}{2} + O(4)$$

Then consider the local biholomorphism $f$ given by:

$$\begin{cases} \widetilde{z} = z \\ \widetilde{w} = \frac{1}{2}w + \frac{1}{8}w^2 \end{cases}$$

If $g = f^{-1}$ is the inverse of the coordinate change we have $g(\widetilde{z}, \widetilde{w}) =$

$$\begin{cases} z = \widetilde{z} \\ w = 2\widetilde{w} - \widetilde{w}^2 + O(3) \end{cases}$$

Now compose $\rho$ with $g$:

$$\widetilde{\rho} = \rho(g(\widetilde{z}, \widetilde{w})) = \frac{2\widetilde{w} - \widetilde{w}^2}{2} + \frac{\overline{2\widetilde{w} - \widetilde{w}^2}}{2} + \left(\frac{2\widetilde{w} - \widetilde{w}^2}{8}\right)^2 + \overline{\left(\frac{2\widetilde{w} - \widetilde{w}^2}{8}\right)}^2 - \frac{\widetilde{z}\overline{\widetilde{z}}}{2} - \frac{\left(2\widetilde{w} - \widetilde{w}^2\right)\left(\overline{2\widetilde{w} - \widetilde{w}^2}\right)}{4} + O(3)$$

$$= \widetilde{w} + \overline{\widetilde{w}} - \frac{\widetilde{z}\overline{\widetilde{z}}}{2} - \widetilde{w}\overline{\widetilde{w}} = w_n + \overline{w}_n + L(\widetilde{\rho}; 0)(w) + O(3) \quad where \quad w_n = \widetilde{w}, \quad \overline{w}_n = \overline{\widetilde{w}}, \quad L(\widetilde{\rho}; 0)(w) = -\frac{\widetilde{z}\overline{\widetilde{z}}}{2} - \widetilde{w}\overline{\widetilde{w}}.$$

This is exactly the form obtained in lemma 4.1.
The notion of Levi convexity can be used to give an important necessarily condition for a domain $D$ to be a domain of holomorphy.

**Theorem 4.3** *Let $D \subset \mathbb{C}^n, n > 1$ be a domain of holomorphy with smooth boundary. Then $\partial D$ is Levi-convex.*

Proof. The proof is now standard in the literature and can be achieved by using lemma 4.1. Let $z^0 \in \partial D$ . By the previous lemma 4.1, up to a holomorphic change of coordinates we may assume that $z^0 = 0$ and we can say that $\partial D$ is locally defined at zero by a function $\rho$ of the form:

$$\rho(z) = z_n + \overline{z}_n + L(\widetilde{\rho}; 0)(z) + O_3(z) \text{ for } z \in \mathbb{C}^n$$

$H_0(\partial D)$ is the coordinate hyperplane $z_n = 0$. Assume by contradiction that there exists $\widetilde{z} \in H_0(\partial D)$ such that $L(\rho; 0)(\widetilde{z}) < 0$. Performing a holomorphic linear change of coordinates in $H_0(\partial D)$, we may assume $\widetilde{z} = (\widetilde{z}_1, 0, ..., 0)$. By hypothesis, we have

$$0 > L(\rho; 0)(\widetilde{z}) = \rho_{z_1\overline{z}_1}(0)|\widetilde{z}_1|^2$$

hence $\rho_{z_1\overline{z}_1}(0) < 0$.
Let

$$D_\epsilon = \{z \in \mathbb{C}^n : |z_1| < \epsilon, z_\alpha = 0, 2 \leq \alpha \leq n\}.$$

And we have that $D = \{\rho < 0\}$ then $\partial D = \{\rho = 0\}$.

In particular, in a neighborhood of the origin, 0 is the only common point between $\partial D$ and $D_\epsilon$.

It follows that there exist $\epsilon_0 > 0$ such that for every $\epsilon < \epsilon_0$, the corona

$$\{z \in \mathbb{C}^n : \frac{\epsilon}{2} \leq |z_1| \leq \epsilon, z_\alpha = 0, 2 \leq \alpha \leq n\}$$

is contained in $D$.

There exist $\eta_0 > 0$ such that, for every $\eta \leq \eta_0$, the domain

$$\Delta_\epsilon = \{z \in \mathbb{C}^n : \frac{\epsilon}{2} < |z_1| < \epsilon, |z_\alpha| < \eta, 2 \leq \alpha \leq n\},$$

i.e the product of the corona

$$\{z_1 \in \mathbb{C} : \frac{\epsilon}{2} < |z_1| < \epsilon\}$$

by the disc

$$B = \{z \in \mathbb{C}^{n-1} : |z_\alpha| < \eta, 2 \leq \alpha \leq n\}$$

is relatively compact in $D$.

Now we add to $\Delta_\epsilon$ a domain $\widetilde{\Delta}$ of type:

$$\widetilde{\Delta} = \{z_1 \in \mathbb{C} : |z_1| < \epsilon\} \times A$$

with $A \subseteq B$, in order to obtain a Hartogs domain $\Delta_\epsilon \cup \widetilde{\Delta}$, whose analytic completion contains the origin, and from here we get a contradiction.
Thus $\Delta = \Delta_\epsilon \cup \widetilde{\Delta}$ is contained in $D$, but $\widehat{\Delta}$ is not, because $0 \in \widehat{\Delta} \setminus D$.
A contradiction, since $D$ is a domain of holomorphy. $\square$

Now we will show how the computations of theorem 4.3 works in a specific example: In $\mathbb{C}^3$ with coordinates $z = (z_1, z_2, z_3)$ consider a domain $D$ of the form $D = \{\rho < 0\}$ with $\rho$ of the following form:

$$\rho(z_1, z_2, z_3) = z_3 + \overline{z}_3 + |z_1|^2 - |z_2|^2 + O_3(z)$$

By definition 4.6 we have that $H_0(\partial D) = \{z_3 = 0\}$. Applying the complex linear holomorphic change of coordinates $(z_1, z_2, z_3) \longrightarrow (z_2, z_1, z_3)$ then $\rho = z_3 + \overline{z}_3 - |z_1|^2 + |z_2|^2 + O_3(z)$ then $\rho_{z_1\overline{z}_1} = -1 < 0$. So

$$0 > L(\rho; 0)(\widetilde{z}) = \rho_{z_1\overline{z}_1}(0)|\widetilde{z}_1|^2$$

For small $\epsilon > 0$ let $D_\epsilon$ be a one dimensional disc in the $z_1$ line in $\mathbb{C}^3$

$$D_\epsilon = \{z = (z_1, z_2, z_3) : |z_1| < \epsilon, z_2 = z_3 = 0\}$$

Recall that $D = \{\rho < 0\}$ then $\partial D = \{\rho = 0\}$.

**Claim 1:** 0 is the only common point between $\partial D$ and $D_\epsilon$.
**Indeed,** First $\rho(0) = 0$ then $0 \in \partial D$ and clearly $0 \in D_\epsilon$, so $0 \in bD \cap D_\epsilon$.

For $\epsilon$ small enough show that 0 is the only common point between $\partial D$ and $D_\epsilon$:

Let $p \in D_\epsilon$ and look at $\rho(p)$:

$$\rho(p) = 0 \iff p \in \partial D \iff p \in \partial D \cap D_\epsilon$$

We now restrict $\rho$ to $D_\epsilon$ and find the points for which $\rho$ is equal to zero,

$p \in D_\epsilon \implies p = (z_1, 0, 0)$ for $|z_1| < \epsilon$, so $\rho(p) = -|z_1|^2 + O(z_1{}^3) = -|z_1|^2 + R(z_1)$ where $R(z_1)$ where $|R(z_1)| \leq c|z_1|^3$ for some constant $c > 0$. Let $t = \frac{1}{2c} \implies$ if $|z_1| < t \implies |R(z_1)| \leq |z_1|^2|z_1|c = \frac{|z_1|^2}{2}$. So if $z_1$ is very small then $\rho(p) \leq -\frac{|z_1|^2}{2} \longrightarrow 0$ as $z_1 \to 0$

Therefore, $D_\epsilon$ will meet $bD$ at zero only.

**Claim 2:** It follows that there exist $\epsilon_0 > 0$ such that for every $\epsilon < \epsilon_0$, the analus

$$\{z = (z_1, z_2, z_3) : \frac{\epsilon}{2} \leq |z_1| \leq \epsilon, z_2 = z_3 = 0\}$$

is contained in $D$.

**Indeed,** If $p$ belong to the analus

$$\rho(p) \leq -\frac{|z_1|^2}{2} \implies \rho(p) < 0 \implies p \in D$$

Therefore, analus is contained in $D$.

**Claim 3:** There exist $\eta_0 > 0$ such that, for every $\eta \leq \eta_0$, the domain

$$\Delta_\epsilon = \{z = (z_1, z_2, z_3) : \frac{\epsilon}{2} < |z_1| < \epsilon, |z_2|, |z_3| < \eta,$$

i.e the product of the analus

$$\{z_1 \in \mathbb{C} : \frac{\epsilon}{2} < |z_1| < \epsilon\}$$

by the disc

$$B = \{(z_2, z_3) \in \mathbb{C}^2 : |z_2|, |z_3| < \eta\},$$

is relatively compact in $D$.

