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Phenomenological aspects of Noncommutative Geometry approach to the standard model

by

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PHENOMENOLOGICAL ASPECTS OF

NONCOMMUTATIVE GEOMETRY APPROACH TO THE STANDARD

MODEL

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AN ABSTRACT OF THE DISSERTATION OF

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The aim of this thesis is to explore the implications that spectral action principle has on the scalar sector of the models derived from the noncommutative geometry approach. After a brief introduction for noncommutative geometry and some short discussions on why it is a promising tool to describe high energy physics in chapter one, the first fruit of this approach is presented in chapter two. This is effectively a singlet extended standard model with some specific features rooted in the settings of the noncommutative geometry approach.

Among these features are the specific scalar potential and the relation between scalar couplings, Yukawa couplings, and the unified gauge coupling. The latter is usually looked at as the initial condition for running of the couplings down from unification scale. Some of the implications of these features such as their consistency with the particle masses and their influence on the running of gauge couplings are discussed in chapter two.

It is shown that there is a range of initial values at the unification scale which is able to produce Higgs and top quark masses at low energies. The stability of the vacuum and the deviation of gauge couplings from experimental values are discussed and compared at the two-loop level with a real scalar singlet and the pure standard model.

In chapter three, the spectral Pati-Salam model is described concisely. We then study the implications of the tight restrictions of the noncommutative geometry settings along with constraints of the spectral action on the scalar potential. As a result, it will be clear that the scalar potential in the spectral Pati-Salam model does not provide a suitable vacuum to break to the standard model. However, this potential is proton decay free up to tree level even though diquark and leptoquark vertices exist.

In the appendix, we introduce some computational tools including a package under Mathematica to find two-loop order renormalization group equations.

Contents

Chapter 1

Introduction

1.1 Standard Model of Elementary Particles

One of the pillars of modern physics is the Standard Model of elementary particles which is based on the quantum field theory. In this model, there are three types of quantum fields which are responsible to describe all the fundamental particles and their interactions. These fields live in spacetime which is a Riemannian manifold with spin structure. The first kind includes fields that describe what is usually referred to as *matter particles* and are called fermionic fields. These are spacetime spinors and essentially can interact with each other only through the other two kinds of fields noted as vector bosons and scalar bosons. The only exception is the mysterious particle called neutrino which might interact with itself with having a what is called "Majorana bare mass".

This model is based on the beautiful idea that if two particles interact with each other, there is kind of symmetry under which they transform to one another. In other words, if two matter fields are not connected together by any kind of symmetry, they are blind to each other and do not interact directly. There is, therefore, an intimate relation between bosonic fields which cause interactions between fermions and the symmetries. It means there are cases in laboratory that we start with one of these particles and end up having the other one plus a bosonic field. The symmetries which are responsible for forces between elementary particles are called gauge symmetries and are described by three symmetry groups, $SU(3)$, $SU(2)$, and $U(1)$. In addition, some of the vector bosons have interactions between themselves which originate from the same symmetries.

If the scalar bosons didn't exist, this picture was very neat. However, it couldn't describe the complexity of our world. The vector bosons associated with the symmetries were massless fields in that case. Fermions were massless as well, and there would have been only three free parameters to tune the strength of the three forces so that the model could predict all the physical phenomenon.

However, nearly all the matter particles are massive and one needs to complexify the noted pure picture to be able to describe this fact. The standard model approach is to assume a certain connection between left-handed and right-handed fields which are two different versions of each fermion. Peculiarly, the standard model does not picture this connection with a new symmetry. Instead, there comes a new scalar field, which is the only one of its kind, to do the job. In other words, interactions between fermions and scalars are not required by the gauge symmetries, but of course they respect them. This part of the standard model, extremely important though it is, is less satisfactory since it looks artificial and has more arbitrary elements compared to the other parts of the model.

This unique scalar boson is called the Higgs. This field is originally consisted of four real components. The Higgs field plays a couple of essential roles in the final picture of the theory. In its potential, it interacts with itself in form of quartic and quadratic self-interactions. Its potential is in a way that it has a minimum at about $246 GeV$ at the classical level. The minimum is symmetric with respect to three of the Higgs components.

