AMERICAN UNIVERSITY OF BEIRUT

Subadditivity of Syzygies in a Minimal Graded Free Resolution

by

GAELLE RIZKALLAH GABRIEL

A thesis
submitted in partial fulfillment of the requirements
for the degree of Master of Science
to the Department of Mathematics
of the Faculty of Arts and Sciences
at the American University of Beirut

Beirut, Lebanon
April 27, 2017

AMERICAN UNIVERSITY OF BEIRUT

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GAELLE RIZAKALLAH GABRIEL

Approved by:

Dr. El Khoury Sabine, Associate Professor Advisor
Mathematics

Dr. Abu Khuzam Hazar, Professor Member of Committee
Mathematics

Dr. Azar Monique, AssistantProfessor Member of Committee
Mathematics

Date of thesis defense: April 27, 2017

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ACKNOWLEDGEMENTS

I would like to thank my advisor, Professor Sabine el Khoury, from the American University of Beirut for making this thesis happen. Thank you for helping me through every step of the way and for teaching me all I need to know.

Thank you Professor Hazar Abu Khuzam and Professor Monique Azar for being part of my committee and second readers for my thesis.

And finally, a special thanks to my family members and loved ones for providing me with unfailing support and encouragement throughout these two years of study. This thesis would have never been possible without any of them.

AN ABSTRACT OF THE THESIS OF

Gaelle Gabriel for Master of Science
Major: Mathematics

Title: Subadditivity of Syzygies in a Minimal Graded Free Resolution

Let $R = k[x_1, \ldots, x_n]$ be the polynomial ring in n variables, and I a homogeneous ideal in $(x_1, \ldots x_n)$. Let $\mathbb{F}$ be a minimal graded free resolution of $S = R/I$. We study the subaddivity of Gorenstein algebras and of monomial ideals in the minimal graded free resolution $\mathbb{F}$.

Contents

Chapter 1

Introduction

Let $R = k[x_1, \ldots, x_n]$ be the polynomial ring in n variables, and I a homogeneous ideal in $(x_1, \ldots x_n)$. Let $\mathbb{F}$ be a minimal graded free resolution of $S = R/I$ given by

$$\mathbb{F} : 0 \to \bigoplus_j R(-j)^{\beta_{sj}} \xrightarrow{\partial_s} \ldots \to \bigoplus_j R(-j)^{\beta_{ij}} \xrightarrow{\partial_i} \ldots \to \bigoplus_j R(-j)^{\beta_{1j}} \xrightarrow{\partial_1} R \to R/I \to 0.$$

For each a, denote by T_a and t_a the maximal and minimal shifts in the resolution $\mathbb{F}$:

$$t_a = min\{j : \beta_{aj} \neq 0\} \text{ and } T_a = max\{j : \beta_{aj} \neq 0\}.$$

$\mathbb{F}$ is said to satisfy the *subadditivity* condition for maximal shifts if for all a and b, we have

$$T_{a+b} \leq T_a + T_b.$$

There is a history of looking at the subadditivity of maximal shifts in a minimal graded free resolution: [1], [2], [4], [7], [8] and [10]. Subadditivity for maximal shifts has been established in a few cases. In [4, Corollary 4.1], the authors proved that $T_p \leq T_a + T_{p-a}$ with p =projdim$S$, in the case where R is of depth zero and of dimension ≤ 1. In [8, Corollary 3], it was shown that $T_p \leq T_1 + T_{p-1}$ for all graded algebras . When $S = R/I$ is Koszul, it was proved that $T_{a+1} \leq T_a + T_1 = T_a + 2$ for $a \leq$ height (I), see [2] for instance. More results were established when I is Gorenstein or monomial, [7], [8].

Furthermore, it is known that the minimal graded free resolution of graded algebras may not satisfy the subadditivity for maximal shifts, as we see in example 5.7 in the thesis. However, no counter examples are known for Gorenstein algebras, nor for monomial ideals. The cases of Gorenstein algebras and monomial ideals were partially tackled by El Khoury-Srinivasan [7], and Herzog-Srinivasan [8] respectively.

In this thesis, we study the subaddivity of Gorenstein algebras and monomial ideals. In the Gorenstein case we get $T_h \leq T_{h-a} + T_a$ with $h \geq p-1$ where p =pdim R/I , and in the monomial case we obtain $T_{a+1} \leq T_a + T_1$ for all a.

Chapter 2

Preliminaries

2.1 Graded Rings

Definition 2.1. A graded ring is a ring R together with a direct sum decomposition $R = R_0 \bigoplus R_1 \bigoplus R_2 \bigoplus \cdots$ as abelian groups, such that: $R_i R_j \subset R_{i+j}$ for $i, j \geq 0$.

Example 2.2. The ring of polynomials $R = k[x_1, ..., x_n]$ is a graded ring, graded by degree: $R = S_0 \bigoplus S_1 \bigoplus \cdots$

Definition 2.3. A homogeneous element of R is an element of one of the groups R_i, and a homogeneous ideal of R is an ideal that is generated by homogeneous elements.

Example 2.4. Let $R = k[x, y, z]$, then $I = (x^3 + y^3, z^2 - xy)$ is an ideal of R generated by homogeneous elements.

Remark 2.5. If $f \in R$, there is a unique expression for f of the form

$$f = f_0 + f_1 + \cdots \text{ with } f_i \in R_i \text{ and } f_j = 0 \text{ for } j \gg 0;$$

the f_i are called the homogeneous components of f.

Example 2.6. Let $R = k[x, y, z]$ be the polynomial ring of two variables, where k is a field. Let $f = x^3 + yz$, then the homogeneous components of f are x^3 and yz.

Definition 2.7. A ring homomorphism $\phi : \mathbb{R}^n \to \mathbb{R}^m$ is called homogeneous if $\deg x = \deg \phi(x)$, *i.e* $\deg \phi = 0$. For that we define $R(a)_d = R(a + d)$. Meaning, if f has degree d in $R(a)$, then it has degree 0 in $R(a + d)$.

Example 2.8. Let g be the following map:

$$g : R(-3) \xrightarrow{x^3} R$$

We have $g(1) = x^3$, then 1 has degree 0 in $R(-3 + 3)$ and degree 3 in $R(-3)$.

Definition 2.9. Let R be a graded ring and M an R-module. M is said to be a graded R-module if there exists a family of subgroups $\{M_n\}_{n \in \mathbb{Z}}$ of M such that:

1. $M = \bigoplus_n M_n$ as abelian groups
2. $R_n.M_m \subseteq M_{n+m}$ $\forall n, m$

If $u \in$ M$\setminus\{0\}$ and $u = u_{i1} + \cdots + u_{ik}$ where $u_{ij} \in R_{ij}\setminus\{0\}$, then $u_{i1}, ..., u_{ik}$ are called the homogeneous components of u.

Definition 2.10. (R-Algebras) Let R be a commutative ring. An R-algebra is a ring A which is also an R-module such that the multiplication map $A \times A \to A$ is R-bilinear, that is,

$$r * (ab) = (r * a).b = a.(r * b) \text{ for any } a, b \in A, r \in R.$$

Theorem 2.11. *(Hilbert's basis theorem) If R is a Noetherian ring, then $R[X]$ is a Noetherian ring.*

Proof. Let I be an ideal of $R[X]$. Let J be the set of the leading coefficients of the polynomials in I. Then, J is an ideal of $R:$ Assume $s > t$. Let $c, d \in J$, then $cx^s + \cdots + c_1x + c_0 \in I$ and $dx^t + \cdots + d_1x + d_0 \in I$. Now, $cx^s + \cdots + c_0 + (dx^t + \cdots + d_1x + d_0)x^{s-t} \in I \implies cx^s + \cdots + c_0 + dx^s + \cdots + d_0x^{s-1} = (c+d)x^s + \text{ lower terms } \in I \implies c + d \in J$. Also, $r(cx^t + \cdots + c_0) = rcx^t + \cdots + rc_0 \in I$. So, $rc \in J$. Therefore, J is an ideal of R. Since J is an ideal of R and R is Noetherian, then J is finitely generated. Say J is generated by: $a_1, a_2, ..., a_n \in R$. For each $i = 1, ..., n$, there is a polynomial $f_i \in R[X]$ with a_i being the leading coefficient $\implies f_i = a_ix^{r_i} + \text{lower terms} \in I$. Let $r = max(r_i)$ and suppose $f_1, ..., f_n$ generate an ideal $I' \subseteq I$ of $R[X]$. Let $f = ax^m + \text{lower degree terms}$ be any polynomial in I.

Case 1: If $m < r$, then we are done.

Case 2: If $m \geq r$, since $a \in J$ and J is generated by $a_1, a_2, ..., a_n \in R$, then $a = c_1a_1 + c_2a_2 + \cdots + c_na_n = \sum_{i=1}^n c_ia_i$. Consider: $f - \sum_{i=1}^n c_if_ix^{m-r_i} = f - (c_1f_1x^{m-r_1} + c_2f_2x^{m-r_2} + \cdots) = f - (c_1a_1 + c_2a_2 + \cdots + c_na_n)x^m + \text{lower terms} = f - ax^m +$

lower terms $\in I$. So this new polynomial is in I and has degree $< m$. Proceeding this way, we continue subtracting elements in I' to get a polynomial g of degree $< r$. So we have $f = g+h$. Let M be the R-module generated by $\{1, x, x^2, ..., x^{r-1}\}$, since $f \in I$, it can be written as $f = g+h$ where $g \in M \cap I$ and $h \in I'$. So $I = M \cap I + I'$. M is finitely generated and hence it is Noetherian $\implies M \cap I$ is finitely generated. So let $g_1, g_2, ..., g_t$ be the generators of $M \cap I$, then $f_1, ..., f_n, g_1, g_2, ..., g_t$ generate I. Hence, I is finitely generated.

