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EXTENSION OF BIHOLOMORPHIC MAPS FROM SPECIAL DOMAINS INTO THEIR BOUNDARIES

by

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Our life on earth as human beings is apparently like a complex number because they both have real and imaginary parts. The imaginary part embellishes the real part of our existence as it does to the real part of a complex number. In other words, complex analysis may be less complicated than real analysis in some special situations.

The first powerful facilitating tool that comes to one's mind is holomorphicity: a holomorphic function is one of the main reasons behind the beauty and easiness of complex analysis. In fact, beginners start to taste the real flavor of an introductory course in complex analysis once they learn that any holomorphic function is analytic, i.e. expressible as a power series. They proceed to reach a higher level when they get to know the Riemann mapping theorem, which states that every non empty proper subset in $\mathbb{C}$ is biholomorphic to the open unit disc. Hence, it follows that any two non-empty proper simply connected domains in $\mathbb{C}$ are conformally equivalent. Then, one may ask himself/herself: is that true in several complex variables? The answer will be provided in this paper, and will be equipped with interesting material.

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Several theorems that hold in the theory of one complex variable cannot be generalized to the theory of several complex variables. One of them is the Riemann Mapping Theorem, which states that every non-empty simply connected domain which is not the entire complex plane is biholomorphic to the open unit disc, and from which follows the fact that any two non-empty proper simply-connected domains in $\mathbb{C}$ are biholomorphic. In this paper, we show that the last statement is not true in $\mathbb{C}^n$ for $n \geq 2$ using the properties of the Levi form. However, the proof entails, assuming the existence of such a biholomorphism f, some boundary information about f. This requirement is fulfilled alluding to Fefferman Theorem, which will be proved for special domains using powerful tools: extremal and stationary maps.

Contents

Chapter 1

Basics

Part of complex analysis in several variables is a generalization from the function theory of one complex variable. However, things get more complicated in higher dimensions, and some theorems that hold in the complex plane $\mathbb{C}$ do not apply in the complex n-space $\mathbb{C}^n$ for $n \geq 2$. In this chapter, we will provide some definitions and recall basic theorems, some of which are generalized from the function theory of one complex variable.

1.1 Holomorphic Functions from subsets of $\mathbb{C}^n$ into $\mathbb{C}$

Etymologically, the word "holomorphic" is derived from two Greek words: "holos" meaning entire, and "morphe" meaning form. Holomorphic functions are central objects in complex analysis. In this section, we define holomorphic functions from subsets of $\mathbb{C}^n$ into $\mathbb{C}$.

Definition 1.1.1

Let Ω be a domain in $\mathbb{C}^n$. A function $f : \Omega \to \mathbb{C}$ is said to be holomorphic if, for each $j = 1, 2, ..., n$ and each $z_1, ..., z_{j-1}, z_{j+1}, ..., z_n$, the function

$$\zeta \to f(z_1, ..., z_{j-1}, \zeta, z_{j+1}, ..., z_n)$$

is holomorphic.
In other words, f is holomorphic on Ω if it is holomorphic in each variable separately.

Next, we show that a holomorphic function $f : \Omega \subseteq \mathbb{C}^n \to \mathbb{C}$ satisfies the Cauchy-Riemann equations.

Theorem 1.1.2

Let Ω be a domain in $\mathbb{C}^n$. A function $f : \Omega \to \mathbb{C}$, written as

$$f(x_1 + iy_1, x_2 + iy_2, ..., x_n + iy_n) = u(x_1, y_1, ..., x_n, y_n) + iv(x_1, y_1, ..., x_n, y_n)$$

is holomorphic if and only if

$$u_{x_j}, u_{y_j}, v_{x_j}, v_{y_j} \text{ are continuous } \forall j = 1, ..., n$$

and

$$\frac{\partial f}{\partial \overline{z_j}} = 0 \qquad \forall j = 1, ..., n,$$

where $\dfrac{\partial}{\partial \overline{z_j}}$ is the partial differential operator on $\mathbb{C}^n$ given by

$$\frac{\partial}{\partial \overline{z_j}} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right), \qquad j = 1, ..., n.$$

Proof. Follows from Definition 1.1.1. and from the one-variable Cauchy-Riemann equations. □

1.2 Cauchy Formula for Polydiscs

In this section, we generalize the Cauchy Integral Formula of the function theory of one complex variable, and we show that any holomorphic function from a subset of $\mathbb{C}^n$ into $\mathbb{C}$ is infinitely differentiable.

We start our section by the definition of an open polydisc.

Definition 1.2.1

Let $a \in \mathbb{C}^n$ and $r > 0$. An open polydisc, denoted by $D^n(a, r)$, is defined to be the set

$$D^n(a, r) = \{z \in \mathbb{C}^n : |z_j - a_j| < r, j = 1, ..., n\}.$$

In other words, an open polydisc is a cartesian product of open discs.
The closure of $D^n(a, r)$ will be denoted by $\overline{D}^n(a, r)$.

Now, we state the Cauchy Formula for Polydiscs.

Theorem 1.2.2 (Cauchy Formula for Polydiscs)

Let $w = (w_1, ..., w_n), r = (r_1, ..., r_n) \in \mathbb{C}^n$ with $r_1, r_2, ..., r_n > 0$.
Suppose f is continuous on $\overline{D}^1(w_1, r_1) \times \overline{D}^1(w_2, r_2) \times ... \times \overline{D}^1(w_n, r_n)$ and holomorphic on $D^1(w_1, r_1) \times D^1(w_2, r_2) \times ... \times D^1(w_n, r_n)$. Then, for any $z = (z_1, ..., z_n) \in D^1(w_1, r_1) \times D^1(w_2, r_2) \times ... \times D^1(w_n, r_n)$,

$$f(z) = f(z_1, ..., z_n) = \frac{1}{(2\pi i)^n} \int\limits_{|\zeta_n - w_n| = r_n} ... \int\limits_{|\zeta_1 - w_1| = r_1} \frac{f(\zeta_1, ..., \zeta_n)}{(\zeta_1 - z_1)...(\zeta_n - z_n)} d\zeta_1 ... d\zeta_n$$

Here is an important consequence of Cauchy Formula:

Corollary 1.2.3

Let Ω be a domain in $\mathbb{C}^n$. Suppose $f : \Omega \to \mathbb{C}$ is a holomorphic function. Then, f is infinitely differentiable.

1.3 Power Series representation

In this section, we define power series for functions of several variables and use them to show important generalizations from the function theory of one complex variable.

Let $\alpha \in \mathbb{N}_0{}^n$ where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. In other words, we can write $\alpha = (\alpha_1, ..., \alpha_n), \alpha_i \in \mathbb{N}_0$.

Notation:

$$z^\alpha = z_1^{\alpha_1}.z_2^{\alpha_2}. \dots .z_n^{\alpha_n}$$

$$|\alpha| = \alpha_1 + ... + \alpha_n$$

$$\frac{1}{z} = \frac{1}{z_1. \dots .z_n}$$

$$\alpha! = \alpha_1! \dots \alpha_n!$$

$$\frac{\partial^{|\alpha|}}{\partial z^\alpha} = \frac{\partial^{\alpha_1}}{\partial z_1^{\alpha_1}}. \dots .\frac{\partial^{\alpha_n}}{\partial z_n^{\alpha_n}}$$

Definition 1.3.1

Let $a \in \mathbb{C}^n$. A power series centered at a is a series of the form

$$\sum_{\alpha \in \mathbb{N}_0^n} c_\alpha (z-a)^\alpha$$

Example 1.3.2

For 3 variables, a power series at the origin is a series of the form

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} c_{jkl} z_1^j z_2^k z_3^l$$

In the theory of one complex variable, if f is a holomorphic function on a domain Ω and $a \in \Omega$, then f is expressible as a power series centered at a in the largest disc of center a lying in Ω. Now, we generalize our result to holomorphic functions from subsets of $\mathbb{C}^n$ into $\mathbb{C}$.

Theorem 1.3.3

Let $D^n(a,r)$ be a polydisc of center $a \in \mathbb{C}^n$ and radius $r > 0$. Suppose $f : \overline{D}^n(a,r) \to \mathbb{C}$ is continuous on $\overline{D}^n(a,r)$ and holomorphic on $D^n(a,r)$. Then, f(z) is expressible as a power series centered at a, for any $z \in D^n(a,r)$.

Cauchy Estimates, an important inequality, is an aftermath of the previous theorem.

Theorem 1.3.4 (Cauchy Estimates)

Let $D^n(a,r)$ be a polydisc of center $a \in \mathbb{C}^n$ and radius $r > 0$. Suppose $f : \overline{D}^n(a,r) \to \mathbb{C}$ is continuous on $\overline{D}^n(a,r)$ and holomorphic on $D^n(a,r)$. Let $C = C_1 \times ... \times C_n$, where C_i is the positively-oriented circle $|\zeta_i - a_i| = r$. Then,

$$\left| \frac{\partial^{|\alpha|} f(a)}{\partial z^\alpha} \right| \leq \frac{\alpha!}{r^\alpha} sup_{z \in C} |f(z)|$$

The identity theorem, one of major theorems in the theory of one complex variable, is also generalized:

Theorem 1.3.5 (Identity Theorem)

Let Ω be a domain in $\mathbb{C}^n$. Let U be an open subset of Ω. Let $f : \Omega \to \mathbb{C}$ be holomorphic. Suppose $f(z) = 0, \forall z \in U$. Then $f(z) = 0, \forall z \in \Omega$.

We end our section by a consequence of the generalized identity theorem, the maximum principle.

Theorem 1.3.6 (Maximum Principle)

Let Ω be a domain in $\mathbb{C}^n$. Let $f : \Omega \to \mathbb{C}$ be a holomorphic function on Ω, Suppose $|f(z)|$ attains a maximum at some $a \in \Omega$, then $f \equiv f(a)$.

Chapter 2

Automorphisms of the unit ball

A bijective holomorphic function whose inverse is holomorphic is said to be biholomorphic. An automorphism of a set U is a biholomorphism from U to U. Automorphisms of a set U form a group under composition, called the automorphism group and denoted by $Aut(U)$. Automorphisms of the unit disc are well-known in the function theory of one complex variable. In this chapter, we will introduce automorphisms of the unit ball in $\mathbb{C}^n$, but this entails Cartan's Uniqueness Theorem which is an analogue of Schwarz's Lemma to several variables.

2.1 Holomorphic Functions from subsets of $\mathbb{C}^n$ into $\mathbb{C}^m$

We start this section by defining holomorphic functions from $\Omega \subseteq \mathbb{C}^n$ into $\mathbb{C}^m$

Definition 2.1.1

Let Ω be a domain in $\mathbb{C}^n$. A function $f : \Omega \to \mathbb{C}^m$, written as $f = (f_1, f_2, ..., f_m)$ is said to be holomorphic if f_i is holomorphic for all $i = 1, 2, ..., m$.

The following theorem shows that a holomorphic function $f : \Omega \subseteq \mathbb{C}^n \to \mathbb{C}^m$ satisfies the Cauchy-Riemann equations.

Theorem 2.1.2

Let Ω be a domain in $\mathbb{C}^n$. A function $f : \Omega \to \mathbb{C}^m$, written as

$$f(z_1, ..., z_n) = (f_1(z_1, ..., z_n), f_2(z_1, ..., z_n), ..., f_m(z_1, ..., z_n))$$

where $f_j : \Omega \to \mathbb{C}$ is written as

$$f_j(x_1 + iy_1, ..., x_n + iy_n) = u_j(x_1, y_1, ..., x_n, y_n) + iv_j(x_1, y_1, ..., x_n, y_n) \forall j = 1, ..., m$$

is holomorphic if and only if

$$u_{ix_j}, u_{iy_j}, v_{ix_j}, v_{iy_j} \text{ are continuous } \forall i = 1, ..., m \qquad \forall j = 1, ..., n$$

and

$$\frac{\partial f_i}{\partial \overline{z_j}} = 0 \qquad \forall i = 1, ..., m \quad \forall j = 1, ..., n$$

Proof. Follows from Definition 2.1.1. and from Theorem 1.1.2.

□

Now, we state and prove a version of the chain rule in the function theory of several complex variables.

Theorem 2.1.3

Suppose $U \subseteq \mathbb{C}^n$ and $V \subseteq \mathbb{C}^m$ are open sets. Let $f : U \to V$, and $g : V \to \mathbb{C}$ be differentiable maps. Write the variables as $z = (z_1, ..., z_n) \in U$ and $w = (w_1, ..., w_m) \in V$. Then for any $j = 1, ..., n$, we have

$$\frac{\partial}{\partial z_j}(g \circ f) = \sum_{l=1}^{m} \left(\frac{\partial g}{\partial w_l}\frac{\partial f_l}{\partial z_j} + \frac{\partial g}{\partial \overline{w_l}}\frac{\partial \overline{f_l}}{\partial z_j} \right)$$

Proof. Write $f_l = u_l + iv_l$ for $l = 1, ..., m$, $z_j = x_j + iy_j$ for $j = 1, ..., n$, $w_l = s_l + it_l$ for $l = 1, ..., m$. Then,

$$\begin{aligned}
\frac{\partial}{\partial z_j}(g \circ f) &= \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right)(g \circ f) \\
&= \frac{1}{2}\sum_{l=1}^{m}\left(\frac{\partial g}{\partial s_l}\frac{\partial u_l}{\partial x_j} + \frac{\partial g}{\partial t_l}\frac{\partial v_l}{\partial x_j} - i\left(\frac{\partial g}{\partial s_l}\frac{\partial u_l}{\partial y_j} + \frac{\partial g}{\partial t_l}\frac{\partial v_l}{\partial y_j}\right)\right) \\
&= \sum_{l=1}^{m}\left(\frac{\partial g}{\partial s_l}.\frac{1}{2}\left(\frac{\partial u_l}{\partial x_j} - i\frac{\partial u_l}{\partial y_j}\right) + \frac{\partial g}{\partial t_l}.\frac{1}{2}\left(\frac{\partial v_l}{\partial x_j} - i\frac{\partial v_l}{\partial y_j}\right)\right) \\
&= \sum_{l=1}^{m}\left(\frac{\partial g}{\partial s_l}\frac{\partial u_l}{\partial z_j} + \frac{\partial g}{\partial t_l}\frac{\partial v_l}{\partial z_j}\right).
\end{aligned}$$

Note that

$$\frac{\partial}{\partial s_l} = \frac{\partial}{\partial w_l} + \frac{\partial}{\partial \overline{w_l}}, \qquad \frac{\partial}{\partial t_l} = i\left(\frac{\partial}{\partial w_l} - \frac{\partial}{\partial \overline{w_l}}\right),$$

for $l = 1, ..., m$. Hence,

$$\begin{aligned}
\frac{\partial}{\partial z_j}(g \circ f) &= \sum_{l=1}^{m}\left(\left(\frac{\partial g}{\partial w_l}\frac{\partial u_l}{\partial z_j} + \frac{\partial g}{\partial \overline{w_l}}\frac{\partial u_l}{\partial z_j}\right) + i\left(\frac{\partial g}{\partial w_l}\frac{\partial v_l}{\partial z_j} - \frac{\partial g}{\partial \overline{w_l}}\frac{v_l}{z_j}\right)\right) \\
&= \sum_{l=1}^{m}\left(\frac{\partial g}{\partial w_l}\left(\frac{\partial u_l}{\partial z_j} + i\frac{\partial v_l}{\partial z_j}\right) + \frac{\partial g}{\partial \overline{w_l}}\left(\frac{\partial u_l}{\partial z_j} - i\frac{\partial v_l}{\partial z_j}\right)\right) \\
&= \sum_{l=1}^{m}\left(\frac{\partial g}{\partial w_l}\frac{\partial f_l}{\partial z_j} + \frac{\partial g}{\partial \overline{w_l}}\frac{\partial \overline{f_l}}{\partial z_j}\right)
\end{aligned}$$

□

We end this section by defining the derivative of a holomorphic function $f : \Omega \subseteq \mathbb{C}^n \to \mathbb{C}^m$ at a point $z \in \Omega$.

Definition 2.1.4

Let Ω be an open subset of $\mathbb{C}^n$. Let $f : \Omega \to \mathbb{C}^m$ be holomorphic. Let $z \in \Omega$. Let $f'(z) : \Omega \to \mathbb{C}^m$ be the linear transformation satisfying

$$f(z+h) = f(z) + f'(z)h + O(|h|^2)$$

for h near the origin of $\mathbb{C}^n$. Then, $f'(z)$ is called the derivative of f at z. For $1 \leq k \leq n$, let $h = \lambda e_k$, where $e_k = (0, ..., 1, ..., 0)$ is the vector in $\mathbb{C}^n$ having as coordinates zeros except *one* at the k^{th} component. Letting $\lambda \to 0$, we get $f'(z)e_k = \dfrac{\partial f}{\partial z_k}(z)$.

2.2 Cartan's Uniqueness Theorem

Cartan's uniqueness theorem is an important tool to deal with the automorphisms of the unit ball in C^n. It was shown by the French mathematician Henri Cartan in 1931. However, its proof requires the notion of homogeneous expansion. So, we start our section by defining homogeneous polynomials from $\mathbb{C}^n$ into $\mathbb{C}$.

Definition 2.2.1

A polynomial $P : \mathbb{C}^n \to \mathbb{C}$ is homogeneous of degree d if $P(sz) = s^d P(z), \forall s \in \mathbb{C}$ and $\forall z \in \mathbb{C}^n$.

Here's an example:

Example 2.2.2

Let $P : \mathbb{C}^2 \to C$ be the polynomial defined by $P(z_1, z_2) = z_1^3 + 2z_1 z_2^2$. Then, P is homogeneous of degree 3.

Now, we show that a holomorphic function $f : \Omega \subseteq \mathbb{C}^n \to \mathbb{C}$ has a homogeneous expansion.

Theorem 2.2.3

Let Ω be a domain in $\mathbb{C}^n$. Let $f : \Omega \to \mathbb{C}$ be a holomorphic function. Let $a \in \Omega$. Then, the power series of f at a can be written as

$$\sum_{j=0}^{\infty} f_j(z-a),$$

where f_j is a homogeneous polynomial of degree j.

Proof. The power series of f at a is given by

$$\sum_{\alpha} c_\alpha (z-a)^\alpha, \qquad z \in D^n(a,r) \subset \Omega$$

with

$$c_\alpha = \frac{1}{(2\pi i)^n} \int_C \frac{f(\zeta)}{(\zeta - a)^{\alpha+1}} d\zeta$$

where C is the boundary of $D^n(a, r)$.
For $j = 1, 2, ...,$ let $f_j(z)$ be the sum of the terms $c_\alpha z^\alpha$ in the power series of f at 0 for which $|\alpha| = j$. See that f_j is a homogeneous polynomial of degree j. $\square$

Next, we provide a more generalized definition of homogeneous polynomials from $\mathbb{C}^n$ into $\mathbb{C}^m$.

Definition 2.2.4

A polynomial $P : \mathbb{C}^n \to \mathbb{C}^m$ is homogeneous if each component is homogeneous.

Accordingly, the next theorem shows that holomorphic functions $f : \Omega \subseteq \mathbb{C}^n \to \mathbb{C}^m$ have homogeneous expansion.

Theorem 2.2.5

Let Ω be a domain in $\mathbb{C}^n$. Let $f : \Omega \to \mathbb{C}^m$ be a holomorphic map. Let $a \in \Omega$. Then, the power series of f at a can be written as

$$\sum_{j=0}^{\infty} f_j(z - a),$$

where f_j is a vector-valued homogeneous polynomial.

Proof. Follows from Theorem 2.2.3 and Definition 2.2.4. $\square$

Finally, we state and prove Cartan's uniqueness theorem.

Theorem 2.2.6 (Cartan's Uniqueness Theorem)

Suppose Ω is a bounded domain in $\mathbb{C}^n$. Let $F : \Omega \to \Omega$ be a holomorphic function. Assume there exists $a \in \Omega$ such that $F(a) = a$ and $F'(a)$ is the identity. Then, $F(z) = z$ for all $z \in \Omega$.