As a consequence of its potential, when energy of the Higgs is at the order of its vacuum expectation value, one of its components gains mass and the rest remain massless. In addition, in its kinetic terms, the Higgs is coupled with gauge fields associated with weak, $SU(2)$, and isospin, $U(1)$, symmetries. One big step to complete this model happened when the Higgs mechanism was proposed. This was due to the works of many physicists such as Goldstone, Higgs, and Englert. They showed that three massless components of the Higgs cannot be detected as independent fields. Instead, three of the originally massless gauge fields appear as massive fields. The massive component of the Higgs field was finally found at LHC in CERN in 2012.

Noncommutative geometry approach to the standard model is an attempt to bring all the above pieces under a unified geometrical picture and reduce the arbitrariness of different parts. As we will see in the coming sections, the gauge sector of the standard model comes in the form of Yang-Mills models which suggest an intimate relation between gauge fields and the spacetime geometry. In addition to making a mathematical ground to clarify this relation, noncommutative geometry has shown that it can bring scalar sector of the standard model under the same unified picture and expand it to include even more useful scalars.

We will discuss briefly in the rest of this chapter why there it is rational to enter new ideas of geometry into physics. In the next chapter, we discuss the phenomenological implications of the extra scalar fields emerging from noncommutative geometry approach. Moreover, this approach is also used as a tool to explore beyond the standard model. In chapter three, we will first discuss the attempts done in this way and then consider the phenomenology of scalar content of the spectral Pati-Salam model.

1.2 Yang-Mills Lagrangian

Description of the gauge symmetries and gauge fields in the standard model is based on Yang-Mills theory. Yang and Mills taught us how to start with a model that describes fields which possess a symmetry between them and then find new fields by localizing that symmetry. Localizing means requiring the transformations of that symmetry to depend on spacetime coordinates. This implies the laws of physics to be the same if at any point of spacetime the original fields are transformed to each other differently. It can happen if other fields, called gauge fields, are present and transform in a way to compensate the difference. The concept of gauge fields has, therefore, an intimate relation with the concept of connections of spacetime which are built by the metric and make the partial derivative covariant. This close relation between geometry and gauge theories is what noncommutative geometry approach attempts to explore more.

Let us start with spinors on the four-dimensional spacetime as a spin manifold. Let ψ be a set of such fields. Its kinetic term in an acceptable action will be in the form of

$$\psi^\dagger \gamma^0 (i\gamma^\mu \partial_\mu)\psi.$$

This term is invariant under any symmetry between fields in ψ, if the symmetry is not spacetime dependent, called global symmetry. To make these symmetries local, the above derivative needs to be changed to make the whole kinetic term invariant again.

Symmetries are represented by groups. Generators of a group show how small changes happen around unity transformation. Therefore, naturally, the fields which are aimed to modify the derivative are living in the generator space of the symmetry groups. These are presented by adjoint representations. If there are n fields in ψ and we want to make the kinetic term invariant under a special unitary transformation, $SU(N)$, of the fields, the covariant derivative will be

$$D_\mu = \partial_\mu - igA_\mu^a T^a, \qquad a = 1 \dots n \ .$$

Here A_μ^a shows n *gauge fields* which modify the derivative and live in the adjoint representation of the group. T^a are $n^2 - 1$ basis for these generators. Therefore, not only the existence of these fields, but also their interactions with spinors is dictated by localizing the global symmetry. The added term to the Lagrangian is now

$$gA_\mu \bar{\psi} \gamma^\mu \psi, \qquad A_\mu = A_\mu^a T^a.$$

Next step is to add kinetic term of gauge fields to the Lagrangian. Roughly speaking, by modifying the derivative, A_μ is just like a spacetime connection. To build higher order covariant tensors and then use them to build spacetime and gauge invariants, therefore, commutator of the covariant derivatives helps:

$$\begin{aligned} F_{\mu\nu} &= [D_\mu, D_\nu] \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu - ig^2 [A_\mu, A_\nu]. \end{aligned} \tag{1.1}$$

This is the same way through which one can find the Ricci tensor for a manifold. The Kinetic term of gauge sector is then proportional to

$$F_{\mu\nu} F^{\mu\nu} \tag{1.2}$$

So, the reason a constant coefficient was singled out of the gauge fields in the beginning is clear now. Kinetic terms are quadratic with respect to the gauge fields, however they couple with the fermions in a linear way. Therefore, by normalizing them to have canonical kinetic terms, constants can appear in the way these fields are coupled with spinors.

