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A GRID-FREE VORTEX METHOD FOR THE QUASI-GEOSTROPHIC SHALLOW-WATER DYNAMICS ON THE SPHERE

by

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An Abstract of the Thesis of

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Navier-Stokes equations are able to explain the dynamics of fluids at all scales. However, small-scale phenomena control the time and space resolution of the problem, making thus impossible to capture the slowly varying large-scale phenomena. Therefore, scale separation becomes a need for the study of large-scale oceanic and atmospheric dynamics. The quasi-geostrophic (QG) equations, derived from Navier-Stokes equations by means of systematic scaling, offer a way to study fluid dynamics at the planetary scale by filtering out fast waves that manifest themselves at the small scale. Their unique feature is that they reduce the dynamic equations to a single prognostic equation by means of the geostrophic relationship. In this study, we aim to describe a numerical implementation of the 'grid-free' vortex method to solve the quasi-geostrophic shallow-water equations, and test the method's ability to simulate fundamental geophysical phenomena. The method appears to be an attractive way for solving the problem. The stability of the method is afforded by the Lagrangian advection of particles. Moreover, conservation of properties carried by particles can be easily expressed in the numerical procedure.

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Nomenclature

α	dissipation coefficient
β	meridional gradient of the Coriolis parameter
δ	ratio of the vertical to the horizontal length scale
η	free-surface elevation
Γ	strength of a vortex element
$\boldsymbol{\Omega}$	Earth's rotation
ω	relative vorticity
ϕ	basis function
ψ	stream function
ρ	fluid density
τ	wind stress
$\tilde{p}$	difference between the total and the hydrostatic pressures
ζ	normal component of the relative vorticity
A_H	horizontal turbulent viscosity coefficient
D	characteristic vertical length scale
F	Froude number $F = (f_o L)^2/gD = (L/L_d)^2$

f Coriolis parameter

f_o characteristic scale for the Coriolis parameter

g acceleration due to gravity

H depth of the fluid $h - h_B$

h height of the surface of the fluid above a reference level

h_B height of the rigid bottom above a reference level

L characteristic horizontal length scale

L_d Rossby radius of deformation $L_d = (gD)^{1/2}/f$

N_o characteristic scale for the free-surface elevation η

P characteristic scale for the pressure field

p total pressure

p_o pressure at the surface of the fluid

Q quasi-geostrophic potential vorticity

q shallow-water potential vorticity

R radius of the Earth

T characteristic time scale

t time

U characteristic scale for the horizontal velocity

u zonal component of the velocity

v meridional component of the velocity

W characteristic scale for the vertical velocity

w vertical component of the velocity

Re Reynolds number $\mathrm{Re} = UL/A_H$

Ro Rossby number $\mathrm{Ro} = U/f_oL$

Chapter 1

Introduction

This thesis deals with the shallow-water quasi-geostrophic equations. Therefore, we found it useful to dedicate a section to show in which context those equations were developed by giving a brief historical background (in subsection Historical Background). We next move on to a humble review from the literature of the numerical methods used to solve those equations (in subsection Literature Review).

1.1 Historical Background

"To understand a science it is necessary to know its history".

Auguste Comte (1799-1857)

Navier-Stokes equations are able to explain the dynamics of fluids at all scales. However, small-scale phenomena control the time and space resolution of the problem, making thus impossible to capture large-scale phenomena. Therefore, scale separation becomes a need for the study of large-scale dynamics in the oceans and the atmosphere. The quasi-geostrophic (QG) equations, derived from Navier-Stokes equations (more precisely from the primitive equations) by means of systematic scaling, offer a way to study fluid dynamics at the planetary scale by filtering out fast waves. The QG equations have evolved in the context of weather prediction and meteorology. Therefore, it is convenient to present a historical background to understand the need that those equations have answered and to trace back the origins of the contributions and efforts made in that field. [1]

Forecasting the weather has always been a need for people in their daily lives. Attempts to predict the weather had been made early (as early as the Babylonians and Greeks) by observing the skies. In 340 B.C. Aristotle wrote his book *Meteorologica* in which he established theories about the formation of rain, clouds, wind, thunder, lightning and hurricanes [15]. In 904, Ibn-Wahshiyya, an Iraqi scientist, translated the book *Nabatean Agriculture* (kitab al-filaha al-nabatiya) which is said to be based on Babylonian sources, in which he used atmospheric changes, lunar phases and wind to predict the rain [15, 22].

Invention of instruments capable of measuring the pressure and temperature in the 16th and 17th centuries helped making progress in the field. In 1592, Galileo Galilei (1564-1642) invented the world's first thermometer. Fifty years later, Galilei's student, Evangelista Torricelli (1608-1647), invented the barometer which allowed to measure the atmospheric pressure.

Gradually, scientists started to notice the influence of large-scale atmospheric processes on weather and that wind and storms follow certain patterns. Benjamin Franklin (1706-1790) was one of the first scientists to study storms. He noticed that storms generally move eastward.

In 1654, the Tuscan nobleman Ferdinand II established one of the first weather observation networks that operated for about 15 years (as cited in [10]). In 1780, the first global weather archive was installed by the Palatine meteorological society (Societas Meteorologica Palatina). For 15 years (1780-1795), 39 weather stations in Northern America, Europe and Russia, collected meteorological data three times per day, using standardized equipment (as cited in [10]). Forecasts have not been produced however; the reason was that the data collected, which was sent by boat, took weeks to arrive to the society's main office.

The invention of the electrical telegraph in 1837 by Samuel Morse facilitated the

[1] This subsection is perhaps more didactic than is proper, and much of the information contained hereafter is taken from [10]. Therefore, except otherwise cited, credits go to [10] and the references within for the information contained in this subsection.

transmission of weather observations to other countries, which allowed the production of weather forecasts. Observation centers began to appear all over Europe and Northern America.

Data from various observation stations, such as temperature and humidity, started to be mapped onto weather charts (see Fig. 1-1), which enabled scientists to study storm systems. Also, comparing the current meteorological situation to past ones, ultimately led to the production of forecasts. This method of weather prediction based on analysis and comparison is called synoptic weather forecasting. The first gale warnings that were issued left no doubt about the financial significance of weather forecasts and the necessity of knowing more about atmospheric processes. However, very few people realized that mathematics could be used to describe these processes and produce more accurate forecasts than synoptic meteorology ever could.

Vilhelm Bjerknes (1862-1951), a Norwegian mathematician, was the first who thought of applying mathematics to weather forecasting. Already at a young age, Bjerknes engaged in mathematics as he assisted his father with his research in hydrodynamics. He then studied mathematics and physics at the University of Christiania (nowadays Oslo). In 1898 he formulated his circulation theorem: in a nutshell, it explains the evolution and the subsequent decay of circulations in fluids. Combining his circulation theorem with hydrodynamics and thermodynamics, Bjerknes discovered that, given initial atmospheric conditions, it is possible to compute the future state of the atmosphere using mathematical formulae. As he put it himself:

> *We must apply the equations of theoretical physics not to ideal cases only, but to the actual existing atmospheric conditions as they are revealed by modern observations... From [these] conditions... we must learn to compute those that will follow.* (as cited in [10])

Bjerknes' equations were not practical for predicting the weather, and although he was aware of this fact, he firmly believed that one day, meteorology would be a proper science and weather forecasts obtained by solving mathematical equations would be

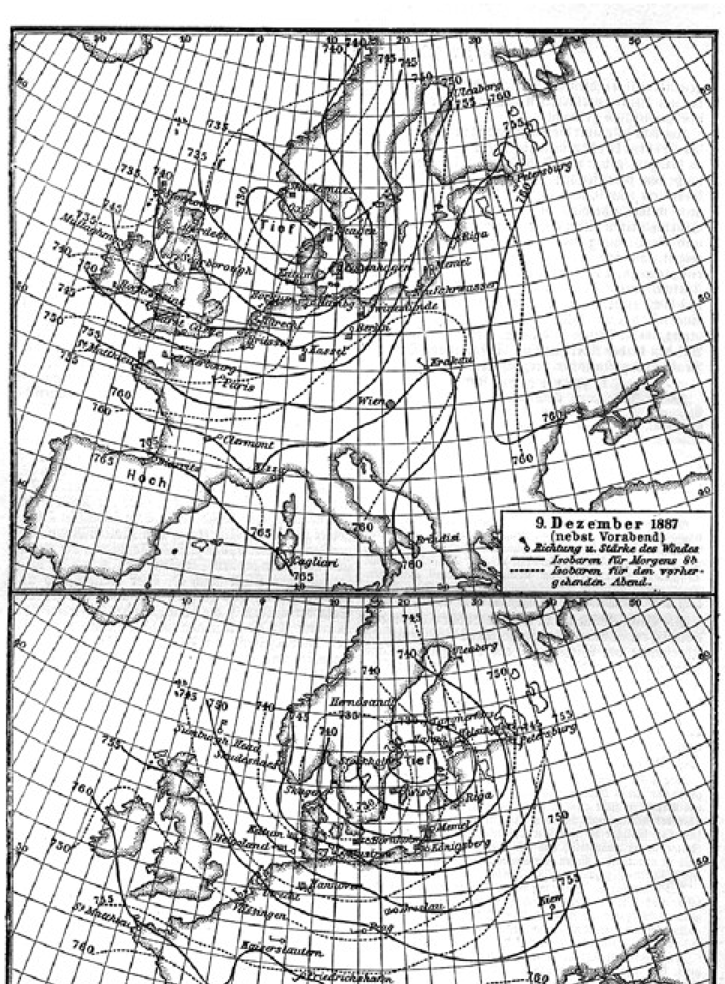


Figure 1-1: Weather maps for Europe 9 and 10 Dec 1887. Reprinted from [10].

feasible:

It could take years to drill a tunnel through a mountain. Many a worker will not live to witness the breakthrough. Nonetheless, this will not bar others from later riding through the tunnel at express train speed. (as cited in [10])

The first to attempt to predict the weather using mathematics was the British mathematician Lewis Fry Richardson (1881-1953). Richardson, who already had invented his method for finite differences, decided to solve dynamical problems of the atmosphere using his method, although at his time, the newly invented method was considered by many to be "approximate mathematics" (as cited in [10]).

Richardson simplified Bjerknes' equations and remodeled them so that they can be solved numerically. Richardson's method consisted of dividing the surface of the Earth into thousands of grid squares, and the atmosphere into several horizontal layers to obtain thousands of square boxes connected to each other by mathematical equations. In fact, Richardson applied Bjerknes' vision that the future state of the atmosphere can be known from observations of the current state at grid elements but Richardson added to it the fact that there is a connection between the grid boxes through his method.

Richardson also had a plan of a weather forecast factory that needs 64,000 human computers to predict the weather, in time with the weather actually happening. Each computer is responsible for a set of calculations at discrete points in a given grid box. In the center of the hall, a chief of operations acts as a "conductor" to make sure that the calculations are being synchronized. Richardson's forecast factory is "remarkably similar to descriptions of modern multiple-processor supercomputers used in weather forecasting today" (as cited in [10]). At the time, all calculations had to be carried out by hand, but Richardson believed that

Perhaps some day in the dim future it will be possible to advance the computations faster than the weather advances and at a cost less than the saving to mankind due

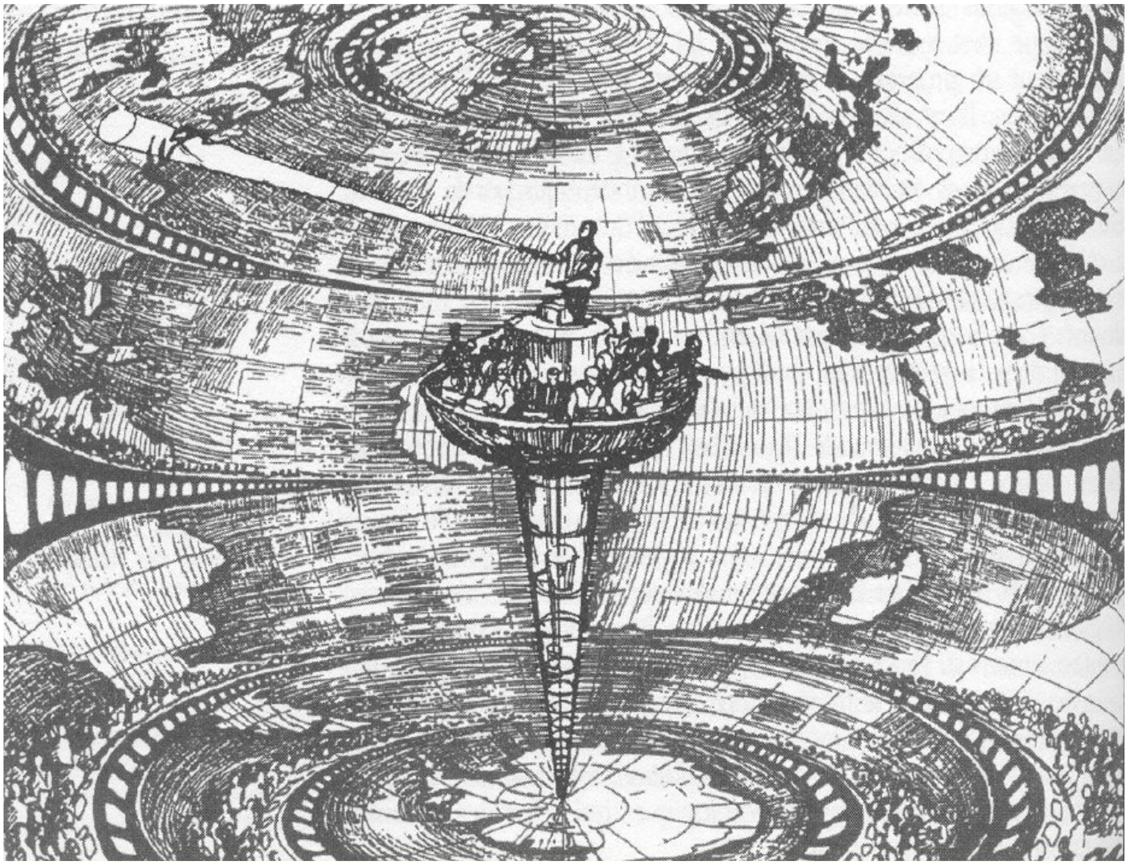


Figure 1-2: Richardson's forecast factory. Drawing by Alf Lannerbaeck. Reprinted from [8].

to the information gained. But that is a dream. (as cited in [10])

In 1916, Richardson tested his model using data provided by Vilhelm Bjerknes and collected on 20th May 1910, which had been an international balloon day when European observatories had taken measurements in the upper air. It took Richardson weeks to produce a six-hour forecast, however, the results he got were absolutely unrealistic. Nonetheless, Richardson decided to include his calculations in his seminal book Weather Prediction by Numerical Process, which was published in 1922. For him, this first failed attempt to predict the weather was only minor compared to the method that he had developed in the first place. (as cited in [10])

Very few appreciated Richardson's work at the time, and his theories were not put into practice again until the mid-1940s, when a team of scientists at the Institute for

Advanced Study in Princeton developed the world's first computer. Meanwhile, the groundbreaking research of meteorologists such as Carl-Gustaf Rossby (1898-1957) promoted scientists' knowledge of the atmosphere and helped making advancements in numerical weather prediction. In 1939, Rossby discovered the so-called Rossby waves, which are meanders of large-scale airflows in the atmosphere.

The mathematician John von Neumann (1903-1957), one of the fathers of computer science, was the first one to think of using computers to predict the weather. In 1946, he presented his ideas to a group of leading meteorologists, including Carl-Gustaf Rossby and the young, gifted mathematician Jule Charney (1917-1981). Charney became one of the leading scientists in von Neumann's Meteorology Project at the Institute for Advanced Study in Princeton. In 1950, the team successfully produced a 24-hour forecast, and from 1955 onwards, numerical forecasts generated by computers were issued on a regular basis.

