AMERICAN UNIVERSITY OF BEIRUT

CENTRAL LIMIT THEOREM ON THE GENERAL LINEAR GROUP

by

HEBA MANSOUR BADAWE

A thesis
submitted in partial fulfillment of the requirements
for the degree of Master of Science
to the Department of Mathematics
of the Faculty of Arts and Sciences
at the American University of Beirut

Beirut, Lebanon
May 2018

AMERICAN UNIVERSITY OF BEIRUT

CENTRAL LIMIT THEOREM ON THE GENERAL LINEAR GROUP

by

Heba Badawe

Approved by:

Prof. Aoun Richard, Assistant Professor Mathematics	Advisor
Prof. Shayya Bassam, Professor Mathematics	Member of Committee
Prof. Bertrand Florian, Assistant Professor Mathematics	Member of Committee

Date of thesis defense: May 1, 2018

AMERICAN UNIVERSITY OF BEIRUT

THESIS, DISSERTATION, PROJECT RELEASE FORM

Student Name: Badawe Heba Mansour

Last First Middle

● Master's Thesis ○ Master's Project ○ Doctoral Dissertation

☒ I authorize the American University of Beirut to: (a) reproduce hard or electronic copies of my thesis, dissertation, or project; (b) include such copies in the archives and digital repositories of the University; and (c) make freely available such copies to third parties for research or educational purposes.

☐ I authorize the American University of Beirut, to: (a) reproduce hard or electronic copies of it; (b) include such copies in the archives and digital repositories of the University; and (c) make freely available such copies to third parties for research or educational purposes after:

One ___ year from the date of submission of my thesis, dissertation or project.
Two ___ years from the date of submission of my thesis , dissertation or project.
Three ___ years from the date of submission of my thesis , dissertation or project.

Heba May 3, 2018

Signature Date

ACKNOWLEDGEMENTS

Sitting down and writing an introductory to my thesis is not an easy task. A lot of thoughts cross my mind while a lot of emotions boil up my chest. This journey has been like a roller coaster from the very first day I got into AUB until this moment. I experienced some ups and downs, cried a river of happy tears and another of sad ones. If you ask me would I change anything, I would definitely say no. The bittersweet taste of it all is what made it so special.

I got introduced to a better learning environment that helped widen the horizon of my thinking. I met a lot of people with great knowledge that didn't fail to influence every aspect of me. And here the first thing to mention is how much I'm grateful for my advisor Professor Richard Aoun who helped me throughout this year to reach this point. His knowledge and mathematical insight encouraged me to start viewing everything in math in a different way and from a different perspective when tackling the problem from various fields. I won't forget his oh so loved sentence "Heba, view it geometrically!"

On the other hand, I would like to thank the professors I took courses with from whom I learned many things. I learned that hard work pays off from Professor Kamal Khuri-Makdisi, mathematical elegance from Professor Faruk Abi-Khuzam, critical thinking from Professor Tamer Tlas, practical thinking from Professor Nabil Nassif and last but not least the beauty of math from Professor Bassam Shayya. I would also like to thank Professor Florian Bertrand for being a member of my thesis defense committee, and the Chairperson Professor Wissam Raji.

In the end I wouldn't forget to thank God, my parents and my friends for believing in my potentials and always pushing me to become a better version of myself, especially Adel, Carl and Elie, without you things would have been much more difficult..

AN ABSTRACT OF THE THESIS OF

Heba Mansour Badawe for Master of Science
Major: Mathematics

Title: Central Limit Theorem on the General Linear Group

Our goal in this thesis is to understand the Central Limit Theorem (CLT) for linear groups proved partially by Le Page in 1982 then fully by Benoist and Quint in 2016. Consider a probability measure μ on the general linear group $\mathrm{GL}(d,\mathbb{R})$, with $d \geq 1$, and $(Y_i)_i$ a sequence of independent and identically distributed random variables on $\mathrm{GL}(d,\mathbb{R})$ of law μ. We are interested in proving , under a natural moment condition on μ and geometric assumptions on the semi-group generated by its support, that the sequence of random variables $\log ||Y_n...Y_1||$, suitably normalized, converges to a Gaussian law. More precisely, assume that the semi-group generated by the support of μ is strongly irreducible and contains a proximal element. Le Page proved in this context the CLT under an exponential moment of μ and Benoist-Quint were able to weaken this assumption to the most natural one (in comparison to the case d=1): that of a moment of order 2. Understanding this question requires the introduction of the

Lyapunov exponents and the notion of stationary measures on the projective space $\mathrm{P}(\mathbb{R}^d)$.

Contents

Chapter 1

Introduction

In this thesis we focus on a specific type of linear groups that is the General Linear Group $\mathrm{GL}(d,\mathbb{R})$, where we consider a Borel probability measure μ on $\mathrm{GL}(d,\mathbb{R})$ and a sequence $(Y_n)_{n\in\mathbb{N}^*}$ of independent and identically distributed random variables of law μ. We define the left random walk as $(S_n = Y_n \cdots Y_1)_{n\in\mathbb{N}^*}$. Our goal is to understand the limit theorems of $\log ||S_n||$ under some geometric assumptions on Γ_μ, the semi-group generated by the support of μ. We are interested namely in the Law of Large Numbers (LLN) and the Central Limit Theorem (CLT). The LLN justifies the existence of the Lyapunov exponent γ in $\mathbb{R}$ where

$$\gamma := \lim_{n\to\infty} \frac{1}{n} \log ||S_n||$$

with μ having a moment of order one. The convergence above has to be understood in the almost sure sense. While the CLT states that, when μ has moment of order two, the sequence of random variables

$$\frac{\log ||S_n|| - n\gamma}{\sqrt{n}}$$

converges in law to a normal law $\mathcal{N}(0,\sigma^2)$, with $\sigma^2 > 0$. More precisely,

Theorem 1.0.1. [B6] *Let μ be a probability measure on the general linear group $GL(d,\mathbb{R})$ such that the semi-group Γ_μ generated by μ is strongly irreducible and contracting (see Definitions 2.1.2 and 2.1.4). Assume that μ has a moment of order 2, i.e $\int_{GL_d(\mathbb{R})} N(g)^2\, d\mu(g) < +\infty$, where $N(g) := \max\{\log ||g||, \log ||g^{-1}||\}$ for $g \in GL(d,\mathbb{R})$ and $||\cdot||$ is any norm on the vector space $M(d,\mathbb{R})$ of $d\times d$ matrices.*
Consider a sequence $(Y_n)_{n\geq 1}$ of random variables on $GL(d,\mathbb{R})$ identically distributed and of same law μ. Then, there exists $\sigma^2 > 0$ such that

$$\frac{\log ||Y_n \cdots Y_1|| - n\gamma}{\sqrt{n}} \underset{n\to+\infty}{\overset{law}{\longrightarrow}} \mathcal{N}(0,\sigma^2).$$

Taking into account the natural action of $\mathrm{GL}(d,\mathbb{R})$ on the projective space $\mathrm{P}(\mathbb{R}^d)$ of $\mathbb{R}^d$, we will show that this general setting differs from the classical case ($d = 1$) in probability theory in the following points :

(i) The non-commutativity of the product operation in $\mathrm{GL}(d,\mathbb{R})$ where $\log ||S_n||$ is no longer a sum of independent and identically distributed random variables. To overcome this obstacle we write $\log ||S_n||$ as the sum of cocycles, more specifically $\log ||S_n|| = \sum_{i=1}^n \sigma(X_i)$, where $(X_i)_{i\in\mathbb{N}^*}$ defines some Markov chain on $\mathrm{GL}(d,\mathbb{R}) \times \mathrm{P}(\mathbb{R}^d)$. However we lose the independency!

(ii) The notion of μ-invariant (or stationary) probability measures on $\mathrm{P}(\mathbb{R}^d)$ plays a crucial role especially in the definition of γ and σ^2.

Now we give a brief overview about the history of the Central Limit Theorem of $\log ||S_n||$. The non-commutative CLT was first introduced by Bellman [44, B54] in 1953 , whose aim was to construct and initiate a general theory for the study of the limiting behavior of systems subjected to non commutative effects. Noting that such frameworks appeared to be the right mathematical model for some physical systems subjected to a number of random effects that are not additive and non-commutative. Furstenberg and Kesten [F0] proved in 1960 the CLT for semi-groups of matrices with positive entries; they strengthened Bellman's results studying the asymptotic behavior of $||S_n||$. Using spectral analysis, Le Page [L2] in 1982 extended the proof for the semi-group generated by the support of μ and allowed the law μ to have a finite exponential moment. In 1980, Guivarc'h and Raugi [G5] gave a detailed proof of Furstenberg's theorem about the almost sure convergence of $S_n^* x$ while assuming that μ is of order 1 and Γ_μ is strongly irreducible and contracting. Recently, in 2016, Benoist and Quint [B6] were able to enhance the assumptions and prove the CLT when μ has a moment of order 2. Such an assumption is optimal. Note that they proved the Classical CLT using martingales.

In this thesis, we follow the spectral approach of Le Page in [L2] to prove Theorem 1.0.1 under an exponential moment of μ (see Definition 2.3.1), and ask whether one can still use this spectral approach to to give another proof of the CLT in the optimal condition (moment of order two) proved by [B6] using martingales.

The basic idea is to mimic the proof of the CLT in classical Probability Theory by showing that the sequence of the Fourier transform of our probability measures (the law of $\frac{\log ||S_n|| - n\gamma}{\sqrt{n}}$) converges pointwise to the Fourier transform of a non degenerate Gaussian law. Once again, the norm being only submultiplicative is an obstacle. The idea of Le Page consists of defining an analytic family $\{T(\xi), \xi \in \mathbb{C}\}$ of operators, called Fourier-Laplace operators, acting on a suitable subspace E of the Banach space $\mathcal{C}_0(P(\mathbb{R}^d))$ of continuous functions on the projective space $\mathrm{P}(\mathbb{R}^d)$ containing the constant function $\mathbb{1}$. When we look at the action of these operators on the function $\mathbb{1}$, then $\xi \mapsto T(\xi)$ will be nothing than the usual Fourier transform of $\log ||Y_1 \frac{x}{||x||}||$. When fixing ξ, $T(\xi)$ is thought of as a perturbation of the Markov operator $T(0)$ of the natural Markov Chain associated to μ on $\mathrm{GL}(d, \mathbb{R}) \times \mathrm{P}(\mathbb{R}^d)$. An important lemma from perturbation theory sheds the lights on the importance of the existence of a spectral gap, namely for some rank one operator N we have $\lim\limits_{n\to\infty} ||T^n(0) - N||^{1/n} < 1$. This would allow the decomposition of the operator $T(\xi)$ in a special way for ξ small enough.

Finally, we present the scheme of this thesis:

- In Chapter 2 we give general definitions and important tools for further use. More specifically, we introduce the framework of Random Matrix Products, Lyapunov exponents and Stationary measures.

- In Chapter 3 we assume that Γ_μ is strongly irreducible and contracting with μ having a moment of order one. We see how S_n has a contracting action on the directions in $\mathrm{P}(\mathbb{R}^d)$.

- In Chapter 4, we first recall some techniques of perturbation of operators. Then, we assume that μ has a density with respect to the Haar measure on $\mathrm{GL}(d, \mathbb{R})$ and an exponential moment. We prove the CLT (Theorem 1.0.1) in this particular case and notice that the whole space of continuous functions on $P(\mathbb{R}^d)$ is an ideal Banach space to work on.

- In Chapter 5 we suppose only that μ has an exponential moment and we resort to some facts from Ergodic theory . We show that one has to restrict to the subspace of $\mathcal{C}_0(P(\mathbb{R}^d))$ (with a suitable exponent) in order to reach the desired result.

Chapter 2

Notation and Terminology

All our variables are defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. We denote by $\mathbb{E}$ the expectation of a random variable with respect to $\mathbb{P}$. We shall refer to the set of $d \times d$ matrices with real entries by $\mathrm{M}(d, \mathbb{R})$ with $d \geq 1$, and we let $\mathrm{GL}(d, \mathbb{R})$ be the set of $d \times d$ invertible matrices of $\mathrm{M}(d, \mathbb{R})$. We represent the transpose matrix of g by g^*. We denote by $\langle \cdot, \cdot \rangle$ the canonical dot product and by $|| \cdot ||$ the Euclidean norm on $\mathbb{R}^d$, i.e. for every $x \in \mathbb{R}^d$:

$$||x|| = \{< x, x >\}^{1/2} = \big\{ \sum_{i=1}^{d} x_i^2 \big\}^{1/2}.$$

To simplify notation, the operator norm on $\mathrm{M}(d, \mathbb{R})$ induced by $|| \cdot ||$ will still be denoted by $|| \cdot ||$, i.e. for every $g \in \mathrm{M}(d, \mathbb{R})$,

$$||g|| = \sup\{||gx||; x \in \mathbb{R}^d, ||x|| = 1\}.$$

2.1 General Linear Group

Definition 2.1.1. *The general linear group of degree d, denoted by $GL(d, \mathbb{R})$, is the set of $d \times d$ invertible matrices with entries in $\mathbb{R}$. This is a group when endowed with the operation of matrix multiplication.*

We recall the polar (or KAK) decomposition of an invertible matrix.

Proposition 2.1.1. *Every matrix $g \in GL(d, \mathbb{R})$ has a factorization of the form KAU with K and U being $d \times d$ orthogonal matrices and A=diag$(a_1, ..., a_d)$ with $a_1 \geq \cdots \geq a_d > 0$ for every i. These a_i's are called the singular values of g .*

Remark 2.1.1. *The a_i's are the square roots of the eigenvalues of the positive definite symmetric matrix g^*g .*

Definition 2.1.2. *Given a subset Γ of $GL(d, \mathbb{R})$, we define the index of Γ as the least integer $p \in \{1, .., d\}$, such that there exists a sequence $\{g_n, n \geq 0\}$ in Γ for which $||g_n||^{-1} g_n$ converges to a rank p matrix. We say Γ is contracting when its index is 1.*

Remark 2.1.2. *The choice of the norm is irrelevant in the definition above as all the norms are equivalent to the finite dimensional vector space $M(d, \mathbb{R})$.*

Example 2.1.1. *The following sequence of matrices in* $GL(d,\mathbb{R})$ *defined by*

$$g_n = \begin{pmatrix} 1 & & & \\ & \frac{1}{n} & & \\ & & \ddots & \\ & & & \frac{1}{n} \end{pmatrix}$$

with $n \in \mathbb{N}$, *is a contracting sequence.*

A standard way to generate a contracting sequence is via proximal elements we define here below.

Definition 2.1.3. *An element* g *in* $GL(d,\mathbb{R})$ *is said to be proximal if and only if it has a unique eigenvalue of maximum modulus.*

Proposition 2.1.2. *Let* g *in* $GL(d,\mathbb{R})$ *be a proximal element. Then the sequence* $(g^n)_{n\in\mathbb{N}}$ *is a contracting sequence.*

Proof. Let g in $\mathrm{GL}(d,\mathbb{R})$ be a proximal element. Then, g has a unique eigenvalue of maximum modulus, say λ. Using the Jordan Decomposition of g, we can write g in a suitable basis of $\mathbb{R}^d$ as

$$\begin{pmatrix} \lambda & 0 \\ 0 & M \end{pmatrix}$$

with M having spectral radius less then λ. By the spectral radius formula, it follows that $\{||g_n||^{-1}g_n\}_{n\geq 1}$ converges to a rank one matrix. □

Definition 2.1.4. *Given a subset* Γ *of* $GL(d,\mathbb{R})$, *we say that*

(i) Γ *is irreducible if there does not exist a proper linear subspace* V *of* $\mathbb{R}^d$, *such that* $gV = V$ *for any* g *in* Γ.

(ii) Γ *is strongly irreducible if there does not exist a finite family of proper linear subspaces* $V_1, V_2, \ldots, V_k$ *of* $\mathbb{R}^d$ *such that*

$$g\left(\bigcup_{i=1}^{k} V_i\right) = \bigcup_{i=1}^{k} V_i$$

for any g *in* Γ.

Example 2.1.2. *Consider the subgroup of Gl(*$3,\mathbb{R}$*) of upper triangular matrices:*

$$S = \left\{ \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & g \end{pmatrix} ;\, a,b,c,d,e,g \in \mathbb{R}^* \right\}.$$

Then S *is not irreducible as it stabilizes the line generated by the vector* $(1,0,0)$.

We have proved in Proposition 2.1.2 then if Γ contains a proximal element, then it is contracting. The converse if true provided Γ is irreducible.

Proposition 2.1.3. *If a subset* Γ *of* $GL(d,\mathbb{R})$ *has a proximal element then* Γ *is contracting. If* Γ *is irreducible and contracting then it has a proximal element.*

2.2 The real projective space

Propositon/Definition 2.2.1. *Let V be a real vector space. The binary relation $\sim$ defined on $V \setminus \{0\}$ by $x \sim y \iff \mathbb{R}x = \mathbb{R}y$ is an equivalence relation on $V \setminus \{0\}$. We call projective space of V, and we denote by $P(V)$, the quotient space $(V \setminus \{0\})/\sim$. Moreover, for every non zero vector x of V, we denote by $[x] \in P(V)$ its equivalence class and by $\pi : (V \setminus \{0\}) \longrightarrow P(V)$ the projection map.*

Remark 2.2.1. *For every $x \in V$, $[x]$ is nothing than the one dimensional space generated by the non zero vector x, hence $P(V)$ is the set of one dimensional spaces of the vector space V.*

Proposition 2.2.1. *The projective space $P(\mathbb{R}^d)$ of $\mathbb{R}^d$ is a compact topological space when endowed with quotient topology.*

Proof. The projection map π is continuous, and a fortiori its restriction $\pi_{|S^{d-1}}$ to the unit sphere S^{d-1} of $\mathbb{R}^d$. Since $\pi_{|S^{d-1}}$ is still surjective, then $\pi(S^{d-1}) = P(\mathbb{R}^d)$ is compact. □

To be able to see $P(\mathbb{R}^d)$ as a metric space, we recall the definition of the angular metric on $P(\mathbb{R}^d)$.

