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APPROXIMATION OF NON-HOLOMORPHIC MAPS

by

NOUR AHMED KHOUDARI

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NOUR KHOUDARI

Approved by:

Prof. Bertrand Florian, Assistant Professor — Advisor
Mathematics

Prof. Abi Khuzam Farouk, Professor — Member of Committee
Mathematics

Prof. Tlas Tamer, Associate Professor — Member of Committee
Mathematics

Prof. Della Sala Giuseppe, Assistant Professor — Member of Committee
Mathematics

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AN ABSTRACT OF THE THESIS OF

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We will study the approximation of nonholomorphic maps from the unit disc to a complex manifold. This starts by generalizations of some theorems from one complex variable to several complex variables like the generalizations of Mittag-leffer and Weirstrass factorization theorem to the famous Cousin problems. Through these generalizations we will face local to global problems, like the $\overline{\partial}$ problem which can be solved by some cohomology conditions. The work is based on a paper by Jean-Pierre Rosay which deals with approximation of nonholomorphic maps and applications to the Poletsky theory of discs. The main question to be answered is whether we can approximate a map with a small $\overline{\partial}$ from the unit disc to a complex manifold by a holomorphic map. Lempert gives an example that negatively answers this question by taking any smooth map from the unit disc to any compact Riemann surface of genus greater than or equal to two. However, by taking a condition on the map to be restricted we will prove that the answer is positive.

Contents

Chapter 1

Preliminaries on Banach Spaces

Throughout this paper Δ will denote the open unit disc in the complex plane.
Banach spaces are used in the main theorem, so we start by introducing Banach spaces based on the refernce [7] and we give some examples that will be mentioned in later sections in this paper.

Definition. *A Banach space is a complete normed vector space, i.e a vector space with a metric that allows the computation of vector length and distance between vectors and is complete in the sense that a Cauchy sequence of vectors always converges to a well defined limit in that space.*

Remark. *One of the examples of Banach spaces is L^p spaces. Another Banach space that we will encounter in this paper in the main theorem is*

$$H^\infty(\Delta) = \mathcal{O}(\Delta) \cap L^\infty(\Delta)$$

the norm defined on this space is

$$\|f\| = sup_{\overline{\Delta}}|f| = sup_{\partial\Delta}|f|$$

note that to show this is a Banach space it is enough to show it is closed since $H^\infty(\Delta) \subset L^\infty(\Delta)$ and $L^\infty(\Delta)$ is a Banach space.
Take a sequence of holomorphic bounded functions f_n that is Cauchy (Cauchy sequence converges in the complete space $L^\infty(\Delta)$) , so $f_n \longrightarrow f$ but since f_n holomorphic then $f \in \mathcal{O}(\Delta)$

Definition. *Let X, Y be two Banach spaces and $F : X \longrightarrow Y$ and $x \in X$:*
We say F is differentiable at x in the direction h if there exist a linear map $D_xF : X \longrightarrow Y$ such that

$$D_xF(h) = lim_{t\to 0}\frac{F(x+th) - F(x)}{t}$$

or

$$F(x+h) = F(x) + D_xF(h) + O(|h|^2)$$

Definition. *Let X, Y be two Banach spaces and $T : X \longrightarrow Y$ linear operator*
the operator norm is defined as

$$\|T\| = sup_{\|v\|_X=1}\|T(v)\|_Y$$

and

$$\|T(v)\|_Y \leq \|T\|\|v\|_X$$

Theorem 1. *(Mean Value Theorem on Banach Spaces) Let X, Y be Banach spaces and $F : X \longrightarrow Y$ Let $[x_1, x_2]$ denote the line segment joining two points x_1, x_2 in an open set $U \subset X$.*
If F is differentiable in U then there exist $x \in [x_1, x_2]$ such that

$$\|F(x_1) - F(x_2)\|_Y \leq \|D_x F(h)\|_Y \|x_1 - x_2\|_X$$

Proof. Define $\varphi(t) = F[(1-t)x_1 + tx_2]$ and apply ordinary Mean Value Theorem on $\varphi(0)$ and $\varphi(1)$ □

Definition. *Let (X, d) be a metric space, then a self-map $T : X \longrightarrow X$ is called a contraction mapping on X if there exist $q \in [0, 1)$ such that $d(T(x), T(y)) \leq qd(x, y)$ for all $x, y \in X$*

Lemma 1. *Let (X, d) be a metric space and $f : X \longrightarrow X$ a contraction map*

$$i.e : d(f(x), f(y)) \leq cd(x, y) \ for \ all \ x, y \in X, 0 \leq c < 1$$

then f is continuous.

Proof. Given $\epsilon > 0$, choose $\delta = \frac{\epsilon}{c}$ and $a \in X$

$$d(f(x), f(a)) \leq cd(x, a) \leq c\delta = c\frac{\epsilon}{c} = \epsilon$$

so f is continuous at a arbitrary
therefore f is continuous on X □

Now the following theorem will be used in the proof of the main theorem.

Theorem 2. *(Banach Fixed point theorem) Let X be a Banach space and $F : X \longrightarrow X$ a contraction mapping on X, then there exist a unique $x^* \in X$ such that $F(x^*) = x^*$.*

Proof. Let $x, y \in X$

$$\begin{aligned} d(x, y) &\leq d(x, f(x)) + d(f(x), f(y)) + d(f(y), y) \\ &\leq d(x, f(x)) + qd(x, y) + d(f(y), y) \end{aligned}$$

so $d(x, y) \leq \frac{d(f(x),x)+d(f(y),y)}{1-q}$
Suppose both x and y are fixed points, then $d(x, y) = 0$, then $x = y$ proving uniqueness of fixed point.

$$\begin{aligned} d(f^n(x), f^m(x)) &\leq \frac{d(f(f^n(x)), f^n(x)) + d(f(f^m(x)), f^m(x))}{1-q} \\ &= \frac{d(f^n(f(x)), f^n(x)) + d(f^m(f(x)), f^m(x))}{1-q} \\ &\leq \frac{q^n d(f(x), x) + q^m d(f(x), x)}{1-q} \\ &= \frac{q^n + q^m}{1-q} d(f(x), x) \end{aligned}$$

so $d(f^n(x), f^m(x)) \longrightarrow_{m,n\to\infty} 0$
therefore $f^n(x)$ is cauchy so it converges to a point $x^* \in X$
$f^n(x)$ generates a sequence $x_n \longrightarrow x^*$ such that:

$$\begin{aligned} & x_0 = x \\ & x_1 = f(x_0) = f(x) \\ & \vdots \\ & x_n = f^n(x) = f(x_{n-1}) \end{aligned}$$

so $lim_{n\to\infty}x_n = lim_{n\to\infty}f^n(x) = lim_{n\to\infty}f(x_{n-1}) = f(lim_{n\to\infty}x_{n-1}) = f(x^*)$
note that since f is a contraction then f is continuous.
therefore $x^* = f(x^*)$ $\square$

Chapter 2

One Complex Variable

2.1 Basic Definitions and Results

In this section we start by a review on some basic definitions and results in complex analysis in one complex variable.
We begin with defining holomorphic functions in $\mathbb{C}$.

Definition. *Let $\Omega \subset \mathbb{C}$ be an open set, and let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be a complex-valued function, if the limit*

$$f'(z_0) = lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists for all $z_0 \in \Omega$, then we say f is holomorphic on Ω.

Remark.

1. *Throughout this paper we will denote by a domain an open connected subset of $\mathbb{C}$.*
2. *We denote by $\mathcal{O}(\Omega)$ the set of all functions that are holomorphic on Ω.*
3. *On an open subset Ω of the complex plane, a function that is holomorphic on all of Ω except for a set of isolated points, which are poles of that function, is called a meromorphic function and the field of meromorphic functions is denoted by $\mathcal{M}(\Omega)$.*
4. *We denote by $\mathcal{O}(\Omega) \cap C(\partial\overline{\Omega})$ the set of all functions that are holomorphic in Ω and continuous in $\overline{\Omega}$*

Now we state a theorem named after Cauchy which gives an important result about line integrals for holomorphic functions in the complex plane.

Theorem 3. *(Cauchy theorem in one variable) Let $\Omega \subseteq \mathbb{C}$ be a domain, $K \subset \Omega$ compact, and $f \in \mathcal{O}(\Omega)$. Then*

$$\int_{\partial K} f(z)dz = 0.$$

A consequence of the above theorem is the Cauchy integral formula which shows that a holomorphic function defined on a disk is completely determined by its values on the boundary of the disk.

Theorem 4. *(Cauchy integral formula in one variable) Let Ω be a disc in $\mathbb{C}$. Suppose $f : \Omega \longrightarrow \mathbb{C}$ and $f \in \mathcal{O}(\Omega) \cap C(\partial\overline{\Omega})$. Then for $z_0 \in \Omega$*

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(z)}{z - z_0} dz.$$

Remark. *For a holomorphic function we have the following operators:*

$$\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}) \qquad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$$

We will now introduce the Pompeiu's formula also known as the generalized Cauchy formula that will be used later in 4.1. This formula is used in case f is not holomorphic. We refer to [3] as a citation for the below:

Lemma 2. *Let $\Omega \subset \mathbb{C}$ be a bounded domain, $\partial\Omega$ piece-wise smooth, $g(z)$ smooth function on $\Omega \cup \partial\Omega$ then:*

$$\int_{\partial\Omega} g(z)dz = 2i \int\int_{\Omega} \frac{\partial g}{\partial \overline{z}} dxdy$$

Proof.

$$dz = dx + idy$$

$$so \qquad \int_{\partial\Omega} g(z)dz = \int_{\partial\Omega} g(z)dx + \int_{\partial\Omega} ig(z)dy$$

By Green's theorem

$$\int_{\partial\Omega} g(z)dz = \int\int_{\Omega} (i\frac{\partial g}{\partial x} - \frac{\partial g}{\partial y})dxdy = 2i \int\int_{\Omega} \frac{\partial g}{\partial \overline{z}} dxdy$$

□

Theorem 5. *(Generalized Cauchy Formula) Let $\Omega \subset \mathbb{C}$ be a bounded domain, $\partial\Omega$ piece-wise smooth, $g(z)$ smooth function on $\Omega \cup \partial\Omega$ then:*

$$g(w) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{g(z)}{z - w} dz - \frac{1}{\pi} \int\int_{\Omega} \frac{\partial g}{\partial \overline{z}} . \frac{1}{z - w} dxdy \qquad \forall w \in \Omega$$

Proof. There exist $\epsilon > 0$ such that $\{|z - w| \leq \epsilon\} \subset \Omega$

Let $\Omega_\epsilon = \Omega - \{|z - w| \leq \epsilon\}$

By the above lemma

$$\begin{aligned}
\int_{\partial\Omega_\epsilon} \frac{g(z)}{z - w} dz &= 2i \int\int_{\Omega_\epsilon} \frac{\partial}{\partial \overline{z}} (\frac{g}{z - w}) dxdy \\
&= 2i \int\int_{\Omega_\epsilon} \frac{\partial g}{\partial \overline{z}} . \frac{1}{z - w} + g\frac{\partial}{\partial \overline{z}} (\frac{1}{z - w}) dxdy \\
&= 2i \int\int_{\Omega_\epsilon} \frac{\partial g}{\partial \overline{z}} . \frac{1}{z - w} dxdy
\end{aligned}$$

but

$$\int_{\partial\Omega_\epsilon} \frac{g(z)}{z - w} dz = \int_{\partial\Omega} \frac{g(z)}{z - w} dz - \int_{\partial\{|z-w|\leq\epsilon\}} \frac{g(z)}{z - w} dz = \int_{\partial\Omega} \frac{g(z)}{z - w} dz - i \int_0^{2\pi} g(w + \epsilon e^{it}) dt$$

$$so \qquad 2i \int\int_{\Omega} \frac{\partial g}{\partial \overline{z}} . \frac{1}{z - w} dxdy = \int_{\partial\Omega} \frac{g(z)}{z - w} dz - i \int_0^{2\pi} g(w + \epsilon e^{it}) dt$$

Letting $\epsilon \longrightarrow 0$ we have $g(w + \epsilon e^{it}) \longrightarrow g(w)$

$$so \qquad 2i \int\int_{\Omega} \frac{\partial g}{\partial \overline{z}} . \frac{1}{z - w} dxdy = \int_{\partial\Omega} \frac{g(z)}{z - w} dz - 2\pi i g(w)$$

□

Now we define the notion of normal families and state without proof the theorem of Arzela-Ascoli [9] which will be used in the proof of Lempert's example in the main section.