The prerequisite for computer-generated forecasts was to simplify the full primitive equations that govern the atmosphere, as early computers were unable to deal with all the equations included in Richardson's model. In 1948, Charney developed the quasi-geostrophic approximation, which reduces several equations of atmospheric motions to only two equations in two unknown variables. These equations are much easier to solve and could be handled by the early computers. Furthermore, this approximation filters out all but the slow long-wave motions that are important in meteorology, so the primitive equations do not have to be solved for acoustic and gravity waves as Richardson did 30 years earlier. Although the computers were fed with simplified equations only, the limited computer power demanded a barotropic (i.e. single-layer) model of the atmosphere. Further research, both in meteorology and in computer science, finally allowed the application of baroclinic (i.e. multi-layer) models.

1.2 Literature Review

The quasi-geostrophic equations are the simplest equations that describe large-scale dynamics. They constitute a simple version of the full dynamical equations by omitting all the terms that do not make a major contribution to large-scale motion. For a first approximation, the motion is considered as adiabatic, and the working fluid as incompressible [9].

To solve those equations, several numerical methods have been used. These include finite difference, finite volume, finite element, Lagrangian/semi Lagrangian, spectral and vortex methods. For instance, Bosler et al. [3] used a Lagrangian method to solve the barotropic vorticity equation on a rotating sphere. In their method, vorticity represented by point vortices is carried by particles. The flow field is obtained from the point vortices using the Biot-Savat law. To update vorticity, conservation of potential vorticity is used to form an equation in the non-conserved vorticity which needs to be solved at every time step, adding by such an additional cost to the computational account. Remeshing, which also introduces an additional cost, is performed to maintain accuracy, but adaptive panel/grid refinement is used to minimize the cost. Cartesian coordinates were used to express particle positions. This allows to avoid singularities at poles which arise when using spherical coordinates but does not constrain particles' movement on the sphere.

Mohammadian and Marshall [16] described an algorithm for using the vortex-in-cell (VIC) method to solve the quasi-geostrophic shallow-water equations. In this method, which was originally developed by Christiansen [5], the flow field is obtained by solving a Poisson equation using an underlying Eulerian grid. Vortex methods have proven to be efficient in the study of wide range of turbulent flows. They present an elegant and robust way to treat the non-linear advection terms. In addition, they offer a natural way of modeling ocecanic and atmospheric flows.

In this study, an attempt to solve the quasi-geostrophic equations is made by means of the 'grid-free' vortex method (see, for example, [14, 11]). In this approach, the flow field is directly obtained from vortex elemets using the Biot-Savat law. Therefore, Green's function of the stream-vorticity equation must be made available for all cases. The method makes advantage of the potential vorticity conservation; particles carry potential vorticity which are materially conserved. To account for dissipation from the friction with the bottom, the redistribution scheme will be used as a step that follows advection.

1.3 Objectives of Present Work

The objective of this study is to use the 'grid-free' vortex method to solve the quasi-geostrophic equations. We will start by exploring the method on a β-plane, and then on a sphere, both in barotropic and baroclinic cases. To allow for the solution of the baroclinic case on the sphere, an expression of Green's function will be derived by means of spherical harmonics. Finally, a formulation using the redistribution scheme will be derived to solve the dissipation equation.

Chapter 2

The Quasi-Geostrophic Shallow-Water Dynamics

"I personally regard the successful reduction of the dynamic equations to a single prognostic equation by means of the geostrophic relationship as the greatest single achievement of twentieth-century dynamic meteorology".

Edward Lorenz (2006)

(Annu. Rev. Earth Planet. Sci., Vol. 34, 37-45)

"I was particularly inspired by the concept of quasi-geostrophy and then fascinated by the fact that even highly simplified dynamical models such as the quasi-geostrophic barotropic model have some relevance to extremely complicated day-to-day weather changes".

Akio Arakawa (2000)

"General Circulation Model Development"

2.1 Introduction

Large-scale flows in the oceans and the atmosphere are characterized by an approximate balance in the vertical direction between the pressure gradient and gravity (hydrostatic balance), and in the horizontal direction between the pressure gradient and the Coriolis force (geostrophic balance). In the following sections, these balances are used, along with some assumptions, to simplify Navier-Stokes equations and derive the shallow-water quasi-geostrophic equations. The process starts by scaling the conservation equations of mass and momentum which results in a set of non-dimensional equations. Next, asymptotic expansion in Rossby number is used for a systematic elimination of the higher order terms. We referred to [17, 20] for the derivation of the shallow-water quasi-geostrophic equations. The reason we included this chapter is that we could not find the complete derivation of the model, in a concise presentation, in any single reference.

2.2 The Shallow-Water Model

We start by considering a sheet of fluid with constant and uniform density ρ. The height of the surface of the fluid above the reference level $z = 0$ is $h(x, y, t)$ as shown in Fig. 2-1. The rotation axis of the fluid coincides with the $z-$axis in the model, i.e., $2\mathbf{\Omega} = f\mathbf{k}$, where f is the Coriolis parameter. The rigid bottom is defined by the surface $z = h_B(x, y)$. The velocity has components u, v and w parallel to the x-, y- and z-axes respectively. The fluid is assumed inviscid, that is, only motions for which viscosity is unimportant are considered for the time being.

Although the depth of the fluid $H = h - h_B$ varies in space and time, we suppose that a characteristic value D for the depth can be sensibly chosen. D could be, for example, the average depth of the layer. We also suppose that D characterizes the vertical scale of the motion as well. Similarly we suppose there exists a characterisitc horizontal length scale for the motion, which we call L. The fundamental parametric condition which characterizes shallow-water theory is

$$\delta = \frac{D}{L} \ll 1 \ . \tag{2.1}$$

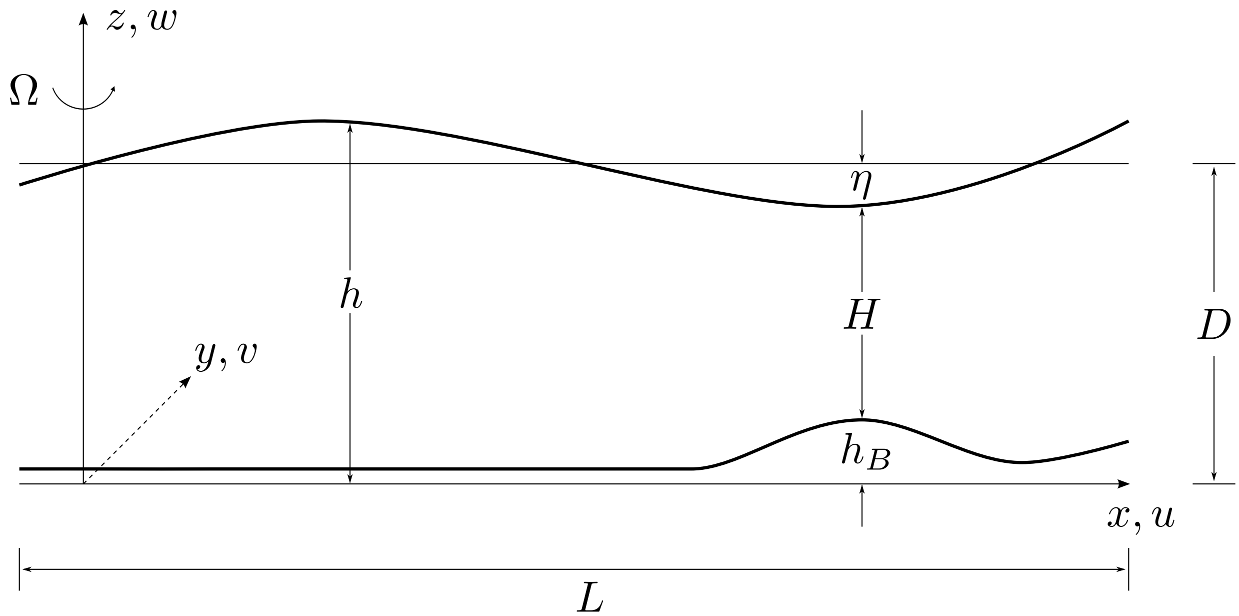


Figure 2-1: The shallow-water model.

2.3 The Shallow-Water Equations

The specification of incompressibility and constant density immediately decouples the dynamics from the thermodynamics and reduces the equation of mass conservation to the condition of incompressibility

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \ . \tag{2.2}$$

Each of the first two terms of Eq. (2.2) is $\mathcal{O}(\mathrm{U/L})$, where U is a characteristic scale for the horizontal velocity. If W is the scale for the vertical velocity, it follows that W/D can be no larger than $\mathcal{O}(\mathrm{U/L})$, i.e., that

$$W \leq \mathcal{O}(\delta U) \ . \tag{2.3}$$

Now let us estimate the terms in the momentum equation

$$
\begin{array}{cccccccc}
\dfrac{\partial u}{\partial t} & +u\dfrac{\partial u}{\partial x} & +v\dfrac{\partial u}{\partial y} & +w\dfrac{\partial u}{\partial z} & -fv & = -\dfrac{1}{\rho}\dfrac{\partial \tilde{p}}{\partial x} & , \\
\dfrac{U^2}{L} & \dfrac{U^2}{L} & \dfrac{U^2}{L} & \dfrac{UW}{D} & fU & \dfrac{P}{\rho L} &
\end{array}
\tag{2.4}
$$

$$
\begin{array}{cccccccc}
\dfrac{\partial v}{\partial t} & +u\dfrac{\partial v}{\partial x} & +v\dfrac{\partial v}{\partial y} & +w\dfrac{\partial v}{\partial z} & +fu & = -\dfrac{1}{\rho}\dfrac{\partial \tilde{p}}{\partial y} & , \\
\dfrac{U^2}{L} & \dfrac{U^2}{L} & \dfrac{U^2}{L} & \dfrac{UW}{D} & fU & \dfrac{P}{\rho L} &
\end{array}
\tag{2.5}
$$

$$
\begin{array}{ccccccc}
\dfrac{\partial w}{\partial t} & +u\dfrac{\partial w}{\partial x} & +v\dfrac{\partial w}{\partial y} & +w\dfrac{\partial w}{\partial z} & = -\dfrac{1}{\rho}\dfrac{\partial \tilde{p}}{\partial z} & . \\
\dfrac{UW}{L} & \dfrac{UW}{L} & \dfrac{UW}{L} & \dfrac{W^2}{D} & \dfrac{P}{\rho D} &
\end{array}
\tag{2.6}
$$

$T = L/U$ is the advection time scale chosen as a characteristic scale for time, while P is the scale for the variable pressure field.

The total pressure p is written as

$$
p(x,y,z,t) = -\rho g z + \tilde{p}(x,y,z,t) \ , \tag{2.7}
$$

the first part of which cancels the constant gravitational force per unit mass in the fluid. Since, by Eq. (2.3)

$$
\frac{UW}{D} = \mathcal{O}\left(\frac{U^2}{L}\right) \ , \tag{2.8}
$$

it follows from Eqs. (2.4) and (2.5) that the pressure scale is given by

$$
P = \rho U \left(U, fL\right)_{\max} \ , \tag{2.9}
$$

in order for the horizontal pressure gradient to enter as a forcing term in the horizontal momentum balance, for otherwise the flow would be unaccelerated. Introducing

Rossby number $\mathrm{Ro} = U/fL$ defined as the ratio of the relative to the Coriolis acceleration, and assuming that

$$\mathrm{Ro} \ll 1 \ , \tag{2.10}$$

we get

$$P = \rho \frac{U^2}{\mathrm{Ro}} \ . \tag{2.11}$$

Similarly, the ratio of the terms on the left-hand side of Eq. (2.6) to the vertical pressure gradient is

$$\rho \frac{dw/dt}{\partial \tilde{p}/\partial z} = \mathcal{O}\left(\rho \frac{UW/L}{P/D}\right) \ , \tag{2.12}$$

or from Eq. (2.11)

$$\rho \frac{dw/dt}{\partial \tilde{p}/\partial z} = \mathcal{O}\left(\delta^2 \mathrm{Ro}\right) \ . \tag{2.13}$$

It follows that $\partial \tilde{p}/\partial z$ is negligible to $\mathcal{O}\left(\delta^2 \mathrm{Ro}\right)$, or, in terms of the total pressure

$$\frac{\partial p}{\partial z} = -\rho g + \mathcal{O}(\delta^2 \mathrm{Ro}) \ , \tag{2.14}$$

which is the hydrostatic approximation. From another point of view, Eq. (2.14) can be taken as the definition of the shallow-water model.

It is possible to immediately integrate Eq. (2.14) and obtain

$$p = \rho g (h - z) + p_o \ , \tag{2.15}$$

so that the pressure in excess of p_o (pressure at the surface of the fluid) at any point is simply equal to the weight of the unit column of fluid above that point.

Next note that the horizontal pressure gradient is independent of z, i.e.,

$$\frac{\partial p}{\partial x} = \rho g \frac{\partial h}{\partial x} \ , \tag{2.16}$$

$$\frac{\partial p}{\partial y} = \rho g \frac{\partial h}{\partial y} \ , \tag{2.17}$$

so that the horizontal acceleration must be independent of z. It is therefore consistent to assume that the horizontal velocities themselves remain z-independent if they are so initially.

The horizontal momentum equations then become

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -g\frac{\partial h}{\partial x} \ , \tag{2.18}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -g\frac{\partial h}{\partial y} \ . \tag{2.19}$$

The condition that u and v are independent of z now allows the incompressibility condition to be integrated in z to yield

$$w(x,y,z,t) = -z\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \tilde{w}(x,y,t) \ . \tag{2.20}$$

Applying boundary conditions at $z = h_B$

$$w\Big|_{z=h_B} = \frac{Dh_B}{Dt} = u\frac{\partial h_B}{\partial x} + v\frac{\partial h_B}{\partial y} \ , \tag{2.21}$$

implies

$$\tilde{w}(x,y,t) = u\frac{\partial h_B}{\partial x} + v\frac{\partial h_B}{\partial y} + h_B\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \ , \tag{2.22}$$

so that

$$w(x,y,z,t) = (h_B - z)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + u\frac{\partial h_B}{\partial x} + v\frac{\partial h_B}{\partial y} \ . \tag{2.23}$$

The corresponding kinematic condition at $z = h$ is

$$w\Big|_{z=h} = \frac{Dh}{Dt} = \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y} \ , \tag{2.24}$$

which, when combined with Eq. (2.23), yields

$$\frac{\partial H}{\partial t} + \frac{\partial (uH)}{\partial x} + \frac{\partial (vH)}{\partial y} = 0 \ , \tag{2.25}$$

or

$$\frac{DH}{Dt} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \ , \tag{2.26}$$

where $H = h - h_B$.

Eq. (2.25) states that if the local horizontal divergence of volume, $\nabla(\mathbf{u}_H \cdot H)$, is positive, it must be balanced by a local decrease of the layer thickness due to a drop in the free surface. The second statement, Eq. (2.26), which is equivalent to the first, is that following the fluid, as the cross section A of a fluid column increases at a rate

$$\frac{1}{A}\frac{DA}{Dt} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \ , \tag{2.27}$$

the total thickness must decrease so that

$$\frac{1}{H}\frac{DH}{Dt} + \frac{1}{A}\frac{DA}{Dt} = 0 \ , \tag{2.28}$$

i.e., so that the volume HA remains constant.

The shallow-water equations are Eqs. (2.18), (2.19) and (2.25) or its equivalent (2.26). Note that the consequences of the condition $\delta \ll 1$ have reduced the number of dynamical equations by one, have reduced the number of dependent variables by one (by eliminating w explicitly from the dynamics), and have reduced the number of independent variables by one (since z no longer explicitly appears in the dynamical equations). The remaining variables are u, v, and h, and they are functions of x, y, and t only.