Proof. WLOG, assume $a = 0$. Since Ω is open, there exists $r_1 > 0$ such that $D^n(0, r_1) \subseteq \Omega$. Since Ω is bounded, there exists $r_2 > 0$ such that $\Omega \subseteq D^n(0, r_2)$. In $D^n(0, r_1)$, F has a homogenous expansion

$$F(z) = z + \sum_{s=2}^{\infty} F_s(z),$$

where F_s is a map from $\mathbb{C}^n$ into $\mathbb{C}^n$ whose components are homogeneous polynomials of degree s. Note that $F_0(z) = 0$ and $F_1(z) = z$ since $F(0) = 0$ and $F'(0)$ is the identity.
Let F^k be defined as follows:

$$\begin{aligned} F^1 &= F \\ F^k &= F^{k-1} \circ F, \qquad k > 1 \end{aligned}$$

For $m \geq 2$, make the induction hypothesis that $F_s = 0$ for $2 \leq s < m$, which is vacuously true for $m = 2$. Then, F^k has a homogeneous expansion

$$F^k(z) = z + kF_m(z) + \sum_{j=k+1}^{\infty} f_j(z), z \in D^n(0, r_1)$$

where f_j is a map from $\mathbb{C}^n$ into $\mathbb{C}^n$ whose components are homogeneous polynomials of degree j. This can be easily proved by induction on k.
Thus, for $\theta \in \mathbb{R}, z \in D^n(0, r_1)$, we have

$$F^k(e^{i\theta}z) = e^{i\theta}z + kF_m(e^{i\theta}z) + \sum_{j=k+1}^{\infty} f_j(e^{i\theta}z)$$
$$F^k(e^{i\theta}z) = e^{i\theta}z + k(e^{i\theta})^m F_m(z) + \sum_{j=k+1}^{\infty} (e^{i\theta})^j f_j(z)$$
$$F^k(e^{i\theta}z)e^{-im\theta} = e^{i\theta}e^{-im\theta}z + kF_m(z) + \sum_{j=k+1}^{\infty} e^{ij\theta}e^{-im\theta} f_j(z)$$

Hence, $\int\limits_{-\pi}^{\pi} F^k(e^{i\theta}z)e^{-im\theta}d\theta = z\int\limits_{-\pi}^{\pi} e^{i\theta}e^{-im\theta}d\theta + 2\pi kF_m(z) + \int\limits_{-\pi}^{\pi} \sum_{j=k+1}^{\infty} (e^{ij\theta}e^{-im\theta})f_j(z)d\theta$
$\implies \int\limits_{-\pi}^{\pi} F^k(e^{i\theta}z)e^{-im\theta}d\theta = 0 + 2\pi kF_m(z) + 0.$
Interchanging the integral and summation symbols is allowed because $\sum\limits_{j=k+1}^{\infty} e^{i(j-m)\theta}f_j(z)$ converges uniformly on $(-\pi; \pi)$.
Therefore,

$$kF_m(z) = \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} F^k(e^{i\theta}z)e^{-im\theta}d\theta, \qquad z \in D^n(0, r_1)$$

Since $F^k(\Omega) \subseteq \Omega$, we have

$$|F^k(e^{i\theta}z)| < r_2 \qquad \forall z \in D^n(0, r_1), \forall\theta \in \mathbb{R}$$

This implies that

$$k|F_m(z)| = \left|\frac{1}{2\pi}\int\limits_{-\pi}^{\pi} F^k(e^{i\theta}z)e^{-im\theta}d\theta\right| \leq \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} |F^k(e^{i\theta}z)|d\theta \leq r_2,$$

$\forall z \in D^n(0, r_1), \forall k = 1, 2, ...$
Hence, $F_m = 0$, and the induction hypothesis holds with $m + 1$ in place of m. Therefore, $F(z) = z\ \forall z \in D^n(0, r_1)$. By the identity theorem, we get $F(z) = z\ \forall z \in \Omega$.

□

2.3 Automorphisms of the unit ball of $\mathbb{C}^n$

In the second chapter of his book "Function Theory in the Unit Ball of $\mathbb{C}^n$", Rudin handled the automorphisms of the unit ball of $\mathbb{C}^n$. In this section, we will define them and list some of their properties.

Definition 2.3.1

A ball in $\mathbb{C}^n$, denoted by $B^n(a,r)$, is defined to be the open set

$$B^n(a,r) = \left\{z \in \mathbb{C}^n : |z_1 - a_1|^2 + ... + |z_n - a_n|^2 < r^2\right\}.$$

The closure of $B^n(a,r)$ will be denoted by $\overline{B}^n(a,r)$.
The unit ball of $\mathbb{C}^n$ is the ball $B^n(0,1)$.

$\mathbb{C}^n$ is turned into an n-dimensional Hilbert space by considering the inner product

$$< z, w >= \sum_{j=1}^{n} z_j \overline{w_j} \qquad (z, w \in \mathbb{C}^n)$$

and the associated norm

$$|z| = \sqrt{< z, z >} \qquad (z \in \mathbb{C}^n)$$

Definition 2.3.2

Let $a \in B^n(0,1)$. Let P_a be the orthogonal projection of $\mathbb{C}^n$ onto the subspace $[a]$ generated by a. Let $Q_a = I - P_a$ be the projection onto the orthogonal complement of $[a]$, where I is the identity operator. Explicitly,

$$P_0 = 0$$

and

$$P_a z = \frac{< z, a >}{< a, a >} a, \qquad a \neq 0.$$

Put $s_a = \sqrt{1 - |a|^2}$ and define

$$\phi_a(z) = \frac{a - P_a z - s_a Q_a z}{1- < z, a >}$$

See that if $n = 1$, then $P_a = I$ and $Q_a = 0$, so we get $\phi_a(z) = \dfrac{a - z}{1 - \overline{a}z}$,which is an automorphism of the unit disc.

Next, we show some properties of the maps ϕ_a to conclude that ϕ_a is a biholomorphism from $B^n(0,1)$ to $B^n(0,1)$.

Theorem 2.3.3 (Properties of the maps ϕ_a)

Let $a \in B^n(0,1)$. Then, ϕ_a has the following properties:

(i) $\phi_a(0) = a$ and $\phi_a(a) = 0$.

(ii) $\phi_a'(0) = -s_a^2 P_a - s_a Q_a$ and $\phi_a'(a) = -\dfrac{P_a}{s_a^2} - \dfrac{Q_a}{s_a}$.

(iii) The identity

$$1- < \phi_a(z), \phi_a(w) >= \frac{(1- < a, a >)(1- < z, w >)}{(1- < z, a >)(1- < a, w >)}$$

holds for all $z, w \in \overline{B}^n(0,1)$.

(iv) The identity

$$1-|\phi_a(z)|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1-<z,a>|^2}$$

holds for all $z \in \overline{B}^n(0,1)$.

(v) ϕ_a is an involution, that is, $\phi_a(\phi_a(z)) = z$ for all $z \in \overline{B}^n(0,1)$.

(vi) ϕ_a is a homeomorphism of $\overline{B}^n(0,1)$ onto $\overline{B}^n(0,1)$, and $\phi_a \in Aut(B)$.

Proof. (i) $\phi_a(0) = \dfrac{a-(P_a0)-s_a(Q_a0)}{1-<0,a>} = \dfrac{a-0-0}{1-0} = a.$

$\phi_a(a) = \dfrac{a-(P_aa)-s_a(Q_aa)}{1-<a,a>} = \dfrac{a-a-0}{1-<a,a>} = 0.$

(ii) Let $z \in \overline{B}^n(0,1)$ be close to the origin of $\mathbb{C}^n$. Then,

$$\begin{aligned}
\phi_a(z) &= \frac{1}{1-<z,a>}(a-P_az-s_aQ_az)\\
&= (1+<z,a>+<z,a>^2+...)(a-P_az-s_aQ_az)\\
&= a+<z,a>a-s_aQ_az-P_az+O(|z|^2)\\
&= \phi_a(0)+|a|^2P_az-s_aQ_az-P_az+O(|z|^2)\\
&= \phi_a(0)+(|a|^2P_a-P_a-s_aQ_a)z+O(|z|^2)\\
&= \phi_a(0)-(s_a^2P_a+s_aQ_a)z+O(|z|^2).
\end{aligned}$$

Therefore, it follows from Definition 2.2.3 that

$$\phi_a'(0) = -s_a^2P_a - s_aQ_a.$$

Now, let $h \in \mathbb{C}^n$ be such that $|<h,a>| < s_a^2$, with h close to the origin of $\mathbb{C}^n$. Then,

$$\begin{aligned}
\phi_a(a+h) &= \frac{a-P_a(a+h)-s_aQ_a(a+h)}{1-<a+h,a>}\\
&= \frac{a-\dfrac{<a+h,a>}{<a,a>}a-s_a\left(a+h-\dfrac{<a+h,a>}{<a,a>}a\right)}{1-<a,a>-<h,a>}\\
&= \frac{a-a-\dfrac{<h,a>}{<a,a>}a-s_a\left(a+h-a-\dfrac{<h,a>}{<a,a>}a\right)}{1-|a|^2-<h,a>}\\
&= \frac{-P_ah-s_aQ_ah}{s_a^2-<h,a>}\\
&= \frac{-P_ah-s_aQ_ah}{s_a^2\left(1-\dfrac{<h,a>}{s_a^2}\right)}\\
&= \frac{1}{s_a^2}\left(1+\frac{<h,a>}{s_a^2}+\frac{<h,a>^2}{s_a^4}+...\right)(-P_ah-s_aQ_ah)\\
&= \frac{1}{s_a^2}(-P_a-s_aQ_a)h+O(|h|^2).
\end{aligned}$$

Therefore, it follows from Definition 2.2.3 that

$$\phi_a'(a) = -\frac{P_a}{s_a^2}-\frac{Q_a}{s_a}.$$

(iii) Let $z, w \in \overline{B}^n(0,1)$. First, see that $P_a^2 = P_a$ and $Q_a^2 = Q_a$. Second, since P_a and Q_a are self-adjoint projections, we have:

(a) $< a, Q_a w > = < Q_a a, w > = < 0, w > = 0.$

(b) $< Q_a z, a > = < z, Q_a a > = < z, 0 > = 0.$

(c) $< a, P_a w > = < P_a a, w > = < a, w >.$

(d) $< P_a z, a > = < z, P_a a > = < z, a >.$

(e) $< P_a z, P_a w > = < z, P_a^2 w > = < z, P_a w >.$

Also, we have the following equalities:

(a) $< z, a >< a, w > = << z, a > a, w > = \; |a|^2 < P_a z, w >.$

(b)

$$\begin{aligned} < P_a z, Q_a w > &= \left\langle \frac{< z, a >}{< a, a >} a, w - \frac{< w, a >}{< a, a >} a \right\rangle \\ &= \frac{< z, a >}{< a, a >} < a, w > - \frac{< z, a >}{< a, a >} . \frac{< a, w >}{< a, a >} < a, a > \\ &= 0. \end{aligned}$$

(c)

$$\begin{aligned} & < P_a z, w > + < Q_a z, Q_a w > \\ & =< P_a z, w > + < z - P_a z, w - P_a w > \\ & =< P_a z, w > + < z, w > + < P_a z, P_a w > - < z, P_a w > - < P_a z, w > \\ & =< P_a z, w > + < z, w > + < z, P_a w > - < z, P_a w > - < P_a z, w > \\ & =< z, w > . \end{aligned}$$

(d) Since $< a - P_a z, -s_a Q_a w > - < s_a Q_a z, a - P_a w >= 0$ (by expanding and using the properties and equalities mentioned above), we have

$$< a - P_a z - s_a Q_a z, a - P_a w - s_a Q_a w >=< a - P_a z, a - P_a w > + s_a^2 < Q_a z, Q_a w >$$

Finally,

$$
\begin{aligned}
&1- < \phi_a z, \phi_a w > \\
&= 1 - \left\langle \frac{a - P_a z - s_a Q_a z}{1- < z, a >}, \frac{a - P_a w - s_a Q_a w}{1- < w, a >} \right\rangle \\
&= 1 - \frac{1}{1- < z, a >} . \frac{1}{\overline{1- < w, a >}} < a - P_a z - s_a Q_a z, a - P_a w - s_a Q_a w > \\
&= 1 - \frac{< a - P_a z, a - P_a w > + s_a^2 < Q_a z, Q_a w >}{(1- < z, a >)(1- < a, w >)} \\
&= \frac{(1- < z, a >)(1- < a, w >) - < a - P_a z, a - P_a w > - s_a^2 < Q_a z, Q_a w >}{(1- < z, a >)(1- < a, w >)} \\
&= \frac{1 + < z, a >< a, w > - < z, a > - < a, w > - < a, a > + < a, P_a w >}{(1- < z, a >)(1- < a, w >)} \\
&+ \frac{< P_a z, a > - < P_a z, P_a w > - s_a^2 < Q_a z, Q_a w >}{(1- < z, a >)(1- < a, w >)} \\
&= \frac{1 + |a|^2 < P_a z, w > - < z, a > - < a, w > - |a|^2 + < a, w >}{(1- < z, a >)(1- < a, w >)} \\
&+ \frac{< z, a > - < z, P_a w > - s_a^2 < Q_a z, Q_a w >}{(1- < z, a >)(1- < a, w >)} \\
&= \frac{s_a^2 - s_a^2 < P_a z, w > - s_a^2 < Q_a z, Q_a w >}{(1- < z, a >)(1- < a, w >)} \\
&= \frac{s_a^2 (1- < P_a z, w > - < Q_a z, Q_a w >)}{(1- < z, a >)(1- < a, w >)} \\
&= \frac{s_a^2 (1- < z, w >)}{(1- < z, a >)(1- < a, w >)}
\end{aligned}
$$

(iv) Take $w = z$ in (iii). This shows that ϕ_a maps $B^n(0,1)$ into $B^n(0,1)$ and the boundary of $B^n(0,1)$ into the boundary of $B^n(0,1)$.

(v) First, note that $P_a Q_a = P_a(I - P_a) = P_a - P_a^2 = P_a - P_a = 0$. Next, let $\psi = \phi_a \circ \phi_a$. ψ is a holomorphic map of $B^n(0,1)$ into $B^n(0,1)$. Also,

$$\psi(0) = \phi_a(\phi_a(0)) = \phi_a(0) = 0.$$

Moreover,

$$\phi_a'(\phi_a(0)).\phi_a'(0) = \phi_a'(a).\phi_a'(0) = \left(-\frac{P_a}{s_a^2} - \frac{Q_a}{s_a} \right) (-s_a^2 P_a - s_a Q_a) = P_a + Q_a = I.$$

Therefore, Cartan's Uniqueness Theorem implies that $\psi(z) = z$.

(vi) Follows from (v). See that $\phi_a^{-1} = \phi_a$.

$\square$

Chapter 3

The Levi Form

The Levi form of a hypersurface was named after the Italian mathematician Eugenio Elia Levi. Levi introduced the concept of pseudoconvextiy, one of the most important concepts in the function theory of several complex variables. In fact, an open connected subset Ω of $\mathbb{C}^n$ with a C^2 boundary is said to be pseudoconvex if and only if the Levi form of its defining function at each point $p \in \partial\Omega$ in the direction of $v \in Tp(\partial\Omega)$ is greater than or equal to zero. In this chapter, we will define these notions, and we will end up showing that the Levi form is invariant under biholomorphic maps.

3.1 Some differential geometry

In this section, we define all the terms related to differential geometry, required for the upcoming sections of this chapter.

We start our section by defining differential manifolds.

Definition 3.1.1

A real differentiable manifold M of real dimension n and of class C^k is a topological space together with a collection of homeomorphisms $\tau_\alpha : U_\alpha \to V_\alpha$ such that:

- $\{U_\alpha\}$ is an open cover of M.
- $V_\alpha \subseteq \mathbb{R}^n$ is open for all α.
- The map $\tau_{\alpha\beta} : \tau_\beta(U_\alpha \cap U_\beta) \to \tau_\alpha(U_\alpha \cap U_\beta)$ with $\tau_{\alpha\beta} = \tau_\alpha \circ \tau_\beta^{-1}$ is C^k diffeomorphism from an open subset of V_β onto an open subset of V_α.

This collection of homeomorphisms is called an atlas of class C^k.

What about smooth manifolds?

Definition 3.1.2

A real smooth manifold is a real differentiable manifold of class C^∞.

We define now smooth maps.

Definition 3.1.3

Let M, N be real differentiable manifolds of dimensions m and n respectively. Then, $F : M \to N$ is said to be a smooth map if, for each $p \in M$, there is an element $U_{M\alpha}$ of the open cover of M containing p and an element $U_{N\beta}$ of the open cover of N containing $F(p)$ with $F(U_{M\alpha}) \subseteq U_{N\beta}$, such that $\tau_{N\beta} \circ F \circ \tau_{M\alpha}^{-1}$ is smooth from $\tau_{M\alpha}(U_{M_\alpha})$ to $\tau_{N\beta}(U_{N_\beta})$ as a map from $\mathbb{R}^m$ to $\mathbb{R}^n$.

Remark 3.1.4

A smooth complex manifold of dimension n has the structure of a real smooth manifold of dimension $2n$.

Tangent spaces can be defined in several ways. One of the interesting ways is the following one:

Definition 3.1.5

Suppose M is a C^k manifold of real dimension n, with $k \geq 1$. Suppose $p \in M$. Let $\tau_\alpha : U_\alpha \to \mathbb{R}^n$ be an element of the atlas of M with U_α containing p. Let

$$C = \{\gamma : (-1, 1) \to U_\alpha : \gamma(0) = p, \tau_\alpha \circ \gamma : (-1, 1) \to \mathbb{R}^n \text{ is differentiable}\}.$$

Define an equivalence relation on C as follows:

$$\gamma_1 \sim \gamma_2 \Leftrightarrow (\tau_\alpha \circ \gamma_1)'(0) = (\tau_\alpha \circ \gamma_2)'(0),$$

and let T_pM be the set of equivalence classes under $\sim$. T_pM is called the tangent space to M at p, and the elements of T_pM are called tangent vectors.

The following theorem shows that the tangent space to a manifold at a point can be turned into a vector space over $\mathbb{R}$ of dimension n.

Theorem 3.1.6

Suppose M is a C^k manifold of real dimension n, with $k \geq 1$. Suppose $p \in M$. Let $\tau_\alpha : U_\alpha \to \mathbb{R}^n$ be an element of the atlas of M with U_α containing p. Consider the equivalence relation defined above, and define $d_p\tau_\alpha : T_pM \to \mathbb{R}^n$ by

$$d_p\tau_\alpha([\gamma]) = (\tau_\alpha \circ \gamma)'(0).$$

Then,

(i) $d_p\tau_\alpha$ is a bijection.

(ii) T_pM can be turned into a vector space over $\mathbb{R}$.

Proof. (i) Let's show that $d_p\tau_\alpha$ is a bijection.
First, $d_p\tau_\alpha$ is $1-1$: Let $[\gamma_1], [\gamma_2] \in T_pM$ such that $(\tau_\alpha \circ \gamma_1)'(0) = (\tau_\alpha \circ \gamma_2)'(0)$, then $\gamma_1 \sim \gamma_2$, and so $[\gamma_1] = [\gamma_2]$.
Second, $d_p\tau_\alpha$ is onto: Let $v \in \mathbb{R}^n$. Define $\alpha : (-1, 1) \to \mathbb{R}^n$ by $\alpha(t) = tv$. Then, $d_p\tau_\alpha([\tau_\alpha^{-1} \circ \alpha]) = \alpha'(0) =$

v, with $\tau_\alpha^{-1} \circ \alpha : (-1,1) \to U_\alpha \in C$.
(ii) Let $[\gamma_1], [\gamma_2] \in T_pM, \lambda \in \mathbb{R}$. Define:

$$[\gamma_1] \oplus [\gamma_2] = (d_p\tau_\alpha)^{-1}(d_p\tau_\alpha[\gamma_1] + d_p\tau_\alpha[\gamma_2]),$$

$$\lambda \odot [\gamma_1] = (d_p\tau_\alpha)^{-1}(\lambda d_p\tau_\alpha[\gamma_1]).$$

Hence, any vector v in T_pM can be written as $v = \sum_{i=1}^{n} v_i \frac{\partial}{\partial x_i}$, where $\left\{ \frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n} \right\}$ is a basis of T_pM. □

Next, we define the differential of a smooth map.

Definition 3.1.7

Let $f : M \to \mathbb{R}^n$ be a smooth map. Let $p \in M$. Define the differential of f at p to be the linear map $d_pf : T_pM \to \mathbb{R}^n$ such that

$$d_pf(v) = \sum_{i=1}^{n} v_i \frac{\partial f}{\partial x_i}(p) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(p) dx_i(v).$$

We proceed to define the tangent and cotangent bundles of a manifold.

Definition 3.1.8

Let M be a smooth manifold. We define the tangent bundle of M, denoted by TM, to be the disjoint union of the tangent spaces at all points of M, and the cotangent bundle of M, denoted by TM^*, to be the disjoint union of the cotangent spaces at all points of M.

Differential forms constitute an important part of the field of differential geometry.

Definition 3.1.9

Let M be a smooth manifold. A differential 1-form $\alpha : M \to TM^*$ is defined as follows:

$$p \to \alpha_p = \sum_j w_j(p) dx_j.$$

If α is a 1-form, then

$$d\alpha = \sum_{i,j} \frac{\partial w_j}{\partial x_i} dx_i \wedge dx_j.$$

Pullbacks can be defined for a k-differential form, and they have some interesting properties. However, we restrict our definition to a 1-form.

Definition 3.1.10

Suppose $F : M \to N$ is a smooth map, and α is a differential 1-form on N. Define the pullback $F^*\alpha$ differential 1-form on M by

$$(F^*\alpha)_p(v) = \alpha_{F(p)}(d_pF(v)).$$

Next, we show that the tangent space to an embedded submanifold with a defining function is the kernel of the differential of that defining map. However, some defintions are required first.

Definition 3.1.11

Suppose M is a smooth manifold. An embedded submanifold of M is a subset S of M that is a manifold in the subspace topology, such that the inclusion map $S \to M$ is a smooth embedding.