Instantly, the commutator in (1.1) reminds us of 2-forms. What we are really doing in this language is to start with a field which has one tensorial index and lives in the space of 1-forms and build a 2-form using the commutator. In noncommutative geometry, p-forms are defined by their commutation with a generalized Dirac operator which is not necessarily a differential operator. This will become clearer in the coming sections. We will see that, astonishingly, gauge fields and even scalar fields in a Yang-Mills theory can be built by commutators of the Dirac operator of a generalized geometrical space with elements of a suitable algebra.

The relation of all these basics with noncommutative geometry is that this mathematical field is a ground to comprehend the connection between gauge fields and geometry in the most straightforward way. In other words, noncommutative geometry aims to expand the notions of geometry so that the gauge fields appear naturally as *connections* along the extra structure of space. This is very similar to the project of Kaluza and Klein in the 1920s. Yet, the difference is that usual projects of this kind probe the extra structure by usual geometrical and topological tools of Riemannian geometry. There have also been attempts in the string theory context, for example, to change spacetime structure with looking at the coordinates as operators which do not commute. Attempts of these kinds are originated from a quantum mechanical point of view. Noncommutative geometry, on the other hand, tries to find a pure geometrical way at the classical level. Obviously, this only happens if we expand our understanding from geometry. The ground for this project was already ready at 1980s when the famous mathematician, Alain Connes, created noncommutative geometry based on ideas in different branches of mathematics such as differential geometry, operator algebra, algebraic geometry, and spectral geometry. In what follows, we briefly discuss some of the basic motivations for this project and outline the reasons to believe noncommutative geometry is likely to be a ground for unifying the forces of nature.

1.3 Gravity

The only fundamental force which is not described by quantum field theory ideas is gravity. Gravitational interactions are clearly very important in forming the universe. This phenomenon is describable by Einstein's general relativity which is a classical field theory. The main idea is to look to space and time as a manifold and describe gravity as the geometry of this manifold which can interact with energy density created by other fields and their interactions. This is formulated in the following action which is based on scalar curvature of spacetime

$$\begin{aligned} I_{total} &= I_{gravity} + I_{matter}, \\ I_{gravity} &= \frac{1}{4\pi G} \int \sqrt{-g}\; R\; d^4x, \end{aligned} \tag{1.3}$$

which leads to the following equation of motion

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = 4\pi G \frac{\delta I_{matter}}{\delta g^{\mu\nu}} = -8\pi G T_{\mu\nu}. \tag{1.4}$$

The right hand side is mass-energy tensor and the left side only depends on the geometry of spacetime.

The difficulty with the above formula is that the ultimate theory to describe matter fields and their associated energies is the quantum field theory introduced before. However, despite the continuous efforts of researchers during decades, still there is no quantum version for gravity. This makes it harder to believe a theory could contain both the standard model and general relativity without fundamental changes in one or the two of them. Different approaches to solve these issues have led to some beautiful models such as string theory and supergravity. These efforts however suffer from the lack of experimental evidence to show the correct path.

Noncommutative geometry has been taking a conservative path in this context by trying to predict the above action at the classical level. Then of course one needs to quantize the theory and one will face with the same problems for gravity.

What is doable for now is to quantize the noncommutative geometry version of the standard model and study the consequences. We follow this path in chapter 2.

Nonetheless, we will see below that noncommutative geometry approach predicts different pieces of the action as various terms in different orders of an expansion. There are geometrical invariants other than curvature which enter in higher orders of the expansion. Up until now, there is no quantum treatment prior to this expansion.

There are however two points which show the strength of noncommutative geometry approach in this regard. First, higher order terms such as the square of curvature are proved to be useful for different purposes and usually are added to the Einstein-Hilbert action to enhance renormalizability [1] or to modify gravity. Second, to be able to apply the action (1.3) for spaces with boundaries, such as when there is a blackhole, the action should be modified as

$$I_g = \frac{1}{4\pi G}\int_M \sqrt{-g}\; R\; d^4x + \frac{1}{8\pi G}\int_{\partial M} \sqrt{h}\; K\; d^3x, \tag{1.5}$$

in order for the variation to be well defined. Induced metric on the boundary is shown with h and K is the extrinsic curvature. This term naturally arises in spectral action of the Dirac operator [2] which is one of the pillars of noncommutative geometry approach to the standard model. The additional boundary term is proved, by works of Bekenstein and Hawking in 1970s, to be very important in the study of the thermodynamics of blackholes.