**Indeed,** The annulus is contained in $D$ which is open, it is well known that if $K$ is a compact set contained in an open set $D$ then a small translation of the compact set will be also contained in $D$. Since $\Delta_\epsilon$ is a union of small translation of the analus it is contained on $D$.

Now we add to $\Delta_\epsilon$ a domain $\widetilde{\Delta}$ of type:

$$\widetilde{\Delta} = \{z_1 \in \mathbb{C} : |z_1| < \epsilon\} \times A$$

with $A \subseteq B$, in order to obtain a Hartogs domain $\Delta_\epsilon \cup \widetilde{\Delta}$, whose analytic completion contains the origin, and from here we get a contradiction.

Thus $\Delta = \Delta_\epsilon \cup \widetilde{\Delta}$ is contained in $D$, but $\widehat{\Delta}$ is not, because $0 \in \widehat{\Delta} \setminus D$.

A contradiction, since $D$ is a domain of holomorphy.

The necessarily condition provided by theorem 4.3 led naturally to the following question known as the Levi problem.

**Levi problem:** Let $D \subseteq \mathbb{C}^n$. Is it true that if $D$ is Levi convex at all points then it is a domain of holomorphy? The Levi problem for domains in $\mathbb{C}^n$ was solved by K. Oka in [7] (we remark however that the problem is still open for domains of a complex analytic variety). For the scope of our discussion it will be enough to recall a local converse of theorem 4.3 which is valid when $D$ is strongly Levi convex and can be proved in an elementary way, similar to the computations used in the proof of theorem 4.3.

**Theorem 4.4** *Let $D \subseteq \mathbb{C}^n$ be a domain of class $C^2$ and $z^0 \in \partial D$ be a point of strong Levi-convexity. Then there exist a neighborhood $U$ of $z^0$ in $\mathbb{C}^n$ such that $U \cap D$ is a domain of holomorphy. More specifically there exist a local change of coordinates $\phi$ on $U$ such that $\phi(U \cap D)$ is strictly convex.*

## 4.2 CR-functions

In this subsection we will discuss the notions of real and complex valued vector fields in $\mathbb{C}^n$ and how that can be used to give a characterizations of holomorphic functions. If $M$ is a hypersurface of $\mathbb{C}^n$, we will use suitable vector fields tangent to $M$ to introduce the notion of CR-functions.

Recall:
A general vector field in $\mathbb{R}^n$ can be represented by:

$$X = a_1 \frac{\partial}{\partial x_1} + a_2 \frac{\partial}{\partial x_2} + ... + a_n \frac{\partial}{\partial x_n},$$

where $a_1, a_2, ..., a_n$ are functions from $\mathbb{R}^n \longrightarrow \mathbb{R}$.

A vector field $X$ as above represents a differential operator acting on differentiable functions $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ as follows:

$$Xf = a_1 \frac{\partial f}{\partial x_1} + a_2 \frac{\partial f}{\partial x_2} + ... + a_n \frac{\partial f}{\partial x_n}.$$

The value of $X$ at a point $p$ is:

$$X(p) = a_1(p) \frac{\partial}{\partial x_1} + a_2(p) \frac{\partial}{\partial x_2} + ... + a_n(p) \frac{\partial}{\partial x_n},$$

which can be identified to the vector $V = a_1(p)e_1 + ... + a_n(p)e_n$. Therefore $X$ differentiates $f$ in the direction of $V$.

A general complex valued vector field in $\mathbb{C}^n$ can be represented by:

$$Z = a_1(z) \frac{\partial}{\partial x_1} + b_1(z) \frac{\partial}{\partial y_1} + a_n(z) \frac{\partial}{\partial x_n} + b_n(z) \frac{\partial}{\partial y_n}$$

where $a_j$ and $b_j$ are functions from $\mathbb{C}^n \longrightarrow \mathbb{C}$.

We can rewrite $Z$ in terms of $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \overline{z}}$ by recalling that

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j}\right) \quad \text{and} \quad \frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j}\right)$$

Then

$$\frac{\partial}{\partial x_j} = \frac{\partial}{\partial z_j} + \frac{\partial}{\partial \overline{z}_j} \quad \text{and} \quad \frac{\partial}{\partial y_j} = i\left(\frac{\partial}{\partial z_j} - \frac{\partial}{\partial \overline{z}_j}\right)$$

So $Z$ can be written as:

$$Z = \alpha_1(z) \frac{\partial}{\partial z_1} + ... + \alpha_n(z) \frac{\partial}{\partial z_n} + \beta_1(z) \frac{\partial}{\partial \overline{z}_1} + ... + \beta_n(z) \frac{\partial}{\partial \overline{z}_n}$$

Where $\alpha_j = a_j + ib_j$ and $\beta_j = a_j - ib_j$. In the following it will be useful to distinguish the part of $Z$ containing $\frac{\partial}{\partial z_j}$ and the part containing $\frac{\partial}{\partial \bar{z}_j}$. More precisely we call the vector field

$$L = \alpha_1(z)\frac{\partial}{\partial z_1} + ... + \alpha_n(z)\frac{\partial}{\partial z_n}$$

the (1,0) part of $Z$, and the vector field

$$N = \beta_1(z)\frac{\partial}{\partial \bar{z}_1} + ... + \beta_n(z)\frac{\partial}{\partial \bar{z}_n}$$

the (0,1) part of $Z$.

We denote by $\mathcal{X}(\mathbb{C}^n)$ the set of complex valued vector fields in $\mathbb{C}^n$. Then $\mathcal{X}(\mathbb{C}^n)$ is a module over the ring $C^\infty(\mathbb{C}^n)$. Defining the sub-modules $\mathcal{X}^{(1,0)}(\mathbb{C}^n) = \left\langle \frac{\partial}{\partial z_1}, ..., \frac{\partial}{\partial z_n} \right\rangle$ and $\mathcal{X}^{(0,1)}(\mathbb{C}^n) = \left\langle \frac{\partial}{\partial \bar{z}_1}, ..., \frac{\partial}{\partial \bar{z}_n} \right\rangle$ of $\mathcal{X}(\mathbb{C}^n)$, the composition defined above corresponds to the fact that

$$\mathcal{X}(\mathbb{C}^n) = \mathcal{X}^{(1,0)}(\mathbb{C}^n) \oplus \mathcal{X}^{(0,1)}(\mathbb{C}^n).$$

Now definition 2.1 can be restated in terms $(0,1)$ vector fields as follows

**Theorem 4.5** *Let $f : \mathbb{C}^n \longrightarrow \mathbb{C}$.*
*$f$ is holomorphic $\iff$ $Lf = 0$ for all $L \in \mathcal{X}^{(0,1)}(\mathbb{C}^n)$.*

We now recall the well known notion of vector field tangent to a hypersurface.

**Definition 4.8** *Let $M = \{\rho = 0\}$ be a hypersurface in $\mathbb{C}^n$, and let $X$ be a vector field. We say that $X$ is* tangent *to $M$ if $X\rho(p) = 0$ for all $p \in M$.*

This allows to define the notion of *CR function* in analogy to Theorem 4.5.

**Definition 4.9** *Let $M = \{\rho = 0\}$ be a hypersurface in $\mathbb{C}^n$ and let $f : M \longrightarrow \mathbb{C}$ be a differentiable function. We say that $f$ is a* CR function *if $Lf = 0$ on $M$ for all $L \in \mathcal{X}^{(0,1)}(\mathbb{C}^n)$ which are tangent to $M$.*

**Remark 4.3** *Let $f : \mathbb{C}^n \longrightarrow \mathbb{C}$ be a holomorphic function and let $M$ be a real hypersurface. Then $f \restriction_M : M \longrightarrow \mathbb{C}$ is a* CR function. *This is a direct consequence of Theorem 4.5 and Definition 4.9, since $Lf$ is actually zero for all $L \in \mathcal{X}^{(0,1)}(\mathbb{C}^n)$ and not only for those tangent to $M$.*

**Constructing tangent vector fields to a manifold**:
In $\mathbb{R}^{n+1}$ consider a hypersurface of the form $M = \{x_{n+1} = l(x_1, ..., x_n)\}$; then a local defining function for $M$ is $\rho(x_1, ..., x_{n+1}) = x_{n+1} - l(x_1, ..., x_n)$.
We can construct $n$ vector fields that are tangent to $M$ as follows:

$$X_1 = \frac{\partial}{\partial x_1} + l_{x_1}\frac{\partial}{\partial x_{n+1}}$$

$$X_2 = \frac{\partial}{\partial x_2} + l_{x_2}\frac{\partial}{\partial x_{n+1}}$$

$$\vdots$$

$$X_n = \frac{\partial}{\partial x_n} + l_{x_n}\frac{\partial}{\partial x_{n+1}}$$

These $X_i's$ generate the module of the vector fields tangent to $M$. In other words, if $X \in \mathcal{X}^{(0,1)}(\mathbb{C}^n)$ tangent to $M$, then there exist functions $g_1, ..., g_n : M \longrightarrow \mathbb{R}$ such that $X \restriction_M = g_1 X_1 \restriction_M + ... + g_n X_n \restriction_M$.