The fact that w is a simple linear function of z has an important further implication. If we use Eq. (2.26) to eliminate $\partial u/\partial x + \partial v/\partial y$ from Eq. (2.23), we obtain

$$w \equiv \frac{Dz}{Dt} = \frac{z - h_B}{H}\frac{DH}{Dt} + u\frac{\partial h_B}{\partial x} + v\frac{\partial h_B}{\partial y} \ , \tag{2.29}$$

which implies that

$$\frac{D}{Dt}\left\{\frac{z-h_B}{H}\right\}=0\ . \tag{2.30}$$

$(z-h_B)/H$ is the relative height from the bottom of each fluid element, i.e., its status ranges from zero at the bottom to unity at the free surface. During the motion of the fluid, the fact that u and v are independent of z implies that the fluid moves as a set of columns oriented parallel to the z-axis. Eq. (2.30) simply states that during the stretching or contraction of each column, the *relative* position of a fluid element in the column is unchanged.

2.4 The Shallow-Water Potential Vorticity

The three components of relative vorticity are, in a Cartesian frame,

$$\omega_x=\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\ , \tag{2.31}$$

$$\omega_y=\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\ , \tag{2.32}$$

$$\omega_z=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\ . \tag{2.33}$$

In the present case, where u and v are independent of z,

$$\omega_x = \frac{\partial w}{\partial y}=\mathcal{O}\left(\frac{W}{L}\right)=\mathcal{O}\left(\delta\frac{U}{L}\right)\ , \tag{2.34}$$

$$\omega_y = -\frac{\partial w}{\partial x}=\mathcal{O}\left(\frac{W}{L}\right)=\mathcal{O}\left(\delta\frac{U}{L}\right)\ , \tag{2.35}$$

$$\omega_z = \frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=\mathcal{O}\left(\frac{U}{L}\right)\ , \tag{2.36}$$

so that the horizontal components of relative vorticity are $\mathcal{O}(\delta)$ smaller than the vertical component of vorticity. Cross-differentiating Eqs. (2.18) and (2.19) to eliminate h yields

$$\frac{D\zeta}{Dt}\equiv\frac{\partial \zeta}{\partial t}+u\frac{\partial \zeta}{\partial x}+v\frac{\partial \zeta}{\partial y}=-(\zeta+f)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\ , \tag{2.37}$$

where the notation

$$\zeta = \omega_z \tag{2.38}$$

has been introduced.

Following fluid columns, the only mechanism which can change the relative vorticity is the convergence of absolute-vorticity filaments. Using Eq. (2.26), we can rewrite Eq. (2.37) as

$$\frac{D\zeta}{Dt} = \frac{\zeta + f}{H}\frac{DH}{Dt} , \tag{2.39}$$

which expresses the same result in terms of vortex-tube stretching, namely, for stretching to occur ($DH/Dt > 0$), the vorticity ζ is intensified by an amount proportional to the product of the column stretching and the absolute vorticity, $\zeta + f$, which is already present. Note that ζ cannot change by vortex tilting, such a mechanism is absent in the columnar motion of shallow-water theory.

If we adopt the f-plane convention and assume that f is constant, Eq. (2.39) can be put in the form

$$\frac{D}{Dt}\left(\frac{\zeta + f}{H}\right) = 0 , \tag{2.40}$$

which is the statement of the conservation of the *shallow-water potential vorticity*

$$q = \frac{\zeta + f}{H} . \tag{2.41}$$

If H increases, the absolute (and hence relative) vorticity must increase to keep q constant for the column. Relative vorticity is produced by column stretching in the field of planetary vorticity, f.

2.5 Quasi-Geostrophic Shallow-Water PV Equation: Case of a β-plane

First, we choose scales L, T, U, f_o and N_o which characterize the magnitudes of length, time, velocity, Coriolis parameter and free-surface elevation η (see Fig. 2-1), respectively.

It is useful to introduce the following notation

$$H = H_o(x,y) + \eta = D + \eta - h_B \ , \tag{2.42}$$

where

$$H_o = D - h_B(x,y) \ . \tag{2.43}$$

As shown in Fig. 2-1, η is the departure of the free surface from its resting level, and h_B is the measure of the bottom variation which produces a departure of H_o, the depth of the resting fluid, from the constant value D.

In this section, we will adopt the β-plane approximation in which the Coriolis parameter f varies linearly with y, i.e.,

$$f = f_o + \beta y \ , \tag{2.44}$$

where $\beta = df/dy$, and assume that variations in the Coriolis parameter are small, that is $|\beta L| \ll |f_o|$. We will recognize the smallness of β compared to f_o/L by letting $\beta = (U/L^2)\beta'$, where β' is assumed to be a non-dimensional parameter of order unity. We will then have

$$f' = f/f_o = 1 + \mathrm{Ro}\beta' y' \ . \tag{2.45}$$

We next use the scales defined earlier to write Eqs. (2.18), (2.19) and (2.25) in terms of non-dimensional (primed) variables

$$\frac{U}{T}\frac{\partial u'}{\partial t'} + \frac{U^2}{L}\left\{u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial y'}\right\} - f_o U f' v' = -g\frac{N_o}{L}\frac{\partial \eta'}{\partial x'} \tag{2.46a}$$

$$\frac{U}{T}\frac{\partial v'}{\partial t'} + \frac{U^2}{L}\left\{u'\frac{\partial v'}{\partial x'} + v'\frac{\partial v'}{\partial y'}\right\} + f_o U f' u' = -g\frac{N_o}{L}\frac{\partial \eta'}{\partial y'} \tag{2.46b}$$

$$\frac{N_o}{T}\frac{\partial \eta'}{\partial t'} + \frac{U}{L}\left\{u'\frac{\partial}{\partial x'}(N_o\eta' - h_B) + v'\frac{\partial}{\partial y'}(N_o\eta' - h_B)\right\} + \frac{U}{L}(D + N_o\eta' - h_B)\left(\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'}\right) = 0 \tag{2.46c}$$

We now insist that the motions to be described should be such that

$$\mathrm{Ro} = \frac{U}{f_o L} \ll 1 \ , \tag{2.47}$$

which is the case for large-scale flows in both the ocean and the atmosphere, and

$$T = \frac{L}{U} \ , \tag{2.48}$$

so to consider cases where the advective time L/U is as short as the time scale for local change.

In this case, the relative acceleration terms in Eqs. (2.46) will be small compared to the Coriolis acceleration terms by $\mathcal{O}(\mathrm{Ro})$. In order for u' and v' to be different from zero, the pressure gradient-terms must be large enough to balance the Coriolis acceleration. We therefore choose the parameter N_o as

$$N_o = \frac{f_o U L}{g} = \frac{U}{f_o L}\frac{f_o^2 L^2}{g} \ , \tag{2.49}$$

which implies that Eq. (2.42) may be written as

$$H = D\left[1 + \mathrm{Ro}\frac{L^2}{L_d^2}\eta' - \frac{h_B}{D}\right] \ , \tag{2.50}$$

where L_d is the Rossby radius of deformation for the layer of depth D, i.e.,

$$L_d = (gD)^{1/2}/f_o \ . \tag{2.51}$$

The parameter

$$F = \frac{f_o^2 L^2}{gD} = \left(\frac{L}{L_d}\right)^2 \ , \tag{2.52}$$

is the square of the ratio of the geometric length scale L to the Rossby radius of deformation L_d. We shall assume that

$$F = \mathcal{O}(1) \tag{2.53}$$

throughout our discussion. Also consider

$$\frac{h_B}{D} = \mathrm{Ro}\,\eta_B(x,y) \ , \tag{2.54}$$

where η_B is $\mathcal{O}(1)$.

If the momentum equations are each divided by $f_o U$, and the mass-conservation statement by U/L, then we obtain

$$\mathrm{Ro}\frac{\partial u}{\partial t} + \mathrm{Ro}\left\{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right\} - (1+\mathrm{Ro}\beta y)v = -\frac{\partial \eta}{\partial x} \ , \tag{2.55a}$$

$$\mathrm{Ro}\frac{\partial v}{\partial t} + \mathrm{Ro}\left\{u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right\} + (1+\mathrm{Ro}\beta y)u = -\frac{\partial \eta}{\partial y} \ , \tag{2.55b}$$

$$\begin{aligned}\mathrm{Ro}F\frac{\partial \eta}{\partial t} + \mathrm{Ro}F\left\{u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y}\right\} - \mathrm{Ro}\left\{u\frac{\partial \eta_B}{\partial x} + v\frac{\partial \eta_B}{\partial y}\right\} \\ + (1+\mathrm{Ro}F\eta - \eta_B)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \ .\end{aligned} \tag{2.55c}$$

For the sake of neatness the primes have been dropped from the nondimensional variables.

We are now in a position to systematically examine the orders of magnitude of the various terms in the equations of motion, but more than that, we can find relationships between terms of like order in Ro. Consider any solution of Eqs. (2.55). It will be a function of x, y, t and Ro. That is, the velocity u, for example, is

$$u = u(x,y,t,\mathrm{Ro}) \ . \tag{2.56}$$

For small Ro, we suppose that u can be expanded in an asymptotic series in Ro. Since only integral powers of Ro appear in Eqs. (2.55), it seems sensible to suppose that the expansion proceeds as

$$u(x, y, t, \mathrm{Ro}) = u_0(x, y, t) + \mathrm{Ro}\, u_1(x, y, t) + \mathrm{Ro}^2 u_2(x, y, t) + \ldots \,, \tag{2.57}$$

where the functions u_0, u_1, etc. are independent of Ro. Each function u, v and η is expanded in this way and then inserted into the equations of motion. Since Ro, while small, is arbitrary, like powers of Ro must balance if the equations are to be satisfied for arbitrary small Ro for all x, y and t. Since Ro is small, our interest is focused mainly on u_0, v_0 and η_0. However, the expansion must in general be carried to higher order than the first to determine the lowest-order fields.

The $\mathcal{O}(1)$ terms from Eqs. (2.55a,b) yield

$$\begin{aligned} v_0 &= \frac{\partial \eta_0}{\partial x} \,, \\ u_0 &= -\frac{\partial \eta_0}{\partial y} \,, \end{aligned} \tag{2.58}$$

so that the lowest-order fields are geostrophic. It follows directly that

$$\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y} = 0 \,. \tag{2.59}$$

We find ourselves at this stage unable to determine the order-one fields u_0, v_0 and η_0. However, we are now able to make progress beyond our earlier crude order-of-magnitude estimates by exploiting the systematic asymptotic expansions of the equations. This is why the nondimensional formulation is so valuable.

The $\mathcal{O}(\mathrm{Ro})$ terms in the equation of motion yield

$$\frac{\partial u_0}{\partial t} + u_0 \frac{\partial u_0}{\partial x} + v_0 \frac{\partial u_0}{\partial y} - v_1 - \beta y v_0 = -\frac{\partial \eta_1}{\partial x} \,, \tag{2.60a}$$

$$\frac{\partial v_0}{\partial t} + u_0 \frac{\partial v_0}{\partial x} + v_0 \frac{\partial v_0}{\partial y} + u_1 + \beta y u_0 = -\frac{\partial \eta_1}{\partial y} \ , \tag{2.60b}$$

$$F\left(\frac{\partial \eta_0}{\partial t} + u_0 \frac{\partial \eta_0}{\partial x} + v_0 \frac{\partial \eta_0}{\partial y}\right) - \left(u_0 \frac{\partial \eta_B}{\partial x} + v_0 \frac{\partial \eta_B}{\partial y}\right) + \left(\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y}\right) = 0 \ . \tag{2.60c}$$

To establish a closed dynamical system which explicitly involves only the $\mathcal{O}(1)$ fields, we eliminate the pressure by cross-differentiating Eqs. (2.60a,b) to obtain

$$\frac{\partial \zeta_0}{\partial t} + u_0 \frac{\partial(\zeta_0 + \beta y)}{\partial x} + v_0 \frac{\partial(\zeta_0 + \beta y)}{\partial y} = -\left(\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y}\right) \ , \tag{2.61}$$

where

$$\zeta_0 = \frac{\partial v_0}{\partial x} - \frac{\partial u_0}{\partial y} = \nabla^2 \eta_0 \ . \tag{2.62}$$

Next, we use Eq. (2.60c) to eliminate $(\partial u_1/\partial x + \partial v_1/\partial y)$ from Eq. (2.61) and get

$$\frac{D(\zeta_0 + \beta y)}{Dt} = F\frac{D\eta_0}{Dt} - \frac{D\eta_B}{Dt} \ , \tag{2.63}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u_0 \frac{\partial}{\partial x} + v_0 \frac{\partial}{\partial y} \ , \tag{2.64}$$

or

$$\frac{D}{Dt}\left(\zeta_0 - F\eta_0 + \beta y + \eta_B\right) = 0 \ . \tag{2.65}$$

Since u_0, v_0 and ζ_0 are related to η_0 by Eqs. (2.58) and (2.62), the conservation statement (2.65) can be written entirely in terms of η_0

$$\left[\frac{\partial}{\partial t} + \frac{\partial \eta_0}{\partial x}\frac{\partial}{\partial y} - \frac{\partial \eta_0}{\partial y}\frac{\partial}{\partial x}\right]\left(\nabla^2 \eta_0 - F\eta_0 + \beta y + \eta_B\right) = 0 \ . \tag{2.66}$$

The equation for η_0 is the quasi-geostrophic potential-vorticity equation, which is a conservation statement for the potential vorticity

$$Q = \zeta_0 - F\eta_0 + \beta y + \eta_B \ . \tag{2.67}$$

That is, it is the potential-vorticity equation in which all terms are evaluated through the use of their geostrophic ($\mathcal{O}(1)$) values.

Since η_0 serves as a streamfunction ψ for the $\mathcal{O}(1)$ velocity field, the notation

$$\eta_0 = \psi(x, y, t) \ , \tag{2.68}$$

is introduced, in terms of which

$$v_0 = \frac{\partial \psi}{\partial x} \ , \tag{2.69a}$$

$$u_0 = -\frac{\partial \psi}{\partial y} \ , \tag{2.69b}$$

while Eq. (2.66) becomes

$$\left[\frac{\partial}{\partial t} + \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y} - \frac{\partial \psi}{\partial y}\frac{\partial}{\partial x}\right]\left(\nabla^2\psi - F\psi + \beta y + \eta_B\right) = 0 \ . \tag{2.70}$$

The quasi-geostrophic approximation to the potential vorticity Q is a linear combination of four terms. The first is the relative vorticity, while the second is the contribution to the potential vorticity due to variations in the free-surface height. The relative importance of the second term to the first term is measured by F, i.e., by the ratio of L , the scale of motion, to the radius of deformation L_d. If L is small compared to L_d, there is, on the scale of the motion, a negligible variation of η and a consequently negligible contribution to the potential vorticity by vortex-tube stretching. Thus if $L \ll L_d$, from the point of view of the vorticity balance, the free surface appears no different than a rigid lid. If $L \gg L_d$, the relative vorticity is unimportant and the fluid velocity appears horizontally uniform. *The Rossby radius of deformation is the scale for which the relative vorticity and the surface height (vortex-tube stretching) make equal contributions to the potential vorticity.*

The dimensional form of Eq. (2.70) is

$$\left[\frac{\partial}{\partial t} + \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y} - \frac{\partial \psi}{\partial y}\frac{\partial}{\partial x}\right]\left(\nabla^2\psi - \frac{1}{L_d^2}\psi + \beta y + \frac{f_0}{D}h_B\right) = 0 \ . \qquad (2.71)$$

Once friction is included in the shallow-water model, along with wind acting as a driving force on the surface of the ocean, the evolution of the quasi-geostrophic potential vorticity will be governed by (refer to [17] for the complete derivation)

$$\begin{aligned}\left[\frac{\partial}{\partial t} + \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y} - \frac{\partial \psi}{\partial y}\frac{\partial}{\partial x}\right]\left(\nabla^2\psi - F\psi + \beta y + \eta_B\right) = \\ \beta \mathbf{k}\cdot\nabla\times\tau - \alpha\nabla^2\psi + \frac{1}{\mathrm{Re}}\nabla^4\psi \ ,\end{aligned} \qquad (2.72)$$

where $\mathbf{k}$ is a unit vector in the z-direction, $\tau = (\tau_x, \tau_y)$ is the wind stress vector obtained from wind speed measured at 10 meters above the sea surface, α is a dissipation coefficient and $\mathrm{Re} = UL/A_H$ is Reynolds number with A_H being the horizontal turbulent viscosity coefficient.