Definition.

1. *Let $\Omega \subset \mathbb{C}$ be a domain. A family $\mathcal{F}$ of complex valued functions on Ω is called a normal family if every sequence $\{f_n\} \in \mathcal{F}$ contains a subsequence that converges uniformly on compact subsets of Ω.*

2. *A family $\mathcal{F}$ of complex valued functions on Ω is said to be point-wise bounded if for each $z \in \Omega$, $sup_{f\in\mathcal{F}}|f(z)| < \infty$.*

3. *A family $\mathcal{F}$ of complex valued functions on Ω is equicontinuous if for every $\epsilon > 0$ and $z \in \Omega$, there exist $\delta > 0$ such that for all $w \in \Omega$*

$$|z - w| < \delta \Longrightarrow |f(z) - f(w)| < \epsilon \ \ \forall f \in \mathcal{F}$$

Theorem 6. *(Arzela-Ascoli) Let $\Omega \subset \mathbb{C}$ be a domain, and let $\mathcal{F}$ be a point-wise bounded, equicontinuous family of complex valued functions on Ω. Then every sequence $\{f_n\} \in \mathcal{F}$ has a subsequence that converges uniformly on compact subsets of Ω.*

The next Lemma will be used later in the proof of a proposition related to the second Cousin problem.

Lemma 3. *Let $\Omega \subset \mathbb{R}^N$ be simply connected*
$f : \Omega \longrightarrow \mathbb{C}$ continuous and non-vanishing, then there is a continuous function g on Ω such that $f = e^g$. If f is C^k, then g is C^k also.

Proof. Let $p_0 \in \Omega$ and let $\gamma : [0, 1] \longrightarrow \Omega$ be a loop i.e, $\gamma(0) = \gamma(1) = p_0$ then $f \circ \gamma(t)$ has continuous logarithm if f has a continuous logarithm.
Suppose f does not have a continuous logarithm (at 1 in particular), i.e Suppose $lim_{t\to 1^-} logf \circ \gamma \neq logf \circ \gamma(0)$
Let $u(s, t)$ be the homotopy between γ and the point p_0:

- u continuous on $[0, 1]x[0, 1]$
- $u(0, t) = \gamma(t)$ for all $t \in [0, 1]$
- $u(s, 0) = u(s, 1) = p_0$ for all $s \in [0, 1]$
- $u(1, t) = p_0$ for all $t \in [0, 1]$

Consider $\rho(s) = \frac{1}{2\pi i}[lim_{t\to 1^-} logf(u(s, t)) - logf(u(s, 0))]$ continuous
but $\rho(0) \neq 0$ and $\rho(1) = 0 \Rightarrow$ contradiction
the C^k result is by implicit differentiation of $f = e^g$ □

2.2 Meromorphic functions with prescribed zeros/poles

In this section we give two important results for finding functions with prescribe zeros or poles and principle parts. The first theorem is in the multiplicative form and is called after Weierstrass. The second is in the additive form and is called after Mittag-Leffler.

Definition. *An infinite product is an expression of the form* $\prod_{j=1}^{\infty} p_j$ *where* p_j *are complex numbers. The infinite product converges if* $p_j \to 1$ *and* $\sum log|p_j|$ *converges where the sum is over all* $p_j \neq 0$. *If the infinite product converges, then its value is zero if one of the* p_j *is zero, otherwise* $\prod_{j=1}^{\infty} p_j = exp(\sum_{j=1}^{\infty} log|p_j|)$.

Remark. *If* $t_j \geq 0$ *then* $\prod(1 \pm t_j)$ *converges if and only if* $\sum t_j$ *converges.*

Lemma 4. *Let* z *be a complex number and* k *a positive integer. Define the canonical factors by* $E_0(z) = 1 - z$ *and* $E_k(z) = (1-z)e^{z+\frac{z^2}{2}+\ldots+\frac{z^k}{k}}$ *for* $k \geq 1$. *If* $|z| \leq \frac{1}{2}$ *then* $|1 - E_k(z)| \leq c|z|^{k+1}$ *for some* $c > 0$.

Proof. since $|z| \leq \frac{1}{2}$ we can use the logarithm to write $1 - z = e^{log(1-z)}$
so $E_k(z) = e^{log(1-z)+z+\frac{z^2}{2}+\ldots+\frac{z^k}{k}} = e^w$ now using Taylor expansion:

$$\begin{aligned} w &= log(1-z) + z + \frac{z^2}{2} + \ldots + \frac{z^k}{k} \\ &= \sum_{n=1}^{\infty}(-1)\frac{z^n}{n} + \sum_{n=1}^{k}\frac{z^n}{n} \\ &= -\sum_{n=k+1}^{\infty}\frac{z^n}{n} \end{aligned}$$

Now $|w| \leq |z|^{k+1}\sum_{n=k+1}^{\infty}\frac{|z|^{n-k-1}}{n} \leq |z|^{k+1}\sum_{j=0}^{\infty}2^{-j} \leq 2|z|^{k+1}$ so

$$\begin{aligned} |1 - E_k(z)| &= |1 - e^w| = |1 - \sum_{n=0}^{\infty}\frac{w^n}{n!}| \\ &= |-\sum_{n=1}^{\infty}\frac{w^n}{n!}| = |-w - \frac{w^2}{2!} - \ldots| \\ &\leq |w|\sum_{n=2}^{\infty}|\frac{w^n}{n!}| \leq |w|\sum_{n=2}^{\infty}|\frac{1}{n!}| \\ &= c^{'}|w| \leq c|z|^{k+1} \end{aligned}$$

□

Theorem 7. *(Weierstrass Factorization Theorem) Given any sequence* a_n *of complex numbers with* $|a_n| \to \infty$ *as* $n \to \infty$ *there exist an entire function* f *that vanishes at all* $z = a_n$ *and nowhere else.*

Proof. Suppose that we are given a zero of order m at the origin, and that $a_n \neq 0$ for all n.
Define the Weierstrass product by

$$f(z) = z^m \prod_{n=1}^{\infty} E_n(\frac{z}{a_n})$$

We claim that this function has the required properties: f is entire with a zero of order m at the origin, zeros at each point of the given sequence, and f vanishes nowhere else.
Fix $R > 0$ and let z belong to the disc $|z| < R$
We prove that f has all the desired properties in the disc, and since R is arbitrary, this will prove the theorem.
case1: $a_n \leq 2R$
There are finitely many a_n's that satisfy $a_n \leq 2R$ since $|a_n| \to \infty$ and the finite product vanishes at all $z = a_n$ with $|a_n| < R$.
case2: $a_n > 2R$

$|\frac{z}{a_n}| \leq \frac{1}{2}$ so we can apply the above lemma:
$|1 - E_n(\frac{z}{a_n})| \leq c|\frac{z}{a_n}|^{n+1} \leq \frac{c}{2^{n+1}}$
$\sum \frac{c}{2^{n+1}} \leq \infty$ so $\sum |1 - E_n(\frac{z}{a_n})|$ converges uniformly on $|z| < R$ then $\prod E_n(\frac{z}{a_n})$ is holomorphic on $|z| < R$. $\square$

A direct result of this factorization theorem is proving that a meromorphic function is the quotient of two holomorphic functions, i.e: $\mathcal{M}$ is a field of fractions of functions in the ring $\mathcal{O}$.

Corollary 1. *Let f be a meromorphic function on Ω then $f = \frac{f_1}{f_2}$ where f_1, f_2 are holomorphic on Ω.*

Proof. f is meromorphic on Ω then f is holomorphic on Ω-$\{a_1, a_2, \ldots\}$ where $\{a_n\}$ are the poles of f.
case1: f has finite number of poles $a_1, a_2, \ldots, a_n$
$f_2 = (z - a_1)^{m_1}(z - a_2)^{m_2} \ldots (z - a_n)^{m_n}$
$f_1 = f f_2 = f(z - a_1)^{m_1}(z - a_2)^{m_2} \ldots (z - a_n)^{m_n}$
f has a pole at $a_1, a_2, \ldots, a_n$ iff $\frac{1}{f}$ has zeros at $a_1, a_2, \ldots, a_n$ iff $\frac{1}{f} = (z - a_1)^{m_1} \ldots (z - a_n)^{m_n} g(z)$ where $g(z)$ is holomorphic and $g(a_1) \neq .. \neq g(a_n) \neq 0$
so $f(z - a_1)^{m_1} \ldots (z - a_n)^{m_n} = \frac{1}{g(z)}$ but $\frac{1}{g(z)}$ is holomorphic at $a_1, a_2, \ldots, a_n$ then f_1 is holomorphic on Ω and f_2 has zeros at $a_1, a_2, \ldots, a_n$
case2: f has an infininte number of poles a_n
By Weierstrass we can find an entire function f_2 that vanishes at $z = a_n$ only, so

$$f_2 = \prod_{n=1}^{\infty} E_n(\frac{z}{a_n})$$

$$f_1 = f.f_2$$

f has poles at a_n iff $\frac{1}{f}$ has zeros at a_n, then $\frac{1}{f} = \prod_{n=1}^{\infty} E_n(\frac{z}{a_n})g(z)$
$g(z)$ holomorphic on Ω and $g(a_n) \neq 0$, $f_1 = \frac{1}{g(z)} = f.f_2$ but $\frac{1}{g(z)}$ is holomorphic on a_n $\square$

Now we state without proof Runge's theorem which is used in the proof of Mittag-Leffler.