Analogously, in $\mathbb{C}^{n+1}$, consider coordinates $(z_1, ..., z_n, w) \in \mathbb{C}^{n+1}$ where $w = u + iv$ and define a real hypersurface as $M = \{v = l(z_1, ..., z_n)\}$.
We can construct $n$ vector fields of type $(0,1)$ that are tangent to $M$ as follows:

$$X_1 = \frac{\partial}{\partial \overline{z}_1} - 2il_{\overline{z}_1}\frac{\partial}{\partial \overline{w}}$$
$$X_2 = \frac{\partial}{\partial \overline{z}_2} - 2il_{\overline{z}_2}\frac{\partial}{\partial \overline{w}}$$
$$\vdots$$
$$X_n = \frac{\partial}{\partial \overline{z}_n} - 2il_{\overline{z}_n}\frac{\partial}{\partial \overline{w}}$$

Just as in the real case the $X_i's$ generate the module of the (restrictions to $M$ of the) vector fields of type $(0,1)$ that are tangent to $M$. We conclude that

$f : M \longrightarrow \mathbb{C}$ is a CR-function

$\iff$ $Lf = 0$ for all $L \in \mathcal{X}^{(0,1)}(\mathbb{C}^n)$ tangent to $M$

$$\iff \begin{cases} X_1 f = 0 \\ X_2 f = 0 \\ \vdots \\ X_n f = 0 \text{ on } M \end{cases}.$$

This fact allows to express the CR condition as a system of partial differential equations. Now we will derive this system explicitly in a particularly important case.

Consider coordinates $z = x + iy$ , $w = u + iv$ in $\mathbb{C}^2$ and define the hypersurface $M \subset \mathbb{C}^2$ as $M = \{v = |z|^2\}$. This is called the *Lewy hypersurface* and much of the work in the rest of the present thesis will revolve around it. Arguing as above we find a vector field $X$ tangent to $M$ having the following form:

$$X = \frac{\partial}{\partial \overline{z}} - 2il_{\overline{z}}\frac{\partial}{\partial \overline{w}}$$

But $l = |z|^2 = z\overline{z} \implies l_{\overline{z}} = z$

$$X = \frac{\partial}{\partial \overline{z}} - 2iz\frac{\partial}{\partial \overline{w}}$$

We want to write this vector field $X$ in terms of $z$ and $u$ which can be used as local coordinates for the manifold $M$. The map giving the local coordinates is the restriction to $M$ of the projection $\pi$ on $\mathbb{C}^n$ defined as $\pi(z, w) = (z, u)$. Then it is immediate to see that the push-forward map $\pi_*$ induced on vector fields satisfies $\pi_* \frac{\partial}{\partial x} = \frac{\partial}{\partial x}$, $\pi_* \frac{\partial}{\partial y} = \frac{\partial}{\partial y}$, $\pi_* \frac{\partial}{\partial u} = \frac{\partial}{\partial u}$ , and $\pi_* \frac{\partial}{\partial v} = 0$. To find the expression of $X$ in the $(z, u)$ coordinates we need to compute $\pi_*(X)$:

$$\pi_*(X) = \pi_* \frac{\partial}{\partial \overline{z}} - 2iz\pi_* \frac{\partial}{\partial \overline{w}} = \frac{\partial}{\partial \overline{z}} - 2iz\frac{1}{2}\left(\pi_* \frac{\partial}{\partial u} + i\pi_* \frac{\partial}{\partial v}\right) = \frac{\partial}{\partial \overline{z}} - iz\frac{\partial}{\partial u}$$

Then a function $f : M \longrightarrow \mathbb{C}$ is a CR-function if and only if $Xf = 0$, that is,

$$\frac{\partial f}{\partial \overline{z}} - iz\frac{\partial f}{\partial u} = 0$$

This is the CR-equation on $M$. This equation had been first studied by Hans Lewy and used in [5] to construct the first example of a partial differential equation with smooth coefficients not admitting local solutions.

**Remark 4.4** *Let $M \subseteq \mathbb{C}^n$ be any hypersurface, $f : M \longrightarrow \mathbb{C}$ a CR-function, and $h : \mathbb{C} \longrightarrow \mathbb{C}$ a holomorphic function. Then $h \circ f : M \longrightarrow \mathbb{C}$ is a CR-function.*

Proof. Verify this using the CR-equation:

Consider

$$L = a_1 \frac{\partial}{\partial \overline{z}_1} + a_2 \frac{\partial}{\partial \overline{z}_2} + ... + a_n \frac{\partial}{\partial \overline{z}_n}$$

to be a $(0,1)$ vector field tangent to $M$.
Then
$$L(h\circ f)=a_1\frac{\partial(h\circ f)}{\partial\overline{z}_1}+a_2\frac{\partial(h\circ f)}{\partial\overline{z}_2}+...+a_n\frac{\partial(h\circ f)}{\partial\overline{z}_n}.$$
In particular by Lemma 2.1
$$\frac{\partial(h\circ f)}{\partial\overline{z}_j}=\frac{\partial h}{\partial z}.\frac{\partial f}{\partial\overline{z}_j}+\frac{\partial h}{\partial\overline{z}}.\frac{\partial\overline{f}}{\partial\overline{z}_j}=\frac{\partial h}{\partial z}.\frac{\partial f}{\partial\overline{z}_j}$$
since $h$ is a holomorphic function. We conclude that
$$L(h\circ f)=a_1\left(\frac{\partial h}{\partial z}.\frac{\partial f}{\partial\overline{z}_1}\right)+...+a_n\left(\frac{\partial h}{\partial z}.\frac{\partial f}{\partial\overline{z}_n}\right)=\frac{\partial h}{\partial z}\left(a_1\frac{\partial f}{\partial\overline{z}_1}+...+a_n\frac{\partial f}{\partial\overline{z}_n}\right)=0$$
since $f$ is a CR-function. Thus $h\circ f$ is a CR-function. $\square$

We want to work out the computations of Remark 4.4 in the particular example of the hypersurface $M=\{v=|z|^2\}$, using the intrinsic point of view rather than the vector fields $L$, that is, using directly the CR equation written in the coordinates of $M$. Let as before $f:M\longrightarrow\mathbb{C}$ be a CR-function, and let $h:\mathbb{C}\longrightarrow\mathbb{C}$ be a holomorphic function.

Proof. Using directly the CR equation,
$$\frac{\partial h\circ f}{\partial\overline{z}}=\frac{\partial h}{\partial z}\frac{\partial f}{\partial\overline{z}}+\frac{\partial h}{\partial\overline{z}}\frac{\partial\overline{f}}{\partial\overline{z}}=\frac{\partial h}{\partial z}\frac{\partial f}{\partial\overline{z}},$$
since $h$ is holomorphic.
$$\frac{\partial h\circ f}{\partial u}=\frac{\partial h}{\partial z}\frac{\partial f}{\partial u}+\frac{\partial h}{\partial\overline{z}}\frac{\partial\overline{f}}{\partial u}=\frac{\partial h}{\partial z}\frac{\partial f}{\partial u},$$
since $h$ is holomorphic.
Then
$$X(h\circ f)=\frac{\partial h\circ f}{\partial\overline{z}}-2i\frac{\partial h\circ f}{\partial u}=\frac{\partial h}{\partial z}\frac{\partial f}{\partial\overline{z}}-2i\frac{\partial h}{\partial z}\frac{\partial f}{\partial u}=\frac{\partial h}{\partial z}\left[\frac{\partial f}{\partial\overline{z}}-iz\frac{\partial f}{\partial u}\right]=0,$$
since $f$ is CR-function.
Therefore $h\circ f$ is a CR-function. $\square$

# Chapter 5

# Extensions of CR functions

As we have seen in the previous section the restriction of a holomorphic function is a CR-function, in this section we will see that not all CR-functions are the restriction of holomorphic functions.

**The square root of $z$:**

• The square root of $z$ can not be defined in $\mathbb{C}$:
Let $w$ be the square root of $z$, then $w^2 = z$. The polynomial $w^2 - z = 0$ has two roots (by the fundamental theorem of algebra). So we can not define the square root of $z$ in $\mathbb{C}$.

• The square root of $z$ is defined and continuous on $\mathbb{C} \setminus \{z \in \mathbb{C} : \Im z = 0 \text{ and } \Re z < 0\}$.