□

Definition 2.12. R is said to be finitely generated as R_0-algebra means that $R \cong R_0[x_1, ..., x_n]$.

Proposition 2.13. *The following are equivalent for a graded ring R:*

i. R is a Noetherian ring;

ii. R_0 is Noetherian and R is finitely generated as an $R_0 - algebra$.

Proof. $i) \implies ii)$: Let $R_+ = \oplus_{n>0} R_n$. $R_0 \cong R/R_+$, hence is Noetherian. R_+ is an ideal in R, hence is finitely generated, say by $x_1, ..., x_s$, which we may take to be homogeneous elements of R, of degrees $k_1, ..., k_s$ say (all > 0). Let R' be the subring of R generated by $x_1, ..., x_s$ over R_0. We shall show that $R_n \subseteq R'$ for all $n \geq 0$, by induction on n. This is certainly true for $n = 0$. Let $n > 0$ and let $y \in R_n$. Since $y \in R_+$, y is a linear combination of the x_t, say $y = \sum_{t=1}^{s} a_t x_t$, where $a_t \in R_{n-k_i}$ (conventionally $R_m = 0$ if $m < 0$). Since each $k_i > 0$, the inductive hypothesis shows that each a_t is a polynomial in the x's with coefficients in R_0. Hence the same is true of y, and therefore $y \in R'$. Hence $R_n \subseteq R'$ and therefore $R = R'$.

ii) $\implies$ *i*): by Hilbert's basis theorem. □

2.2 Notions in Algebra

Lemma 2.14. *(Nakayama) Suppose M is a finitely generated graded R-module and $m_1, ..., m_n \in M$ generate M/mM. Then $m_1, ..., m_n$ generate M.*

Proof. Let $\overline{M} = M/\sum Rm_i$, show that $\overline{M} = 0$. First, if the m_i generate M/mM, then $\overline{M}/m\overline{M} = (M/\sum Rm_i)/(mM/\sum Rm_i) \cong M/(mM + \sum Rm_i) = 0 \implies \overline{M} = m\overline{M}$. Suppose $\overline{M} \neq 0$, take ξ of least degree in $\overline{M}$ and show that $\xi \notin m\overline{M}$. Since M is finitely generated, $\overline{M}$ is finitely generated. Then, there will be a non-zero element ξ of least degree in $\overline{M}$. Suppose $\xi \in m\overline{M}$, then $\xi = \sum u_i v_i$ with deg $u_i > 0$. Contradiction. □

Definition 2.15. (Local Ring) A local ring is a ring R that has a unique maximal ideal.

Example 2.16. We know that for p prime, $\mathbb{Z}_p$ is a field and the only ideals of a field are $\{0\}$ and itself. Therefore, $\mathbb{Z}_p$ is a local ring.

Example 2.17. Let $R = k[x_1, ..., x_n]$, then the maximal ideals of R are of the following form: $(x_1 - a_1, ..., x_n - a_n)$ since $R/(x_1 - a_1, ..., x_n - a_n) \cong k$. Since R is a graded ring then R can be viewed as a local ring with the only maximal ideal to be $m = (x_1, ..., x_n)$ with respect to homogeneous ideals. For that we set $a_1 = \cdots = a_n = 0$.

Definition 2.18. (Non-zero Divisors) For a commutative ring R and an R-module M, an element r in R is called a non-zero-divisor on M if $rm = 0$ implies $m = 0$ for m in M.

Example 2.19. In the ring $\mathbb{Z}/4\mathbb{Z}$, $\bar{2}$ is a zero divisor since $\bar{2} \times \bar{2} = \bar{4} = \bar{0}$.

Definition 2.20. Let R be a ring and let M be an R-module. A sequence of elements $x_1, ..., x_n \in R$ is called a regular sequence on M if:

1. $(x_1, ..., x_n)M \neq M$, and
2. For $i = 1, ..., n, x_i$ is a non-zero divisor on $M/(x_1, ..., x_{i-1})M$.

Example 2.21. $\{x^2, y^3, z^3\}$ is a regular sequence on the polynomial ring $k[x, y, z]$. So, y^3 is a non-zero divisor on M/x^2M and z^3 is a non-zero divisor on $M/(x^2, y^3)M$.

Definition 2.22. (Height) The height of a proper prime ideal P of R is the maximum of the lengths n of the chains of prime ideals contained in P,

$$P_0 \subset P_1 \subset \cdots \subset P_n = P.$$

The height of any proper ideal I is the minimum of the heights of the prime ideals containing I.

Example 2.23. Let $R = k[x, y, z]$ and $P = (x, y)$. The homogeneous prime ideals of R are: $(0), (x)$, and (x, y). We have, $(0) \subset (x) \subset (x, y) = P$. Hence, $ht(P) = 2$.

2.3 Exterior Algebras

Definition 2.24. Let R be a commutative ring, and let M be an R-module. Define $T^0(M) = R, T^1(M) = M$, and $T^p(M) = M \otimes_R \cdots \otimes_R M \quad (p \text{ tensor times}) \quad \text{if } p \geq 2.$

Proposition 2.25. *If M is an R-module, then there is a graded R-algebra*

$$T(M) = \sum_{p \geq 0} T^p(M)$$

with the action of $r \in R$ on $T^q(M)$ given by

$$r(y_1 \otimes \cdots \otimes y_q) = (ry_1) \otimes y_2 \otimes \cdots \otimes y_q = (y_1 \otimes \cdots \otimes y_q)r,$$

and with the multiplication $T^p(M) \times T^q(M) \to T^{p+q}(M)$, for $p, q \geq 1$, given by $(x_1 \otimes \cdots \otimes x_p, y_1 \otimes \cdots \otimes y_q) \mapsto x_1 \otimes \cdots \otimes x_p \otimes y_1 \otimes \cdots \otimes y_q$.

Definition 2.26. If R is a commutative ring and M is an R-module, then $T(M)$ is called the tensor algebra on M.

Definition 2.27. If M is an R-module, then its exterior algebra is $\bigwedge(M) = T(M)/J$, where J is the ideal generated by all $x \otimes x$ with $x \in M$. J is generated by homogeneous elements (of degree 2), so it is a graded ideal. Hence, $\bigwedge(M)$ is a graded R-algebra,

$$\bigwedge(M) = R \oplus M \oplus \wedge^2(M) \oplus \cdots \oplus \wedge^n(M).$$

This direct sum decomposition gives the exterior algebra the additional structure of a graded algebra, that is

$$\wedge^k(M). \wedge^p (M) \subset \wedge^{k+p}(M).$$

Moreover, if k is the basis field, we have $\wedge^0(M) = k$ and $\wedge^1(M) = M$.

Lemma 2.28. *Let R be a commutative ring, and let M be a R-module.*

1) If $x, y \in M$, then in $\bigwedge^2(M)$, we have

$$x \wedge y = -y \wedge x.$$

2) If $p \geq 2$ and $x_i = x_j$ for some $i \neq j$, then $x_1 \wedge \cdots \wedge x_p = 0$ in $\bigwedge^p(M)$.

Proof. 1) $0 = (x+y) \wedge (x+y) = x \wedge x + x \wedge y + y \wedge x + y \wedge y = x \wedge y + y \wedge x$ hence, $x \wedge y = -(y \wedge x)$.

2) Recall from the definition, that $\bigwedge^p(M) = T^p(M)/J^p$, where $J^p = J \cap T^p(M)$ consists of all elements of degree p in the ideal J generated by all elements in $T^2(M)$ of the form $x \otimes x$. In more detail, J^p consists of all sums of homogeneous elements $\alpha \otimes x \otimes x \otimes \beta$, where $x \in M, \alpha \in T^q(M), \beta \in T^r(M)$, and $q + r + 2 = p$. Since the multiplication in $\bigwedge(M)$ is associative, we can (anti)commute a factor x_i of $x_1 \wedge \cdots \wedge x_p$ several steps away at the possible cost of a change in sign, and so we can force any pair of factors to be adjacent, *i.e.* $x_1 \wedge \cdots \wedge x_p = 0$ if there are two equal adjacent factors, say $x_i = x_{i+1}$.

□

Definition 2.29. (Basis and dimension) If the dimension of V is n and $\{e_1, ..., e_n\}$ is a basis of V, then the set $\{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k} / 1 \leq i_1 < i_2 < \cdots < i_k \leq n\}$ is a basis for $\bigwedge^k(V)$.

