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OPTIMAL TIME AND COST BALANCE IN PROJECT RISK MANAGEMENT

by
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OPTIMAL TIME AND COST BALANCE IN PROJECT

RISK MANAGEMENT

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In almost all real-life projects, activity durations are not known with certainty. Therefore, adopting a schedule that ensures the successful completion of a project on time with a high probability is critical. This often requires, starting the project activities as soon as possible, while abiding by precedence constraints, as advocated by the classical methods of CPM and PERT. However, from a cost control and financing prospective, performing an activity in early stages of a project may not be the best course of action. On the contrary, the present worth criterion advocates starting activities as late as possible, to save on financing costs. The present research is aimed at determining the optimal start time of activities in a project network while accounting for both time value of costs and the risk of scheduling delays. A mathematical program is developed in order to find the optimal starting time of each activity under different structures of the project network topology, while assuming that the activity durations approximately follow independent normal distributions. We find that the optimal delay structure advocates delaying activities at the beginning of the project. In addition, we observe that it might be optimal to delay activities that fall on the path with the longest expected duration, which are deemed "critical" in the classic CPM/PERT approach. A bi-product of our work is a new method to estimate schedule risk, PERT-X, which is found to be more accurate than the classic PERT by comparison with Monte Carlo Simulation results. PERT-X approximates the duration of the maximum of all the project paths by a Normal random variable.

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CHAPTER I

INTRODUCTION AND MOTIVATION

"Time is money" as Benjamin Franklin said. In project management, this quotation is supported by the Triple Constraint Triangle shown in Figure 1, e.g. Haughey (2011). Time, Cost and Scope constitute these constraints that every project should meet to be successful. Keeping scope aside and assuming infinite resources, time and cost are negatively correlated. To finish early, as in point E of Figure 1 costs need to be sacrificed. Reciprocally, to cut on costs as in point L of Figure 1, scheduling delays are bound to occur. This makes finding the optimal start time of project activities a challenging problem that has been largely ignored in the project management literature. Finding a simpler approach to measure the amount by which each activity should be delayed in a way that balances time and cost would be a significant contribution to the literature. Developing such a balancing approach is the main objective of this work. We assume a fixed scope and ample resources, and focus on determining the optimal time-cost trade-off schedule as shown in Figure 1. With our assumption of fixed scope and unlimited resources, the main trade-off we exploit is specifically related to time value of money in terms of the present value of the expected project cost.

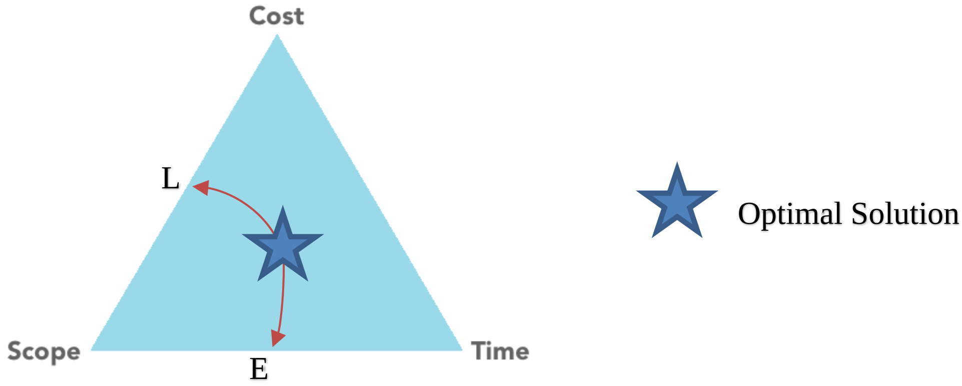

Figure 1. Triple Constraint Triangle assuming constant scope

For “simple” project networks, involving parallel non-overlapping paths, we develop a tractable optimization model (a nonlinear program) that determines the start time of each activity that minimizes the expected present value of cost subject to delay constraints. For these parallel-path networks, we establish the quasiconvexity of the cost function, and develop bounds on the optimal delay of each activity. This allows determining the optimal schedule easily in practice, using, for example, basic spreadsheets. For more complex networks, with paths having common activities and experiencing “merge-bias[1]”, we propose an extension of the classical PERT method to carry the desired analysis. Specifically, we develop PERT-X, where the duration of the maximum path in the project is approximated by a well-calibrated normal distribution, which depends, among other things, on the path covariance matrix[2]. The remainder of the analysis then proceeds in a similar manner to more simple networks. In Chapter 2 of this paper, we review the literature, in Chapter 3 we present a stylized two-activity model. In Chapter 4, we discuss a network with multiple activities in series. In Chapter

[1] See Hullet (2009) for an interesting discussion of the merge-bias phenomenon resulting from paths crossing in project network.
[2] The path durations in a project network are positively correlated due to the commonality of activities among paths. The path covariance matrix is developed base on the commonality structure. See Sculli and Shum (1991) for a good example.

5, we address a network combining series and parallel activities. In Chapter 6, we propose PERT-X a new method to compute the delay probability in a complex project network. In Chapters 7, we apply our optimization method to cover general networks. In Chapter 8, we apply our optimization on a real-life network. Finally, in Chapter 9, we present conclusions and ideas for future research.

CHAPTER II

LITERATURE REVIEW

This chapter is divided into two major sections. The first is about time-cost trade-off and the second is about PERT and its improvements.

A. PERT Variants

Project scheduling is a key for the success of any project. The Critical Path Method (CPM) and Project Evaluation and Review Technique (PERT) are the two major tools in project scheduling. While CPM assumes activities durations are known with certainty, PERT is an improvement of CPM, where it is a probabilistic approach for activity durations, (Paul and Banerjee 2008). While both tools focus on the critical path, the one having the highest mean duration, PERT assumes activity durations that follow PERT-Beta distribution based on three inputs; minimum, maximum and most likely. In PERT, the critical path is used to compute the probability of being late, assuming that the central limit theorem holds and the critical path duration follows a normal distribution with mean and variance as the sum of the means and variances of activities on the critical path, (Robillard and Trahan 1977). One major drawback of PERT is that it underestimates the real completion time of a project, since it neglects the fact that some “noncritical” paths may become “critical”. In the literature, many previous attempts are made to improve the outcome of PERT by utilizing the multivariate normal distribution. Monhor (2011) discusses using the normal distribution assumption for all the paths. He takes into consideration the correlation by introducing a bivariate normal distribution method to compute bounds for the probability. Others express a method to compute the multivariate normal distribution to a specific extent and with much computational efforts. For example, Anklesaria and Drezner (1986) consider a limited number of paths in a specific network. PERT-M method consists of finding the probability of being late using the multivariate normal distribution. To compute the probability using the multivariate normal distribution without the help of some programs although possible, it is difficult and requires so much time to complete. In addition, Sculli and Shum (1991) express a method to approximate the mean and the variance for the whole network taking into consideration all the paths one at a time and continuously updating the covariance matrix. Using a different approach, Hmadh (2016) suggests PERT-IA a new method to approximate the probability of being late by using the weighted average between PERT and IA[3]. Some of the authors used simulation techniques to compute the probabilities e.g. Sculli (1989) and Haga and Marold (2004). We propose a new method, PERT-X, which is based on successive approximations of the durations of parallel paths/activities two at a time with a normal distribution. PERT-X is based on an approximation of the maximum of two normal random variables by Nadarajah and Kotz (2008) in Electrical Engineering. PERT-X consistently produce results.