**The $\log z$ function:**

• The $\log z$ function can not be defined in $\mathbb{C}$:
Let $w = \log z$, then $e^w = z$. Suppose $w = u + iv$, and $z = \rho e^{i\theta}$. Then the equation $e^w - z = 0$ will have infinitely many solutions:

$$e^{u+iv} = \rho e^{i\theta} \implies e^u e^{iv} = \rho e^{i\theta} \implies \begin{cases} \rho = e^u \implies u = \log \rho \\ e^{iv} = e^{i\theta} \implies v = \theta + 2\pi n \end{cases}$$

Since the log function is not injective, we can not define $\log z$ on $\mathbb{C}$.

• The $\log z$ function can be defined on $\mathbb{C} \setminus \{z \in \mathbb{C} : \Im z = 0 \text{ and } \Re z \leq 0\}$.

Once we define the log function, we can define any real power of $z$:
For $\gamma \in \mathbb{R}, z^\gamma$ can be defined since $z^\gamma = e^{\gamma \log z}$ on $\mathbb{C} \setminus \{z \in \mathbb{C} : \Im z = 0 \text{ and } \Re z \leq 0\}$.

**Note:** Let $H = \{z \in \mathbb{C} : \Re z > 0\}$ be the upper half-plane in $\mathbb{C}$. Then on $H$ the log, square-roots, and powers of $z$ are all defined.

## 5.1 CR functions that are not the restriction of holomorphic functions

We want to study on $H = \{z \in \mathbb{C} : \Re z > 0\}$ a function $h : H \longrightarrow \mathbb{C}$ defined as:

$$h(z) = e^{\frac{-1}{z^\gamma}} \quad for \quad \gamma \in \mathbb{R} \quad and \quad \gamma > 0$$

Note that $h(z)$ is a holomorphic function on $H$ since it is the composition of two holomorphic functions the exponential and $\frac{-1}{z^\gamma}$.

We want to check if the function $h$ extends continuously to the boundary of $H$.

• Check first for $\gamma = 1$, $h(z) = e^{\frac{-1}{z}}$:

(i)If $z = x$ is real, then $h(z) = e^{\frac{-1}{x}}$. This function extends continuously to the boundary of $H$ since as $x \to 0$, $h(z) \to 0$.

(ii)If $z = iy$ is pure imaginary, then $h(z) = e^{\frac{-1}{iy}} = e^{\frac{i}{y}} = \cos\frac{1}{y} + i\sin\frac{1}{y}$.

But both the $\lim\limits_{y\to 0}\cos\dfrac{1}{y}$ and the $\lim\limits_{y\to 0}\sin\dfrac{1}{y}$ do not exist. Thus $h(z)$ does not extend continuously to the boundary of $H$.

If instead of $H$ we take the domain $V_c = \{z = x+iy \in \mathbb{C} : x > c|y|\}$ for $c$ a constant.

Claim: $h(z) = e^{\frac{-1}{z}}$ is continuous restricted to the boundary of $V_c$. In other words, $|h(z)| \longrightarrow 0$ as $z \longrightarrow 0$ where $z \in V_c$.

Indeed, we know that for any exponential $e^w$, its modulus is equal to $= |e^w| = e^{\Re w}$.

Then $|h(z)| = e^{\Re\left(\frac{-1}{z}\right)}$. But

$$\frac{-1}{z} = \frac{-1}{x+iy} = \frac{-x+iy}{x^2+y^2} \text{ then } \Re\left(\frac{-1}{z}\right) = \frac{-x}{x^2+y^2}$$

Then

$$|h(z)| = e^{\frac{-x}{x^2+y^2}}.$$

But on $V_c$

$$\frac{-x}{x^2+y^2} > \frac{-x}{x^2+\frac{x^2}{c^2}} \to -\infty \text{ for } (x,y)\to(0,0) \implies |h(z)| = e^{\frac{-x}{x^2+y^2}} \to e^{-\infty} \to 0 \text{ for } (x,y)\to(0,0).$$

(End Claim)

• Then work with $\gamma = \frac{1}{2}$, $h(z) = e^{\frac{-1}{\sqrt{z}}}$ for $z \in H$ :

(i)If $z = x$ is real, then $h(z) = e^{\frac{-1}{\sqrt{x}}}$. This function extends continuously to the boundary of $H$ since as $x \longrightarrow 0$, $h(z) \longrightarrow 0$.

(ii)If $z = x+iy$, $h(z) = e^{\frac{-1}{\sqrt{z}}}$ for $z \in H$.

**Lemma 5.1** *The function $h(z) = e^{\frac{-1}{\sqrt{z}}}$ is continuous up to the boundary of $H$. In other words, $|h(z)| \to 0$ as $z \to 0$ where $z \in H$.*

Proof. We know that $|h(z)| = e^{\Re\left(\frac{-1}{\sqrt{z}}\right)}$. So try to find a bound for $\Re\left(\frac{1}{\sqrt{z}}\right)$.

Let $p = \frac{1}{\sqrt{z}}$, then $p \in V_1$. So $p$ can be estimated by its real part. Indeed, for $z \in V_c$ and any $c \neq 0$, $c|y| < x \implies |z| = \sqrt{x^2+y^2} < \sqrt{x^2+\frac{x^2}{c^2}} = xc'$. In particular for $c = 1$, $|y| < x \implies |z| = \sqrt{x^2+y^2} < \sqrt{x^2+x^2} < \sqrt{2x^2} = x\sqrt{2}$. Then for $p \in V_1 \implies |p| < \sqrt{2}\Re p \implies |\frac{1}{\sqrt{z}}| < \sqrt{2}\Re p \implies \Re p > \frac{1}{\sqrt{2}\sqrt{|z|}}$.

So $|h(z)| = e^{-\Re\left(\frac{1}{\sqrt{z}}\right)} = e^{-\Re p} < e^{\frac{-1}{\sqrt{2}\sqrt{|z|}}} \to 0$ as $z \to 0$ for $z \in H$. $\square$

We are going to use the function $h$ of the lemma 5.1 in order to construct a function $\varphi$ on the hypersurface $M = \{v = |z|^2\}$ which can not be extended to a holomorphic function on $\mathbb{C}^n$.

Consider the coordinate function $w = u+iv$ on $\mathbb{C}^2$, then $w \restriction_M = u + i|z|^2$ is a CR function. let $\psi = -iw \restriction_M$, then define the function $\varphi = e^{\frac{-1}{\sqrt{\psi}}}$.

We want to prove the following things:

(i)That the function $\varphi$ is well defined.

(ii)That the function $\varphi$ is $C^\infty$.

(iii)That the function $\varphi$ is a CR function which is not the restriction of a holomorphic function.

In order to show that $\varphi$ is $C^\infty$ we should study the function $h$ deeply. In particular we have to look at the behavior of the derivatives of $h$ at the boundary of $H$.

**Lemma 5.2** *The derivatives of $h = e^{\frac{-1}{\sqrt{z}}}$ are continuous up to the boundary of $H$.*

Proof. Consider the first derivative of $h$. We have proved before that $|e^{\frac{-1}{\sqrt{z}}}| = e^{-\Re\left(\frac{1}{\sqrt{z}}\right)} \leq e^{\frac{-c}{\sqrt{z}}}$. Then the first derivative of $h$ is equal to $h'(z) = \frac{e^{\frac{-1}{\sqrt{z}}}}{2z^{\frac{3}{2}}} \implies |h'(z)| = \frac{|e^{\frac{-1}{\sqrt{z}}}|}{|2z^{\frac{3}{2}}|} = \frac{|e^{\frac{-1}{\sqrt{z}}}|}{2|z|^{\frac{3}{2}}} \leq \frac{e^{\frac{-c}{\sqrt{z}}}}{|2z^{\frac{3}{2}}|} \to 0$ as $z \to 0$, so the first derivative of $h$ extends continuously to the boundary of $H$.

By induction on $n$ we can show that all the derivatives of $h$ have the following form:

$$(a_1 z^{\gamma_1} + a_2 z^{\gamma_2} + ... + a_k z^{\gamma_k})e^{\frac{-1}{\sqrt{z}}}$$

Then $h^{(n)}(z) = (a_1 z^{\gamma_1} + a_2 z^{\gamma_2} + ... + a_k z^{\gamma_k})e^{\frac{-1}{\sqrt{z}}}$ for $a_1, \ldots, a_k \in \mathbb{R}$ and $\gamma_1, \ldots, \gamma_k \in \mathbb{Q}$ that depend on $n$.
So $|h^{(n)}(z)| \leq (|a_1||z|^{\gamma_1} + |a_2||z|^{\gamma_2} + ... + |a_k||z|^{\gamma_k})e^{\frac{-c}{\sqrt{|z|}}} \to 0$ as $z \to 0$ for any integer $n$. $\square$

**Proof of (i):** Look at $-iw \restriction_M = |z|^2 - iu$, let $\psi = -iw \restriction_M$, then $\Re\psi = |z|^2 \geq 0 \implies$ the range of $\psi$ lives in $H$.
Since every thing is defined on $H$, we can work with $\varphi = e^{\frac{-1}{\sqrt{\psi}}}$.
Therefore, $\varphi$ is well defined since the values of $\psi$ are in the domain of the function $h$ which is $H$.