Definition 3.1.12

Suppose M is a smooth manifold, and $\rho : M \to \mathbb{R}$ is a smooth map. A level set $\rho^{-1}(c) = \{p \in M : \rho(p) = c\}$ is called a regular level set if for every point $p \in \rho^{-1}(c)$, the map $d_p\rho : T_pM \to \mathbb{R}$ is surjective. In particular, if $\rho^{-1}(c) = \phi$, then $\rho^{-1}(c)$ is a regular level set of ρ.

Definition 3.1.13

Suppose M is a smooth manifold. Assume $S \subseteq M$ is an embedded submanifold. A smooth map $\rho : M \to \mathbb{R}$ such that S is a regular level set of ρ is a defining map for S.

Theorem 3.1.14

Assume $S \subseteq \mathbb{R}^m$ is a hypersurface, and $\rho : \mathbb{R}^m \to \mathbb{R}$ is a defining map for S. Then, $T_pS = ker\ d_p\rho$, for each $p \in S$.

Proof. Consider the inclusion map $i : S \to \mathbb{R}^m$. Then, $d_pi(T_pS) \subseteq \mathbb{R}^m$. Note that $\rho \circ i$ is constant on S, so $d_p\rho \circ d_pi$ is the zero map from T_pS to $\mathbb{R}$, hence $Im\ d_pi \subseteq ker\ d_p\rho$. Since $d_p\rho$ is surjective, then, by the rank-nullity law, we have

$$dim(Ker\ d_p\rho) = dim(\mathbb{R}^m) - dim(\mathbb{R}) = dim(T_pS) = dim(Im\ d_pi).$$

Therefore, $Im\ d_pi = ker\ d_p\rho$.

$\square$

We define now the complex tangent space and we provide some useful examples.

Definition 3.1.15

Assume $\Omega = \{\rho < 0\}$, where $\rho : \mathbb{C}^n \to \mathbb{R}$ is a defining function for $\partial\Omega$. Let $p \in \partial\Omega$. We define the complex tangent space to $\partial\Omega$ at p to be

$$T_p^{\mathbb{C}}(\partial\Omega) = T_p(\partial\Omega) \cap iT_p(\partial\Omega),$$

where $T_p(\partial\Omega) = ker\ \nabla\rho(p)$ is the real tangent space at p.

Example 3.1.16

Consider the open unit ball $B^2(0,1) = \{z = (z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 < 1\}$. So, $\rho(z_1, z_2) = |z_1|^2 + |z_2|^2 - 1$ or $\rho(x_1, y_1, x_2, y_2) = x_1^2 + y_1^2 + x_2^2 + y_2^2 - 1$. Let $p = (p_1, p_2) \in \mathbb{C}^2$ satisfy $|p_1|^2 + |p_2|^2 = 1$. Write $p_1 = a_1 + ib_1, p_2 = a_2 + ib_2$. Then,
$ker\ \nabla\rho(a_1, b_1, a_2, b_2) = \{(x_1, y_1, x_2, y_2) \in \mathbb{R}^4 : a_1x_1 + b_1y_1 + a_2x_2 + b_2y_2 = 0\}$.
$\Rightarrow T_p\{\rho = 0\} = ker\ \nabla\rho(p_1, p_2) = \{(z_1, z_2) \in \mathbb{C}^2 : \mathcal{R}e(\overline{p_1}z_1 + \overline{p_2}z_2) = 0\}$.
Consequently, $iT_p\{\rho = 0\} = ker\ \nabla\rho(p_1, p_2) = \{(z_1, z_2) \in \mathbb{C}^2 : \mathcal{I}m(\overline{p_1}z_1 + \overline{p_2}z_2) = 0\}$.
Therefore, $T_p^{\mathbb{C}}\{\rho = 0\} = \{(z_1, z_2) \in \mathbb{C}^2 : \overline{p_1}z_1 + \overline{p_2}z_2 = 0\}$.

Example 3.1.17

Consider the bidisc $D^2(0,1) = \{z \in \mathbb{C}^2 : |z_1| < 1, |z_2| < 1\}$. So,

$$\rho(z_1, z_2) = max\{|z_1|^2, |z_2|^2\} - 1 = \begin{cases} |z_2|^2 - 1, & |z_2| \geq |z_1| \\ |z_1|^2 - 1, & |z_1| \geq |z_2| \end{cases},$$

or

$$\rho(x_1, y_1, x_2, y_2) = \begin{cases} x_2^2 + y_2^2 - 1, & x_2^2 + y_2^2 \geq x_1^2 + y_1^2 \\ x_1^2 + y_1^2 - 1, & x_2^2 + y_2^2 \leq x_1^2 + y_1^2 \end{cases}.$$

Let $p = (p_1, p_2) \in \mathbb{C}^2$ satisfies $max\{|p_1|^2, |p_2|^2\} = 1$.
Write $p_1 = a_1 + ib_1, p_2 = a_2 + ib_2$. Then, 2 cases come into play:
Case 1: $|p_2| > |p_1|$. In such a case,
$ker\ \nabla\rho(a_1, b_1, a_2, b_2) = \{(x_1, y_1, x_2, y_2) \in \mathbb{R}^4 : a_2x_2 + b_2y_2 = 0\}$.
$\Rightarrow T_p\{\rho = 0\} = ker\ \nabla\rho(p_1, p_2) = \{(z_1, z_2) \in \mathbb{C}^2 : \mathcal{R}e(\overline{p_2}z_2) = 0\}$.
Consequently, $iT_p\{\rho = 0\} = \{(z_1, z_2) \in \mathbb{C}^2 : \mathcal{I}m(\overline{p_2}z_2) = 0\}$.
Therefore, $T_p^{\mathbb{C}}\{\rho = 0\} = \{(z_1, z_2) \in \mathbb{C}^2 : \overline{p_2}z_2 = 0\} = \{(z_1, 0) \in \mathbb{C}^2\}$.
Case 2: $|p_1| > |p_2|$.
Similarly, we get $T_p^{\mathbb{C}}\{\rho = 0\} = \{(z_1, z_2) \in \mathbb{C}^2 : \overline{p_1}z_1 = 0\} = \{(0, z_2) \in \mathbb{C}^2\}$.

3.2 J_{st} - holomorphic maps

J-holomorphic curves are smooth maps, introduced by Mikhail Gromov in 1985, and satisfying the Cauchy-Riemann equations. In this section, we will handle J-holomorphic maps, where J is the standard complex structure, denoted by J_{st}.

We start our section by defining complex structures.

Definition 3.2.1

A complex structure on a real vector space V is a real linear transformation $J : V \to V$ such that $J^2 = -Id_V$.

We define the standard complex structure on $\mathbb{R}^{2n}$ as an example.

Example 3.2.2

Let $J_{st} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be the linear transformation defined by

$$J_{st}\begin{pmatrix} x_1 \\ y_1 \\ \vdots \\ x_n \\ y_n \end{pmatrix} = \begin{pmatrix} 0 & -1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & -1 \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ \vdots \\ x_n \\ y_n \end{pmatrix} = \begin{pmatrix} -y_1 \\ x_1 \\ \vdots \\ -y_n \\ x_n \end{pmatrix}$$

See that $J_{st}^2 = -Id$, hence J_{st} is a complex structure. It is called the standard complex structure.

Next, we show that $(\mathbb{R}^{2n}, J)$ is a $\mathbb{C}$-vector space.

Theorem 3.2.3

Let $J : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a complex structure. Then, $(\mathbb{R}^{2n}, J)$ is a $\mathbb{C}$-vector space.

Proof. Let $v_1 = \begin{pmatrix} x_1 \\ y_1 \\ \vdots \\ x_n \\ y_n \end{pmatrix}, v_2 = \begin{pmatrix} a_1 \\ b_1 \\ \vdots \\ a_n \\ b_n \end{pmatrix} \in \mathbb{R}^{2n},\ c = \alpha + i\beta \in \mathbb{C}$.

Define: $v_1 \oplus v_2 = \begin{pmatrix} x_1 + a_1 \\ y_1 + b_1 \\ \vdots \\ x_n + a_n \\ y_n + b_n \end{pmatrix}$ and $c \odot v = \alpha \begin{pmatrix} x_1 \\ y_1 \\ \vdots \\ x_n \\ y_n \end{pmatrix} + \beta J \begin{pmatrix} x_1 \\ y_1 \\ \vdots \\ x_n \\ y_n \end{pmatrix}$.

□

Next, we provide a definition of a J_{st}-holomorphic map.

Definition 3.2.4

Let $\Omega \subseteq \mathbb{R}^{2n}$ and $\Omega' \subseteq \mathbb{R}^{2m}$ be domains. Let $f : \Omega \to \Omega'$ be a real differentiable function. We say that f is J_{st}-holomorphic $\Leftrightarrow d_pf \circ J_{st} = J_{st} \circ d_pf$.

We end our section by showing that J_{st}-holomorphic maps satisfy the Cauchy-Riemann equations.

Theorem 3.2.5

Let $\Omega \subseteq \mathbb{R}^{2n}$ and $\Omega' \subseteq \mathbb{R}^{2m}$. Let $f : \Omega \to \Omega'$ be a real differentiable function. We say that f is J_{st}-holomorphic $\Leftrightarrow f$ satisfies the Cauchy-Riemann equations on Ω.

Proof. Write $f\begin{pmatrix} x_1 \\ y_1 \\ \vdots \\ x_n \\ y_n \end{pmatrix} = \begin{pmatrix} u_1(x_1, y_1, ..., x_n, y_n) \\ v_1(x_1, y_1, ..., x_n, y_n) \\ \vdots \\ u_m(x_1, y_1, ..., x_n, y_n) \\ v_m(x_1, y_1, ..., x_n, y_n) \end{pmatrix}$.

Then, it is easy to check that f is J_{st}-holomorphic $\Leftrightarrow d_p f \circ J_{st} = J_{st} \circ d_p f \Leftrightarrow \dfrac{\partial u_k}{\partial x_j} = \dfrac{\partial v_k}{\partial y_j}$ and $\dfrac{\partial u_k}{\partial y_j} = -\dfrac{\partial v_k}{\partial x_j}$, $k = 1, ..., m$, $j = 1, ..., n$. □

3.3 The Levi Form

In this section, we will introduce the Levi form and show that it's invariant under biholomorphism maps.

We start our section by the definition of the Levi form.

Definition 3.3.1

Suppose $\rho : \mathbb{C}^n \to \mathbb{R}$ is a C^2 function. Define the Levi form of ρ at $p \in \mathbb{C}^n$ in the direction $v \in \mathbb{C}^n$ to be

$$\mathcal{L}_{\rho(p)}(v) = \sum_{i,j=1}^{n} \frac{\partial^2 \rho(p)}{\partial z_i \partial \overline{z_j}} v_i \overline{v_j}.$$

Next, we show that the Levi form is real-valued.

Theorem 3.3.2

If $\rho : \mathbb{C}^n \to \mathbb{R}$ is a C^2 function, $p, v \in \mathbb{C}^n$, then $\mathcal{L}_{\rho(p)}(v) \in \mathbb{R}$.

Proof. $\overline{\dfrac{\partial^2 \rho(p)}{\partial z_i \partial \overline{z_j}} v_i \overline{v_j}} = \dfrac{\partial^2 \overline{\rho}(p)}{\partial \overline{z_i} \partial z_j} \overline{v_i} v_j = \dfrac{\partial^2 \rho(p)}{\partial \overline{z_i} \partial z_j} \overline{v_i} v_j.$

□

Let's compute the Levi form of the defining function of the open unit ball in $\mathbb{C}^2$.

Example 3.3.3

Consider $B^2(0,1) = \{z \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 < 1\}$.
Then, $\rho(z_1, z_2) = |z_1|^2 + |z_2|^2 - 1$. Let $p = (p_1, p_2) \in \mathbb{C}^2$, $v = (v_1, v_2) \in \mathbb{C}^2$. Hence,

$$\begin{aligned}
\mathcal{L}_{\rho(p)}(v) &= \sum_{i,j=1}^{2} \frac{\partial^2 \rho(p)}{\partial z_i \partial \overline{z_j}} v_i \overline{v_j} \\
&= \begin{pmatrix} \overline{v_1} & \overline{v_2} \end{pmatrix} \begin{pmatrix} \frac{\partial^2 \rho(p)}{\partial z_1 \partial \overline{z_1}} & \frac{\partial^2 \rho(p)}{\partial z_2 \partial \overline{z_1}} \\ \frac{\partial^2 \rho(p)}{\partial z_1 \partial \overline{z_2}} & \frac{\partial^2 \rho(p)}{\partial z_2 \partial \overline{z_2}} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\
&= \begin{pmatrix} \overline{v_1} & \overline{v_2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\
&= |v_1|^2 + |v_2|^2.
\end{aligned}$$

What about the Levi form of the defining function of the bidisc?

Example 3.3.4

Consider $D^2(0,1) = \{z \in \mathbb{C}^2 : |z_1| < 1 \textit{ and } |z_2| < 1\} = \{z \in \mathbb{C}^2 : max\{|z_1|^2, |z_2|^2\} < 1\}$.
Then,

$$\rho(z_1, z_2) = max\{|z_1|^2, |z_2|^2\} - 1 = \begin{cases} |z_2|^2 - 1, & |z_2| \geq |z_1| \\ |z_1|^2 - 1, & |z_1| \geq |z_2| \end{cases}.$$

Let $p = (p_1, p_2) \in \mathbb{C}^2$ be such that $|p_1| \neq |p_2|$. Let $v = (v_1, v_2) \in \mathbb{C}^2$. Two cases come into play:
Case 1: $|p_2| > |p_1|$. In that case, we have

$$\begin{aligned}
\mathcal{L}_{\rho(p)}(v) &= \begin{pmatrix} \overline{v_1} & \overline{v_2} \end{pmatrix} \begin{pmatrix} \frac{\partial^2 \rho(p)}{\partial z_1 \partial \overline{z_1}} & \frac{\partial^2 \rho(p)}{\partial z_2 \partial \overline{z_1}} \\ \frac{\partial^2 \rho(p)}{\partial z_1 \partial \overline{z_2}} & \frac{\partial^2 \rho(p)}{\partial z_2 \partial \overline{z_2}} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\
&= \begin{pmatrix} \overline{v_1} & \overline{v_2} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\
&= |v_2|^2.
\end{aligned}$$

Case 2: $|p_1| > |p_2|$. In that case, we have $\mathcal{L}_{\rho(p)}(v) = |v_1|^2$.

We proceed now to show, via Theorems 3.3.6, 3.3.7 and 3.3.8, that the Levi form is invariant under biholomorphic maps.

Definition 3.3.5

Let $\rho : \mathbb{R}^{2n} \to \mathbb{R}$ be a C^2 function.
Define $d_J^{\mathbb{C}}\rho : \mathbb{R}^{2n} \to \mathbb{R}$ by $d_J^{\mathbb{C}}\rho(v) = -d\rho(J_{st}v)$.

From now on, J stands for J_{st}.

Theorem 3.3.6

Let $v \in \mathbb{R}^{2n}$. Then,

$$d_J^{\mathbb{C}}\rho(v) = \sum \frac{\partial \rho}{\partial x_i} dy_i - \sum \frac{\partial \rho}{\partial y_i} dx_i.$$

Proof. Let $v = (a_1, b_1, ..., a_n, b_n) \in \mathbb{R}^{2n}$. Then,

$$\begin{aligned} d_J^{\mathbb{C}}\rho(v) &= -d\rho(Jv) \\ &= d\rho\left(\sum b_i \frac{\partial}{\partial x_i} - \sum a_i \frac{\partial}{\partial y_i}\right) \\ &= \sum b_i \frac{\partial \rho}{\partial x_i} - \sum a_i \frac{\partial \rho}{\partial y_i} \\ &= \sum \frac{\partial \rho}{\partial x_i} dy_i - \sum \frac{\partial \rho}{\partial y_i} dx_i. \end{aligned}$$

The last equality is justified by the fact that

$$-dx_i(v) = d_J^{\mathbb{C}} x_i(J^{-1}v) = -a_i \forall i$$

and

$$-dy_i(v) = d_J^{\mathbb{C}} y_i(J^{-1}v) = -b_i \forall i$$

□

Theorem 3.3.7

Let $\rho : \mathbb{R}^{2n} \to \mathbb{R}$ be a C^2 function. Let $p, v \in \mathbb{R}^{2n}$. Then,

$$dd_J^{\mathbb{C}}\rho_p(v, Jv) = 4\mathcal{L}_{\rho(p)}(v).$$

Proof. Let $v = (a_1, b_1, ..., a_n, b_n) \in \mathbb{R}^{2n}$.
$dd_J^{\mathbb{C}}\rho = d\left(\sum \frac{\partial \rho}{\partial x_i} dy_i - \sum \frac{\partial \rho}{\partial y_i} dx_i\right)$

$$= -\sum \frac{\partial^2 \rho}{\partial x_j \partial y_l} dx_j \wedge dx_l \; - \; \sum \frac{\partial^2 \rho}{\partial y_j \partial y_l} dy_j \wedge dx_l \; + \; \sum \frac{\partial^2 \rho}{\partial x_j \partial x_l} dx_j \wedge dy_l \; + \; \sum \frac{\partial^2 \rho}{\partial y_j \partial x_l} dy_j \wedge dy_l.$$

Then,

$$
\begin{aligned}
dd^{\mathbb{C}}_J\rho(v,Jv) = & -\sum \frac{\partial^2\rho}{\partial x_j \partial y_l}(dx_j(v)dx_l(Jv) - dx_l(v)dx_j(Jv)) \\
& -\sum \frac{\partial^2\rho}{\partial y_j \partial y_l}(dy_j(v)dx_l(Jv) - dx_l(v)dy_j(Jv)) \\
& +\sum \frac{\partial^2\rho}{\partial x_j \partial x_l}(dx_j(v)dy_l(Jv) - dx_j(Jv)dy_l(v)) \\
& +\sum \frac{\partial^2\rho}{\partial y_j \partial x_l}(dy_j(v)dy_l(Jv) - dy_j(Jv)dy_l(v)) \\
= & -\sum \frac{\partial^2\rho}{\partial x_j \partial y_l}(a_l b_j - a_j b_l) + \sum \frac{\partial^2\rho}{\partial y_j \partial y_l}(a_l a_j + b_j b_l) \\
& +\sum \frac{\partial^2\rho}{\partial x_j \partial x_l}(a_j a_l + b_j b_l) + \sum \frac{\partial^2\rho}{\partial y_j \partial x_l}(a_l b_j - a_j b_l) \\
= & \sum \left(\frac{\partial^2\rho}{\partial x_j \partial x_l} + \frac{\partial^2\rho}{\partial y_j \partial y_l} \right)(a_l a_j + b_l b_j) \\
& +\sum \left(\frac{\partial^2\rho}{\partial y_j \partial x_l} - \frac{\partial^2\rho}{\partial x_j \partial y_l} \right)(a_l b_j - a_j b_l).
\end{aligned}
$$

But,

$$
\begin{aligned}
\frac{\partial^2}{\partial x_j \partial x_l} &= \frac{\partial}{\partial x_j}\left[\frac{\partial}{\partial z_l} + \frac{\partial}{\partial \overline{z_l}}\right] = \frac{\partial^2}{\partial z_j \partial z_l} + \frac{\partial^2}{\partial z_j \partial \overline{z_l}} + \frac{\partial^2}{\partial \overline{z_j} \partial z_l} + \frac{\partial^2}{\partial \overline{z_j} \partial \overline{z_l}}, \\
\frac{\partial^2}{\partial y_j \partial y_l} &= -i\frac{\partial}{\partial y_j}\left[\frac{\partial}{\partial \overline{z_l}} - \frac{\partial}{\partial z_l}\right] = -\left[\frac{\partial^2}{\partial \overline{z_j} \partial \overline{z_l}} - \frac{\partial^2}{\partial \overline{z_j} \partial z_l} - \frac{\partial^2}{\partial z_j \partial \overline{z_l}} + \frac{\partial^2}{\partial z_j \partial z_l}\right], \\
\frac{\partial^2}{\partial y_j \partial x_l} &= \frac{\partial}{\partial y_j}\left[\frac{\partial}{\partial z_l} + \frac{\partial}{\partial \overline{z_l}}\right] = -i\left[\frac{\partial^2}{\partial \overline{z_j} \partial z_l} + \frac{\partial^2}{\partial \overline{z_j} \partial \overline{z_l}} - \frac{\partial^2}{\partial z_j \partial z_l} - \frac{\partial^2}{\partial z_j \partial \overline{z_l}}\right], \\
\frac{\partial^2}{\partial x_j \partial y_l} &= -i\frac{\partial}{\partial x_j}\left[\frac{\partial}{\partial \overline{z_l}} - \frac{\partial}{\partial z_l}\right] = -i\left[\frac{\partial^2}{\partial z_j \partial \overline{z_l}} - \frac{\partial^2}{\partial z_j \partial z_l} + \frac{\partial^2}{\partial \overline{z_j} \partial \overline{z_l}} - \frac{\partial^2}{\partial \overline{z_j} \partial z_l}\right].
\end{aligned}
$$