B. Time-Cost Trade-Off

Many works aim to minimize the project duration while maintaining resource constraint and not taking into consideration the time value of money, according to Elmaghraby and Herroelen (1990). Others aim to find an optimal way to maximize the net present value of a project without taking into consideration duration uncertainty e.g.

[3] IA: (Independence Analysis) is a method that takes into consideration all the paths in a network and assumes that these paths are independent.

Russell (1970). Also, Vanhoucke, Demeulemeester and Herroelen (2001) use the resource constraint to maximize the net present value. Similarly, Vanhoucke, Demeulemeester and Herroelen (1999) discuss an unconstrained project scheduling problem to maximize the net present value of the project without accounting for project uncertainty. On the other hand, some took into consideration this uncertainty but without giving much importance for the probability of being late. Buss and Rosenblatt (1997) discuss how to find the optimal starting time of each activity assuming exponential distribution, showing the importance of delaying an activity and how to compute the duration in which each activity is delayed and what is its NPV accounting for costs and revenues at the end of the project. They assume continuous compounding and assume that activity durations follow the exponential distribution. In addition, Buss and Rosenblatt (1997) claim of being the first paper studying the expected net present value with stochastic activity durations. Furthermore, Buss and Rosenblatt (1997) consider no constraint in his method to compute the delays. According to our experience, this paper is the first discussing the issue of minimizing cost while maintaining a specific constraint which limits the risk that the project is delayed. Our approach which focuses on costs appears to be more practical than the one proposed by Buss and Rosenblatt (1997), since revenues on a project are hard to assess. In addition, our method present more general results such as convexity of the Cost function and structural property of the optimal activity delays.

CHAPTER III

PROJECT WITH TWO PARALLEL ACTIVITIES

Starting with a simple network, in this chapter we discuss a project with only two parallel activities as represented in Figure 2. In Section 3.1, we formulate our model. Then, in Section 3.2, we discuss our solution methodology. Finally, in Section 3.3, we present some numerical results, and draw useful insights.

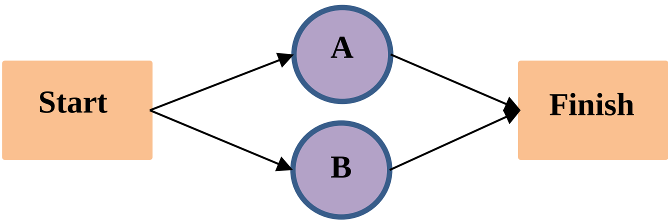

Figure 2. Network with Two Parallel Activities

A. Formulation

The network discussed in this chapter is shown in Figure 2. The duration, T_i of both activities A and B are assumed to be normally distributed with mean and variance $E(T_i) = \mu_i$ and $Var(T_i) = \sigma_i^2$ for $i = A, B$. It is also assumed that T_A and T_B are independent. The cost of each activity is $C_i, i = A, B$ and it is paid at the beginning of the activity. The objective is to minimize the present value of the costs. For this project the expected present value of cost is $C = \sum_{i=A,B} C_i e^{rd_i} = C_A e^{-rd_A} + C_B e^{-rd_B}$, where r is the discounting rate and d_i is the delay time of Activity i which is also the start time of

Activity i for this simple network. The project needs to be completed by a deadline t_0. The optimization problem of this network is formulated as follows:

$$\min_{d_A, d_B} C = \sum_{i=A,B} C_i e^{rd_i} = C_A e^{-rd_A} + C_B e^{-rd_B} \tag{1}$$

Subject to

$$\Phi\left(\frac{t_0 - (\mu_A + d_A)}{\sigma_A}\right) \times \Phi\left(\frac{t_0 - (\mu_B + d_B)}{\sigma_B}\right) \geq \alpha \tag{2}$$

$$d_i \geq 0, i = A, B$$

where α is the probability that the project is completed without any delay and Φ is the cumulative density function of the standard normal distribution $\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$. Constraint (2) ensures that the project is finished on time $\alpha\%$ of the time. The left hand side (l.h.s) of this constraint is the probability of finishing on time which is $P\{T_A \le t_0\} \times P\{T_B \le t_0\}$.

B. Solution Method

For the project in Figure 2, the completion time of path P_i, $i = A, B$, is assumed to be normally distributed with mean $\mu_i + d_i$ and variance σ_i^2. Lemma 1 ensures that (2) is binding this is a result from the l.h.s of (2) being decreasing in d_i, $i = A, B$.

Lemma 1. *Constraint (2) is binding at optimality.*

Proof. See appendix A.

Lemma 1 simplifies the analysis significantly. In particular, it allows determining high quality upper bounds as shown in Lemma 2. In addition, Lemma 1 allows replacing the two-variable non-linear constrained optimization problem with a quasiconvex single-variable problem, which can be carried-out easily as shown in the sequel.

Lemma 2. *for the two parallel paths network,*

$$t_0 - \mu_i - 3\sigma_i \le d_i \le t_0 - \mu_i - \sigma_i \times \Phi^{-1}\left(\frac{\alpha}{\Phi\left(\frac{t_0 - \mu_j}{\sigma_j}\right)}\right), \quad i = A, B,\ i \ne j \tag{3}$$

A more simplified upper bound is $t_0 - \mu_i - \sigma_i \times \Phi^{-1}(\alpha)$

Proof. See Appendix A.

Back to the optimization problem, Zogheib and Hlynka provide an approximation to the standard normal distribution CDF as follows

$$\Phi(z) = 1 - 0.5e^{-1.2z^{1.3}} \tag{4}$$

In Appendix B we argue that approximation (4) is accurate with an average relative error of 1.8%. Utilizing Lemma 1 and Approximation (4), the delay d_A could be found in terms of d_B as shown in (5), leading to an unconstrained optimization problem with the objective function described in (6),

$$d_A = t_0 - \mu_A - \sigma_A \times \left[\frac{-1}{1.2} \times \ln \left(2 - \frac{\alpha}{\left(0.5 - 0.25 e^{-1.2\left(\frac{t_0 - (\mu_B + d_B)}{\sigma_B}\right)^{1.3}} \right)} \right) \right]^{\frac{1}{1.3}}, \tag{5}$$

$$\min_{d_B} C = C_A e^{-r\left(t_0 - \mu_A - \sigma_A \times \left[\frac{-1}{1.2} \times \ln \left(2 - \frac{\alpha}{\left(0.5 - 0.25 e^{-1.2\left(\frac{t_0 - (\mu_B + d_B)}{\sigma_B}\right)^{1.3}} \right)} \right) \right]^{\frac{1}{1.3}} \right)} + C_B e^{-r d_B} \tag{6}$$

The following theorem provides a generalized convexity result on cost in (6).

Theorem 1. The cost $C(d_B)$ *in (6) is quasiconvex.*

Proof. See Appendix A.

Theorem 1 and (5) imply that the optimal delays of activities can be found easily, e.g. with basic spreadsheets.

C. Example

As an example, the values in Table 1 give the necessary parameters for Activities A and B. In addition, it was assumed that $t_0 = 30$, discount rate $r = 1.5\%$ and $\alpha = 95\%$.