**Remark 5.1** *If we consider the function $iw : \mathbb{C}^2 \longrightarrow \mathbb{C}$ instead of $iw \restriction_M : M \longrightarrow \mathbb{C}$, then $h \circ (iw)$ is not well defined.*
*Indeed, $w = u + iv \implies iw = -v + iu$, but $(-v + iu)$ might not be in $\mathbb{C} \setminus \{z \in \mathbb{C} : \Im z = 0$ and $\Re z > 0\}$.*
*So we should take the function $\psi = -iw \restriction_M$ instead of $-iw$, since $-iw \restriction_M = |z|^2 - iu$*
*$\implies \Re\psi = |z|^2 > 0 \implies \psi$ is always in the domain $H$.*

**Proof of (ii):** $\varphi = h \circ \psi$, $\psi$ is $C^\infty$ and by Lemma 5.2 $h$ is also $C^\infty$. Thus $\varphi$ is $C^\infty$ since the composition of $C^\infty$ functions is $C^\infty$.

**Proof of (iii):** Define $L = \{(u, z) \in \mathbb{R} \times \mathbb{C} : z = 0\}$
$h$ is a holomorphic function on $H$ and $\psi$ is a CR function everywhere. Then $\varphi = h \circ \psi$ is a CR function in $(\mathbb{C} \times \mathbb{R}) \setminus L$ by Remark 4.4. I still want to say that $\varphi$ is a CR function everywhere on $(\mathbb{C} \times \mathbb{R})$:
Consider the point $p = (u, z) \in (\mathbb{C} \times \mathbb{R}) \setminus L$. And $\Re\psi = 0$ on $(\mathbb{C} \times \mathbb{R}) \setminus L \implies |z|^2 = 0 \implies z = 0$ then $p = (u, 0)$.
$\varphi$ is CR for $q \in (\mathbb{C} \times \mathbb{R}) \setminus L$, implies

$$\frac{\partial\varphi}{\partial\bar{z}}(q) - i\frac{\partial\varphi}{\partial u}(q) = 0 \text{ for } q \in (\mathbb{C} \times \mathbb{R}) \setminus L,$$

so it remains to prove that $\varphi$ is CR for $q \in (\mathbb{C} \times \mathbb{R})$, i.e it remains to prove that

$$\frac{\partial\varphi}{\partial\bar{z}}(q) - i\frac{\partial\varphi}{\partial u}(q) = 0 \text{ for } q \in (\mathbb{C} \times \mathbb{R}).$$

**Remark 5.2** *Let $l : \mathbb{C} \times \mathbb{R} \longrightarrow \mathbb{C}$ be a continuous function such that $l = 0$ on $\mathbb{C} \times \mathbb{R} \setminus L$. Then $l = 0$ everywhere on $\mathbb{C} \times \mathbb{R}$.*

Indeed, $(\mathbb{C} \times \mathbb{R}) \setminus L$ is dense in $\mathbb{C} \times \mathbb{R}$.

Now consider the continuous function $l$ to be equal to $\frac{\partial\varphi}{\partial\bar{z}}(q) - i\frac{\partial\varphi}{\partial u}(q)$. And $l = 0$ on $(\mathbb{C} \times \mathbb{R}) \setminus L$. Thus $l = \frac{\partial\varphi}{\partial\bar{z}}(q) - i\frac{\partial\varphi}{\partial u}(q) = 0$ everywhere on $\mathbb{C} \times \mathbb{R}$.
Therefore, $\varphi$ is a CR-function everywhere.

**Lemma 5.3** *Consider $F : \mathbb{C}^2 \longrightarrow \mathbb{C}$ to be a a holomorphic function, and $\varphi = F \restriction_M : M \longrightarrow \mathbb{C}$. Then if $\varphi$ is flat, $F$ is flat.*

We will give the proof of Lemma 5.3 in the case of the hypersurface $M = \{\Im w = |z|^2\}$ that we are studying.

Proof. Look at $M = \{\Im w = |z|^2\}$ and $\varphi(z,u) = F(z,w) \restriction_M = F(z, u + i|z|^2)$. Relate the derivatives of $F$ to the derivatives of $\varphi$:

According to the first derivative:

- $\frac{\partial \varphi}{\partial u}(z,u) = \frac{\partial F}{\partial w}.\frac{\partial(u+i|z|^2)}{\partial u} + \frac{\partial F}{\partial \overline{w}}.\frac{\partial(u-i|z|^2)}{\partial u} + \frac{\partial F}{\partial z}.\frac{\partial z}{\partial u} + \frac{\partial F}{\partial \overline{z}}.\frac{\partial \overline{z}}{\partial u} = \frac{\partial F}{\partial w}.$

- $\frac{\partial \varphi}{\partial z}(z,u) = \frac{\partial F}{\partial w}.\frac{\partial(u+i|z|^2)}{\partial z} + \frac{\partial F}{\partial \overline{w}}.\frac{\partial(u-i|z|^2)}{\partial z} + \frac{\partial F}{\partial z}.\frac{\partial z}{\partial z} + \frac{\partial F}{\partial \overline{z}}.\frac{\partial \overline{z}}{\partial z} = \frac{\partial F}{\partial z} + \frac{\partial F}{\partial w} i\overline{z}.$

- $\frac{\partial \varphi}{\partial \overline{z}}(z,u) = \frac{\partial F}{\partial w}.\frac{\partial(u+i|z|^2)}{\partial \overline{z}} + \frac{\partial F}{\partial \overline{w}}.\frac{\partial(u-i|z|^2)}{\partial \overline{z}} + \frac{\partial F}{\partial z}.\frac{\partial z}{\partial \overline{z}} + \frac{\partial F}{\partial \overline{z}}.\frac{\partial \overline{z}}{\partial \overline{z}} = \frac{\partial F}{\partial w} iz.$

According to the second derivative: Call $\frac{\partial F}{\partial w} = g(z, u + i|z|^2)$,

- $\frac{\partial^2 \varphi}{\partial u^2} = \frac{\partial g}{\partial u} = \frac{\partial g}{\partial z}.\frac{\partial z}{\partial u} + \frac{\partial g}{\partial \overline{z}}.\frac{\partial \overline{z}}{\partial u} + \frac{\partial g}{\partial w}.\frac{\partial(u+i|z|^2}{\partial u} + \frac{\partial g}{\partial \overline{w}}.\frac{\partial g(u-i|z|^2}{\partial u} = \frac{\partial g}{\partial w} = \frac{\partial^2 F}{\partial w^2}$

- $\frac{\partial^2 \varphi}{\partial u \partial z} = \frac{\partial g}{\partial z} + \frac{\partial g}{\partial w}.\frac{\partial(u+i|z|^2}{\partial z} = \frac{\partial g}{\partial z} + \frac{\partial g}{\partial w}.i\overline{z} = \frac{\partial^2 F}{\partial z \partial w} + \frac{\partial^2 F}{\partial w^2} i\overline{z}$

- $\frac{\partial^2 \varphi}{\partial u \partial \overline{z}} = \frac{\partial g}{\partial w}.iz = \frac{\partial^2 F}{\partial w^2} iz$

- $\frac{\partial^2 \varphi}{\partial z^2} = \frac{\partial^2 F}{\partial z^2} + \frac{\partial^2 F}{\partial z \partial w}.i\overline{z} + \frac{\partial^2 F}{\partial w^2}.(-\overline{z}^2)$

- $\frac{\partial^2 \varphi}{\partial \overline{z}^2} = \frac{\partial^2 F}{\partial \overline{z} \partial w}.iz + \frac{\partial^2 F}{\partial w^2}.(-z^2)$

- $\frac{\partial^2 \varphi}{\partial z \partial \overline{z}} = \frac{\partial^2 F}{\partial w \partial z}.iz + \frac{\partial^2 F}{\partial w^2}.(-z\overline{z} + \frac{\partial F}{\partial w}.i$

So a common pattern for the derivatives is:

- $\frac{\partial^n \varphi}{\partial z^n} = \frac{\partial^n F}{\partial z^n}$+something that will be zero at $z = 0$.

- $\frac{\partial^n \varphi}{\partial u^n} = \frac{\partial^n F}{\partial w^n}$+something that will be zero at $z = 0$.

- $\frac{\partial^n \varphi}{\partial z^{n-1} \partial u} = \frac{\partial^n F}{\partial z^{n-1} \partial w}$+something that will be zero at $z = 0$.

- $\frac{\partial^n \varphi}{\partial z^{n-k} \partial u^k} = \frac{\partial^n F}{\partial z^{n-k} \partial w^k}$+something that will be zero at $z = 0$.