The presence of friction has produced in (2.72) three important changes in this generalization of the potential-vorticity equation. In the absence of friction, the right-hand side of (2.72) is zero and (2.72) reduces to the statement of potential-vorticity conservation, i.e., (2.71). Friction now allows the potential vorticity of each fluid column to change with time. The curl of the applied stress acts as a source of potential vorticity, while the frictional dissipation in the lower Ekman layer (second term on the right-hand side) acts as a sink of potential vorticity. The presence of a small amount ($\mathcal{O}(\mathrm{Re}^{-1})$) of friction in the interior, although generally negligible, acts to diffuse vorticity laterally.

2.6 Quasi-Geostrophic Shallow-Water PV Equation: Case of a Sphere

The derivation of the quasi-geostrophic shallow-water potential vorticity equation on the sphere is taken from [19]. Consider the shallow-water primitive equations in spherical coordinates (λ is longitude and θ is latitude)

$$\frac{Du}{Dt} - \left(2\Omega\sin\theta + \frac{u\tan\theta}{R}\right)v + \frac{g}{R\cos\theta}\frac{\partial\eta}{\partial\lambda} = 0\ , \tag{2.73a}$$

$$\frac{Dv}{Dt} + \left(2\Omega\sin\theta + \frac{u\tan\theta}{R}\right)u + \frac{g}{R}\frac{\partial\eta}{\partial\theta} = 0\ , \tag{2.73b}$$

$$\frac{DH}{Dt} + H\left(\frac{1}{R\cos\theta}\frac{\partial u}{\partial\lambda} + \frac{1}{R\cos\theta}\frac{\partial}{\partial\theta}(v\cos\theta)\right) = 0\ , \tag{2.73c}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \frac{1}{R\cos\theta}\frac{\partial}{\partial\lambda} + \frac{1}{R}\frac{\partial}{\partial\theta} \tag{2.74}$$

is the material derivative.

u and v are the velocities along increasing λ and θ directions respectively, R is the radius of the Earth and Ω is the magnitude of its angular velocity, g is the acceleration due to gravity and $H = \eta + D$ is the height of a column of water above a certain reference level (the height of the rigid bottom h_B is assumed to be zero); D is a characteristic vertical length scale and η is the deviation of the water free surface from its resting position.

Cross-differentiating Eqs. (2.73a) and (2.73b)

$$\frac{1}{R\cos\theta}\frac{\partial}{\partial\lambda}(2.73b) - \frac{1}{R\cos\theta}\frac{\partial}{\partial\theta}(\cos\theta \times (2.73a))$$

to eliminate pressure terms, we obtain

$$\frac{D}{Dt}(\zeta + f) + (\zeta + f)\left(\frac{1}{R\cos\theta}\frac{\partial u}{\partial\lambda} + \frac{1}{R\cos\theta}\frac{\partial}{\partial\theta}(v\cos\theta)\right) = 0\ , \tag{2.75}$$

where

$$\zeta = \frac{1}{R\cos\theta}\frac{\partial v}{\partial\lambda} - \frac{1}{R\cos\theta}\frac{\partial}{\partial\theta}(u\cos\theta) \tag{2.76}$$

is the local normal component of the relative vorticity, and

$$f = 2\Omega \sin\theta \tag{2.77}$$

is the Coriolis parameter.

Substituting the divergence term of Eq. (2.73c) in Eq. (2.75), we obtain

$$\frac{DQ}{Dt} = 0 \ , \tag{2.78}$$

where

$$Q = \frac{\zeta + f}{H} \tag{2.79}$$

is the potential vorticity.

Eq. (2.78) is the statement of the conservation of potential vorticity. The potential vorticity can also be written as

$$\begin{aligned} Q &= \frac{D}{\eta + D}\left(\frac{1}{R\cos\theta}\frac{\partial v}{\partial\lambda} - \frac{1}{R\cos\theta}\frac{\partial}{\partial\theta}(u\cos\theta) + 2\Omega\sin\theta\right) \\ &= \frac{D}{\eta + D}\nabla_s^2\psi - 2\Omega\sin\theta\left(\frac{\eta}{\eta + D}\right) + 2\Omega\sin\theta \ , \end{aligned} \tag{2.80}$$

where ψ denotes the streamfunction of the non-divergent part of the flow

$$(u_\psi, v_\psi) = \left(-\frac{1}{R}\frac{\partial\psi}{\partial\theta}, \frac{1}{R\cos\theta}\frac{\partial\psi}{\partial\lambda}\right) \ , \tag{2.81}$$

and

$$\nabla_s^2 = \frac{1}{R^2\cos^2\theta}\frac{\partial^2}{\partial\lambda^2} + \frac{1}{R^2\cos\theta}\left(\cos\theta\frac{\partial}{\partial\theta}\right) \ . \tag{2.82}$$

Assuming that $\eta \ll D$, Eq. (2.80) then becomes

$$Q = \nabla_s^2\psi - \frac{2\Omega\sin\theta}{D}\eta + 2\Omega\sin\theta \ . \tag{2.83}$$

Following the arguments of Kuo (1959) [13] and Charney and Stern (1962) [4], we now assume (i) that ψ and η are related by the linear balance condition

$$\nabla \cdot (2\Omega \sin\theta \nabla\psi) = g\nabla_s^2 \eta \ , \tag{2.84}$$

and (ii) that $2\Omega \sin\theta$ can be considered as slowly varying, so that Eq. (2.84) can be simplified to

$$\nabla_s^2 (g\eta - 2\Omega \sin\theta \psi) = 0 \ , \tag{2.85}$$

from which the local linear balance condition

$$g\eta = 2\Omega \sin\theta \psi \tag{2.86}$$

then follows.

Finally, the expression of the potential vorticity becomes

$$Q = \nabla_s^2 \psi - \frac{\psi}{L_d^2} + f \ , \tag{2.87}$$

where

$$L_d = \frac{\sqrt{gD}}{f} \tag{2.88}$$

is Rossby radius of deformation.

Chapter 3

Applying the Grid-Free Vortex Method

In this chapter, we will consider the quasi-geostrophic equations describing the evolution of a shallow-water layer, on a β-plane and a sphere, in the absence of topography and wind stress. The only dissipating term that is retained is friction with the bottom. In general, such a model can be expressed as (see, for example, [21] and [16])

$$Q = f + \nabla^2\psi - \frac{\psi}{L_d^2} , \tag{3.1}$$

$$\frac{DQ}{Dt} = -\alpha\zeta , \tag{3.2}$$

where $\zeta = \nabla^2\psi$ is the relative vorticity, Q is the potential vorticity, f is the Coriolis parameter, ψ is the streamfunction, α is a dissipation coefficient and L_d is Rossby radius of deformation. The Coriolis paramtere f will be expressed as βy or $2\Omega \sin\theta$, depending on whether the model is applied on a β-plane or a sphere, respectively.

Now rewrite Eq. (3.1) as

$$\nabla^2\psi - \frac{\psi}{L_d^2} = Q - f , \tag{3.3}$$

and write Q as

$$Q = \sum_{i=1}^{N} \Gamma_i \phi(|\mathbf{r'} - \mathbf{r_i}|) \ , \tag{3.4}$$

where ϕ is a basis function, Γ_i is the strength of the i^{th} point vortex at $\mathbf{r_i}$, and $\mathbf{r'}$ is an integration vector. Different choices of the basis function are available, with one possibility being the Gaussian distribution [1].

The solution of Eq. (3.3) is

$$\begin{aligned} \psi &= G * (Q - f) \\ &= G * \left(\sum_{i=1}^{N} \Gamma_i \phi(|\mathbf{r'} - \mathbf{r_i}|) - f \right) \ , \end{aligned} \tag{3.5}$$

where $*$ is the convolution operator and G is Green's function. Since G is a function of the distance between the target and the integration point only (i.e., $|\mathbf{r} - \mathbf{r'}|$), Eq. (3.5) can be written as

$$\psi = \sum_{i=1}^{N} \Gamma_i G * \phi(|\mathbf{r'} - \mathbf{r_i}|) - G * f \ . \tag{3.6}$$

Note that the second term on the right hand side of Eq. (3.6) is the contribution of the Coriolis force to the streamfunction and represents a background state independent of any disturbance.

The velocity field can be expressed in terms of particle positions and the streamfunction as

$$(u^x, u^y) = \left(\frac{Dx}{Dt}, \frac{Dy}{Dt} \right) = \left(-\frac{\partial \psi}{\partial y}, \frac{\partial \psi}{\partial x} \right) \ , \tag{3.7}$$

where u^x and u^y are, respectively, the zonal (eastward) and meridional (northward) velocity components of the flow in the rotating reference frame.

In spherical coordinates, the angular velocity components (in rad/s) in longitude

λ and latitude θ directions are given by

$$(u^\lambda, u^\theta) = \left(\frac{D\lambda}{Dt}, \frac{D\theta}{Dt}\right) = \frac{1}{R^2\cos\theta}\left(-\frac{\partial\psi}{\partial\theta}, \frac{\partial\psi}{\partial\lambda}\right) \ , \tag{3.8}$$

where R is the radius of the sphere.

Changes in latitude ($\mathrm{d}\theta$) and longitude ($\mathrm{d}\lambda$) are given by

$$\mathrm{d}\theta = \frac{\mathrm{d}y}{R} \ , \tag{3.9}$$

$$\mathrm{d}\lambda = \frac{\mathrm{d}x}{R\cos\theta} \ , \tag{3.10}$$

where $\mathrm{d}x$ and $\mathrm{d}y$ are changes in distance along latitude and longitude circles, respectively.

The algorithm that will be used in this study is as follows

1. Calculate ψ by solving Eq. (3.3)
2. Compute the flow field using either Eq. (3.7) (for the β-plane) or Eq. (3.8) (for the sphere)
3. Solve Eq. (5.5)
   (a) Advect the particles using the computed flow field
   (b) Solve for $Q(t + \Delta t)$ by integrating the dissipation equation

$$\frac{\partial Q}{\partial t} = -\alpha\nabla^2\psi \tag{3.11}$$

   (c) Solve for $\Gamma_i(\mathbf{r}_i(t + \Delta t), t + \Delta t)$ by inverting Eq. (3.4)

4. Go to Step 1

Another approach is to consider replacing Steps (b) and (c) by redistribution (see Chapter 5).

3.1 Solution on the β-plane

3.1.1 Barotropic Case

For the barotropic case ($L_d = \infty$) on the β-plane, Eq. (3.3) reduces to Poisson's equation

$$\nabla^2 \psi = Q - \beta y \ . \tag{3.12}$$

Its Green's function is

$$G = \frac{1}{2\pi} \log |\mathbf{r} - \mathbf{r}'| \ . \tag{3.13}$$

Here, $|\mathbf{r} - \mathbf{r}'|$ is the distance between the points $\mathbf{r}(x, y)$ and $\mathbf{r}'(x', y')$ on the β-plane

$$|\mathbf{r} - \mathbf{r}'| = \sqrt{(x - x')^2 + (y - y')^2} \ . \tag{3.14}$$

Using Eq. (3.6), the streamfunction can be written as

$$\begin{aligned} \psi = & \frac{1}{2\pi} \sum_{i=1}^{N} \Gamma_i \log |\mathbf{r} - \mathbf{r}'| * \phi(|\mathbf{r}' - \mathbf{r_i}|) - \frac{1}{2\pi} \log |\mathbf{r} - \mathbf{r}'| * \beta y' \\ = & \frac{1}{2\pi} \sum_{i=1}^{N} \Gamma_i \int_0^{L_x} \int_0^{L_y} \log |\mathbf{r} - \mathbf{r}'| \phi(|\mathbf{r}' - \mathbf{r_i}|) \mathrm{d}x' \mathrm{d}y' \\ & - \frac{\beta}{2\pi} \int_0^{L_x} \int_0^{L_y} \log |\mathbf{r} - \mathbf{r}'| y' \mathrm{d}x' \mathrm{d}y' \ , \end{aligned} \tag{3.15}$$

where $L_x \times L_y$ is the size of the β-plane periodic domain.

The velocity field can then be obtained using Eq. (3.7)

$$u^x = -\frac{1}{2\pi} \sum_{i=1}^{N} \Gamma_i \int_0^{L_x} \int_0^{L_y} \frac{y - y'}{|\mathbf{r} - \mathbf{r}'|^2} \phi(|\mathbf{r}' - \mathbf{r_i}|) \mathrm{d}x' \mathrm{d}y' + \frac{\beta}{2\pi} \int_0^{L_x} \int_0^{L_y} \frac{y - y'}{|\mathbf{r} - \mathbf{r}'|^2} y' \mathrm{d}x' \mathrm{d}y', \tag{3.16}$$

$$u^y = \frac{1}{2\pi} \sum_{i=1}^{N} \Gamma_i \int_0^{L_x} \int_0^{L_y} \frac{x - x'}{|\mathbf{r} - \mathbf{r}'|^2} \phi(|\mathbf{r}' - \mathbf{r_i}|) \mathrm{d}x' \mathrm{d}y' - \frac{\beta}{2\pi} \int_0^{L_x} \int_0^{L_y} \frac{x - x'}{|\mathbf{r} - \mathbf{r}'|^2} y' \mathrm{d}x' \mathrm{d}y'. \tag{3.17}$$

3.1.2 Baroclinic Case

For the baroclinic case (finite L_d) on the β-plane, Eq. (3.3) is the screened Poisson's equation

$$\nabla^2\psi - \frac{\psi}{L_d^2} = Q - \beta y \ . \tag{3.18}$$

Its Green's function is

$$G = -\frac{1}{2\pi} K_0\left(\frac{|\mathbf{r}-\mathbf{r}'|}{L_d}\right) \ , \tag{3.19}$$

where K_0 is the modified Bessel function of the second kind of order zero.

Note that

$$\begin{aligned} &K_0(z) \sim -\ln(z) \\ &\text{as } z \to 0 \end{aligned} \tag{3.20}$$

and so Eq. (3.19) reduces to Eq. (3.13) as $L_d \to \infty$.