Hence,

$$
\begin{aligned}
dd^{\mathbb{C}}_J\rho(v,Jv) = & \sum 2\left[\frac{\partial^2\rho}{\partial \overline{z_j} \partial z_l} + \frac{\partial^2\rho}{\partial z_j \partial \overline{z_l}}\right](a_l a_j + b_l b_j) \\
& +\sum 2i\left[\frac{\partial^2\rho}{\partial z_j \partial \overline{z_l}} - \frac{\partial^2\rho}{\partial \overline{z_j} \partial z_l}\right](a_l b_j - a_j b_l) \\
= & \sum 2\frac{\partial^2\rho}{\partial \overline{z_j} \partial z_l}[(a_l a_j + b_l b_j) - i(a_l b_j - a_j b_l)] \\
& +\sum 2\frac{\partial^2\rho}{\partial z_j \partial \overline{z_l}}[(a_l a_j + b_l b_j) + i(a_l b_j - a_j b_l)] \\
= & \sum 2\frac{\partial^2\rho}{\partial \overline{z_j} \partial z_l} v_l \overline{v_j} + \sum 2\frac{\partial^2\rho}{\partial z_j \partial \overline{z_l}} \overline{v_l} v_j \\
= & \sum 2\left(\frac{\partial^2\rho}{\partial \overline{z_j} \partial z_l} + \frac{\partial^2\rho}{\partial z_l \partial \overline{z_j}}\right) v_l \overline{v_j} \\
= & \, 4\sum \frac{\partial^2\rho}{\partial z_l \partial \overline{z_j}} v_l \overline{v_j}.
\end{aligned}
$$

Therefore,

$$
dd^{\mathbb{C}}_J\rho_p(v,Jv) = 4\mathcal{L}_{\rho(p)}(v).
$$

□

Theorem 3.3.8

Suppose $\rho : \mathbb{C}^n \to \mathbb{R}$ be a C^2 function. Suppose $F : \Omega \subseteq \mathbb{R}^{2n} \to \Omega' \subseteq \mathbb{R}^{2m}$ is a biholomorphism. If $p \in \Omega, v \in \mathbb{C}^n$, then

$$\mathcal{L}_{\rho(p)}(v) = \mathcal{L}_{\rho\circ F^{-1}(F(p))}(d_pF(v)).$$

Proof. It is enough to show that $dd^{\mathbb{C}}_J\rho_p(v, Jv) = dd^{\mathbb{C}}_J(\rho\circ F^{-1})_{F(p)}(d_pF(v), d_pF(Jv))$, and the rest follows from Theorem 3.3.7. Note that

$$\begin{aligned}
d^{\mathbb{C}}_J(\rho\circ F^{-1})_{F(p)}(d_pF(v)) &= -d(\rho\circ F^{-1})_{F(p)}(Jd_pF(v)) = -d(\rho\circ F^{-1})_{F(p)}(d_pF(Jv)) \\
&= -F^*d(\rho\circ F^{-1})_p(Jv) = -dF^*(\rho\circ F^{-1})_p(Jv) \\
&= -d(\rho\circ F^{-1}\circ F)_p(Jv) = -d\rho_p(Jv) \\
&= d^{\mathbb{C}}_J\rho(v).
\end{aligned}$$

□

3.4 Plurisubharmonic Functions and Pseudoconvex Domains

Pseudoconvex domains are extensively used in the theory of several complex variables. In this section, we will just define them, and mention a useful test for domains having C^2 boundaries. Moreover, we will show that two non-empty proper simply-connected domains in $\mathbb{C}^2$ are not necessarily biholomorphic.

Definition 3.4.1

Let $f : A \to \mathbb{R}\cup\{-\infty; +\infty\}$ be a function defined on a subset A of $\mathbb{R}^m$, and let $x_0 \in A$. Then:

a. f is said to be upper semi-continuous at x_0 if $f(x_0) \geq \limsup\limits_{x\to x_0} f(x)$.

b. f is said to be lower semi-continuous at x_0 if $f(x_0) \leq \liminf\limits_{x\to x_0} f(x)$.

In particular, if $f(x_0) = +\infty$ (or $f(x_0) = -\infty$), then f is upper (or lower) semi-continuous at x_0. Finally, f is upper (lower) semi-continuous on A if it is upper (lower) semi-continuous at every point of A.

Definition 3.4.2

A function $u(z)$ is said to be subharmonic in a domain $G \subset \mathbb{C}$ if the followings are satisfied:

1. $-\infty \leq u(z) < +\infty$ in G.
2. $u(z)$ is upper semi-continuous in G.
3. For any subdomain G' of G and any function $U(z)$ that is harmonic in G' and continuous on $\overline{G'}$, the inequality $u(z) \leq U(z)$ on $\partial G'$ implies $u(z) \leq U(z)$ in G'.

A function $u(z)$ such that $-u(z)$ is subharmonic is called superharmonic.

Definition 3.4.3

A function $u(z)$ is said to be plurisubharmonic in a domain $G \subset \mathbb{C}^n$ if the followings are satisfied:

1. $-\infty \leq u(z) < +\infty$ in G.
2. $u(z)$ is upper semi-continuous in G.
3. For any $z_0 \in \mathbb{C}^n$, $a \in \mathbb{C}^n$, the function $u(z_0 + \lambda a)$ is subharmonic on $G = \{\lambda \in \mathbb{C} : z_0 + \lambda a \in G\}$.

A function $u(z)$ is said to be plurisuperhamonic if $-u(z)$ is a plurisubharmonic function.

Definition 3.4.4

Let G be a domain in $\mathbb{C}^n$, $z \in G$. Define

$$d(z, \partial G) := \sup_{B^n(z,r) \subset G} r$$

G is said the be pseudoconvex if the function $-ln(d(z, \partial G))$ is plurisubharmonic in G.

If G has a C^2 boundary, a very special test allows us to check whether a domain is pseudoconvex or not. Our next goal is to state the test, but no proof will be provided.

Theorem 3.4.5

Suppose G has a C^2 boundary. Then, it can be shown that G has a defining function, i.e. there exists $\rho : \mathbb{C}^n \to \mathbb{R}$ which is C^2 so that $G = \{\rho < 0\}$ and $\partial G = \{\rho = 0\}$. In such a case, G is pseudoconvex if and only if for every $p \in \partial G$ and $w \in T_p \partial G$, we have

$$\mathcal{L}_{\rho(p)}(w) = \sum_{i,j=1}^{n} \frac{\partial^2 \rho(p)}{\partial z_i \partial \overline{z_j}} w_i \overline{w_j} \geq 0.$$

Recall that two non-empty proper simply-connected domains in $\mathbb{C}$ are biholomorphic, and this is a consequence of the Riemann mapping Theorem. Does this theorem hold in the theory of several complex variables?

Well, we end our chapter by showing that the complex ellipsoid

$$E = \left\{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^4 < 1\right\}$$

and the open unit ball in $\mathbb{C}^2$ are not biholomorphic. However, we will use the following theorem proved by Bell in 1981: If $F : D_1 \to D_2$ is a biholomorphic map between two pseudoconvex domains with C^2 boundaries, then F extends to a diffeomorphism of $\overline{D_1}$ and $\overline{D_2}$ if at least one of them is a smooth bounded strictly pseudoconvex domain.

Theorem 3.4.6

The complex ellipsoid E and the open unit ball in C^2 are not biholomorphic.

Proof. Let $\rho(z_1, z_2) = |z_1|^2 + |z_2|^4 - 1$ be the defining function of E, and note that

$$\mathcal{L}_{\rho(p)}(v) = |v_1|^2 + 4|p_2|^2|v_2|^2$$

for all $p = (p_1, p_2), v = (v_1, v_2) \in \mathbb{C}^2$.
Now, suppose there exists a biholomorphism $F : E \to B^2(0, 1)$. By the theorem mentioned above, F extends to a diffeomorphism of $\overline{E}$ and $\overline{B}^2(0, 1)$. Let $p = (1, 0)$ and $v = (0, 1)$. Obviously,

$$\mathcal{L}_{\rho(p)}(v) = 0$$

whereas

$$\mathcal{L}_{\rho\circ F^{-1}(F(p))}(d_pF(v)) = |d_pF(v)|^2 \neq 0$$

because $v \neq 0$ and F is a diffeomorphism. Hence, we get a contradiction! $\square$

Chapter 4

Stationary and Extremal Maps

In his paper, the Hungarian-American mathematician Laszlo Lempert defined stationary maps and used them to get some useful applications in the field of complex analysis. In this chapter, we will provide some definitions and prove Fefferman Theorem for special domains.

4.1 The Kobayashi Metric

The kobayashi metric, introduced by the Japanese-American mathematician Shoshichi Kobayashi in 1967, is a pseudometric defined on any complex manifold. This metric plays an important role in complex geometry, since it is invariant under a biholomorphic map. In this section, we will show this latter idea.

We start our section by recalling the Poincaré metric. we will denote the open unit disc in $\mathbb{C}$ by U.

Definition 4.1.1

Let $z \in U, v \in \mathbb{C}$. Define the Poincaré metric at (z, v) to be

$$\rho_U(z,v) = \frac{|v|}{1-|z|^2}.$$

If $\Omega \subseteq \mathbb{C}^n$ and $\Omega' \subseteq \mathbb{C}^m$, we denote the set of all holomorphic functions from Ω into Ω' by $Holo(\Omega, \Omega')$.

Let's define now the kobayashi metric.

Definition 4.1.2

Let $\Omega \subseteq \mathbb{C}^n$.For $p \in \Omega$, $v \in T_P\Omega$, define the Kobayashi metric at (p, v) to be

$$K_\Omega(p,v) := inf\left\{\frac{1}{r} > 0 : f \in Holo(U,\Omega), f(0) = p, f'(0) = rv\right\}.$$

The following theorem shows that the Kobayashi metric of $\mathbb{C}^n$ vanishes.

Theorem 4.1.3

Let $z, v \in \mathbb{C}^n$. Then $K_{\mathbb{C}^n}(z, v) = 0$.

Proof. Let $M > 0$. Let $f : U \to \mathbb{C}^n$ be the function defined by $f(\zeta) = z + M\zeta v$. It is clear that $f \in Holo(U, \mathbb{C}^n), f(0) = z$ and $f'(0) = Mv$. Hence, $K_{\mathbb{C}^n}(z, v) \leq \dfrac{1}{M}$. This is true for any $M > 0$. Therefore, $K_{\mathbb{C}^n}(z, v) = 0$.

□

Next, we show that the Kobayashi metric on the open unit disc is equal to the poincaré metric.

Theorem 4.1.4

Let $p \in U, v \in \mathbb{C}$. Then, $K_U(p, v) = \rho_U(p, v)$.

Proof. If $v = 0$, the result is obtained directly. So, assume $v \neq 0$.
Let $\varphi_p : U \to U$ be an automorphism of U defined by

$$\varphi_p(z) = \frac{p - z}{1 - \overline{p}z}.$$

If $f \in Holo(U, U)$ with $f(0) = p$ and $f'(0) = rv$ for some $r > 0$, then $\varphi_p \circ f \in Holo(U, U)$ and $(\varphi_p \circ f)(0) = 0$. By the classical Schwarz lemma, we have

$$|(\varphi_p \circ f)'(0)| \leq 1.$$

This implies that

$$\frac{r|v|}{1 - |p|^2} \leq 1.$$

Hence,

$$\frac{1}{r} \geq \frac{|v|}{1 - |p|^2}.$$

On the other hand, let $f : U \to U$ be the function defined by

$$f(z) = \frac{\frac{v}{|v|}z + p}{1 + \frac{v}{|v|}\overline{p}z}.$$

It is clear that $f \in Holo(U, U), f(0) = p$ and $f'(0) = \dfrac{1 - |p|^2}{|v|}v$. Therefore,

$$K_U(p, v) = \frac{|v|}{1 - |p|^2} = \rho_U(p, v).$$

□

The following theorem shows that the Kobayashi metric has the distance-decreasing property.

Theorem 4.1.5

Let $f : \Omega \subseteq \mathbb{C}^n \to \Omega' \subseteq \mathbb{C}^m$ be a holomorphic function on Ω. Then,

$$K_{\Omega'}(f(z), d_z f(v)) \leq K_\Omega(z, v), \qquad \forall z \in \Omega, \forall v \in T_z\Omega.$$

Proof. Let $z \in \Omega, v \in T_z\Omega$. Let $\phi : U \to \Omega$ be a holomorphic function such that $\phi(0) = z$ and $\phi'(0) = rv$ for some $r > 0$. Then, $f \circ \phi \in Holo(U, \Omega')$ satisfying $(f \circ \phi)(0) = f(z)$ and $(f \circ \phi)'(0) = f'(\phi(0)).\phi'(0) = rf'(z).v = rd_z f(v)$. Hence, $K_{\Omega'}(f(z), d_z f(v)) \leq \frac{1}{r}$. Therefore,

$$K_{\Omega'}(f(z), d_z f(v)) \leq K_\Omega(z, v).$$

□

As a consequence of Theorem 4.1.5, we show that the Koyabashi metric is invartiant under a biholomorphism.

Corollary 4.1.6

Let $f : \Omega \subseteq \mathbb{C}^n \to \Omega' \subseteq \mathbb{C}^m$ be a biholomorphic function. Then,

$$K_{\Omega'}(f(z), d_z f(v)) = K_\Omega(z, v), \qquad \forall z \in \Omega, \forall v \in T_z\Omega.$$

Proof. Let $z \in \Omega, v \in T_z\Omega$. By applying Theorem 4.1.5 twice, we get

$$K_{\Omega'}(f(z), d_z f(v)) \leq K_\Omega(z, v) = K_\Omega(f^{-1}(f(z)), d_{f(z)} f^{-1}(d_z f(v))) \leq K_{\Omega'}(f(z), d_z f(v)).$$

□

We will denote the unit ball in $\mathbb{C}^n$ by B^n and the unit polydisc by D^n. We proceed now to compute the Kobayashi metric of the unit ball and the unit polydisc in $\mathbb{C}^n$. This requires the following version of the Schwarz Lemma:

Theorem 4.1.7

Let $f : U \to B^n$ be a holomorphic function. If $f(0) = 0$, then $|f'(0)| \leq 1$.

Proof. Let $0 < r < 1$. Then,

$$\begin{aligned}
1 &> \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{it})|^2 dt \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(|f_1(re^{it})|^2 + ... + |f_n(re^{it})|^2\right) dt \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} |f_1(re^{it})|^2 dt + ... + \frac{1}{2\pi} \int_{-\pi}^{\pi} |f_n(re^{it})|^2 dt.
\end{aligned}$$

But Parseval's identity implies that

$$\sum_{n=-\infty}^{\infty} |c_{in}|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f_i(re^{it})|^2 dt, \qquad i = 1, ..., n$$

where

$$c_{in} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f_i(re^{it})e^{-int} dt, \qquad i = 1, ..., n.$$

Since each f_i is holomorphic on U, we have

$$\begin{aligned} c_{in} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\sum_{m=0}^{\infty} \frac{f_i^{(m)}(0)}{m!} r^m e^{imt}\right) e^{-int} dt \\ &= \frac{1}{2\pi}\sum_{m=0}^{\infty}\int_{-\pi}^{\pi} \frac{f_i^{(m)}(0)}{m!} r^m e^{i(m-n)t} dt \end{aligned}$$

Hence, $c_{in} = 0$ if $n < 0$ and $c_{in} = \dfrac{f_i^{(n)}(0)}{n!} r^n$ if $n \geq 0$. Note that interchanging the summation and integral symbols is allowed since $\displaystyle\sum_{m=0}^{\infty} \frac{f_i^{(m)}(0)}{m!} r^m e^{imt} e^{-int}$ converges uniformly on $(-\pi, \pi)$. Therefore,

$$\begin{aligned} 1 &> \frac{1}{2\pi}\int_{-\pi}^{\pi} |f_1(re^{it})|^2 dt + ... + \frac{1}{2\pi}\int_{-\pi}^{\pi} |f_n(re^{it})|^2 dt \\ &= (|f_1(0)|^2 + r^2|f_1'(0)|^2 + ...) + ... + (|f_n(0)|^2 + r^2|f_n'(0)|^2 + ...) \\ &\geq (|f_1(0)|^2 + r^2|f_1'(0)|^2) + ... + (|f_n(0)|^2 + r^2|f_n'(0)|^2) \\ &= |f(0)|^2 + r^2|f'(0)|^2. \end{aligned}$$

Since $f(0) = 0$, then $|f'(0)| \leq 1$. □

Note that one can also show that $|f(z)| \leq |z|$ for all $z \in U$, but this requires more tools. A sophisticated proof is available in Chapter 8 of Rudin's book 'Function Theory in the Unit Ball of $\mathbb{C}^n$'.

Theorem 4.1.8

Let $p \in B^n, v \in T_pB^n$. Then, $K_{B^n}(p, v) = |d_p\phi_p(v)|$, where ϕ_p is an automorphism of the unit ball.

Proof. Since ϕ_p is a biholomorphism, then by Corollary 4.1.6, we have

$$K_{B^n}(p, v) = K_{B^n}(\phi_p(p), d_p\phi_p(v)) = K_{B^n}(0, d_p\phi_p(v)).$$

If $d_p\phi_p(v) = 0$, then the equality is reached. So, assume $d_p\phi_p(v) \neq 0$. If $f \in Holo(U, B^n)$ is such that $f(0) = 0$ and $f'(0) = rd_p\phi_p(v)$ for some $r > 0$, then by the Schwarz Lemma, we have

$$|f'(0)| = r|d_p\phi_p(v)| \leq 1.$$

Hence,

$$\frac{1}{r} \geq |d_p\phi_p(v)|.$$

On the other hand, write $d_p\phi_p(v) = (d_p\phi_p(v)_1, ..., d_p\phi_p(v)_n)$, and let $f : U \to B^n$ be the function defined by

$$f(z) = \left(\frac{d_p\phi_p(v)_1}{|d_p\phi_p(v)|}z, ..., \frac{d_p\phi_p(v)_n}{|d_p\phi_p(v)|}z\right).$$

Clearly, f is holomorphic on U. Also, $f(0) = 0$ and $f'(z) = \dfrac{1}{|d_p\phi_p(v)|}d_p\phi_p(v)$. Therefore, $K_{B^n}(p, v) = |d_p\phi_p(v)|$. $\square$

Theorem 4.1.9

Let $p = (p_1, ..., p_n) \in D^n, v = (v_1, ..., v_n) \in T_pD^n$. Then,

$$K_{D^n}(p, v) = max\left\{\frac{|v_1|}{1-|p_1|^2}, ..., \frac{|v_n|}{1-|p_n|^2}\right\}.$$

Proof. Let $p = (p_1, ..., p_n) \in D^n, v = (v_1, ..., v_n) \in T_pD^n$. If $v = 0$, we get our result. So, assume that $v \neq 0$. Let $f \in Holo(D^1, D^n)$ be a holomorphic function such that $f(0) = p$ and $f'(0) = rv$ for some $r > 0$. Let $\phi : D^n \to D^n$ be the map defined by

$$\phi(z_1, ..., z_n) = \left(\frac{p_1 - z_1}{1 - \overline{p_1}z_1}, ..., \frac{p_n - z_n}{1 - \overline{p_n}z_n}\right).$$

Let $h = \phi \circ f : D^1 \to D^n$. Then for any $z \in D^1$, we have

$$(\phi \circ f)(z) = \phi(f(z)) = \phi(f_1(z), f_2(z), ..., f_n(z)) = \left(\frac{p_1 - f_1(z)}{1 - \overline{p_1}f_1(z)}, ..., \frac{p_n - f_n(z)}{1 - \overline{p_n}f_n(z)}\right).$$

Hence, $(\phi \circ f)(0) = (0, ..., 0)$. Let

$$h_i(z) = \frac{p_i - f_i(z)}{1 - \overline{p_i}f_i(z)}, \qquad i = 1, ..., n.$$

By the classical Schwarz lemma applied to $h_1, ..., h_n$, we get

$$|h_i'(0)| \leq 1 \ \forall i.$$

Hence,

$$\frac{r|v_i|}{1-|p_i|^2} \leq 1 \ \forall i.$$

Therefore,

$$\frac{1}{r} \geq max\left\{\frac{|v_1|}{1-|p_1|^2}, ..., \frac{|v_n|}{1-|p_n|^2}\right\}.$$

Let $M = max\left\{\dfrac{|v_1|}{1-|p_1|^2}, ..., \dfrac{|v_n|}{1-|p_n|^2}\right\}$. Let $f : D^1 \to D^n$ be the map defined by

$$f(z) = \left(\frac{p_1 - \dfrac{v_1}{M(|p_1|^2-1)}z}{1 - \dfrac{v_1}{M(|p_1|^2-1)}\overline{p_1}z}, ..., \frac{p_n - \dfrac{v_n}{M(|p_n|^2-1)}z}{1 - \dfrac{v_n}{M(|p_n|^2-1)}\overline{p_n}z}\right).$$

Obviously,f is holomorphic on D^1. Also, it is easy to show that $f(0) = p$ and $f'(0) = \dfrac{1}{M}v$. Therefore,

$$K_{D^n}(p, v) = max \left\{ \frac{|v_1|}{1 - |p_1|^2}, ..., \frac{|v_n|}{1 - |p_n|^2} \right\}.$$

$\square$

4.2 A metric for convex domains

In this section, we define a new metric for bounded convex domains and show that it is indeed a metric.