Propositon/Definition 2.2.2. *For every $[x], [y] \in P(\mathbb{R}^d)$, let*

$$\delta([x],[y]) := 1 - \frac{\langle x, y\rangle^2}{||x||^2\,||y||^2},$$

i.e. the sine of the angle between the lines $\mathbb{R}x$ and $\mathbb{R}y$. It is easily seen that δ is a well-defined map from $P(\mathbb{R}^d) \times P(\mathbb{R}^d)$ to $[0, +\infty)$. One can also check that it defines a metric on $P(\mathbb{R}^d)$ and induces the quotient topology on $P(\mathbb{R}^d)$. We call δ the angular metric, or the Fubini-Study metric on $P(\mathbb{R}^d)$.

Proposition 2.2.2. *The general linear group acts naturally on $P(\mathbb{R}^d)$ by the following map*

$$\begin{aligned} Gl(d,\mathbb{R}) \times P(\mathbb{R}^d) &\longmapsto P(\mathbb{R}^d) \\ (g\,,\,[x]) \qquad &\longmapsto g\cdot[x] = \frac{gx}{||gx||} \end{aligned}$$

We recall the notion of an additive cocycle.

Definition 2.2.1. *Let G be a topological semi-group acting on a topological X. A continuous map $\sigma : G \times X \to \mathbb{R}$ is said to be an additive cocycle if*

$$\sigma(gh, x) = \sigma(g, hx) + \sigma(h, x)$$

For all $g, h \in G$ and $x \in X$.

Example 2.2.1. *The following map is well-defined and is an additive cocycle:*

$$\begin{aligned} \sigma : Gl(d,\mathbb{R}) \times P(\mathbb{R}^d) &\longmapsto \mathbb{R} \\ (g\,,\,[x]) \qquad &\longmapsto \sigma(g,[x]) = \log\frac{||gx||}{||x||}. \end{aligned}$$

Indeed, let $g, h \in GL(d,\mathbb{R})$ and $[x] \in P(\mathbb{R}^d)$. We have,

$$\begin{aligned} \sigma(gh,[x]) &= \log\frac{||ghx||}{||x||} \\ &= \log\left(\frac{||ghx||}{||x||}.\frac{||hx||}{||hx||}\right) \\ &= \log\frac{||ghx||}{||hx||} + \log\frac{||hx||}{||x||} \\ &= \sigma(g, h[x]) + \sigma(h,[x]). \end{aligned}$$

We recall finally the definition of a proper measure (or also non-degenerate) on the projective space.

Definition 2.2.2. *We say that a Borel probability measure ν on the projective space $P(V)$ of V is proper if and only if for every hyperplane H of V we have $\nu([H]) = 0$, where $[H]$ denotes the projective subspace $\pi(H)$ of $P(V)$.*

2.3 Random Matrix Product and Lyapunov Exponents

The Lyapunov Exponent plays an important role in a number of different contexts. It's a major problem to find both an explicit clear expression for this identity, often referred to as γ, and a useful method of accurate approximation.

Definition 2.3.1. *For every $g \in GL(d, \mathbb{R})$, let $N(g) = \max\{\log ||g||, \log ||g^{-1}||\}$. Consider a probability measure μ on $GL(d, \mathbb{R})$. We say that μ has*

1. *a moment of order $p \geq 1$, if $\int_{GL(d,\mathbb{R})} N(g)^p \, d\mu(g) < +\infty$.*
2. *an exponential moment, if there exists $\tau > 0$ such that $\int_{GL(d,\mathbb{R})} e^{\tau N(g)} \, d\mu(g) < +\infty$.*

Propositon/Definition 2.3.1. *Let $\{Y_n, n \geq 1\}$ be a sequence of independent identically distributed random variables on the general linear group $GL_d(\mathbb{R})$ with common distribution μ. We suppose that μ has a moment of order one, i.e.*

$$\max\{\mathbb{E}(\log ||Y_1||), \mathbb{E}(\log ||Y_1^{-1}||)\} < \infty.$$

Then the numerical sequence $\left(\frac{1}{n}\mathbb{E}\left(\log ||Y_n ... Y_1||\right)\right)_{n\in\mathbb{N}}$ converges in $\mathbb{R}$. Its limit is called the upper Lyapunov exponent associated with μ and will be denoted by γ, i.e.

$$\gamma := \lim_{n\to\infty} \frac{1}{n}\mathbb{E}\left(\log ||Y_n ... Y_1||\right)$$

with $\gamma \in \mathbb{R}$.

Remark 2.3.1. *The upper Lyapunov exponent is independent of the norm chosen since all norms are equivalent on $M(d, \mathbb{R})$, the latter being finite dimensional real vector space.*

To prove the existence of the limit in the Proposition/Definition 2.3.1, we recall the following classical lemma in real analysis.

Lemma 2.3.1. *(Fekete's lemma) Let $\{a_n\}_{n\in\mathbb{N}}$ be a subadditive sequence of non-negative terms, i.e. $a_{n+m} \leq a_n + a_m$ for every integers n and m. Then, the sequence $\left(\frac{a_n}{n}\right)_{n\in\mathbb{N}}$ converges to $\inf\left\{\frac{a_m}{m}, m \in \mathbb{N}\right\}$.*

Proof of Proposition/Definition 2.3.1: The operator norm $||\cdot||$ is a matrix norm (submultiplicative), i.e. $||AB|| \leq ||A|| \, ||B||$ for every $A, B \in M(d, \mathbb{R})$. It follows readily that the $\{\mathbb{E}\left(\log ||S_n||\right)\}_{n\in\mathbb{N}^*}$ is a subadditive sequence. Thus the upper Lyapunov exponent is well defined as a consequence of Fekete's Lemma. $\square$

We are interested in a stronger mode of convergence in the definition of the upper Lyapunov exponent, namely an almost sure convergence. We are referring to a thereom of Furstenberg and Kesten we state here below (Theorem 2.3.2). To motivate this result, we recall first the strong law of large numbers in classical probability theory which says roughly that the probability that the average of the observations converges to the expected value is equal to one.

Theorem 2.3.1. *(Strong Law of Large numbers)*
Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. and $\{X_i\}_{i\in\mathbb{N}^*}$ an infinite sequence of independent identically distributed real random variables all having the same law. Assume that X_1 has a moment of order 1, i.e. $\mathbb{E}(|X_1|) < +\infty$. Then for $\mathbb{P}$-almost every $\omega \in \Omega$:

$$\frac{(X_1 + X_2 + ... + X_n)(\omega)}{n} \underset{n\to\infty}{\longrightarrow} \mathbb{E}\left(X_1(\omega)\right).$$

Theorem 2.3.2. *(The Theorem of Furstenburg and Kesten* [F0]*)*
Let $\{Y_n, n \geq 1\}$ be a sequence of independent identically distributed random matrices in $GL(d, \mathbb{R})$ with common distribution μ. With the same assumptions as in Theorem 2.3.1, we have with probability one,

$$\lim_{n\to\infty} \frac{1}{n} \log ||Y_n...Y_1|| = \gamma.$$

Remark 2.3.2. *For $d = 1$, the previous theorem is a straight forward implication of the Strong Law of Large numbers (Theorem 2.3.1).*
Indeed, let $Y_1, Y_2, ...$ be independent identically distributed non-zero real numbers with common distribution μ such that $\mathbb{E}(\log|Y_1|) < +\infty$. Let $X_i(\omega) = \log|Y_i(\omega)|$. Since the Y_i's are independent and identically distributed then the X_i's are as well. Moreover, μ has a moment of order one. Hence, by the Strong Law of large Numbers one gets :

$$\frac{(X_1 + ... + X_n)(\omega)}{n} \underset{n\to\infty}{\longrightarrow} \mathbb{E}(X_1)$$

for $\mathbb{P}$-almost all $\omega \in \Omega$.
This means that $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \log|Y_i(\omega)| = \lim_{n\to\infty} \frac{1}{n} \log(|Y_n...Y_1(\omega)|) = \mathbb{E}(\log|Y_1(\omega)|) = \gamma$.

Definition 2.3.2. *Let μ be a probability measure on $Gl(d, \mathbb{R})$. Let $\{Y_i\}_{i\in\mathbb{N}*}$ be a family of independent identically distributed random variables in $GL(d, \mathbb{R})$ with the same law μ. We define the following random variables on $GL(d, \mathbb{R})$:*

$$S_n = X_n...X_1 \ \ ; \ \ M_n = X_1...X_n$$

We call them respectively the left and right random walks. We also denote by $(R_n)_{n\in\mathbb{N}}$ the family of random walks defined for every integer n by:

$$R_n := X_n^* \cdots X_1^*,$$

which is nothing than the left random walk but for the probability measure μ^* on $GL(d, \mathbb{R})$, pushforward measure of μ by the map $g \mapsto g^*$.

Remark 2.3.3. *For every $n \in \mathbb{N}^*$, S_n and M_n have the same law (by independence of the X_i's). Similarly, R_n and S_n^* have the same.*

2.4 Stationary Measures

Definition 2.4.1. *Let G be a topological group acting on a topological space X. Let μ be a Borel probability measure on G and ν a Borel probability measure on X.*

1. *We define $\mu * \nu$ to be the probability measure on X given by*

$$\int_X f(x)\,\mathrm{d}(\mu * \nu)(x) = \int_G \int_X f(g.x)\mathrm{d}\mu(g)d\nu(x),$$

for any continuous function f on X.

2. *We say that ν is μ-invariant (or stationary) if $\mu * \nu = \nu$.*

3. *When $X = G$ and G acts on itself by conjugation, then the $\mu \star \mu$ is denoted by μ^2 and is called the second convolution power of μ. More generally, one can define the nth convolution power of μ by itself for any integer n.*

Remark 2.4.1. *Let G acting on X, μ a probability measure on G and ν a μ-invariant probability measure on X.*

1. *If g is a random variable of G with law μ and Z is a random variable on X with law ν and independent of X, then the random variable $Z' := g \cdot Z$ on X has law ν.*

2. *In particular, the nth step of the left and right random walks $S_n = Y_n \cdots Y_1$ and $M_n = Y_1 \cdots Y_n$ have law μ^n.*

Proposition 2.4.1. *If X is compact, then for any probability measure μ on G, there exists at least one μ-invariant probability measure on X.*

Example 2.4.1. *Despite its simplicity, this example will be a guiding one for the next section. Consider a proximal element $g \in GL(2, \mathbb{R})$ (see Definition 2.1.3) as for instance the diagonal matrix $diag(2, \frac{1}{2})$. Let $v_g^+, v_g^{-1} \in P(\mathbb{R}^2)$ be the points in the projective line corresponding respectively to the eigenvectors of the highest and least eigenvalue (in modulus). Let $\mu := \delta_g$ be the Dirac delta measure on g. We claim that the only μ-invariant stationary measures on the projective line $P(\mathbb{R}^2)$ are convex combination of $\nu_1 := \delta_{v_g^+}$ and $\nu_2 := \delta_{v_g^-}$, the Dirac delta measures on v_g^+ and v_g^-. Indeed, first notice that these probability measures are indeed μ-invariant as g stabilizes each eigenspace (each being a line in this case). Hence any convex combination of these probability measures remains a stationary measure. Consider now a μ-stationary probability measure on $P(\mathbb{R}^2)$ that gives zero mass to v_g^-, i.e. $\nu(\{v_g^-\}) = 0$. It is enough to prove that $\nu = \delta_{v_g^+}$. Indeed, since g is proximal, we see by Proposition 2.1.2 that*

$$\forall [x] \in P(\mathbb{R}^2) \setminus v_g^-,\ g^n \cdot [x] \underset{n\to+\infty}{\longrightarrow} v_g^+.$$

Since ν is g-invariant, and since $\nu(\{v_g^-\}) = 0$, we deduce by the dominated convergence theorem that for any continuous function on $P(\mathbb{R}^2)$, one has:

$$\int_{P(\mathbb{R}^2)} f([x])\,d\nu([x]) = \int_{P(\mathbb{R}^2)} f(g^n[x])\,d\nu([x]) = \int_{P(\mathbb{R}^2)\setminus v_g^-} f(g^n[x])\,d\nu([x]) \underset{n\to+\infty}{\longrightarrow} f(v_g^+).$$

Hence $\nu(\{v_g^+\}) = 1$, i.e. $\nu = \delta_{\{v_g^+\}}$.

We end by a result of Furstenberg.

Proposition 2.4.2. [33, F63] *Let μ be a probability measure on $GL(d, \mathbb{R})$ such that Γ_μ is strongly irreducible. Then, any μ-invariant probability measure on $P(\mathbb{R}^d)$ is proper (see Definition 2.2.2).*

2.5 Exterior Products

Given a vector space V, we define $V \wedge V$, called the exterior product of two copies of V. This space is a subspace of $V \otimes V$ consisting of all linear combinations of tensors of the form $v_1 \otimes v_2 - v_2 \otimes v_1$, with $v_1, v_2 \in V$. The exterior product of k-copies of V is denoted by $\wedge^k V$, and it is the space spanned by expressions of the form $v_1 \wedge v_2 \wedge ... \wedge v_k$ with $v_i \in V$ for all $i = 1, ..., k$.

Remark 2.5.1. *$||v_1 \wedge ... \wedge v_k||$ is the k-dimensional volume of the parallelogram generated by $v_1, ..., v_k$.*

The expression of the angular metric δ on the projective space of $\mathbb{R}^d$ defined in Proposition/Definition 2.2.2 can be expressed using exterior products:

Definition 2.5.1. *For every $[x], [y] \in P(\mathbb{R}^d)$, one has:*

$$\delta([x], [y]) = \frac{||x \wedge y||}{||x||\, ||y||}.$$

Proposition 2.5.1. *Let $g \in GL(d, \mathbb{R})$ and E be a d-dimensional vector space. Let $u_1, ..., u_p \in E$. Then, $(\wedge^p g)(u_1 \wedge ... \wedge u_p) = gu_1 \wedge ... \wedge gu_p$.*

Definition 2.5.2. *We define $|| \wedge^p g|| = \sup \{||(\wedge^p g)\, w\,, w \in \wedge^p \mathbb{R}^d,\ ||w|| = 1\}$.*

Remark 2.5.2. *Note that $\wedge^p(gh) = (\wedge^p g)(\wedge^p h)$ then , $|| \wedge^p (gh)|| \leq || \wedge^p g|| || \wedge^p h||$.*

Proposition 2.5.2. *Let g be a matrix in $Gl(d, \mathbb{R})$. Let $a_1 \geq a_2 \geq ... \geq a_d > 0$ be the square roots of the eigenvalues of $g^* g$. Then, for any p such that $1 \leq p \leq d$ we have,*

$$|| \wedge^p g|| = a_1 \cdot ... \cdot a_p$$

Proof. First, write g in polar decomposition as $g = KAU$ with K and $U \in \mathcal{O}(d)$ which is the set of $d \times d$ orthogonal matrices and A is equal to $\mathrm{diag}(a_1, ..., a_d)$. Since $\wedge^p K$ and $\wedge^p U$ are isometries, then $|| \wedge^p g|| = || \wedge^p A||$. The set $\{e_{i_1}, ..., e_{i_p}\,,\ 1 \leq i_1 < ... < i_p \leq d\}$ represents an orthonormal basis of $\wedge^p \mathbb{R}^d$. We have, $\wedge^p A(e_{i_1} \wedge ... \wedge e_{i_p}) = a_{i_1 e_{i_1}} \wedge ... \wedge a_{i_p} e_{i_p} = a_{i_1} \cdot ... \cdot a_{i_p}(e_{i_1} \wedge ... \wedge e_{i_p})$, for $1 \leq i_1 < ... < i_p \leq d$. Therefore, $|| \wedge^p A|| = \sup\{a_{i_1} \cdot ... \cdot a_{i_p}\,;\ 1 \leq i_1 < ... < i_p \leq d\} = a_1 \cdot ... \cdot a_p$. □

Proposition 2.5.3. *Let g be a matrix in $Gl(d, \mathbb{R})$. Let $a_1 \geq a_2 \geq ... \geq a_d > 0$ be the square roots of the eigenvalues of $g^* g$. Then,*

$$\frac{\delta(g[x], g[y])}{\delta([x], [y])} \leq a_1 a_2 \frac{||x||\, ||y||}{||gx||\, ||gy||}$$

Proof. This follows readily from the definition of the distance δ and some simple computations:

$$\begin{aligned} \frac{\delta(g[x], g[y])}{\delta([x], [y])} &= \frac{||gx \wedge gy||.||x||\, ||y||}{||gx||\, ||gy||.||x \wedge y||} \\ &\leq \frac{|| \wedge^2 g||.||x||\, ||y||}{||gx||\, ||gy||} \\ &= a_1 a_2 \frac{||x||\, ||y||}{||gx||\, ||gy||}. \end{aligned}$$

□

Chapter 3

Random Matrix Products in the strongly irreducible case

The results of this section are present in [G5], see also [B5, chapter 3]. We now present a main result on the behavior of the random matrix product S_n without moment hypotheses on the measure μ. We will always assume that Γ_μ is strongly irreducible and contracting (See Definitions 2.1.2, 2.1.4). We will see that for every [x] in P($\mathbb{R}^d$), S_n contracts almost surely the angular distance δ and $S_n^*[x]$ converges in probability to a random direction independent of $[x]$. This important result is due to Guivarc'h and Raugi [G5]. And it is remarkable that it holds under a condition which depends solely on Γ_μ. In some sense this result is the "random version of Proposition 2.1.2"

Theorem 3.0.1. *Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. Let $Y_1, Y_2, ...$ be independent identically distributed random matrices in $GL(d, \mathbb{R})$ with common distribution μ. We suppose that Γ_μ is strongly irreducible. let p be the index of Γ_μ (See Definition 2.1.2)*
Then, if $g_n = Y_1...Y_n$, then for almost all $\omega \in \Omega$ there exists a p-dimensional subspace $V(\omega)$ of $\mathbb{R}^d$ such that any limit point of $\{||g_n(\omega)||^{-1}g_n(\omega); n \geq 1\}$ is a rank p matrix with range $V(\omega)$. Also, for any non-zero vector x we have,

$$\mathbb{P}\{x \textit{ is orthogonal to } V(\omega)\} = 0.$$

When Γ_μ is contracting there exists a unique μ-invariant distribution ν on $P(\mathbb{R}^d)$, so that $g_n(\omega)\nu$ converges weakly to $\delta_{[z](\omega)}$, where $z(\omega)$ is any non-zero vector.