Using Eq. (3.6), the strean function can be written as

$$\begin{aligned} \psi &= -\frac{1}{2\pi}\sum_{i=1}^{N}\Gamma_i K_0\left(\frac{|\mathbf{r}-\mathbf{r}'|}{L_d}\right) * \phi(|\mathbf{r}'-\mathbf{r_i}|) + \frac{1}{2\pi}K_0\left(\frac{|\mathbf{r}-\mathbf{r}'|}{L_d}\right) * \beta y' \\ &= -\frac{1}{2\pi}\sum_{i=1}^{N}\Gamma_i \int_0^{L_x}\int_0^{L_y} K_0\left(\frac{|\mathbf{r}-\mathbf{r}'|}{L_d}\right)\phi(|\mathbf{r}'-\mathbf{r_i}|)\mathrm{d}x'\mathrm{d}y' \\ &\quad + \frac{\beta}{2\pi}\int_0^{L_x}\int_0^{L_y} K_0\left(\frac{|\mathbf{r}-\mathbf{r}'|}{L_d}\right) y'\mathrm{d}x'\mathrm{d}y' \\ &= -\frac{1}{2\pi}\sum_{i=1}^{N}\Gamma_i K_0\left(\frac{|\mathbf{r}-\mathbf{r_i}|}{L_d}\right)\left(1+\frac{1}{4}\left(\frac{\sigma}{L_d}\right)^2+\frac{1}{32}\left(\frac{\sigma}{L_d}\right)^4+\dots\right) \\ &\quad + \beta y L_d^2 \ . \end{aligned} \tag{3.21}$$

The velocity field can then be obtained using Eq. (3.7)

$$\begin{aligned} u^x &= \frac{1}{2\pi}\sum_{i=1}^{N}\Gamma_i \int_0^{L_x}\int_0^{L_y} \frac{y-y'}{L_d|\mathbf{r}-\mathbf{r}'|}K_1\left(\frac{|\mathbf{r}-\mathbf{r}'|}{L_d}\right)\phi(|\mathbf{r}'-\mathbf{r_i}|)\mathrm{d}x'\mathrm{d}y' \\ &\quad -\frac{\beta}{2\pi}\int_0^{L_x}\int_0^{L_y} \frac{y-y'}{L_d|\mathbf{r}-\mathbf{r}'|}K_1\left(\frac{|\mathbf{r}-\mathbf{r}'|}{L_d}\right) y'\mathrm{d}x'\mathrm{d}y' \ , \end{aligned} \tag{3.22}$$

$$\begin{aligned} u^y &= -\frac{1}{2\pi}\sum_{i=1}^{N}\Gamma_i \int_0^{L_x}\int_0^{L_y} \frac{x-x'}{L_d|\mathbf{r}-\mathbf{r}'|}K_1\left(\frac{|\mathbf{r}-\mathbf{r}'|}{L_d}\right)\phi(|\mathbf{r}'-\mathbf{r_i}|)\mathrm{d}x'\mathrm{d}y' \\ &\quad +\frac{\beta}{2\pi}\int_0^{L_x}\int_0^{L_y} \frac{x-x'}{L_d|\mathbf{r}-\mathbf{r}'|}K_1\left(\frac{|\mathbf{r}-\mathbf{r}'|}{L_d}\right) y'\mathrm{d}x'\mathrm{d}y' \ . \end{aligned} \tag{3.23}$$

3.2 Solution on the Sphere

3.2.1 Barotropic Case

For the barotropic case ($L_d = \infty$) on the sphere, Eq. (3.3) reduces to Poisson's equation

$$\nabla_s^2\psi = Q - f \ , \tag{3.24}$$

where $f = 2\Omega\sin\theta$. Here, ∇_s^2 is the Laplace-Beltrami operator expressed as

$$\nabla_s^2 = \frac{1}{R^2\cos^2\theta}\frac{\partial^2}{\partial\lambda^2} + \frac{1}{R^2\cos\theta}\frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial}{\partial\theta}\right) \ . \tag{3.25}$$

Its Green's function on the spherical surface is [2, 12]

$$G = \frac{1}{4\pi}\log(1-\cos\gamma) \ , \tag{3.26}$$

where

$$\cos\gamma = \frac{\mathbf{r}\cdot\mathbf{r}'}{|\mathbf{r}|\cdot|\mathbf{r}'|} = \sin\theta\sin\theta' + \cos\theta\cos\theta'\cos(\lambda-\lambda') \ , \tag{3.27}$$

and γ is the central angle between the two points $\mathbf{r}(\theta,\lambda)$ and $\mathbf{r}'(\theta',\lambda')$ on the sphere as in Fig.3-1.

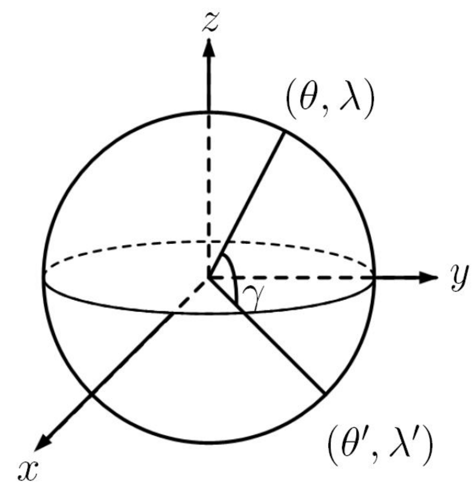


Figure 3-1: Diagram for showing the central angle between two points on the sphere.

Green's function (3.26) satisfies

$$\nabla^2 G = \left(\delta(\theta, \lambda, \theta', \lambda') - \frac{1}{4\pi R^2} \right) , \tag{3.28}$$

where $\delta(\theta, \lambda, \theta', \lambda')$ is the delta function. The right hand side of Eq. (3.28) is such that

$$\int_{S^2} \left(\delta - \frac{1}{4\pi R^2} \right) dS = \int_{S^2} \delta dS - \int_{S^2} \frac{1}{4\pi R^2} dS = 0 , \tag{3.29}$$

for the solution to satisfiy Gauss condition (total circulation should be conserved according to Kelvin's theorem)

$$\int_S \zeta \mathrm{d}s = 0 , \tag{3.30}$$

where the integration is taken over the whole spherical surface.

Using Eq. (3.6), the strean function can be written as

$$
\begin{aligned}
\psi = & \frac{1}{4\pi}\sum_{i=1}^{N}\Gamma_i \log(1-\cos\gamma) * \phi(|\mathbf{r'}-\mathbf{r_i}|) - \frac{1}{4\pi}\log(1-\cos\gamma) * f \\
= & \frac{R^2}{4\pi}\sum_{i=1}^{N}\Gamma_i \int_0^{2\pi}\int_{-\pi/2}^{\pi/2} \log(1-\cos\gamma)\phi(|\mathbf{r'}-\mathbf{r_i}|)\cos\theta' \mathrm{d}\theta' \mathrm{d}\lambda' \\
& - \frac{\Omega R^2}{4\pi}\int_0^{2\pi}\int_{-\pi/2}^{\pi/2} \log(1-\cos\gamma)\sin 2\theta' \mathrm{d}\theta' \mathrm{d}\lambda' \ .
\end{aligned}
\tag{3.31}
$$

The velocity field can then be obtained using Eq. (3.8)

$$
\begin{aligned}
u^\lambda = & -\frac{1}{4\pi\cos\theta}\Bigg\{\sum_{i=1}^{N}\Gamma_i \int_0^{2\pi}\int_{-\pi/2}^{\pi/2} \frac{(1-\cos\gamma)_\theta}{1-\cos\gamma}\phi(|\mathbf{r'}-\mathbf{r_i}|)\cos\theta' \mathrm{d}\theta' \mathrm{d}\lambda' \\
& -\Omega \int_0^{2\pi}\int_{-\pi/2}^{\pi/2} \frac{(1-\cos\gamma)_\theta}{1-\cos\gamma}\sin 2\theta' \mathrm{d}\theta' \mathrm{d}\lambda' \Bigg\} \ ,
\end{aligned}
\tag{3.32}
$$

$$
\begin{aligned}
u^\theta = & \frac{1}{4\pi\cos\theta}\Bigg\{\sum_{i=1}^{N}\Gamma_i \int_0^{2\pi}\int_{-\pi/2}^{\pi/2} \frac{(1-\cos\gamma)_\lambda}{1-\cos\gamma}\phi(|\mathbf{r'}-\mathbf{r_i}|)\cos\theta' \mathrm{d}\theta' \mathrm{d}\lambda' \\
& -\Omega \int_0^{2\pi}\int_{-\pi/2}^{\pi/2} \frac{(1-\cos\gamma)_\lambda}{1-\cos\gamma}\sin 2\theta' \mathrm{d}\theta' \mathrm{d}\lambda' \Bigg\} \ ,
\end{aligned}
\tag{3.33}
$$

where

$$
(1-\cos\gamma)_\theta = \frac{\partial}{\partial\theta}(1-\cos\gamma) = -\cos\theta\sin\theta' + \sin\theta\cos\theta'\cos(\lambda-\lambda') \ , \tag{3.34}
$$

$$
(1-\cos\gamma)_\lambda = \frac{\partial}{\partial\lambda}(1-\cos\gamma) = -\sin\theta\sin\theta' + \cos\theta\cos\theta'\sin(\lambda-\lambda') \ . \tag{3.35}
$$

3.2.2 Baroclinic Case

For the baroclinic case (finite L_d) on the sphere, Eq. (3.3) is again the screened Poisson's equation

$$\nabla_s^2 \psi - \frac{\psi}{L_d^2} = Q - f \ . \tag{3.36}$$

A closed-form for Green's function does not exist in this case. In Chapter 4, we attempt to find an expression of Green's function in terms of Legendre Polynomial.

Chapter 4

The Missing Green's Function

In this chapter, we attempt to find Green's function of the screened Poisson equation on the spherical shell, which allows to compute the stream function ψ from the difference between the potential vorticity Q and the planetary vorticity f.

Eigenfunctions of the Laplace-Beltrami Operator

We define $L_2(S^2)$ as the space of all functions $f : S^2 \to S^2$ satisfying

$$\int_{S^2} |f|^2 d\beta < \infty \ , \tag{4.1}$$

$L_2(S^2)$ is a Hilbert space. The Laplace-Beltrami operator on the unit sphere

$$\nabla_s^2 = \frac{1}{\cos^2\theta}\frac{\partial^2}{\partial\lambda^2} + \frac{1}{\cos\theta}\frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial}{\partial\theta}\right) \tag{4.2}$$

admits a set of eigenfunctions that is complete and orthogonal. These eigenfunctions define a basis for the space defined above.

The eigenvalues of the spherical harmonics are of the form $l(l+1)$, $l \in \mathbb{N}$. For each l, there is $2l + 1$ linearly independent eigenfunctions having all the same eigenvalue $l(l + 1)$. Each eigenfunction is defined by its order m, where $m = -l, ..., l$.

The normalized eigenfunctions are given by (θ^* is colatitude and λ is longitude)

$$Y_{l,m}(\theta^*, \lambda) = \sqrt{\frac{(2l+1)}{4\pi}\frac{(l-|m|)!}{(l+|m|)!}}P_l^{|m|}(\cos\theta^*)e^{im\lambda} \tag{4.3}$$

where $P_l^m(x)$ is the l-th **Associated Legendre Polynomial** given by

$$P_l^m(x) = \frac{(-1)^m}{2^l\ l!}(1-x^2)^{m/2}\frac{\mathrm{d}^{l+m}}{\mathrm{d}x^{l+m}}([x^2-1]^l) \tag{4.4}$$

and

$$e^{im\lambda} = \cos(m\lambda) + i\sin(m\lambda)\ \text{(Euler's Identity)} \tag{4.5}$$

Since these eigenfunctions are orthonormal, we have

$$\int_{\theta^*=0}^{\pi}\int_{\lambda=0}^{2\pi} Y_{l,m}(\theta^*, \lambda)Y_{l,m}(\theta^{*\prime}, \lambda')\sin\theta^*\,\mathrm{d}\theta^*\,\mathrm{d}\lambda = \delta((\theta^*, \lambda) - (\theta^{*\prime}, \lambda')) \tag{4.6}$$

Where the point $(\theta^{*\prime}, \lambda')$ is the point source on the unit sphere. The spherical harmonic addition theorem states that

$$\frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y_{l,m}(\theta^*, \lambda)[Y_{l,m}(\theta^{*\prime}, \lambda')]^* = P_l(\cos\gamma) \tag{4.7}$$

where $\cos\gamma = \cos\theta^*\cos\theta^{*\prime} + \sin\theta^*\sin\theta^{*\prime}\cos(\lambda-\lambda')$. The closure relationship for the spherical harmonics is

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y_{l,m}(\theta^*, \lambda)[Y_{l,m}(\theta^{*\prime}, \lambda')]^* = \frac{1}{\sin\theta^*}\delta((\theta^*, \lambda) - (\theta^{*\prime}, \lambda')) \tag{4.8}$$

Also, we have [18]

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y_{l,m}(\theta^*, \lambda)[Y_{l,m}(\theta^{*\prime}, \lambda')]^* = \sum_{l=0}^{\infty}\frac{2l+1}{4\pi}P_l(\cos\gamma) \tag{4.9}$$

Proposed Solution for $\nabla_s^2 u = f$

Since the spherical harmonics are basis for the space $L_2(S^2)$, any function f on the sphere can be expanded into a summation of spherical harmonics. Thus, f can be written as

$$f(\theta, \lambda) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} f_{lm}(r) Y_{l,m}(\theta, \lambda) \tag{4.10}$$

and the solution u is given by

$$u(\theta, \lambda) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} u_{lm}(r) Y_{l,m}(\theta, \lambda) \tag{4.11}$$

Using the orthogonality of the eigenfunctions, and since the space is equipped with the inner product operator

$$< f, g >= \int_{S^2} f.g^* d\beta \tag{4.12}$$

where g^* is the complex conjugate of the function g, f_{lm} is given by

$$f_{lm} = \int Y_{l,m}^* \; f \, \mathrm{d}\Omega = \int_{\theta=0}^{\pi} \int_{\lambda=0}^{2\pi} Y_{l,m}^*(\theta, \lambda) f(\theta, \lambda) \cos\theta \, \mathrm{d}\lambda \, \mathrm{d}\theta \tag{4.13}$$

where the integration is taken over the unit sphere S^2.

Using the fact that

$$\nabla_s^2 Y_{l,m} = -l(l+1) Y_{l,m} \text{ (Eigenfunction definition)} \tag{4.14}$$

we can write

$$\sum_{l=0}^{\infty} \sum_{m=-l}^{l} u_{lm}(r)(-l)(l+1) Y_{l,m}(\theta, \lambda) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} f_{lm}(r) Y_{l,m}(\theta, \lambda) \tag{4.15}$$

Since this equality holds $\forall r \in \mathbb{R}$

$$u_{lm} = \frac{f_{lm}}{-l(l+1)} = \frac{\int Y_{l,m}^*(\theta', \lambda') f \mathrm{d}\Omega}{-l(l+1)} \tag{4.16}$$

and

$$u(\theta,\lambda) = \sum_{l=1}^{\infty}\sum_{m=-l}^{l} \frac{f_{lm}}{-l(l+1)} Y_{l,m}(\theta,\lambda) = \sum_{l=1}^{\infty}\sum_{m=-l}^{l} \frac{\int Y_{l,m}^{*}(\theta',\lambda') f(\theta',\lambda') \mathrm{d}\Omega}{-l(l+1)} Y_{l,m}(\theta,\lambda) \tag{4.17}$$

Since $u = G * f$, Green's function is then

$$G = \sum_{l=1}^{\infty}\sum_{m=-l}^{l} \frac{Y_{l,m}^{*}(\theta',\lambda') Y_{l,m}(\theta,\lambda)}{-l(l+1)} = \sum_{l=1}^{\infty} \frac{-1}{l(l+1)} \sum_{m=-l}^{l} Y_{l,m}^{*}(\theta',\lambda') Y_{l,m}(\theta,\lambda) \tag{4.18}$$

Using the **Spherical Harmonic Addition Theorem**

$$G = \sum_{l=1}^{\infty} \frac{-1}{l(l+1)} \frac{2l+1}{4\pi} P_l(\cos(\gamma)) = \frac{-1}{4\pi} \sum_{l=1}^{\infty} \frac{2l+1}{l(l+1)} P_l(\cos(\gamma)) \tag{4.19}$$

Noting that

$$\frac{2l+1}{l(l+1)} = \frac{1}{l} + \frac{1}{l+1} \text{ (Partial Fraction Expansion)} \tag{4.20}$$

Green's function may be written as

$$G = \frac{-1}{4\pi} \sum_{l=1}^{\infty} \left(\frac{1}{l} + \frac{1}{l+1} \right) P_l(\cos(\gamma)) \tag{4.21}$$

Since

$$\frac{-1}{4\pi} \sum_{l=1}^{\infty} \frac{1}{l} P_l(\cos(\gamma)) \text{ and } \frac{-1}{4\pi} \sum_{l=1}^{\infty} \frac{1}{l+1} P_l(\cos(\gamma))$$

both converge, G becomes

$$G = \frac{-1}{4\pi} \left(\sum_{l=1}^{\infty} \frac{1}{l} P_l(\cos(\gamma)) + \sum_{l=1}^{\infty} \frac{1}{l+1} P_l(\cos(\gamma)) \right) \tag{4.22}$$

Closed Form Solution

$$\sum_{l=1}^{\infty} \frac{1}{l} P_l(\cos(\gamma)) = \log\left(\frac{2}{1-\cos(\gamma)+\sqrt{2-2\cos(\gamma)}} \right) \tag{4.23}$$

$$\sum_{l=1}^{\infty} \frac{1}{l+1} P_l(\cos(\gamma)) = \log\left(1 + \frac{2}{\sqrt{2-2\cos(\gamma)}} \right) - 1 \tag{4.24}$$

Hence

$$
\begin{aligned}
G = & -\frac{1}{4\pi}\log\left(\frac{2}{1-\cos(\gamma)+\sqrt{2-2\cos(\gamma)}}\right) \\
& -\frac{1}{4\pi}\left(\log\left(1+\frac{2}{\sqrt{2-2\cos(\gamma)}}\right)-1\right) \\
= & -\frac{1}{4\pi}\log\left(\frac{2}{1-\cos(\gamma)+\sqrt{2-2\cos(\gamma)}}\times\left(1+\frac{2}{\sqrt{2-2\cos(\gamma)}}\right)\times\frac{1}{e}\right)
\end{aligned}
\tag{4.25}
$$

Simplifying, we get

$$
\begin{aligned}
G = \frac{1}{4\pi}\log\left(e\sin^2\left(\frac{\gamma}{2}\right)\right) &= \frac{1}{4\pi}\log\left(\frac{e}{2}(1-\cos\gamma)\right) \\
&= \frac{1}{4\pi}\log(1-\cos\gamma)+\frac{1}{4\pi}\log\left(\frac{e}{2}\right)
\end{aligned}
\tag{4.26}
$$

Dropping the constant term $\frac{1}{4\pi}\log\left(\frac{e}{2}\right)$, we get

$$
G = \frac{1}{4\pi}\log(1-\cos(\gamma)) \tag{4.27}
$$

for a unit sphere.