The reasoning behind definition 2.29 is the following:

When given any exterior product of the form $v_1 \wedge \cdots \wedge v_k$, every vector v_j can be written as a linear combination of the basis vectors e_i; using bilinearity of the exterior product, this can be expanded to a linear combination of exterior products of those basis vectors. Any exterior product in which the same basis vector appears more than once is zero; any exterior product in which the basis vectors do not appear in the proper order can be reordered, changing the sign whenever two basis vectors change places. In general, the resulting coefficients of the basis k-vectors can be computed as the minors of the matrix that describes the vectors v_j in terms of the basis e_i. By counting the basis elements, the dimension of $\bigwedge^k(V)$ is equal to a binomial coefficient:

$$dim \wedge^k (V) = \binom{n}{k}.$$

In particular, $\bigwedge^k(V) = \{0\}$ for $k > n$. Any element of the exterior algebra can be written as a sum of k-vectors. Hence, as a vector space the exterior algebra is a direct sum

$$\wedge(V) = \wedge^0(V) \oplus \wedge^1(V) \oplus \wedge^2(V) \oplus \cdots \oplus \wedge^n(V),$$

and therefore its dimension is equal to the sum of the binomial coefficients, which is 2^n.

Example 2.30. (Cross and triple products) For vectors in R^3, the exterior algebra is closely related to the cross product and triple product. Using the standard basis $\{e_1, e_2, e_3\}$, the exterior product of a pair of vectors $u = u_1e_1 + u_2e_2 + u_3e_3$ and $v =$

$v_1e_1 + v_2e_2 + v_3e_3$ is:

$$
\begin{aligned}
u \wedge v &= (u_1e_1 + u_2e_2 + u_3e_3) \wedge (v_1e_1 + v_2e_2 + v_3e_3) \\
&= u_1v_2e_1 \wedge e_2 + u_1v_3e_1 \wedge e_3 + u_2v_1e_2 \wedge e_1 + u_2v_3e_2 \wedge e_3 + u_3v_1e_3 \wedge e_1 + u_3v_2e_3 \wedge e_2 \\
&= (u_1v_2 - u_2v_1)(e_1 \wedge e_2) + (u_3v_1 - u_1v_3)(e_3 \wedge e_1) + (u_2v_3 - u_3v_2)(e_2 \wedge e_3)
\end{aligned}
$$

where $\{e_1 \wedge e_2, e_3 \wedge e_1, e_2 \wedge e_3\}$ is the basis for the three dimensional space $\bigwedge^2(R^3)$. The scalar coefficient is the triple product of the three vectors.

Theorem 2.31. *If M is an R-module, $x \in \bigwedge^p(M)$, and $y \in \bigwedge^q(M)$, then*

$$x \wedge y = (-1)^{pq} y \wedge x.$$

Corollary 2.32. *If M can be generated by n-elements, then $\bigwedge^p(M) = \{0\}$ for all $p > n$.*

2.3.1 Koszul Complex

Definition 2.33. (Chain Complex) A chain complex $(M_\bullet, \delta_\bullet)$ is a sequence of abelian groups or modules $(M_i)_{i\in\mathbb{Z}}$ connected by homomorphisms $\delta_n : M_{n+1} \to M_n$, such that the composition of any two consecutive maps is the zero map: $\delta_n \circ \delta_{n+1} = 0$ for all n ($Im\delta_{n+1} \subset Ker\delta_n$). A chain complex is usually written down like this:

$$\cdots M_{i+1} \xrightarrow{\delta_{i+1}} M_i \xrightarrow{\delta_i} M_{i-1} \xrightarrow{\delta_{i-1}} \cdots$$

Definition 2.34. Let R be a commutative ring and M a free R-module of finite rank r. Let $\bigwedge^i M$ be the i^{th} exterior power of M, then, given an R-linear map $s : M \to R$, the

Koszul complex associated to s is the chain complex of R-modules:

$$\mathbb{K}(s) : 0 \to \wedge^r M \xrightarrow{\delta_r} \wedge^{r-1} M \to \cdots \to \wedge^1 M \xrightarrow{\delta_1} R \to 0$$

where the differential δ_k is given, for any $e_i \in M$, by

$$\delta_k(e_1, \cdots, e_k) = \sum_{i=1}^{k} (-1)^{i+1} \delta_1(e_i) e_1 \wedge \cdots \wedge e_{i-1} \wedge e_{i+1} \wedge \cdots \wedge e_k.$$

Notice that e_i is omitted, showing that $\delta_k \circ \delta_{k+1} = 0$.

Example 2.35. Let $R = k[x, y, z]$ and $I = (x^2, y^3, z^3)$ a homogeneous ideal. Let $\mathbb{K}$ be the Koszul complex associated to I:

$$\mathbb{K}(x^2, y^3, z^3) : 0 \to \mathbb{R} \xrightarrow{\delta_3} \mathbb{R}^3 \xrightarrow{\delta_2} \mathbb{R}^3 \xrightarrow{\delta_1} R \to R/I \to 0.$$

$$\delta_1(e_1) = x^2, \delta_1(e_2) = y^3, \text{ and } \delta_1(e_3) = z^3$$

$$\begin{aligned} \text{Hence, } \delta_2(e_1 \wedge e_2) &= \delta_1(e_1)e_2 - \delta_1(e_2)e_1 \\ &= x^2 e_2 - y^3 e_1 \\ \delta_3(e_1 \wedge e_2 \wedge e_3) &= \delta_1(e_1)e_2 \wedge e_3 - \delta_1(e_2)e_1 \wedge e_3 + \delta_1(e_3)e_1 \wedge e_2 \\ &= x^2 e_2 \wedge e_3 - y^3 e_1 \wedge e_3 + z^3 e_1 \wedge e_2 \end{aligned}$$

Example 2.36. In example 2.35, we constructed the Koszul complex associated to the regular sequence $\{x^2, y^3, z^3\}$. Let $I = (x^2, y^3, z^3)$, then

$$\mathbb{K}(x^2, y^3, z^3) : 0 \to \mathbb{R} \xrightarrow{\delta_3} \mathbb{R}^3 \xrightarrow{\delta_2} \mathbb{R}^3 \xrightarrow{\delta_1} R \to R/I \to 0.$$

is a Koszul complex associated to this regular sequence.

Remark 2.37. A Koszul complex complex is a minimal free resolution if and only if the ideal I is generated by a regular sequence. We will discuss in details minimal free resolutions in section 3.1.

2.4 Exact Sequences

Definition 2.38. Consider a sequence of R-modules and homomorphisms

$$\cdots M_{i+1} \xrightarrow{\delta_{i+1}} M_i \xrightarrow{\delta_i} M_{i-1} \to \cdots$$

The sequence is *exact* at M_i if $Im(\delta_{i+1}) = ker(\delta_i)$

Example 2.39.

$$0 \to M_1 \xrightarrow{f} M \xrightarrow{g} M_2 \to 0$$

is exact iff: f is injective, g is surjective, and $ker g = Im f$.

Properties 2.40.

1. *$\delta : M \to N$ is onto if and only if the sequence $M \xrightarrow{\delta} N \to 0$ is exact, where $N \to 0$ is the homomorphism sending every element of N to 0.*

2. *$\delta : M \to N$ is one-to-one if and only if the sequence $0 \to M \xrightarrow{\delta} N$ is exact, where $0 \to M$ is the homomorphism sending 0 to the additive identity of M.*

3. *$\delta : M \to N$ is an isomorphism if and only if $0 \to M \xrightarrow{\delta} N \to 0$ is exact. This follows from the above since δ is an isomorphism if and only if it is one-to-one and onto.*

Properties 2.41. *1. For any R-module homomorphism $\delta : M \to N$, we have an exact sequence*

$$0 \to ker(\delta) \to M \xrightarrow{\delta} N \to coker(\delta) \to 0$$

where $ker(\delta) \to M$ is the inclusion mapping and $N \to coker(\delta) = N/Im(\delta)$ is the natural homomorphism onto the quotient module.

2. If $Q \subset P$ is a submodule of an R-module P, then we have an exact sequence $0 \to Q \to P \xrightarrow{v} P/Q \to 0$, where $Q \to P$ is the inclusive mapping, and v is the natural homomorphism onto the quotient module.

Next we state the following theorem by Buchsbaum-Eisenbud without its proof.

Theorem 2.42. *(Buchsbaum-Eisenbud) A complex of free modules*

$$\mathbb{F} : 0 \to F_m \xrightarrow{\delta_m} F_{m-1} \to \cdots \to F_1 \xrightarrow{\delta_1} F_0$$

over a Noetherian ring R is exact if and only if $rank\delta_{i+1} + rank\delta_i = rankF_i$ and $depthI(\delta_i) \geq i$, for every i.

Chapter 3

Free Resolutions

3.1 Minimal Free Resolutions

The properties of the polynomial ring $R = k[x_1, ..., x_n]$ and its ideals play a fundamental role in the study of the homogeneous coordinate rings of projective varieties and the modules over them in algebraic geometry. In order to study ideals effectively we need to study more general graded modules over R. The simplest way to describe a module is by generators and relations which are called syzygies. Finding syzygies is what characterizes minimal free resolutions.

Definition 3.1. (Free resolutions) Let R be a Noetherian ring and M an R-module. A free resolution of M is an exact sequence of the form: $\cdots F_2 \xrightarrow{\delta_2} F_1 \xrightarrow{\delta_1} F_0 \xrightarrow{\delta_0} M \to 0$, where for all i, F_i is a free R-module. If there exists l such that $F_l = F_{l+1} = \cdots = 0$, then the resolution is finite. In other words, a free resolution can give us exactly the structure of the ideal. It is a complex that resolves I.