Table 1. Values for the Activities of the Network Described

Activity	Mean	Variance	Cost
A	2	1	15

B	20	4	10

The upper and lower bounds for the delay along with the optimal value of the delay as described in Section 2.2, are given in Table 2. The optimal cost according to our model is 19.27 and the probability of being on time is 0.95. On the other hand, the cost according to CPM and PERT is 21.45. That is our model reduces cost by around 10%.

Table 2. Values for the Activities Delays

Activity	Lower Bound	Upper Bound	Optimal	CPM/PERT Optimal
A	25	26.36	25.34	18
B	4	6.71	6.64	0

Another example regarding the same network would be assigning cost proportional to the mean duration of each activity as presented in Table 3.

Table 3. Values for the Activities of the Network Described

Activity	Mean	Variance	Cost
A	2	1	3
B	20	4	30

The results are summarized in Table 4. The optimal cost according to our method is 29.2 and the probability of being on time is 0.95. The cost according to CPM and PERT is 32.29. Therefore, our model reduces cost also by around 10%.

Table 4. Values for the Activities Delays

Activity	Lower Bound	Upper Bound	Optimal	CPM/PERT Optimal
A	25	26.36	24.97	18
B	4	6.71	6.71	0

CPM and PERT do not allow delaying the “critical” activity, having the highest mean duration, whereas our model shows that this activity should be delayed. Therefore, our model provides better results regarding the present value of the costs. In addition, the two examples presented here demonstrate that our model is effective in cutting costs regardless of the cost structure, proportional to the mean activity duration or not.

CHAPTER IV
PROJECTS WITH ONE PATH FOR MULTIPLE ACTIVITIES

In this chapter, we deal with another project network with one path having multiple activities in series as shown in Figure 3. In Section 4.1 we formulate our model. Then, in Section 4.2 we discuss our solution methodology. Finally, in Section 4.3 we present numerical results. Similar assumptions to Chapter 3 on the normality of activity duration, continuous compounding, and cost structure are maintained in this chapter.

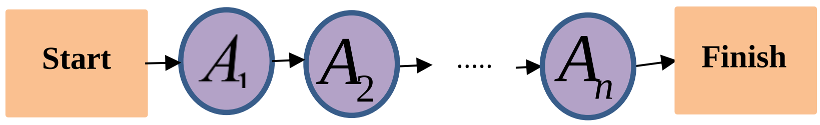

Figure 3. Single Path Network with Multiple Activities

A. Formulation

In this network, all the activities will be treated as a single activity of mean and variance, that is the summation of the means and variances of all activities A_i to compute the probability of being late. The moment generating function of a normal distribution states that

$$E\left(e^{tx}\right)=e^{t\mu_x+\frac{1}{2}\sigma_x^2t^2} \tag{7}$$

Based on (7) and assuming $t=-r$ and based on the fact that the activities are assumed to be normally distributed the optimization problem can be formulated in the following manner

$$\min_{d_{A_1},\ldots,d_{A_n}} \mathrm{C} = \sum_{i=1}^{n} C_{A_i} e^{-r\left(\sum_{j=1}^{i-1} d_{A_J} + \mu_{A_J}\right) + 0.5\left(\sum_{j=1}^{i-1} \sigma_{A_J}^2\right) r^2} \tag{8}$$

Subject to

$$\Phi\left(\frac{t_0 - \left(\sum_{i=1}^{n} d_{A_i} + \mu_{A_i}\right)}{\sqrt{\sum_{i=1}^{n} \sigma_{A_i}^2}}\right) \geq \alpha \tag{9}$$

B. Solution Method

Theorem 2. *For multiple activities in series, only the first activity may be delayed. That is,* $d_i = 0$, $i = 2, \ldots, n$.

Proof. See Appendix A

With the result in Theorem 2 and a similar result to Lemma 1 indicating that (9) is binding, the following theorem is obtained

Theorem 3. *For the network in figure 3, the delay is computed by*

$$d_{A_1} = t_0 - \sum_{i=1}^{n} \mu_{A_i} - \sqrt{\sum_{i=1}^{n} \sigma_{A_i}^2} \times \Phi^{-1}(\alpha) \tag{10}$$

Proof. See Appendix A

C. Example

As an example, we chose a network compromising four activities in series. The necessary parameters are presented in Table 5. In addition, it was assumed that $t_0 = 50$, discount rate $r = 1.5\%$ and $\alpha = 95\%$.

Table 5. Values for the Activities of the Network Described

Activity	Mean	Variance	Cost
A	4	1	6
B	12	4	16
C	8	3	11
D	17	6	23

The optimal value of the delay as described in Section 4.2, are given in Table 6. The optimal cost according to our model is 43.88 and the probability of being on time is 0.95. On the other hand, the cost according to CPM and PERT is 45.79. That is our model reduces cost by around 5%.

Table 6. Values for the Activities Delays

Activity	Optimal	CPM/PERT Optimal
A	2.85	0
B	0	0
C	0	0
D	0	0

CHAPTER V

PROJECTS WITH MULTIPLE PARALLEL PATHS AND MULTIPLE ACTIVITIES PER PATH

In this chapter, a model which is a combination of those in Chapter 3 and 4 is treated, where the network has multiple parallel paths and on each path, there are multiple activities in series as shown in Figure 4. In Section 5.1 we formulate our model. Then in Section 5.2 we discuss our solution methodology. Finally, in Section 5.3 we present some numerical results.

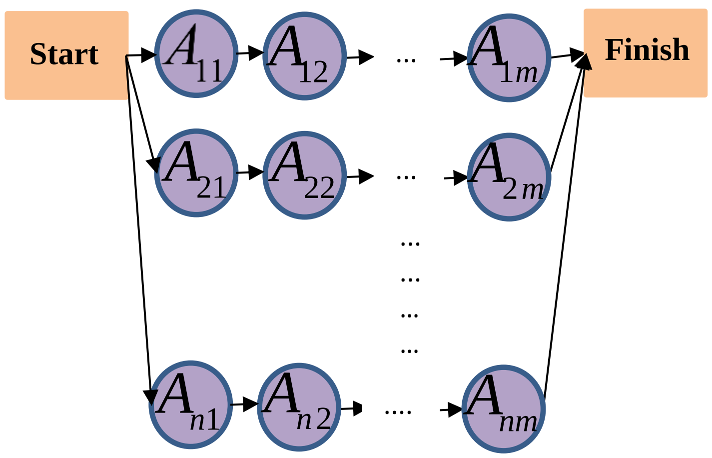

Figure 4.Network with Multiple Parallel Paths

A. Formulation

On each path the activities are in series. Therefore, according to Theorem 2, only the first activity should be delayed. Therefore, the optimization problem can be formulated in the following manner

$$\min_{\substack{d_{A_{ij}}\\ \forall i,\forall j}} C = \sum_{i=1}^{n}\sum_{j=1}^{m} C_{A_{ij}} e^{-r\left(d_{A_{i1}}+\sum_{k=1}^{j-1}\mu_{A_{ik}}\right)+0.5\left(\sum_{k=1}^{j-1}\sigma_{A_{ik}}^{2}\right)r^{2}} \quad (11)$$