There are two more observations which can give us motivations to expand the notions of geometry for including other forces. First, it is a known fact that some useful geometrical information of a manifold, such as its total curvature, is embedded in the spectrum of operators like Laplacian and Dirac operator. This is interesting because after quantum mechanics arena we have learned how much representing physical observables with operators is useful. Second, the study of commutative algebra of functions on a manifold can give some very basic information about its geometry.

1.4 Spectral Geometry

Spectral geometry tries to drag geometrical information about a manifold out of the spectrum of some operators on it. As a very simple example, consider a three dimensional flat Euclidean manifold with boundaries in all directions. Eigenfunctions of the Laplacian in a suitable coordinate system are periodic harmonics and the eigenvalues or wavelengths are in the form of

$$\lambda = \frac{2\pi}{a}(n_1^2 + n_2^2 + n_3^2), \tag{1.6}$$

which n_i is used for integer numbers and a indicates dimension of the space. One important point is the way the number of these wavelengths grow. When λ is very big, the number of all wavelengths smaller than it is approximated by the volume of the sphere with radius $(n_1^2 + n_2^2 + n_3^2)^{1/2}$, more precisely

$$N(\lambda) = \frac{V}{(4\pi)^{\frac{3}{2}}\frac{3\sqrt{\pi}}{4}}\lambda^{\frac{3}{2}}. \tag{1.7}$$

This is a special case of a general theorem called Weyl's law. According to this law, asymptotic behavior of the growth rate of Laplacian eigenvalues on a manifold always reveals volume and dimension of the manifold

$$\lambda \to \infty : \qquad N(\lambda) \propto V\ \lambda^{\frac{d}{2}}. \tag{1.8}$$

This nice phenomenon of connection between geometry and operator algebra is not restricted to Weyl's law. It turns out that there is many more geometrical information that can be derived from this spectrum. At this stage, the famous question comes forward, are we able to hear the shape of a drum? Put another way, how much of the geometrical information of a manifold is derivable from such spectrums. In general, for example, one cannot determine the metric of a manifold by only having this spectrum. So there are not geometrically equivalent spaces with the same spectrum. We do not peruse this nice and advanced topic in its general and abstract form any further here. Instead, we focus on what the spectrum of Dirac operator can revile about a *Riemannian manifold*.

Let M be a Riemannian manifold with a *Clifford algebra structure*. Dirac operator is defined to be $D = \gamma^\mu \partial_\mu$. This operator acts on the spinors living on the manifold which form a Hilbert space denoted by $L^2(M, S)$. These are square integrable functions on manifold M which has the spin structure. Now there are some methods to obtain the spectrum of this operator on the manifold. Heat kernel is one of these method. The heat equation

$$\partial_t + D^2 = 0,$$

has a, symbolic, solution which is the kernel of e^{-tD^2}. This solution has an asymptotic expansion for very small t [3]

$$\frac{1}{(4\pi t)^{m/2}} \left(a_0(x, D^2) + a_1(x, D^2)t + a_2(x, D^2)t^2 + ... \right). \tag{1.9}$$

Coefficients of this expansion are known due to the works of Gilkey and others such as Seeley and De Witt (e.g. refer to [4, 5]). These coefficients, naturally, depend on the manifold points. However, quite amazingly, they are spacetime invariants:

$$\begin{aligned} a_0(x, D^2) &= 1, \\ a_2(x, D^2) &= \frac{1}{6} R, \\ a_4(x, D^2) &= \frac{1}{360} \left(5R^2 + 2R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 2R_{\mu\nu} R^{\mu\nu} \right). \end{aligned} \tag{1.10}$$