Definition 3.2. A complex of graded R-modules

$$\cdots \to F_i \xrightarrow{\delta_i} F_{i-1} \to \cdots$$

is called minimal if for each i, the image of δ_i is contained in mF_{i-1}. *i.e.* $\delta_i(F_i) \subset mF_{i-1}$.

Notation 3.3. We set up the notation for the rest of the thesis as follows:

- $R = k[x_1, ..., x_n]$ the polynomial ring over the field k
- $m = (x_1, ..., x_n)$ the maximal ideal of R
- I a homogeneous ideal in R

Construction of a Minimal Free Resolution

1. Let M as an R-module and m_i be a generator of M. Then, define a map $F_0 = \oplus_i R_i \to M$ by sending the i^th generator to m_i. Let $M_1 \subset F_0$ be the kernel of F_0.

$$\begin{tikzcd} F_1 \arrow[r, "\delta_1"] \arrow[dr, "s_1"'] & F_0 \arrow[r, "\delta_0"] & M \\ & M_1 \arrow[u, "i_1"'] & \end{tikzcd}$$

   Since R is Noetherian, by the Hilbert Basis theorem, M_1 is finitely generated and the elements of M_1 are called the syzygies on m_i.

2. Choosing finitely many homogeneous syzygies that generate M_1, we define a map from a graded free module $F_1 \to F_0$ with image M_1.

3. Continuing this way, we construct a sequence of maps of graded free modules, called a graded free resolution of M:

$$\cdots \to F_i \xrightarrow{\delta_i} F_{i-1} \to \cdots \to F_1 \xrightarrow{\delta_1} F_0$$

since δ_i preserves degrees, we get an exact sequence of finite dimensional vector spaces.

Example 3.4. Let $R = k[x, y, z]$ and $I = (x^2, y^2, xz)$, then a minimal free resolution for I is given by:

$$0 \to \mathbb{R} \xrightarrow{\delta_3} \mathbb{R}^3 \xrightarrow{\delta_2} \mathbb{R}^3 \xrightarrow{\delta_1} R \xrightarrow{\delta_0} R/I \to 0$$

with $\delta_1 = I = (x^2, y^2, xz)$ and δ_2 is constructed in the following way: we need to have $Im\delta_2 = Ker\delta_1$. For that we find the first syzygy. *i.e.* $\begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} \in \mathbb{R}^3$ such that

$$\begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} (x^2, y^2, xz) = 0 \implies \delta_2 = \begin{pmatrix} -y^2 & -z & 0 \\ x^2 & 0 & xz \\ 0 & x & -y^2 \end{pmatrix}$$

Now construct δ_3 in a similar way. So we need

$$Im\delta_3 = Ker\delta_2 \implies \begin{pmatrix} -y^2 & -z & 0 \\ x^2 & 0 & xz \\ 0 & x & -y^2 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \implies \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} = \begin{pmatrix} -z \\ y^2 \\ x \end{pmatrix}.$$

Next, we give an example of a non-minimal free resolution.

Example 3.5. Let $R = k[x]$ and $I = (x^2, x^3)$, then the following is a non-minimal free resolution of R/I:

$$0 \to \mathbb{R} \xrightarrow{\begin{pmatrix} -x \\ 1 \end{pmatrix}} \mathbb{R}^2 \xrightarrow{(x^2,x^3)} R \xrightarrow{\delta_0} R/I \to 0$$

This free resolution is not minimal because $1 \in \delta_2$. i.e: $\delta_2(\mathbb{R}) \nsubseteq m(\mathbb{R}^2)$.

Next, we state the Hilbert Syzygy theorem without the proof.

Theorem 3.6. *(Hilbert Syzygy Theorem) Any finitely generated graded R-module M has a finite graded free resolution*

$$0 \to F_m \xrightarrow{\delta_m} F_{m-1} \to \cdots \to F_1 \xrightarrow{\delta_1} F_0.$$

Theorem 3.7. *Let M be a finitely generated graded R-module, if $\mathbb{F}$ and $\mathbb{G}$ are two minimal free resolutions of M, then there is an isomorphism of complexes $\mathbb{F} \to \mathbb{G}$ inducing the identity map on M.*

Proof. Consider:

$$\begin{tikzcd}
\mathbb{F}:\cdots & F_1 \arrow[r] & F_0 \arrow[r, "d_0"] & M \arrow[r] \arrow[d, "id_M"] & 0 \\
\mathbb{G}:\cdots & G_1 \arrow[r] & G_0 \arrow[r, "\delta_0"] & M \arrow[r] & 0
\end{tikzcd}$$

We first start by constructing the identity map on M. Now, since δ_0 is surjective, F_o is free and every free module is a projective module. Then, there exists $f_0 : F_0 \to G_0$ such that:

$$\begin{tikzcd}
F_0 \arrow[d, "f_0"] & \\
G_0 \arrow[r, "\delta_0"] & M
\end{tikzcd}$$

the diagram commutes. We need to show that f_0 is an isomorphism. To do so, we tensor both $\mathbb{F}$ and $\mathbb{G}$ with $k = R/m$ and we show that $f_0 \otimes id$ is an isomorphism.

$$\begin{tikzcd}
\mathbb{F}:\cdots & F_1 \otimes k \arrow[r] & F_0 \otimes k \arrow[r, "d_0 \otimes id"] & M \otimes k \arrow[r] \arrow[d, "id_M \otimes k"] & 0 \\
\mathbb{G}:\cdots & G_1 \otimes k \arrow[r] & G_0 \otimes k \arrow[r, "\delta_0 \otimes id"] & M \otimes k \arrow[r] & 0
\end{tikzcd}$$

Since $\mathbb{F}$ and $\mathbb{G}$ are minimal, $F_0 \otimes k \cong F_0/MF_0$ and $G_0 \otimes k \cong G_0/MG_0$ which are k-vector spaces, then by theorem $d_0 \otimes id$ and $\delta_0 \otimes id$ are isomorphisms. Hence, $f_0 \otimes id$ is an isomorphism. We will now show that f_0 is an isomorphism. Let $f_0 = (a_{ij})$ be the matrix, then $f_0 \otimes id = (a_{ij} \otimes 1) = (a'_{ij})$ is invertible. Thus, $det(a'_{ij})$ is a unit in k and $det(a_{ij})$ is not in M. This implies that $det(a_{ij})$ is a unit in R and the matrix is invertible, so, f_0 is an isomorphism. Now, to construct f_1 we proceed the same way. f_0 induces an isomorphism between $Kerd_0$ and $Ker\delta_0$. As we have seen earlier in the construction of a minimal free resolution, we map F_1 onto $Kerd_0$, so we obtain a surjective map: $F_1 \to Kerd_0$. Similarly with G_1 and $Ker\delta_0$. We then follow the same procedure as above. □

Definition 3.8. (*Projective Dimension*) *The projective dimension is the length of a minimal free resolution denoted by pdim.*

Example 3.9. In example 3.4, $pdim(R/I) = 3$.

3.2 Graded Minimal Free Resolutions

Definition 3.10. Let $R = k[x_1, ..., x_n]$ be the polynomial ring in n-variables, and I a homogeneous ideal of R. The minimal free resolution of I is said to be graded if the maps are degree preserving at every step of the resolution, that is $\deg\delta_i = 0$ for every i.

Example 3.11. In example 3.4, we have $R = k[x, y, z]$ and $I = (x^2, y^2, xz)$, then a graded minimal free resolution of I is given by:

$$0 \to R(-5) \xrightarrow{\delta_3} \mathbb{R}(-4) \oplus \mathbb{R}(-3) \oplus \mathbb{R}(-4) \xrightarrow{\delta_2} \mathbb{R}^3(-2) \xrightarrow{\delta_1} R \xrightarrow{\delta_0} R/I \to 0.$$

To do so, first consider $\mathbb{R}^3 \xrightarrow{\delta_1} R$. Take $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3$, we have $\begin{pmatrix} a \\ b \\ c \end{pmatrix} (x^2, y^2, xz) = ax^2 + by^2 + cxz$ a homogeneous expression. We split $\mathbb{R}^3$ into $\mathbb{R}(-k) \oplus \mathbb{R}(-l) \oplus \mathbb{R}(-m)$. Hence, a has degree $a + k$ in $\mathbb{R}(-k)$, b has degree $b + l$ in $\mathbb{R}(-l)$, and c has degree $c + m$ in $\mathbb{R}(-m)$.

Since the above expression must be homogeneous then degree $a+2$ = degree $b+2$ = degree $c+2$. On the other hand, the maps are degree preserving *i.e. degree* $\delta_1 = 0$. *We obtain,*

$$\text{degree } a + k = \text{degree } a + 2 \implies k = 2,$$

$$\text{degree } b + l = \text{degree } b + 2 \implies l = 2, \text{ and}$$

$$\text{degree } c + m = \text{degree } c + 2 \implies m = 2.$$

So $\mathbb{R}^3$ *is represented by the graded module* $\mathbb{R}^3(-2)$.