Lemma 3.0.1. *If μ^* denotes the distribution of Y_1^*, and Γ_μ is strongly irreducible then, Γ_{μ^*} is strongly irreducible as well.*

Proof. $\Gamma_{\mu^*} = \{g^*, g \in \Gamma_\mu\}$.
Suppose by contradiction that Γ_{μ^*} is not strongly irreducible then, there exists a family of proper subspaces $V_1, ..., V_k$ of $\mathbb{R}^d$ such that

$$g(\bigcup_{i=1}^k V_i) = \bigcup_{i=1}^k V_i$$

Note that if $gV = V$ then $g^*V^\perp = V^\perp$:
Let $x \in V^\perp$ and $y \in V$ then, $< g^*x, y >=< x, gy >= 0$ because $gy \in V$. Therefore, $g^*V^\perp = V^\perp$.
So, if W_i is the subspace orthogonal to V_i for all i then,

$$g^*(\bigcup_{i=1}^k W_i) = \bigcup_{i=1}^k W_i \qquad \text{for } g \in \Gamma_{\mu^*}$$

$$\Rightarrow g^*(\bigcup_{i=1}^{k} W_i) = \bigcup_{i=1}^{k} W_i \qquad \text{for } g^* \in \Gamma_\mu$$

This implies that Γ_μ is not strongly irreducible. A contradiction.

$\square$

Proposition 3.0.4. *Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. Let $Y_1, Y_2, ...$ be independent identically distributed random matrices in $GL(d, \mathbb{R})$ with common distribution μ, and $S_n = Y_n...Y_1$. Consider a polar decomposition $S_n = K_n A_n U_n$ with K_n, U_n in $\mathcal{O}(d)$ (See Proposition 2.1.1), and A_n =diag($a_1(n), ..., a_d(n)$) with $a_1(n) \geq ... \geq a_d(n) > 0$. If Γ_μ is strongly irreducible and contracting then,*

a) The subspace spanned by $\{U^*(\omega)e_1\}$ converges almost surely to a one-dimensional subspace V(ω).

b) With probability one,

$$\lim_{n\to\infty} \frac{a_2(n)}{||S_n||} = \lim_{n\to\infty} \frac{a_2(n)}{a_1(n)} = 0$$

c) For any sequence $\{x_n, n \geq 1\}$ in $\mathbb{R}^d$ which converges to a non-zero vector,

$$\sup_{n\geq 1} \frac{||S_n(\omega)||}{||S_n(\omega)x_n||} < +\infty \qquad a.s$$

Proof. Γ_μ is strongly irreducible and contracting then so is Γ_{μ^*}. Then, for almost all $\omega \in \Omega$, each limit point of the sequence $\{||S_n^*(\omega)||^{-1}S_n^*(\omega), n \geq 1\}$ is a rank one matrix $g(\omega)$ with range $V(\omega)$. Knowing that $||S_n|| = a_1(n)$ we get,

$$S_n^* = U_n^* A_n^* K_n^* \Longrightarrow \frac{S_n^*}{||S_n||} = U_n^* \text{diag}\left(1, \frac{a_2(n)}{a_1(n)}, ..., \frac{a_d(n)}{a_1(n)}\right) K_n^*.$$

If we denote by $U_\infty(\omega), K_\infty(\omega), a_2(\omega), ..., a_d(\omega)$ the limit points of $U_n(\omega), K_n(\omega)$, and $\frac{a_2(n)}{a_1(n)}(\omega), ..., \frac{a_d(n)}{a_1(n)}(\omega)$ we get,

$$g(\omega) = U_\infty^*(\omega)\, \text{diag}(1, a_2(\omega), ..., a_d(\omega)) K_\infty^*(\omega)$$

We know that, with probability one, the rank of $g(\omega) = 1$.
Then, $a_2(\omega) = ... = a_d(\omega) = 0$ with $a_1(\omega) = 1 \neq 0$. Thus proving part (b).
Let $y \in \mathbb{R}^d$. Then,

$$g(\omega)y = U_\infty^* \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix} K_\infty^* \, y$$

If $K_\infty^* y \in \mathbb{R}e_1$ then, $g(\omega)y \in U_\infty^* \mathbb{R}e_1$. So, $\text{Im}(g(\omega)) = \mathbb{R}U_\infty^* e_1$.
Therefore $[z](\omega) = \left[U_\infty^* e_1\right]$ ie $U_\infty^* e_1$ is the range of $g(\omega)$.
For part (c), let $\{x_n\}$ be a sequence in $\mathbb{R}^d$ which converges to a non-zero vector x. Using Frobenius norm we get,

$$\frac{||S_n x_n||^2}{||S_n||^2} = \left(\frac{||A_n U_n x_n||}{||A_n||}\right)^2 = \sum_{i=1}^{d} \left(\frac{a_i}{a_1}\right)^2 < U_n x_n, e_i >$$

$$\Rightarrow \frac{||S_n x_n||^2}{||S_n||^2} \geq \left(\frac{a_2}{a_1}\right)^2 < x_n, U_n^* e_2 >.$$

Let $\{y_n(\omega)\}$ be the sequence which converges to $y(\omega)$, with $y(\omega)$ being the orthogonal projection of x onto $V(\omega)$. Then,

$$\begin{aligned}
\liminf_{n\to\infty} \frac{||S_n x_n||}{||S_n||} &\geq \inf_{n\geq 1}\left(\frac{a_2}{a_1}\right)||y_n(\omega)|| \\
\implies \frac{1}{\limsup\limits_{n\to\infty} \frac{||S_n||}{||S_n x_n||}} &\geq \frac{1}{\sup\limits_{n\geq 1}\left(\frac{a_1}{a_2}\right)}||y_n(\omega)|| \\
\implies \limsup_{n\to\infty} \frac{||S_n||}{||S_n x_n||} &\leq \frac{1}{||y_n(\omega)||}\sup_{n\geq 1}\left(\frac{a_1}{a_2}\right)
\end{aligned}$$

But $\sup\limits_{n\geq 1}\left(\frac{a_1}{a_2}\right) = +\infty$, and $\mathbb{P}(x$ is orthogonal to $V(\omega)) = \mathbb{P}\left(||y(\omega)|| = 0\right) = 0$.
Thus almost-surely,

$$\limsup \frac{||S_n(\omega)||}{||S_n(\omega)x_n||} < +\infty.$$

This implies that the sequence $\dfrac{||S_n(\omega)||}{||S_n(\omega)x_n||}$ is bounded above. Therefore, its sup is finite a.s.

□

Corollary 3.0.1. *Consider the sequence $\{Y_n, n \geq 1\}$ of independent identically distributed random matrices in $GL(d,\mathbb{R})$ with common distribution μ. We suppose $\mathbb{E}\left(\log ||Y_1||\right) < +\infty$ and Γ_μ is strongly irreducible. Then, for any sequence $\{x_n, n \geq 1\}$ that converges to a non-zero vector we have,*

$$\lim_{n\to\infty} \frac{1}{n}\log ||Y_n ... Y_1 x_n|| = \gamma \qquad a.s.$$

In particular,

$$\lim_{n\to\infty} \sup_{[x]\in P(\mathbb{R}^d)} \frac{1}{n}\mathbb{E}(\log ||Y_n ... Y_1 x_n||) = \gamma \tag{3.1}$$

Proof. Let $S_n = Y_n ... Y_1$ and $a_1(n) \geq ... \geq a_d(n) > 0$ be the square roots of the eigenvalues of $S_n^* S_n$. Then, for any sequence $\{x_n\}$ which converges to a non-zero vector x, we get $\inf\limits_{n\geq 1} \dfrac{||S_n x_n||}{||S_n||} > 0$ a.s. Thus, there exists a constant c such that

$$0 < c \leq \frac{||S_n x_n||}{||S_n||} \leq \frac{||S_n||}{||S_n||} = 1 \qquad \text{a.s}$$

So,

$$\frac{1}{n}\log c + \frac{1}{n}\log ||S_n|| \leq \frac{1}{n}\log ||S_n x_n|| \leq \frac{1}{n}\log ||S_n||$$

Hence,

$$\lim_{n\to\infty} \frac{1}{n}\log ||S_n|| \leq \lim_{n\to\infty} \frac{1}{n}\log ||S_n x_n|| \leq \lim_{n\to\infty} \frac{1}{n}\log ||S_n||$$

Therefore,

$$\lim_{n\to\infty} \frac{1}{n}\log ||S_n x_n|| = \gamma \qquad \text{a.s}$$

Using the LLN, one can easily prove for any sequence of vectors $\{x_n\}_n$ in $\mathbb{R}^d$ of norm 1 that the sequence of random variables $\{\frac{1}{n}\log ||S_n x_n||\}_n$ is uniformly integrable so that the previous convergence is also true in L^1. Thus proving 3.1. □

Theorem 3.0.2. *Let $S_n = Y_n ... Y_1$, where Y_i's are iid matrices in $GL(d,\mathbb{R})$ with common distribution μ. Suppose that Γ_μ is strongly irreducible and contracting. Then,*

(i) For any $[x], [y]$ in $P(\mathbb{R}^d)$, $\lim\limits_{n\to\infty} \delta(S_n.[x], S_n.[y]) = 0 \qquad a.s$

(ii) There exists a random direction $[z]$ such that $S_n^*.[x]$ converges in probability to $[z]$, uniformly in $[x] \in P(\mathbb{R}^d)$.

(iii) There is a unique μ-invariant distribution ν on $P(\mathbb{R}^d)$ and for any continuous function f on $P(\mathbb{R}^d)$,

$$\sup_{[x]\in P(\mathbb{R}^d)} \left|\mathbb{E}\{f(S_n \cdot [x])\} - \int f d\nu\right| \to 0 \qquad \text{as } n \to \infty.$$

Proof. Let a_1, a_2 be the square roots of the eigenvalues of $S_n^* S_n$.
Let $[x], [y] \in \mathrm{P}(\mathbb{R}^d)$. Then,

$$\begin{aligned}
\frac{\delta(S_n[x], S_n[y])}{\delta([x],[y])} &\leq \frac{|| \wedge^2 S_n||\, ||x \wedge y|| ||x||\, ||y||}{||S_n x||\, ||S_n y||\, ||x \wedge y||} \\
&= \frac{a_1.a_2\, ||x||\, ||y||}{||S_n x||\, ||S_n y||} \\
&= \frac{a_2}{||S_n||} \frac{||S_n||}{||S_n x||} \frac{||S_n||}{||S_n y||}.||x||\, ||y||
\end{aligned}$$

From Proposition 3.0.4 we get that almost surely,

$$\lim_{n\to\infty} \frac{\delta(S_n[x], S_n[y])}{\delta([x],[y])} = 0.$$

And since $\delta([x],[y]) \leq 1$, we get that $\lim\limits_{n\to\infty} \delta(S_n.[x], S_n.[y]) = 0$ a.s
Thus proving (i).
The above can be done when taking any two sequences $\{x_n\}$ and $\{y_n\}$ of unit vectors that converge in direction. So we get, $\lim\limits_{n\to\infty} \delta(S_n.\bar{x_n}, S_n.\bar{y_n}) = 0$ almost surely. Let μ^* be the common distribution of the random matrices $X_i = Y_i^*$ and let $R_n = X_n ... X_1$.
We claim that $\sup\limits_{[x],[y]} \mathbb{E}\left(\delta(R_n[x], R_n[y])\right) \xrightarrow[n\to\infty]{} 0.$
Suppose not, then there exists an $\epsilon > 0$ and subsequences $\{x_{n_i}\}, \{y_{n_i}\}$ for which

$$\mathbb{E}\left(\delta(R_n[x]_{n_i}, R_n[y]_{n_i})\right) > \epsilon \qquad (*)$$

Without loss of generality, we can assume that $\{x_{n_i}\}$ and $\{y_{n_i}\}$ converge. Then,

$$\lim_{i\to\infty} \frac{\delta(R_n[x]_{n_i}, R_n[y]_{n_i})}{\delta([x]_{n_i}, [y]_{n_i})} = 0$$

It would imply that $\lim\limits_{i\to\infty} \delta(R_n[x]_{n_i}, R_n[y]_{n_i}) = 0$. This contradicts $(*)$.
Therefore,

$$\sup_{[x],[y]} \mathbb{E}\left(\delta(R_n[x], R_n[y])\right) \xrightarrow[n\to\infty]{} 0.$$

We know that if the probability measure m is the unique μ^*-invariant distribution on $\mathrm{P}(\mathbb{R}^d)$ then, $g_n m \xrightarrow[n\to\infty]{} \delta_{[z]}$ weakly, with $g_n = X_1 .... X_n$.
Then we have,

$$\mathbb{E}\left(\delta(S_n^*[x], [z])\right) \leq \mathbb{E}\left(\delta(S_n^*[x], S_n^*[y])\right) + \mathbb{E}\left(\delta(S_n^*[y], [z])\right)$$

But S_n^* and R_n have the same law so,

$$\mathbb{E}\left(\delta(S_n^*[x], [z])\right) \leq \mathbb{E}\left(\delta(R_n[x], R_n[y])\right) + \mathbb{E}\left(\delta(S_n^*[y], [z])\right)$$

$$\implies \int \mathbb{E}\left(\delta(S_n^*[x], [z])\right) dm([y]) \leq \int \mathbb{E}\left(\delta(R_n[x], R_n[y])\right) dm([y]) + \int \mathbb{E}\left(\delta(S_n^*[y], [z])\right) dm([y])$$

$$\implies \sup_{[x]\in \mathrm{P}(\mathbb{R}^d)} \mathbb{E}\left(\delta(S_n^*[x], [z])\right) \leq \sup_{[x],[y]\in \mathrm{P}(\mathbb{R}^d)} \mathbb{E}\left(\delta(R_n[x], R_n[y])\right) + \mathbb{E}\left(\int \delta(S_n^*[y], [z]) dm([y])\right)$$

But,

$$\sup_{[x],[y]\in \mathrm{P}(\mathbb{R}^d)} \mathbb{E}\left(\delta(R_n[x],R_n[y])\right) \underset{n\to\infty}{\longrightarrow} 0$$

$$\text{and } \mathbb{E}\left(\int \delta(g_n[y],[z])dm([y])\right) = \mathbb{E}\left(\int \delta([y],[z])dg_n m([y])\right) \xrightarrow[n\to\infty]{} \mathbb{E}\left(\delta([z],[z])\right) = 0$$

Therefore, $S_n^*[x]$ converges in probability to $[z]$, uniformly in $[x] \in \mathrm{P}(\mathbb{R}^d)$.
Thus proving (ii).
Let f be a continuous function on $\mathrm{P}(\mathbb{R}^d)$. We have, $S_n^*[x]$ converges in probability to $[z]$, uniformly in $[x] \in \mathrm{P}(\mathbb{R}^d)$ then, $f(S_n^*[x])$ converges in probability to $f([z])$, uniformly in $[x] \in \mathrm{P}(\mathbb{R}^d)$. So,

$$\sup_{[x]\in \mathrm{P}(\mathbb{R}^d)} \mathbb{E}\big|\big\{f(S_n^*.[x]) - f([z])\big|\big\} \underset{n\to\infty}{\longrightarrow} 0$$

But $\sup_{[x]\in \mathrm{P}(\mathbb{R}^d)} \big|\mathbb{E}\left(f(R_n.[x])\right) - \int f dm\big| \leq \sup_{[x]\in \mathrm{P}(\mathbb{R}^d)} \mathbb{E}\big\{\big|f(S_n^* \cdot [x]) - f([z])\big|\big\} \underset{n\to\infty}{\longrightarrow} 0$
where m is the unique μ^*-invariant distribution on $\mathrm{P}(\mathbb{R}^d)$.
Similarly, we can work with Y_i instead of X_i and ν being the unique μ-invariant distribution on $\mathrm{P}(\mathbb{R}^d)$ to get,

$$\sup_{[x]\in \mathrm{P}(\mathbb{R}^d)} \big|\mathbb{E}\{f(S_n \cdot [x])\} - \int f d\nu\big| \to 0 \qquad \text{as } n \to \infty.$$

Thus proving (iii). □

Chapter 4

Central Limit theorem of $\log ||S_n||$ when the measure has a density

The reference for this section is [L2] and Chapter 5 in [B5].
In this section we allow μ to have an exponential moment and a density with respect to the Haar measure on $\mathrm{GL}(d, \mathbb{R})$. We prove Theorem 1.0.1 in this setting and observe that he density assumption will simplify our task. We get rid of this assumption in the next section.

4.1 Central Limit Theorem in $\mathbb{R}$

From a certain population of interest one can pick random samples of the same size then calculate the mean for each one of these samples. These samples are thought of as being independent from one another. The Central Limit Theorem states that regardless of what the original population distribution looked like, our sampling distribution will have a normal distribution. Of course for the theorem to hold we do need a sample size that is large enough.