This is all nice, but we have not seen yet the spectrum of square of the Dirac operator. Here, a *trace formula* helps to make the connection. The trace of e^{-tD^2}

can be computed in two different ways. First, by integral of the answer of the above heat equation, and second, by summing over the spectrum of square of the Dirac operator

$$Tr(e^{-tD^2}) = \sum_n e^{-t\lambda_n} \approx \frac{1}{(4\pi t)^{m/2}} \sum_n a_n t^n \quad t \to 0 \tag{1.11}$$

a_n coefficients are for different t orders and therefore are all spectral invariants. By $t = \frac{1}{\Lambda^2}$, the following relations between spacetime invariants as spectral invariants is achieved for large enough Λ

$$\begin{aligned} a_0 &= \int_M \sqrt{-g}\ d^3x = V, \\ a_2 &= \frac{1}{6}\int_M \sqrt{-g}\ R\ d^3x, \\ a_4 &= \frac{1}{360}\int_M \sqrt{-g}\ \left(5R^2 + 2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu}\right)\ d^3x\ . \end{aligned} \tag{1.12}$$

As will be discussed more in the coming sections, the *spectral action principle* proposes the following action for gravity

$$I_g = Tr\left(\frac{D}{\Lambda}\right), \tag{1.13}$$

instead of Einstein-Hilbert action which was introduced in the previous section [6]. To write 1.13 in terms of 1.3, this relation helps:

$$Tr(D^{-s}) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\ Tr(e^{-tD})\ dt\ .$$

We noted before that the boundary term in 1.3 comes automatically from this action [2] and higher order terms are in general useful. It is shown that the spectral action for gravity is very accurate up to very high energies [7, 8].

1.5 Algebraic Geometry

We introduce a few ideas in this vast branch of mathematics which are useful to understand the rationale behind noncommutative geometry. The goal is to relate topological and geometrical properties of a manifold to the algebra of functions on it and then hopefully reverse this process to be able to develop geometrical spaces by algebraic tools.

Usually one starts to work with algebras which have some structure. A *Banach algebra* is an associative algebra with a norm defined for its members. For any two members of the algebra, their norms have to satisfy

$$||xy|| \leqslant ||x||\ ||y|| \tag{1.14}$$

Next is the notion of a star structure.

$$(x+y)^* = x^* + y^*, \qquad (xy)^* = y^* x^*, \qquad (x^*)^* = x. \tag{1.15}$$

A Banach algebra with this property is called a **-algebra*. Next useful tool is the notion of a *C*-algerba* which has the extra property of

$$||x^* x|| \leqslant ||x^*||\ ||x||. \tag{1.16}$$

One example is the algebra of functions on a finite set with pointwise addition and multiplication. Let X be a finite set and $C(X)$ the algebra of functions from X to the ring of complex numbers,

$$\begin{aligned} C(X) &= \{f : X \to \mathbb{C}\}, \\ (f+g)(x) &= f(x) + g(x), \\ (f.g)(x) &= f(x) g(x). \end{aligned} \tag{1.17}$$

The star structure is simply provided by the complex conjugate operation

$$f^*(x) \equiv \bar{f}(x). \tag{1.18}$$

Another important example of star algebras are *matrix algebras*. Let H be a finite inner product space over $\mathbb{C}^n$, and let $B(H)$ be the space of all the linear operators on H. Naturally $B(H)$ is a matrix algebra and it forms a C*-algebra by complex conjugate plus transpose as its star structure. Now, two remarks show these concepts are much more useful than being only examples. First, any finite dimensional space is isomorphic to $\mathbb{C}^n$. Second, any algebra which possesses a unitary member, called unital algebra, and is isomorphic to a *-subalgebra of $B(H)$ is a C*-algebra.

next, we are ready for a simplified version of *Gelfand-Niamark theorem* which is the starting point of making a relationship between pure algebraic concepts and algebraic tools on a manifold. According to this theorem, any commutative C*-algebra is isomorphic to the algebra of functions over some finite set of points. So

$$\forall \quad A \subseteq B(H) \qquad \exists\, X, \psi \ \mid\ \psi : A \to C(X). \tag{1.19}$$

A is commutative and C is the algebra of functions on X. If the algebra is unital, X is compact.

Now, everything gets more serious when we learn that all closed involutive subalgebras of the algebra of bounded operators on a Hilbert space are C*-algebras. There is therefore a natural connection between, in more technical words, algebra of bounded operators in any Hilbert space with the commutative algebra of complex valued functions on a Hausdorff space which is basically a topological object.