We start our section by the following definition:

Definition 4.2.1

Let D be a bounded domain in $\mathbb{C}^n$. Let $U = D^1(0, 1)$. Define the function $\delta_D : D \times D \to \mathbb{R}$ by

$$\delta_D(z_1, z_2) = \inf \left\{ d_H(\zeta_1, \zeta_2) : \exists f : U \to D : f \in Holo(U, D), f(\zeta_1) = z_1, f(\zeta_2) = z_2 \right\},$$

where

$$d_H(\zeta_1, \zeta_2) = \log \left(\frac{1 + \left| \frac{\zeta_2 - \zeta_1}{1 - \overline{\zeta_1}\zeta_2} \right|}{1 - \left| \frac{\zeta_2 - \zeta_1}{1 - \overline{\zeta_1}\zeta_2} \right|} \right).$$

We note that δ_D has the following property:

Theorem 4.2.2

δ_D has the distance decreasing property.

Proof. Let $g : D_1 \to D_2$ be a holomorphic function. Let $z_1, z_2 \in D_1$. Let $f : U \to D_1$ be a holomorphic function such that $f(\zeta_1) = z_1$ and $f(\zeta_2) = z_2$. Then $g \circ f \in Holo(U, D_2)$ with $(g \circ f)(\zeta_1) = g(z_1)$ and $(g \circ f)(\zeta_2) = g(z_2)$. Hence,

$$\delta_{D_2}(g(z_1), g(z_2)) \leq d_H(\zeta_1, \zeta_2).$$

Therefore,

$$\delta_{D_2}(g(z_1), g(z_2)) \leq \delta_{D_1}(z_1, z_2).$$

$\square$

Here is another straightforward property:

Theorem 4.2.3

Suppose $D' \subset D$ and $z_1, z_2 \in D'$. Then $\delta_D(z_1, z_2) \leq \delta_{D'}(z_1, z_2)$.

Proof. Suppose there exists $f \in Holo(U, D')$ such that $f(\zeta_1) = z_1, f(\zeta_2) = z_2$. Let $A = \{d_H(\zeta_1, \zeta_2) : \exists f \in Holo(U, D'), f(\zeta$ and $B = \{d_H(\zeta_1, \zeta_2) : \exists f \in Holo(U, D), f(\zeta_1) = z_1, f(\zeta_2) = z_2\}$. Define $g : U \to D$ by $g(\zeta) = f(\zeta)$. Thus, $A \subset B$. Therefore, $\delta_D(z_1, z_2) \leq \delta_{D'}(z_1, z_2)$. $\square$

Our next goal is to show the following interesting axiom whose proof is not available in Lempert's paper:

Theorem 4.2.4

Let $z_1, z_2 \in D$. Then, $\delta_D(z_1, z_2) = 0 \Leftrightarrow z_1 = z_2$.

Proof. $\Leftarrow$: Suppose that $z_1 = z_2$. Define $f : U \to D$ by $f(\zeta) = z_1$. Note that $f \in Holo(U, D)$ and $f(0) = z_1 = z_2$. Since $d_H(0, 0) = 0$, we have $\delta_D(z_1, z_2) = 0$.

$\Rightarrow$: Suppose now that $\delta_D(z_1, z_2) = 0$. Then, there is a sequence of functions $\{f_n\}$ such that $f_n \in Holo(U, D)$, $f_n(\zeta_n) = z_1, f_n(\eta_n) = z_2$ and $\lim\limits_{n\to\infty} d_H(\zeta_n, \eta_n) = 0$, where $\{\zeta_n\}, \{\eta_n\} \subset U$.
By Bolzano-Weierstrass, $\{\zeta_n\}$ has a subsequence $\{\zeta_{n_k}\}$ that converges to $\zeta \in \overline{U}$, and $\{\eta_{n_k}\}$ has a subsequence $\{\eta_{n_{km}}\}$ that converges to $\eta \in \overline{U}$, and so $f_{n_{km}}(\zeta_{n_{km}}) = z_1, f_{n_{km}}(\eta_{n_{km}}) = z_2$ and $d_H(\zeta_{n_{km}}, \eta_{n_{km}}) \longrightarrow 0$.
Since D is bounded, then $\{f_{n_{km}}\}$ is a sequence of holomorphic functions that is uniformly bounded, and by Arzela-Ascoli it has a subsequence $\{f_{n_{kml}}\}$ that converges uniformly, on compact subsets of U, to a function f.
Two cases come into play:
Case 1: $|\zeta| < 1$. Since $\lim\limits_{l\to\infty} d_H(\zeta_{n_{kml}}, \eta_{n_{kml}}) = 0$, then $\lim\limits_{l\to\infty} (\zeta_{n_{kml}} - \eta_{n_{kml}}) = 0$. So, $\lim\limits_{l\to\infty} \zeta_{n_{kml}} = \lim\limits_{l\to\infty} \eta_{n_{kml}}$. Hence, $\zeta = \eta$, and $|\eta| < 1$.
Let ϵ and ϵ' be small enough so that $\overline{D}^1(\zeta, \epsilon) \subset U$ and $\overline{D}^1(\eta, \epsilon') \subset U$.
Since $\lim\limits_{l\to\infty} \zeta_{n_{kml}} = \zeta$, then there exists $N \in \mathbb{N}$ such that

$$l \geq N \Rightarrow |\zeta_{kml} - \zeta| < \epsilon \Rightarrow \zeta_{kml} \in \overline{D}^1(\zeta, \epsilon).$$

Since $f_{n_{kml}}(\zeta_{n_{kml}}) = z_1$, then $\lim\limits_{n\to\infty} f_{n_{kml}}(\zeta_{n_{kml}}) = z_1$. By uniform convergence on $\overline{D}^1(\zeta, \epsilon)$, we get $f(\zeta) = z_1$. Similarly, $f(\eta) = z_2$. But $\zeta = \eta$, so $z_1 = z_2$.
Case 2: $\zeta = c, |c| = 1$. In such a case, we get $\lim\limits_{l\to\infty} d_H(\zeta_{n_{kml}}, \eta_{n_{kml}}) = \infty$, which is a contradiction. So this case is rejected.

$\square$

Next, we note that if D is convex, then δ_D satisfies the triangle inequality:

Theorem 4.2.5 [Lempert, 81]

If D is convex, then $\delta_D(z, s) \leq \delta_D(z, w) + \delta_D(w, s) \forall z, s, w \in D$.

Proof. Let $z, w, s \in D$. Let $\epsilon > 0$. There exist $f, g \in Holo(U, D), \zeta, \eta, \eta'\sigma, \in U$ such that

$$f(\zeta) = z, f(\eta) = w, \delta_D(z, w) > d_H(\zeta, \eta) - \frac{\epsilon}{2},$$

$$g(\eta') = w, g(\sigma) = s, \delta_D(w, s) > d_H(\eta', \sigma) - \frac{\epsilon}{2}.$$

WLOG, assume $\zeta = 0, \eta = \eta' > 0, \sigma > \eta$ (If not, compose f and g with suitable automorphisms of the unit disc and proceed). Assume also that f and g are continuous on $\overline{U}$ (If not, let $R = \max\{|\zeta|, |\eta|, |\eta'|, |\sigma|\}$

and let $r < R < 1$. Define $u : \overline{U} \to D$ by $u(a) = f\left(\frac{a}{r}\right)$ and $v : \overline{U} \to D$ by $v(a) = g\left(\frac{a}{r}\right)$ and proceed).
Let

$$h(\xi) = \lambda(\xi)f(\xi) + (1 - \lambda(\xi))g(\xi)$$

with

$$\lambda(\xi) = (\xi - \sigma)\left(\xi - \frac{1}{\sigma}\right)\left(\frac{1}{\xi - \eta}\right)\left(\frac{1}{\xi - \frac{1}{\eta}}\right).$$

Note that $\lambda(0) = 1, \lambda(\sigma) = 0$, λ is holomorphic on $\overline{U} - \{\eta\}$ and λ is real on ∂U with $0 < \lambda < 1$:

$$(\xi - \sigma)\left(\xi - \frac{1}{\sigma}\right)\left(\frac{1}{\xi - \eta}\right)\left(\frac{1}{\xi - \frac{1}{\eta}}\right) = \frac{\eta}{\sigma}\frac{|\xi - \sigma|^2}{|\xi - \eta|^2} < 1,$$

for all $\xi \in \partial U$.
Hence, $h : U \to \mathbb{C}^n$ is holomorphic on U (η is a removable singularity) with $h(0) = z$, $h(\sigma) = s$ and $h(\partial U) \subset \overline{D}$ (because D is convex). By the maximum principle, $h(U) \subset D$.
Therefore,

$$\delta_D(z, s) \le d_H(0, \sigma) = d_H(0, \eta) + d_H(\eta, \sigma) \le \delta_D(z, w) + \delta_D(w, s) + \epsilon.$$

□

Corollary 4.2.6

If D is convex, then δ_D is a metric.

The following example shows that if D is not convex, then δ_D is not necessarily a metric.

Example 4.2.7

Let $D_\epsilon = \{z \in \mathbb{C}^2 : |z_1| < 2, |z_2| < 2, |z_1 z_2| < \epsilon\}$. Let $P = (1, 0)$ and $Q = (0, 1)$. See that $\delta_{D_\epsilon}(0, P)$ and $\delta_{D_\epsilon}(0, Q)$ do not depend on ϵ but $\delta_{D_\epsilon}(P, Q) \to \infty$ as $\epsilon \to 0$.

Royden has shown that the Kobayashi distance on D is given by

$$\kappa_D(Z_1, Z_2) = \inf\left\{\sum_{j=1}^{k} \delta_D(w_{j-1}, w_j) : w_j \in D(j = 0, ..., k), w_0 = Z_1, w_k = Z_2\right\}$$

for all $Z_1, Z_2 \in D$. It turns out that $\kappa_D = \delta_D$ if D is a bounded convex domain:

Theorem 4.2.8

If D is convex, then $\delta_D(Z_1, Z_2) = \kappa_D(Z_1, Z_2)$.

Proof. First, note that if D is a bounded domain in $\mathbb{C}^n$, we have

$$\kappa_D(Z_1, Z_2) \le \delta_D(Z_1, Z_2) \qquad \forall Z_1, Z_2 \in D.$$

Moreover, if D is convex, then δ_D satisfies the triangle inequality. Hence, if $Z_1, Z_2 \in D$, we have

$$\delta_D(Z_1, Z_2) \leq \delta_D(Z_1, w_1) + ... + \delta_D(w_{k-1}, Z_2) \qquad \forall w_1, ..., w_{k-1} \in D$$

and so

$$\delta_D(Z_1, Z_2) \leq \kappa_D(Z_1, Z_2) \quad \forall Z_1, Z_2 \in D.$$

□

4.3 Stationary and Extremal Maps

In this section, we will define stationary and extremal maps and prove that stationary maps are extremal.

We start our section by the definition of a strongly convex domain.

Definition 4.3.1

A domain $D \subset \mathbb{C}^n$ is said to be strongly convex if it is bounded, with C^2 boundary, and whose defining function $\rho(z)$ satisfies

$$2Re\left(\sum_{i,j=1}^{n} \frac{\partial^2 \rho}{\partial z_i \partial z_j}(p) v_i v_j\right) + 2\mathcal{L}_{\rho(p)}(v) > 0 \qquad \forall p \in \partial D \forall v \in T_p(\partial D)(v \neq 0).$$

Let's fix some notations we are going to use throughout this chapter.

Notations

1. D will be used to denote a strongly convex domain.
2. If D is strongly convex with defining function ρ, $SC_D(p, v)$ will be used to denote

$$2Re\left(\sum_{i,j=1}^{n} \frac{\partial^2 \rho}{\partial z_i \partial z_j}(p) v_i v_j\right) + 2\mathcal{L}_{\rho(p)}(v).$$

3. $v(z) = (v_1(z), ..., v_n(z))$ will be used to denote the exterior normal of ∂D at $z \in \partial D$. Note that since D has a C^2 boundary, it has a defining function ρ, and in such a case $v(z) = \nabla \rho(z)$.
4. For $0 < p < \infty$, $C^p(K)$ will be used to denote the set of all $\lfloor p \rfloor$-differentiable functions on the interior of a compact set K such that:
   (a) If p is an integer, $h^{(k)}$ extends continuously to ∂K for all $k \leq p$.
   (b) If not, let $\theta = p - \lfloor n \rfloor > 0$. Then, $h^{(\alpha)}$ satisfies $|h^{(\alpha)}(z) - h^{(\alpha)}(w)| \leq C|z - w|^\theta, \forall z, w \in K^\circ, \forall \alpha \leq \lfloor n \rfloor$.
5. $C^\infty(K)$ will be used to denote $\cap_{p<\infty} C^p(K)$
6. If $E \subset \mathbb{C}^n$ is a set, then $C^p(E)$ will denote $\cap \{C^p(K) : K \subset E\text{compact}\}$.

7. C^ω will be used to denote the set of real analytic functions.

8. P^*_{n-1} will be used to denote the projective space of hyperplanes in $\mathbb{C}^n$.

What is an extremal map?

Definition 4.3.2

Let $f : U \to D$ be a holomorphic map.

1. f is said to be extremal with respect to $Z_1, Z_2 \in D (Z_1 \neq Z_2)$ if $f(0) = Z_1$, $f(\xi) = Z_2$ (ξ being a positive real number), and $\delta_D(Z_1, Z_2) = d_H(0, \xi)$.

2. f is said to be extremal with respect to $Z \in D, v \neq 0 \in \mathbb{C}^n$ if $f(0) = Z$, $f'(0) = \lambda v$ ($\lambda > 0$), and for each map $g : U \to D \in Holo(U, D)$ such that $g(0) = Z$, $g'(0) = \mu v$ ($\mu > 0$), we have $\mu \leq \lambda$.

Finally, if f is extremal, then $f(U)$ will be called the extremal disc with respect to Z_1, Z_2 (resp. Z, v).

Here is an example of an extremal map:

Example 4.3.3

Let's show that the map $f : U \to B^2(0, 1)$ defined by $f(z) = (z, 0)$ is extremal with respect to $f(0) = (0, 0)$ and $f'(0) = (1, 0)$. To do so, let $g : U \to B^2(0, 1)$ be a map such that $g(0) = (0, 0)$ and $g'(0) = \lambda(0, 1)$ for some $\lambda > 0$. By the Schwarz Lemma, we have $|g'(0)| = \lambda \leq 1$. Hence, we get our result.

What is a stationary map?

Definition 4.3.4

A holomorphic map $f : U \to D$ is said to be stationary if it can be extended to a 1/2-Holder continuous map on $\overline{U}$ (which will be called f also), $f(\partial U) \subset \partial D$, and if there exists $p : \partial U \to \mathbb{R}^+$ such that p is 1/2-Holder continuous and the map $\zeta p(\zeta)\overline{v(f(\zeta))}$ defined on ∂U extends to a continuous map $\widehat{f}$ on $\overline{U}$ that is holomorphic on U.
If f is stationary, $f(\overline{U})$ will be called stationary disc.
Remark: By a Theorem of Hardy and Littlewood, $\widehat{f}$ is 1/2-Holder continuous on $\overline{U}$.

Our first goal is to show that a stationary map f is extremal with respect to $f(0)$ and $f'(0)$.

Theorem 4.3.5 [Lempert, 81]

A stationary map f is the unique extremal map with respect to $z = f(0)$ and $v = f'(0)$. In particular, $f'(0) \neq 0$.

Proof. Let $g : U \to D \in Holo(U, D)$ with $g(0) = f(0) = z$ and $g'(0) = \lambda f'(0), \lambda \geq 0$. Since g is bounded on U, then

$$g(\zeta) = \lim_{r \to 1^-} g(r\zeta)$$

exists for a.e. $\zeta \in \partial U$ by Fatou's Theorem. Since D is strictly convex, then

$$Re < f(\zeta) - g(\zeta), v(f(\zeta)) >\geq 0 \qquad \text{for a.e. } \zeta \in \partial U.$$

Assume $f \neq g$. Since $f(\zeta) - g(\zeta)$ is holomorphic and bounded on U, then $f(\zeta) - g(\zeta) \neq 0$ a.e. on ∂U by a theorem of F. and M. Riesz. Hence,

$$Re < f(\zeta) - g(\zeta), v(f(\zeta)) >> 0 \qquad \text{for a.e. } \zeta \in \partial U.$$

Multiplying by $p(\zeta)$, we get

$$0 < Re < f(\zeta) - g(\zeta), p(\zeta)v(f(\zeta)) >= Re \left\langle \frac{f(\zeta) - g(\zeta)}{\zeta}, \overline{\widehat{f}(\zeta)} \right\rangle \qquad \text{for a.e. } \zeta \in U.$$

Since $\left\langle \frac{f(\zeta) - g(\zeta)}{\zeta}, \overline{\widehat{f}(\zeta)} \right\rangle$ is holomorphic and bounded on U, then
$Re \left\langle \frac{f(\zeta) - g(\zeta)}{\zeta}, \overline{\widehat{f}(\zeta)} \right\rangle$ is harmonic and bounded on U. Hence,

$$0 < Re < f'(0) - g'(0), \overline{\widehat{f}(0)} >= (1 - \lambda) Re < f'(0), \overline{\widehat{f}(0)} > .$$

This is true for any g satisfying the conditions above. In particular, take $g(\zeta) = z$, so $\lambda = 0$ and hence $Re < f'(0), \overline{\widehat{f}(0)} >> 0$. Therefore, $1 - \lambda > 0$, and so $|g'(0)| < |f'(0)|$. □

Here is an example of a stationary map:

Example 4.3.6

Let's find all stationary maps $f : U \to B^2(0,1)$ such that $f(0) = (0,0)$ and $f'(0) = (1,0)$. Suppose there exists a stationary map $f : U \to B^2(0,1)$ such that $f(0) = (0,0)$ and $f'(0) = (1,0)$. Then, f is the unique extremal map with respect to $f(0)$ and $v = f'(0)$. However, the map $f : U \to B^2(0,1)$ defined by $f(z) = (z,0)$ is extremal with respect to $f(0) = (0,0)$ and $f'(0) = (1,0)$. By uniqueness, $f(z) = (z,0)$. Note that one can show, following the same procedure, that if $f : U \to B^2(0,1)$ is a stationary map such that $f(0) = (0,0)$ and $f'(0) = (a_1, a_2)$ with $|a_1|^2 + |a_2|^2 = 1$, then $f(z) = (a_1 z, a_2 z)$.

One can see that the composition of a stationary map and an automorphism of the unit disc is a stationary map:

Theorem 4.3.7 [Lempert, 81]

If f is a stationary map and a is an automorphism of U, then $f_a = f \circ a$ is a stationary map.

Proof. We have to show that there exists a 1/2-holder continuous map $p_a : \partial U \to \mathbb{R}^+$ such that $\zeta p_a(\zeta)\overline{v(f_a(\zeta))}$ extends to a map that is holomorphic on U and continuous on $\overline{U}$.
Let ϕ be a holomorphic function on U such that

$$Im(\phi(\zeta)) = Im \log \left(\frac{\zeta}{a(\zeta)} \right) \qquad \forall \zeta \in \partial U.$$

Define $q : \partial U \to \mathbb{R}^+$ by

$$q(\zeta) = e^{\phi(\zeta)} \frac{a(\zeta)}{\zeta}.$$

Note that q is well-defined because

$$q(\zeta) = e^{Re(\phi(z))} e^{iIm(\phi(\zeta))} \frac{a(\zeta)}{\zeta} = e^{Re(\phi(\zeta))} e^{log(\frac{\zeta}{a(\zeta)})} \frac{a(\zeta)}{\zeta} = e^{Re(\phi(\zeta))} \qquad \forall \zeta \in \partial U.$$

Define $p_a : \partial U \to \mathbb{R}^+$ by

$$p_a(\zeta) = q(\zeta)p(a(\zeta)).$$

Note that

$$\zeta p_a(\zeta)\overline{v(f_a(\zeta))} = e^{\phi(\zeta)} a(\zeta)p(a(\zeta))\overline{v(f_a(\zeta))}$$

which extends to $e^{\phi(\zeta)}\widehat{f}(a(\zeta))$, a holomorphic function on U, continuous on $\overline{U}$. □

Note that $\widehat{f}_a(\zeta) = e^{\phi(\zeta)}\widehat{f}(a(\zeta)) = 0 \Leftrightarrow (\widehat{f} \circ a)(\zeta) = 0$.