Theorem 8. *(Runge's theorem) Let $K \subset \mathbb{C}$ be a compact set and $P \subset \mathbb{C} - K$ contains at least one point from each connected component of $\mathbb{C} - K$. If $f(z)$ is holomorphic on an open set containing K, then for any $\epsilon > 0$, f can be approximated uniformly on K by rational function R with poles in P such that*

$$max_{z \in K}|f(z) - R(z)| < \epsilon$$

Theorem 9. *(Mittag-Leffler Theorem) Suppose Ω is a domain, and $A \subset \Omega$, A has no accumulation point in Ω, and to each $\alpha \in A$ there are associated a positive integer $m(\alpha)$ and a rational function*

$$P_\alpha(z) = \sum_{j=1}^{m(\alpha)} c_j (z - \alpha)^{-j}$$

Then there exists a meromorphic function f in Ω, whose poles are α and whose principle parts at each α is P_α

Proof. Let $K_m = \{z \in \Omega : |z| \leq m\}$ and the distance from z to $\partial\Omega$ is at least $\frac{1}{m}$. K_m is a sequence of compact sets such that $\Omega \in \cup K_m$ and $K_m \subset K_{m+1}$ and each component $\mathbb{C} - K_m$ contains a component of $\mathbb{C} - \Omega$.
Let $A_1 = A \cap K_1 \ldots A_m = A \cap K_m$. Since $A_m \subset K_m$ and A has no accumulation point in Ω and hence

in K_m also, each A_m is a finite set.
Put

$$Q_m(z) = \sum_{\alpha \in A_m} P_\alpha(z)$$

Since each A_m is finite, each Q_m is rational. The poles of Q_m lie in $K_m - K_{m-1}$ for $m \geq 2$. In particular, Q_m is holomorphic in an open set containing K_{m-1}. Now by Runge's theorem there exist rational functions R_m whose poles are all in $\mathbb{C} - \Omega$ such that

$$|R_m(z) - Q_m(z)| < 2^{-m}$$

Let $f(z) = \sum_{m=1}^{\infty} [Q_m(z) - R_m(z)]$.
f converges uniformly on each compact subset of Ω by the Weierstrass M-test.
Fix $N > 0$, $\sum_{m=N+1}^{\infty} [Q_m(z) - R_m(z)]$ is holomorphic on K_N, and $\sum_{m=1}^{N} [Q_m(z) - R_m(z)]$ has poles at the points α that are in K_N, with prescribed principle parts, $f(z)$ has the prescribed poles and principle parts in Ω. $\square$

Chapter 3

Several Complex Variables

3.1 Basic Definitions and Theorems

In this section we will list some important definitions and results to be compared with the one variable case.
Now we define holomorphic functions in several variables.

Definition. *Let $\Omega \subseteq \mathbb{C}^n$ be a domain, and let $f : \Omega \longrightarrow \mathbb{C}^m$ be a map. We say $f \in \mathcal{O}(\Omega) \Leftrightarrow f_1, \dots f_m \in \mathcal{O}(\Omega)$ where $f_j : \Omega \longrightarrow \mathbb{C} \Leftrightarrow f$ is smooth and for all $j = 1, \dots, m$ and for all $k = 1, \dots, n$ $\frac{\partial f_j}{\partial \bar{z}_k} = 0$*

In one complex variable, we deal with a model domain called the unit disc

$$\Delta = \{z \in \mathbb{C}; |z| < 1\}$$

The importance of the unit disc becomes clear in the Riemann mapping theorem. However, in several complex variables, let $w \in \mathbb{C}^n$ and $r = (r_1, \dots, r_n)$ an n-tuple of positive real numbers and $R > 0$, there are two different analogues of the unit disc:

- the ball: $\mathbb{B}^n(w, R) = \{z \in \mathbb{C}^n; |z - w| < R\}$
- the polydisc: $D^n(w, r) = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n; |z_i - w_i| < r_i\}$

and it is well known, since Poincare that $\mathbb{B}^n$ and D^n are not biholomorphic to each other.

As we will see, the Cauchy formula can be generalized in several complex variables on polydiscs.

Theorem 10. *(Cauchy Formula for Polydiscs) Let $w \in \mathbb{C}^n$ and $r_1, \dots, r_n > 0$. Suppose f continuous on $\overline{D}^n(w, r) = \overline{D}(w_1, r_1) \times \dots \times \overline{D}(w_n, r_n)$ and holomorphic on $D(w_1, r_1) \times \dots (w_n, r_n)$ then*

$$f(z) = \frac{1}{2\pi i^n} \int_{|\zeta_n - w_n| = r_n} \dots \int_{|\zeta_1 - w_1| = r_1} \frac{f(\zeta_1, \dots, \zeta_n)}{(\zeta_1 - z_1) \dots (\zeta_n - z_n)} d\zeta_1 \dots d\zeta_n \qquad \forall z \in D^n(w, r)$$

Proof. By repeated application of the one variable Cauchy integral formula, we obtain

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{|\zeta_n - w_n| = r_n} \frac{f(z_1, z_2, \dots, z_{n-1}, \zeta_n)}{\zeta_n - z_n} d\zeta_n \\ &\vdots \\ &= \frac{1}{2\pi i^n} \int_{|\zeta_n - w_n| = r_n} \dots \int_{|\zeta_1 - w_1| = r_1} \frac{f(\zeta_1, \dots, \zeta_n)}{(\zeta_1 - z_1) \dots (\zeta_n - z_n)} d\zeta_1 \dots d\zeta_n \end{aligned}$$

□

3.2 Manifolds

We know introduce a special type of topological spaces.

Definition. *A differentiable manifold M of real dimension m and of class C^k is a topological space, which we shall always assume Hausdorff and second countable, equipped with an atlas of class C^k with values in $\mathbb{R}^m$.*
An atlas of class C^k is a collection of homeomorphisms $\tau_\alpha : U_\alpha \longrightarrow V_\alpha$, $\alpha \in I$, where the pair (U_α, τ_α) is called a coordinate chart, such that $\{U_\alpha\}_{\alpha\in I}$ is an open covering of M and V_α an open subset of $\mathbb{R}^m$, and such that for all $\alpha, \beta \in I$ the transition map

$$\tau_{\alpha\beta} = \tau_\alpha \circ \tau_\beta^{-1} : \tau_\beta(U_\alpha \cap U_\beta) \longrightarrow \tau_\alpha(U_\alpha \cap U_\beta)$$

is a C^k diffeomorphism from an open subset of V_β onto an open subset of V_α

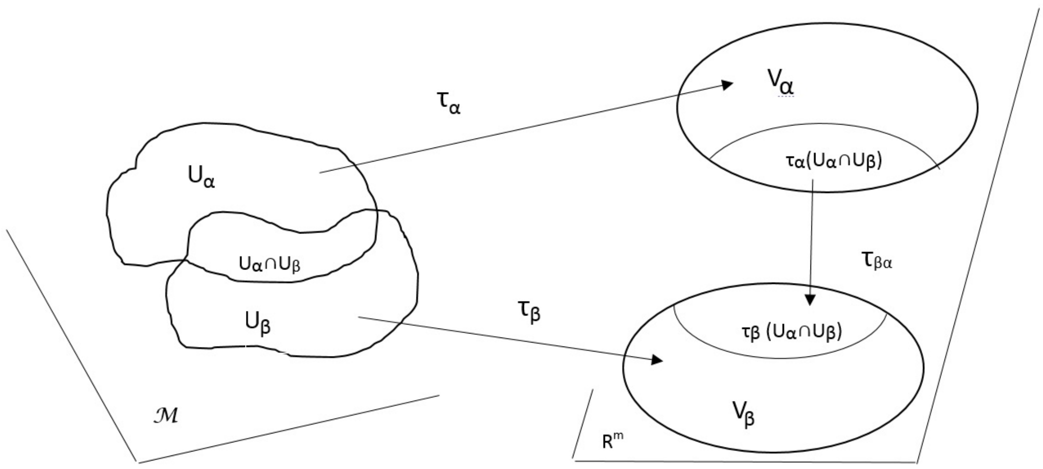

Figure 3.1: Charts and transition maps

Remark. *A smooth manifold is a space that looks locally like an open set in $\mathbb{R}^n$, and a complex manifold is a manifold whose coordinate charts are open subsets of $\mathbb{C}^n$ and the transition functions between charts are holomorphic functions.*

We define now a special type of open sets in $\mathbb{C}^n$ called pseudoconvex sets.

Definition.

1. *Let G be a domain, we say G has a defining function if there exist a function $\rho : \mathbb{C}^n \longrightarrow \mathbb{R}$ of class C^2 so that $G = \{\rho < 0\}$ and $\partial G = \{\rho = 0\}$ and $\nabla \rho \neq 0$.*
2. *The tangent space $T_{p,M}$ at p on the $n-$ dimensional manifold M is a vector space of all the tangent vectors at p, i.e if $w \in T_{p,M}$ then $\sum_{i=1}^n \frac{\partial}{\partial z_i}|_p w_i = 0$*
3. *Let $G \subset \mathbb{C}^n$ be a domain with C^2 boundary, G has a defining function ρ of class C^2,*
   Let $p \in \partial G$ and $w \in T_{p,M}$ we say G is pseudoconvex if for all such p and w we have

$$\sum_{i,j=1}^{n} \frac{\partial^2 \rho(p)}{\partial z_i \partial \overline{z_j}} w_i \overline{w_j} \geq 0$$

Now we introduce partitions of unity that will be used in the proofs of the Cousin problems and consequently contributes towards the proof of the main theorem.