Subject to

$$\prod_{k=1}^{n}\Phi\left(\frac{t_0-\left(d_{A_{k1}}+\sum_{i=1}^{m}\mu_{A_{ki}}\right)}{\sqrt{\sum_{i=1}^{m}\sigma_{A_{ki}}^{2}}}\right)\geq\alpha \quad (12)$$

B. Solution Method

Let $\mu_i=\sum_{j=1}^{m}\mu_{A_{ij}}$ and $\sigma_i^2=\sum_{j=1}^{m}\sigma_{A_{ij}}^2$, Lemma 2 can be used to compute lower and upper bounds for the delays as shown in (13).

$$t_0-\mu_i-3\sigma_i\leq d_{A_{i1}}\leq t_0-\mu_i-\sigma_i\times\Phi^{-1}\left(\frac{\alpha}{\prod_{j\neq i}\Phi\left(\frac{t_0-(\mu_j)}{\sigma_j}\right)}\right) \quad (13)$$

Similar to Lemma 1 it can be shown that (12) is binding. Using the normal distribution approximation in (4) it can be stated that:

$$d_{A_{i1}}=t_0-\mu_i-\sigma_i\times\left[\frac{-1}{1.2}\times\ln\left(2-\frac{\alpha}{\prod_{\substack{j=1\\ j\neq i}}^{n}\left(0.5-0.25e^{-1.2\left(\frac{t_0-(d_{A_{j1}}+\mu_j)}{\sigma_j}\right)^{1.3}}\right)}\right)\right]^{\frac{1}{1.3}}, \; i=1,\ldots,\text{n} \quad (14)$$

replacing d_{A_1} in the unconstrained objective function and solving numerically to find the optimal solution for the delays. Theorem 1 proves that $C\left(d_{A_{11}},d_{A_{21}},\ldots,d_{A_{n1}}\right)$ is quasiconvex.

C. Example

As an example, the values in Table 7 were assigned for all activities in the network. Also it is assumed that $t_0 = 30$ and a discount rate $r = 1.5\%$.

Table 7. Values for the Activities of the Described Network

Path	Activity	Mean	Variance	Cost
1	A11	4	1	6
	A12	9	4	13
2	A21	1	1	2
	A22	5	2	7
	A23	8	3	12
3	A31	3	2	5
	A32	6	4	9
	A33	1	1	2
	A34	9	4	13
4	A41	18	5	26

The upper and lower bounds for the delay along with the optimal value of the delay as described in Section 5.2, are found in Table 8.

Table 8. Values for the Activities Delays

Path	Mean	Variance	Lower	Upper	Optimal	CPM/PERT Optimal
1	24.29	5.00	10.29	13.32	11.57	6

2	23.98	6.00	8.65	11.97	10.26	5
3	23.35	11.00	1.05	5.53	4.30	0
4	25.21	5.00	5.29	8.32	6.96	1

The optimal expected present value of the cost is found to be 80.69 and the probability of being on time is 0.95. Applying CPM and PERT method while using equation (7) the optimal present value of the cost is 87.27. Our model reduced cost by around 10%. Therefore, our method shows the importance of delaying activities on the critical path opposing to PERT assumption.

CHAPTER VI

PERT-X

This chapter highlights a new method to compute the probability of being late in a project network. This method is based on a result for electrical engineers suggested by Nadarajah and Kotz (2008). In Section 6.1 we describe the basic steps of PERT-X. Then, in Section 6.2 we present some numerical results.

A. PERT-X Steps

In their paper Nadarajah and Kotz (2008) express a method to analyze a digital circuit. This method consists of finding the expected value of maximum and minimum of correlated Gaussian random variables. We tailor this method to estimate an approximation of the distribution of the longest path in project network. This works as follow. First, assuming normality of all activities compute the mean and variance of each one. Second, choose a specific loop in this network. Define all the parallel paths in this loop. Third, using convolution, replace each path in the loop by a single activity by summing up the means and variances of the activities on this specific path. Fourth, applying approximations (15), (16) and (17) this loop will vanish and the two parallel activities in it will turn into one activity having the following mean and variance.

$$\text{Mean: } E(X) = \mu_1 \times \Phi\left(\frac{\mu_1 - \mu_2}{\theta}\right) + \mu_2 \times \Phi\left(\frac{\mu_2 - \mu_1}{\theta}\right) + \theta \times \phi\left(\frac{\mu_1 - \mu_2}{\theta}\right) \tag{15}$$

$$E\left(X^2\right) = \left(\sigma_1^2 + \mu_1^2\right) \times \Phi\left(\frac{\mu_1 - \mu_2}{\theta}\right) + \left(\sigma_2^2 + \mu_2^2\right) \times \Phi\left(\frac{\mu_2 - \mu_1}{\theta}\right) + \left(\mu_1 + \mu_2\right)\theta \times \phi\left(\frac{\mu_1 - \mu_2}{\theta}\right) \tag{16}$$

$$\text{Variance: } E\left(X^2\right) - \left[E\left(X\right)\right]^2 \tag{17}$$

Where μ_i and σ_i^2 are the mean and variance of path i in the loop and $\theta = \sqrt{\sigma_1^2 + \sigma_2^2}$. Repeat steps two through four until ending up with a single path. Finally, using convolution, the last-mentioned path can be turned into a single activity as described in step three.

In case of a path that belong to two loops at the same time, apply the beforementioned steps simultaneously for both loops.

The reason of the choice of this method is the ease of its computations and that it provides a mean and variance which facilitate solving the optimization problem for complex networks.

PERT-X will be useful in the computation of the present value of the costs since it can compute the mean and variance of any loop which can directly be applied to the moment generating function and the path will be treated as suggested by (8).

B. Example

As an example, the network presented in Figure 5 will be analyzed.

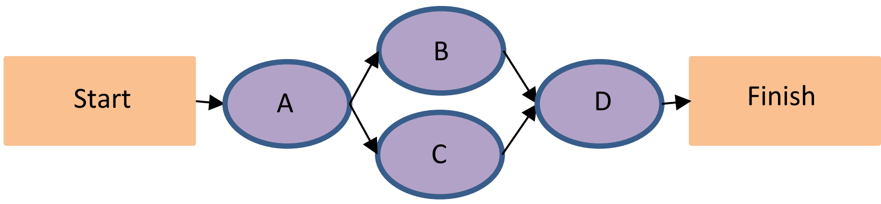

Figure 5. Network with Two Crossing Paths

The required parameters are presented in Table 9.

Table 9. Values for the Network Described in Figure 5

Path	Activity	Mean	Variance

1,2	A	4	1
1	B	10	4
2	C	9	3
1,2	D	5	2

To verify the results of this method, the probability of being late for different project durations, the assigned time was compared between PERT-X results, the results from the simulations and the results using one of the methods that Hmadh (2017) proposed PERT-IA. The results are summarized in Table 10. Simulation results in the second column of the table were obtained using Arena (e.g.). For brief details on the PERT-M and PERT-IA methods reported in Table 10, refer to section 2.1. PERT-X produce very accurate results with respect to the simulation, which outperform PERT and PERT-IA.

Table 10. Comparison Between the Results of PERT-X and Other Methods

t	Simulation	PERT-M	PERT-X	PERT	Pert-IA
15	0.97	0.98	0.97	0.93	0.94
17	0.87	0.87	0.87	0.78	0.80
19	0.61	0.60	0.60	0.50	0.52
21	0.27	0.28	0.28	0.22	0.24
23	0.08	0.08	0.08	0.07	0.07

As another example, network from Hillier and Lieberman (2005) will be discussed. This network is presented in Figure 6.