Theorem 4.1.1. *Let $X_1, X_2, ..., X_n$ be real independent identically distributed random variables in a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, with common distribution μ with moment of order 2 and finite variance $\sigma^2 > 0$.*
Then, $Z_n = \dfrac{\sum_{i=1}^n X_i - n\mu}{\sqrt{n}\sigma}$ has a limiting cumulative distribution function which approaches a Normal Distribution. This means,

$$\mathbb{P}(Z_n \in [a, b]) \underset{n\to\infty}{\longrightarrow} \frac{1}{\sqrt{2\pi}} \int_a^b e^{-x^2/2} dx.$$

Proof. In $\mathbb{R}$, the X_i's are independent identically distributed real random variables of law μ and variance σ^2. Assume μ is finite and of order 2 with $\sigma^2 < \infty$.
Let t in $\mathbb{R}$ and n in $\mathbb{N}$. Let $Z_n = \dfrac{\sum_{j=1}^n X_j - n\mu}{\sqrt{n}\sigma}$.
Define the characteristic function of Z_n by

$$\Phi_{Z_n}(t) = \mathbb{E}(e^{itZ_n}).$$

Then,

$$\begin{aligned}
\Phi_{Z_n}(t) &= \mathbb{E}\left(e^{\frac{it}{\sqrt{n}\sigma}\left(\sum_{j=1}^n X_j - n\mu\right)}\right) \\
&= \mathbb{E}\left(e^{\frac{it}{\sqrt{n}\sigma}\sum_{j=1}^n (X_j-\mu)}\right) \\
&= \mathbb{E}\left(\prod_{j=1}^n e^{\frac{it}{\sqrt{n}\sigma}(X_j-\mu)}\right) \\
&\overset{(*)}{=} \prod_{j=1}^n \mathbb{E}\left(e^{\frac{it}{\sqrt{n}\sigma}(X_j-\mu)}\right) \\
&\overset{(*)}{=} \left[\mathbb{E}\left(e^{\frac{it}{\sqrt{n}\sigma}(X_1-\mu)}\right)\right]^n \\
&= \left[\Phi_{Z_1}\left(\frac{t}{\sqrt{n}}\right)\right]^n
\end{aligned}$$

$(*)$ Since the X_j's are independent and have the same law.
Then, the Taylor expansion of Φ_{Z_1} around 0 gives:

$$\Phi_{Z_1}\left(\frac{t}{\sqrt{n}}\right) = \Phi_{Z_1}(0) + \frac{t}{\sqrt{n}}\Phi'_{Z_1}(0) + \frac{\left(\frac{t}{\sqrt{n}}\right)^2}{2}\Phi''_{Z_1}(0) + \frac{t^2}{n}\epsilon(\frac{t}{\sqrt{n}})$$

With $\epsilon(\frac{t}{\sqrt{n}}) \underset{n\to\infty}{\longrightarrow} 0$.
So,

$$\begin{aligned}
\Phi_{Z_1}\left(\frac{t}{\sqrt{n}}\right) &= 1 + \frac{it}{\sqrt{n}}\mathbb{E}(Z_1) - \frac{t^2}{2n}\mathbb{E}(Z_1^2) + \frac{t^2}{n}\epsilon(\frac{t}{\sqrt{n}}) \\
&= 1 - \frac{t^2}{2n} + \frac{t^2}{n}\epsilon(\frac{t}{\sqrt{n}})
\end{aligned}$$

Since $\mathbb{E}(Z_1) = 0$ and $\mathbb{E}(Z_1^2) = \mathbb{E}\left(\frac{(X_1-\mu)^2}{\sigma^2}\right) = \frac{1}{\sigma^2}\mathbb{E}\left((X_1 - \mathbb{E}(X_1))^2\right) = \frac{\sigma^2}{\sigma^2} = 1$
Then, $\Phi_{Z_1}\left(\frac{t}{\sqrt{n}}\right) = \exp\left(-\frac{t^2}{2n} + \frac{t^2}{n}\epsilon(\frac{t}{\sqrt{n}})\right)$, with $\epsilon(\frac{t}{\sqrt{n}}) \underset{n\to\infty}{\longrightarrow} 0$.
Hence, $\lim\limits_{n\to\infty}\Phi_{Z_n}(t) = \lim\limits_{n\to\infty}\left(\Phi_{Z_1}\left(\frac{t}{\sqrt{n}}\right)\right)^n = e^{-t^2/2}$.
Notice that $\Phi_{Z_n}(t) = \mathbb{E}\left(e^{itZ_n}\right) = \hat{\mu_n}(-t)$, where $\hat{\mu}(t) = \int e^{-itx}\mathrm{d}\mu(x)$.
Then, $\hat{\mu_n}(t) = \Phi_{Z_n}(-t)$ and $\lim\limits_{n\to\infty}\hat{\mu_n}(t) = e^{-t^2/2} = \hat{\nu}(t)$.
Levy's continuity theorem states that if a sequence of functions $\hat{f}_n(t)$ converges to a function $f(t)$ for all t in $\mathbb{R}^d$ with f being a constant at 0 then, there exists a probability measure ν on $\mathbb{R}^d$ such that $\hat{\nu}(t) = f(t)$ and $f_n \longrightarrow \nu$ weakly. Let ν be the fourier inverse of $\hat{\nu}$. Then,

$$\nu(t) = \frac{1}{\sqrt{2\pi}}e^{-t^2/2}.$$

Since $\mu_n \longrightarrow \nu$ weakly, then for every continuous and bounded function φ we have,

$$\int \varphi(t)\mathrm{d}\mu_n(t) \underset{n\to\infty}{\longrightarrow} \int \varphi(t)\mathrm{d}\nu(t).$$

Since ν is absolutely continuous with respect to the Lebesgue measure, where it gives mass zero to the boundary of $[a, b]$ we have that,

$$\mathbb{E}\left(\mathbb{1}_{(a,b)}(Z_n)\right) = \mathbb{P}\left(Z_n \in (a,b)\right) \underset{n\to\infty}{\longrightarrow} \frac{1}{\sqrt{2\pi}}\int_a^b e^{-t^2/2}\mathrm{dt}.$$

□

4.2 A Lemma of Perturbation Theory

This section is only about perturbation theory of operators. We refer to [D].
We work on an abstract Banach space E over $\mathbb{C}$. We consider an analytical family of bounded operators $\{T(\xi), \xi \in \mathbb{C}\}$ on E. We suppose that there exists a rank-one operator $N(0)$ such that $T^n(0)$ converges exponentially fast to $N(0)$ in the following sense ie $\lim_{n\to\infty} ||T^n(0) - N(0)||^{1/n} < 1$. our goal is to find a suitable decomposition of $T(\xi)$ for ξ small enough.

Definition 4.2.1. *The spectrum of an operator T is the set of complex numbers λ such that the operator $\lambda I - T$ is not invertible.*

Definition 4.2.2. *The resolvent set of an operator T defied on a Banach space E, denoted by $r(T)$, is the set of complex numbers λ for which $(\lambda I - T)^{-1}$ exists as a bounded operator. It is the compliment of the spectrum of T, denoted by $\sigma(T)$. We define the resolvent function of T by*

$$R(z,T) = (zI - T)^{-1}$$

Which is well-defined outside the spectrum of T.

Remark 4.2.1. *The resolvent set $r(T)$ is open and $R(z,T)$ is analytic in $r(T)$.*

Lemma 4.2.1. *Let U_1 and U_2 be two open sets in $\mathbb{C}$ such that $\overline{\mathrm{U}_1} \cap \overline{\mathrm{U}_2} = \emptyset$. Suppose that the spectrum of the operator T given by $\sigma(T)$ is contained in $\mathrm{U}_1 \cup \mathrm{U}_2$. Define*

$$N_1 = \frac{1}{2\pi i} \int_{\partial \mathrm{U}_1} R(z,T)\, dz$$

$$N_2 = \frac{1}{2\pi i} \int_{\partial \mathrm{U}_2} R(z,T)\, dz$$

Then, N_1 and N_2 are two projections such that $N_1 T = T N_1$ with $N_1 + N_2 = I$ and $N_1 N_2 = N_2 N_1$.

Theorem 4.2.1. *Let E be a complex Banach space and V a neighborhood of 0 in $\mathbb{C}$. Let $\{T(\xi), \xi \in V\}$ be an analytic family of bounded operators on E. Suppose there exists a rank-one operator $N(0)$ such that*

$$\rho = \lim_{n\to\infty} ||T^n(0) - N(0)||^{1/n} < 1.$$

Then, one can find an $\epsilon > 0$ such that for $|\xi| < \epsilon$ we have,

$$T(\xi) = \lambda(\xi) N(\xi) + Q(\xi)$$

where

(i) *$\lambda(\xi)$ is the unique eigenvalue of $T(\xi)$ of maximum modulus.*

(ii) *$N(\xi)$ is a rank-one projection such that $N(\xi)Q(\xi) = Q(\xi)N(\xi) = 0$.*

(iii) *The maps $\xi \longmapsto \lambda(\xi)$, $\xi \longmapsto N(\xi)$, and $\xi \longmapsto Q(\xi)$ are analytic.*

(iv) *$|\lambda(\xi)| \geq \dfrac{2+\rho}{3}$, and for some $p \in \mathbb{N}$, and there exists $c > 0$ such that for every $n \in \mathbb{N}$ we have,*

$$\left|\left| \frac{d^p}{d\xi^p} Q^n(\xi) \right|\right| \leq c \left(\frac{1+2\rho}{3} \right)^{n+1}.$$

(v) Let $t \in \mathbb{R}$. We have,

$$e^{-it\sqrt{n}\lambda'(0)}T^n(\frac{it}{\sqrt{n}}) \underset{n\to\infty}{\longrightarrow} e^{\frac{-t^2}{2}(\lambda''(0)-\lambda'^2(0))}N(0) \tag{4.1}$$

And for a fixed $n \in \mathbb{N}$, let $\xi \in \mathbb{C}$ so that we have,

$$\frac{1}{n}\frac{d^2}{d\xi^2}\left(e^{-n\xi\lambda'(0)}T^n(\xi)\right) \underset{n\to\infty}{\longrightarrow} \left(\lambda''(0) - \lambda'^2(0)\right)N(0). \tag{4.2}$$

Proof. Since $\lim\limits_{n\to\infty} T^n(0) = N(0)$ then, $N(0) = T^n(0) + R_n$ with $R_n \underset{n\to\infty}{\longrightarrow} 0$.

$$\text{So, } N(0)T(0) = T(0)^{n+1} + R_nT(0)$$
$$\text{and } T(0)N(0) = T(0)^{n+1} + T(0)R_n.$$

Thus, $N(0)T(0) - T(0)N(0) = R_nT(0) - T(0)R_n$.
As $n \to \infty$ we get $T(0)N(0) = N(0)T(0)$. Notice that $\lim\limits_{n\to\infty} N(0)T(0) = \lim\limits_{n\to\infty}\left(T^{n+1}(0) + R_nT(0)\right)$.
Therefore, $N(0)T(0) = N(0)$.
Then, the restriction of $T(0)$ on $\mathrm{Im}N(0)$ is the identity and its restriction on $\mathrm{Ker}N(0)$ is $T(0) - N(0)$.
The spectral radius of $T(0)\big|_{\mathrm{Im}N(0)}$ is 1.
Let ρ be the spectral radius of $T(0)\big|_{\mathrm{Ker}N(0)}$. Then, $\lim\limits_{n\to\infty} ||T^n(0) - N(0)||^{1/n} < 1$. Thus,

$$\sigma\left(T(0)\right) = \sigma\left(T(0)\big|_{\mathrm{Ker}N(0)}\right)\bigcup\sigma\left(T(0)\big|_{\mathrm{Im}N(0)}\right) \subset B(0,\rho)\cup\{1\}.$$

Now, let γ be a small enough circle around 1. Define P to be an operator on E given by

$$P = \frac{1}{2\pi i}\int_\gamma R(z,T(0))\mathrm{d}z$$

where $R(z,T(0)) = (zI - T(0))^{-1}$ is the Rezolvent function of $T(0)$, defined on the compliment of the spectrum of $T(0)$, called the resolvent set of $T(0)$.
Now, for all $x \in E$ we have,

$$Px = \frac{1}{2\pi i}\int_\gamma R(z,T(0))x\mathrm{d}z$$

Notice that Px is well-defined since $z \notin \sigma(T(0))$. We have two cases:

Case 1: $x \in \mathrm{KerN}(0)$. Then, $T(0) = T(0) - N(0)$ and $1 \notin \sigma(T(0))$. So, $R(z,T(0))x$ is analytic in the disc of boundary γ. Thus, by Cauchy's formula,

$$Px = \frac{1}{2\pi i}\int_\gamma R(z,T(0))x\mathrm{d}z = 0.$$

Case 2: $x \in \mathrm{Im}N(0)$. Then, $T(0) = I$ and $R(z,T(0)) = (z-1)^{-1}$. So, by Cauchy's formula,

$$Px = \frac{1}{2\pi i}\int_\gamma \frac{x}{z-1}\mathrm{d}z = x$$

Hence, $P = N(0)$ and it's a rank-one projection on E.
Since $R(z,T(0))T(0) = T(0)R(z,T(0)) = -I + R(z,T(0))$, we get $T(0)N(0) = N(0)T(0) = N(0)$.
Let $\epsilon > 0$ and $\xi \in V$ such that $|\xi| < \epsilon$. The map $\xi \longmapsto T(\xi)$ is continuous , so for $|\xi| < \epsilon$ we have,

$$||T(\xi) - T(0)|| < \epsilon' = ||R(z,T(0))||^{-1}$$

This ensures that $\sum[(T(\xi) - T(0))\,R(z,T(0))]^n$ defines a geometric series with inverse $I - (T(\xi) - T(0))\,R(z,T(0))$.
But,

$$\begin{aligned}
I - (T(\xi) - T(0))\,R(z,T(0)) &= (zI - T(0))R(z,T(0)) - (T(\xi) - T(0))R(z,T(0))\\
&= (zI - T(0) - T(\xi) + T(0))R(z,T(0))\\
&= (zI - T(\xi))R(z,T(0))
\end{aligned}$$

Then, $R(z,T(\xi)) = \sum (T(\xi) - T(0))^n \, R(z,T(0))^{n+1}$.
This shows that whenever $||T(\xi) - T(0)|| < ||R(z,T(0))||^{-1}$ we have, $R(z,T(\xi))$ is analytic and well-defined with $z \notin \sigma(T(\xi))$.
Consider the 2 discs $D_1\left(1, \dfrac{1-\rho}{3}\right)$ and $D_2\left(0, \dfrac{1+2\rho}{3}\right)$. Let $\gamma_1 = \partial D_1$ and $\gamma_2 = \partial D_2$. Let $\mathrm{U} = \mathrm{U}_1 \cup \mathrm{U}_2$, where $\mathrm{U}_1 = \mathrm{B}(1, \dfrac{1-\rho}{3})$ and $\mathrm{U}_2 = \mathrm{B}(0, \dfrac{1+2\rho}{3})$.
So that $\overline{\mathrm{U}} = D_1 \cup D_2$. We have $\sigma(T(0)) \subset \mathrm{U}$.

D_1 D_2

Now let $M = \sup\{||R(z,T(0))||, z \notin \mathrm{U}\}$. For $||T(\xi) - T(0)|| < \dfrac{1}{M}$ we have $R(z,T(\xi))$ is well-defined and analytic with $z \notin \sigma(T(\xi))$. The continuity of the map $\xi \longmapsto T(\xi)$ ensures the continuity of the map $\xi \longmapsto \lambda(\xi)$ where $\lambda(\xi) \in \sigma(T(\xi))$. Since $\sigma(T(0)) \subset \mathrm{U}$, we have $\sigma(T(\xi)) \subset \mathrm{U}$, and for all $\epsilon' > 0$ there exists $\epsilon > 0$ such that for $|\xi| < \epsilon$ we have $|\lambda(\xi) - \lambda(0)| \leq \epsilon'$, where $\lambda(0)$ is the spectral radius of $T(0)$ that is equal to 1.
For $\epsilon' = \dfrac{1-\rho}{3} > 0$ we get $|\lambda(\xi)| \geq \dfrac{2+\rho}{3}$ with $\lambda(\xi)$ being the unique eigenvalue of $T(\xi)$ with maximum modulus. We have U_1 and U_2 are open sets with $\overline{\mathrm{U}_1} \cap \overline{\mathrm{U}_2} = D_1 \cap D_2 = \emptyset$ such that $\sigma(T(\xi)) \subset \mathrm{U}$.
Then, by the previous lemma, we can define

$$N(\xi) = \frac{1}{2\pi i} \int_{\gamma_1} R(z,T(\xi)) \mathrm{d}z$$

and

$$I - N(\xi) = \frac{1}{2\pi i} \int_{\gamma_2} R(z,T(\xi)) \mathrm{d}z$$

Both being rank-one projections and $T(\xi)N(\xi) = N(\xi)T(\xi)$.
Claim: $N(\xi)$ is of rank one if $||N(\xi) - N(0)|| < 1$:

$N(0)$ is of rank one so there exists some $x \in E$ such that for all $y \in E$, there exists some $\lambda_y \in \mathbb{C}$, so that $N(0)y = \lambda_y x$. In particular, we take $y \in \mathrm{Im}N(0)$ then, $y = \lambda_y x$.
Let $x, y \in E$ and $\lambda \in \mathbb{C}$ such that $N(\xi)x = x$ and $N(\xi)y = y$. Look at $||x - \lambda y||$:

$$\begin{aligned} ||x - \lambda y|| = ||N(\xi)(x - \lambda y)|| &= ||\left(N(\xi) - N(0)\right)(x - \lambda y)|| \\ &\leq ||N(\xi) - N(0)||\, ||x - \lambda y|| \\ &< ||(x - \lambda y)|| \end{aligned}$$

This is true if and only if $x = \lambda y$. Thus, $N(\xi)$ is of rank one.
Now, let $x_0 \in \mathrm{Im}N(\xi)$. $N(\xi)$ is of rank one then, for all $y \in E$ the exists $\lambda \in \mathbb{C}$ so that $N(\xi)y = \lambda x_0$. In particular for $y = T(\xi)x_0 \in E$. Then, $T(\xi)N(\xi)x_0 = N(\xi)T(\xi)x_0 = \lambda x_0$. So, $T(\xi)x_0 = \lambda(\xi)x_0$. Thus, $\lambda(\xi)$ is an eigenvalue of $T(\xi)$ and x_0 is its corresponding eigen function. Hence, we get $N(\xi)T(\xi) = T(\xi)N(\xi) = \lambda(\xi)N(\xi)$.
Now, define $Q(\xi) = T(\xi)(I - N(\xi)) = T(\xi) - \lambda(\xi)N(\xi)$. Then,

$$T(\xi) = \lambda(\xi)N(\xi) + Q(\xi).$$

Notice that $N(\xi)Q(\xi) = Q(\xi)N(\xi) = 0$. So, for all $n \geq 1$,

$$T^n(\xi) = \lambda^n(\xi)N(\xi) + Q^n(\xi).$$

Claim: $Q^n(\xi) = \frac{1}{2\pi i} \int\limits_{\gamma_2} z^n R(z, T(\xi)) \mathrm{d}z$:

We have,

$$\begin{aligned} T^n(\xi)R(z, T(\xi)) &= \left(T^n(\xi) - z^n I + z^n I\right) R(z, T(\xi)) \\ &= z^n R(z, T(\xi)) - (z^n I - T^n(\xi))R(z, T(\xi)) \\ &= z^n R(z, T(\xi)) - (z^n I - T^n(\xi))(zI - T(\xi))^{-1} \\ &= z^n R(z, T(\xi)) - \left(z^{n-1}I + z^{n-2}T(\xi) + ... + zT^{n-2}(\xi) + T^{n-1}(\xi)\right). \end{aligned}$$

So,

$$\begin{aligned} Q^n(\xi) = T^n(\xi)(I - N(\xi)) &= \frac{1}{2\pi i} \int\limits_{\gamma_2} T^n(\xi)R(z, T(\xi)) \mathrm{d}z \\ &= \frac{1}{2\pi i} \int\limits_{\gamma_2} z^n R(z, T(\xi)) \mathrm{d}z \\ &\quad - \frac{1}{2\pi i} \int\limits_{\gamma_2} z^{n-1}I + z^{n-2}T(\xi) + ... + zT^{n-2}(\xi) + T^{n-1}(\xi) \mathrm{d}z. \end{aligned}$$

Then [1], since the map $z \longmapsto z^k$ is analytic, we get

$$Q^n(\xi) = \frac{1}{2\pi i} \int\limits_{\gamma_2} z^n R(z, T(\xi)) \mathrm{d}z.$$

Now, for $\xi \in V$ and some $p \in \mathbb{N}$ we have,

$$\frac{\mathrm{d}^p}{\mathrm{d}\xi^p} Q^n(\xi) = \frac{1}{2\pi i} \int\limits_{\gamma_2} z^n \frac{\mathrm{d}^p}{\mathrm{d}\xi^p} R(z, T(\xi)) \mathrm{d}z.$$

[1]To integrate a Banach space-valued function f over a contour C in $\mathbb{C}$ is the same as in Complex Analysis where we parameterize C as $z = z(t)$ with $a \leq t \leq b$ and $\int\limits_C f \mathrm{d}z = \int\limits_a^b f(z(t))z'(t)\mathrm{d}t$. And we can "pull out" any constant outside the integral.