Definition. *A partition of unity on a smooth manifold M is a collection $\{\varphi_i\}$ of smooth real valued functions on M such that:*

- $\varphi_i \geq 0$ *for all* i
- *for all* $x \in M$ *there exist a neighborhood* U *such that* $U \cap supp(\varphi_i) = \phi$ *for all but finitely many* φ_i
- *for all* $x \in M$ $\sum \varphi_i(x) = 1$

Remark. *We say that a partition of unity $\{\varphi_i\}$ on M is subordinate to an open cover $\{U_i\}$ if for all φ_i there exist $U_i \in \{U_i\}$ such that $supp(\varphi_i) \subset U_i$*

Theorem 11. *Let M be a manifold, then given any open cover $\{U_\alpha\}$ there exists a partition of unity $\{\phi_i\}$ subordinate to $\{U_\alpha\}$*

Proof.
Since M is a manifold, it has a countable basis $\{B_\alpha\}$.
Consider a local refinement $\{B_i\}$ such that for each i there exists a coordinate ball $B_i^{'}$ where $\overline{B_i} \subset B_i^{'} \subset U_\alpha$ for some α, and $\varphi_i : B_i^{'} \longrightarrow \mathbb{R}^n$ smooth and let $\varphi_i(\overline{B_i}) = \overline{B_{r_i}}(0)$ and $\varphi_i(B_i^{'}) = B_{r_i^{'}}(0)$ for $r_i < r_i^{'}$.
Note that a coordinate ball is a compact subset of M such that there exists U open in M with $B_i^{'} \subset U \subset M$ and a homeomorphism $\varphi_i : U \longrightarrow \mathbb{R}^n$ such that $\varphi_i(B_i^{'}) = \{x : |x| \leq 1\} \subset \mathbb{R}^n$
Let $H_i : \mathbb{R}^n \longrightarrow \mathbb{R}$ smooth function that is positive in $B_{r_i}(0)$ and zero elsewhere.
Define $f_i : M \longrightarrow \mathbb{R}$ by

$$f_i = \begin{cases} H_i \circ \varphi_i & B_i^{'} \\ 0 & M - \overline{B_i} \end{cases}$$

On $B_i^{'} - \overline{B_i}$ where the two definitions overlap, both lead to zero so f_i is well defined and smooth and *supp* $f_i = \overline{B_i}$.
Define $f : M \longrightarrow \mathbb{R}$ by $f(x) = \sum_i f_i(x)$
Since each f_i is nonnegative everywhere and positive on B_i, and for all $x \in M$, $x \in B_i$ for some i, then $f(x) > 0$ on M.
Define $\phi_i : M \longrightarrow \mathbb{R}$ such that $\phi_i(x) = \frac{f_i(x)}{f(x)}$ smooth, so $\sum_i \phi_i = 1$.
We can reindex to match the index of the function with the index of the open set in the cover. □

3.3 Differential Calculus on Complex Manifolds

We begin by introducing some complex differentials [2] that are important in the understanding of Dolbeault Cohomology and Cousin problem.

Definition. *Let Ω be n-dimensional complex manifold. Locally in a coordinate chart we can write $z_j = x_j + iy_j$ for all $1 \leq j \leq n$ and*

$$dz_j = dx_j + idy_j \qquad d\overline{z}_j = dx_j - idy_j$$

We define a $(p, q) - form$ to be a map defined on Ω with the following local form:

$$\alpha = \sum_{|I|=p, |J|=q} \alpha_{IJ} dz_I \wedge d\overline{z}_J$$

where $dz_I = dz_{i_1} \wedge \ldots \wedge dz_{i_p}$ *and* $d\overline{z}_J = d\overline{z}_{j_1} \wedge \ldots \wedge d\overline{z}_{j_q}$
We denote by $\Lambda^{p,q}(\Omega)$ *the space of all* $(p,q) - form$ *on* Ω. *We also define the following operators on these spaces:*

$$\partial : \Lambda^{p,q} \longrightarrow \Lambda^{p+1,q}, \qquad \partial\alpha = \sum_{|I|=p,|J|=q} \sum_{i=1}^{n} \frac{\partial \alpha_{IJ}}{\partial z_i} dz_i \wedge dz_I \wedge d\overline{z}_J$$

$$\overline{\partial} : \Lambda^{p,q} \longrightarrow \Lambda^{p,q+1}, \qquad \overline{\partial}\alpha = \sum_{|I|=p,|J|=q} \sum_{i=1}^{n} \frac{\partial \alpha_{IJ}}{\partial \overline{z}_i} d\overline{z}_i \wedge dz_I \wedge d\overline{z}_J$$

The differential of a C^1 *function is defined as:*

$$df = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} dx_j + \frac{\partial f}{\partial y_j} dy_j = \sum_{j=1}^{n} \frac{\partial f}{\partial z_j} dz_j + \frac{\partial f}{\partial \overline{z}_j} d\overline{z}_j$$

where

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right) \qquad \frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right)$$

Remark. *Note that* $d = \partial + \overline{\partial}$. *Moreover, we have* $d^2 = 0$ *(1.B.3 in* [2]*).*
Since for a $(p,q) - form$

$$d^2 = \underbrace{\partial^2}_{(p+2,q)} + \underbrace{\overline{\partial}^2}_{(p,q+2)} + \underbrace{\partial\overline{\partial} + \overline{\partial}\partial}_{(p+1,q+1)} = 0$$

Thus, each of the above must be zero, so $\overline{\partial}^2 = 0$

In this paper we focus only on $(0,q) - form$ and more specifically on the first Dolbeault cohomology related to the space $\Lambda^{0,1} = \{\alpha = \sum_i \alpha_i d\overline{z}_i\}$

The above allows us to consider the Dolbeault complex chain as we will see in the next section.

3.4 Cohomology

To every complex manifold we can associate a system of cohomology groups called the Dolbeault Cohomology.
Consider Ω a complex manifold and define the following spaces on it:

$$C^\infty(\Omega) = \{f : \Omega \longrightarrow \mathbb{C} \ \ smooth\}$$

$$\Lambda^{0,1}(\Omega) = \{(0,1) - forms \ \ on \ \Omega\}$$

$$\vdots$$

$$\Lambda^{0,n}(\Omega) = \{(0,n) - forms \ \ on \ \Omega\}$$

and for a $(0,q) - form$ $\alpha = \sum \alpha_I d\overline{z}_I$ define the $\overline{\partial}$ operator to be:

$$\overline{\partial}\alpha = \sum \frac{\partial \alpha_I}{\partial \overline{z}_j} d\overline{z}_j \wedge dz_I$$

so we can form the following long sequence:

$$C^\infty(\Omega) \xrightarrow{\overline{\partial}_0} \Lambda^{0,1}(\Omega) \xrightarrow{\overline{\partial}_1} \Lambda^{0,2}(\Omega) \xrightarrow{\overline{\partial}_2} \ldots$$

since $\overline{\partial}_j \circ \overline{\partial}_{j-1} = 0$ then $Im(\overline{\partial}_{j-1}) \subset Ker(\overline{\partial}_j)$

Definition. *Define the Dolbeault Cohomology on* $\Lambda^{0,j}(\Omega)$ *to be:*

$$H^{0,j}(\Omega) = \frac{Ker\overline{\partial}_j}{Im\overline{\partial}_{j-1}}$$

Take $w_1, w_2 \in Ker\overline{\partial}_j$, *to see if those two* $(0,j) - form$ *are in the same equivalence class, we define a relation* $\sim$ *on* $\Lambda^{0,j}(\Omega)$*:*

$$\begin{aligned} w_1 \sim w_2 &\Leftrightarrow w_1 = w_2 + \overline{\partial}_{j-1}(\alpha) \;\; \alpha \in \Lambda^{0,j-1}(\Omega) \\ &\Leftrightarrow w_1 - w_2 \in Im\overline{\partial}_{j-1} \end{aligned}$$

Our main purpose is to find conditions to solve the following:

$$Given\ w \in \Lambda^{0,1}(\Omega),\ find\ u \in C^\infty(\Omega)\ such\ that\ w = \overline{\partial}u \tag{3.1}$$

Now we notice the following:

$$\begin{aligned} H^{0,1}(\Omega) = \frac{Ker\overline{\partial}_1}{Im\overline{\partial}_0} = 0 &\Leftrightarrow Im\overline{\partial}_0 = Ker\overline{\partial}_1 \\ &\Leftrightarrow \forall w \in Ker\overline{\partial}_1, w \in Im\overline{\partial}_0 \\ &\Leftrightarrow \forall w \in Ker\overline{\partial}_1, \exists u \in C^\infty(\Omega)\ \ st.\ \ w = \overline{\partial}u \end{aligned}$$

So the condition we are searching for to solve (3.1) is $H^{0,1}(\Omega) = 0$

Chapter 4

The Cousin Problems

In this section we deal with the two famous Cousin problems. We recall that we found in the Cohomology section conditions on the Dolbeault cohomology of the domain on which the function is defined. This condition will help us to determine on which domains the Cousin problems can be solved on.

4.1 Cauchy-Green Operator and the $\overline{\partial}$ Problem

Given g, a big question is to solve the following linear non-homogeneous partial differential equation:

$$\frac{\partial f}{\partial \overline{z}} = g$$

The solution will be useful towards proving the main theorem, so in this section we introduce an operator that helps solve this equation.

Definition. *Let $f : \Omega \longrightarrow \mathbb{C}^n$ and $f \in C^1$ define the following operators:*

- *Cauchy transform operator:*
$$Cf(z) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{f(\zeta)}{\zeta - z} d\zeta$$
If f is holomorphic then $f(z) = Cf(z) \qquad \forall z \in \Omega$

- *Cauchy-Green operator:*
$$Tf(z) = \frac{1}{2\pi i}\int\int_{\Omega} \frac{f(\zeta)}{\zeta - z} d\zeta \wedge d\overline{\zeta}$$

then the General Cauchy formula (see page 5) translates as

$$f = Cf + T\frac{\partial f}{\partial \overline{z}}$$

Remark. *Consider the following Banach spaces:*

$$L^p(\Omega) = \{f : \Omega \longrightarrow \mathbb{C}^n; \int\int_{\Omega} |f|^p < \infty\} \ where \ \|f\|_p = \Big(\int\int_{\Omega} |f|^p\Big)^{\frac{1}{p}}$$

$$W^{1,p}(\Omega) = \{f : \Omega \longrightarrow \mathbb{C}^n; f \in C^1, f^{'} \in L^p\} \ where \ \|f\|_{1,p} = \Big(\int\int_{\Omega} |f|^p + |f^{'}|^p\Big)^{\frac{1}{p}}$$

By Vekua theorem [10]*, which we will not prove, the Cauchy-green operator above is a well defined operator which maps continuous functions in $L^p(\Omega)$ to $W^{1,p}(\Omega)$,namely there exists $c > 0$ such that*

$$\|Tf\|_{1,p} \leq c\|f\|_p \tag{4.1}$$

Property. *Notice that* $\frac{\partial}{\partial \bar{z}} \circ T(u) = u$ *for* $u = \frac{\partial f}{\partial \bar{z}}$

Proof.
Apply $\frac{\partial}{\partial \bar{z}}$ on $f = Cf + T\frac{\partial f}{\partial \bar{z}}$ we get:

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial}{\partial \bar{z}} Cf + \frac{\partial}{\partial \bar{z}} T \frac{\partial f}{\partial \bar{z}} = \frac{\partial}{\partial \bar{z}} T \frac{\partial f}{\partial \bar{z}}$$

□

Remark. *In general we have* $\overline{\partial} \circ T = Id$ *on* $L^p(\Omega)$.

Now we will see the use of the Cauchy-green operator in approximation. The following proposition is concerned with the approximation of non-holomorphic functions by holomorphic ones.