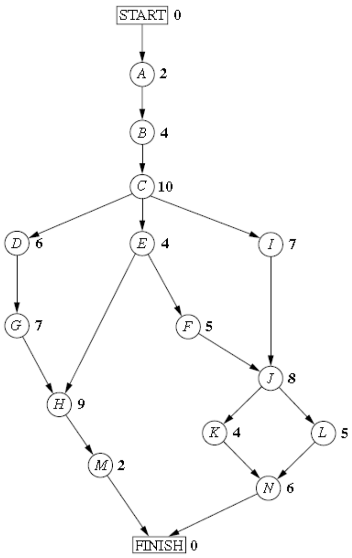

Figure 6. Networks proposed by Hillier and Lieberman (2005)

Table 11 summarizes the parameters of the proposed network.

Table 11. The Paths of the Network in Figure 6

Path	Name	Mean	Variance
1	ABCDGHM	40	11.22
2	ABCEHM	31	9.67
3	ABCEFJKN	43	8.00
4	ABCEFJLN	44	9.00
5	ABCIJKN	41	7.56
6	ABCIJLN	42	8.56

Loop JKLN was chosen to start with, activity K and L take the form of two parallel activities in this loop. Therefore, applying PERT-X these two activities will be converted into a single activity KL, resulting in a network of 4 paths only. Next step would be loop CEFIJ and loop CEDGH, since both loops contains activity E therefore we need to treat the simultaneously. Applying PERT-X each of these loops is replaced by a single activity. Activity IEF and EDG are the resulting ones. Finally, only one loop is left with two path EDGHM and IEFJKLN. Applying PERT-X those paths are replace by a single activity with mean 44.26 and a variance 8.32.

To verify the results of this method, the probability of being late for different project durations, the assigned time was compared similar to the previous example. The results are summarized in Table 12. Again Table 12 shows that PERT-X produce very accurate results with respect to the simulation, which outperform PERT and PERT-IA.

Table 12. Comparison Between the Results of PERT-X and Other Methods

T	Simulation	PERT-M	PERT-X	PERT	IA	Pert-IA
40	0.94	0.93	0.93	0.91	1.00	0.92
42	0.77	0.78	0.78	0.75	0.98	0.78
44	0.53	0.54	0.54	0.50	0.82	0.55
46	0.26	0.27	0.27	0.25	0.46	0.28
48	0.10	0.10	0.10	0.09	0.16	0.10

CHAPTER VII

GENERAL NETWORK TIME AND COST MODE

The main concern of this Chapter is to combine the previous results in order to analyze a network with crossing paths. Till now only networks with independent paths were discussed, this chapter deals with a network of two intersecting paths as shown in Figure 5 of Chapter 6. In Section 7.1 we formulate our model. Then in Section 7.2 we discuss our solution methodology. Finally, in Section 7.3 we present some numerical results.

A. Formulation

To formulate the optimization problem, the moment generating function was used to compute the expected net present value of the costs in a similar way as Chapters 4 and 5. Furthermore, to compute the probability of not being late it is necessary to find the mean and the variance as stated in Section 6.1. The optimization problem can be stated as follows

$$\min \mathrm{C} = C_A e^{-rd_A} + C_B e^{-r(\mu_A+d_A+d_B)+\frac{\sigma_A^2 r^2}{2}} + C_C e^{-r(\mu_A+d_A+d_C)+\frac{\sigma_A^2 r^2}{2}} + C_D e^{-r(\mu_A+d_A+E(T_B,T_C)+d_D)+\frac{\left(\sigma_A^2+\sigma_{T_B,T_C}^2\right)r^2}{2}} \quad (18)$$

Subject to

$$\Phi\left(\frac{t_0-E}{\sigma}\right) \geq \alpha\,, \quad (19)$$

Where, from Nadarajah and Kotz (2008), $E = \mu_A + d_A + \mu_D + d_D + E(T_B, T_C)$,

$\theta = \sqrt{\sigma_B^2 + \sigma_C^2}$,

$$E(T_B, T_C) = (\mu_C + d_C)\Phi\left(\frac{\mu_C + d_C - \mu_B - d_B}{\theta}\right) + (\mu_B + d_B)\Phi\left(\frac{\mu_B + d_B - \mu_C - d_C}{\theta}\right) + \theta\phi\left(\frac{\mu_C + d_C - \mu_B - d_B}{\theta}\right)$$

$$E\left[(T_B, T_C)^2\right] = \left(\sigma_B^2 + (\mu_B + d_B)^2\right)\Phi\left(\frac{\mu_B + d_B - \mu_C - d_C}{\theta}\right) + \left(\sigma_C^2 + (\mu_C + d_C)^2\right)\Phi\left(\frac{\mu_C + d_C - \mu_B - d_B}{\theta}\right)$$
$$+ \left((\mu_B + d_B) + (\mu_C + d_C)\right)\theta\phi\left(\frac{\mu_B + d_B - \mu_C - d_C}{\theta}\right)$$

$\sigma_{T_B, T_C}^2 = E\left[(T_B, T_C)^2\right] - \left[E(T_B, T_C)\right]^2$ and $\sigma^2 = \sigma_{T_B, T_C}^2 + \sigma_A^2 + \sigma_D^2$.

B. Solution Method

To solve this network, it is necessary to find a way to simplify the long computations. Therefore, the following theorem was proven.

Theorem 4. *In any network, where the activities taking the parallel form and making a loop, only the activity with the lowest mean is delayed. Also, in the activities taking the series form without making any loop, only the first activity should be delayed. For this network, Activity D should not be delayed and either B or C should be delayed (the one with the lowest mean).*

Proof. See Appendix A.

Therefore, since $d_B = 0$ and $d_D = 0$, based on the constraint function it can be found that

$$d_A = t_0 - \sigma_{T_1, T_2}\Phi^{-1}(\alpha) - (\mu_1)\Phi\left(\frac{\mu_B - \mu_C - d_C}{\theta}\right) - (\mu_2 + d_C)\Phi\left(\frac{\mu_C + d_C - \mu_B}{\theta}\right) - \theta\phi\left(\frac{\mu_B - \mu_C - d_C}{\theta}\right) \quad (20)$$

Thus, if the value of d_A is replaced in the objective function, an unconstrained single variable optimization problem is obtained.

C. Example

As an example, the values in Table 11 were assigned for the activities of the network in Figure 5. Also $t_0 = 30$ and $r = 1.5\%$.

Table 13. Values for the Network Described in Section 7

Path	Activity	Mean	Variance	Cost
1,2	A	4	1	6
1	B	10	4	13
2	C	6	3	8
1,2	D	5	2	7

The optimal delay of Activities A and C were found to be 6.38 and 3.31 and the optimal expected present value of the costs is 28.19. Using CPM and PERT method the present value of the costs is 31.45. This indicates that our method performs better than the traditional methods since it decreased costs by around 10%.