Therefore, if $\varphi$ is flat, i.e. all its derivatives at $(0,0)$ are zero, then all the derivatives of $F$ at $(0,0)$ will be zero. Thus $F$ will be also flat. $\square$

Thus $\varphi$ by lemma 5.3 is not the restriction of a holomorphic function $F$ since all the derivatives of $\varphi$ at $(0,0)$ are zero, i.e. $\varphi$ is flat , then $F$ is flat, which implies that $F$ is identically zero. Then $\varphi$ is identically zero. This gives a contradiction. $\square$

**Remark 5.3** *Consider the hypersurface $M = \{\Im w = |z|^2\}$, let $F : \mathbb{C}^2 \longrightarrow \mathbb{C}$ be a holomorphic function, and let $\varphi = F \restriction_M : M \longrightarrow \mathbb{C}$. Since $M$ is a real analytic hypersurface, $\varphi$ is also be analytic.*

Note that Remark 5.3 is another way to show that $\varphi$ is not the restriction of a holomorphic function.

$\diamond$ Let $\Omega = \{\Im w > |z|^2\} = \{v > |z|^2\} \subset \mathbb{C}^2$ and $\varphi = e^{\frac{-1}{\sqrt{\psi}}}$. We proved previously that $\varphi$ is a CR-function which is not the restriction of a holomorphic function defined on $\mathbb{C}^2$. Instead, $\varphi$ is the restriction of a holomorphic function $F$ defined on $\Omega$ and smooth up to $\partial\Omega$. Indeed, we can choose $F(z,w) = e^{\frac{-1}{\sqrt{-iw}}}$: $F$ is well defined because for any $(z,w) \in \Omega$ we have that $\Re(-iw) = \Im w > |z|^2 > 0$.
In the discussion above we found an example of a CR function on $M$ which does not extend holomorphically to $\mathbb{C}^2$ but extends holomorphically to $\Omega$. This happens more in general; the following fundamental result was proved by Hans Lewy in [4].

**Theorem 5.1** *Let $M$ be a smooth hypersurface of $\mathbb{C}^2$. Let $p \in M$. Suppose that $M$ is strongly Levi-convex at $p$. Then there exist one-sided neighborhood $\Omega$ of $p$ such that every CR function on $\Omega \cap M$ extends holomorphically to $\Omega$.*

Remark: Note that $\Omega$ can be mapped by a biholomorphism to the sphere, and the sphere is a domain of holomorphy. Then $\Omega$ is a domain of holomorphy since the notion of domain of holomorphy is invariant under biholomorphisms. Thus there exist holomorphic functions in it that can not be extended to a bigger domain. So it is clear that in $\Omega$ we can find more holomorphic functions than in $\mathbb{C}^2$.

**Our goal** is to prove Theorem 5.1 in the particular case where $M = \{\Im w = |z|^2\}$. All the main ideas of the proof given in [4] already appear in this case and up to some geometrical complication it is not too hard to extend the argument we give to the general case,
the proof will be achieved in three main steps:

(i) We show that the intersection of $\overline{\Omega}$ with complex lines of the form $\{w = c\}$ is compact.

(ii) By solving a one variable problem we extend the function $f$ holomorphically to each domain $\Omega \cap \{w = c\}$. The resulting extension $F : \Omega \to \mathbb{C}$ is holomorphic w.r.t. $z$.

(iii) Finally we show that $F$ is holomorphic in $\Omega$ and smooth up to the boundary and $F \restriction_M = f$.

We first consider point (i), so we need to find the intersection of $M = \{\Im w = |z|^2\}$ and $\Omega = \{\Im w > |z|^2\}$ with a complex line $w = c = c_1 + ic_2 \in \mathbb{C}^2$. This results in the following systems:

$$M \cap \{w = c\} = \begin{cases} w = c_1 + ic_2 \\ \Im w = |z|^2 \end{cases} \qquad \Omega \cap \{w = c\} = \begin{cases} w = c_1 + ic_2 \\ \Im w > |z|^2 \end{cases}$$

We distinguish the following cases:

. If $c_2 < 0 \implies$ no intersection since $c_2 = |z|^2 > 0$.

. If $c_2 = 0 \implies z = 0$ and $w = c_1 \implies$ the intersection $M \cap \{w = c\}$ is a single point $(0, c_1)$ and $\Omega \cap \{w = c\} = \emptyset$ since $z = 0$ and $|z|^2 < 0$.

. If $c_2 > 0 \implies |z|^2 = c_2 \implies r^2 = c_2$ where $|z| = r \implies x^2 + y^2 = c_2 \implies M \cap \{w = c\}$ is a circle of radius $\sqrt{c_2}$ and $\Omega \cap \{w = c\}$ is a disc $\Delta_c$ of radius $\sqrt{c_2}$ of raduis $c_2$ and center zero.

Then we consider point(ii):
If a function is defined on $M$, then it will be defined on $\partial\Delta_c$ for all $c$ for which $c_2 > 0$. So we still need to extend it inside $\Delta_c$. If $f \restriction_{\partial\Delta_c}$ admits a holomorphic extension to $\Delta_c$ then we can find the values of the extension $F$ by using the Cauchy formula.

$$f(z) = \frac{1}{2\pi i} \int_{\partial\Delta_c} \frac{f(\rho)}{\rho - z} d\rho, \text{ where } \rho \in \partial\Delta_c \text{ and } z \in \Delta_c.$$

However not every continuous function defined on the unit circle extends holomorphically to the unit disc $\Delta_c$. For instance for this to happen its Fourier Series its Fourier series must have only non-negative indexed coefficients.

Indeed, consider for example the holomorphic function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ whose radius of convergence is larger than one and restrict $f$ to the unit circle. Then $f(e^{i\theta}) = \sum_{n=0}^{\infty} a_n e^{in\theta}$ this is the Fourier Series of $f \restriction_{\partial\Delta}$. Thus the negative indexed Fourier coefficients are zero for holomorphic functions. Then the restriction of $f$ to $\partial\Delta_c$ is $f(e^{i\theta}) = \sum_{n=0}^{\infty} c_n e^{in\theta}$.

Example: The function $\sin\theta : [0, 2\pi] \longrightarrow \mathbb{C}$ does not extend holomorphically inside the unite circle.
$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \implies a_{-1} = \frac{-1}{2i} \neq 0 \implies$ the function $\sin\theta$ can not extend holomorphically on the unit disc.

Example: The function $\cos\theta - isin\theta : [0, 2\pi] \longrightarrow \mathbb{C}$ does not extend holomorphically inside the unite circle.
$\cos\theta - i\sin\theta = e^{-i\theta} \implies a_{-1} = 1 \neq 0 \implies$ the function $\cos\theta - i\sin\theta$ can not extend holomorphically on the unit disc.

We are now recalling few basic facts of Fourier Series. First of all you observe that:

$$\int_0^{2\pi} e^{in\theta} d\theta = \begin{cases} 0 & n \neq 0 \\ 2\pi & n = 0 \end{cases}$$

Suppose the function $f$ admits a uniformly convergent Fourier Series

$$f(\theta) = \sum_{j=-\infty}^{\infty} a_j e^{ij\theta}.$$

Computing $\int_0^{2\pi} f(\theta)e^{-in\theta} d\theta$ we get:

$$\int_0^{2\pi} f(\theta)e^{-in\theta} d\theta = \int_0^{2\pi} \sum_{j=-\infty}^{\infty} a_j e^{(j-n)i\theta} d\theta = \sum_{j=-\infty}^{\infty} a_j \int_0^{2\pi} e^{(j-n)i\theta} d\theta = 2\pi a_n$$

because

$$\int_0^{2\pi} e^{(j-n)i\theta} d\theta = \begin{cases} 0 & j \neq n \\ 2\pi & j = n \end{cases} .$$

Then we can recover the Fourier coefficients of $f$ as follows

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} f(\theta)e^{-in\theta} d\theta.$$

In particular, the coefficient $a_0$ corresponds to the average of $f$ on the unit circle:

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) d\theta.$$

Thus if $f$ extends holomorphically to the unit disc $\Delta$ we must have

$$a_{-n} = \frac{1}{2\pi} \int_0^{2\pi} f(\theta)e^{in\theta} d\theta = 0 \text{ for } n \geq 0.$$

We can express the coefficient $a_n$ in another way. Using the parametrization of the unit circle by $e^{i\theta}$ for $0 \leq \theta \leq 2\pi$, we get

$$\frac{1}{2\pi i} \int_{\partial\Delta} f(z)z^n dz = \frac{1}{2\pi i} \int_0^{2\pi} f(e^{i\theta})ie^{i\theta}e^{in\theta} dz = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta})e^{i(n+1)\theta} d\theta =$$
$$= \frac{1}{2\pi} \int_0^{2\pi} \widetilde{f}(\theta)e^{i(n+1)\theta} d\theta = a_{-(n+1)}.$$

Then the general form of the negative indexed Fourier coefficients is as follows:

$$a_{-(n+1)} = \frac{1}{2\pi i} \int_{\partial\Delta} f(z)z^n dz \text{ for } n \geq 0.$$

The integrals $\int_{\partial\Delta} f(z)z^n dz, n \geq 0$, are called the *moments* of $f$. Thus, if $f$ extends holomorphically to $\Delta$ then all its moments must vanish. As it turns out the converse is also true: if $a_n = 0$ for $n < 0$, then $f$ extends holomorphically to the unit disc $\Delta$. This is a particular case of the following result.