Proposition 1. *Given* $\epsilon > 0$, *there exist* $\delta > 0$ *such that if* $h : \Delta \longrightarrow \mathbb{C}^n$ *smooth, such that* $\|\frac{\partial h}{\partial \bar{z}}\|_p < \delta$ *then there exist* $f : \Delta \longrightarrow \mathbb{C}^n$ *holomorphic such that* $\|f - h\|_\infty < \epsilon$

Proof.
Let $\epsilon > 0$, c be the constant in equation 4.1 in case $\Omega = \Delta$
Take $\delta = \frac{\epsilon}{c}$ and $f = h - T\frac{\partial h}{\partial \bar{z}}$

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial h}{\partial \bar{z}} - \frac{\partial}{\partial \bar{z}} T \frac{\partial h}{\partial \bar{z}} = \frac{\partial h}{\partial \bar{z}} - \frac{\partial h}{\partial \bar{z}} = 0$$

$\Rightarrow$ f is holomorphic

$$\|f - h\|_\infty = \|T\frac{\partial h}{\partial \bar{z}}\|_\infty \leq \|T\frac{\partial h}{\partial \bar{z}}\|_{1,p} \leq c.\|\frac{\partial h}{\partial \bar{z}}\|_p < c.\delta = c.\frac{\epsilon}{c} = \epsilon$$

□

We state now an important result that helps in proving the Cousin problems. We refer to Corollary (4.6.10) in [4].

Theorem 12. *(Cauchy-green) If* $\Omega \subset \mathbb{C}^n$ *is pseudoconvex and* f *is* $(p, q+1)form$ *on* Ω *with* C^∞ *coefficients and satisfying* $\overline{\partial} f = 0$, *then there is a* $(p, q) - form$ u *on* Ω *with* C^∞ *coefficients satisfying* $\overline{\partial} u = f$.

We end this section by stating a special case of the above theorem called Dolbeault-Grothendieck lemma in [2].

If we define a closed form to be a differential form α such that $d\alpha = 0$, and an exact form to be to be a differential form α such that $\alpha = d\beta$ for some differential form β. Since $d^2 = 0$ we automatically have that any exact form is closed. The Poincare lemma determines on which topological domains we have that every closed form is exact.

We notice that this lemma is the analogue for $\overline{\partial}$ of the Poincare lemma.

Lemma 5. *(Dolbeault-Grothendieck lemma) Let* $v = \sum_{|J|=q} v_J d\bar{z}_J$ *with* $q \geq 1$, *be a* $(0, q) - form$ *on a polydisc* $\Omega \subset \mathbb{C}^n$. *Then there exist a smooth* $(0, q-1) - form$ u *on* Ω *such that* $\overline{\partial} u = v$.

4.2 The First Cousin Problem

Problem. *(First Cousin Problem) Let $\Omega \subset \mathbb{C}$ be a domain. Let U_i be a an covering of Ω. Suppose that for each U_j, U_k with non-empty intersection there is a holomorphic $g_{jk} : U_j \cap U_k \longrightarrow \mathbb{C}$ satisfying:*

$$\begin{aligned} g_{jk} &= -g_{kj} \\ g_{jk} + g_{kl} + g_{lj} &= 0 \qquad U_j \cap U_k \cap U_l \end{aligned}$$

Find holomorphic functions g_j on U_j such that

$$g_{jk} = g_k - g_j$$

on $U_j \cap U_k$ whenever this intersection is not empty.

An important tool in dealing with the Cousin problems is the Dolbeault-Grothendieck in 5
We show in the following proposition that the First Cousin problem can be solved on a pseudoconvex domain.

Theorem 13. *Let $\Omega \subset \mathbb{C}^n$ be a pseudoconvex domain. Let U_i be an open covering of Ω. Suppose for each U_j, U_k with non-empty intersection there is a holomorphic $g_{jk} : U_j \cap U_k \longrightarrow \mathbb{C}$ satisfying:*

$$\begin{aligned} g_{jk} &= -g_{kj} \\ g_{jk} + g_{kl} + g_{lj} &= 0 \qquad U_j \cap U_k \cap U_l \end{aligned}$$

then there exist holomorphic functions g_j on U_j such that

$$g_{jk} = g_k - g_j$$

on $U_j \cap U_k$ whenever this intersection is not empty.

Proof. Ω is a pseudoconvex domain, then it is manifold and all manifolds have a partition of unity subordinate to any open covering.
Let φ_i be a partition of unity subordinate to U_i.
Define $h_i = \sum_k \varphi_k g_{ki}$ on U_i. Note that h_i may not be holomorphic.
On $U_i \cap U_j$: $h_j - h_i = \sum_k \varphi_k (g_{kj} - g_{ki}) = \sum_k \varphi_k g_{ij} = g_{ij}$
but g_{ij} is holomorphic, then $\overline{\partial} g_{ij} = 0 \Longrightarrow \overline{\partial} h_j = \overline{\partial} h_i$ on $U_i \cap U_j$
Let $f = \overline{\partial} h_j$: f is well defined (agrees on intersection), $\overline{\partial}$ closed, and C^∞ on Ω (since h_j is C^∞).
By Cauchy-green theorem, there is a $u \in C^\infty(\Omega)$ such that $\overline{\partial} u = f$
let $g_j = h_j - u$ on U_j
then on $U_i \cap U_j$:

$$\begin{aligned} g_j - g_i &= (h_j - u) - (h_i - u) \\ &= h_j - h_i = g_{ij} \end{aligned}$$

and g_j is holomorphic since on U_j:

$$\begin{aligned} \overline{\partial} g_j &= \overline{\partial} h_j - \overline{\partial} u \\ &= \overline{\partial} h_j - f \\ &= \overline{\partial} h_j - \overline{\partial} h_j = 0 \end{aligned}$$

∎

We now give an alternative equivalent statement for the first Cousin problem.

Proposition 2. *A solution of the above theorem implies a solution of the following formulation of the first cousin problem:*
Let U_i be an open covering of a pseudoconvex domain $\Omega \subset \mathbb{C}^n$. On each define f_j meromorphic on U_j such that $f_i - f_j$ holomorphic on $U_i \cap U_j$, then there exist f meromorphic on Ω such that $f - f_j$ holomorphic on U_j for all j.

Proof.
Let $g_{jk} = f_j - f_k$ so g_{jk} holomorphic on $U_i \cap U_j$
and $g_{jk} = -g_{kj}$ and $g_{ij} + g_{jk} + g_{ki} = f_i - f_j + f_j - f_k + f_k - f_i = 0$
then g_{jk} satisfy the first cousin data.
Let g_j be the holomorphic solution of the above theorem then $g_{jk} = g_k - g_j$ but also $g_{jk} = f_j - f_k$ on $U_j \cap U_k$
then $f_j + g_j = f_k + g_k$
let $f = f_j + g_j \in \mathcal{M}(U_j)$
and $f - f_j = g_j \in \mathcal{O}(U_j)$ □

Remark. *A generalization of the Mittag-Leffler theorem into higher dimensions is the First Cousin problem.*
To elaborate this we show that the first Cousin problem in $\mathbb{C}^n$ is the equivalent of Mittag-Leffler in $\mathbb{C}$, by proving Mittag-Leffler using Cousin.

Proof. using the first cousin problem:
Denote by $\{w_i\}$ the set of given points in $\Omega\mathbb{C}$ and p_i the given polynomials.
Let $U_j = \Omega$-$\{w_k$ such that $k \neq j\}$
then U_j is an open cover of Ω
Let $f_j = \sum_{n=1}^{m} (z - w_i)^{-n} \in \mathcal{M}(U_j)$
$f_j - f_k \in \mathcal{O}(U_j \cap U_k)$ where $U_j \cap U_k = \Omega - (w_i)$ for all i.
then by the above proposition there exist $f \in \mathcal{M}(\Omega)$ such that $f - f_j \in \mathcal{O}(U_j)$. □

4.3 The First Cousin Problem with Bounds

In this section we will prove the first cousin problem with bounds on the unit disc, and this result will be used in the proof of our main theorem:

Theorem 14. *(The First Cousin Problem with Bounds) Let $\overline{\Delta} \subset \cup_{i=1}^{R} U_i$, with no triple intersection. Given $\epsilon > 0$, there exist $\delta > 0$ such that if $g_{jk} : U_j \cap U_k \cap \Delta \longrightarrow \mathbb{C}^n$ and $g_{jk} \in \mathcal{O}(U_j \cap U_k \cap \Delta)$ such that:*

$$\begin{aligned} g_{jk} &= -g_{kj} \\ g_{jk} + g_{kl} + g_{lj} &= 0 \qquad U_j \cap U_k \cap U_l \cap \Delta \\ \|g_{kj}\|_\infty &< \delta \end{aligned}$$

then there exist $g_j \in \mathcal{O}(U_j \cap \Delta)$ such that $g_{jk} = g_k - g_j$ on $U_j \cap U_k \cap \Delta$ and $\|g_j\|_\infty < \epsilon$

Proof.
Denote by $V_i = U_i \cap \Delta$
Let $\delta = \frac{\epsilon}{L+C.M}$
Let $\{\varphi_i\}$ be a partition of unity subordinate to $\{V_i\}$.
Define $h_j = \sum_i \varphi_i g_{ij}$ on V_i.
On $V_j \cap V_k$ we have:
$\overline{\partial}(h_k - h_j) = \overline{\partial}(\sum_i \varphi_i(g_{ik} - g_{ij})) = \overline{\partial}(\sum_i \varphi_i g_{jk}) = \overline{\partial} g_{jk} = 0$

$$\begin{aligned}
\|h_j\|_\infty = \|\sum_i \varphi_i g_{ij}\|_\infty &\leq \sum_i \|\varphi_i\|_\infty \|g_{ij}\|_\infty \\
&< \sum_i \|\varphi_i\|_\infty \delta \qquad let \sum_i \|\varphi_i\|_\infty = L \\
&= L.\delta
\end{aligned}$$

Let $f = \overline{\partial} h_j$: f is well defined $(0,1)-form$, $\overline{\partial}$ closed, and $\overline{\partial} h_j = \sum(\overline{\partial}\varphi_i) g_{ij} + \sum \varphi_i(\overline{\partial} g_{ij}) = \sum(\overline{\partial}\varphi_i) g_{ij}$ so

$$\begin{aligned}
\|f\|_\infty &\leq \sum_{j=1}^{R} \|\overline{\partial} h_j\|_\infty \\
&\leq \sum \|\overline{\partial}\varphi_i\|_\infty \|g_{ij}\|_\infty \\
&< \sum \|\overline{\partial}\varphi_i\|_\infty \delta \qquad let \sum \|\overline{\partial}\varphi_i\|_\infty = M \\
&= M.\delta
\end{aligned}$$

By Cauchy-green theorem, there is a $u \in C^\infty(\Delta)$ such that $\overline{\partial} u = f$ and $\|u\|_\infty \leq \|u\|_{1,p} \leq c\|f\|_p \leq C\|f\|_\infty$ then $g_j = h_j - u$ solves the cousin problem and

$$\begin{aligned}
\|g_j\|_\infty = \|h_j - u\|_\infty &\leq \|h_j\|_\infty + \|u\|_\infty \\
&\leq L.\delta + C.M.\delta \\
&= (L + C.M)\delta \\
&= \epsilon
\end{aligned}$$

□

4.4 The Second Cousin Problem

In this section we deal with the multiplicative analogue of the First Cousin problem.