This connection is deep from any aspect. For example, vector fields and differential forms depend directly on the algebra of functions on a manifold. This is how derivations of the algebra elements can be defined and the algebra admits a grading structure.

At this point, the natural question arises that whether more general algebras, which are not necessarily commutative, have anything to do with geometrical spaces? Well, this can be the way one will expand the notions of geometry to beyond ordinary manifolds by using these very powerful algebraic tools.

1.6 Noncommutative Geometry

So far, we have seen that many geometrical information about a manifold M can be obtained from the algebra of functions on it and from the spectrum of the Dirac operator. Also, one can always find an algebra of operators, $\mathcal{A}$, on a Hilbert space, H, which is isomorphic to this algebra of operators. These together form the concept of *spectral triple* shown by $(H, \mathcal{A}, D)$.

One question is that whether the spectral triple has all the geometrical information of a manifold, or in other words, is it possible to build a manifold based on its associated spectral triple. There are manifolds with different metrics on which the Dirac operator has the same spectrum. However, when the algebra is also given, one can recover all the information including the concept of geodesic distance from a spectral triple [9].

Representing a manifold in this form has the privilege that one can generalize this notion to spaces which correspond to algebras other than the algebra of functions. Other algebras might be noncommutative such as algebra of matrices. The only requirements are that the algebra should be associative, possess a unity member, and it should be involutive which means to have a map to itself, *, such that

$$\forall a, b \in \mathcal{A} \quad \& \quad \lambda \in \mathbb{C} \qquad \begin{aligned} &(a^*)^* = a \\ &(ab)^* = b^* a^* \\ &(\lambda a + b)^* = \bar{\lambda} a^* + b^*. \end{aligned} \tag{1.20}$$

this is in harmony with what we saw in the last section for connection between C*-algebras and Hausdorff topological spaces.

So, with this algebra, a Hilbert space which admits a faithful representation of the algebra, and a self-adjoint operator D, the concept of a noncommutative geometrical space can be defined which contains usual geometry as its special case.

As a simple but pedagogically very useful example, consider an algebra which is a direct sum of two algebras of functions on two distinct points, that is $\mathbb{C} \oplus \mathbb{C}$. Components of this algebra act on a Hilbert space which is consisted of the direct sum of vectors of arbitrary finite dimensions on the points. Let us assume two one-dimensional vector spaces for simplicity. The Dirac operator is a Hermitian matrix that we take it for now to be an offdiagonal matrix. The spectral triple is then

$$\begin{aligned} &\mathcal{A} = \mathbb{C} \oplus \mathbb{C}, \\ &\qquad a \in \mathcal{A} \; \Rightarrow \; a = \begin{pmatrix} f_1 & 0 \\ 0 & g_1 \end{pmatrix}, \quad x, y \in \mathbb{C}, \\ &H = H_1 + H_2, \\ &\qquad \psi \in H \; \Rightarrow \; \psi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \\ &D = \begin{pmatrix} 0 & \lambda \\ \bar{\lambda} & 0 \end{pmatrix}, \qquad \lambda \in \mathbb{C}. \end{aligned} \tag{1.21}$$

The geometric space associated with this algebraic information can be thought of to be two distinct points which have no neighborhood and are not connected with any more points. However, as we pointed in the previous section, the notion of distance is now generalized and it is applicable in this case as well. Without writing the simple formula, we just express that the distance between the points is equal to $|\lambda|^{-1}$. Very interestingly, the Dirac operator elements have direct geometrical meanings in the generalized sense. However, λ is yet a number and does not possess transformational properties. What will be even more interesting is to build tensors out of Dirac operator elements as one can do for gravity and for gauge theories by making the partial derivative covariant under relevant transformations. The *spectral action principle* gives dynamic to these fields by proposing that the action is the sum over all the spectrum of the Dirac operator. Thus, the geometrical degrees of freedom will show themselves as physical fields.