**Theorem 5.2** *Let $U \subset \mathbb{C}$ be a domain whose boundary $\partial U = \gamma$ is a closed simple curve of class $C^1$. Let $f : \gamma \to \mathbb{C}$ be a continuous function. Then $f$ admits a holomorphic extension to $U$, continuous up to $\gamma$, iff all the moments of $f$ vanish, that is $\int_\gamma f(z)z^n dz = 0$ for all $n \geq 0$.*

Thus $f$ extends holomorphically to the disc $\Delta_c$ with $c = c_1 + ic_2$ where $c_2 > 0$ if

$$\frac{1}{2\pi i}\int_{\partial\Delta_c} f(z)z^n dz = 0 \text{ for } n \geq 0.$$

For $c = c_1 + ic_2$ with $c_2 \geq 0$ and $M = \{v = |z|^2\}$, $\{w = c\} \cap M = \{|z| = \sqrt{c_2}\} = \gamma_c$. The restriction of $f : M \longrightarrow \mathbb{C}$ to the curve $\gamma_c \subset M$ is $f_c(z) : \gamma_c \longrightarrow \mathbb{C}$. Fix $n$ and consider

$$\int_{\gamma_c} f_c(z)z^n dz = K(c).$$

Our plan is to show that $K(c) = 0, \quad \forall n \geq 0$ and $\forall c \in \widetilde{H}$ where $\widetilde{H} = \{z \in \mathbb{C} : \Im z \geq 0\}$. In view of the next result it is enough to show that $K$ is holomorphic on $\widetilde{H}^\circ$ and continuous up to the boundary.

**Lemma 5.4** *If $f$ is a holomorphic function on $\widetilde{H}^\circ = \{z \in \mathbb{C} : \Im z > 0\}$, extends continuously up to the boundary, and is zero on the boundary. Then $f$ is identically zero everywhere in $\mathbb{C}$.*

Lemma 5.4 is a direct consequence of the Schwarz reflection principle:

**Lemma 5.5** *(Schwarz reflection principle) Suppose $f : \widetilde{H} \longrightarrow \mathbb{C}$ a holomorphic function on $\widetilde{H}^o$, extends continuously up to the boundary, and is real valued on the real line. Then $f$ extends holomorphically on $\mathbb{C}$.*

Lemma 5.5 is of course well known but we provide a proof for convenience. Define a function $g : \mathbb{C} \longrightarrow \mathbb{C}$ in the following way:
If $z \in \widetilde{H}$ we put $g(z) = f(z)$.
For $z \in \mathbb{C} \setminus \widetilde{H}$ we have $\overline{z} \in \widetilde{H}$, hence we can define $g$ as $g(z) = \overline{f(\overline{z})}$. To show that $g$ is holomorphic on $\mathbb{C} \setminus \widetilde{H}$ we should check that $\frac{\partial g}{\partial \overline{z}} = 0$:

$$\frac{\partial g}{\partial \overline{z}} = \left(\overline{\frac{\partial f(\overline{z})}{\partial z}}\right) = \overline{\left(\frac{\partial f(\overline{z})}{\partial \overline{z}}\right)} = 0$$

since $f$ is holomorphic.
Next we show that $g$ extends continuously to the boundary of $\mathbb{C} \setminus \widetilde{H}$. Let $z_0 \in \partial(\mathbb{C} \setminus \widetilde{H})$ then

$$\lim_{z \to z_0} g(z) = \lim_{z \to z_0} \overline{f(\overline{z})} \text{ for } z \in \mathbb{C} \setminus \widetilde{H}.$$

Let $w = \overline{z}$, we know that $f$ is continuous up to the boundary then

$$\lim_{z \to z_0} f(z) = f(z_0) \implies \lim_{w \to z_0} \overline{f(w)} = \overline{f(z_0)} = f(z_0)$$

since $z_0 \in \mathbb{R}$ and $f$ is real on $\mathbb{R}$.
Then

$$\lim_{z \to z_0} g(z) = f(z_0) \text{ for } z \in \mathbb{C} \setminus \widetilde{H}.$$

Thus $g$ extends continuously to the boundary and its limit is $f(z_0)$ as $z \to z_0$.
Consider $\widetilde{f} : \mathbb{C} \longrightarrow \mathbb{C}$ defined as:

$$\widetilde{f} = \begin{cases} f(z) & z \in \widetilde{H} \\ g(z) & z \in \mathbb{C} \setminus \widetilde{H} \end{cases}$$

Both $f$ and $g$ extend continuously to the boundary and $\lim_{z \in \mathbb{C} \setminus \widetilde{H} \to z_0} g(z) = \lim_{z \in \widetilde{H} \to z_0} f(z) = f(z_0) \implies \widetilde{f}$ is continuous everywhere on $\mathbb{C}$.

$f$ is holomorphic on $\widetilde{H}^o$ and $g$ is holomorphic on $\mathbb{C} \setminus \widetilde{H} \implies \widetilde{f}$ is holomorphic on $\mathbb{C} \setminus$(real line). Then by the Riemann extension theorem $\widetilde{f}$ is holomorphic on $\mathbb{C}$. □

Proof of Lemma 5.4. $f$ is a holomorphic function on $\widetilde{H}^o = \{z \in \mathbb{C} : \Im z > 0\}$, extends continuously up to the boundary, and is zero on the boundary. So $f$ is real valued on the real line. Then by Lemma 5.5 $f$ extends holomorphically to $\widetilde{f}$ on $\mathbb{C}$.

$\widetilde{f} : \mathbb{C} \longrightarrow \mathbb{C}$ is holomorphic everywhere on $\mathbb{C}$ and $\widetilde{f} = f = 0$ on the real line. Then $\widetilde{f}$ has a non-isolated zero. Thus $\widetilde{f}$ is identically zero.

Therefore, $f$ is identically zero on all of $\mathbb{C}$. □

To prove that $K(c) = 0$ we want to use Lemma 5.4 and check first that $K(c)$ is a holomorphic function. To achieve this, rather than trying to differentiate $K$ it will be more convenient to use Morera's Theorem:

**Theorem 5.3** *(Morera's Theorem) If f(z) is continuous function in a region D and satisfies*

$$\int_\Gamma f(z)dz = 0$$

*for all closed curves $\Gamma$ in D, then $f(z)$ is holomorphic in D.*

Thus let us fix any simple closed curve $\Gamma$ of class $C^1$ on $\widetilde{H}^o$, and integrate $K(c)$ over $\Gamma$:

$$\int_\Gamma K(c)dc = \int_\Gamma \int_{\gamma_c} f(z,c)z^n dzdc$$

Consider $V$ to be the interior of the curve $\Gamma$, then define $\Lambda = \bigcup_{c \in V} \gamma_c$ and $\omega = \partial\Lambda$. Then $\Lambda$ is a three dimensional submanifold of $M$, that is, it is an open domain since $M$ itself has dimension three. Moreover, $\omega = \partial\Lambda = \bigcup_{c \in \Gamma} \gamma_c$ is a two dimensional compact surface of $M$, homeomorphic to a torus.

Since the $\int_\Gamma K$ is equal to the integral of $fz^n$ on $\omega$, to prove that the former vanishes we will employ Stoke's theorem:

**Theorem 5.4** *(Stoke's Theorem) Consider a k differential form $\mu$ and an orientable manifold $\Omega$ of dimension $k+1$. Then:*

$$\int_{\partial\Omega} \mu = \int_\Omega d\mu.$$

Apply Theorem 5.4 to $\Omega = \bigcup_{c \in V} \gamma_c$ and $\mu = f(z,c)z^n dzdc$. Then:

$$\int_\omega \mu = \int_\Lambda d\mu.$$

Consider first the following integral:

$$I = \int_\omega f(z,c)z^n dzdc$$

But $dc = dc_1 + idc_2$ and $c_2 = |z|^2 = z\overline{z}$ in $M \implies dc_2 = \overline{z}dz + zd\overline{z} \implies dc = dc_1 + i(\overline{z}dz + zd\overline{z}) = dc_1 + i\overline{z}dz + izd\overline{z}$.