Proposed Solution for $\nabla_s^2 u - \frac{1}{\alpha}u = f$

We start by expressing the screened Poisson equation ($\alpha > 0$) in terms of the spherical harmonics of u and f

$$
\begin{aligned}
\sum_{l=0}^{\infty}\sum_{m=-l}^{l} u_{lm}(r)(-l)(l+1)Y_{l,m}(\theta,\lambda) - \frac{1}{\alpha}\sum_{l=0}^{\infty}\sum_{m=-l}^{l} u_{lm}(r)Y_{l,m}(\theta,\lambda) = \\
\sum_{l=0}^{\infty}\sum_{m=-l}^{l} f_{lm}(r)Y_{l,m}(\theta,\lambda)
\end{aligned}
\tag{4.28}
$$

$$
\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(-l(l+1)-\frac{1}{\alpha}\right)u_{lm}(r)Y_{l,m}(\theta,\lambda) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} f_{lm}(r)Y_{l,m}(\theta,\lambda) \tag{4.29}
$$

Thus

$$u_{lm} = \frac{f_{lm}}{-l(l+1) - 1/\alpha} \tag{4.30}$$

Hence, Green's function of **Screened Poisson Equation** on the unit sphere S^2 is

$$G = \frac{-1}{4\pi} \sum_{l=0}^{\infty} \frac{2l+1}{l(l+1)+1/\alpha} P_l(\cos(\gamma)) \tag{4.31}$$

This infinite series converge except for $\cos\gamma = 1$.

Green's Function on the Sphere of Radius R

The Laplace-Beltrami operator is then

$$\nabla_s^2 = \frac{1}{R^2\cos^2\theta}\frac{\partial^2}{\partial\lambda^2} + \frac{1}{R^2\cos\theta}\frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial}{\partial\theta}\right) \tag{4.32}$$

Hence

$$\nabla_s^2 Y_{l,m} = \frac{-l(l+1)}{R^2} Y_{l,m} \tag{4.33}$$

and the expansion coefficient f_{lm} is given by

$$f_{lm} = \frac{1}{R^2}\int Y_{l,m}^* \, f \, \mathrm{d}\Omega = \frac{1}{R^2}\int_{\theta=0}^{\pi}\int_{\lambda=0}^{2\pi} Y_{l,m}^*(\theta,\lambda) f(\theta,\lambda)\cos\theta \, \mathrm{d}\lambda \, \mathrm{d}\theta \tag{4.34}$$

Thus, Green's function of the **Screened Poisson Equation** on the sphere of radius R is

$$G = \frac{-1}{4\pi} \sum_{l=0}^{\infty} \frac{2l+1}{l(l+1)+R^2/\alpha} P_l(\cos(\gamma)) \tag{4.35}$$

Integral Form of Green's Function

Consider Green's Function of the Screened Poisson Equation on the unit sphere

$$G = \frac{-1}{4\pi} \sum_{l=0}^{\infty} \frac{2l+1}{l(l+1)+1/\alpha} P_l(\cos(\gamma)) \tag{4.36}$$

Tools used

$$\sum_{l=0}^{\infty} t^l P_l(x) = \frac{1}{\sqrt{t^2 - 2tx + 1}} \quad \text{Legendre Generating Function} \tag{4.37}$$

$$\frac{1}{l+R} = \int_0^{+\infty} e^{-z(l+R)}\, dz \tag{4.38}$$

Method

$$l(l+1) + \frac{1}{\alpha} = l^2 + l + \frac{1}{\alpha} = (l - R_1)(l - R_2) \tag{4.39}$$

where R_1 and R_2 are the roots of the quadratic equation (real or complex).

$$\frac{2l+1}{l(l+1) + 1/\alpha} = \frac{r_1}{l - R_1} + \frac{r_2}{l - R_2} \tag{4.40}$$

r_1 and r_2 are real or complex coefficients.

$$\frac{r_1}{l - R_1} = r_1 \int_0^{+\infty} e^{-z(l-R_1)}\, dz \tag{4.41}$$

and

$$\frac{r_2}{l - R_2} = r_2 \int_0^{+\infty} e^{-z(l-R_2)}\, dz \tag{4.42}$$

Hence

$$G = \frac{-1}{4\pi} \sum_{l=0}^{\infty} \left(\int_0^{+\infty} r_1 e^{-z(l-R_1)} + r_2 e^{-z(l-R_2)}\, \mathrm{d}z \right) P_l(\cos(\gamma)) \tag{4.43}$$

$$= \frac{-1}{4\pi} \sum_{l=0}^{\infty} \left(\int_0^{+\infty} e^{-zl} [r_1 e^{zR_1} + r_2 e^{zR_2}]\, \mathrm{d}z \right) P_l(\cos(\gamma)) \tag{4.44}$$

$$= \frac{-1}{4\pi} \int_0^{+\infty} (r_1 e^{zR_1} + r_2 e^{zR_2}) \sum_{l=0}^{\infty} (e^{-z})^l P_l(\cos(\gamma))\, \mathrm{d}z \tag{4.45}$$

$$= \frac{-1}{4\pi} \int_0^{+\infty} \frac{(r_1 e^{zR_1} + r_2 e^{zR_2})}{\sqrt{e^{-2z} - 2e^{-z}\cos(\gamma) + 1}} \mathrm{d}z \tag{4.46}$$

Using the change of variable $t = e^{-z}$

$$G = \frac{-1}{4\pi} \int_0^{1} \frac{(r_1 t^{-R_1} + r_2 t^{-R_2})}{t\sqrt{t^2 - 2t\cos(\gamma) + 1}} \mathrm{d}t \tag{4.47}$$

Since $t=0$ is a singularity of the integrand

$$G = \lim_{\epsilon\to 0}\frac{-1}{4\pi}\int_{\epsilon}^{1}\frac{(r_1 t^{-R_1}+r_2 t^{-R_2})}{t\sqrt{t^2-2t\cos(\gamma)+1}}\mathrm{d}t \tag{4.48}$$

Proof of the Convergence of Green's Function

Let

$$f(w,\gamma) = \sum_{l\geq 0}\frac{(2l+1)}{l(l+1)+w}P_l(\cos(\gamma)),\ w\in\mathbb{R}^+. \tag{4.49}$$

If $\cos(\gamma)=1$, and since $P_l(1)=1,\ \forall l\in\mathbb{N}$

$$\sum_{l\geq 0}\frac{2l+1}{l(l+1)+w}\sim\sum_{l\geq 0}\frac{1}{l+w} \tag{4.50}$$

so the series diverges like the harmonic series.

Assume that $\cos(\gamma)\neq 1$

$$f(w,\gamma) = 2\sum_{l\geq 0}\frac{lP_l(\cos(\gamma))}{l(l+1)+w}+\sum_{l\geq 0}\frac{P_l(\cos(\gamma))}{l(l+1)+w} \tag{1}$$

the second series is convergent since

$$\sum_{l\geq 0}\left|\frac{P_l(\cos(\gamma))}{l(l+1)+w}\right|\leq\sum_{l\geq 0}\frac{1}{l^2+w}<\infty,\ \forall w\in\mathbb{R}^+ \tag{4.51}$$

and the first series

$$\sum_{l\geq 0}\frac{lP_l(\cos(\gamma))}{l(l+1)+w}\leq\sum_{l\geq 0}\frac{P_l(\cos(\gamma))}{l+1} \tag{4.52}$$

Using the Legendre generating function

$$\sum_{l\geq 0}\frac{P_l(\cos(\gamma))}{l+1}=\int_0^1\frac{1}{\sqrt{1-2\cos(\gamma)t+t^2}}dt \tag{4.53}$$

and

$$\int_0^1 \frac{1}{\sqrt{1-2\cos(\gamma)\,t+t^2}}dt = \log\left(2+2\sqrt{2-\cos(\gamma)}-\cos(\gamma)\right)-\log(2-\cos(\gamma)) \tag{4.54}$$

which is convergent, hence the series converges.

Chapter 5

The Redistribution Scheme

In general, the redistribution method is used to solve the diffusion equation. The method accounts for diffusion by distributing fractions f_{ij} of the circulation of each vortex element i to neighboring elements j. Neighboring elements are those situated within a certain distance from the element to be diffused. This distance is chosen to be of the order of the diffusion distance covered during a time step Δt. More precisely, this distance is defined as

$$h_\nu \equiv \sqrt{\nu \Delta t} \tag{5.1}$$

where ν is the coefficient of kinematic viscosity.

The fractions of the circulation of element i to be distributed are determined by solving the moment equations.

In this chapter, the redistribution method is used to solve the dissipation equation of the quasi-geostrophic shallow-water equations. The kinematic viscosity coefficient ν is therefore made analogous to the dissipation coefficient α that will be used to define an upper bound of the distance between an element i and its neighbor j. This distance will be defined as

$$|\mathbf{r}_j - \mathbf{r}_i| \leq a\, h_\alpha \tag{5.2}$$

where a is a chosen constant.

In what follows, we present a formulation of the redistribution scheme that leads to the moment equations, the conservation of which allows to write a system of linear equations in the unknown fractions f_{ij}.

Consider the quasi-geostrophic equations of a shallow-water layer on a β-plane of dimensions $L_x \times L_y$

$$Q = f + \nabla^2\psi - \frac{\psi}{L_d^2} \tag{5.3}$$

$$\frac{DQ}{Dt} = -\alpha\zeta \tag{5.4}$$

Here, $\zeta = \nabla^2\psi$ and $f = \beta y$.

After we advect the particles using the computed flow field, we attempt to solve the following dissipation equation

$$\begin{aligned}
\frac{\partial Q}{\partial t} &= -\alpha\nabla^2\psi \\
\frac{\partial Q}{\partial t} &= -\alpha\left(Q - f + \frac{\psi}{L_d^2}\right) \\
\frac{\partial Q}{\partial t} + \alpha Q &= \alpha\left(f - \frac{\psi}{L_d^2}\right) \\
\mathrm{e}^{\alpha t}\frac{\partial Q}{\partial t} + \alpha\mathrm{e}^{\alpha t}Q &= \alpha\mathrm{e}^{\alpha t}\left(f - \frac{\psi}{L_d^2}\right)
\end{aligned} \tag{5.5}$$

or

$$\frac{\partial}{\partial t}\left(\mathrm{e}^{\alpha t}Q\right) = \alpha\mathrm{e}^{\alpha t}\left(\beta y - \frac{\psi}{L_d^2}\right) \tag{5.6}$$

Integrating Eq. (5.6) in time from t to $t+\Delta t$ leads to

$$\begin{aligned}
\mathrm{e}^{\alpha(t+\Delta t)}Q(t+\Delta t) - \mathrm{e}^{\alpha t}Q(t) &= \alpha\beta y\int_t^{t+\Delta t}\mathrm{e}^{\alpha t'}\,\mathrm{d}t' - \frac{\alpha}{L_d^2}\int_t^{t+\Delta t}\mathrm{e}^{\alpha t'}\psi(t')\,\mathrm{d}t' \\
Q(t+\Delta t) - \mathrm{e}^{-\alpha\Delta t}Q(t) &= \alpha\mathrm{e}^{-\alpha(t+\Delta t)}\beta y\int_t^{t+\Delta t}\mathrm{e}^{\alpha t'}\,\mathrm{d}t' - \frac{\alpha\mathrm{e}^{-\alpha(t+\Delta t)}}{L_d^2}\int_t^{t+\Delta t}\mathrm{e}^{\alpha t'}\psi(t')\,\mathrm{d}t'
\end{aligned} \tag{5.7}$$

Expressing

$$Q(t) = \sum_{i=1}^{N(t)}\Gamma_i(t)\,\phi\left(\mathbf{r},\mathbf{r}_i(t)\right) \tag{5.8}$$

and substituting ψ by $G * (Q - f)$, Eq. (5.7) becomes

$$\sum_{i=1}^{N(t+\Delta t)} \Gamma_i(t+\Delta t)\,\phi\left(\mathbf{r}, \mathbf{r}_i(t+\Delta t)\right) - \mathrm{e}^{-\alpha\Delta t} \sum_{i=1}^{N(t)} \Gamma_i(t)\,\phi\left(\mathbf{r}, \mathbf{r}_i(t)\right) = \beta y\,\left(1 - \mathrm{e}^{-\alpha\Delta t}\right)$$
$$+ \frac{\alpha \mathrm{e}^{-\alpha(t+\Delta t)}}{L_d^2} \int_t^{t+\Delta t} \mathrm{e}^{\alpha t'} \frac{1}{2\pi} \sum_{i=1}^{N(t')} \Gamma_i(t') \int_0^{L_x}\int_0^{L_y} K_0\left(\frac{|\mathbf{r}-\mathbf{r}'|}{L_d}\right) \phi(|\mathbf{r}' - \mathbf{r}_i(t')|)\mathrm{d}x'\mathrm{d}y'\mathrm{d}t'$$
$$- \frac{\alpha \mathrm{e}^{-\alpha(t+\Delta t)}}{L_d^2} \int_t^{t+\Delta t} \mathrm{e}^{\alpha t'} \frac{\beta}{2\pi} \int_0^{L_x}\int_0^{L_y} K_0\left(\frac{|\mathbf{r}-\mathbf{r}'|}{L_d}\right) y'\mathrm{d}x'\mathrm{d}y'\mathrm{d}t' \tag{5.9}$$

$$\sum_{i=1}^{N(t+\Delta t)} \Gamma_i(t+\Delta t)\,\phi\left(\mathbf{r}, \mathbf{r}_i(t+\Delta t)\right) - \mathrm{e}^{-\alpha\Delta t} \sum_{i=1}^{N(t)} \Gamma_i(t)\,\phi\left(\mathbf{r}, \mathbf{r}_i(t)\right) = \beta y\,\left(1 - \mathrm{e}^{-\alpha\Delta t}\right)$$
$$+ \frac{\alpha \mathrm{e}^{-\alpha(t+\Delta t)}}{L_d^2} \int_t^{t+\Delta t} \mathrm{e}^{\alpha t'} \frac{1}{2\pi} \sum_{i=1}^{N(t')} \Gamma_i(t') \int_0^{L_x}\int_0^{L_y} K_0\left(\frac{|\mathbf{r}-\mathbf{r}'|}{L_d}\right) \phi(|\mathbf{r}' - \mathbf{r}_i(t')|)\mathrm{d}x'\mathrm{d}y'\mathrm{d}t'$$
$$- \frac{\beta\,\left(1 - \mathrm{e}^{-\alpha\Delta t}\right)}{2\pi L_d^2} \int_0^{L_x}\int_0^{L_y} K_0\left(\frac{|\mathbf{r}-\mathbf{r}'|}{L_d}\right) y'\mathrm{d}x'\mathrm{d}y' \tag{5.10}$$