Problem. *(Second Cousin Problem) Let $\Omega \subset \mathbb{C}$ be a domain. Let $\{U_i\}$ be a an covering of Ω. Suppose that for each U_j, U_k with non-empty intersection there is a non-vanishing holomorphic $g_{jk} : U_j \cap U_k \longrightarrow \mathbb{C}$ satisfying:*

$$\begin{aligned}
g_{jk} \cdot g_{kj} &= 1 \\
g_{jk} \cdot g_{kl} \cdot g_{lj} &= 1 \qquad U_j \cap U_k \cap U_l
\end{aligned}$$

Find a non-vanishing holomorphic functions g_j on U_j such that

$$g_{jk} = g_k/g_j$$

on $U_j \cap U_k$ whenever this intersection is not empty.

We show in the following proposition that the Second Cousin problem can be solved on a pseudo-convex domain.

Theorem 15. *Let $\Omega \subset \mathbb{C}$ be a domain. Let $\{U_i\}$ be a an covering of Ω. Suppose that for each U_j, U_k with non-empty intersection there is a non-vanishing holomorphic $g_{jk} : U_j \cap U_k \longrightarrow \mathbb{C}$ satisfying:*

$$\begin{aligned} g_{jk} . g_{kj} &= 1 \\ g_{jk} . g_{kl} . g_{lj} &= 1 \qquad U_j \cap U_k \cap U_l \end{aligned}$$

If there exist a non-vanishing continuous function $g_i^{'} : U_i \longrightarrow \mathbb{C}$ such that $g_{ij} = g_j^{'} g_i^{'-1}$ on $U_j \cap U_i$ then there exist a non-vanishing holomorphic g_i such that $g_{ij} = g_j g_i^{-1}$ on $U_j \cap U_i$

Proof.
case 1: U_i is a polydisc $\Rightarrow$ simply connected:
then by the above lemma we can write $g_i^{'} = e^{h_i^{'}}$ on U_i where $h_i^{'}$ is continuous.
Let $h_{ij} = h_j^{'} - h_i^{'}$ then $g_{ij} = g_j^{'} g_i^{'-1} = e^{h_j^{'}} . e^{-h_i^{'}} = e^{h_{ij}}$ Note that g_{ij} is non-vanishing holomorphic, then h_{ij} is also holomorphic.
then $\{h_{ij}\}$ satisfy the first cousin data for the cover $\{U_i\}$
then there exist holomorphic functions $h_i : U_i \longrightarrow \mathbb{C}$ such that $h_{ij} = h_j - h_i$ on $U_j \cap U_i$.
then $g_i = e^{h_i} \in \mathcal{O}(U_i)$ and non-vanishing

$$g_j g_i^{-1} = e^{h_j - h_i} = e^{h_{ij}} = g_{ij} = e^{h_j^{'} - h_i^{'}} = g_j^{'} g_i^{'-1}$$

case 2: U_i are not all polydiscs:
Let $\{\tilde{U}_j\}$ be a refinement of the open covering $\{U_j\}$ such that $\tilde{U}_j$ is a polydisc.
This refinement is done by a function $\rho : \mathbb{N} \longrightarrow \mathbb{N}$ such that $\tilde{U}_i \subset U_{\rho(i)}$ for all i.
Define $\tilde{g}_{ij} : \tilde{U}_i \cap \tilde{U}_j \longrightarrow \mathbb{C}$ by

$$\tilde{g}_{ij} = g_{\rho(i)} g_{\rho(j)}$$

then $\tilde{g}_{ij}$ is holomorphic satisfying the second cousin data for the covering $\{\tilde{U}_i\}$, by case 1, we can find $\tilde{g}_i$ non-vanishing holomorphic on polydisc $\tilde{U}_i$ such that

$$\tilde{g}_{ij} = \tilde{g}_j \tilde{g}_i^{-1}$$

Now on $U_i \cap \tilde{U}_j \cap \tilde{U}_k$ we have

$$\tilde{g}_k \tilde{g}_j^{-1} g_{\rho(k)i} g_{i\rho(j)} = \tilde{g}_k \tilde{g}_j^{-1} g_{\rho(k)\rho(j)} = \tilde{g}_k \tilde{g}_j^{-1} \tilde{g}_{kj} = \tilde{g}_k \tilde{g}_j^{-1} \tilde{g}_j \tilde{g}_k^{-1} = 1$$

then on $U_i \cap \tilde{U}_j \cap \tilde{U}_k$ $\tilde{g}_k g_{\rho(k)i} = \tilde{g}_j g_{\rho(j)i}$
Let $g_i = \tilde{g}_k g_{\rho(k)i}$ on $U_i \cap \tilde{U}_k$
then g_i is well defined non-vanishing holomorphic on U_i
and $g_j g_i^{-1} = \tilde{g}_k g_{\rho(k)j} \tilde{g}_k^{-1} g_{\rho(k)i}^{-1} = g_{ij}$ □

Another equivalent formulation for the Second Cousin problem is the following:

Proposition 3. *A solution of the above theorem implies a solution of the following formulation of the second cousin problem:*
Let $\{U_i\}$ be an open covering of a pseudoconvex domain $\Omega \subset \mathbb{C}^n$. On each define f_j meromorphic on U_j such that $f_i.f_j^{-1}$ holomorphic on $U_i \cap U_j$ and non-vanishing, then there exist f meromorphic on Ω such that $f.f_j^{-1}$ holomorphic and non-vanishing on U_j for all j.

Proof.
Let $g_{jk} = f_j.f_k^{-1}$ so g_{jk} holomorphic on $U_i \cap U_j$
and $g_{jk}.g_{kj} = 1$ and $g_{ij}.g_{jk}.g_{ki} = 1$
then g_{jk} satisfy the second cousin data.
Let g_j be the non-vanishing holomorphic solution of the above theorem then $g_{jk} = g_k.g_j^{-1}$ but also $g_{jk} = f_j.f_k^{-1}$ on $U_j \cap U_k$
then $f_j.g_j = f_k.g_k$ on $U_j \cap U_k$
let $f = f_j.g_j \in \mathcal{M}(U_j)$
and $f.f_j^{-1} = g_j \in \mathcal{O}(U_j)$ □

Remark. *A generalization of the Weierstrass theorem in higher dimensions is the second Cousin problem.*
To elaborate this we show that the Second Cousin problem in $\mathbb{C}^n$ is the equivalent of Weierstrass in $\mathbb{C}$, by proving Weierstrass using Cousin.

Proof. using the second cousin problem:
Denote by $\{w_i\}$ the given set of points in the open set $\Omega \subset \mathbb{C}$
Let $U_j = \Omega - \{w_k \text{ such that } k \neq j\}$
then $\{U_j\}$ is an open cover of Ω
Let $f_j = (z - w_i)^{-n_j} \in \mathcal{M}(U_j)$
$f_i.f_j^{-1} \in \mathcal{O}(U_j \cap U_i)$ where $U_j \cap U_k = \Omega - (w_i)$ for all i.
then by the above proposition there exist $f \in \mathcal{M}(\Omega)$ such that $f.f_j^{-1} \in \mathcal{O}(U_j)$ and non-vanishing. □

Chapter 5

Approximation of Non-holomorphic Maps

In this section we will tackle the main theorem from the work of Rosay [6]. The main idea that Rosay focused on in his paper "Approximation of non-holomorphic maps and poletsky theory of discs" is approximation of non-holomorphic functions by holomorphic ones.

In his paper, Rosay tries to find specific conditions for a non-holomorphic function so that the following works:

Let $\mathcal{M}$ be a complex manifold equipped with some metric, and let Ω be a relatively compact region in $\mathcal{M}$. For every $\epsilon > 0$, does there exist $\delta > 0$ such that if u is a map - with some conditions- from the unit disc Δ in $\mathbb{C}$ into Ω , with $|\overline{\partial}| < \delta$, then there exist a holomorphic map h:$\Delta \longrightarrow \Omega$ such that $|h-u| < \epsilon$.

We start by the notations that will be used in the main theorem. As before, Δ denotes the open unit disc in $\mathbb{C}$ and $\overline{\Delta}$ the closed unit disc. $\mathcal{M}$ will denote a complex manifold with dimension n, $\Omega_1, \ldots, \Omega_R$ some open sets in $\mathcal{M}$ with $K_1, \ldots, K_R$ compact subsets of $\Omega_1, \ldots, \Omega_R$ respectively. Each Ω_j is biholomorphic to some open set in $\mathbb{C}^n$. Denote, as before, by $U_1, \ldots, U_R$ open sets in $\mathbb{C}$ such that $\overline{\Delta} \subset \bigcup_{j=1}^{R} U_j$.
We assume that all triple intersections are empty, i.e. $U_j \cap U_k \cap U_l = \phi$ if j, k, l are all distinct.

Definition. *A map $\varphi : \Delta \longrightarrow \mathcal{M}$ is called restricted if for every $j \in \{1, \ldots, R\}, \varphi(U_j \cap \Delta) \subset K_j$.*

We equip $\mathcal{M}$ with some metric to make sense of $|\overline{\partial}\varphi| \leq \delta$. We also define the distance between two maps from Δ into $\mathcal{M}$ by

$$d(f, g) = sup_{\zeta \in \Delta} dist(f(\zeta), g(\zeta))$$

Theorem 16. *(Main Theorem)*
For every $\epsilon > 0$, there exits $\delta > 0$ such that if u is a restricted map from Δ into $\mathcal{M}$ satisfying $|\overline{\partial}u| \leq \delta$, there exists a holomorphic map h from Δ into $\mathcal{M}$ such that $d(u, h) \leq \epsilon$.

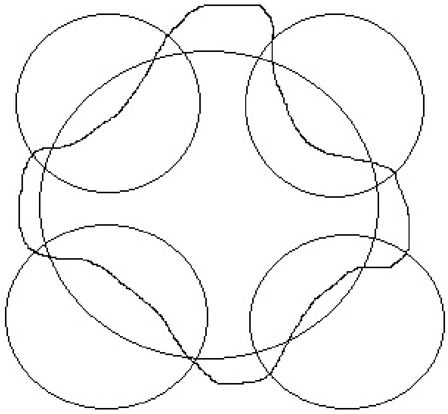

Figure 5.1: An example of a covering with no triple intersections

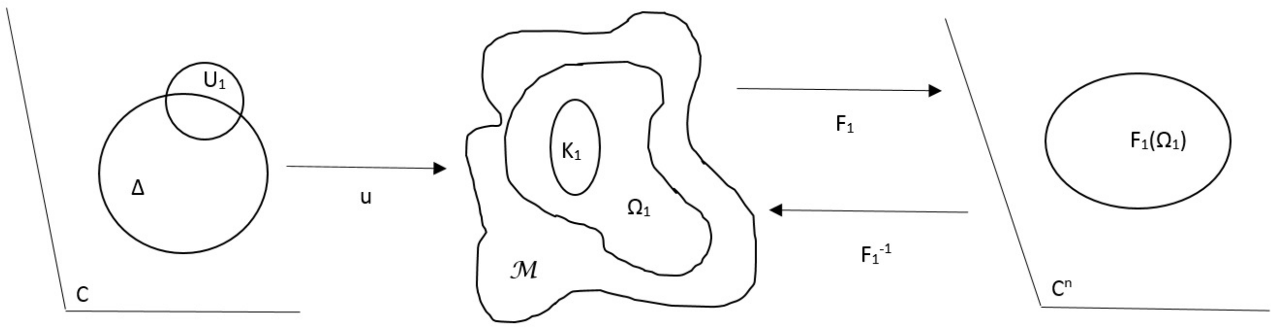

Figure 5.2: Maps between the disc the manifold and $\mathbb{C}^n$

5.1 Lempert's Example

A general question that Rosay raised at the beginning of his paper is the following: *Is every map from the unit disc into a complex manifold, with a small* $\overline{\partial}$*, close to a holomorphic map?* The answer to Rosay's question is negative as it was shown by Lempert: it will not work for any non-holomorphic function in general.