As was noted in section 1.2, the concept of p-forms helps here and is a natural choice to generalize. A 1-form is defined as

$$A = \sum_i a_i[D, b_i] \tag{1.22}$$

This 1-form is also called *linear fluctuation* of the Dirac operator. The sum is over any suitable set of elements of the algebra. We need to add these fluctuations to the Dirac operator to make it covariant under transformations of automorphisms of the algebra. This will promote Dirac operator elements to tensors and therefore, generate new fields.

$$D_A = D + A \ .$$

For the above example, fluctuations are

$$A = \sum_{i,j} \begin{pmatrix} 0 & f_j\,(f_i - g_i)\,\lambda \\ -g_j\,(f_i - g_i)\,\bar{\lambda} & 0 \end{pmatrix} \tag{1.23}$$

Because of the sum and because it should respect the self-adjointness of the Dirac operator, one gets

$$A = \begin{pmatrix} 0 & \phi\,\lambda \\ \bar{\phi}\,\bar{\lambda} & 0 \end{pmatrix} \tag{1.24}$$

This is a 1-form now. In this simple example transformations are trivial due to the commutativity. The covariant Dirac operator is equal to

$$D_A = \begin{pmatrix} 0 & (1+\phi)\,\lambda \\ (1+\bar{\phi})\,\bar{\lambda} & 0 \end{pmatrix} \tag{1.25}$$

Since ϕ is a generic complex number, effectively, Dirac operator is not changed. If Dirac operator here has diagonal elements, they would not appear in the commutator and, therefore, would not contribute in 1-forms.

Next step is to think how to produce fields in this way. To make 1-forms spacetime functions, one can think of direct product between this discrete space with the Riemannian spacetime manifold. So the concepts of the product space and its spectral triple are needed. First, let us introduce two more useful operators.

Reality operator, J, and *grading operator*, γ, are defined as operators on the Hilbert space with the following properties

$$\begin{aligned}
&\gamma \ \text{ is a unitarry operator and} \\
&\qquad \gamma^2 = 1, \quad a \in \mathcal{A} \quad [\gamma, a] = 0, \\
&J \ \text{ is an anti-unitary operator and} \\
&\qquad J^2 = 1, \\
&DJ = JD, \quad J\gamma = -\gamma J, \quad D\gamma = -\gamma D.
\end{aligned} \tag{1.26}$$

We will talk about the physics behind these creatures in the next chapter. Let us assume for now two spaces with spectral triples which are decorated by these additional operators

$$F_1 = (\mathcal{A}_1, H_1, D_1, J_1, \gamma_1), \qquad F_2 = (\mathcal{A}_2, H_2, D_2, J_2, \gamma_2). \tag{1.27}$$

The characteristics of the product space, $F_1 \times F_2$ is

$$\begin{aligned}
\mathcal{A} &= \mathcal{A}_1 \otimes \mathcal{A}_2, \\
H &= H_1 \otimes H_2, \\
J &= J_1 \otimes J_2, \\
\gamma &= \gamma_1 \otimes \gamma_2, \\
D &= D_1 \otimes \gamma_2 + \gamma_1 \otimes D_2.
\end{aligned} \tag{1.28}$$

Now, there are enough tools in hand to see how new fields emerge in physics from geometrical features of noncommutative spaces. As an example, the direct product of the two spectral triples introduced so far is illustrative. Let

$(C^\infty(M), L^2(M,S), \gamma^\mu\partial_\mu, \gamma^5)$ be the spectral triple of four dimensional spacetime as a Riemannian manifold. L^2 is the Hilbert space of square integrable spinors on the manifold. Let $(\mathcal{A}, H, D)$ be the triple in (1.21). The product space is then characterized by

$$\mathcal{A} = C^\infty(M) \otimes (\mathbb{C} \oplus \mathbb{C}), \tag{1.29}$$

$$a \in \mathcal{A} \;\Rightarrow\; a = \begin{pmatrix} f_1(x) & 0 \\ 0 & g_1(x) \end{pmatrix}, \quad x \in M,$$

$$H = L^2 \otimes (H_1 + H_2),$$

$$\psi \in H \;\Rightarrow\; \psi = \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix},$$

$$D = \begin{pmatrix} \gamma^\mu\partial_\mu & \gamma^5\lambda \\ \gamma^5\bar{\lambda} & \gamma^\mu\partial_\mu \end{pmatrix}, \qquad \lambda \in \mathbb{C}.$$

Using (1.22), two kinds of 1-forms are obtained. With spacetime indices and with the algebraic indices

$$D = \begin{pmatrix} \gamma^\mu A_\mu & \gamma^5\phi(x) \\ \gamma^5\bar{\phi}(x) & \gamma^\mu B_\mu \end{pmatrix}. \tag{1.30}$$

In the next chapter, we see how in a more realistic model these are the origin of gauge fields and scalar fields. For now, since the discrete space algebra is commutative, the scalars do not get tensorial indices and are in the trivial representation of $U(1) \times U(1)$ group which is the unitary symmetry group of the algebra. For more complicated models, the indices of scalars in the covariant Dirac operator indicate their representations.