Moving onto δ_2, *we want* $\begin{pmatrix} -y^2 & -z & 0 \\ x^2 & 0 & xz \\ 0 & x & -y^2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ *which is equal to* $\begin{pmatrix} -ay^2 - zb \\ ax^2 + cxz \\ bx - cy^2 \end{pmatrix}$ *to belong to* $\mathbb{R}^3(-2)$. $\mathbb{R}^3$ *is split into* $\mathbb{R}(-n) \oplus \mathbb{R}(-p) \oplus \mathbb{R}(-q)$. *Since the maps are degree preserving and the polynomials are homogeneous, then*

$$\deg a + n = \deg a + 4 \implies n = 4,$$

$$\deg b + p = \deg b + 3 \implies p = 3, \text{ and}$$

$$\deg c + q = \deg c + 4 \implies q = 4.$$

So, $\mathbb{R}^3$ *is represented by the following graded module:* $\mathbb{R}^2(-4) \oplus \mathbb{R}(-3)$.

Now, $\delta_3 : \mathbb{R}(-r) \to \mathbb{R}(-4) \oplus \mathbb{R}(-3) \oplus \mathbb{R}(-4)$, *so* $\deg a + r = \deg a + 5 \implies r = 5$. *Hence,* R *is represented by the following graded module:* $R(-5)$.

Corollary 3.12. *A graded free resolution* $\mathbb{F} : \cdots \to F_i \xrightarrow{\delta_i} F_{i-1} \to \cdots$ *is minimal if and only if* $\forall i, \delta_i$ *takes a basis of* F_i *to a minimal set of generators of the image of* δ_i.

Proof. Consider the right exact sequence $F_{i+1} \xrightarrow{\delta_{i+1}} F_i \xrightarrow{\delta_i} Im\delta_i \to 0$.

$$\begin{aligned}
\mathbb{F} \text{ is minimal} &\iff \forall i, \delta_{i+1}(F_{i+1}) \subseteq mF_i \\
&\iff F_{i+1} \xrightarrow{\bar{\delta}_{i+1}} F_i/mF_i \text{ is the zero map} \\
&\iff F_{i+1}/mF_{i+1} \xrightarrow{\bar{\delta}_{i+1}} F_i/mF_i \text{ is the zero map} \\
&\iff F_i/mF_i \xrightarrow{\bar{\phi}} Im\delta_i/m(Im\delta_i) \text{ is an isomorphism}
\end{aligned}$$

(because $\bar{\delta}_{i+1}$ is the zero map, and by exactness $Ker\bar{\phi} = Im\bar{\delta}_{i+1} = 0$ and $\bar{\phi}$ is surjective). Suppose now $\bar{\phi}$ is an isomophism. We will show that $\forall i, \delta_i$ takes a basis of F_i to a minimal set of generators of the image of δ_i.

$\{f_1, ..., f_n\}$ is a basis of F_i (minimal set of generators), then $\{\bar{f}_1, ..., \bar{f}_n\}$ is a minimal set of generators of F_i/mF_i. By Nakayama's lemma, with $M/mM = F_i/mF_i$, we have $\{\bar{f}_1, ..., \bar{f}_n\}$ generate F_i. Now, $\bar{\phi}(\bar{f}_i) = m_i$ is a minimal set of generators of $Im\delta_i/m(Im\delta_i) \implies$ by Nakayam's lemma, m_i generates $Im\delta_i$ and $\{m_i\}$ is a minimal set of generators.

Suppose now that $\forall i, \delta_i$ takes a basis of F_i to a minimal set of generators of the image of δ_i, we show that $\bar{\phi}$ is an isomorphism. Since $R = k[x_0, ..., x_n]$ and M is an R-module, then M/mM is an $R/m = k$ vector space. Consider the following diagram

$$\begin{tikzcd}
F_i/mF_i \arrow[r, "\bar{\phi}"] & Im\delta_i/mIm\delta_i \\
F_i \arrow[u] \arrow[r, "\delta_i"] & Im\delta_iF_i \arrow[u]
\end{tikzcd}$$

We know that a basis in F_i is sent to a minimal set of generators of F_i/mF_i, and a minimal set of generators of $Im\delta_i F_i$ is sent to a minimal set of generators of $mIm\delta_i$. Therefore, $\bar{\phi}$ is an isomorphism.

$\square$

3.3 Betti Diagrams

Definition 3.13. Let $R = k[x_1, ..., x_n]$ be the polynomial ring in n-variables. A compact way to describe minimal free resolutions is the Betti diagram. Let $(\mathbb{F}, \delta)$ be a minimal graded free resolution of R over S with $\mathbb{F}_a = \oplus_j S(-j)^{\beta_{aj}}$. The β_{aj} are called Betti numbers. The elements $(\beta_{ij})_{\substack{0 \leq i \leq n \\ j \in \mathbb{N}}}$ satisfy:

i. for all $0 \leq i \leq n, \beta_{i,j} = 0$ for all j such that $|j| \gg 0$.

ii. for all $i > 0$ and for all j, if $\beta_{i,j} \neq 0$, then there exists $j' < j$ such that $\beta_{i-1,j'} \neq 0$.

Definition 3.14. *(Betti Diagram) Let $(\mathbb{F}, \delta)$ be a minimal graded free resolution of R over S with $\mathbb{F}_a = \oplus_j S(-j)^{\beta_{aj}}$. Each F_i requires $\beta_{i,j}$ minimal generators of degree j. The Betti diagram of $\mathbb{F}$ has the form:*

	0	*1*	$\cdots$	s
0	β_0	β_1	$\ldots$	β_s
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
i	$\beta_{0,i}$	$\beta_{1,i+1}$	$\cdots$	$\beta_{s,i+s}$
i+1	$\beta_{0,i+1}$	$\beta_{1,i+2}$	$\cdots$	$\beta_{s,i+s+1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
j	$\beta_{0,j}$	$\beta_{1,j+1}$	$\cdots$	β_{s,j_s}

- *the column labeled i describes the free module F_i*
- *s+1 columns correspond to the free modules $F_0, ..., F_s$*
- *rows labeled with consecutive integers correspond to the degrees*
- $\beta_k = \sum_r \beta_{kr}$

Example 3.15. Let $R = \mathbb{Q}[x, y, z, w]$, $I = (x^2y, xy^2, z^3, w^2 - y^2, w^3)$, and the following minimal free resolution:

$$0 \to R^2 \to R^7 \to R^9 \to R^5 \to R$$

which can be extracted from the Betti diagram below.

	0	1	2	3	4
0	1	5	9	7	2
1	-	1	-	-	-
2	-	4	2	-	-
3	-	-	4	2	-
4	-	-	3	2	-
5	-	-	-	3	2

We will extract the graded minimal free resolution:

Step 0: 1 minimal generator of degree 0 $\implies$ $R = R$.

Step 1: 1 minimal generator of degree 2 and 4 minimal ones of degree 3 $\implies$ R^5 is represented by the following graded module: $R^4(-3) \oplus R(-2)$.

Step 2: 2 minimal generators of degree 4, 4 of degree 5, and 3 of degree 6 $\implies$ R^9 is represented by the following graded module: $R^3(-6) \oplus R^4(-5) \oplus R^2(-4)$.

Step 3: 2 minimal generators of degree 6, 2 of degree 7, and 3 of degree 8 $\implies$ R^7 is represented by the following graded module: $R^3(-8) \oplus R^2(-7) \oplus R^2(-6)$.

Step 4: 2 minimal generators of degree 9 $\implies$ $R^2 = R^2(-9)$.

So the graded minimal free resolution is given by:

$$0 \to R^2(-9) \to R^3(-8) \oplus R^2(-7) \oplus R^2(-6) \to R^3(-6) \oplus R^4(-5) \oplus R^2(-4)$$

$$\to R^4(-3) \oplus R(-2) \to R.$$

Chapter 4

Syzygies of Gorenstein Algebras

Let $\mathbb{F}$ be a minimal graded free resolution of $S = R/I$

$$0 \to \bigoplus_{j=t_s}^{T_s} R(-j)^{\beta_{sj}} \xrightarrow{\partial_s} \ldots \to \bigoplus_{j=t_i}^{T_i} R(-j)^{\beta_{ij}} \xrightarrow{\partial_i} \ldots \to \bigoplus_{j=t_1}^{T_1} R(-j)^{\beta_{1j}} \xrightarrow{\partial_1} R \to R/I \to 0.$$

$\mathbb{F}$ is said to satisfy the *subadditivity* condition for maximal shifts if for all a and b, we have $T_{a+b} \leq T_a + T_b$. Subadditivity has been established in several cases, see [1], [2], [4], [7], [8] and [10]. In this section, we study the Gorenstein case done by [7], they show that subadditivity holds for $a + b = h$ and $h - 1$ respectively, where h is the height of I.

4.1 Gorenstein Algebras

Definition 4.1. Let R be a commutative Noetherian ring with finite projective dimension. The grade of I is the length of a maximal R-sequence contained in I. If the grade of I is equal to the height of I, then R is said to be Cohen-Macauly.

Definition 4.2. An ideal I is called perfect if the grade of $I = pdim(R/I)$.

Definition 4.3. An ideal I of grade g is called a Gorenstein ideal if I is perfect and $rankF_g = 1$.