Proposition 4. *Let* $\mathcal{M}$ *be a compact Riemann surface of genus*≥ 2*, equipped with some metric. There exist* $\epsilon > 0$ *such that for every* $\delta > 0$ *there exists a smooth map* $\rho : \Delta \longrightarrow \mathcal{M}$ *such that* $|\overline{\partial}\rho| < \delta$*, but such that for every holomorphic map* $\lambda : \Delta \longrightarrow \mathcal{M}$ $sup\ _{z\in\Delta} dist(\rho(z), \lambda(z)) \geq \epsilon$*.*

Proof. Let P be a covering of $\mathcal{M}$, by the unit disk. Let d denote the distance function on $\mathcal{M}$, and let d_0 denote the Poincare distance on Δ.
We mention the following lifting fact: there exist $\epsilon > 0$ such that if f and g are continuous maps from Δ into $\mathcal{M}$, and $sup_{z\in\Delta} d(f(z), g(z)) \leq \epsilon$, then f and g can be lifted to continuous maps $\tilde{f}$ and $\tilde{g}$ where $f = P \circ \tilde{f}$ and $g = P \circ \tilde{g}$ and $sup_{z\in\Delta} d_0(\tilde{f}(z), \tilde{g}(z)) \leq 1$.
Let B be a function defined on a neighborhood of $\overline{\Delta}$ in $\mathbb{C}$ with the following properties:

1. $|B(e^{i\theta})| \equiv 1$,

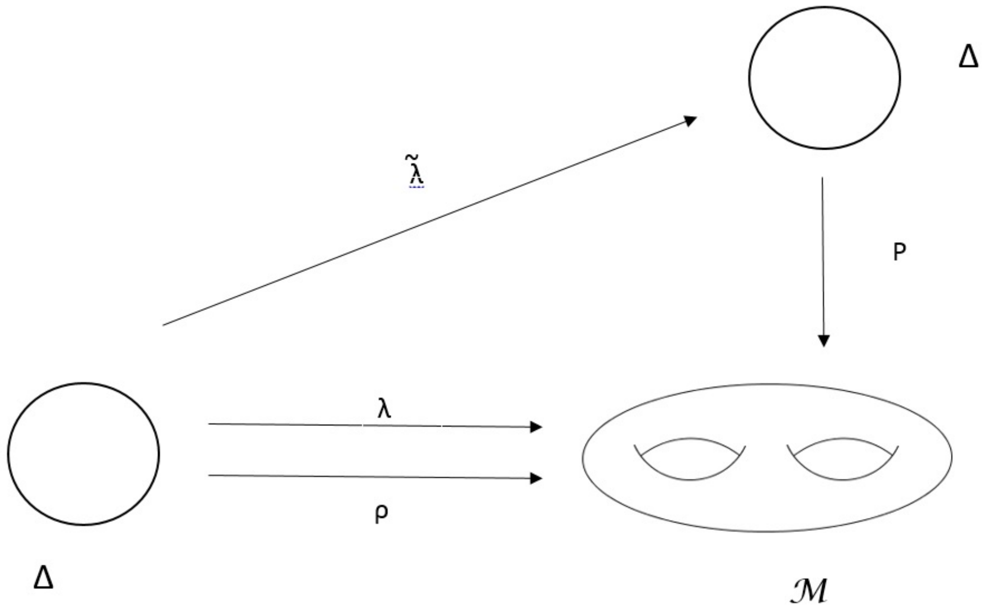

Figure 5.3: Maps and lifts

2. $|B| < 1$ on Δ,

3. B is holomorphic on a neighborhood of the unit circle,

4. for every $k \in \mathbb{N}$, $z^k B$ restricted to the unit circle does not extend holomorphically to the unit disk.

By conditions (1),(2), and (4), for every holomorphic map $h : \Delta \longrightarrow \Delta$, $sup_{z\in\Delta} d_0(h(z), z^k B(z)) = +\infty$.
By a normal family argument, there exists α_k, $0 < \alpha_k < 1$, such that for every holomorphic map $h : \Delta \longrightarrow \Delta$, $sup_{z\in\Delta} d_0(h(z), (1-\alpha_k) z^k B(z)) > 1$.
Take $\rho_k : \Delta \longrightarrow \mathcal{M}$ defined by

$$\rho_k = P \circ ((1-\alpha_k) z^k B(z))$$

Then $|\overline{\partial}\rho_k| \to 0$ uniformly (by Arzela-Ascoli) on $\overline{\Delta}$ as $k \to +\infty$ due to condition (3).
Suppose for every holomorphic map $\lambda : \Delta \longrightarrow \mathcal{M}$, $sup_{z\in\Delta} d(\rho_k(z), \lambda(z)) < \epsilon$. Lift ρ_k to the map $(1-\alpha_k) z^k B(z)$ and lift λ to a map $\tilde{\lambda}$. Now by the lifting fact stated at the beginning, $sup_{z\in\Delta} d_0((1-\alpha_k) z^k B(z), \tilde{\lambda}) \le 1$ contradicting the choice of α_k.
Thus,for every holomorphic map $\lambda : \Delta \longrightarrow \mathcal{M}$, $sup_{z\in\Delta} d(\rho_k(z), \lambda(z)) > \epsilon$. □

5.2 The Cartan Lemma with Bounds

We start by recalling the usual Cartan Lemma without proof:

Lemma 6. *Let $a_1 < a_2 < a_3 < a_4$ and $b_1 < b_2$ and define rectangles in the complex plane by*

$$K_1 = \{z_1 = x_1 + iy_1 : a_2 < x_1 < a_3, b_1 < y_1 < b_2\}$$

$$K_1^{'} = \{z_1 = x_1 + iy_1 : a_1 < x_1 < a_3, b_1 < y_1 < b_2\}$$

$$K_1^{"} = \{z_1 = x_1 + iy_1 : a_2 < x_1 < a_4, b_1 < y_1 < b_2\}$$

so that $K_1 = K_1^{'} \cap K_1^{"}$. *Let* $K_2, \ldots, K_n$ *be simply connected domains in* $\mathbb{C}$ *and let*

$$K = K_1 \times K_2 \times \ldots \times K_n$$

$$K^{'} = K_1^{'} \times K_2 \times \ldots \times K_n$$

$$K^{"} = K_1^{"} \times K_2 \times \ldots \times K_n$$

so that again $K = K^{'} \cap K^{"}$. *Suppose that* $F(z)$ *is a complex holomorphic matrix-valued function on a rectangle* $K \in \mathbb{C}^n$ *such that* $F(z)$ *is an invertible matrix. Then there exist holomorphic functions* $F^{'} \in K^{'}$ *and* $F^{"} \in K^{"}$ *such that*

$$F(z) = F^{'}(z)F^{"}(z) \qquad in \quad K$$

An important result that is needed in the proof of the main theorem is the following Cartan Lemma with bounds [1] which we state without proof:

Lemma 7. *Let* $(V_j)_{j=1}^{N}$ *be a covering of the closed unit disc* $\overline{\Delta}$ *by open subsets of* $\mathbb{C}$. *For each* $(j,k) \in \{1, \ldots, N\}^2$ *let* g_{jk} *be a holomorphic* $(n \times n)$ *matrix bounded and with bounded inverse defined on* $(V_j \cap V_k) \cap \Delta$, *with the conditions:* $g_{jj} = 1$, $g_{jk} = g_{kj}^{-1}$, $g_{jk}g_{kl}g_{lj} = 1$. *Then there exist bounded holomorphic matrices* g_j *with bounded inverses defined on* $V_j \cap \Delta$ *with* $j = 1, \ldots, N$ *such that* $g_{jk} = g_k^{-1}g_j$ *on* $(V_j \cap V_k) \cap \Delta$, *with bounds for the* g_j*'s and their inverses depending only on the covering and the sup norm of the* g_{jk}*'s and of their inverses.*

5.3 A non-linear Cousin problem

In this section we will prove a proposition that will be used to reduce the main theorem into a non-linear cousin problem.

Let $U_1, \ldots, U_R$ be open sets in $\mathbb{C}$ that cover $\overline{\Delta}$ and with empty triple intersections. For $1 \leq j < k \leq R$ we shall introduce subsets of $\mathbb{C}^n$, $\omega_{jk}^{'}$ and ω_{jk} such that $\omega_{jk} \subset \omega_{jk}^{'}$. Let F_{jk} be a holomorphic immersion from $\omega_{jk}^{'}$ into $\mathbb{C}^n$. Note that we define F_{jk} and ω_{jk} only for $j < k$.

Proposition 5. *With the above notations: For every* $\epsilon > 0$ *there exist* $\delta > 0$ *such that if for every* $j \in 1, \ldots, R$, u_j *is a holomorphic map from* $U_j \cap \Delta$ *into* $\mathbb{C}^n$ *such that for* $1 \leq j < k \leq R$, $u_j[(U_j \cap U_k) \cap \Delta] \subset \omega_{jk}$ *and* $\|u_k - F_{jk} \circ u_j\|_\infty \leq \delta$, *then there exist holomorphic maps* $v_1, \ldots, v_R$ *respectively from* U_j *into* $\mathbb{C}^n$ *such that:*

$$\|v_j\|_\infty \leq \epsilon$$

$$and$$

$$u_k + v_k = F_{jk}(u_j + v_j) \ on \ (U_j \cap U_k) \cap \Delta$$

Proof.

case 1: F_{jk} *being the identity map*

We have $\{U_1, \ldots, U_R\}$ open cover of $\Delta \subset \mathbb{C}$ where Δ is pseudoconvex and $u_j : U_j \cap \Delta \longrightarrow \mathbb{C}^n$ is holomorphic with $\|u_k - u_j\|_\infty < \delta$. Let $\epsilon > 0$, then by the standard additive cousin problem there exist a meromorphic function v on Δ such that $v - u_j$ is holomorphic on $U_j \cap \Delta$.