Then,

$$\left|\left|\frac{\mathrm{d}^p}{\mathrm{d}\xi^p}Q^n(\xi)\right|\right| \le \frac{1}{2\pi}\int\limits_{\gamma_2} ||z^n|| \left|\left|\frac{\mathrm{d}^p}{\mathrm{d}\xi^p}R(z,T(\xi))\right|\right| ||\mathrm{d}z||.$$

Denote by $c = \sup\limits_{\substack{|z|=\frac{1+2\rho}{3} \\ \xi|<\epsilon}} \left|\left|\frac{\mathrm{d}^p}{\mathrm{d}\xi^p}R(z,T(\xi))\right|\right|$.

For any $\xi \in V$ we get,

$$\begin{aligned}\left|\left|\frac{\mathrm{d}^p}{\mathrm{d}\xi^p}Q^n(\xi)\right|\right| &\le \frac{c}{2\pi}\int\limits_{\gamma_2} ||z^n||\,||\mathrm{d}z|| \\ &= c\left(\frac{1+2\rho}{3}\right)^{n+1}.\end{aligned}$$

Hence,

$$\sup_{|\xi|<\epsilon}\left|\left|\frac{\mathrm{d}^p}{\mathrm{d}\xi^p}Q^n(\xi)\right|\right| \le c\left(\frac{1+2\rho}{3}\right)^{n+1}.$$

Now, let $t \in \mathbb{R}$ and $\xi = \dfrac{it}{\sqrt{n}}$. For $|\xi| < \epsilon$ we have,

$$T^n(\frac{it}{\sqrt{n}}) = \lambda^n(\frac{it}{\sqrt{n}})N(\frac{it}{\sqrt{n}}) + Q^n(\frac{it}{\sqrt{n}})$$

Taylor expansion of $\lambda(.)$ near zero gives:

$$\begin{aligned}\lambda(\frac{it}{\sqrt{n}}) &= \lambda(0) + \frac{it}{\sqrt{n}}\lambda'(0) - \frac{t^2}{2n}\lambda''(0) + \frac{t^2}{n}\epsilon(\frac{t}{\sqrt{n}}) \\ &= \lambda(0) + \frac{it}{\sqrt{n}}\lambda'(0) - \frac{t^2}{2n}\lambda''(0) + \frac{t^2}{2n}\lambda'^2(0) - \frac{t^2}{2n}\lambda'^2(0) + \frac{t^2}{n}\epsilon(\frac{t}{\sqrt{n}}) \\ &= 1 + \frac{it}{\sqrt{n}}\lambda'(0) - \frac{t^2}{2n}\lambda''(0) + \frac{t^2}{2n}\lambda'^2(0) - \frac{t^2}{2n}\lambda'^2(0) + \frac{t^2}{n}\epsilon(\frac{t}{\sqrt{n}}) \\ &= \exp\left(\frac{it}{\sqrt{n}}\lambda'(0) - \frac{t^2}{2n}\lambda''(0) + \frac{t^2}{2n}\lambda'^2(0) + \frac{t^2}{n}\epsilon(\frac{t}{\sqrt{n}})\right)\end{aligned}$$

where $\epsilon(\dfrac{t}{\sqrt{n}}) \underset{n\to\infty}{\longrightarrow} 0$.

This is because

$$\begin{aligned}&\exp\left(\frac{it}{\sqrt{n}}\lambda'(0) - \frac{t^2}{2n}\lambda''(0) + \frac{t^2}{2n}\lambda'^2(0) + \frac{t^2}{n}\epsilon(\frac{t}{\sqrt{n}})\right) \\ &\quad= 1 + \frac{it}{\sqrt{n}}\lambda'(0) - \frac{t^2}{2n}\lambda''(0) + \frac{t^2}{2n}\lambda'^2(0) + \frac{\left(\frac{it}{\sqrt{n}}\lambda'(0) - \frac{t^2}{2n}\lambda''(0) + \frac{t^2}{2n}\lambda'^2(0)\right)^2}{2} + \frac{t^2}{n}\epsilon(\frac{t}{\sqrt{n}}) \\ &\quad= 1 + \frac{it}{\sqrt{n}}\lambda'(0) - \frac{t^2}{2n}\lambda''(0) + \frac{t^2}{2n}\lambda'^2(0) - \frac{t^2}{2n}\lambda'^2(0) + \frac{t^2}{n}\epsilon(\frac{t}{\sqrt{n}})\end{aligned}$$

Thus, $\lambda^n(\dfrac{it}{\sqrt{n}}) = \exp\left(it\sqrt{n}\lambda'(0) - \dfrac{t^2}{2}\lambda''(0) + \dfrac{t^2}{2}\lambda'^2(0) + \dfrac{t^2}{n}\epsilon(\dfrac{t}{\sqrt{n}})\right)$.

Hence,

$$T^n(\frac{it}{\sqrt{n}}) = \exp\left(it\sqrt{n}\lambda'(0) - \frac{t^2}{2}\lambda''(0) + \frac{t^2}{2}\lambda'^2(0) + \frac{t^2}{n}\epsilon(\frac{t}{\sqrt{n}})\right)N(\frac{it}{\sqrt{n}}) + Q^n(\frac{it}{\sqrt{n}})$$

where $\epsilon(\dfrac{t}{\sqrt{n}}) \underset{n\to\infty}{\longrightarrow} 0$.

Therefore,

$$e^{-it\sqrt{n}\lambda'(0)}T^n(\frac{it}{\sqrt{n}}) \underset{n\to\infty}{\longrightarrow} e^{\frac{-t^2}{2}\left(\lambda''(0)-\lambda'^2(0)\right)}N(0)$$

Let $n \in \mathbb{N}$. For every $\xi \in \mathbb{C}$ we have,

$$T^n(\xi) = \lambda^n(\xi)N(\xi) + Q^n(\xi)$$

Then,

$$\frac{\mathrm{d}}{\mathrm{d}\xi}T^n(\xi) = n\lambda^{n-1}(\xi)\lambda'(\xi)N(\xi) + \lambda^n(t)\frac{\mathrm{d}}{\mathrm{d}\xi}N(\xi) + \frac{\mathrm{d}}{\mathrm{d}\xi}Q^n(\xi)$$

Knowing that $\lambda(0) = 1$ we get,

$$\frac{\mathrm{d}}{\mathrm{d}\xi}T^n(\xi)\big|_{\xi=0} = n\lambda'(0)N(0) + \frac{\mathrm{d}}{\mathrm{d}\xi}N(t)\big|_{\xi=0} + \frac{\mathrm{d}}{\mathrm{d}\xi}Q^n(\xi)\big|_{\xi=0}$$

Now consider $e^{-n\xi\lambda'(0)}T^n(\xi)$. We have,

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left(e^{-n\xi\lambda'(0)}T^n(\xi)\right) = \frac{\mathrm{d}}{\mathrm{d}\xi}T^n(\xi)e^{-n\xi\lambda'(0)} - n\lambda'(0)e^{-n\xi\lambda'(0)}T^n(\xi).$$

Then,

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\left(e^{-n\xi\lambda'(0)}T^n(\xi)\right) = &\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}T^n(\xi)e^{-n\xi\lambda'(0)}\\ &- n\lambda'(0)e^{-n\xi\lambda'(0)}\frac{\mathrm{d}}{\mathrm{d}\xi}T^n(\xi)\\ &+ n^2\lambda'^2(0)e^{-n\xi\lambda'(0)}\left(\lambda^n(\xi)N(\xi) + Q^n(\xi)\right)\\ &- n\lambda'(0)e^{-n\xi\lambda'(0)}\left(n\lambda^{n-1}(\xi)\lambda'(\xi)N(\xi) + \lambda^n(\xi)\frac{\mathrm{d}}{\mathrm{d}\xi}N(\xi) + \frac{\mathrm{d}}{\mathrm{d}\xi}Q^n(\xi)\right).\end{aligned}$$

Evaluating at zero we get:

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\left(e^{-n\xi\lambda'(0)}T^n(\xi)\right)\big|_{\xi=0} = &(n\lambda''(0) - n\lambda'(0)^2)N(0) + n^2\lambda'^2(0)Q^n(0)\\ &+ \frac{\mathrm{d}^2}{\mathrm{d}\xi^2}Q^n(\xi)\big|_{\xi=0} - 2n\lambda'(0)\frac{\mathrm{d}}{\mathrm{d}\xi}Q^n(\xi)\big|_{\xi=0} + \frac{\mathrm{d}^2}{\mathrm{d}\xi^2}N(\xi)\big|_{\xi=0}.\end{aligned}$$

Hence,

$$\frac{1}{n}\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\left(e^{-n\xi\lambda'(0)}T^n(\xi)\right)\big|_{\xi=0} \underset{n\to\infty}{\longrightarrow} (\lambda''(0) - \lambda'^2(0))N(0).$$

□

4.3 The Fourier-Laplace Transform

We go back to our framework and we are concerned with the central limit behavior of $\log ||S_n||$. Here we give a specific form for our Banach space E and our family of operators $\{T(\xi)\}$.

Definition 4.3.1. *Let $E = \mathcal{C}_0(P(\mathbb{R}^d))$. Define formally the operator*

$$T(\xi)f([x]) = \mathbb{E}\left(e^{\xi\log\frac{||gx||}{||x||}}f(g\cdot[x])\right)$$

for $g \in GL(d,\mathbb{R})$, $[x] \in P(\mathbb{R}^d)$ and $\xi \in \mathbb{C}$ with f being a complex-valued function on $P(\mathbb{R}^d)$.

Proposition 4.3.1. *Consider the following Markov chain on $GL(d,\mathbb{R})\times P(\mathbb{R}^d)$ defined by $X_1 = (Y_1,[x])$, $X_2 = (Y_2, Y_1[x]), ..., X_n = (Y_n, S_{n-1}[x])$. The following defines an additive cocycle $\sigma(g,[x]) = \log \frac{||gx||}{||x||}$, where for $||x|| = 1$ we have,*

$$\log ||S_n x|| = \sigma(S_n,[x]) = \sum_{i=1}^{n} \sigma(X_i).$$

Remark 4.3.1. *Note that $T(0)$ is the Markov operator of the Markov chain presented in the previous Proposition 4.3.1, where for any complex-valued function f and any $[x] \in P(\mathbb{R}^d)$ we get*

$$T(0)f([x]) = \int f(g\cdot[x])d\mu(g).$$

Proposition 4.3.2. *Let $Y_1, Y_2, ...$ be independent identically distributed random matrices in $GL(d,\mathbb{R})$ with common distribution μ.*
Let $T(\xi)f([x]) = \mathbb{E}\left(e^{\xi \log \frac{||Y_1 x||}{||x||}} f(Y_1\cdot[x])\right)$. Then, for $||x|| = 1$ and $n \geq 1$ we get

$$T^n(\xi)f([x]) = \mathbb{E}\left(e^{\xi \log ||S_n x||} f(S_n\cdot[x])\right).$$

Proof. We prove this by induction. It is true for the case $n = 1$.
Suppose it's true for $n-1$ that is

$$T^{n-1}(\xi)f([x]) = \mathbb{E}\left(e^{\xi \log ||S_{n-1} x||} f(S_{n-1}\cdot[x])\right).$$

We prove it for n:

$$\begin{aligned}
T^n(\xi)f([x]) &= T^{n-1}(\xi)\left(T(\xi)f\right)([x]) \\
&= \mathbb{E}\left(e^{\xi\sigma(S_{n-1},[x])} T(\xi)f(S_{n-1}\cdot[x])\right) \\
&= \mathbb{E}\left(e^{\xi\sigma(S_{n-1},[x])} e^{\xi\sigma(Y_n,S_{n-1}[x])} f(S_n\cdot[x])\right) \\
&= \mathbb{E}\left(e^{\xi\sigma(S_n,[x])} f(S_n\cdot[x])\right) \\
&= \mathbb{E}\left(e^{\xi\left(\sum_{i=1}^{n}\sigma(\mathrm{X}_i)\right)} f(S_n\cdot[x])\right) \\
&= \mathbb{E}\left(e^{\xi \log ||S_n x||} f(S_n\cdot[x])\right).
\end{aligned}$$

□

Proposition 4.3.3. *The family of operators $\{T(\xi)\}$ for ξ in a neighborhood V of 0 is bounded.*

Proof. Let ξ in in a neighborhood V of 0. Let f be a complex-valued function on $\mathrm{P}(\mathbb{R}^d)$. We have,

$$\begin{aligned}
||T(\xi)f|| &\leq ||f||_\infty \sup_x \mathbb{E}\left(e^{\xi \log ||g\frac{x}{||x||}||}\right) \\
&\leq ||f||_\infty \sup_x \mathbb{E}\left(e^{\mathfrak{Re}(\xi) \log ||g\frac{x}{||x||}||}\right) \\
&\leq ||f||_\infty \sup_x \mathbb{E}\left(\frac{||gx||}{||x||}\right)^{\mathfrak{Re}(\xi)} \\
&\leq ||f||_\infty \sup_x \mathbb{E}\left(||g||^{\mathfrak{Re}(\xi)}\right) \\
&< \infty.
\end{aligned}$$

Only when $\mathfrak{Re}(\xi) < \tau$, where $\tau > 0$ and $\mathbb{E}(||g||^\tau) < \infty$ given that μ has an exponential moment. □

4.4 Density with respect to The Haar Measure

In what follows we assume that μ has a density Φ with respect to the Haar measure m, that is $\Phi = \dfrac{\mathrm{d}\mu}{\mathrm{d}m}$.

Definition 4.4.1. *We say an operator T is compact if it is a linear operator from a Banach space X to another Banach space Y, such that the image under T of any bounded subset of X is a relatively compact subset of Y (has compact closure). Such an operator would necessarily be bounded and continuous.*

Proposition 4.4.1. *Let μ be a probability measure on $GL(d, \mathbb{R})$ that has density Φ with respect to the Haar measure m. Let $T(0)$ be a bounded operator on $\mathcal{C}_0(P(\mathbb{R}^d))$ defined by*

$$T(0)f([x]) = \int f(g \cdot [x]) d\mu(g)$$

for every continuous function f on $P(\mathbb{R}^d)$. Then, $T(0)$ is a compact operator.