CHAPTER VIII

REAL-LIFE PROJECTS

A. Formulation

In this chapter, a real-life complex network with multiple correlated paths suggested by Stevenson, Nsakanda and Ozgur (2009) is discussed and presented in Figure 7. As per Chapters 4 and 5 the optimization problem can be formulated in the following manner:

$$\min \mathrm{E}(C) = \sum_{\forall i} C_i e^{-rE(w_i)+\frac{\sigma_{w_i}^2 r^2}{2}} \tag{21}$$

Subject to

$$\Phi\left(\frac{t_0 - E}{\sigma}\right) \geq \alpha \tag{22}$$

Where w_i and $\sigma_{w_i}^2$ are the mean and variance of the total time till the starting date of Activity i, C_i is the cost of Activity i. E and σ are the mean and standard deviation for the total duration of the project.

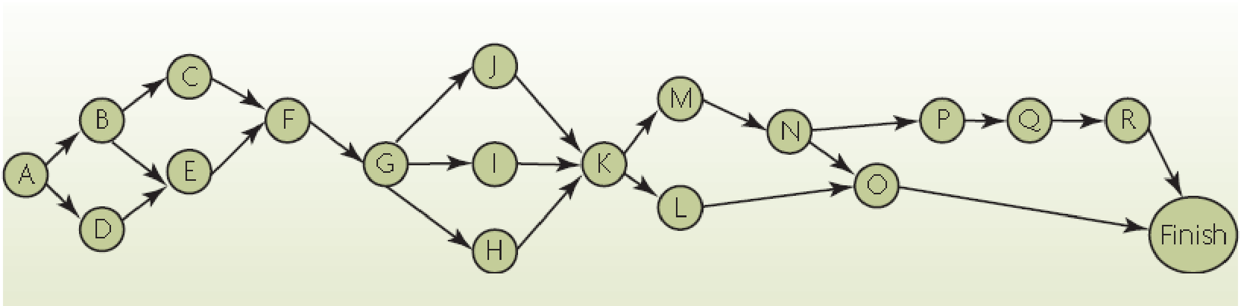

Figure 7. Real Life Network

B. Solution Method

Using PERT-X, $E(T_i)$, $\sigma_{T_i}^2$, E and σ were computed for all activities as well as the entire project.

Let d_i be the delay of Activity i, then activities P, Q and R have together higher mean than Activity O, activities M and N have together higher mean than Activity L, Activity I has higher mean than Activities H and J, Activity E has higher mean than Activity C and finally Activity B has higher mean than Activity D, therefore, according to Theorem 4, activities B, E, I, M, N, P, Q and R should not be delayed. In addition, activities F, G and K are in series with Activity A and other loops therefore should not be delayed. Thus, only Activities A, C, D, H, J, L and O can be delayed. Similar to Chapter 7, d_A could be computed in terms of all other d_i and could be replaced in the objective function to obtain an unconstrained optimization problem.

C. Example

Taking $t_0 = 50$, $r = 1.5\%$ and $C_i = 12 \times \mu_i$, and the values in Table 14.

Table 14. Values for the Network Described in Figure 7

ACTIVITY	a	m	b	Mean, $t_e = \frac{a+4m+b}{6}$	Variance, $\sigma^2 = \left(\frac{b-a}{6}\right)^2$
A	1	2	3	2	0.111
B	1	2	3	2	0.111
C	0.5	1	1.5	1	0.028
D	0.5	1	1.5	1	0.028
E	3	4	5	4	0.111
F	1	2	3	2	0.111
G	3	5	7	5	0.444
H	1	2	3	2	0.111
I	10	12	14	12	0.444
J	1	2	3	2	0.111
K	0.5	1	1.5	1	0.028
L	0.5	1	1.5	1	0.028
M	4	6	8	6	0.444
N	0.5	1	1.5	1	0.028
O	1	2	3	2	0.111
P	1.5	3	4.5	3	0.250
Q	1	2	3	2	0.111
R	2	4	6	4	0.444

The optimal delays were found and summarized in Table 15.

Table 15. Summary of the Delays of the Network Described in Chapter 8

Activity	PERT-X d_i	PERT d_i	Activity	PERT-X d_i	PERT d_i
A	3.13	0.00	J	9.39	10.00
B	0.00	0.00	K	0.00	0.00
C	2.86	3.00	L	5.25	6.00
D	1.02	1.00	M	0.00	0.00
E	0.00	0.00	N	0.00	0.00

F	0.00	0.00	O	6.21	7.00
G	0.00	0.00	P	0.00	0.00
H	9.47	10.00	Q	0.00	0.00
I	0.00	0.00	R	0.00	0.00

The present value for some of the activities are presented in Table 16.

Table 16. Present Value of the Cost for Some Activities

Activity	Present Value of Cost (PVC)	Numerical Value for PVC
E	$C_E e^{-r\left(\mu_A+d_A+E(T_B,T_D)+d_E\right)+\frac{\left(\sigma_A^2+\sigma_{T_B,T_D}^2\right)r^2}{2}}$	43.03
F	$C_F e^{-r\left(\mu_A+d_A+E(T_B,T_D)+E(T_C,T_E)+d_F\right)+\frac{\left(\sigma_A^2+\sigma_{T_B,T_D}^2+\sigma_{T_C,T_E}^2\right)r^2}{2}}$	20.23
J	$C_J e^{-r\left(\mu_A+d_A+E(T_B,T_D)+E(T_C,T_E)+\mu_F+d_F+\mu_G+d_G+d_J\right)+\frac{\left(\sigma_A^2+\sigma_{T_B,T_D}^2+\sigma_{T_C,T_E}^2+\sigma_F^2+\sigma_G^2\right)r^2}{2}}$	15.83
N	$C_N e^{\left(-r\left(\mu_A+d_A+E(T_B,T_D)+E(T_C,T_E)+\mu_F+d_F+\mu_G+d_G+E\left(E(T_J,T_I),E(T_I,T_H)\right)+\mu_K+d_K+\mu_M+d_M+d_N\right)+\frac{\left(\sigma_A^2+\sigma_{T_B,T_D}^2+\sigma_{T_C,T_E}^2+\sigma_F^2+\sigma_G^2+\sigma_{E(T_J,T_I),E(T_I,T_H)}^2+\sigma_K^2+\sigma_M^2\right)r^2}{2}\right)}$	6.84

According to PERT-X the optimal present value of the project costs is found to be 456.22. Whereas according to CPM and PERT and using (7) to compute the present value which is found to be 476.8. In addition, using the fact proposed by CPM that the activity durations are certain the present value is 479.39. Therefore, the PERT-X method showed its efficiency in a real-life project since it decreases cost by around 5%.

CHAPTER IX

CONCLUSION

This paper showed first the importance of delaying activities while maintaining a certain probability of being on time. Furthermore, a method used in electric circuits was applied in project management to facilitate the computation of the probabilities. In addition, this paper presented a new method to compute the optimal delay for each activity in different types of networks. Moreover, it is to note that the different methods presented in this paper could be combined to solve more complex networks. This paper's results are sought to make a useful contribution in understanding the trade-off between time and cost in project risk management. Additionally, the analysis of PERT-X, constitutes a side-contribution to project schedule risk analysis, which offers computational improvements on the commonly-used Monte Carlo simulation.

Proving the convexity of the objective function of a general network would make a good extension of this paper. In addition, finding a heuristic method to easily approximate the delays of the activities in any network without a mathematical optimization constitutes another useful future extension. The bounds on the delay we present in this research offer a good starting point. In addition, taking a penalty for the delay along with a probabilistic model will constitute a great extension for this work.