Lempert has mentioned the following two corollaries without proof:

Corollary 4.3.8

If f is a stationary map, then $\widehat{f} : U \to \mathbb{C}^n$ is never zero.
In fact, $< f'(\zeta), \overline{\widehat{f}(\zeta)} > \neq 0$ for all $\zeta \in U$.

Proof. We have already shown that if f is stationary, then

$$Re < f'(0), \overline{\widehat{f}(0)} >> 0,$$

hence $\widehat{f}(0) \neq 0$.
Next, let $\zeta \in U$. Let a be the automorphism of the unit disc defined by

$$a(z) = \frac{\zeta + z}{1 + \bar{\zeta} z}.$$

Hence, $\widehat{f}_a(0) \neq 0$ and so $(\widehat{f} \circ a)(0) \neq 0$. Therefore, $\widehat{f}(\zeta) \neq 0$. Moreover, we know that

$$Re < (f \circ a)'(0), \overline{\widehat{f}_a(0)} >= Re \left((1 - |\zeta|^2) e^{\phi(0)} < f'(\zeta), \overline{\widehat{f}(\zeta)} > \right) > 0.$$

Therefore, $< f'(\zeta), \overline{\widehat{f}(\zeta)} > \neq 0$. □

Corollary 4.3.9

A stationary disc $f(\overline{U})$ is the unique extremal disc with respect to $z = f(\zeta)$ and $v = f'(\zeta)$, for any $\zeta \in U$.

Proof. Let $\zeta \in U$. Let $a(z) = \dfrac{z + \zeta}{1 + \bar{\zeta} z}$. Since $f \circ a$ is stationary, then $f \circ a$ is the unique extremal map with respect to $(f \circ a)(0) = f(\zeta)$ and $(f \circ a)'(0) = (1 - |\zeta|^2) f'(\zeta)$. In fact, $f \circ a$ is also extremal with respect to $f(\zeta)$ and $f'(\zeta)$:

1. $(f \circ a)(0) = f(\zeta)$.

2. $(f \circ a)'(0) = (1 - |\zeta|^2) f'(\zeta)$, with $1 - |\zeta|^2 > 0$.

3. Suppose there exists $g : U \to D \in Holo(U, D)$ such that $g(0) = f(\zeta)$ and $g'(0) = \lambda f'(\zeta)$ with $\lambda > 0$. Write $g'(0) = \frac{\lambda}{1 - |\zeta|^2}(1 - |\zeta|^2) f'(\zeta)$, and use the fact that $f \circ a$ is extremal with respect to $(f \circ a)(0) = f(\zeta)$ and $(f \circ a)'(0) = (1 - |\zeta|^2) f'(\zeta)$ to conclude that $\lambda \leq 1 - |\zeta|^2$.

Moreover, $f \circ a$ is the unique extremal map with respect to $f(\zeta)$ and $f'(\zeta)$. If not, then there exists an extremal map F with respect to $f(\zeta)$ and $f'(\zeta)$. One can show in a very similar way that F is extremal with respect to $(f \circ a)(0) = f(\zeta)$ and $(f \circ a)'(0) = (1 - |\zeta|^2) f'(\zeta)$ and conclude that $F = f \circ a$. Therefore, $(f \circ a)(\overline{U}) = f(\overline{U})$ is the unique extremal disc with respect to $z = f(\zeta)$ and $v = f'(\zeta)$. $\square$

We proceed to show that a stationary disc $f(\overline{U})$ is the unique extremal disc with respect to any two elements in the image of f. Note that the proof of the following theorem is more detailed than the one done by Lempert:

Theorem 4.3.10 [Lempert, 81]

A stationary disc $f(\overline{U})$ is the unique extremal disc with respect to $z_1 = f(\zeta_1), z_2 = f(\zeta_2)$ for any $\zeta_1 \neq \zeta_2, \zeta_1, \zeta_2 \in U$.

Proof. Suppose $z_1 = f(\zeta_1), z_2 = f(\zeta_2)$. Let $\phi_1(z) = \frac{\zeta_1 + z}{1 + \overline{\zeta_1} z}$. Let $\phi_2(z)$ be the rotation of U that maps $\left|\frac{\zeta_1 - \zeta_2}{1 - \overline{\zeta_1}\zeta_2}\right|$ to $\frac{\zeta_2 - \zeta_1}{1 - \overline{\zeta_1}\zeta_2}$. See that

$$(f \circ \phi_1 \circ \phi_2)(0) = f(\zeta_1)$$

and

$$(f \circ \phi_1 \circ \phi_2)\left(\left|\frac{\zeta_1 - \zeta_2}{1 - \overline{\zeta_1}\zeta_2}\right|\right) = f(\zeta_2).$$

We will show that $f \circ \phi_1 \circ \phi_2$ is the unique extremal map with respect to $f(\zeta_1)$ and $f(\zeta_2)$.

Suppose $g : U \to D \in Holo(U, D)$ with $g(0) = f(\zeta_1)$ and $g(w) = f(\zeta_2)$, $w > 0$. Suppose also that $f \neq g$. Let $l = \left|\frac{\zeta_1 - \zeta_2}{1 - \overline{\zeta_1}\zeta_2}\right|$. It is enough to show that $w > l$ and it follows that $d_H(0, w) > d_H(0, l)$ which means that:

1. g is not extremal with respect to $z_1 = f(\zeta_1)$ and $z_2 = f(\zeta_2)$.

2. $f \circ \phi_1 \circ \phi_2$ is extremal with respect to $z_1 = f(\zeta_1)$ and $z_2 = f(\zeta_2)$ because

$$d_H(0, w) > d_H(0, l) \Rightarrow d_H(0, l) \leq \delta_D(f(\zeta_1), f(\zeta_2)) \Rightarrow d_H(0, l) = \delta_D(f(\zeta_1), f(\zeta_2))$$

since $\delta_D(f(\zeta_1), f(\zeta_2)) \leq d_H(0, l)$.

Suppose that $w \leq l$. Define $G : U \to D$ by $G(\zeta) = g\left(\frac{\zeta w}{l}\right)$. Note that $G(0) = z_1$ and $G(l) = z_2$. Since $G \neq f$ and D is strictly convex, we have

$$Re < f(\zeta) - G(\zeta), v(f(\zeta)) >> 0 \qquad \text{for a.e. } \zeta \in \partial U,$$

and hence

$$Re \left\langle \frac{f(\zeta) - G(\zeta)}{\zeta}, \overline{\widehat{f}(\zeta)} \right\rangle > 0 \qquad \text{for a.e. } \zeta \in U.$$

But,

$$Re\left\langle \frac{f(\zeta_2)-G(\zeta_2)}{\zeta_2}, \overline{\widehat{f}(\zeta_2)} \right\rangle = 0$$

and $Re\left\langle \dfrac{f(\zeta)-G(\zeta)}{\zeta}, \overline{\widehat{f}(\zeta)} \right\rangle$ is harmonic, so we get a contradiction by the minimum principle. $\square$

It follows that:

Corollary 4.3.11

Stationary maps are injective.

Proof. Let f be a stationary map. Let $\zeta_1, \zeta_2 \in U$ with $\zeta_1 \neq \zeta_2$. From the previous proof, we see that

$$\delta_D(f(\zeta_1), f(\zeta_2)) = d_H(0, l) > 0$$

so $f(\zeta_1) \neq f(\zeta_2)$. $\square$

4.4 Regularity of Stationary Maps

The goal of this section is to show that $< f'(\zeta), \widehat{f}(\zeta) >$ is a positive constant for any stationary map f.

We start our section by the definition of a totally real submanifold:

Definition 4.4.1

A real submanifold M of a complex manifold is said to be totally real at $z \in M$ if $T_zM \cap iT_zM = 0$.

Next, we state without proof the following theorem:

Theorem 4.4.2 ([7])

Let S be a real hypersurface in $\mathbb{C}^n$ of class C^2. Define $\Psi : S \to \mathbb{C}^n \times P^*_{n-1}$ by $\Psi(z) = (z, T_z^{\mathbb{C}}(S))$. Then $\Psi(S)$ is totally real at $\Psi(z_0)$ if and only if the Levi form of S at z_0 is not zero.

We proceed to state and prove the required theorems to reach the main goal of the section:

Lemma 4.4.3

Let $\psi \in C^1(\mathbb{C})$ such that $supp(\psi)$ is compact. Then the function defined by

$$u(\zeta) = \frac{-1}{2\pi i} \int\limits_{\mathbb{C}} \frac{\psi(\xi)}{\xi - \zeta} d\overline{\xi} \wedge d\xi$$

satisfies

$$\frac{\partial u(\zeta)}{\partial \overline{\zeta}} = \psi(\zeta).$$

Proof. Let $D^1(0, r)$ be a disc that contains $supp(\psi)$. Then:

$$\begin{aligned}
\frac{\partial u(\zeta)}{\partial \overline{\zeta}} &= \frac{-1}{2\pi i} \frac{\partial}{\partial \overline{\zeta}} \int\limits_{\mathbb{C}} \frac{\psi(\xi)}{\xi - \zeta} d\overline{\xi} \wedge d\xi \\
&= \frac{-1}{2\pi i} \frac{\partial}{\partial \overline{\zeta}} \int\limits_{\mathbb{C}} \frac{\psi(\xi + \zeta)}{\xi} d\overline{\xi} \wedge d\xi \\
&= \frac{-1}{2\pi i} \int\limits_{\mathbb{C}} \frac{\frac{\partial}{\partial \overline{\xi}} \psi(\xi + \zeta)}{\xi} d\overline{\xi} \wedge d\xi \\
&= \frac{-1}{2\pi i} \int\limits_{D^1(0,r)} \frac{\frac{\partial}{\partial \overline{\xi}} \psi(\xi)}{\xi - \zeta} d\overline{\xi} \wedge d\xi \\
&= \psi(\zeta) - \frac{1}{2\pi i} \int\limits_{\partial D^1(0,r)} \frac{\psi(\xi)}{\xi - \zeta} d\xi \qquad (Pompeiu's Formula) \\
&= \psi(\zeta).
\end{aligned}$$

□

Theorem 4.4.4 [Lempert, 81]

Let:

1. $E \subset U$ be a bounded domain such that ∂E is a simply connected curve with an open subarc $A \subset \partial E$ contained in ∂U.
2. Let M be a totally real submanifold of a complex manifold X such that $dim_{\mathbb{R}} M = dim_{\mathbb{C}} X = m$.
3. $g : E \cup A \to X$ be a 1/2-Holder-continuous map, holomorphic on E, such that $g(A) \subset M$.

If M is of class $C^r (r = 2, 3, ..., \omega)$, then $g \in C^{r-\epsilon}(E \cup A), \epsilon > 0$.

Proof. WLOG, suppose $X = \mathbb{C}^m$, $E \subset U^+ = \{\zeta \in \mathbb{C} : Im(\zeta) > 0\}$ and $A \subset \partial E \cup \partial U^+$ is a segment.

Case 1 : $r = \omega$: Since M is of class C^r, then there exists an analytic diffeomorphism $\phi : V \to M$ where V is an open subset of $\mathbb{R}^m$. Consequently, there exists a biholomorphism $\Phi : N_V \to \mathbb{C}^m$ from a neighborhood N_V of V into $\mathbb{C}^m$ such that $\Phi|_V = \phi$. Let $\Psi = \Phi^{-1} \circ g$. Note that Ψ is holomorphic on E, continuous on A and is real-valued on A since $g(A) \subset M$, so $\Psi(A) \subset V$. By the reflection principle, Ψ extends analytically across A, and so does g.

Case 2: $r < \omega$: There exists a C^r-diffeomorphism $\phi' : V \to M$ where V is an open subset of $\mathbb{R}^m$. By HARVEY and WELLS Jr, there exists a C^r-extension $\Phi : N_V \to \mathbb{C}^m$ of ϕ' from a neighborhood N_V into $\mathbb{C}^m$, such that all derivatives of order $\leq r - 1$ of $\overline{\partial}\Phi$ are 0 on V.Hence, Φ is a diffeomorphism from N_V into a neighborhood N_M of M, such that all derivatives of order $\leq r - 1$ of $\overline{\partial}\Phi^{-1}$ are 0 on M. Let h be the function defined on $\Omega \cap U^+$ by

$$h(\zeta) = (\Phi^{-1} \circ g)(\zeta)$$

where Ω is a neighborhood of A symmetric with respect to A. Let

$$H(\zeta) = \begin{cases} h(\zeta) & if \zeta \in \Omega, Im(\zeta) \geq 0 \\ \overline{h(\overline{\zeta})} & if \zeta \in \Omega, Im(\zeta) \leq 0 \end{cases}.$$

Note that H is continuous on Ω since $h(A) \subset V$ and of class C^r on $\Omega - \mathbb{R}$ since g is holomorphic on E. Let

$$\phi(\zeta) = \begin{cases} H_{\overline{\zeta}}(\zeta) & if \zeta \in \Omega - A \\ 0 & if \zeta \in A \end{cases}.$$

Note that ϕ is continuous on $\Omega - A$ because H is of class C^r there. We need to investigate the continuity of ϕ on A. Well, on $\Omega \cap U^+$,

$$H_{\overline{\zeta}}(\zeta) = \frac{\partial \Phi^{-1}}{\partial \overline{z}}(g(\zeta))g'(\zeta).$$

Note that

$$\frac{\partial \Phi^{-1}}{\partial \overline{z}}(g(\zeta)) = o(1)(1 - |\zeta|)^{1/2}(\zeta \to \zeta_0 \in A).$$

By a Theorem of Hardy and Littlewood,

$$g'(\zeta) = O(1)(1 - |\zeta|)^{-1/2}(\zeta \to \zeta_0 \in A).$$

Consequently, $\phi(\zeta) \to 0$ as $\zeta \to \zeta_0 \in A$. So, ϕ is continuous on Ω, and hence for any $\Omega' \subset\subset \Omega$, there exists $\Psi \in C^{1-\epsilon}(\Omega')$ such that $\Psi_{\overline{\zeta}} = \phi$. Thus, $H_{\overline{\zeta}} - \Psi_{\overline{\zeta}} = 0$ on $\Omega' - A$, which implies that $H - \Psi$ is holomorphic on $\Omega' - A$. Since $H - \Psi$ is continuous on Ω', then $H - \Psi$ is holomorphic on Ω' by the symmetry principle. Therefore, $H \in C^{1-\epsilon}(\Omega')$. This is true for any $\Omega' \subset\subset \Omega$, so $h \in C^{1-\epsilon}(\Omega \cap \overline{U^+})$ and $g \in C^{1-\epsilon}(\Omega \cap \overline{U^+})$. Now,

$$\frac{\partial \Phi^{-1}}{\partial \overline{z}}(g(\zeta)) = o(1)(1 - |\zeta|)^{1-\epsilon}(\zeta \to \zeta_0 \in A)$$

and

$$g'(\zeta) = O(1)(1 - |\zeta|)^{-\epsilon}(\zeta \to \zeta_0 \in A).$$

So, $\phi \in C^{1-\epsilon}(\Omega')$ and $\Psi \in C^{2-\epsilon}(\Omega')$, hence $h, g \in C^{2-\epsilon}(\Omega \cap \overline{U^+})$.
If $r = 2$, stop. If $r > 2$, then $g' \in C^{1-\epsilon}(\Omega \cap \overline{U^+})$, and by the same theorem of Hardy and Littlewood,

$$g''(\zeta) = O(1)(1 - \zeta)^{-\epsilon}(\zeta \to \zeta_0 \in A)$$

and so $\phi \in C^{2-\epsilon}(\Omega')$. But

$$\phi \in C^{2-\epsilon}(\Omega') \Rightarrow \Psi \in C^{3-\epsilon}(\Omega') \Rightarrow h \in C^{3-\epsilon}(\Omega \cap \overline{U^+}) \Rightarrow g \in C^{3-\epsilon}(\Omega \cap \overline{U^+}).$$

If $r = 3$, stop. If not, proceed until you get $g \in C^{r-\epsilon}(E \cup A)$. □

Theorem 4.4.5 [Lempert, 81]

Suppose ∂D is of class $C^k(3 \leq k \leq \omega)$ and $f : \overline{U} \to \overline{D}$ be a stationary map. Then, $f, \widehat{f} \in C^{k-2}(\overline{U})$.

Proof. Let $F = \pi \circ \widehat{f} : \overline{U} \to P^*_{n-1}$, where π is the cannonical projection. Note that $\widehat{f} \neq 0$ on $\overline{U}$, so F is well-defined.
The map $(f, F) : \overline{U} \to \mathbb{C}^n \times P^*_{n-1}$ is 1/2-Holder continuous, holomorphic on U, and $(f, F)(\partial U) \subset \Psi(\partial D)$,

where $\Psi(\partial D)$ is totally real since D is strongly convex. By Theorem 4.4.4, $(f,F) \in C^{k-1-\epsilon}(\overline{U})$, hence $f \in C^{k-2}(\overline{U})$. Since $v \circ f \in C^{k-1-\epsilon}(\partial U)$, we just have to show that $p \in C^{k-2}(\partial U)$.

Let $\zeta_0 \in \partial U$ and suppose that $v_1(f(\zeta_0)) \neq 0$. Let $\phi : \partial U \to \mathbb{C}$ be Holder-continuous of order $k-1-\epsilon$ such that $\overline{v_1(f(\zeta))} = e^{\phi(\zeta)}$ in a neighborhood $V \cap \partial U$ of $\zeta_0 \in \partial U$.
Let $\gamma : \partial U \to \mathbb{R}$ be such that $\gamma + iIm(\phi)$ extends to a holomorphic function on U. By Privaloff, $\gamma \in C^{k-1-\epsilon}(\partial U)$.

Since $\overline{v_1(f(\zeta))}e^{\gamma(\zeta)-Re(\phi(\zeta))}$ and $p(\zeta)\overline{v_1(f(\zeta))}$ extend to holomorphic functions on $U \cap V$, so does $p(\zeta)e^{Re(\phi(\zeta))-\gamma(\zeta)}$ which is real on $V \cap \partial U$, so it extends to an analytic function on $V \cap \partial U$. The regularity of $e^{Re(\phi)-\gamma}$ implies that $p \in C^{k-1-\epsilon}(V \cap \partial U)$.

Since ζ_0 is arbitrary, $p \in C^{k-1-\epsilon}(\partial U)$, so $\widehat{f} \in C^{k-1-\epsilon}(\overline{U})$. □

Theorem 4.4.6 [Lempert, 81]

If $f : U \to D$ is a stationary map, then $< f'(\zeta), \overline{\widehat{f}(\zeta)} >$ is a constant positive map.

Proof. Since $f(\partial U) \subset \partial D$, then the tangent to the curve $\{f(w) : w \in \partial U\}$ at $f(\zeta)$ is orthogonal to $v(f(\zeta))$. Hence, for $\zeta \in \partial U$, we have

$$\begin{aligned} 0 &= p(\zeta)Re < i\zeta f'(\zeta), v(f(\zeta) >= p(\zeta)Im < \zeta f'(\zeta), v(f(\zeta)) > \\ &= Im < f'(\zeta), \overline{\zeta}p(\zeta)v(f(\zeta)) >= Im < f'(\zeta), \overline{\widehat{f}(\zeta)} > . \end{aligned}$$

Note that $Im < f'(\zeta), \overline{\widehat{f}(\zeta)} >$ is harmonic in U and continuous on $\overline{U}$ by Theorem 4.4.5 . Therefore, $Im < f'(\zeta), \overline{\widehat{f}(\zeta)} > = 0$ on $\overline{U}$ by the maximum principle. But $< f'(\zeta), \overline{\widehat{f}(\zeta)} >$ is holomorphic on U, so $< f'(\zeta), \overline{\widehat{f}(\zeta)} >$ is a real constant c on U. Moreover, $< f'(0), \overline{\widehat{f}(0)} >> 0$, which implies that $c > 0$.
From now on, we will assume that $< f'(\zeta), \overline{\widehat{f}(\zeta)} >= 1$ so that

$$p(\zeta) = \frac{1}{< \zeta f'(\zeta), v(f(\zeta)) >} \qquad (\zeta \in \partial U).$$

□

4.5 Holder Estimation of Stationary Maps

In this section, we prove some Holder estimates of stationary maps.

Theorem 4.5.1 [Lempert, 81]

Let $f : \overline{U} \to \overline{D}$ be a stationary map. Suppose that the diameter of ∂D (denoted by $diamD$), $SC_D(p,v)(p \in \partial D, v \in T_p(\partial D))$, and the distance from $f(0)$ to ∂D are bounded below and above by positive numbers a and b respectively. Then, there exists a uniform constant $C_1 > 0$ such that

$$dist(f(\zeta), \partial D) \leq C_1(1 - |\zeta|), \qquad \zeta \in U.$$

Proof. Let $\rho > 0$ be smaller than a. There exists a constant $C_2 > 0$ such that

$$z \in D, dist(z, \partial D) \geq \rho \Rightarrow \delta_D(f(0), z) < C_2.$$

Let $\zeta \in U$.