Proof. For simplicity we write $T(0) = T$. To prove that T is a compact operator we prove that the set $S = \{Tf; |f| \leq 1\}$ is a relatively compact subset of $\mathcal{C}_0(\mathrm{P}(\mathbb{R}^d))$. Let f be a function in $\mathcal{C}_0(\mathrm{P}(\mathbb{R}^d))$. Let $[x]$ and $[y]$ in $\mathrm{P}(\mathbb{R}^d)$. The action of $\mathcal{O}(d)$ on $\mathrm{P}(\mathbb{R}^d)$ is transitive so there exists k_1, k_2 in $\mathcal{O}(d)$ so that $[x] = k_1[e_1]$ and $[y] = k_2[e_1]$.
Without loss of generality we have, $\delta([x], [y]) \approx ||k_2 - k_1|| = ||k_2^{-1} - k_1^{-1}||$. Let $\mathcal{C}_c(\mathbb{R})$ be the set of continuous functions in $\mathbb{R}$ with compact support. Note that $\Phi \in \mathrm{L}^1(\mathbb{R})$ and $\mathcal{C}_c(\mathbb{R})$ is dense in $\mathrm{L}^1(\mathbb{R})$ then, for $\epsilon > 0$ there exists Ψ in $\mathcal{C}_c(\mathbb{R})$ such that $||\Psi - \Phi||_1 < \dfrac{\epsilon}{3}$.
Let $K = \mathrm{Supp}(\Psi)$ and $C = \max_{g \in K}\{||g||\}$. Since Ψ is uniformly continuous on $\mathrm{GL}(d, \mathbb{R})$ then for g and h in $\mathrm{GL}(d, \mathbb{R})$, there exists some $\beta > 0$ such that $||g - h|| < \beta$ implies $|\Psi(g) - \Psi(h)| < \dfrac{\epsilon}{3}$.
So for k_1, $k_2 \in \mathcal{O}(d)$. For $||k_2^{-1} - k_1^{-1}|| < \dfrac{\beta}{C}$ and for all $g \in k_1K \cap k_2K$ we have, $||gk_2^{-1} - gk_1^{-1}|| < \beta$.
Hence $\displaystyle\int |\Psi(gk_2^{-1}) - \Psi(gk_1^{-1})| dm(g) < \frac{\epsilon}{3} m(Kk_1 \cap Kk_2) < \infty$. We have S is closed and bounded. By Arzela-Ascoli , we only still need to show that this set is equicontiuous. Let k_1, $k_2 \in \mathcal{O}(d)$. For all $\epsilon > 0$, there exists $0 < \delta = \frac{\beta}{C}$; for all f in $\mathcal{C}_0(\mathrm{P}(\mathbb{R}^d))$ and $[x], [y]$ in $\mathrm{P}(\mathbb{R}^d)$ we have,

$$\begin{aligned}
\big|Tf(k_2 \cdot [e_1]) - Tf(k_1 \cdot [e_1])\big| &= \Big| \int f(gk_2 \cdot [e_1]) - f(gk_1 \cdot [e_1]) d\mu(g) \Big| \\
&= \Big| \int f(gk_2 \cdot [e_1]) \Phi(g) dm(g) - \int f(gk_1 \cdot [e_1]) \Phi(g) dm(g) \Big| \\
&= \Big| \int f(g \cdot [e_1]) \Phi(gk_2^{-1}) dm(gk_2^{-1}) - \int f(g \cdot [e_1]) \Phi(gk_1^{-1}) dm(gk_1^{-1}) \Big| \\
&= \Big| \int f(g \cdot [e_1]) \left(\Phi(gk_2^{-1}) - \Phi(gk_1^{-1})\right) dm(g) \Big| \\
&\leq \int \big|f(g \cdot [e_1])\big| \big|\Phi(gk_2^{-1}) - \Phi(gk_1^{-1})\big| dm(g) \\
&\leq \int \big|\Phi(gk_2^{-1}) - \Phi(gk_1^{-1})\big| dm(g)
\end{aligned}$$

So,

$$
\begin{aligned}
\left|Tf(k_2\cdot[e_1]) - Tf(k_1\cdot[e_1])\right| &\le \int \big|\Phi(gk_2^{-1}) - \Psi(gk_2^{-1}) + \Psi(gk_2^{-1}) - \Phi(gk_1^{-1}) \\
&\qquad\qquad + \Psi(gk_1^{-1}) - \Psi(gk_1^{-1})\big|\, dm(g) \\
&\le \int \left|\Phi(gk_2^{-1}) - \Psi(gk_2^{-1})\right| + \left|\Psi(gk_1^{-1}) - \Phi(gk_1^{-1})\right| \\
&\qquad\qquad + \left|\Psi(gk_2^{-1}) - \Psi(gk_1^{-1})\right| dm(g) \\
&\le \int \left|\Phi(gk_2^{-1}) - \Psi(gk_2^{-1})\right| dm(g) + \int \left|\Psi(gk_1^{-1}) - \Phi(gk_1^{-1})\right| dm(g) \\
&\qquad\qquad + \int \left|\Psi(gk_2^{-1}) - \Psi(gk_1^{-1})\right| dm(g) \\
&\le \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\
&= \epsilon.
\end{aligned}
$$

Therefore, T is a compact operator. □

Theorem 4.4.1 (CLT)**.** *Let μ be a probability measure on $GL(d,\mathbb{R})$ that has density Φ with respect to the Haar measure m. Let $Y_1, Y_2, ...$ be independent identically distributed random matrices in $GL(d,\mathbb{R})$ with common distribution μ. Suppose Γ_μ is strongly irreducible and contracting, and ν is the unique μ-invariant distribution on $P(\mathbb{R}^d)$. If γ represents the upper Lyapunov exponent then, $\dfrac{\log ||S_n x|| - n\gamma}{\sqrt{n}}$ converges in distribution to $\mathcal{N}(0,\sigma^2)$, with $\sigma^2 > 0$.*

Proof. We choose $\mathcal{C}_0(\mathrm{P}(\mathbb{R}^d))$ to be our Banach space and define on it

$$T(\xi)f([x]) = \mathbb{E}\left(e^{\xi \log \frac{||Y_1 x||}{||x||}} f(Y_1\cdot[x])\right).$$

Then, by Proposition 4.3.2 and for $||x|| = 1,$ we get

$$T^n(\xi)f([x]) = \mathbb{E}\left(e^{\xi \log ||S_n x||} f(S_n\cdot[x])\right).$$

Notice that $T(0)f([x]) = \displaystyle\int f(g\cdot[x])\mathrm{d}\mu(g)$ for $g \in \mathrm{GL}(d,\mathbb{R})$.
Let $N(0)f([x]) = \displaystyle\int f([y])\mathrm{d}\nu([y]) = \nu(f)\mathbb{1}$. It is immediate that $N^2(0) = N(0)$, and $T(0)N(0) = N(0)T(0) = N(0)$ since $\mu * \nu = \nu$.
$N(0)$ is a constant operator and $||T(0)f([x])||_\infty \le ||f||_\infty$ for all f continuous on $\mathrm{P}(\mathbb{R}^d)$, so $T(0)$ and $N(0)$ are bounded operators on $\mathcal{C}_0^\infty$. By Theorem 3.0.2 we have that

$$\sup_{[x]\in\mathrm{P}(\mathbb{R}^d)} \left|\mathbb{E}\left(f(S_n\cdot[x])\right) - \int f\mathrm{d}\nu\right| \underset{n\to\infty}{\longrightarrow} 0$$

Using operators we write

$$||T^n(0) - N(0)||_\infty \underset{n\to\infty}{\longrightarrow} 0.$$

Let $Q^n(0) = T^n(0) - N(0)$. So, we have $||Q^n(0)||_\infty \underset{n\to\infty}{\longrightarrow} 0$. This means that for every continuous function f and every $[x] \in \mathrm{P}(\mathbb{R}^d)$ we have,

$$Q^n(0)f([x]) \underset{n\to\infty}{\longrightarrow} 0$$

Let $\lambda \in \sigma(Q(0))$ be the eigenvalue of $Q(0)$ associated with f in $\mathcal{C}_0(\mathrm{P}(\mathbb{R}^d))$.
Then, $\lambda^n f([x]) \underset{n\to\infty}{\longrightarrow} 0$. This implies that $|\lambda| < 1$ for all eigenvalues λ in $\sigma(Q(0))$.
Working on $\mathcal{C}_0(\mathrm{P}(\mathbb{R}^d))$ having density with respect to the Haar measure ensure that $T(0)$ is a compact

operator. Thus, $Q(0)$ is also a compact operator; Hence, $\sigma(Q(0))$ only contains eigenvalues.
Thus, $\rho(Q(0)) := \sup\{|\lambda|, \lambda \in \sigma(Q(0))\}$ would be less that one as well. Then, by Gelfand's formula, we deduce that $\rho(Q(0)) = \lim_{n\to\infty} ||Q^n(0)||_\infty^{1/n} < 1$.
So, by Theorem 4.2.1 and for $\epsilon > 0$ and $\xi \in \mathbb{C}$ satisfying $|\xi| < \epsilon$ we have,

$$T^n(\xi) = \lambda^n(\xi)N(\xi) + Q^n(\xi).$$

But

$$\begin{aligned} T^n(\xi)\mathbb{1}([x]) &= \mathbb{E}\left(e^{\xi \log ||S_n x||}\right) \\ &= \lambda^n(\xi)N(\xi)\mathbb{1}([x]) + Q^n(\xi)\mathbb{1}([x]). \end{aligned}$$

Then,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\xi}T^n(\xi)\mathbb{1}([x])\big|_{\xi=0} &= \mathbb{E}\left(\log ||S_n x||\right) \\ &= n\lambda'(0)N(0)\mathbb{1}([x]) + \frac{\mathrm{d}}{\mathrm{d}\xi}N(\xi)\mathbb{1}([x])\big|_{\xi=0} + \frac{\mathrm{d}}{\mathrm{d}\xi}Q^n(\xi)\mathbb{1}([x])\big|_{\xi=0} \end{aligned}$$

Knowing that $N(0)\mathbb{1}([x]) = \mathbb{1}([x])$ we get,

$$\begin{aligned} \lim_{n\to\infty}\frac{1}{n}\mathbb{E}\left(\log ||S_n x||\right) &= \lambda'(0) + \lim_{n\to\infty}\frac{1}{n}\frac{\mathrm{d}}{\mathrm{d}\xi}N(\xi)\mathbb{1}([x])\big|_{\xi=0} + \lim_{n\to\infty}\frac{1}{n}\frac{\mathrm{d}}{\mathrm{d}\xi}Q^n(\xi)\mathbb{1}([x])\big|_{\xi=0} \\ &= \lambda'(0) \end{aligned}$$

But $\lim_{n\to\infty}\frac{1}{n}\mathbb{E}\left(\log ||S_n x||\right) = \gamma$. Therefore $\lambda'(0) = \gamma$.

We also know that

$$\begin{aligned} \frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\left(e^{-n\xi\gamma}T^n(\xi)\mathbb{1}([x])\right)\big|_{\xi=0} &= (n\lambda''(0) - n\gamma^2)N(0)\mathbb{1}([x]) + n^2\gamma^2(0)Q^n(0)\mathbb{1}([x]) \\ &+ \frac{\mathrm{d}^2}{\mathrm{d}\xi^2}Q^n(\xi)\mathbb{1}([x])\big|_{\xi=0} - 2n\gamma\frac{\mathrm{d}}{\mathrm{d}\xi}Q^n(\xi)\mathbb{1}([x])\big|_{\xi=0} + \frac{\mathrm{d}^2}{\mathrm{d}\xi^2}N(\xi)\mathbb{1}([x])\big|_{\xi=0}. \end{aligned}$$

But

$$\begin{aligned} \frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\left(e^{-n\xi\gamma}T^n(\xi)\mathbb{1}([x])\right)\big|_{\xi=0} &= \frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\mathbb{E}\left(e^{\xi\log ||S_n x|| - n\xi\gamma}\right)\big|_{\xi=0} \\ &= \mathbb{E}\left((\log ||S_n x|| - n\gamma)^2\right). \end{aligned}$$

Then, $\lim_{n\to\infty}\frac{1}{n}\mathbb{E}\left((\log ||S_n x|| - n\gamma)^2\right) = \lambda''(0) - \gamma^2$.

Thus, there exists a $\sigma^2 \geq 0$ so that $\lambda''(0) - \gamma^2 = \sigma^2$.

Hence, using Theorem 4.2.1, and for $\xi = \frac{it}{\sqrt{n}}$ we get,

$$\mathbb{E}\left(e^{\frac{it}{\sqrt{n}}(\log ||S_n x|| - n\gamma)}\right) \underset{n\to\infty}{\longrightarrow} e^{\frac{-t^2}{2}\sigma^2}.$$

Therefore, $\frac{\log ||S_n x|| - n\gamma}{\sqrt{n}}$ converges in distribution to $N(0, \sigma^2)$, with $\sigma^2 \geq 0$.

Note that by Corollary 3.0.1 we know that $\dfrac{||S_n x||}{||S_n||}$ is bounded for every non-zero x. And since almost-surely $\gamma = \lim_{n\to\infty} \dfrac{1}{n} \log ||S_n||$, then we can talk about a central limit theorem for $\log ||S_n||$ only.

Therefore, $\dfrac{\log ||S_n|| - n\gamma}{\sqrt{n}}$ converges in distribution to $\mathcal{N}(0, \sigma^2)$, with $\sigma^2 \geq 0$. One can further prove that σ^2 is actually strictly positive ($\sigma^2 > 0$). For this See [L2].

□

Chapter 5

Central Limit theorem of $\log ||S_n||$ when the measure has an exponential moment

In this section our measure μ is not assumed to have density with respect to the Haar measure, only an exponential moment is enough. The idea is to find a suitable Banach space so that the operators we defined previously satisfy the properties of Theorem 4.2.1. But first we present some consequences of the Ergodic theorem that would help establish the fact that the two upper lyapunov exponents satisfy $\gamma_1 > \gamma_2$.

5.1 Consequences of the Ergodic Theorem

Ergodic Theory is the study of the long term average behavior of systems evolving in time. The collection of all states of the system form a space X and the evolution is represented by a transformation T. Here, and in our case, the space is a probability space $(X, \mathcal{F}, \mu)$, and the evolution is described by a measurable transformation $T : X \longmapsto X$, where T is measure preserving.

Definition 5.1.1. *Let $(X, \mathcal{F}, \mu)$ be a probability space and $T : X \longmapsto X$ measurable. We say that T is measure preserving with respect to μ if for all $A \in \mathcal{F}$ we have, $\mu(T^{-1}A) = \mu(A)$. Same as saying μ is T-invariant.*

Theorem 5.1.1 (The Ergodic Theorem)**.** *Let $(X, \mathcal{F}, \mu)$ be a probability space and let $T : X \longmapsto X$ be a measure preserving transformation. Then, for any f in $L^1(\mu)$,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i(x)) = f^*$$

exists almost surely. Moreover, it is T-invariant and $\int_X f\, d\mu = \int_X f^*\, d\mu$.

Proposition 5.1.1. *Let $(E, \mathcal{F}, m)$ be a probability space and $\theta : E \longmapsto E$ be a measure preserving transformation.*

If $f : E \longmapsto \mathbb{R}$ is such that $\displaystyle\int f^+ \, dm < \infty$ and $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} f\,\theta^i = +\infty$ almost everywhere then,

$$f \in L^1(dm) \;\; \textit{and} \;\; \int f \, dm > 0.$$

Proof. By the Ergodic Theorem we have $\displaystyle\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f(\theta^i(x)) = f^*$.

Since $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} f\,\theta^i = +\infty$ almost everywhere then, $f^* \geq 0$ and f is integrable. We have $\displaystyle\int_X f \mathrm{d}\mu = \int_X f^* \mathrm{d}\mu$.
Assume $\displaystyle\int_X f \mathrm{d}\mu = 0 = \int_X f^* \mathrm{d}\mu$. Then, $f^* = 0$ a.e. Thus $\displaystyle\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f(\theta^i(x)) = 0$ a.e. This means that for m-almost all x in E and for all $\delta > 0$, there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have, $|S_n(x)| \leq n\delta$, where $\displaystyle S_n = \sum_{i=1}^{n} f\theta^i$.
Set $I_\epsilon(t) = [t-\epsilon, t+\epsilon]$ and $R_n^\epsilon(x) = \lambda\left(\cup_{i=1}^n I_\epsilon(S_i(x))\right)$, where λ denotes the lebesgue measure on $\mathbb{R}$.
Let $n \geq n_0$. We have,

$$\begin{aligned}
R_n^\epsilon(x) - R_{n_0}^\epsilon(x) &= \lambda\left(\cup_{i=1}^n I_\epsilon(S_i(x))\right) - \lambda\left(\cup_{i=1}^{n_0} I_\epsilon(S_i(x))\right) \\
&\leq \lambda\left(\cup_{i=n_0}^n I_\epsilon(S_i(x))\right) \\
&= \lambda\left(\cup_{i=n_0}^n [S_i(x)-\epsilon, S_i(x)+\epsilon]\right)
\end{aligned}$$

But,

$$\begin{aligned}
[S_i(x)-\epsilon, S_i(x)+\epsilon] &\subset [-i\delta-\epsilon, i\delta+\epsilon] \\
\Longrightarrow \cup_{i=n_0}^n [S_i(x)-\epsilon, S_i(x)+\epsilon] &\subset \cup_{i=n_0}^n [-i\delta-\epsilon, i\delta+\epsilon] \\
&= [-n\delta-\epsilon, n\delta+\epsilon]
\end{aligned}$$

So, $R_n^\epsilon(x) - R_{n_0}^\epsilon(x) \leq \lambda\left([-n\delta-\epsilon, n\delta+\epsilon]\right) = 2n\delta + 2\epsilon$. Then,

$$\frac{1}{n} R_n^\epsilon(x) \leq \frac{1}{n} R_{n_0}^\epsilon(x) + 2\delta + \frac{2\epsilon}{n}.$$

Thus,

$$\limsup_{n\to\infty} \frac{1}{n} R_n^\epsilon(x) \leq 2\delta$$

For all $\delta > 0$. Therefore, $\displaystyle\lim_{n\to\infty} \frac{1}{n} R_n^\epsilon(x) = 0$ for m-almost all x in E.
Hence, we have that $\displaystyle\lim_{n\to\infty} \frac{1}{n} \int R_n^\epsilon \mathrm{d}m = 0$, that is $\displaystyle\lim_{n\to\infty} \frac{1}{n} \mathbb{E}(R_n^\epsilon) = 0$.