We can solve equation (5.10) by considering the dissipation of element i from t to $t + \Delta t$. Since dissipation is a local phenomenon, we assume that the set of neighbors involved in the dissipation process is $\mathcal{N}_i$. Then, for each element i,

$$\sum_{j\in\mathcal{N}_i} \Gamma_j(t+\Delta t)\,\phi\left(\mathbf{r}, \mathbf{r}_j(t+\Delta t)\right) - \mathrm{e}^{-\alpha\Delta t} \Gamma_i(t)\,\phi\left(\mathbf{r}, \mathbf{r}_i(t)\right) = \beta y\,\left(1 - \mathrm{e}^{-\alpha\Delta t}\right)$$
$$+ \frac{\alpha \mathrm{e}^{-\alpha(t+\Delta t)}}{L_d^2} \int_t^{t+\Delta t} \mathrm{e}^{\alpha t'} \frac{1}{2\pi} \Gamma_i(t') \int_0^{L_x}\int_0^{L_y} K_0\left(\frac{|\mathbf{r}-\mathbf{r}'|}{L_d}\right) \phi(|\mathbf{r}' - \mathbf{r}_i(t')|)\mathrm{d}x'\mathrm{d}y'\mathrm{d}t'$$
$$- \frac{\beta\,\left(1 - \mathrm{e}^{-\alpha\Delta t}\right)}{2\pi L_d^2} \int_0^{L_x}\int_0^{L_y} K_0\left(\frac{|\mathbf{r}-\mathbf{r}'|}{L_d}\right) y'\mathrm{d}x'\mathrm{d}y' \tag{5.11}$$

If we choose a first order approximation for the RHS, i.e,

$$\int_{t}^{t+\Delta t} f(t')\mathrm{d}t' \simeq f(t)\Delta t$$

then we can write

$$\begin{aligned}
\sum_{j\in\mathcal{N}_i} \Gamma_j(t+\Delta t)\,\phi\left(|\mathbf{r}-\mathbf{r}_j(t+\Delta t)|\right) - \mathrm{e}^{-\alpha\Delta t}\Gamma_i(t)\,\phi\left(|\mathbf{r}-\mathbf{r}_i(t)|\right) = \\
\frac{\alpha\Delta t\,\mathrm{e}^{-\alpha\Delta t}}{2\pi L_d^2}\Gamma_i(t)\int_0^{L_x}\int_0^{L_y} K_0\left(\frac{|\mathbf{r}-\mathbf{r}'|}{L_d}\right)\phi(|\mathbf{r}'-\mathbf{r}_i(t)|)\mathrm{d}x'\mathrm{d}y' \\
+\beta y\left(1-\mathrm{e}^{-\alpha\Delta t}\right) - \frac{\beta\left(1-\mathrm{e}^{-\alpha\Delta t}\right)}{2\pi L_d^2}\int_0^{L_x}\int_0^{L_y} K_0\left(\frac{|\mathbf{r}-\mathbf{r}'|}{L_d}\right) y'\mathrm{d}x'\mathrm{d}y'
\end{aligned} \qquad (5.12)$$

Translating the reference frame to the center of element i, so that $\chi = x - x_i = \rho\cos\theta$

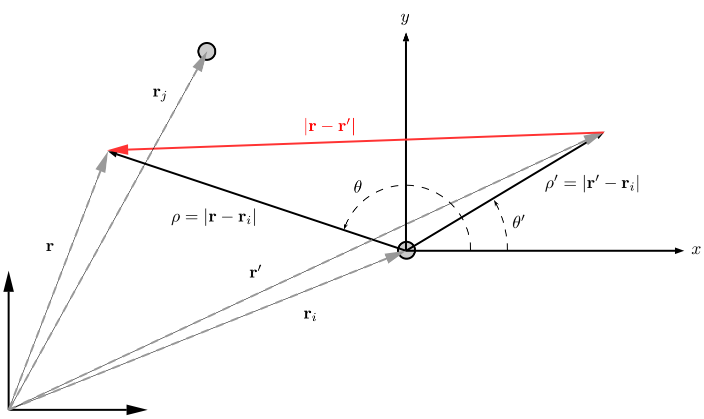


Figure 5-1: Schematic showing position vectors of source, target and integration points.

and $\eta = y - y_i = \rho\sin\theta$, and referring to Figure 5-1, (also let $x'' = x' - x = \rho''\cos\theta''$

and $y'' = y' - y = \rho'' \sin\theta''$)

$$
\begin{aligned}
\sum_{j\in\mathcal{N}_i} \Gamma_j(t+\Delta t)\,\phi\left(|\mathbf{r}-\mathbf{r}_j(t+\Delta t)|\right) - \mathrm{e}^{-\alpha\Delta t}\Gamma_i(t)\,\phi\left(\rho\right) = \\
\frac{\alpha\Delta t\,\mathrm{e}^{-\alpha\Delta t}}{2\pi L_d^2}\Gamma_i(t)\int_0^\infty\int_0^{2\pi} K_0\left(\frac{\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos(\theta-\theta')}}{L_d}\right)\phi(\rho')\,\rho'\mathrm{d}\rho'\mathrm{d}\theta' \\
+\beta y\,\left(1-\mathrm{e}^{-\alpha\Delta t}\right) \\
-\frac{\beta\,\left(1-\mathrm{e}^{-\alpha\Delta t}\right)}{2\pi L_d^2}\int_0^\infty\int_0^{2\pi} K_0\left(\frac{\rho''}{L_d}\right)(y+\rho''\sin\theta'')\rho''\mathrm{d}\rho''\mathrm{d}\theta''
\end{aligned}
\tag{5.13}
$$

The integral in the last term on the RHS of Eq. (5.13) can be simplified as follows

$$
\begin{aligned}
\int_0^\infty\int_0^{2\pi} K_0\left(\frac{\rho''}{L_d}\right)(y+\rho''\sin\theta'')\rho''\mathrm{d}\rho''\mathrm{d}\theta'' = \\
y\int_0^\infty K_0\left(\frac{\rho''}{L_d}\right)\rho''\mathrm{d}\rho''\mathrm{d}\theta''\int_0^{2\pi}\mathrm{d}\theta'' + \int_0^\infty K_0\left(\frac{\rho''}{L_d}\right)\rho''^2\mathrm{d}\rho''\int_0^{2\pi}\sin\theta''\mathrm{d}\theta'' = \\
y\,2\pi L_d^2 + \frac{\pi}{2}L_d^3\int_0^{2\pi}\sin\theta''\mathrm{d}\theta'' = \\
y\,2\pi L_d^2
\end{aligned}
\tag{5.14}
$$

and so the last two terms on the RHS of Eq. (5.13) vanish identically, and the equation becomes

$$
\begin{aligned}
\sum_{j\in\mathcal{N}_i} \Gamma_j(t+\Delta t)\,\phi\left(|\mathbf{r}-\mathbf{r}_j(t+\Delta t)|\right) - \mathrm{e}^{-\alpha\Delta t}\Gamma_i(t)\,\phi\left(\rho\right) = \\
\frac{\alpha\Delta t\,\mathrm{e}^{-\alpha\Delta t}}{2\pi L_d^2}\Gamma_i(t)\int_0^\infty\int_0^{2\pi} K_0\left(\frac{\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos(\theta-\theta')}}{L_d}\right)\phi(\rho')\,\rho'\mathrm{d}\rho'\mathrm{d}\theta'
\end{aligned}
\tag{5.15}
$$

We choose to satisfy equation (5.15) by conserving the moments

$$\int\int \chi^m \eta^n \left\{ \sum_{j\in\mathcal{N}_i} \Gamma_j(t+\Delta t)\, \phi\left(|\mathbf{r}-\mathbf{r}_j(t+\Delta t)|\right) - \mathrm{e}^{-\alpha\Delta t}\Gamma_i(t)\, \phi(\rho) \right\} \mathrm{d}\chi \mathrm{d}\eta =$$
$$\frac{\alpha\Delta t\, \mathrm{e}^{-\alpha\Delta t}}{2\pi L_d^2}\Gamma_i(t) \int\int \chi^m \eta^n \left\{ \int_0^\infty \int_0^{2\pi} K_0\left(\frac{\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos(\theta-\theta')}}{L_d} \right) \phi(\rho')\, \rho' \mathrm{d}\rho' \mathrm{d}\theta' \right\} \mathrm{d}\chi \mathrm{d}\eta \quad (5.16)$$

or (after dividing by Γ_i)

$$\int\int \chi^m \eta^n \left\{ \sum_{j\in\mathcal{N}_i} \mathrm{f}_{ij}\, \phi\left(|\mathbf{r}-\mathbf{r}_j(t+\Delta t)|\right) - \mathrm{e}^{-\alpha\Delta t}\phi(\rho) \right\} \mathrm{d}\chi \mathrm{d}\eta = \frac{\alpha\Delta t\, \mathrm{e}^{-\alpha\Delta t}}{2\pi L_d^2} \times$$
$$\int\int \rho^{m+n} \cos\theta^m \sin\theta^n \left\{ \int_0^\infty \int_0^{2\pi} K_0\left(\frac{\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos(\theta-\theta')}}{L_d} \right) \phi(\rho')\, \rho' \mathrm{d}\rho' \mathrm{d}\theta' \right\} \rho \mathrm{d}\rho \mathrm{d}\theta \quad (5.17)$$

where $\mathrm{f}_{ij} = \Gamma_j(t+\Delta t)/\Gamma_i(t)$.

We note that the first term of the above RHS integrand decays as $\mathbf{r}'$ goes away from $\mathbf{r}_i$ due to the behavior of ϕ. This allows us to express K_0 as a Taylor series expansion in $|\mathbf{r}'-\mathbf{r_i}|$ (ρ' in the reference frame where i is the center) around $|\mathbf{r}_i|$ (0 in the reference frame where i is the center), then $\mathbf{r}-\mathbf{r}' = \mathbf{r}-\mathbf{r}_i-(\mathbf{r}'-\mathbf{r}_i)$, and

$$\int_0^\infty \int_0^{2\pi} K_0\left(\frac{\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos(\theta-\theta')}}{L_d} \right) \phi(\rho')\, \rho' \mathrm{d}\rho' \mathrm{d}\theta' \simeq K_0\left(\frac{\rho}{L_d}\right) F(S) \quad (5.18)$$

where for $\phi(\rho) = \frac{1}{\pi\sigma^2}\mathrm{e}^{-\rho^2/\sigma^2}$, $F(S) = \left(1+\frac{1}{4}S^2+\frac{1}{32}S^4+\ldots\right)$ and $S \equiv \sigma/L_d$. For a $\phi = \frac{\delta(\rho)}{2\pi}$, we get $F(S) = \left(1+\frac{1}{4}S^2+\frac{1}{64}S^4+\ldots\right)$.

The moment equations then look like

$$\int\int \chi^m \eta^n \left\{ \sum_{j\in\mathcal{N}_i} \mathrm{f}_{ij}\, \phi\left(|\mathbf{r} - \mathbf{r}_j(t+\Delta t)|\right) - \mathrm{e}^{-\alpha\Delta t}\phi(\rho) \right\} \mathrm{d}\chi \mathrm{d}\eta = \frac{\alpha\Delta t\, \mathrm{e}^{-\alpha\Delta t}}{2\pi L_d^2} \int\int \rho^{m+n} \cos\theta^m \sin\theta^n K_0\left(\frac{\rho}{L_d}\right) F(S)\rho \mathrm{d}\rho \mathrm{d}\theta \tag{5.19}$$

$$\sum_{j\in\mathcal{N}_i} \mathrm{f}_{ij} \underbrace{\int\int \chi^m \eta^n \phi\left(|\mathbf{r} - \mathbf{r}_j(t+\Delta t)|\right) \mathrm{d}\chi \mathrm{d}\eta}_{A_{m,n}} - \mathrm{e}^{-\alpha\Delta t} \underbrace{\int\int \chi^m \eta^n \phi(\rho) \mathrm{d}\chi \mathrm{d}\eta}_{B_{m,n}} = \frac{\alpha\Delta t\, \mathrm{e}^{-\alpha\Delta t}}{2\pi L_d^2} \underbrace{\int\int \rho^{m+n} \cos\theta^m \sin\theta^n K_0\left(\frac{\rho}{L_d}\right) F(S)\rho \mathrm{d}\rho \mathrm{d}\theta}_{C_{m,n}} \tag{5.20}$$

Let $x - x_j = r\cos\alpha$ and $y - y_j = r\sin\alpha$, then $\chi = r\cos\alpha + x_j - x_i$ and $\eta = r\sin\alpha + y_j - y_i$, and noting that

$$\int_0^\infty \rho^p\, K_0\left(\frac{\rho}{L_d}\right) \mathrm{d}\rho = 2^{p-1} L_d^{p+1} \left[\Gamma\left(\frac{p+1}{2}\right)\right]^2$$

$$\int_0^\infty \rho^p\, \phi(\rho)\, \mathrm{d}\rho = \frac{1}{2\pi}\sigma^{p-1}\Gamma\left(\frac{p+1}{2}\right) \text{ for } \phi = \frac{1}{\pi\sigma^2}\mathrm{e}^{-\rho^2/\sigma^2}$$

and

$$\int_0^{2\pi} \cos\alpha^a \sin\alpha^b \mathrm{d}\alpha = 0 \text{ for } a \text{ odd or } b \text{ odd}$$

$$\int_0^{2\pi} \cos\alpha^a \sin\alpha^b \mathrm{d}\alpha = 2\pi, \pi/4, 3\pi/64, 5\pi/512, 35\pi/16384 \text{ for } a = b = 0, 2, 4, 6, 8$$

we can write

$$\begin{aligned} A_{m,n} &= \int\int \chi^m \eta^n \phi\left(|\mathbf{r} - \mathbf{r}_j(t+\Delta t)|\right) \mathrm{d}\chi \mathrm{d}\eta \\ &= \int\int (r\cos\alpha + x_j - x_i)^m (r\sin\alpha + y_j - y_i)^n \phi(r)\, r\, \mathrm{d}r \mathrm{d}\alpha \end{aligned} \tag{5.21}$$

$$
\begin{aligned}
B_{m,n} &= \int\int \chi^m \eta^n \phi(\rho) \mathrm{d}\chi \mathrm{d}\eta \\
&= \int\int (\rho\cos\theta)^m (\rho\sin\theta)^n \phi(\rho)\, \rho\, \mathrm{d}\rho \mathrm{d}\theta \\
&= \int\int \rho^{m+n+1} \cos\theta^m \sin\theta^n \phi(\rho)\, \mathrm{d}\rho \mathrm{d}\theta \\
&= \int_0^{2\pi} \cos\theta^m \sin\theta^n \mathrm{d}\theta \int_0^{\infty} \rho^{m+n+1} \phi(\rho) \mathrm{d}\rho
\end{aligned}
\tag{5.22}
$$

$$
\begin{aligned}
C_{m,n} &= \int\int \rho^{m+n} \cos\theta^m \sin\theta^n K_0\left(\frac{\rho}{L_d}\right) F(S) \rho\, \mathrm{d}\rho \mathrm{d}\theta \\
&= F(S) \int_0^{2\pi} \cos\theta^m \sin\theta^n \mathrm{d}\theta \int_0^{\infty} \rho^{m+n+1} K_0\left(\frac{\rho}{L_d}\right) \mathrm{d}\rho
\end{aligned}
\tag{5.23}
$$