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APPENDIX A

Proof of Lemma 1.

$P=\sum_{i=A,B} C_i e^{rd_i}$ and $\frac{\partial P}{\partial d_i}=-rC_i e^{-rd_i}<0$. Therefore, P is strictly decreasing with respect to d_i. Assume d_i^* are the optimal solutions s.t: $\prod_{\forall i}\Phi\left(\frac{t_0-(\mu_i+d_i^*)}{\sigma_i}\right)>\alpha$.

But since d_i^* can take any continuous value. Therefore, we can find d'^*_i such as $d'^*_i > d_i^*$, $\prod_{\forall i}\Phi\left(\frac{t_0-(\mu_i+d_i^*)}{\sigma_i}\right)>\prod_{\forall i}\Phi\left(\frac{t_0-(\mu_i+d'^*_i)}{\sigma_i}\right)=\alpha$ and $\sum_{\forall i} C_i e^{rd'^*_i}<\sum_{\forall i} C_i e^{rd_i^*}$ which contradicts the fact that d_i^* are the optimal solutions. Therefore, the constraint can be switched to: $\prod_{\forall i}\Phi\left(\frac{t_0-(\mu_i+d_i^*)}{\sigma_i}\right)=\alpha$. This proof can be applied for a network with more than 2 parallel path as well as network with crossing paths.

Proof of Lemma 2.

First assume that $d_B=0$ therefore using Equations (5) and (6),

$C(d_B)=C_A e^{-r\left(t_0-\mu_A-\sigma_A\times\Phi^{-1}\left(\frac{\alpha}{\Phi\left(\frac{t_0-\mu_B}{\sigma_B}\right)}\right)\right)}+C_B$. Now assume that $d_B=t_0-\mu_B-3\sigma_B$ and $t_0-\mu_B-3\sigma_B\geq 0$ we get $C(d_B)=C_A e^{-r\left(t_0-\mu_A-\sigma_A\times\Phi^{-1}\left(\frac{\alpha}{\Phi\left(\frac{t_0-\mu_B-(t_0-\mu_B-3\sigma_B)}{\sigma_B}\right)}\right)\right)}+C_B e^{-r(t_0-\mu_B-3\sigma_B)}$ and equivalently $C(d_B)=C_A e^{-r\left(t_0-\mu_A-\sigma_A\times\Phi^{-1}\left(\frac{\alpha}{\Phi(3)}\right)\right)}+C_B e^{-r(t_0-\mu_B-3\sigma_B)}$. Since

$\Phi\left(\frac{t_0-\mu_B}{\sigma_B}\right)\approx\Phi(3)\approx 1$, then d_A is not affected. But the present value of an activity costs decrease if this activity was delayed, therefore, $\mu_i+d_i+3\sigma_i\geq t_0$ and equivalently $d_i\geq t_0-\mu_i-3\sigma_i$. In addition, based on (2) it can be stated that d_A is decreasing in d_B. Thus, it can be stated that the maximum value for d_A is when d_B is zero.

Proof of Theorem 1.

Let $f\left(d_B\right)=0.5-0.25e^{-1.2\left(\frac{t_0-(\mu_B+d_B)}{\sigma_B}\right)^{1.3}}$.

Therefore, $f^{'}(d_B)=-(0.25)(1.2)(1.3)\left(\frac{1}{\sigma_B}\right)\left(\frac{t_0-\left(\mu_B+d_B\right)}{\sigma_B}\right)^{0.3}e^{-1.2\left(\frac{t_0-(\mu_B+d_B)}{\sigma_B}\right)^{1.3}}$

and

$$f^{''}(d_B)=(0.3)(0.25)(1.2)(1.3)\left(\frac{1}{\sigma_B}\right)^2\left(\frac{t_0-\left(\mu_B+d_B\right)}{\sigma_B}\right)^{-0.7}e^{-1.2\left(\frac{t_0-(\mu_B+d_B)}{\sigma_B}\right)^{1.3}}-(0.25)(1.2)^2(1.3)^2\left(\frac{1}{\sigma_B}\right)^2\left(\frac{t_0-\left(\mu_B+d_B\right)}{\sigma_B}\right)^{0.6}e^{-1.2\left(\frac{t_0-(\mu_B+d_B)}{\sigma_B}\right)^{1.3}}$$

Assume $\alpha\geq 0.611$ and $f^{''}(d_B)\leq 0$.

Therefore, equivalently $1-5.2\left(\frac{t_0-\left(\mu_B+d_B\right)}{\sigma_B}\right)^{1.3}\leq 0$ and $d_B\leq t_0-\mu_B-0.281\sigma_B$.

According to Lemma 2 it can be stated that

$$d_B\leq t_0-\mu_B-\sigma_B\times\Phi^{-1}\left(\frac{\alpha}{\prod_{i\neq B}\Phi\left(\frac{t_0-(\mu_i)}{\sigma_i}\right)}\right).$$

Therefore, for $f''(d_B) \le 0$ to hold the following inequation should hold

$$t_0 - \mu_B - \sigma_B \times \Phi^{-1}\left(\frac{\alpha}{\Phi\left(\frac{t_0 - (\mu_A + d_A)}{\sigma_A}\right)}\right) \le t_0 - \mu_B - 0.281\sigma_B .$$

Equivalently $\Phi\left(\frac{t_0 - (\mu_A + d_A)}{\sigma_A}\right) \le \frac{\alpha}{0.611}$ should hold, this statement will hold if and only if $\alpha \ge 0.611$, since a probability is always less than 1. Therefore, it can be concluded that $f\left(d_B\right) = 0.5 - 0.25e^{-1.2\left(\frac{t_0 - (\mu_B + d_B)}{\sigma_B}\right)^{1.3}}$ is concave.

Since, $0.5 - 0.25e^{-1.2\left(\frac{t_0 - (\mu_i + d_i)}{\sigma_i}\right)^{1.3}}$ is concave and ln(x) is a non-decreasing concave function therefore per Bazaraa, Sherali and Shetty (2006), $\ln\left(0.5 - 0.25e^{-1.2\left(\frac{t_0 - (\mu_i + d_i)}{\sigma_i}\right)^{1.3}}\right)$ is concave. In addition, the sum of concave functions is concave. Thus,

$\ln\left(\prod_{\forall i, i \ne A}\left(0.5 - 0.25e^{-1.2\left(\frac{t_0 - (\mu_i + d_i)}{\sigma_i}\right)^{1.3}}\right)\right)$ is concave and equivalently it is quasiconcave.