Every Gorenstein ideal admits a symmetric minimal free resolution of the following form:

If I is of height $2k+1$, then

$$0 \to R(-c) \to \sum_{j=1}^{b_1} R(-(c-a_{1j}) \to \cdots \to \sum_{j=1}^{b_k} R(-(c-a_{kj})) \to \sum_{j=1}^{b_k} R(-a_{kj}) \to$$

$$\cdots \to \sum_{j=1}^{b_1} R(-a_{1j}) \to R$$

and if I is of height $2k$, then

$$0 \to R(-c) \to \sum_{j=1}^{b_1} R(-(c-a_{1j}) \to \cdots \to \sum_{j=1}^{b_k/2=r_k} R(-(c-a_{kj}) \oplus \sum_{j=1}^{b_k/2=r_k} R(-a_{kj}) \to$$

$$\cdots \to \sum_{j=1}^{b_1} R(-a_{1j}) \to R$$

Example 4.4.

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i110 : R = QQ[x,y,z,w]
```

```
o110 = R

o110 : PolynomialRing

o111 = ideal (w^2, y*w, x*w, y*z, x*z, x^2 - y^2, z^3, y^3 + z^2 w, x*y^2)

o111 : Ideal of R

i112 : betti res o111


              0 1  2 3 4

o112 = total: 1 9 16 9 1

           0: 1 .  . . .

           1: . 6  8 3 .

           2: . 3  8 6 .

           3: . .  . . 1

o112 : BettiTally
```

Hence, I admits the following graded minimal free resolution:

$$0 \to R(-7) \to R^6(-5) \oplus R^3(-4) \to R^8(-4) \oplus R^8(-3) \to R^3(-3) \oplus R^6(-2) \to R.$$

4.2 Syzygies

In this subsection, we first prove a general result on the syzygies of homogeneous algebras, then we show a partial result on the subadditivity of graded Gorenstein algebras R/I with height $I = h$.

Theorem 4.5. *Let $S = R/I$ be a graded algebra with t_i and T_i being the minimal and maximal shifts in the minimal graded free R-resolution of S at degree i, then $t_n \leq t_1 + T_{n-1}$, for all n.*

Proof. We show the theorem by induction on n, where n is the n^{th} step of the resolution. For $n = 1$, $T_0 = 0$ and hence $t_1 = t_1$. We need to prove the theorem for $1 < n \leq s$, where

$$\mathbb{F} : 0 \to F_s \to \cdots \to F_2 \xrightarrow{\delta_2} F_1 \xrightarrow{\delta_1} S$$

is the graded resolution of S. Let $I = (g_1, ..., g_{\beta_1})$ where $\{g_1, ..., g_{\beta_1}\}$ is a set of minimal generators of I and $F_i = \oplus_{j=1}^{\beta_i} R f_{ij}$ with $t_1 = \deg f_{11} \leq \deg f_{12} \leq ... \leq \deg f_{1\beta_1} = T_1$. Suppose $\delta_1(f_{1j}) = g_j$ and $\delta_n(f_{nt}) = \sum_{i=1}^{\beta_{n-1}} r_{ti} f_{(n-1)i}$ for all n.

We show the theorem for $n = 2$, and $n = 3$. Consider the following diagram:

$$\begin{tikzcd}
\mathbb{F} : \cdots \arrow[r] & F_3 \arrow[r, "\delta_3"] & F_2 \arrow[r, "\delta_2"] & F_1 \arrow[r] & F_0 \\
\mathbb{K} : \cdots \arrow[r] & F_2 \wedge F_1 \arrow[r] \arrow[u] & F_1 \wedge F_1 \arrow[r] \arrow[u] & F_1 \arrow[r] \arrow[u] & F_0 \arrow[u]
\end{tikzcd}$$

where $\mathbb{K}$ is the Koszul complex associated to $(g_1, ..., g_{\beta_1})$.

Let $n = 2$ and $2 \leq i \leq \beta_1$. Construct $Z(f_{11}, f_{1i}) \in F_1$ where $Z(f_{11}, f_{1i}) = g_1 f_{1i} - g_i f_{11}$ is a non-zero second syzygy. We have

$$\delta_1(Z(f_{11}, f_{1i})) = g_1 \delta_1(f_{1i}) - g_i \delta_1(f_{11}) = g_1 g_i - g_i g_1 = 0$$

$\implies$ $Z(f_{11}, f_{1i}) \in Ker\delta_1 = Im\delta_2$, by exactness. So, there exists an element $f_{11} * f_{1i} \in F_2$ such that $\delta_2(f_{11} * f_{1i}) = Z(f_{11}, f_{1i}) \neq 0$. So we get, $\deg(f_{11} * f_{1i})$=$\deg(\delta_2(f_{11} * f_{1i})$ =$\deg(Z(f_{11}, f_{1i})$ = $\deg(g_1)$+ $\deg(g_i) \leq t_1 + T_1$. Since $Z(f_{11}, f_{1i})$ is a syzygy, then degg_1+degg_i should be greater than or equal to the minimum of the degrees of the syzygies in F_2 which is t_2, $\Longrightarrow$ $t_1 + T_1 \geq$ degg_1+deg$g_i \geq t_2$.

Let $n = 3$, by the induction assumption, we construct the following syzygy $f_{11} * f_{1i} \in F_2/\ \delta_2(f_{11} * f_{1i}) = Z(f_{11}, f_{1i})$. On the other hand, $\delta_2(f_{11} * f_{1i}) = \delta_1(f_{11}).f_{1i} - f_{11} * \delta_1(f_{1i})$ by $\mathbb{K}$. Let $f_{11} * f_{1i} = \sum_{j=1}^{\beta_2} s_{ij} f_{2j}$.

$Z(f_{11}, f_{2t}) = \delta_1(f_{11}) f_{2t} - \sum\limits_{i=1}^{\beta_1} \sum\limits_{j=1}^{\beta_2} r_{tj} s_{ij} f_{2j} \in F_2$. We get

$$\begin{aligned}
\delta_2(Z(f_{11}, f_{2t})) &= \delta_1(f_{11})\delta_2(f_{2t}) - \sum_{i=1}^{\beta_1} r_{ti}\delta_2(\sum_{j=t}^{\beta_2} s_{ij} f_{2j}) \\
&= \delta_1(f_{11}) \sum_{i=1}^{\beta_1} r_{ti} f_{1i} - \sum_{i=1}^{\beta_1} r_{ti}\delta_2(f_{11} * f_{1i}) \\
&= \sum_{i=1}^{\beta_1} [\delta_1(f_{11}) r_{ti} f_{1i} - \delta_1(f_{11}) r_{ti} f_{1i} + r_{ti} f_{11} * \delta_1(f_{1i})] \\
&= \sum_{i=1}^{\beta_1} r_{ti} f_{11} * \delta_1(f_{1i}) \\
&= f_{11} * \delta_1(\sum_{i=1}^{\beta_1} r_{ti} f_{1i}) \\
&= f_{11} * (\delta_1 \circ \delta_2(f_{2t})) \qquad\qquad \text{(by exactness)} \\
&= f_{11} * 0 \\
&= 0
\end{aligned}$$

We show $Z(f_{11}, f_{2t}) \neq 0$. For that, we suppose $Z(f_{11}, f_{2t}) = 0$ for all t, then

$\delta_1(f_{11})f_{2t} = \sum\limits_{i=1}^{\beta_1}\sum\limits_{j=1}^{\beta_2} r_{ti}s_{ij}f_{2j}$. So for all t, we have