Then $dzdc = dz(dc_1 + i\overline{z}dz + izd\overline{z}) = dzdc_1 + izdzd\overline{z}$. So:

$$I = \int_\omega f(z,c_1)z^n dzdc = \int_\omega f(z,c_1)z^n(dzdc_1 + izdzd\overline{z}) = \int_\omega f(z,c_1)z^n dzdc_1 + i\int_\omega f(z,c_1)z^{n+1}dzd\overline{z}.$$

But

$$d(f(z,c_1)z^n dzdc_1) = d(f(z,c_1)z^n)dzdc_1 = \left(\frac{\partial(f(z,c_1)z^n)}{\partial z}dz + \frac{\partial(f(z,c_1)z^n)}{\partial \overline{z}}d\overline{z} + \frac{\partial(f(z,c_1)z^n)}{\partial c_1}dc_1\right)dzdc_1$$

$$= \frac{\partial(f(z,c_1)z^n)}{\partial\overline{z}} d\overline{z}dzdc_1 = z^n \frac{\partial(f(z,c_1))}{\partial\overline{z}} d\overline{z}dzdc_1.$$

And

$$d(z^{n+1}f(z,c_1)dzd\overline{z}) = d(z^{n+1}f(z,c_1))dzd\overline{z}$$

$$= \left( \frac{\partial(z^{n+1}f(z,c_1))}{\partial z} dz + \frac{\partial(z^{n+1}f(z,c_1))}{\partial\overline{z}} d\overline{z} + \frac{\partial(z^{n+1}f(z,c_1))}{\partial c_1} dc_1 \right) dzd\overline{z} = \frac{\partial(z^{n+1}f(z,c_1))}{\partial c_1} dc_1 dzd\overline{z}$$

$$= z^{n+1}\frac{\partial f}{\partial u} dc_1 dzd\overline{z}.$$

Then by Theorem 5.4

$$I = \int_\Lambda d(f(z,c_1)z^n dzdc_1) + i\int_\Lambda d(f(z,c_1)z^{n+1}dzd\overline{z}) = \int_\Lambda z^n \frac{\partial f}{\partial\overline{z}} d\overline{z}dzdc_1 + i\int_\Lambda z^{n+1}\frac{\partial f}{\partial u} dc_1 dzd\overline{z}$$

$$= \int_\Lambda z^n \frac{\partial f}{\partial\overline{z}} d\overline{z}dzdc_1 - i\int_\Lambda z^{n+1}\frac{\partial f}{\partial u} d\overline{z}dzdc_1 = \int_\Lambda z^n \left[\frac{\partial f}{\partial\overline{z}} - iz\frac{\partial f}{\partial u}\right] d\overline{z}dzdc_1 = 0.$$

Since $f$ is a CR-function.

Then

$$\int_\Gamma K(c) = 0 \text{ on every } \Gamma.$$

Therefore, Theorem 5.3 $K$ is holomorphic in $\mathbb{C}$.

Then the function $K$ is a holomorphic function on $\widetilde{H}^o = \{z \in \mathbb{C} : \Im z > 0\}$, extends continuously up to the boundary, and is zero on the boundary. So by Theorem 5.4 $K$ is identically zero everywhere in $\mathbb{C}$. This implies that

$$\int_{\gamma_c} f(z,c)z^n dz = 0.$$

Thus $f$ has no negative indexed Fourier coefficients. Then $f$ can be extended holomorphically inside the disc $\Delta_c$ for every $c$.

Now consider point(iii):

Consider the open domain $\Omega = \{v > |z|^2\}$, define the function $F(b,c) : \Omega \longrightarrow \mathbb{C}$ for $b = b_1 + ib_2, c = c_1 + ic_2 \in \Omega$ by

$$F(b,c) = \frac{1}{2\pi i}\int_{\gamma_c} \frac{f(z,c)}{z-b} dz.$$

For a fixed $c$, $F$ is holomorphic with respect to $b$. However, we need to show that $F$ is holomorphic on $\Omega$ with respect to both variables. In view of the following well known result it is enough to show that $F$ is also holomorphic with respect to $c$ for any fixed $b$:

**Theorem 5.5** *(Hartog's Separate holomorphicity theorem)*
*Consider a continuous function $f(z_1,z_2) : \mathbb{C}^2 \longrightarrow \mathbb{C}$ such that $f\restriction_{z_2=c}$ and $f\restriction_{z_1=c'}$ are holomorphic for every $c$ and $c'$. Then $f$ is holomorphic on all of $\mathbb{C}^2$.*

Now for fixed $b$, consider a closed curve $\Gamma$ in the complex plane of the $c$ variable, and integrate $F$ over $\Gamma$ as follows:

$$\int_\Gamma F(b,c)dc = \frac{1}{2\pi i}\int_\Gamma\int_{\gamma_c} \frac{f(z,c)}{z-b} dzdc.$$

Then by Theorem 5.4

$$\int_\omega \frac{f(z,c)}{z-b} dzdc = \int_\Lambda d\left(\frac{f(z,c)}{z-b}\right) dzdc$$

Similar to the work we did before,

$$J = \int_\omega \frac{f(z,c)}{z-b} dzdc = \int_\omega \frac{f(z,c)}{z-b} dz(dc_1 + i\overline{z}dz + izd\overline{z}) = \int_\omega \frac{f(z,c)}{z-b} dzdc_1 + i\int_\omega \frac{f(z,c)}{z-b} zdzd\overline{z} =$$

$$\int_\Lambda \frac{1}{z-b}\frac{\partial f}{\partial\overline{z}} d\overline{z}dzdc_1 - i\int_\Lambda \frac{z}{z-b}\frac{\partial f}{\partial\overline{z}} d\overline{z}dzdc_1 = \int_\Lambda \frac{1}{z-b}\left[\frac{\partial f}{\partial\overline{z}} - iz\frac{\partial f}{\partial u}\right] d\overline{z}dzdc_1 = 0.$$

Since $f$ is a CR-function.

Thus, by Theorem 5.3 $F$ is holomorphic with respect to $c$. Therefore, by Theorem 5.5 $F$ is holomorphic on all of $\Omega$.

The fact that $F$ is continuous on $\overline{\Omega}$ can be proved by standard arguments since the integral defining $F$ depends continuously on its parameters.

# Chapter 6

# Further developments on the extension theorem

In the previous section we studied a manifold $M$ with the property that a CR function defined on $M$ can be extended to a holomorphic function defined on a domain of $\mathbb{C}^2$. More in general Lewy's theorem provides a one-sided extension result in a neighborhood of $p \in \partial D \subset \mathbb{C}^2$ whenever the Levi form is positive definite at $p$.
On the other hand, if $D$ is a relatively compact domain and $f$ is a CR function defined on the boundary of $D$ then $f$ extends to $D$:

**Theorem 6.1** *Let $D \subset \mathbb{C}^n$ be a relatively compact domain with smooth, connected boundary $\partial D$, and let $f \in C^1(\partial D)$ be a CR function. Then there exists $F \in C^1(\overline{D}) \cap \mathcal{O}(D)$ such that $F \restriction_{\partial D} = f$. The following theorem will in some way generalize the extension to an appropriate whole domain of $\mathbb{C}^n$.*

The function $F$ of Theorem 6.1 can also be expressed in an explicit way by means of an appropriate kernel. The following representation formula was discovered independently by Bochner [1] and Martinelli [6].

**Definition 6.1** *For $z \in \mathbb{C}^n$, the* Bochner-Martinelli *kernel is the following form defined on $\mathbb{C}^n \setminus \{z\}$: $K_{BM}(z, \zeta)$*

$$= \frac{(n-1)!}{(2\pi i)^n} \sum_{\alpha=1}^{n} \frac{(-1)^\alpha (\overline{\zeta}_\alpha - \overline{z}_\alpha)}{|z-\zeta|^{2n}} d\zeta_1 \wedge \ldots \wedge d\zeta_n \wedge d\overline{\zeta}_1 \wedge \ldots \wedge d\hat{\overline{\zeta}}_\alpha \wedge \ldots \wedge d\overline{\zeta}_n$$

*where $d\hat{\overline{\zeta}}_\alpha$ means that the differential $\overline{\zeta}_\alpha$ is missing.*

**Theorem 6.2** *Let $D$ be a bounded domain in $\mathbb{C}^n$, with connected boundary of $C^1$. Let $f$ be holomorphic in $D$ and continuous up to $\overline{D}$. Then, for every $z \in D$,*

$$f(z) = \int_{\partial D} f K_{BM}(z, .).$$

Then the statement of Theorem 6.1 can be made precise by saying that $F$ can be obtained by means of integration of the form $fK_{BM}$ over $\partial D$. The fact that the result is a holomorphic function depends on the fact that $f$ satisfies the CR condition.
This theorem generalizes to higher dimensions. The similar extension results which we used in one variable: the Bochner- Martinelli kernel replaces the Cauchy kernel $\frac{dz}{z-\zeta}$ and the CR condition on $f$ replaces the moments condition.
Going back to the local extension problem, it turns out that a sufficient condition for one-sided extension in a neighborhood of $p \in M \subset \mathbb{C}^n$ is the presence of at least one non-vanishing eigenvalue in the Levi form at $p$: this is proved in [2]. As a consequence, if the Levi form has eigenvalues of both signs, one obtains the extension to a whole neighborhood of $p$ in $\mathbb{C}^n$.

Subsequent work has provided local extension results under progressively weaker assumptions on the geometry of $M$. The most far reaching generalizations have been obtained by Treprean ([8]) and Tumanov ([9]): the one-sided local extension of CR functions holds iff $M$ is *minimal* at $p$. In the setting we are interested in (i.e. real hypersurfaces) the minimality condition just means that $M$ does not contain any complex hypersurface passing through $p$.

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