Let $v_j = v - u_j|_{U_j}$ and by the bounded first cousin problem 14 $\|v_j\|_\infty \leq \epsilon$.
Now $v_j - v_k = (v - u_j) - (v - u_k) = u_k - u_j$
so $u_k + v_k = u_j + v_j$ on $(U_j \cap U_k) \cap \Delta$.

case 2: *F_{jk} is different than the identity map*

To prove $u_k + v_k = F_{jk}(u_j + v_j)$ we can prove an equivalent equality.
We linearize using Taylor:

$$F_{jk}(u_j(z) + t) = F_{jk}(u_j(z)) + (F_{jk}^{'}(u_j(z))t + \mathcal{O}(|t|^2)$$

for $t \in \mathbb{C}^n$. Letting $t = v_j(z)$ we get:

$$F_{jk}(u_j(z) + v_j(z)) = F_{jk}(u_j(z)) + [F_{jk}^{'}(u_j(z)]v_j(z)$$

but if $u_k + v_k = F_{jk}(u_j + v_j)$ is true, then

$$[F_{jk}^{'}(u_j(z)]v_j(z) - v_k(z) = -F_{jk}(u_j(z)) + u_k(z)$$

now using Cartan lemma with bounds we can write

$$F_{jk}^{'}(u_j(z)) = g_k^{-1}(z)g_j(z)$$

for $j < k$ where g_j is holomorphic matrix on U_j. Now multiply by $g_k(z)$:

$$g_k(z)\{(g_k^{-1}(z)g_j(z))v_j(z) - v_k(z) = -F_{jk}(u_j(z)) + u_k(z)\}$$

so we should prove that

$$g_j(z)v_j(z) - g_k(z)v_k(z) = g_k(z)[-F_{jk}(u_j(z)) + u_k(z)]$$

inorder to prove $u_k + v_k = F_{jk}(u_j + v_j)$.

Let $\alpha_{jk} = -F_{jk}(u_j(z)) + u_k(z)$
so we have a family $\alpha = (\alpha_{jk})_{1 \leq j < k \leq R}$ of n-tuples of bounded holomorphic functions $\alpha_{jk} \in [H^\infty((U_j \cap U_k) \cap \Delta)]^n$
by the first cousin problem there exist $\alpha_j \in [H^\infty(U_j \cap \Delta)]^n$ for all j such that for $j < k$ on $(U_j \cap U_k) \cap \Delta$

$$\alpha_{jk} = \alpha_j - \alpha_k$$

Define a new family $\beta = (\beta_{jk})_{j<k}$ where $\beta_{jk} = g_k \alpha_{jk}$
Similarly by the first cousin problem there exist β_j such that on $(U_j \cap U_k) \cap \Delta$

$$\beta_{jk} = \beta_j - \beta_k$$

Now let $v_j = g_j^{-1}\beta_j$
so $g_j v_j - g_k v_k = g_j g_j^{-1}\beta_j - g_k g_k^{-1}\beta_k = \beta_j - \beta_k = \beta_{jk} = g_k \alpha_{jk}$
so $g_j v_j - g_k v_k = g_k[-F_{jk}(u_j) + u_k]$ therefore

$$u_k + v_k = F_{jk}(u_j + v_j)$$

We still need to show that $\|v_j\|_\infty < \epsilon$:
Set the linear operator $S_j(\alpha) = v_j = g_j^{-1}\beta_j$ so

$$[F_{jk}^{'}(u_j)]S_j(\alpha) - S_k(\alpha) = \alpha_{jk}$$

Define the Banach space $H = \oplus[H^\infty((U_j \cap U_k) \cap \Delta)]^n$ and the map $\Phi : H \longrightarrow H$ by

$$(\Phi(\alpha))_{jk} = \alpha_{jk} - [F_{jk}(u_j + S_j(\alpha)) - u_k - S_k(\alpha)]$$

Claim: Given $\epsilon > 0$, if all $\|u_k - F_{jk} \circ u_j\|_\infty \leq \delta$, Φ has a fixed point α with $\|S(\alpha)\| < \epsilon$:
On $[H^\infty((U_j \cap U_k) \cap \Delta)^n]$ consider the norm:

$$\|f\|_{jk} = max_m sup|f_m|$$

for $f = (f_1, \ldots, f_n)$
On H consider the norm:

$$\|\alpha\| = max\|\alpha_{jk}\|_{jk}$$

for $\alpha = (\alpha_{jk})$

Note that $\Phi^{'}(0) = 0$:

$$(\Phi(\alpha))_{jk} = \alpha_{jk} - F_{jk}(u_j + S_j(\alpha)) + u_k + S_k(\alpha)$$

$$\begin{aligned}(\Phi(\alpha + th))_{jk} = \alpha_{jk} + th_{jk} - F_{jk}(u_j + S_j(\alpha) + tS_j(h))\\ + u_k + S_k(\alpha) + tS_k(h)\end{aligned}$$

$$\begin{aligned}(\Phi(\alpha + th))_{jk} - \Phi(\alpha))_{jk} &= t(h_{jk} + S_k(h))\\ &\quad - F_{jk}(u_j + S_j(\alpha) + tS_j(h)) + F_{jk}(u_j + S_j(\alpha))\end{aligned}$$

Now since
$h_{jk} = F_{jk}^{'}(u_j)S_j(h) - S_k(h)$ and $F(x + th) = F(x) + tF^{'}(x)h + \mathcal{O}(t^2\|h\|^2)$

we get

$$\frac{(\Phi(\alpha + th))_{jk} - \Phi(\alpha))_{jk}}{t} = h_{jk} + S_k(h) - F_{jk}^{'}(u_j + S_j(\alpha))S_j(h)$$

$$\begin{aligned}\Phi^{'}(0) &= lim_{t\to 0}\frac{\Phi(0 + th) - \Phi(0)}{t}\\ &= h_{jk} + S_k(h) - F_{jk}^{'}(u_j + S_j(0))S_j(h)\\ &= F_{jk}^{'}(u_j)S_j(h) - S_k(h) + S_k(h) - F_{jk}^{'}(u_j)S_j(h)\\ &= 0\end{aligned}$$

Denote by β_ρ the ball of radius ρ such that $\beta_\rho \subset H$
Let $\alpha \in \beta_\rho$, choose ρ small enough such that $\|\Phi(\alpha)\| \leq \rho$ and thus $\Phi(\beta_\rho) \subset \beta_\rho$.
By mean value theorem on Banach spaces and for $\alpha, \beta \in \beta_\rho$

$$\|\Phi(\alpha) - \Phi(\beta)\| \leq \|\Phi^{'}(\gamma)\|\|(\alpha - \beta)\|$$

for some $\gamma \in \beta_\rho$ By continuity of $\Phi^{'}$ at 0:
Given $0 < c < 1$, there exist $r > 0$ (in this case $r = \rho$) such that if $\|\gamma - 0\| \leq r$ then
$\|\Phi^{'}(\gamma) - \Phi^{'}(0)\| \leq c$
but $\Phi^{'}(0) = 0$ so if $\|\gamma\| \leq r$ then $\|\Phi^{'}(\gamma)\| \leq c$
so $\|\Phi(\alpha) - \Phi(\beta)\| \leq \ c\|\alpha - \beta\|$
so Φ is a contraction map.

By Banach fixed point theorem there exist $\alpha^* \in \beta_\rho$ such that $\Phi(\alpha^*) = \alpha^*$
Let $\delta = (1+c)\frac{\epsilon}{\|S\|}$ and $\rho \leq \frac{\epsilon}{\|S\|}$
$\|\Phi(\alpha^*) - \Phi(0)\| \leq c\|\alpha^*\|$ but $\Phi(0) = (-F_{jk}(u_j) + u_k)_{jk}$
so $\|\alpha^* + (-F_{jk}(u_j) + u_k)_{jk}\| \leq c\|\alpha^*\|$
$\|F_{jk}(u_j) - u_k)_{jk}\| - \|\alpha^*\| \leq c\|\alpha^*\|$

$$\begin{aligned}
\|F_{jk}(u_j) - u_k)_{jk}\| &= (1+c)\|\alpha^*\| \\
&\leq (1+c)\rho \\
&\leq (1+c)\frac{\epsilon}{\|S\|} = \delta
\end{aligned}$$

Now using operator norm

$$\begin{aligned}
|S(\alpha)| &\leq \|S\|\|\alpha\| \\
&\leq \|S\|\rho \\
&\leq \|S\|\frac{\epsilon}{\|S\|} = \epsilon
\end{aligned}$$

so $\|v_j\| < \epsilon$ for all j □

5.4 Proof of The Main Theorem

In this section we prove the Main theorem

Proof.
Denote by F_j a biholomorphism from Ω_j into an open set $F_j(\Omega_j) \subset \mathbb{C}^n$ for all $j = 1, \ldots, R$.
Define $u_j = F_j \circ u|_{U_j \cap \Delta}$ from $U_j \cap \Delta$ into $F_j(K_j) \subset \mathbb{C}^n$ such that on $(U_j \cap U_k) \cap \Delta$ we have $u_k = F_k \circ F_j^{-1} \circ u_j$ where the map $F_k \circ F_j^{-1}$ is defined on a neighborhood of $F_j(K_j \circ K_k)$.
We want to find a holomorphic map $h : \Delta \longrightarrow \mathcal{M}$. So we seek R holomorphic maps $h_1, \ldots, h_R$ respectively from $U_j \cap \Delta$ into $\mathbb{C}^n$ for all $j = 1, \ldots, R$ satisfying $h_k = F_k \circ F_j^{-1} \circ h_j$ on $(U_j \cap U_k) \cap \Delta$ and such that $d(h_j, u_j)$ is small as desired.
By the proposition 1 proved in 4.1, if $u_j : U_j \cap \Delta \longrightarrow \mathbb{C}^n$ is such that $|\overline{\partial} u_j| \leq \delta$, there exist $w_j : U_j \cap \Delta \longrightarrow \mathbb{C}^n$ where $w_j = -T\overline{\partial} u_j$ and such that $h_j = u_j + w_j$ is holomorphic, and $|w_j| \leq c\delta$ for some appropriate c.
Due to non-linearity, we do not have

$$u_k + w_k = F_k \circ F_j^{-1} \circ (u_j + w_j)$$

So we must perturb the holomorphic maps $u_j + w_j$ in order to get holomorphic maps $u_j + w_j + v_j$ defined on $U_j \cap \Delta$ such that $u_k + w_k + v_k = F_k \circ F_j^{-1} \circ (u_j + w_j + v_k)$ on $(U_j \cap U_k) \cap \Delta$
Now we use the proposition in the previous section to complete the proof. We now match the corresponding notations from the proposition to the ones in the main theorem.

- $u_j = u_j + w_j$
- $w^{'}_{jk} = F_j(\Omega_j \cap \Omega_k)$
- w_{jk} is the image under F_j of the intersection of given neighborhoods of K_j and K_k
- $F_{jk} = F_k \circ F_j^{-1}$

$\square$

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