$$\sum_{i=1}^{\beta_1}\sum_{j=1}^{\beta_2} r_{ti}s_{ij} = \begin{cases} 0 & j \neq t \\ \\ \delta_1(f_{11}) & j = t. \end{cases} \tag{4.1}$$

We set $\bar{r} = \bar{r}_2 = (r_{ti})_{\beta_2\times\beta_1}$ and $\bar{s} = \bar{s}_2 = (s_{ij})_{\beta_1\times\beta_2}$:

$$\begin{tikzcd}
\mathbb{F}:\cdots \arrow[r] & F_3 \arrow[r, "\bar{r}_3"] & F_2 \arrow[r, "\bar{r}_2"] & F_1 \arrow[r] & F_0 \\
\mathbb{K}:\cdots \arrow[r] & F_2\wedge F_1 \arrow[r] \arrow[u, "\bar{s}_3"'] & F_1\wedge F_1 \arrow[r] \arrow[u, "\bar{s}_2"'] & F_1 \arrow[r] \arrow[u] & F_0 \arrow[u]
\end{tikzcd}$$

By 4.1, we get $\bar{r}\bar{s} = \delta_1(f_{11})I$ which implies that the rank $\bar{r}\bar{s} = \beta_2$. By the exactness of the resolution, we have rank$(\bar{r}_3)$+rank$(\bar{r}_2) = \beta_2$ where $\bar{r}_3$ is the matrix representing δ_3. Since $\delta_3 \neq 0$, then rank $\bar{r}_2 < \beta_2$ which is a contradiction.

We showed there exists a non zero cycle in F_2 of degree t_1+deg$f_{2t} \leq t_1 + T_2$. Again, $Z(f_{11}, f_{2t}) \in ker\delta_2 = im\delta_3$ which implies the existence of an element $f_{11} * f_{2i} \in F_3$ such that $\delta_3(f_{11} * f_{2i}) = Z(f_{11}, f_{2t})$. The degree of $f_{11} * f_{2i}$=deg$(Z(f_{11}, f_{2t})) = t_1 + T_2$ which is $\geq t_3$.

The cases $n = 2, 3$ are established. We proceed by supposing that the statement is true for $n-1$. For that, there exists an element $f_{11} * f_{(n-2)i} \in F_{n-1}$ for all $1 \leq i \leq \beta_{n-2}$ such that

$$\delta_{n-1}(f_{11} * f_{(n-2)i}) = \delta_1(f_{11})f_{(n-2)i} - f_{11} * \delta_2(f_{(n-2)i})),$$

$$\text{and } t_{n-1} \leq t_1 + T_{n-2} \text{ for } 4 \leq n \leq s.$$

Let $(f_{11} * f_{(n-2)i}) = \sum\limits_{j=1}^{\beta_{n-1}} s_{ij}f_{(n-1)j}$. We also have $\delta_{n-1}(f_{(n-1)t}) = \sum\limits_{i=1}^{\beta_{n-2}} r_{ti}f_{(n-2)i}$ for all

$1 \leq t \leq \beta_{n-1}$. Consider $Z(f_{11}, f_{(n-1)t}) = \delta_1(f_{11})f_{(n-1)t} - \sum\limits_{i=1}^{\beta_{n-2}} \sum\limits_{j=1}^{\beta_{n-1}} r_{ti}s_{ij}f_{(n-1)j}$, then $Z(f_{11}, f_{(n-1)t})$ is a cycle in F_{n-1}, since

$$\begin{aligned}
\delta_{n-1}(Z(f_{11}, f_{(n-1)t})) &= \delta_1(f_{11})\delta_{n-1}(f_{(n-1)t}) - \sum_{i=1}^{\beta_{n-2}} r_{ti}\delta_{n-1}(\sum_{j=t}^{\beta_{n-1}} s_{ij}f_{(n-1)j}) \\
&= \delta_1(f_{11}) \sum_{i=1}^{\beta_{n-2}} r_{ti}f_{(n-1)i} - \sum_{i=1}^{\beta_{n-2}} r_{ti}\delta_{n-1}(f_{11} * f_{(n-2)i})) \\
&= \sum_{i=1}^{\beta_{n-2}} [\delta_1(f_{11})r_{ti}f_{(n-2)i} - \delta_1(f_{11})r_{ti}f_{(n-2)i} + r_{ti}f_{11} * \delta_{n-2}(f_{(n-2)i})] \\
&= \sum_{i=1}^{\beta_(n-2)} r_{ti}f_{11} * \delta_{n-2}(f_{(n-2)i}) \\
&= f_{11} * \delta_{n-2}(\sum_{i=1}^{\beta_{n-2}} r_{ti}f_{(n-2)i}) \\
&= f_{11} * (\delta_{n-2} \circ \delta_{n-1}(f_{(n-1)t})) \\
&= f_{11} * 0 \\
&= 0
\end{aligned}$$

We show that there is at least one t such that one of the cycles $Z(f_{11}, f_{(n-1)t})$ is not identically zero. Suppose that $Z(f_{11}, f_{(n-1)t}) = 0$ for all t, then $\delta_1(f_{11})f_{(n-1)t} = \sum\limits_{i=1}^{\beta_{n-2}} \sum\limits_{j=1}^{\beta_{n-1}} r_{ti}s_{ij}f_{(n-1)j}$. So for all t, we have

$$\sum_{i=1}^{\beta_{n-2}} \sum_{j=1}^{\beta_{n-1}} r_{ti}s_{ij} = \begin{cases} 0 & j \neq t \\ \delta_1(f_{11}) & j = t. \end{cases}$$

By setting $\bar{r} = \bar{r}_{n-1} = (r_{ti})_{\beta_{n-1} \times \beta_{n-2}}$ and $\bar{s} = (s_{ij})_{\beta_{n-2} \times \beta_{n-1}}$, we get $\bar{r}\bar{s} = \delta_1(f_{11})I$ and hence the rank $\bar{r}_{n-1}\bar{s} = \beta_{n-1}$. By the exactness of the resolution, we have rank$(\bar{r}_n)$+rank$(\bar{r}_{n-1}) =$

β_{n-1} where $\bar{r}_n$ is the matrix representing δ_n. Since $\delta_n \neq 0$, this implies that rank $\bar{r}_{n-1} < \beta_{n-1}$ which is a contradiction.

Thus, for every t, $1 \leq t \leq \beta_{n-1}$, there exists an element $f_{11} * f_{(n-1)t} \in F_n$ of the same degree as $\deg Z(f_{11}, f_{(n-1)t})$ that is mapped by δ_n onto $Z(f_{11}, f_{(n-1)t})$. This means there is an element $f_{11} * f_{(n-1)t} \in F_n$ of degree $t_1 + \deg f_{(n-1)t} \leq t_1 + T_{n-1}$ such that $\delta_n(f_{11}) * f_{(n-1)t}) = Z(f_{11}, f_{(n-1)t}) \neq 0$. Hence, we get $t_1 + T_{n-1} \geq t_n$. □

Remark 4.6. If $S = R/I$ is a Gorenstein algebra, with height $I = h$, then $T_h \leq T_a + T_{h-a}$. Since $c = T_h = t_h$, then by the duality of the minimal graded free resolution $\mathbb{F}$ we get $c - t_{h-a} = T_a$ for all $a = 1, ..., h-1$. This implies that $c = T_a + t_{h-a} \leq T_a + T_{h-a}$.

Theorem 4.7. *For any graded Gorenstein algebra R/I with height $I = h$, we have $T_{h-1} \leq T_a + T_{h-1-a}$. Thus, $T_n \leq T_a + T_{n-a}$ for $n \geq h-1$.*

Proof. Since R/I is Gorenstein, then $T_{h-1} = T_h - t_1$ and $T_{h-1-a} = T_h - t_{a+1}$. So, $T_{h-a-1} = T_h - t_{a+1} \geq T_h - (t_1 + T_a)$ by theorem 4.5. So, $T_{h-a-1} \geq T_h - t_1 - T_a = T_{h-1} - T_a$ and hence $T_{h-1} \leq T_a + T_{h-a-1}$ as desired. □

Example 4.8.