For $m + n \leq 2$, we obtain

$$
\begin{aligned}
A_{0,0} &= 1 \\
A_{1,0} &= x_j - x_i \\
A_{0,1} &= y_j - y_i \\
A_{1,1} &= (x_j - x_i)(y_j - y_i) \\
A_{2,0} &= \frac{\sigma^2}{2} + (x_j - x_i)^2 \\
A_{0,2} &= \frac{\sigma^2}{2} + (y_j - y_i)^2
\end{aligned}
\tag{5.24}
$$

$$
\begin{aligned}
B_{0,0} &= 1 \\
B_{1,0} &= 0 \\
B_{0,1} &= 0 \\
B_{1,1} &= 0 \\
B_{2,0} &= \sigma^2/2 \\
B_{0,2} &= \sigma^2/2
\end{aligned} \tag{5.25}
$$

$$
\begin{aligned}
C_{0,0} &= 2\pi L_d^2 F(S) \\
C_{1,0} &= 0 \\
C_{0,1} &= 0 \\
C_{1,1} &= 0 \\
C_{2,0} &= 4\pi L_d^4 F(S) \\
C_{0,2} &= 4\pi L_d^4 F(S)
\end{aligned} \tag{5.26}
$$

The following equations are obtained

$$\sum_{j \in \mathcal{N}_i} \mathrm{f}_{ij} A_{m,n} = \mathrm{e}^{-\alpha \Delta t} \left(B_{m,n} + \frac{\alpha \Delta t}{2\pi L_d^2} C_{m,n} \right) \tag{5.27}$$

Zeroth moment

$$\sum_{j \in \mathcal{N}_i} \mathrm{f}_{ij} = \mathrm{e}^{-\alpha \Delta t} \left(1 + \alpha \Delta t F(S)\right) \tag{5.28}$$

First χ moment

$$\sum_{j \in \mathcal{N}_i} (x_j - x_i)\, \mathrm{f}_{ij} = 0 \tag{5.29}$$

First η moment

$$\sum_{j \in \mathcal{N}_i} (y_j - y_i)\, \mathrm{f}_{ij} = 0 \tag{5.30}$$

Cross χη moment

$$\sum_{j\in\mathcal{N}_i}(x_j - x_i)(y_j - y_i)\,\mathrm{f}_{ij} = 0 \tag{5.31}$$

Second χ moment

$$\begin{aligned}\sum_{j\in\mathcal{N}_i}\left(\frac{\sigma^2}{2} + (x_j - x_i)^2\right)\,\mathrm{f}_{ij} &= \mathrm{e}^{-\alpha\Delta t}\left(\frac{\sigma^2}{2} + 2\alpha\Delta t L_d^2 F(S)\right)\\ \sum_{j\in\mathcal{N}_i}(x_j - x_i)^2\,\mathrm{f}_{ij} &= \alpha\Delta t\,\mathrm{e}^{-\alpha\Delta t}\left(2L_d^2 - \frac{\sigma^2}{2}\right)F(S)\end{aligned} \tag{5.32}$$

Second η moment

$$\begin{aligned}\sum_{j\in\mathcal{N}_i}\left(\frac{\sigma^2}{2} + (y_j - y_i)^2\right)\,\mathrm{f}_{ij} &= \mathrm{e}^{-\alpha\Delta t}\left(\frac{\sigma^2}{2} + 2\alpha\Delta t L_d^2 F(S)\right)\\ \sum_{j\in\mathcal{N}_i}(y_j - y_i)^2\,\mathrm{f}_{ij} &= \alpha\Delta t\,\mathrm{e}^{-\alpha\Delta t}\left(2L_d^2 - \frac{\sigma^2}{2}\right)F(S)\end{aligned} \tag{5.33}$$

Chapter 6

Results & Discussion

The algorithm described in Chapter 3 was implemented using Fortran 95. The method was tested on a β-plane for the baroclinic case (finite L_d) using the inviscid form of the quasi-geostrophic equations. Details of the implementation are as follows. First of all, a uniform grid of user-defined resolution is used to compute the streamfunction at grid vertices. Nonetheless, the method is still grid-free since the flow can be directly computed at particle positions. Next, the domain is discretized using a finite number of vortex elements or material particles. Grid cells contain a user-defined number of particles. The total number of particles is then equal to the grid size times the number of particles per cell. Each particle i carries a strength Γ_i equivalent to the circulation of potential vorticity over an area A_i, i.e.,

$$\Gamma_i = \int_{A_i} Q \, \mathrm{d}\mathbf{\Omega} \ .$$

A_i is the area of a grid cell divided by the number of particles per cell. Γ_i can then be approximated as

$$\Gamma_i = Q(\mathbf{r}_i) \frac{\Delta x \Delta y}{n_p} \ ,$$

where Q is evaluated at the particle position $\mathbf{r}_i$, $\Delta x \Delta y$ is cell dimension and n_p is the number of particles per cell. Particles are then advected (transported) for a single time step using an initial flow field. We used first-order Euler integration scheme.

Since Q is materially conserved in our case, particles conserve their strengths and the streamfunction, as well as the flow field, can be computed directly from the new particle positions. The process is then continued for the remaining time steps.

Boundary Conditions

The β-plane of dimensions $L \times L$ ($L = 8000$ km) has periodic boundary conditions. The fact that the domain is periodic has its implications on the effect of particles through boundaries. For instance, a particle at the top of the domain will have its effect on the bottom through the top boundary. And since potential vorticity depends on the planetary vorticity (through the Coriolis parameter), which depends by its turn on the y-position on the plane, potential vorticity carried by particles must be assigned accordingly. In other words, a particle at the top will affect regions at the bottom as if it was placed below the bottom boundary and therefore its potential vorticity must be defined accordingly.

Test Case

The inviscid form of the quasi-geostrophic equations on the β-plane for the baroclinic case is

$$\begin{aligned} Q &= \nabla^2\psi - \frac{\psi}{L_d^2} + f \\ \frac{DQ}{Dt} &= 0 \end{aligned} \tag{6.1}$$

A solution of (6.1) is Rossby wave given by

$$\psi = a\sin(kx - \omega t)\sin(ly) \tag{6.2}$$

We initialize the system using $k = l = 4\pi/L$ and $a = 10$ m^2/s. The initial condition is shown in Fig. 6-1. We next set the number of particles per cell to $n_p = 9$ and change the grid resolution to test the minimum number of particles that can represent the initial streamfunction. This is done by solving for the streamfunction before advecting

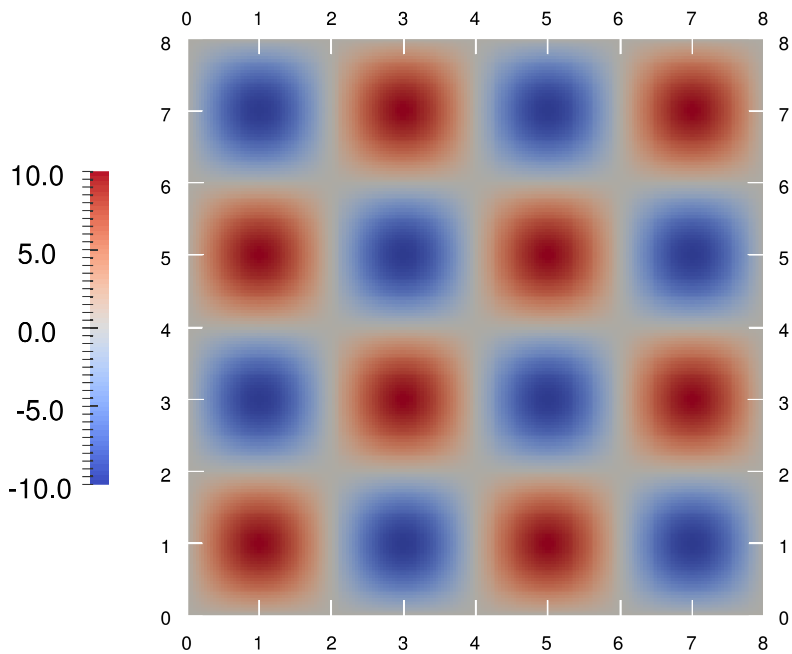
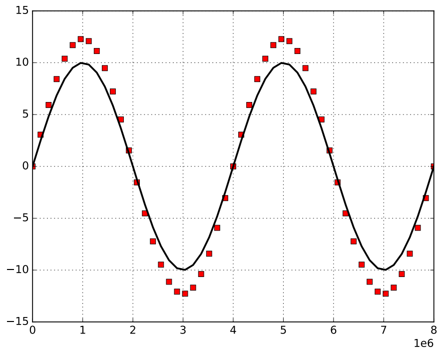
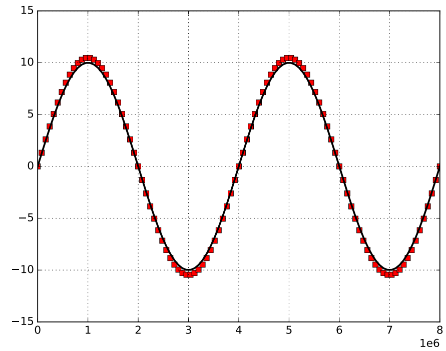


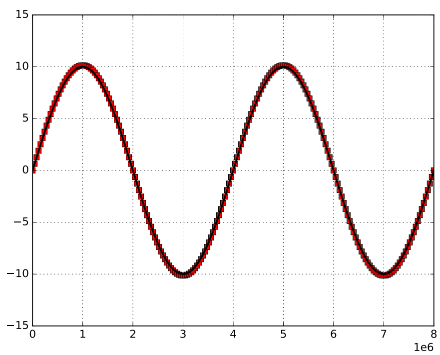
Figure 6-1: Initial Rossby waves at $t = 0$. Colors show streamfunction in m^2/s. Horizontal and vertical axes represent distance in 10^6 m.



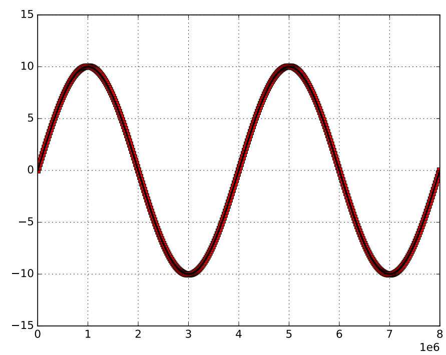
(a) 50×50



(b) 100×100



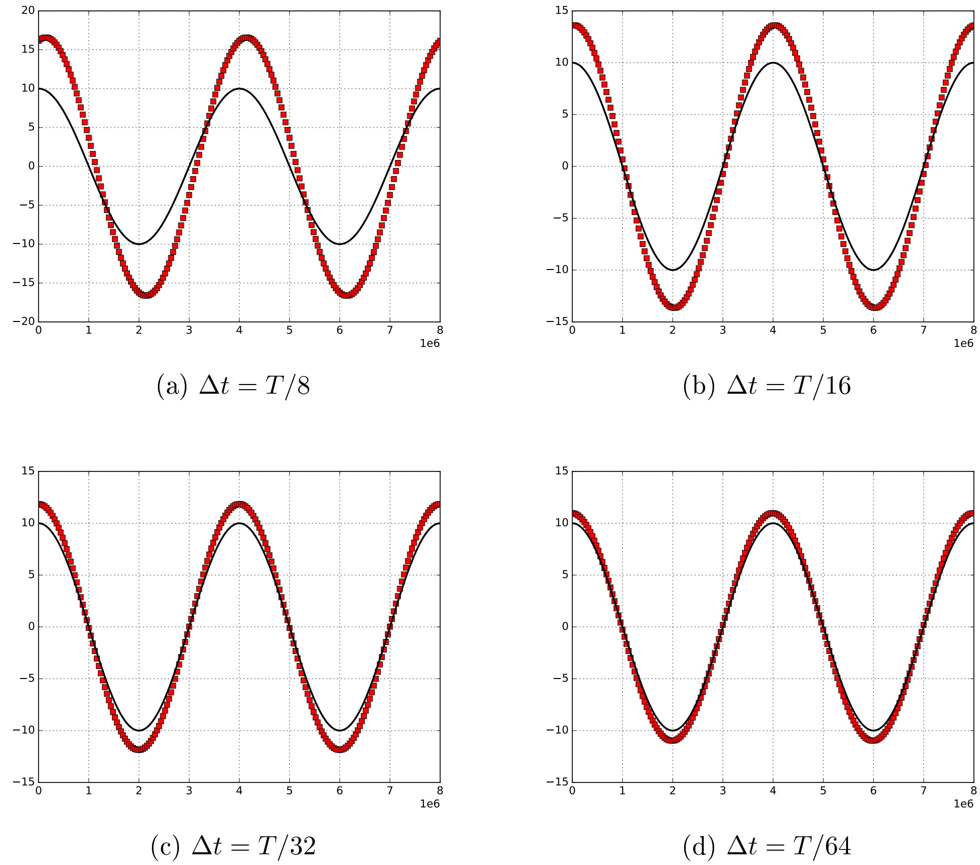
(c) 200×200



(d) 400×400

Figure 6-2: Comparison of analytical and numerical solutions for the Rossby wave at $t = 0$. Vertical axis represent streamfunction in m^2/s at $y = L/8$ and horizontal axis represent distance in m. Squares show numerical solution and continuous curves show analytical solution for a (a) 50×50, (b) 100×100, (c) 200×200 and (d) 400×400 grid sizes.

the particles. Comparisons of analytical and numerical solutions for the Rossby wave at $t = 0$ for 50×50, 100×100, 200×200 and 400×400 grids are shown in Fig. 6-2. The number of particles corresponding to the 200×200 grid appears to give a good representation of the initial wave. This number (3.6×10^5 particles) is used in the next experiment to study the effect of the time step on the accuracy of the method. Comparisons of analytical and numerical solutions for the Rossby wave at $t = T/4$ (where T is the wave period) for different time steps are shown in Fig. 6-3. It is clear that for smaller time steps, the method gives more accurate results. Higher order integration schemes such as RK2 and RK4 are expected to also give more accurate results. It should be noted that the amplitude of the streamfunction used in this



(a) $\Delta t = T/8$ (b) $\Delta t = T/16$

(c) $\Delta t = T/32$ (d) $\Delta t = T/64$

Figure 6-3: Comparison of analytical and numerical solutions for the Rossby wave at $t = T/4$. Vertical axis represent streamfunction in m^2/s at $y = L/8$ and horizontal axis represent distance in m. Squares show numerical solution and continuous curves show analytical solution for a (a) $T/8$, (b) $T/16$, (c) $T/32$ and (d) $T/64$ time steps.

experiment is too small and does not reflect real conditions. Of course, the higher the amplitude of the wave (streamfunction), the lower the time step should be.

The results shown in this chapter are preliminary, however, they seem promising. We did not include information about computational time since the code written to solve the problem can be further optimized reducing thus the time of the computations.

Chapter 7

Conclusion

The grid-free vortex method appears to be an attractive method for solving vorticity-streamfunction equations. It is a particle-based (as opposed to grid-based) method in which stability is afforded by the Lagrangian transport of particles, which eliminates difficulties in the representation of advection, such as nonlinear instabilities associated with Eulerian methods. In this thesis, we built an infrastructure to solve the quasi-geostrophic equations using the grid-free vortex method. We presented an algorithm with all the tools needed to solve the inviscid form of the quasi-geostrophic equations. The missing Green's function of the screened Poisson equation encountered when solving the baroclinic form of the equations over a sphere was derived. The method was tested for the baroclinic case on a β-plane and promising results were obtained. The code that was written can be easily adjusted to solve the barotropic (one-layer model) form. Also, having considered a periodic domain for the β-plane over which the method was tested, it is now easier to test the method over a sphere which exhibits periodicity in the zonal direction, especially that Green's function is now available for both the barotropic and the baroclinic cases.

Future Work

The work presented in this thesis is a first step in the journey of a thousand-mile. Once the method is fully tested on both the β-plane and the sphere, future research can be made to include the effect of topography which appears as a term in the expression of potential vorticity. Furthermore, the formulation of the redistribution scheme that was developed here to solve the dissipation equation on a β-plane can be implemented and further formulations can be made to include the effect of wind stress and diffusion.

Future research can be also carried out in another direction to account for the effect of density stratification by considering the two- (or multi-) layer model.

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