If $dist(f(\zeta), \partial D) \geq \rho$, then

$$\begin{aligned}\delta_D(f(0), f(\zeta)) &< C_2 \\ &= C_2 + \log(diamD) - \log(diamD) \\ &= C_3 - \log(diamD) \\ &< C_3 - \log(dist(f(\zeta), \partial D)).\end{aligned}$$

If $dist(f(\zeta), \partial D) < \rho$, let Z be the closest point on ∂D to $f(\zeta)$ and let z be the center of the ball $b \subset D$ of radius ρ tangent to ∂D at Z. Then,

$$\begin{aligned}\delta_D(f(0), f(\zeta)) &\leq \delta_D(f(0), z) + \delta_D(z, f(\zeta)) \\ &< C_2 + \delta_b(z, f(\zeta)) \qquad\qquad (dist(z, Z) = \rho, b \subset D) \\ &< C_4 - \log(dist(f(\zeta), \partial D)).\end{aligned}$$

But $f(\overline{U})$ is the extremal disk determined by $f(0), f(\zeta)$, i.e. $f \circ \phi_2$ is the unique extremal map with respect to $f(0)$ and $f(\zeta)$, where ϕ_2 is the rotation mapping $|\zeta|$ to ζ. Hence,

$$\delta_D(f(0), f(\zeta)) = \log\left(\frac{1 + |\zeta|}{1 - |\zeta|}\right) \geq -log(1 - |\zeta|).$$

Hence,

$$\log(dist(f(\zeta), \partial D) < max\{C_3, C_4\} + \log(1 - |\zeta|).$$

Therefore,

$$dist(f(\zeta), \partial D) \leq C_1(1 - |\zeta|),$$

where $C_1 = e^{max\{C_3, C_4\}}$. □

Lemma 4.5.2

Suppose $g \in Holo(U, B^n(z_0, R))$ and $|g(0) - z_0| = r$. Then $|g'(0)| \leq \sqrt{R^2 - r^2}$.

Proof. If $r = 0$, we get the result by the Schwarz Lemma. If not, we can compose g with an automorphism of the ball $B^n(z_0, R)$ and then apply the Schwarz Lemma. □

Theorem 4.5.3 [Lempert, 81]

Let $f : \overline{U} \to \overline{D}$ be a stationary map. Suppose that the diameter of ∂D,$SC_D(p, v)(p \in \partial D, v \in T_p(\partial D))$, and the distance from $f(0)$ to ∂D are bounded below and above by positive numbers. Then there exists a uniform constant $C_5 > 0$ such that

$$|f(\zeta_1) - f(\zeta_2)| \leq C_5|\zeta_1 - \zeta_2|^{1/2} \qquad (\zeta_1, \zeta_2 \in U).$$

Proof. Let $\zeta_0 \in U$. Let $Z \in \partial D$ be the closest point to $f(\zeta_0)$. Let B be the smallest ball tangent to ∂D at Z such that $D \subset B$. Let z_0 be the center of B. Let

$$h(\zeta) = f\left(\frac{\zeta_0 - \zeta}{1 - \overline{\zeta_0}\zeta}\right).$$

Hence, $h(U) \subset B$ and $h(0) = f(\zeta_0)$. By Lemma 4.5.2, we have

$$|h'(0)| \leq \sqrt{|Z - z_0|^2 - |f(\zeta_0) - z_0|^2} \leq C_6|f(\zeta_0) - Z|^{1/2}.$$

Hence,

$$|f'(\zeta_0)| = \frac{1}{1 - |\zeta_0|^2}|h'(0)| \leq C_7\frac{\sqrt{dist(f(\zeta_0), \partial D)}}{1 - |\zeta_0|}.$$

By Theorem 4.5.1,

$$|f'(\zeta)| \leq C_8(1 - |\zeta|)^{-1/2} \qquad (\zeta \in U).$$

By a theorem of Hardy and Littlewood, the last inequality is equivalent to

$$|f(\zeta_1) - f(\zeta_2)| \leq C_5|\zeta_1 - \zeta_2|^{1/2} \qquad (\zeta_1, \zeta_2 \in U).$$

□

Lemma 4.5.4

Suppose $f : U \to \{z \in \mathbb{C} : Re(z) < 0\}$ is a holomorphic map. Let $\zeta \in U$. Then,

$$|f'(\zeta)| \leq \frac{2|Re(f(\zeta))|}{1 - |\zeta|^2}.$$

Proof. Define $B : U \to U$ by

$$B(z) = \frac{z + \zeta}{1 + \bar{\zeta}z}$$

and $h : \{z \in \mathbb{C} : Re(z) < 0\} \to U$ by

$$h(z) = \frac{z - f(\zeta)}{z + \overline{f(\zeta)}}.$$

By the Schwarz Lemma,

$$|(h \circ f \circ B)'(0)| \leq 1,$$

hence

$$|f'(\zeta)| \leq \frac{2|Re(f(\zeta))|}{1 - |\zeta|^2}.$$

□

Theorem 4.5.5 [Lempert, 81]

Let $f : \overline{U} \to \overline{D}$ be a stationary map. Suppose that the diameter of ∂D, $SC_D(p, v)(p \in \partial D, v \in T_p(\partial D))$, and the distance from $f(0)$ to ∂D are bounded below and above by positive numbers. Then, there exist uniform constants C_9, C_{10} such that

$$0 < C_9 << f'(\zeta), \bar{\zeta}v(f(\zeta)) >= p(\zeta)^{-1} < C_{10} \qquad (\zeta \in \partial U).$$

Proof. Let $\zeta \in U, \epsilon > 0, \zeta_1 = (1-\epsilon)\zeta$. WLOG, assume that $v(f(\zeta)) = (1, 0, ..., 0)$ and $f(\zeta) = 0$. Hence,

$$z = (z_1, ..., z_n) \in D \Rightarrow Re(z_1) < 0.$$

Write $f = (f_1, ..., f_n)$. Hence,

$$f_1(U) \subset \{z_1 \in \mathbb{C} : Re(z_1) < 0\}.$$

By Lemma 4.5.4,

$$|f_1'(\zeta_1)| \leq \frac{2|Re(f_1(\zeta_1))|}{1 - |\zeta_1|^2}.$$

But there exists $\epsilon > 0$ such that

$$|\zeta_1 - \zeta| < \epsilon \Rightarrow |Re(f_1(\zeta_1))| \sim dist(f(\zeta_1), \partial D).$$

By Theorem 4.5.1,

$$|\zeta_1 - \zeta| < \epsilon \Rightarrow |f_1'(\zeta_1)| \leq \frac{2dist(f(\zeta_1), \partial D)}{1 - |\zeta_1|^2} \leq 2C_1.$$

In particular,

$$|f_1'(\zeta)| = < f'(\zeta), v(f(\zeta)) > \leq 2C_1.$$

Next, consider the subharmonic function $V : \overline{U} \to \mathbb{R}$ defined by

$$V(\zeta) = -dist(f(\zeta), \partial D).$$

Note that V attains its maximum on ∂U where it is differentiable.
Let d be the diameter of D and $b = dist(f(0), \partial D)$. Let B be the ball of center $f(0)$ and radius d. Note that

$$|f(\zeta) - f(0)| \leq d \qquad \forall \zeta \in \overline{U},$$

so $f(\overline{U}) \subset B$. By the Schwarz Lemma,

$$|f(\zeta) - f(0)| \leq d|\zeta| \leq \frac{b}{2} \qquad \text{if } |\zeta| \leq \frac{b}{2d}.$$

Let

$$M(x) = max_{0 \leq t \leq 2\pi} V(e^{x+it}).$$

If $x \leq \log \frac{b}{2d}$, then $e^x \leq \frac{b}{2d}$ and so $|e^{x+it}| \leq \frac{b}{2d}$ for all t. Hence,

$$|f(e^{x+it}) - f(0)| \leq \frac{b}{2} \qquad \text{if } x \leq \log \frac{b}{2d}.$$

Therefore,

$$dist(f(e^{x+it}), \partial D) \geq \frac{b}{2} \qquad \text{if } x \leq \log \frac{b}{2d}.$$

Now, M is a convex function, so for $\log \frac{b}{2d} \le x \le 0$,

$$\frac{M(x) - M\left(\log \frac{b}{2d}\right)}{x - \log \frac{b}{2d}} \le \frac{M(0) - M(x)}{0 - x}.$$

Since $M(0) = 0$ and $M\left(\log \frac{b}{2d}\right) \le -\frac{b}{2}$, then we have

$$M(x)\left(\frac{1}{x}\log \frac{b}{2d}\right) \le M\left(\log \frac{b}{2d}\right) \le -\frac{b}{2}$$

and therefore

$$M(x) \le \frac{-xb}{2\log \frac{b}{2d}} \qquad \text{if} \quad \log \frac{b}{2d} \le x \le 0.$$

Finally, if $\zeta \in \partial U$, then $V(\zeta) = 0$ and so

$$V(\zeta e^x) - V(\zeta) \le \frac{-xb}{2\log \frac{b}{2d}} \qquad \text{if} \quad \log \frac{b}{2d} \le x \le 0.$$

Hence,

$$\frac{-b}{2\log \frac{b}{2d}} \le \frac{V(\zeta e^x) - V(\zeta)}{x - 0} \qquad \text{if} \quad \log \frac{b}{2d} \le x \le 0.$$

Therefore,

$$\frac{-b}{2\log \frac{b}{2d}} \le \left[\frac{dV(\zeta e^x)}{dx}\right]_{x=0} = \zeta V'(\zeta) =< \zeta f'(\zeta), v(f(\zeta)) > .$$

□

Note that the proof of the following theorem is more detailed than the one done by Lempert:

Theorem 4.5.6 [Lempert, 81]

Let $f : \overline{U} \to \overline{D}$ be a stationary map. Suppose that the diameter of ∂D, $SC_D(p, v)(p \in \partial D, v \in T_p(\partial D))$, and the distance from $f(0)$ to ∂D are bounded below and above by positive numbers. Then, there exists a uniform constant $C_{11} > 0$ such that

$$|p(\zeta_1) - p(\zeta_2)| \le C_{11}|\zeta_1 - \zeta_2|^{1/2} \qquad (\zeta_1, \zeta_2 \in \partial U).$$

Proof. Let's prove that it is enough to show that there exist uniform constants $0 < C_{12} < 1, C_{11} > 0$ such that

$$\zeta_1, \zeta_2 \in \partial U, |\zeta_1 - \zeta_2| < C_{12} \Rightarrow |p(\zeta_1) - p(\zeta_2)| \le C_{11}|\zeta_1 - \zeta_2|^{1/2}.$$

First, fix $0 < k < C_{12}$. Suppose, in the worst case, that $|\zeta_1 - \zeta_2| = 2$ and partition the semi-circle whose endpoints are ζ_1 and ζ_2 into n points, $\zeta_{1,1} = \zeta_1, \zeta_{1,2}, ...\zeta_{1,n} = \zeta_2$ such that

$$\begin{cases} |\zeta_{1,i} - \zeta_{1,i+1}| = k, & if\ 1 \leq i \leq n-2 \\ |\zeta_{1,i} - \zeta_{1,i+1}| \leq k, & if\ i = n-1 \end{cases}.$$

Hence, if $\zeta_1, \zeta_2 \in \partial U$ satisfy $|\zeta_1 - \zeta_2| \geq C_{12}$, then partition the part of the unit circle whose endpoints are ζ_1 and ζ_2 into l points, $\zeta_{1,1} = \zeta_1, ...\zeta_{1,l} = \zeta_2$ such that

$$\begin{cases} |\zeta_{1,i} - \zeta_{1,i+1}| = k, & if\ 1 \leq i \leq l-2 \\ |\zeta_{1,i} - \zeta_{1,i+1}| \leq k, & if\ i = l-1 \end{cases}.$$

Note that $l \leq n$ and n depends only on C_{12}. Now,

$$\begin{aligned} |p(\zeta_1) - p(\zeta_2)| &= |p(\zeta_{1,1}) - p(\zeta_{1,2}) + p(\zeta_{1,2}) - p(\zeta_{1,3}) + ... + p(\zeta_{1,l-1}) - p(\zeta_2)| \\ &\leq C_{11}|\zeta_1 - \zeta_{1,2}|^{1/2} + C_{11}|\zeta_{1,2} - \zeta_{1,3}|^{1/2} + ... + C_{11}|\zeta_{1,l-1} - \zeta_2|^{1/2} \\ &\leq C_{11}C_{12}^{1/2} + C_{11}C_{12}^{1/2} + ... + C_{11}C_{12}^{1/2} \\ &\leq lC_{11}C_{12}^{1/2} \\ &\leq nC_{11}C_{12}^{1/2} \\ &\leq nC_{11}|\zeta_1 - \zeta_2|^{1/2} \\ &= C|\zeta_1 - \zeta_2|^{1/2} \end{aligned}$$

where $C = nC_{11}$ is uniform. Hence we get,

$$|p(\zeta_1) - p(\zeta_2)| \leq C|\zeta_1 - \zeta_2|^{1/2} \qquad (\zeta_1, \zeta_2 \in \partial U).$$

Now, let's prove the theorem.
Let $\zeta_1 \in \partial U$. Suppose that $v_1(f(\zeta_1)) = 1$.
Let $\epsilon = \frac{1}{2}$. Since ∂U is compact and v_1 is continuous on ∂U, then there exists $\delta > 0$ such that

$$\zeta, \zeta' \in \partial U, |\zeta - \zeta'| < \delta \Rightarrow |v_1(\zeta_1) - v_1(\zeta_2)| \leq \frac{1}{2}.$$

Note that δ does not depend on f. Let $\zeta \in \partial U$. Since there exists a uniform constant C_5 such that

$$|f(\zeta) - f(\zeta_1)| \leq C_5|\zeta - \zeta_1|^{1/2},$$

then,

$$|\zeta - \zeta_1| < \frac{\delta}{C_5} \Rightarrow |v_1(f(\zeta)) - 1| < \frac{1}{2}.$$

Note that $\frac{\delta}{C_5}$ is uniform. Therefore, one can find $C_{12} \in (0, 1/4)$ such that

$$|\zeta - \zeta_1| \leq 2C_{12} \Rightarrow |v_1(f(\zeta)) - 1| < \frac{1}{2}.$$

Next, construct a function $\phi : \partial U \to \mathbb{C}$ such that:

1. $\phi(\zeta) = \overline{v_1(f(\zeta))}$ if $|\zeta - \zeta_1| \leq 2C_{12}$;

2. $|\phi(\zeta) - 1| < \frac{1}{2}$.

3. The 1/2-Holder norm of ϕ is equal to that of $v \circ f$.

Let $r : \partial U \to \mathbb{R}$ be the harmonic conjugate of $Im(\log(\phi))$, hence $r + iIm(\log(\phi))$ extends to a holomorphic function on U. By a theorem of Privaloff, since $\log(\phi)$ is 1/2-Holder continuous, then r is too. Moreover, the 1/2-Holder norm of r is uniformly bounded. Hence, if $q = r - Re(\log(\phi))$, then

$$|q(\zeta) - q(\zeta')| \leq C_{13}|\zeta - \zeta'|^{1/2} \qquad (\zeta, \zeta' \in \partial U)$$

and $q + \log(\phi) = r + iIm(\log(\phi))$ extends to a function $h : \overline{U} \to \mathbb{C}$, 1/2-Holder continuous, holomorphic on U.
Let $g(\zeta) = \widehat{f_1(\zeta)}e^{-h(\zeta)}$ and $G(\zeta) = \dfrac{g(\zeta)}{\zeta}$. Note that g is holomorphic on U and G is holomorphic on $U - \{0\}$. Moreover, g is uniformly bounded because p is uniformly bounded. Also, G is uniformly bounded on $B = \overline{U} \cap U_1$ where $U_1 = \{\zeta \in \mathbb{C} : |\zeta - \zeta_1| < 2C_{12}\}$ because $|\zeta| > 1 - 2C_{12} > 0$ on B.
If $\zeta \in \partial U \cap U_1$, then

$$\begin{aligned} G(\zeta) = \frac{g(\zeta)}{\zeta} = \frac{\widehat{f_1(\zeta)}e^{-h(\zeta)}}{\zeta} &= p(\zeta)\overline{v_1(f(\zeta))}e^{-r(\zeta)}e^{-iIm(\log(\phi(\zeta))} \\ &= p(\zeta)\phi(\zeta)e^{-r(\zeta)}\frac{1}{\phi(\zeta)} = p(\zeta)e^{-r(\zeta)}, \end{aligned}$$

so G is real on $\partial U \cap U_1$, hence G extends holomorphically across $\partial U \cap U_1$ and the extended function is uniformly bounded on U_1. Hence, G is uniformly Lipschitz on compact subsets of U_1. In particular,

$$|\zeta_2 - \zeta_1| \leq C_{12} \Rightarrow |G(\zeta_1) - G(\zeta_2)| \leq C_{14}|\zeta_1 - \zeta_2|^{1/2}.$$

Finally, since e^h and $v_1 \circ f$ are uniformly Holder-continuous, then we get the desired inequality for $p = \dfrac{Ge^h}{\overline{v_1 \circ f}}$. □

Theorem 4.5.7 [Lempert, 81]

Let $f : \overline{U} \to \overline{D}$ be a stationary map. Suppose that the diameter of ∂D, $SC_D(p, v)(p \in \partial D, v \in T_p(\partial D))$, and the distance from $f(0)$ to ∂D are bounded below and above by positive numbers. Then, there exists a uniform constant $C_{15} > 0$ such that

$$|\widehat{f}(\zeta_1) - \widehat{f}(\zeta_2)| \leq C_{15}|\zeta_1 - \zeta_2|^{1/2} \qquad (\zeta_1, \zeta_2 \in \overline{U}).$$

Proof. If $\zeta_1, \zeta_2 \in \partial U$, then the desired inequality follows from the previous theorems of this section. By a Theorem of Hardy and Littlewood, the inequality holds for any $\zeta_1, \zeta_2 \in \overline{U}$. □

4.6 Discs attached to a small perturbation of the boundary of the unit ball in $\mathbb{C}^2$

In this section, the spaces $\mathcal{C}^{k+1-\epsilon}(\partial U)$, $\epsilon > 0$, $k \in \mathbb{N}$ are equipped with their usual norm:

$$\|\boldsymbol{h}\|_{\mathcal{C}^{k+1-\epsilon}(\partial U)} = \sum_{l=0}^{k} \|\boldsymbol{h}^{(l)}\|_\infty + \sup_{\zeta \neq \eta \in \partial U} \frac{\|\boldsymbol{h}^{(k)}(\zeta) - \boldsymbol{h}^{(k)}(\eta)\|}{|\zeta - \eta|^\epsilon},$$

where $\|\boldsymbol{h}^{(l)}\|_\infty := \max\limits_{\partial U} \|\boldsymbol{h}^{(l)}\|$.

We start our section by stating, without proof, the following theorem:

Theorem 4.6.1 (Birkhoff Factorization)

Consider a map $G : \partial U \to GL_N(\mathbb{C})$. Define

$$B(\zeta) = -\overline{G(\zeta)^{-1}} G(\zeta), \qquad \zeta \in \partial U.$$

Then, there exist two continuous maps $B^+ : \overline{U} \to GL_N(\mathbb{C})$ and $B^- : (\mathbb{C} \cup \infty) - U \to GL_N(\mathbb{C})$ such that

$$B(\zeta) = B^+(\zeta)\Lambda(\zeta)B^-(\zeta)$$

where B^+ and B^- are holomorphic on U and $\mathbb{C} - U$ respectively, and

$$\Lambda(\zeta) = \begin{pmatrix} \zeta^{\kappa_1} & & (0) \\ & \ddots & \\ (0) & & \zeta^{\kappa_N} \end{pmatrix}$$

with $\kappa_1 \geq ... \geq \kappa_N$.