Since, e^x is a non-decreasing function therefore $\prod_{\forall i, i \ne A}\left(0.5 - 0.25e^{-1.2\left(\frac{t_0 - (\mu_i + d_i)}{\sigma_i}\right)^{1.3}}\right)$ is quasiconcave and since it is the product of probabilities then it is positive. Therefore, it can be stated that $\frac{1}{\prod_{\substack{\forall i \\ i \ne A}}\left(0.5 - 0.25e^{-1.2\left(\frac{t_0 - (\mu_i + d_i)}{\sigma_i}\right)^{1.3}}\right)}$ is quasiconvex. In addition, the negative of a quasiconvex function is quasiconcave therefore

$$\left(2-\frac{\alpha}{\prod_{\forall i,i\neq A}\left(0.5-0.25e^{-1.2\left(\frac{t_0-(\mu_i+d_i)}{\sigma_i}\right)^{1.3}}\right)}\right)$$ is quasiconcave. It follows that

$$\ln\left(2-\frac{\alpha}{\prod_{\forall i,i\neq A}\left(0.5-0.25e^{-1.2\left(\frac{t_0-(\mu_i+d_i)}{\sigma_i}\right)^{1.3}}\right)}\right)$$ is quasiconcave. Therefore, according to

Bazaraa, Sherali and Shetty (2006), the following can also be considered quasiconvex,

$$\frac{-1}{1.2}\ln\left(2-\frac{\alpha}{\prod_{\forall i,i\neq A}\left(0.5-0.25e^{-1.2\left(\frac{t_0-(\mu_i+d_i)}{\sigma_i}\right)^{1.3}}\right)}\right)$$. In addition, $f(x)=x^{\frac{1}{1.3}}$ is a non-decreasing

function, combined to a quasiconvex function provide a quasiconvex function,

according to Bazaraa, Sherali and Shetty (2006),

$$r\sigma_A\left(\frac{-1}{1.2}\times\ln\left(2-\frac{\alpha}{\prod_{\forall i,i\neq A}\left(0.5-0.25e^{-1.2\left(\frac{t_0-(\mu_i+d_i)}{\sigma_i}\right)^{1.3}}\right)}\right)\right)^{\frac{1}{1.3}}$$ is quasiconvex. Furthermore, e^x is

a non-decreasing function therefore, $C_A e^{-r\left(t_0-\mu_A-\sigma_A\times\left(\frac{-1}{1.2}\times\ln\left(2-\frac{\alpha}{\prod_{\forall i,i\neq A}\left(0.5-0.25e^{-1.2\left(\frac{t_0-(\mu_i+d_i)}{\sigma_i}\right)^{1.3}}\right)}\right)\right)^{\frac{1}{1.3}}\right)}$ is quasiconvex.

Since the sum of 2 quasiconvex functions is quasiconvex, therefore,

$$P = C_A e^{-r\left(t_0-\mu_A-\sigma_A\times\left(\frac{-1}{1.2}\times\ln\left(2-\frac{\alpha}{\prod_{\forall i,i\neq A}\left(0.5-0.25e^{-1.2\left(\frac{t_0-(\mu_i+d_i)}{\sigma_i}\right)^{1.3}}\right)}\right)\right)^{\frac{1}{1.3}}\right)} + \sum_{\forall i,i\neq A} C_i e^{-rd_i}$$ is quasiconvex..

Proof of Theorem 2.

Let $W=\sum_{i=1}^{n} T_i + d_i$. Therefore, $P\left\{completing\, the\, project\, on\, time\right\} = P\left\{W \le t_0\right\}$.

Assume $d_1 = \sum_{i=1}^{n} d_i$. Thus $W=\sum_{i=1}^{n} T_i + d_i = d_1 + \sum_{i=1}^{n} T_i$. Therefore, the expected value and the variance of the path are independent of the timing of the delay, but the present value of the cost will decrease if the delay happened early on.

Proof of Theorem 3.

Using Lemma 1 we can say that constraint (9) is binding. Therefore,

$\Phi\left(\frac{t_0-\left(\sum_{i=1}^{n} d_{A_i} + \mu_{A_i}\right)}{\sqrt{\sum_{i=1}^{n}\sigma_{A_i}^2}}\right)=\alpha$. Using the fact that only the first activity should be delayed we can state that $\Phi\left(\frac{t_0-\left(d_{A_1}+\sum_{i=1}^{n}\mu_{A_i}\right)}{\sqrt{\sum_{i=1}^{n}\sigma_{A_i}^2}}\right)=\alpha$ and equivalently

$$d_{A_1}=t_0-\sum_{i=1}^{n}\mu_{A_i}-\sqrt{\sum_{i=1}^{n}\sigma_{A_i}^2}\times\Phi^{-1}(\alpha).$$

Proof of Theorem 4.

First Assume $d_A=a$, $d_B=b$, $d_C=c$, $d_D=d$ and $\mu_B>\mu_C$, then assume $d_A=a+b+d$, $d_B=0$, $d_C=c-b$, $d_D=0$. $E(T_1,T_2)$, $E\left[(T_1,T_2)^2\right]$, $E(T_B,T_C)$ and $E\left[(T_B,T_C)^2\right]$ are not affected, but the expected net present value of the cost decreased.

APPENDIX B

Approximation for the Normal Distribution CDF

To test Approximation (4), we compare it with the real normal distribution CDF.

Using the sum of absolute error, $\int_0^\infty \frac{\left|\Phi(z)-\hat{\Phi}(z)\right|}{\Phi(z)}$ is computed numerically and found to be 0.018. Therefore, we can state that this approximation acts well with high accuracy. Zogheib and Hlynka discussed the accuracy by providing two graphs. The first one is for the difference between the normal distribution CDF and its approximation and the other one present the two plots overlaid as shown in Figure 8.

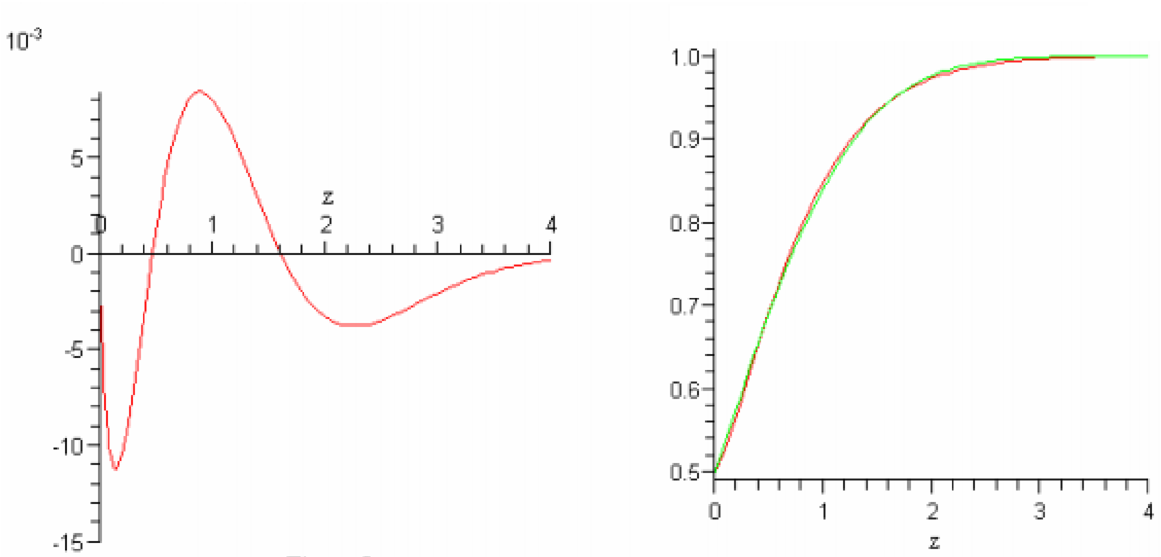

Figure 8. Difference Between the Normal Distribution CDF and Its Approximation.