Notice that $S_n\theta = S_{n+1} - S_1$. So,

$$\begin{aligned}
R_{n+1}^\epsilon - R_n^\epsilon\theta &= \lambda\left(\cup_{i=1}^{n+1} I_\epsilon(S_i)\right) - \lambda\left(\cup_{i=1}^n I_\epsilon(S_{i+1} - S_1)\right) \\
&= \lambda\left(\cup_{i=1}^{n+1} I_\epsilon(S_i)\right) - \lambda\left(\cup_{i=2}^{n+1} I_\epsilon(S_i)\right) \\
&\geq 2\epsilon \mathbb{1}_{\{|S_i - S_1| > 2\epsilon\,,\, i=2,\ldots,n+1\}}
\end{aligned}$$

Then, $\mathbb{E}(R_{n+1}^\epsilon) - \mathbb{E}(R_n^\epsilon\theta) \geq 2\epsilon\, m\left(\{x\,,\, |S_i(x) - S_1(x)| > 2\epsilon,\, i = 2, ..., n+1\}\right)$.
But θ is m-preserving so, $\mathbb{E}(R_{n+1}^\epsilon) - \mathbb{E}(R_n^\epsilon) \geq 2\epsilon\, m\left(\{x\,,\, |S_i(x)| > 2\epsilon,\, i = 1, ..., n\}\right)$. Hence,

$$\begin{aligned}
\mathbb{E}(R_2^\epsilon) - \mathbb{E}(R_1^\epsilon) &\geq 2\epsilon\, m\left(\{x\,,\, |S_1(x)| > 2\epsilon\}\right) \\
\mathbb{E}(R_3^\epsilon) - \mathbb{E}(R_2^\epsilon) &\geq 2\epsilon\, m\left(\{x\,,\, |S_1(x)| > 2\epsilon \text{ and } |S_2(x)| > 2\epsilon\}\right) \\
&\vdots \\
\mathbb{E}(R_{n+1}^\epsilon) - \mathbb{E}(R_n^\epsilon) &\geq 2\epsilon\, m\left(\{x\,,\, |S_1(x)| > 2\epsilon \text{ and}\,, ..., |S_n(x)| > 2\epsilon\}\right)
\end{aligned}$$

Now, let $A_k = \{x : |S_i(x)| > 2\epsilon, \forall i = 1, ..., k\}$. Then we have,

$$\begin{aligned}
\mathbb{E}(R_2^\epsilon) - \mathbb{E}(R_1^\epsilon) &\geq 2\epsilon\, m(A_1) \\
\mathbb{E}(R_3^\epsilon) - \mathbb{E}(R_2^\epsilon) &\geq 2\epsilon\, m(A_2) \\
&\vdots \\
\mathbb{E}(R_{n+1}^\epsilon) - \mathbb{E}(R_n^\epsilon) &\geq 2\epsilon\, m(A_n)
\end{aligned}$$

Notice that $\{A_n\}_{n\geq 1}$ is decreasing where $A_{n+1} \subset A_n$ for all $n \geq 1$. So,

$$m\left(\cap_{n\geq 1} A_n\right) = \lim_{n\to\infty} m(A_n).$$

For all $n \geq 1$ we have,

$$\mathbb{E}(R_{n+1}^\epsilon) - \mathbb{E}(R_1^\epsilon) \geq 2n\epsilon\, m(A_n).$$

Thus,

$$\frac{\mathbb{E}(R_{n+1}^\epsilon)}{n} - \frac{\mathbb{E}(R_1^\epsilon)}{n} \geq 2\epsilon\, m(A_n).$$

Hence,

$$\begin{aligned}
\lim_{n\to\infty} \frac{\mathbb{E}(R_{n+1}^\epsilon)}{n} &\geq 2\epsilon\, m\left(\cap_{n\geq 1} A_n\right) \\
&= 2\epsilon m\left(\{x, |S_i(x)| > 2\epsilon, \forall\, i \geq 1\}\right).
\end{aligned}$$

But $\lim_{n\to\infty} \dfrac{\mathbb{E}(R_n^\epsilon)}{n} = 0$. So, $m\left(\{x, |S_i(x)| > 2\epsilon, \forall i \geq 1\}\right) = 0$.

Therefore, for any integer $p > 0$ and $\epsilon > 0$, we have,

$$m\left(\{x, |S_{i+p}(x) - S_p(x)| > \epsilon, \forall i \geq 1\}\right) = 0$$

Let $\epsilon > 0, \epsilon \in \mathbb{Q}$. Let $A_\epsilon = \{x : \exists\, i \geq 1\,, \forall\, p \in \mathbb{N}^*\,, |S_{i+p}(x) - S_p(x)| \leq \epsilon\}$.
Then, $m(A_\epsilon) = 1$ for all $\epsilon > 0, \epsilon \in \mathbb{Q}$. Hence, $m(\cap A_\epsilon) = 1$. This shows that for m-almost every x in E and for all $\epsilon > 0$ with $\epsilon \in \mathbb{Q}$, there exists $i = i(\epsilon, \omega) \geq 1$, so that for all $p \in \mathbb{N}^*$ we have,

$$|S_{i+p}(x) - S_i(x)| \leq \epsilon.$$

This is equivalent to saying that $\{S_n\}_{n\geq 1}$ is Cauchy in $\mathbb{R}$. So, $\{S_n\}_{n\geq 1}$ is convergent in $\mathbb{R}$. A contradiction with the fact that $\lim_{n\to\infty} S_n = +\infty$ a.e.
Therefore, $\displaystyle\int f \mathrm{d}m > 0$.

□

Corollary 5.1.1. *Let G be a topological group acting on some space B. Let σ be an additive cocycle on $G \times B$. Let $\{Y_n\}_{n\geq 1}$ be a sequence of independent random elements of G with common distribution μ. Suppose that ν is a μ-invariant distribution on B such that*

(i) $\displaystyle\int\int \sigma^+(g, x) d\mu(g) d\nu(x) < +\infty.$

(ii) For $\mathbb{P} \otimes \nu$-almost all (ω, x) we have, $\lim_{n\to\infty} \sigma(Y_n(\omega)...Y_1(\omega), x) = +\infty.$

Then,

$$\sigma \in L^1(\mathbb{P} \otimes \nu) \;\; \textit{and} \;\; \int\int \sigma(g, x) d\mu(g) d\nu(x) > 0.$$

Proof. Let $\Omega = \{\omega = \{Y_n\}_{n\geq 1}\,;\, Y_n \in G\}$ and $\mathcal{A}^{\otimes\mathbb{N}}$ be the Borel σ-algebra of Ω. Denote by $\mathbb{P}$ the probability measure for which the Y_i's are independent of common distribution μ. Let $\mathcal{A}'$ be the Borel σ-algebra of B.
Define $\theta : \Omega \times B \longmapsto \Omega \times B$ given by

$$\theta(\{Y_n\}_{n\geq 1}, x) = (\{Y_{n+1}\}_{n\geq 1}, Y_1 \cdot x)$$

Consider the following dynamical system $\left(\Omega \times B,\, \mathcal{A}^{\otimes\mathbb{N}} \times \mathcal{A}'\,, \mathbb{P} \otimes \nu\,, \theta\right)$. We show that θ is measure preserving:
Let $A_0 \in \mathcal{A}'$ and $A_1, A_2, ... \in \mathcal{A}^{\otimes\mathbb{N}}$, and let $A = (A_1 \times A_2 \times ...) \times A_0$.
Then,

$$\begin{aligned}
(\mathbb{P} \otimes \nu)(\theta^{-1}A) &= (\mathbb{P} \otimes \nu)\left(\{(\omega, x) \in \Omega \times B\,;\ \theta(\omega, x) \in A\}\right) \\
&= (\mathbb{P} \otimes \nu)\left(\{(\omega, x) \in \Omega \times B\,;\ (\{Y_{n+1}\}_{n\geq 1}, Y_1 \cdot x) \in A_1 \times ... \times A_0)\}\right. \\
&= (\mathbb{P} \otimes \nu)\left(\{(\omega, x) \in \Omega \times B\,;\ Y_2 \in A_1, ..., Y_{n+1} \in A_n, ..., Y_1 \cdot x \in A_0\}\right)
\end{aligned}$$

But ν is μ-invariant so,

$$\begin{aligned}
(\mathbb{P} \otimes \nu)(\theta^{-1}A) &= (\mathbb{P} \otimes \nu)\left(\{(\omega, x) \in \Omega \times B\,;\ Y_1 \in A_1, ..., Y_n \in A_n,\, x \in A_0\}\right) \\
&= (\mathbb{P} \otimes \nu)(A).
\end{aligned}$$

Thus θ preserves $\mathbb{P} \otimes \nu$.
Let $p \in \mathbb{N}$. Notice that since $\theta(\{Y_n\}, x) = (\{Y_{n+1}\}, Y_1 \cdot x)$. Then, $\theta^2(\{Y_n\}, x) = (\{Y_{n+2}\}, Y_2Y_1 \cdot x)$. So we get,

$$\theta^{p-1}(\{Y_n\}, x) = (\{Y_{n+p-1}\}, Y_{p-1}...Y_1 \cdot x).$$

Define $f : \Omega \times B \longmapsto \mathbb{R}$ given by $f(\omega, x) = \sigma(Y_1, x)$, where σ is an additive cocycle. We have,

$$\begin{aligned}
\sigma(Y_n...Y_1, x) &= \sigma(Y_n, Y_{n-1}...Y_1 \cdot x) + \sigma(Y_{n-1}...Y_1, x) \\
&= \sum_{p=1}^{n} \sigma(Y_p, Y_{p-1}...Y_1 \cdot x) \\
&= \sum_{p=1}^{n} f(\theta^{p-1}(\omega, x)) \\
&= \sum_{p=0}^{n-1} f(\theta^{p}(\omega, x)).
\end{aligned}$$

Then, $\displaystyle\lim_{n\to\infty} \sigma(Y_n(\omega)...Y_1(\omega), x) = \lim_{n\to\infty} \sum_{p=0}^{n-1} f(\theta^p(\omega, x)) = +\infty$ for $\mathbb{P} \otimes \nu$-almost all (ω, x) in $\Omega \times B$.
Thus, by Proposition 5.1.1 we get that $\sigma \in L^1(\mathbb{P} \otimes \nu)$ and $\displaystyle\int\int \sigma(g, x)\mathrm{d}\mu(g)\mathrm{d}\nu(x) > 0.$

□

5.2 Comparison of the Top Lyapunov Exponents

Definition 5.2.1. *Let $Y_1, Y_2, ...$ be independent identically distributed random matrices in $GL(d, \mathbb{R})$ with $\mathbb{E}(\log^+ ||Y_1||) < +\infty$. Inductively, we define the lyapunov exponents $\gamma_1, ..., \gamma_d$ associated with*

$\{Y_n, n \geq 1\}$ by $\gamma = \gamma_1$ and for $p \geq 2$, we write $\displaystyle\sum_{i=1}^{p} \gamma_i = \lim_{n\to\infty} \frac{1}{n}\mathbb{E}(\log || \wedge^p Y_n...Y_1||)$ which is equal to $\displaystyle\lim_{n\to\infty} \frac{1}{n}(\log || \wedge^p Y_n...Y_1||)$ a.s.

If for some p we have $\displaystyle\sum_{i=1}^{p} \gamma_i = -\infty$ then we put $\gamma_p = \gamma_{p+1} = ... = -\infty$.

Proposition 5.2.1. *If $a_1(n) \geq ... \geq a_d(n) > 0$ are the square roots of the eigenvalues of $S_n^* S_n$, then almost-surely,*

$$\gamma_p = \lim_{n\to\infty} \frac{1}{n} \mathbb{E}\left(\log a_p(n)\right) = \lim_{n\to\infty} \frac{1}{n} \log a_p(n)$$

Proof. We distinguish two cases:

Case 1: If $\gamma_1 + ... + \gamma_{p-1} = -\infty$. We have $a_1(n) \cdot ... \cdot a_{p-1}(n) \geq a_p^{p-1}(n)$. Using Proposition 2.5.2 we get,

$$\begin{aligned}
(p-1) \lim_{n\to\infty} \frac{1}{n} \log a_p(n) &\leq \lim_{n\to\infty} \frac{1}{n} \log\left(a_1(n) \cdot ... \cdot a_{p-1}(n)\right) \\
&= \lim_{n\to\infty} \frac{1}{n} \log || \wedge^p Y_n...Y_1|| \\
&= \sum_{i=1}^{p-1} \gamma_i \\
&= -\infty
\end{aligned}$$

Therefore, $\displaystyle\lim_{n\to\infty} \frac{1}{n} \log a_p(n) = -\infty = \gamma_p = \lim_{n\to\infty} \frac{1}{n} \mathbb{E}\left(\log a_p(n)\right)$ a.s.

Case 2: If $\gamma_1 + ... + \gamma_{p-1} \neq -\infty$. Write

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n} \log a_p(n) &= \lim_{n\to\infty} \frac{1}{n} \log \frac{a_1(n) \cdot ... \cdot a_p(n)}{a_1(n) \cdot ... \cdot a_{p-1}(n)} \\
&= \lim_{n\to\infty} \frac{1}{n} \log || \wedge^p Y_n...Y_1|| - \lim_{n\to\infty} \frac{1}{n} \log || \wedge^{p-1} Y_n...Y_1|| \\
&= \sum_{i=1}^{p} \gamma_i - \sum_{i=1}^{p-1} \gamma_i \\
&= \gamma_p.
\end{aligned}$$

□

Lemma 5.2.1. *Consider a sequence $\{Y_n\}_{n\geq 1}$ of independent and identically distributed random matrices in $GL(d, \mathbb{R})$ with common distribution μ. We suppose that $\mathbb{E}(\log^+ ||Y_1||) < \infty$ and Γ_μ is strongly irreducible. If ν is a μ-invariant distribution on $P(\mathbb{R}^d)$ then,*

$$\gamma = \int \int \log \frac{||gx||}{||x||} \, d\mu(g) d\nu([x]).$$

where γ represents the Lyapunov exponent.

Proof. Let ν be a μ-invariant distribution on $\mathrm{P}(\mathbb{R}^d)$.

Let $\sigma(g, [x]) = \log \dfrac{||gx||}{||x||}$ define an additive cocycle. Then, for $||x|| = 1$ we get $\log ||S_n x|| = \log ||Y_n...Y_1 x|| = \sigma(S_n, [x])$, and $\displaystyle\lim_{n\to\infty} \frac{1}{n}\sigma(S_n, [x]) = \lim_{n\to\infty} \frac{1}{n} \log ||S_n x|| = \gamma$ almost surely. Define

$$\theta : \mathrm{Gl}\,(d, \mathbb{R}) \times \mathrm{P}(\mathbb{R}^d) \longmapsto \mathrm{Gl}\,(d, \mathbb{R}) \times \mathrm{P}(\mathbb{R}^d)$$

Where $\theta(\omega, [x]) = \theta(\{Y_n\}_{n\geq 1}, [x]) = (\{Y_{n+1}\}_{n\geq 1}, Y_1 \cdot [x])$. As seen before this θ preserves $\mathbb{P} \otimes \nu$. Define

$$f : \mathrm{Gl}(d, \mathbb{R}) \times \mathrm{P}(\mathbb{R}^d) \longmapsto \mathbb{R}$$

Where $f(\{Y_n\}_{n\geq 1},[x]) = \sigma(Y_1,[x])$. Then, by Corollary 5.1.1 we get,

$$\lim_{n\to\infty}\frac{1}{n}\sigma(S_n,[x]) = \lim_{n\to\infty}\frac{1}{n}\sum_{p=0}^{n-1} f\theta^p(w,[x]) = f^*, \text{ for } \mathbb{P}\otimes\nu \text{ almost all } (\omega,[x]).$$

By the Ergodic Theorem we get,

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\mathbb{E}(\log||S_n x||) &= \gamma \\ &= \mathbb{E}(f^*) \\ &= \mathbb{E}(f) \\ &= \int\int \sigma(g,[x])\mathrm{d}\mu(g)\mathrm{d}\nu([x]) \\ &= \int\int \log\frac{||gx||}{||x||}\mathrm{d}\mu(g)\mathrm{d}\nu([x]).\end{aligned}$$

□

Theorem 5.2.1. *Let μ be a probability measure on $GL(d,\mathbb{R})$. Suppose $\int \log^+||Y_1||d\mu$ is finite. If Γ_μ is strongly irreducible and contracting then $\gamma_1 > \gamma_2$.*

Proof. Suppose that Γ_μ is strongly irreducible and contracting. Consider the following dynamical system $(\Gamma_\mu \times B, \mathcal{F}, m, \theta)$ where,

•$B = B_1 \times B_2 =$P$(\mathbb{R}^d)\times$P$(\wedge^2\mathbb{R}^d)$.

• $\mathcal{F}$ is the σ-algebra of $\Gamma_\mu \times B$.

• Take $m = \mathbb{P}\otimes\nu$, where $\nu = \nu_1\otimes\nu_2$, with ν_1 being the unique μ-invariant distribution on B_1 such that

$$\int\int \log\frac{||gx||}{||x||}\mathrm{d}\mu(g)\mathrm{d}\nu_1([x]) = \gamma_1.$$

By Furstenberg and Kifer we know that there exists some ν_2, a μ-invariant distribution on B_2 such that

$$\int\int \log\frac{||\wedge^2 ga||}{||a||}\mathrm{d}\mu(g)\mathrm{d}\nu_2([a]) = \gamma_1+\gamma_2.$$

Note that $\nu_1\otimes\nu_2$ is a μ-invariant probability measure on B.

• $\theta : \mathrm{Gl}(d,\mathbb{R})\times B \longmapsto \mathrm{Gl}(d,\mathbb{R})\times B$, given by

$$\theta(\omega,([x],[a])) = \theta\left(\{Y_n\},([x],[a])\right) = \left(\{Y_{n+1}\},(Y_1[x],Y_1[a])\right).$$

Note that $\mathrm{GL}(d,\mathbb{R})$ acts on B via the following action $g\cdot([x],[a]) = (g[x],\wedge^2 g[a])$.

Let σ be an additive cocycle given by

$$\sigma(g,([x],[a])) = \log\frac{||g\frac{x}{||x||}||^2}{||\wedge^2 g\frac{a}{||a||}||}.$$

We have,

$$\begin{aligned}\sigma(S_n,([x],[a])) &= \log\frac{||S_n\frac{x}{||x||}||^2}{||\wedge^2 S_n\frac{a}{||a||}||} \\ &\geq \log\frac{||S_n\frac{x}{||x||}||^2}{||\wedge^2 S_n||} \\ &= \log\left(\frac{||S_n\frac{x}{||x||}||}{||S_n||}\right)^2 + \log\frac{||S_n||^2}{||\wedge^2 S_n||}.\end{aligned}$$

From Corollary 3.0.1 we know that there exists a constant c such that $\dfrac{||S_n x||}{||S_n||} \geq c > 0$. Then, $\log\left(\dfrac{||S_n \frac{x}{||x||}||}{||S_n||}\right)^2 \geq 2\log c$.

We also know that $\lim_{n\to\infty} \dfrac{a_2}{||S_n||} = \lim_{n\to\infty} \dfrac{|| \wedge^2 S_n||}{||S_n||^2} = 0$ almost-surely.

Hence, $\lim_{n\to\infty} \dfrac{||S_n||^2}{|| \wedge^2 S_n||} = +\infty$ almost-surely. Taking these into consideration we get,

$$\lim_{n\to\infty} \sigma(S_n, ([x],[a])) = +\infty.$$

Now define

$$f : \mathrm{Gl}(d,\mathbb{R}) \times B \longmapsto \mathbb{R}$$

By $f(\omega, ([x],[a])) = \sigma(Y_1, ([x],[a]))$. Notice that $\sigma(S_n, ([x],[a])) = \sum_{p=0}^{n-1} f\theta^p(\omega, ([x],[a]))$. Thus,

$$\lim_{n\to\infty} \sigma(S_n, ([x],[a])) = \lim_{n\to\infty} \sum_{p=0}^{n-1} f\theta^p(\omega, ([x],[a])) = +\infty.$$

Then, by Corollary 5.1.1, we get $\int_{\mathrm{Gl}(d,\mathbb{R})} \int_B \sigma(g, ([x],[a]))\mathrm{d}\mu(g)\mathrm{d}\nu([x],[a]) > 0$.