The integers $\kappa_1, ..., \kappa_N$ are called the partial indices of B. We define the Maslov index of B to be $\kappa := \sum\limits_{j=1}^{N} \kappa_j$, and show that it is given by a winding number:

Theorem 4.6.2 ([9])

Suppose that the determinant of B is of class C^1 on ∂U. Then

$$\kappa = \frac{1}{2\pi i} \int\limits_{\partial U} \frac{(detB)'(\zeta)}{detB(\zeta)} d\zeta.$$

Proof. Extend the map B^- antiholomorphically to U by defining

$$\tilde{B}^-(\zeta) = B^-\left(\frac{1}{\overline{\zeta}}\right), \qquad \forall \zeta \in U.$$

Fix $0 < r < 1$, and let

$$b_r^+(\theta) = det(B^+(re^{i\theta})),$$

$$b_r^-(\theta) = det(\tilde{B}^-(\overline{re^{i\theta}})) = det(\tilde{B}^-(re^{-i\theta})),$$

$$\beta_r(\theta) = b_r^+(\theta) r^\kappa e^{i\kappa\theta} b_r^-(\theta).$$

Since $\beta_r(\theta) \neq 0$ on $[0, 2\pi]$, then the curve $\gamma_r = \beta_r([0, 2\pi])$ does not pass by 0. Hence,

$$2\pi i Ind_{\gamma_r}(0) = \int_{\gamma_r} \frac{d\zeta}{\zeta} = \int_0^{2\pi} \frac{{b_r^+}'(\theta)}{b_r^+(\theta)} d\theta + \int_0^{2\pi} i\kappa d\theta + \int_0^{2\pi} \frac{{b_r^-}'(\theta)}{b_r^-(\theta)} d\theta.$$

But

$$\int_0^{2\pi} \frac{{b_r^+}'(\theta)}{b_r^+(\theta)} d\theta = \int_{r\partial U} \frac{det(B^+(\zeta))'}{det(B^+(\zeta))} d\zeta = 2\pi i(N_0 - N_\infty)$$

where N_0 and N_∞ are the number of zeros and poles of $det(B^+(\zeta))$ in rU. Since B^+ is invertible, we get

$$\int_0^{2\pi} \frac{{b_r^+}'(\theta)}{b_r^+(\theta)} d\theta = 0.$$

Similarly,

$$\int_0^{2\pi} \frac{{b_r^-}'(\theta)}{b_r^-(\theta)} d\theta = 0.$$

Therefore,

$$Ind_{\gamma_r}(0) = \kappa.$$

Now since the compact set $\{\beta_r(\theta) \mid r \in [1/2, 1],\ \theta \in [0, 2\pi]\}$ does not contain 0, it is then contained in an open set Ω that does not contain 0. Moreover, the closed curves $\gamma_{1/2}$ and γ_1 are of class $\mathcal{C}^1$ and homotopic in Ω by the application

$$(t, \theta) \mapsto \beta_{1/(2-t)}(\theta).$$

Since any two homotopic curves have the same index, we get:

$$Ind_{\gamma_{1/2}}(0) = \kappa = Ind_{\gamma_1}(0) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{\beta_1'(\theta)}{\beta_1(\theta)} d\theta$$

where $\beta_1(\theta) = det(B(e^{i\theta}))$. □

Our next goal is to give a more general definition of stationary maps. To do so, we need to provide the following definitions:

Definition 4.6.3

Let $\Gamma = \{\rho = 0\}$ be a smooth real hypersurface of $\mathbb{C}^n$. Let $p \in \Gamma$. Define the conormal fiber at p, denoted by $N_p^*\Gamma$,to be the real line generated by

$$\nabla\rho(p) = \left(\frac{\partial\rho}{\partial z_1}(p), \ldots, \frac{\partial\rho}{\partial z_n}(p)\right).$$

That is,

$$N_p^*\Gamma = \operatorname{span}_{\mathbb{R}}\{\nabla\rho(p)\} \subset \mathbb{C}^n.$$

Definition 4.6.4

A holomorphic disc f is a holomorphic map $f : U \to \mathbb{C}^n$.

Definition 4.6.5

A holomorphic disc f is said to be attached to a submanifold $M \subset \mathbb{C}^n$ if f extends to a continuous map on $\overline{U}$ (which will be denoted by f) such that $h(\partial U) \subset M$.

Definition 4.6.6

Let Γ be a real hypersurface in $\mathbb{C}^n$. We define the conormal fibration to be set

$$\mathscr{N}\Gamma = \bigcup_{\zeta \in \partial U} \mathscr{N}\Gamma(\zeta),$$

where

$$\mathscr{N}\Gamma(\zeta) = \{(z, \zeta w) \mid z \in \Gamma,\ w \in N_z^*\Gamma - \{0\}\}.$$

Definition 4.6.7

A holomorphic disc f attached to a a real hypersurface Γ is said to be stationary for Γ if there exists a holomorphic lift $\boldsymbol{f} = (f, \widehat{f}) : U \to C^{2n}$ of f continuous up to the boundary of U such that $\boldsymbol{f}(\zeta) \in \mathscr{N}\Gamma(\zeta)$ for all $\zeta \in \partial U$.

In case $\Gamma = \{\rho = 0\}$, one can show that Definition 4.6.7 is equivalent to the existence of a continuous function $p : \partial U \to \mathbb{R}^+$ such that $\zeta p(\zeta)\nabla\rho(f(\zeta))$ is continuous on ∂U and extends to a holomorphic function on U.

We proceed now to compute the defining equations of conormal fibration $\mathscr{N}\Gamma$, where Γ is the boundary of the open unit ball in $\mathbb{C}^2$.

Example 4.6.8

Let Γ be the boundary of the unit ball in $\mathbb{C}^2$ defined by

$$\rho(z) = |z_1|^2 + |z_2|^2 - 1.$$

Let $(z_1, z_2) \in \Gamma$. WLOG, suppose $z_1 \neq 0$. Then $\nabla\rho(z) = (\overline{z_1}, \overline{z_2})$. Hence, the conormal fiber at z is given by

$$N_z^*\Gamma = \operatorname{span}_{\mathbb{R}} \{(\overline{z_1}, \overline{z_2})\}.$$

Let $\zeta \in \partial U$. Then, $(z, w) = (z_1, z_2, w_1, w_2) \in \mathscr{N}\Gamma(\zeta) \subset \mathbb{C}^4$ if and only if $\rho(z) = 0$ and $w = c\zeta(\overline{z_1}, \overline{z_2})$ for some $c \in \mathbb{R}$. Therefore,

$$(z, w) \in \mathscr{N}\Gamma \Leftrightarrow \rho(z) = 0, \frac{w_1}{\zeta\overline{z_1}} \in \mathbb{R}, w_2 = \frac{w_1\overline{z_2}}{\overline{z_1}}.$$

We get the following defining equations of $\mathscr{N}\Gamma$:

$$\begin{cases} \tilde\rho_1(\zeta)(z,w) &= |z_1|^2+|z_2|^2-1=0, \\ \tilde\rho_2(\zeta)(z,w) &= i\dfrac{w_1}{\zeta\overline{z_1}}-i\dfrac{\zeta\overline{w_1}}{z_1}=0, \\ \tilde\rho_3(\zeta)(z,w) &= w_2-\dfrac{w_1\overline{z_2}}{\overline{z_1}}+\overline{w_2}-\dfrac{\overline{w_1}z_2}{z_1}=0, \\ \tilde\rho_4(\zeta)(z,w) &= i\left(w_2-\dfrac{w_1\overline{z_2}}{\overline{z_1}}\right)-i\left(\overline{w_2}-\dfrac{\overline{w_1}z_2}{z_1}\right)=0. \end{cases} \tag{4.1}$$

In other words,

$$(z,w)\in\mathscr{N}\Gamma \Leftrightarrow \tilde\rho_1(\zeta)(z,w)=\tilde\rho_2(\zeta)(z,w)=\tilde\rho_3(\zeta)(z,w)=\tilde\rho_4(\zeta)(z,w)=0.$$

We note that if f is stationary for Γ, then $\tilde\rho(\zeta)(\boldsymbol{f}(\zeta))=0$.

Next, we want to construct stationary discs for small perturbations of Γ. To do so, we need the following version of the implicit function theorem:

Theorem 4.6.9 ([2])

Let X,Y,Z be Banach spaces. Let U an open neighborhood of 0 in $X\times Y$, and let $\mathcal{F}:U\mapsto Z$ be a $\mathcal{C}^1$ map such that $\mathcal{F}(0)=0$. Assume that $d_Y\mathcal{F}(0):Y\to Z$ is onto and that $\ker d_Y\mathcal{F}(0)$ is complemented in Y, namely $Y=\ker d_Y\mathcal{F}(0)\oplus H$ where H is a closed subspace of Y. Identify $X\times Y$ with $X\times \mathrm{Ker} d_Y\mathcal{F}(0,0)\times H$ in the canonical way. Then there are neighborhoods V_1 of 0 in X, V_2 of 0 in $\mathrm{Ker} d_Y\mathcal{F}(0)$, V_3 of 0 in H and a $\mathcal{C}^1$ map $g:V_1\times V_2\mapsto V_3$ such that

$$(x_1,x_2,x_3)\in U \text{ and } \mathcal{F}(x_1,x_2,x_3)=0$$

if and only if

$$(x_1,x_2)\in V_1\times V_2 \quad\text{and}\quad x_3=g(x_1,x_2).$$

In particular the set $\{x\in V_1\times V_2\times V_3 \mid \mathcal{F}(x)=0\}$ is a $\mathcal{C}^1$ submanifold of $V_1\times V_2\times V_3$ and for each $x_1\in V_1$ the set $\{(x_2,x_3)\in V_2\times V_3 \mid \mathcal{F}(x_1,x_2,x_3)=0\}$ is a $\mathcal{C}^1$ submanifold of $V_2\times V_3$.

One can show that if $\ker d_Y\mathcal{F}(0)$ has finite dimension N then $\ker d_Y\mathcal{F}(0)$ is complemented in Y and therefore $\{(x_2,x_3)\in V_2\times V_3 \mid \mathcal{F}(x_1,x_2,x_3)=0\}$ is a $\mathcal{C}^1$ submanifold of finite dimension N.

Now, in order to apply this theorem, we define first the following Banach spaces for $0<\epsilon<1$:

1. $X=\mathcal{C}^{2-\epsilon}(\partial U,\mathcal{C}^3(\mathbb{C}^4,\mathbb{R}^4))$,
2. $Y=\mathcal{A}^{2-\epsilon}(U,\mathbb{C}^4)$,
3. $Z=\mathcal{C}^{2-\epsilon}(\partial U,\mathbb{R}^4)$,

where $\mathcal{A}^{2-\epsilon}(U,\mathbb{C}^4)$ is the set of lifts of class $C^{2-\epsilon}$ up to the boundary of U.

Let U be a neighborhood of $\tilde\rho$ in X. Consider the lift $\boldsymbol{f_0}=(\zeta,0,1,0)$ and let V be a neighborhood of $\boldsymbol{f_0}$ in Y. Define the map

$$\mathcal{F} : U \times V \to Z$$

by

$$\mathcal{F}(\tilde{r}, \boldsymbol{f}) = \tilde{r}(.)(\boldsymbol{f}).$$

One can show that $\mathcal{F}$ is well defined and is of class C^1 [3]. Moreover,

$$d_Y\mathcal{F}(\tilde{\rho}, \boldsymbol{f_0})(\boldsymbol{f}) = 2\Re e\left(\overline{G}\boldsymbol{f}\right)$$

where $G : \partial\Delta \to GL_4(\mathbb{C})$ is given by

$$G(\zeta) = \left(\frac{\partial\tilde{\rho}}{\partial\overline{z_1}}(\boldsymbol{f_0}(\zeta)), \frac{\partial\tilde{\rho}}{\partial\overline{z_2}}(\boldsymbol{f_0}(\zeta)), \frac{\partial\tilde{\rho}}{\partial\overline{w_1}}(\boldsymbol{f_0}(\zeta)), \frac{\partial\tilde{\rho}}{\partial\overline{w_2}}(\boldsymbol{f_0}(\zeta))\right) = \begin{pmatrix} \zeta & 0 & 0 & 0 \\ -i\zeta & 0 & -i & 0 \\ 0 & -\zeta & 0 & 1 \\ 0 & -i\zeta & 0 & -i \end{pmatrix}.$$

Our next step is to show that $d_Y\mathcal{F}(\tilde{\rho}, \boldsymbol{f_0})$ is onto and compute the dimension of its kernel. However, we need first the following lemma:

Lemma 4.6.10 ([1])

Let $A : \partial U \to GL_{2n}(\mathbb{C})$ of class $\mathcal{C}^\alpha$ $(0 < \alpha < 1)$, and denote by $\kappa_1 \geq \ldots \geq \kappa_{2n}$ the partial indices of the map $\zeta \mapsto A(\zeta)\overline{A(\zeta)^{-1}}$. Then there exists a map $\Theta : \bar{U} \to GL_{2n}(\mathbb{C})$ of class $\mathcal{C}^\alpha$, holomorphic on U, such that

$$\Theta(\zeta)A(\zeta)\overline{A(\zeta)^{-1}} = \begin{pmatrix} \zeta^{\kappa_1} & & (0) \\ & \ddots & \\ (0) & & \zeta^{\kappa_{2n}} \end{pmatrix}\overline{\Theta(\zeta)}, \qquad \forall\zeta \in \partial U.$$

Theorem 4.6.11

Let $B(\zeta) = -\overline{G(\zeta)^{-1}}G(\zeta)$. Then, the partial indices of B are nonnegative and the Maslov index of B is equal to 4.

Proof. Tedious calculation leads to

$$B(\zeta) = \begin{pmatrix} -\zeta^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \zeta \\ 2\zeta & 0 & 1 & 0 \\ 0 & \zeta & 0 & 0 \end{pmatrix}.$$

Apply Lemma 4.6.10 to the matrix $i\overline{G(\zeta)^{-1}}$ to get a continuous map $\Theta : \bar{U} \to GL_4(\mathbb{C})$, holomorphic on U such that

$$\Theta(\zeta)B(\zeta) = \begin{pmatrix} \zeta^{\kappa_1} & & (0) \\ & \ddots & \\ (0) & & \zeta^{\kappa_4} \end{pmatrix}\overline{\Theta(\zeta)} \qquad \forall\zeta \in \partial U.$$

Denote by $l = (l_1, l_2, l_3, l_4)$ the last row of the matrix Θ. Then, for all $\zeta \in \partial U$, we have:

$$\begin{cases} l_3(\zeta) = \zeta^{\kappa_4}\overline{l_3(\zeta)} \\ -\zeta^2 l_1(\zeta) + 2\zeta l_3(\zeta) = \zeta^{\kappa_4}\overline{l_1(\zeta)} \\ \zeta l_4(\zeta) = \zeta^{\kappa_4}\overline{l_2(\zeta)} \\ \zeta l_2(\zeta) = \zeta^{\kappa_4}\overline{l_4(\zeta)} \end{cases} \tag{4.2}$$

If $l_3 \not\equiv 0$, then $\kappa_4 \geq 0$ by holomorphicity of l_3. If $l_3 \equiv 0$ and $l_1 \not\equiv 0$, then $\kappa_4 \geq 2$. If $l_3 \equiv 0$ and $l_1 \equiv 0$, then we must have $l_2 \not\equiv 0$ by invertibility of Θ, and so $\kappa_4 \geq 1$.

Since $\kappa_1 \geq ... \geq \kappa_4$, then the partial indices of B are nonnegative. Therefore, the linear map $d_Y\mathcal{F}(\tilde{\rho}, \boldsymbol{f_0})$ is onto by a result of Globevnik [2,3]. Moreover, by Theorem 4.6.2,

$$\kappa = \frac{1}{2\pi i}\int\limits_{\partial U} \frac{(detB)'(\zeta)}{detB(\zeta)}d\zeta = \frac{1}{2\pi i}\int\limits_{\partial U} \frac{4\zeta^3}{\zeta^4}d\zeta = 4.$$

It follows again by a result of Globevnik that the kernel of $d_Y\mathcal{F}(\tilde{\rho}, \boldsymbol{f_0})$ has dimension $\kappa + \dim_{\mathbb{C}} \mathbb{C}^4 = 4 + 4 = 8$.

$\square$

Applying Theorem 4.6.9, we get the following corollary:

Corollary 4.6.12

There are neighborhoods V_1 of $\tilde{\rho}$ in X and V_2 of $\boldsymbol{f_0}$ in Y such that for each $\tilde{r} \in V_1$, the set

$$\{\boldsymbol{f} \in V_2 \mid \mathcal{F}(\tilde{r}, \boldsymbol{f}) = 0\}$$

is a $\mathcal{C}^1$ submanifold of Y of real dimension 8.

Notation:

We write $D \sim B$ if D is a domain having a defining function r such that $\tilde{r} \in V_1$.

Theorem 4.6.13

Let $D \subset \mathbb{C}^2$ be a strongly convex domain such that $D \sim B$. Let $Z \in D, v \in \mathbb{C}^2$ with $v \neq 0 (Z, z \in D, Z \neq z)$. Then, there exists a unique extremal map $f : U \to D$ with respect to Z, v (or Z, z). Moreover, $f(\overline{U})$ is extremal with respect to any couple of points $w_1, w_2 \in f(\overline{U}) (w_1 \neq w_2)$ and with respect to any point $w = f(\omega)$ and direction $f'(\omega) \in \mathbb{C}^2$.

Proof. Let $D \subset \mathbb{C}^2$ be a strongly convex domain such that $D \sim B$. Let r be the defining function of D, so $\tilde{r} \in V_1$. Let $Z \in D, v \in \mathbb{C}^2$ with $v \neq 0$. Then, there exists a neighborhood U of $0 \in \mathbb{R}^8$ such that for each $t \in U$, there is a unique $H(\tilde{r}, t) \in \left\{\boldsymbol{f} = (f, \widehat{f}) : \mathcal{F}(\tilde{r}, \boldsymbol{f}) = 0, ||\boldsymbol{f} - \boldsymbol{f_0}|| << 1\right\}$. It follows that the map $\Psi : U \to \mathbb{R}^8$ defined by

$$\Psi(t) = ((\pi \circ H(\tilde{r}, t))(0), (\pi \circ H(\tilde{r}, t))'(0))$$

is a bijection of U onto $D \times \mathbb{C}^2$, where π be the projection onto the first component. Hence, there exists a unique $t \in U$ such that $(\pi \circ H(\tilde{r}, t))(0) = Z$ and $(\pi \circ H(\tilde{r}, t))'(0) = v$, with $\pi \circ H(\tilde{r}, t)$ a stationary map, so extremal with respect to Z, v. □

4.7 Applications

In this last section, we state and prove some applications.

We start our section by showing that there is a bijection between a strongly convex domain D such that $D \sim B$ and B:

Theorem 4.7.1

Let $D \subset \mathbb{C}^2$ be a strongly convex domain such that $D \sim B$. There is a bijection between D and B.

Proof. Fix $Z \in D$. Define $\Phi_Z : D \to B$ as follows:
Let $z \in D$. Let $f : U \to D$ be the extremal map with respect to Z and z, so

$$f(0) = Z, f(\xi) = z(\xi > 0), \delta_D(Z, z) = d_H(0, \xi).$$

Let

$$\Phi_Z(z) = \xi \frac{f'(0)}{|f'(0)|}.$$

Define $\Psi : B \to D$ by

$$v \mapsto f_{Z,v}(|v|)$$

where $f_{Z,v}$ is the unique extremal map with respect to Z and v.

1. Let $v \in B$.
   Then, $\Phi_Z(\Psi(v)) = \Phi_Z(f_{Z,v}(|v|)) = |v| \dfrac{f'_{Z,v}(0)}{|f'_{Z,v}(0)|} = |v| \dfrac{\lambda v}{\lambda |v|} = v.$
2. Let $z \in D$.
   Then, $\Psi(\Phi_Z(z)) = \Psi\left(\xi \dfrac{f'(0)}{|f'(0)|}\right) = f_{Z,z_*}(\xi)$ where f is the extremal map with respect to Z, z and $z_* = \xi \dfrac{f'(0)}{|f'(0)|}$.
   By Theorem 4.3.5, f is the unique extremal map with respect to $f(0) = Z$ and $f'(0)$. But f_{Z,z_*} is extremal with respect to Z and $f'(0)$, and hence $\Psi(\Phi_Z(z)) = f(\xi) = z$.

□

In fact D and B are homeomorphic. Moreover, one can show using advanced tools the following stronger statement:

Theorem 4.7.2 ([7])

Let $D \subset \mathbb{C}^2$ be a strongly convex domain such that ∂D is of class C^6 and $D \sim B$. Let $Z \in D$. The map Φ_Z extends to a homeomorphism between $\overline{D}$ and $\overline{B}$, and this extended map is a diffeomorphism of class C^2 between $\overline{D} - \{Z\}$ and $\overline{B} - \{0\}$.

Finally, we state and prove Fefferman Theorem for strongly convex domains D such that $D \sim B$.

Theorem 4.7.3 [Fefferman, 74 ; Lempert, 81]

Let D_1, D_2 be domains in $\mathbb{C}^2$ such that $\partial D_1, \partial D_2$ are of class C^6, $D_1 \sim B$ and $D_2 \sim B$. Let $F : D_1 \to D_2$ be a biholomorphic map. Then, $F \in C^2(\overline{D_1})$

Proof. WLOG, suppose that $0 \in D_1, D_2, F(0) = 0, F'(0) = I_2$.
Let $\Phi_0^{(i)} : D_i \to B$ be the maps defined in Theorem 4.7.1. We will show that $\Phi_0^{(1)} = \Phi_0^{(2)} \circ F$.
Let $z \in D_1$. Suppose $f : U \to D_1$ is extremal with respect to $0, z$. Consider $F \circ f : U \to D_2$, and note that $F \circ f$ is extremal with respect to 0 and $F(z)$. Hence

$$\Phi_0^{(2)}(F(z)) = \xi \frac{(F \circ f)'(0)}{|(F \circ f)'(0)|} = \xi \frac{F'(f(0))f'(0)}{|F'(f(0))f'(0)|} = \xi \frac{f'(0)}{|f'(0)|} = \Phi_0^{(1)}(z).$$

But by Theorem 4.7.2, $\Phi_0^{(i)}$ extends to a diffeomorphism of class C^2 between $\overline{D_i} - \{0\}$ and $\overline{B} - \{0\}$. Therefore, $F \in C^2(\overline{D_1})$. ☐

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