```
i197 : R= QQ[a,b,c,d,e]

o197 = R

o197 : PolynomialRing

i198 : I = ideal fromDual matrix{{a^2*b*c^2-c^4*d+d^3*e^2+e^5}}
```

```
o198=ideal(c*e,b*e, a*e,b*d,a*d,b^2,d^3-e^3,c*d^2,a^2b+c^2d,a^3,d*e^3,

                   c^4+d^2e^2,b*c^3,a*c^3)

o198 : Ideal of R

i199 : betti res o198

              0  1  2  3  4 5

o199 = total: 1 14 35 35 14 1

           0: 1  .  .  .  . .

           1: .  6  8  3  . .

           2: .  4 12 12  4 .

           3: .  4 12 12  4 .

           4: .  .  3  8  6 .

           5: .  .  .  .  . 1

o199 : BettiTally
```

Then by theorem 4.7,

$$T_4 \leq T_3 + T_1 \implies 8 \leq 11 \text{ and } T_4 \leq 2T_2 \implies 8 \leq 12.$$

Chapter 5

Syzygies of Monomial Ideals

In this section, we study the subadditivity of monomial ideals that was shown by [8] for $b = 1$. Before stating the main theorems, we recall the definition of a dual basis of a vector spaces over a field k.

Definition 5.1. (Dual Basis) Let V be a vector space over the field k. We view k as a one-dimensional vector space over itself. The set of all linear maps of V into k is called the dual space, and will be denoted by V^*. Elements of the dual space are usually called functionals.

Let V be finite dimensional of dimension n. Let $\{v_1, ..., v_n\}$ be a basis. Write each element v in terms of its coordinate vector $v = x_1v_1 + ... + x_nv_n$. For each i we let

$$\phi_i : V \to k$$

be the functional such that

$$\phi_i(v_i) = 1 \qquad \text{and} \qquad \phi_i(v_j) = 0 \qquad \text{if } i \neq j.$$

Then

$$\phi_i(v) = x_i.$$

The functionals $\{\phi_1, ..., \phi_n\}$ form a basis of V^*, called the dual basis of $\{v_1, ..., v_n\}$.

Definition 5.2. If $f : V \to W$ is a linear map, then the transpose (or dual) $f^* : W^* \to V^*$ is defined by:

$$f^*(\phi) = \phi \circ f \qquad \text{for every } \phi \in W^*.$$

The resulting functional $f^*(\phi)$ in V^* is called the pullback of ϕ along f.

If the linear map f is represented by the matrix A with respect to two bases of V and W, then f^* is represented by the transpose matrix A^T with respect to the dual bases of W^* and V^*.

Notation 5.3. Let k be a field, $R = k[x_1, ..., x_n]$ the polynomial ring over k in the indeterminates $(x_1, ..., x_n)$ and $I \subset R$ a graded ideal. Let $(\mathbb{F}, \delta)$ be a graded R-resolution of $S = R/I$. Each free module $\mathbb{F}_a$ in the resolution of the form $\mathbb{F}_a = \oplus_j R(-j)^{b_{aj}}$. We set

$$T_a = max\{j : b_{aj} \neq 0\}.$$

Proposition 5.4. *Let $I \subset R$ be a graded ideal, $\mathbb{F}$ the graded minimal free resolution of R/I. Suppose there exists a homogeneous basis $f_1, ..., f_r$ of F_a such that*

$$\delta(\mathbb{F}_{a+1}) \subset \bigoplus_{i=1}^{r-1} Rf_i.$$

Then $degf_r \leq T_{a-1} + T_1$.

Proof. We denote by $(\mathbb{F}^*, \delta^*)$ the complex $\mathrm{Hom}_R(\mathbb{F}, R)$ which is dual to $\mathbb{F}$. For any basis $h_1, ..., h_l$ of $\mathbb{F}_b$ we denote by h_i^* the basis element of $\mathbb{F}_b^*$ with

$$h_i^*(h_j) = \begin{cases} 1 & \text{if } j = i \\ \\ 0 & \text{if } j \neq i. \end{cases}$$

Then, $h_1^*, ..., h_l^*$ is a basis of $\mathbb{F}_b^*$, the so-called dual basis of $h_1, ..., h_l$.

$$\mathbb{F} : \cdots \to F_{a+1} \xrightarrow{\delta} F_a \xrightarrow{\delta} F_{a-1} \cdots$$

$$\mathbb{F}^* : \cdots \to F_{a-1}^* \xrightarrow{\delta^*} F_a^* \xrightarrow{\delta^*} F_{a+1}^* \cdots$$

Our assumption implies that $\delta^*(f_r^*) = 0$. This implies that f_r^* is a generator of $H^a(\mathbb{F}^*) = Ker\delta^*/Im\delta^*$, which is an R/I module.

On the other hand, if $g_1, ..., g_m$ is a basis of $\mathbb{F}_{a-1}$ and $\delta(f_r) = c_1 g_1 + ... + c_m g_m$, then $\delta^*(g_i^*) = c_i f_r^* + m_i$ where each m_i is a linear combination of the remaining basis elements of $\mathbb{F}_a^*$. Let $c \in I$ be a generator of maximal degree. Then by definition, $\deg c = T_1(I)$. Since $If_r^* = 0$ in $H^a(\mathbb{F}^*)$, there exist homogeneous elements $s_i \in S$ such that $cf_r^* = \sum_{i=1}^m s_i(c_i f_r^* + m_i)$. This is only possible if $T_1 = \deg c_i + \deg s_i$ for some i. In particular, $\deg c_i \leq T_1$. It follows that $\deg f_r = \deg c_i + \deg g_i \leq T_1 + T_{a-1}$, as desired. □

Corollary 5.5. *Let I be a monomial ideal. Then $T_a \leq T_{a-1} + T_1$ for all $a \geq 1$.*

For the proof of this result, we will use the restriction lemma as given in [4, Lemma 4.4]: let I be a monomial ideal with multigraded (minimal) free resolution $\mathbb{F}$ and let $\alpha \in \mathbb{N}^n$.

Then the restricted complex $\mathbb{F}^{\leq\alpha}$*which is the subcomplex of* $\mathbb{F}$ *for which* $(\mathbb{F}^{\leq\alpha})_i$ *is spanned by those basis elements of* $\mathbb{F}_i$ *whose multidegree is componentwise less than or equal to* α*, is a (minimal) multigraded free resolution of the monomial ideal* $I^{\leq\alpha}$ *which is generated by all monomials* $x^b \in I$ *with* $b \leq \alpha$*, componentwise.*

Proof. Let $\mathbb{F}$ the minimal multigraded free R-resolution of R/I, and let $f \in F_a$ be a homogeneous element of multidegree $\alpha \in \mathbb{N}^n$ whose total degree is $T_a(I)$. We apply the restriction lemma and consider the restricted complex $\mathbb{F}^{\leq\alpha}$.

Let $f_1, ..., f_r$ be a homogeneous basis of $(\mathbb{F}^{\leq\alpha})_a$ with $f_r = f$. Since there is no basis element of $(\mathbb{F}^{\leq\alpha})_{a+1}$ of multidegree whose coefficient is bigger than α, and since the resolution $\mathbb{F}^{\leq\alpha}$ is minimal, it follows that $\delta((\mathbb{F}^{\leq\alpha})_{a+1}) \subset \bigoplus_{i=1}^{r-1} Rf_i$. Thus, we may apply the above proposition and deduce that $T_a(I^{\leq\alpha}) \leq T_{a-1}(I^{\leq\alpha}) + T_1(I^{\leq\alpha})$. Since $T_a(I) \leq T_a(I^{\leq\alpha}), T_{a-1}(I^{\leq\alpha}) \leq T_{a-1}$ and $T_1(I^{\leq\alpha}) \leq T_1$, the assertion follows. $\square$

Example 5.6. `i2 : R = QQ[a,b,c,d,e,f,g,h]`

```
o2 = R

o2 : PolynomialRing

i3 : ideal(a^2*b^3*c, a^3*b^3*c^3, d*f^2, e*f*c*a, g*h^2, g^3*h^3*a^2, g*h*a*c)

o3 = ideal(a^2b^3c, a^3b^3c^3, d*f^2, a*c*e*f, g*h^2, a^2 g^3h^3, a*c*g*h)

o3 : Ideal of R

i4 : betti res o3

            0 1 2 3 4

o4 = total: 1 5 8 5 1
```

```
0: 1 . . . .
1: . . . . .
2: . 2 . . .
3: . 2 1 . .
4: . . 3 . .
5: . 1 1 2 .
6: . . 2 . .
7: . . 1 2 .
8: . . . 1 1
o4 : BettiTally
```

Hence,

$$T_1 \le T_0 + T_1 \implies 6 \le 6$$

$$T_2 \le 2T_1 \implies 9 \le 12$$

$$T_3 \le T_2 + T_1 \implies 11 \le 15$$

$$T_4 \le T_3 + T_1 \implies 12 \le 17.$$

As was stated at the beginning of the thesis, subadditivity does not work in general, but no counter examples were found for Gorenstein and monomial ideals. Next, we give an example for a non Gorenstein algebra and a non monomial ideal where the subadditivity fails.

Example 5.7. `i1 : R = QQ[x,y,z]`

```
o1 = R

o1 : PolynomialRing

i2 : I = ideal(x^12, y^12, z^12, x^5*y^5*z^2-x^6*y^6-y^6*z^6+x^6*z^6)

o2 = ideal(x^12, y^12, z^12, -x^6y^6 + x^5y^5z^2 + x^6z^6 - y^6z^6 )

o2 : Ideal of R

i3 : betti res o2

            0 1  2 3

o3 = total: 1 4 10 7

         0: 1 .  . .

         1: . .  . .

         2: . .  . .

         3: . .  . .

         4: . .  . .

         5: . .  . .

         6: . .  . .

         7: . .  . .

         8: . .  . .

         9: . .  . .

        10: . .  . .

        11: . 4  . .

        12: . .  . .
```

```
        13: . .  . .
        14: . .  . .
        15: . .  . .
        16: . .  . .
        17: . .  . .
        18: . .  1 .
        19: . .  2 1
        20: . .  1 .
        21: . .  2 .
        22: . .  3 2
        23: . .  . 1
        24: . .  1 2
        25: . .  . 1

o3 : BettiTally
```

Subadditivity fails in this case since $t_2 \nleq 2t_1$.

Bibliography

[1] L.L. Avramov, A. Conca, and S. B. Iyengar, *Free resolutions over commutative Koszul algebras*, Math. Res. Lett. **17** (2010), no. 2, 197–210.

[2] L.L. Avramov, A. Conca, and S. B. Iyengar, *Subadditivity of Syzygies of Koszul algebras*, Math. Ann. **361** (2015), no. 1-2, 511–534.

[3] M.Atiyah and I.Macdonald. *Introduction to Commutative Algebra*, University of Oxford, (1969).

[4] D. Eisenbud, C. Huneke and B. Ulrich, *The regularity of Tor and graded Betti numbers*, Amer. J. Math. **128** (2006), no. 3, 573–605.

[5] D.Eisunbud. *Commutative Algebra with a View Toward Algebraic Geometry*, Library of Congress, (1995).

[6] D.Eisunbud. *The Geometry of Syzygies*, Graduate Texts in Mathematics, (2003).

[7] S. El Khoury and H. Srinivasan. *A Note on the Subadditivity of Syzygies*, Journal of Algebra and Its Applications, (2016): 1750177.

[8] J. Herzog and H. Srinivasan. *A Note on the Subadditivity Problem for Maximal Shifts in Free Resolutions*, arXiv preprint arXiv:1303.6214, (2013).

[9] J. Herzog, T. Hibi, and X. Zheng. *Monomial ideals whose powers have a linear resolution*, Math. Scand. 95, (2004), no. 1, 23-32.

[10] J. McCullough, *A polynomial bound on the regularity of an ideal in terms of half the syzygies*, Math. Res. Lett. **19** (2012), no. 3, 555–565.

[11] J.Rotman. *Advanced Modern Algebra*, Vol. 114. American Mathematical Soc., (2010).

[12] S.Lang. *Linear Algebra*, Undergraduate Texts in Mathematics, (1987).

[13] Rana Sabbagh. *Minimal free resolution, Hilbert function, and Graded Betti Numbers* , American University of Beirut, (2009).