Hence, $\int\int \sigma(g, ([x],[a]))\mathrm{d}\mu(g)\mathrm{d}\nu([x],[a])$

$$\begin{aligned}
&= 2\int\int \log ||g\frac{x}{||x||}||\mathrm{d}\mu(g)\mathrm{d}\nu_1([x]) - \int\int \log || \wedge^2 g\frac{a}{||a||}||\mathrm{d}\mu(g)\mathrm{d}\nu_2([a]) \\
&= 2\gamma_1 - (\gamma_1 + \gamma_2) \\
&= \gamma_1 - \gamma_2 \\
&> 0
\end{aligned}$$

Therefore, $\gamma_1 > \gamma_2$.

□

5.3 The Space of Holder Continuous Functions $\mathbb{L}(\alpha)$

Now we can define the Banach space we work on. Le Page suggested it to be a space of Holder Continuous Functions, and called it $\mathbb{L}(\alpha)$. It requires finding the suitable α that would make the work flow smoothly. This search highly depends on the moment of μ.

Lemma 5.3.1. *Let $g \in Gl(d,\mathbb{R})$ and $w \in \wedge^p\mathbb{R}^d$ with p being an integer satisfying $1 \leq p \leq d$. Then,*

$$\begin{aligned}
&\big|\log || \wedge^p g||\big| \leq p\,N(g) \\
\textit{and} \qquad &\big|\log || \wedge^p gw||\big| \leq p\,N(g)\,||w||.
\end{aligned}$$

Proposition 5.3.1. *Suppose $\{Y_n, n \geq 1\}$ is a sequence of independent identically distributed random matrices in $GL(d,\mathbb{R})$ with common distribution μ. If μ has an exponential moment and Γ_μ is strongly irreducible and contracting then,*

(i) For $[x],[y] \in P(\mathbb{R}^d)$, we have $\lim_{n\to\infty} \dfrac{1}{n} \log \dfrac{\delta(S_n\cdot[x], S_n\cdot[y])}{\delta([x],[y])} < 0 \qquad a.s$

(ii) $\overline{\lim_{n\to\infty}} \sup_{[x],[y]} \dfrac{1}{n}\mathbb{E}\left(\log \dfrac{\delta(S_n\cdot[x], S_n\cdot[y])}{\delta([x],[y])}\right) < 0.$

Proof. (i) We have,

$$\frac{\delta(S_n\cdot[x],S_n\cdot[y])}{\delta([x],[y])} = \frac{||S_nx\wedge S_ny||}{||S_nx||\,||S_ny||}.\frac{||x||\,||y||}{||x\wedge y||}$$
$$\leq ||\wedge^2 S_n||.\frac{||x||}{||S_nx||}.\frac{||y||}{||S_ny||}.$$

Using Corollary 3.0.1 we get,

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\log\frac{\delta(S_n\cdot[x],S_n\cdot[y])}{\delta([x],[y])} &\leq \lim_{n\to\infty}\frac{1}{n}\log||\wedge^2 S_n|| + \lim_{n\to\infty}\frac{1}{n}\log\frac{||x||}{||S_nx||} + \lim_{n\to\infty}\frac{1}{n}\log\frac{||y||}{||S_ny||}\\ &\leq \lim_{n\to\infty}\frac{1}{n}\log||\wedge^2 S_n|| - \lim_{n\to\infty}\frac{1}{n}\log||S_n\frac{x}{||x||}|| - \lim_{n\to\infty}\frac{1}{n}\log||S_n\frac{y}{||y||}||\\ &\leq \gamma_1+\gamma_2-\gamma_1-\gamma_1 \qquad \text{a.s}\\ &= \gamma_2-\gamma_1.\end{aligned}$$

But Γ_μ is strongly irreducible and contracting then by Theorem 5.2.1 we get $\gamma_2-\gamma_1<0$. Therefore, $\lim_{n\to\infty}\frac{1}{n}\log\frac{\delta(S_n[x],S_n[y])}{\delta([x],[y])}<0$ a.s.

(ii) We have, $\overline{\lim_{n\to\infty}}\sup_{[x],[y]}\frac{1}{n}\mathbb{E}\left(\log\frac{\delta(S_n\cdot[x],S_n\cdot[y])}{\delta([x],[y])}\right)$

$$\begin{aligned}&\leq \overline{\lim_{n\to\infty}}\frac{1}{n}\sup_{[x],[y]}\mathbb{E}\left(\log||\wedge^2 S_n||+\log\frac{||x||}{||S_nx||}+\log\frac{||y||}{||S_ny||}\right)\\ &= \gamma_1+\gamma_2-2\gamma_1\\ &< 0.\end{aligned}$$

□

Lemma 5.3.2. *Let T be a topological semi-group acting on a set X and σ be an additive cocycle on $T \times X$. Consider a probability measure μ on T such that for some $g\in T$, $x\in X$ and positive integer p, one has $\sup_{x\in X}\int\sigma(g,x)d\mu^p(g)<0$. Then for $\alpha>0$ small enough we get,*

$$\lim_{n\to\infty}\frac{1}{n}\log\{\sup_{x\in X}\int e^{\alpha\sigma(g,x)}d\mu^n(g)\}<0.$$

Proposition 5.3.2. *Let μ be a probability measure on $GL(d,\mathbb{R})$ such that Γ_μ is strongly irreducible and contracting. If μ has an exponential moment then there exists $\alpha_0>0$ such that for each $\alpha\in(0,\alpha_0]$ one has*

$$\lim_{n\to\infty}\Big[\sup_{[x],[y]\in P(\mathbb{R}^d)}\int\left\{\frac{\delta(g\cdot[x],g\cdot[y])}{\delta([x],[y])}\right\}^\alpha d\mu^n(g)\Big]^{1/n}<1.$$

Proof. Let X=$\{([x],[y]),[x],[y]\in \mathrm{P}(\mathbb{R}^d),[x]\neq[y]\}$. Define a cocycle s on $\mathrm{Gl}(d,\mathbb{R})\times$X given by

$$\sigma(g,([x],[y]))=\log\frac{\delta(g\cdot[x],g\cdot[y])}{\delta([x],[y])}.$$

By Proposition 5.3.1 we know that $\sup_{([x],[y])\in X}\int\sigma(g,x)\mathrm{d}\mu^p(g)<0$. This means that

$$\sup_{([x],[y])\in X}\mathbb{E}\left(\log\frac{\delta(g\cdot[x],g\cdot[y])}{\delta([x],[y])}\right)<0.$$

Then, by the previous Lemma, we have a positive and small enough α such that

$$\lim_{n\to\infty}\frac{1}{n}\log\Big[\sup_{([x],[y])\in X}\int\{\frac{\delta(g\cdot[x],g\cdot[y])}{\delta([x],[y])}\}^\alpha\mathrm{d}\mu^n(g)\Big]<0.$$

□

Definition 5.3.1. *Let $\alpha > 0$. Let f be a continuous function on $P(\mathbb{R}^d)$. Set*

$$|f|_\infty = \sup_{[x]\in P(\mathbb{R}^d)} |f([x])| \text{ and } m_\alpha(f) = \sup_{[x],[y]\in P(\mathbb{R}^d)} \frac{|f([x]) - f([y])|}{(\delta([x],[y]))^\alpha}.$$

We define the space of Holder continuous functions, denoted by $\mathbb{L}(\alpha)$, where for any continuous function f on $P(\mathbb{R}^d)$ we have, $||f||_\alpha = |f|_\infty + m_\alpha(f)$ is finite.

Proposition 5.3.3. *$\mathbb{L}(\alpha)$ equipped with the $||.||_\alpha$ norm is a Banach Space.*

Theorem 5.3.1. *Let μ be a probability measure on $GL(d,\mathbb{R})$ such that Γ_μ is strongly irreducible and contracting. Let ν be the μ-invariant probability measure on $P(\mathbb{R}^d)$. Suppose that μ has an exponential moment ie there exists some $\tau > 0$ such that $\int e^{\tau\ell(g)}\,d\mu(g)$ is finite for $g \in GL(d,\mathbb{R})$.*
Then, there exists $\alpha_0 > 0$ such that for all $0 < \alpha \leq \alpha_0$, the operators T and N defined on $\mathbb{L}(\alpha)$ by

$$T(0)f([x]) = \int f(g\cdot[x])\,d\mu(g)$$

$$N(0)f([x]) = \int f([y])\,d\nu([y])$$

are bounded and satisfy

$$\lim_{n\to\infty} ||T^n(0) - N(0)||_\alpha^{1/n} < 1$$

Proof. We have $|T(0)f|_\infty \leq |f|_\infty$ for every continuous function f on $\mathrm{P}(\mathbb{R}^d)$. Since $\mathrm{P}(\mathbb{R}^d)$ is compact then f is bounded and attains its boundaries. So, $|f|_\infty < \infty$.
Also $m_\alpha(T(0)f) = \sup\limits_{[x],[y]\in \mathrm{P}(\mathbb{R}^d)} \dfrac{|T(0)f([x]) - T(0)f([y])|}{(\delta([x],[y]))^\alpha}$
Notice that

$$\begin{aligned}
\frac{|T(0)f([x]) - T(0)f([y])|}{(\delta([x],[y]))^\alpha} &\leq \int \frac{|f(g\cdot[x]) - f(g\cdot[y])|}{(\delta([x],[y]))^\alpha}\mathrm{d}\mu(g)\\
&\leq \int \left(\frac{\delta(g\cdot[x], g\cdot[y])}{\delta([x],[y])}\right)^\alpha m_\alpha(f)\mathrm{d}\mu(g)\\
&\leq m_\alpha(f)\int e^{4\alpha\ell(g)}\mathrm{d}\mu(g)
\end{aligned}$$

This is because

$$\begin{aligned}
\log\frac{\delta(g\cdot[x], g\cdot[y])}{\delta([x],[y])} &\leq \log\left(||\wedge^2 g||.\frac{||x||}{||gx||}.\frac{||y||}{||gy||}\right)\\
&\leq \log||\wedge^2 g|| + \log\frac{||x||}{||gx||} + \log\frac{||y||}{||gy||}\\
&\leq 4\mathbb{N}(g) \qquad (*)
\end{aligned}$$

$(*)$ Using Lemma 5.3.1.
Since μ has an exponential moment then, there exists $\tau > 0$ such that $\int e^{\tau N(g)}\mathrm{d}\mu(g) < \infty$. Thus $m_\alpha(T(0)f) < \infty$ for $\alpha \in (0, \frac{\tau}{4})$.
Hence, $T(0)$ is bounded on $\mathbb{L}(\alpha)$.
We have $N(0)f = \nu(f)\mathbb{1}$. So, $N(0)$ is a constant thus it is bounded on $\mathbb{L}(\alpha)$.

$$\begin{aligned}
|(T^n(0) - N(0))f([x])| &= |T^n(0)f([x]) - N(0)f([x])|\\
&\overset{(*)}{=} \left|\int f(g\cdot[x])\mathrm{d}\mu^n(g) - \int\int f(g\cdot[y])\mathrm{d}\mu^n(g)\mathrm{d}\nu([y])\right|\\
&\leq \int_{\mathrm{Gl}(d,\mathbb{R})\times\mathrm{P}(\mathbb{R}^d)} \big|f(g\cdot[x]) - f(g\cdot[y])\big|\mathrm{d}\mu^n(g)\mathrm{d}\nu([y])\\
&\leq m_\alpha(f)\int(\delta(g\cdot[x], g\cdot[y]))^\alpha\mathrm{d}\mu^n(g)\mathrm{d}\nu([y])
\end{aligned}$$

$(*)$ since $\mu * \nu = \nu$.

Then,

$$\begin{aligned}\left|(T^n(0)-N(0))f\right|_\infty &= \sup_{[x]\in \mathrm{P}(\mathbb{R}^d)} \left|(T^n(0)-N(0))f([x])\right| \\ &\leq m_\alpha(f) \sup_{[x],[y]\in \mathrm{P}(\mathbb{R}^d)} \int \{\frac{\delta(g\cdot[x], g\cdot[y])}{\delta([x],[y])}\}^\alpha \mathrm{d}\mu^n(g)\mathrm{d}\nu([y]).\end{aligned}$$

From Proposition 5.3.2 we know that $\sup_{[x],[y]\in \mathrm{P}(\mathbb{R}^d)} \int \{\frac{\delta(g\cdot[x], g\cdot[y])}{\delta([x],[y])}\}^\alpha \mathrm{d}\mu^n(g) = \rho^n < 1.$

Then,

$$\left|(T^n(0)-N(0))f\right|_\infty \leq m_\alpha(f)\rho^n$$

Look at

$$\begin{aligned}\frac{\left|(T^n(0)-N(0))f([x])-(T^n(0)-N(0))f([y])\right|}{(\delta([x],[y]))^\alpha} &= \frac{\left|T^n(0)f([x])-T^n(0)f([y])\right|}{(\delta([x],[y]))^\alpha} \\ &\leq \int m_\alpha(f)\left(\frac{\delta(g\cdot[x], g\cdot[y])}{\delta([x],[y])}\right)^\alpha \mathrm{d}\mu^n(g) \\ &\leq \rho^n m_\alpha(f)\end{aligned}$$

Thus, for every continuous function f on $\mathrm{P}(\mathbb{R}^d)$ we have,

$$\begin{aligned}||(T^n(0)-N(0))f||_\alpha &= m_\alpha\left((T^n(0)-N(0))f\right) + \left|(T^n(0)-N(0))f\right|_\infty \\ &\leq 2\rho^n m_\alpha(f) \\ &\leq 2\rho^n ||f||_\alpha\end{aligned}$$

Therefore,

$$\lim_{n\to\infty} ||T^n(0)-N(0)||_\alpha^{1/n} \leq \rho < 1$$

□

Therefore, Using the $||\cdot||_\alpha$ norm, we obtained the important identity that is

$$\lim_{n\to\infty} ||T^n(0)-N(0)||_\alpha^{1/n} < 1.$$

We define a family of operators $T(\xi)$ and prove in a similar way that it is bounded and analytical. This would allow us to proceed as we did previously in Section 4 to prove that $\log ||S_n x||$ satisfies a central limit theorem, where we get

$$\mathbb{E}\left(e^{\frac{it}{\sqrt{n}}(\log ||S_n x|| - n\gamma)}\right) \underset{n\to\infty}{\longrightarrow} e^{\frac{-t^2}{2}\sigma^2}$$

with $\sigma^2 \geq 0$. Again to show strict positivity of σ^2 refer to [L2].

Chapter 6

Conclusion

In this thesis, we have proved the Central Limit Theorem on the general linear group under an exponential moment and under the assumption of strong irreducibility and contraction of the semi-group generated by the support of the probability measure μ. For this goal we have followed the spectral method used by Le Page [L2]. This theorem was then improved by Benoist-Quint in [B6] with the natural moment condition (that of order two) using a different approach (that of Martingales). An interesting question would be to try to mix both methods and give a spectral approach for the CLT with a moment of order 2. Moreover, another natural question would be to treat the non-irreducible case in the line with the recent work of Aoun-Guivarc'h [A].

Bibliography

[AG] R. Aoun and Y. Guivarc'h. Random matrix products when the top lyapunov exponent is simple. *preprint*.

[Bel54] R. Bellman. Limit theorems for non-commutative operations. I. *Duke Math. J.*, 21:491–500, 1954.

[BL85] P. Bougerol and J. Lacroix. *Products of random matrices with applications to Schrödinger operators*, volume 8 of *Progress in Probability and Statistics*. Birkhäuser Boston Inc., Boston, MA, 1985.

[BQ16] Y. Benoist and J-F. Quint. Central limit theorem for linear groups. *Ann. Probab.*, 44(2):1308–1340, 2016.

[Bre68] L. Breiman. *Probability*. Addison-Wesley Publishing Company, Reading, Mass., 1968.

[DS] N. Dunford and J. Schwartz. *Linear Operators, Vol. 1*. Interscience.

[FK60] H. Furstenberg and H. Kesten. Products of random matrices. *Ann. Math. Statist.*, 31:457–469, 1960.

[FK83] H. Furstenberg and Y. Kifer. Random matrix products and measures on projective spaces. *Israel J. Math.*, 46(1-2):12–32, 1983.

[Fur63] H. Furstenberg. Noncommuting random products. *Trans. Amer. Math. Soc.*, 108:377–428, 1963.

[Fur73] H. Furstenberg. *Boundary theory and stochastic processes on homogeneous spaces*. Harmonic analysis on homogeneous spaces (Proc. Sympos. Pure Math., Vol. XXVI, Williams Coll., Williamstown, Mass., 1972). 1973.

[GG96] I.Y. Goldsheid and Y. Guivarc'h. Zariski closure and the dimension of the gaussian law of the product of random matrices. 105 (1):109–142, 1996.

[GM89a] I. Ya. Goldsheid and G. A. Margulis. Lyapunov exponents of a product of random matrices. *Russian Math. Surveys*, 44(5):11–71, 1989.

[GM89b] I. Ya. Goldsheid and G. A. Margulis. Lyapunov exponents of a product of random matrices. *Uspekhi Mat. Nauk*, 44(5(269)):13–60, 1989.

[GR85] Y. Guivarc'h and A. Raugi. Frontière de Furstenberg, propriétés de contraction et théorèmes de convergence. *Z. Wahrsch. Verw. Gebiete*, 69(2):187–242, 1985.

[Gui90] Y. Guivarc'h. Produits de matrices aléatoires et applications aux propriétés géométriques des sous-groupes du groupe linéaire. *Ergodic Theory Dynam. Systems*, 10(3):483–512, 1990.

[LP82] É. Le Page. Théorèmes limites pour les produits de matrices aléatoires. In *Probability measures on groups (Oberwolfach, 1981)*, volume 928 of *Lecture Notes in Math.*, pages 258–303. Springer, Berlin, 1982.