As it comes to physics, the original idea of Alain Connes and Ali Chamseddine proposed in [6] is to not only produce scalar fields and gauge fields in this way, but also to give them dynamic and bring them to life by assuming their action to be the spectrum of the covariant Dirac operator. By now, we already know this idea works for gravity. The triple is what was mentioned at the beginning of this section and the action is written based on the spectrum of Dirac operator (1.13). We will introduce the settings for that project in the next chapter and try to find some of the phenomenological implications of the scalar sector obtained in this approach.

In this point of view, the number of fermions and their representations are due to the discrete structure of the hyperspace that they are living in, in the same exact way that their spinorial structure is due to the transformational properties they have by living on a Riemannian manifold. So it is the geometry which dictates all without any physics yet. Physics comes forth when the spectral action principle requires the least for the total spectrum of the Dirac operator. This way, all the different bosonic fields are promoted from geometrical features to physical fields. This is the very straightforward path to find roots for the intimate relationship between geometry and the gauge fields. The validity of this picture depends a great deal on the predictions it has from the scalar sector of the emerging effective field theory. Another advantage of this approach is that it implies the unification of couplings without bringing the gauge groups under one single group. This also has phenomenological implications which will be discussed more in the coming chapters. One extra benefit of this is to not face with tight proton decay restrictions on the intermediate scale and breaking energies. We study this matter more closely in the last chapter.

For more on the topics discussed here refer to [10, 11, 12, 13, 14, 5, 9, 15, 16, 17, 18, 19]

Chapter 2

Spectral Singlet Extended Standard Model

2.1 Introduction

The noncommutative geometry approach to the structure of spacetime has been able to produce the standard model coupled with gravity, almost uniquely, by using very weak constraints [6]. In this model, spacetime is taken to consist of a continuous $4D$ Riemannian manifold tensored with a finite noncommutative space. One of the defining ingredients of this hyperspace is an operator which coincides with the Dirac operator in the commutative $4D$ part of the space and can be considered as the generalized noncommutative version of it. This operator has all the useful geometrical information of the space, and just like the Dirac operator in the standard model, its structure reveals the fermionic content of the model. Moreover, in the noncommutative geometry, other information like gauge field interactions and the scalar sector are embedded in the spectrum of this operator. In the work of Chamseddine and Connes in [6] and the papers that followed, it was shown that the simplest possible noncommutative structure has the correct fermionic content and also leads to the gauge symmetry of the standard model.

The Lagrangian of this model comes from the most general form of the Dirac operator consistent with axioms of noncommutative geometry plus an additional constraint called the first order condition. This Lagrangian possesses three important features distinguishing it from the minimal standard model. First, the couplings of the model are not totally arbitrary and there are relations between them at the unification scale. These relations are consistent with grand unified theories such as $SU(5)$ unified theory. Second, in addition to the Higgs, there is a singlet scalar field present in the spectral action. It is shown that this field can help the situation with the low Higgs mass which is not otherwise consistent with the unification of spectral action in high energies [20]. We will see in this letter that the results improve if the extra singlet scalar field is taken to be complex. It is also seen that such an extra scalar field can be responsible for dark matter particle [21, 22]. Finally, right-handed neutrino appears into the picture automatically as well as its Yukawa interaction. These terms are needed to give a small mass to the left-handed neutrino by seesaw mechanism and usually are added to the standard model by hand.

In [20], the singlet scalar field was assumed to be real. Then using 1-loop renormalization group equations, it was shown that the model with the singlet can accommodate a Higgs field with the mass of order $125 GeV$. In fact, the reality condition on the singlet field is not necessary and we assume the singlet to be a complex field in this work. Our consideration shows the model in its most general form is consistent with the current experimental values of the Higgs and top quark masses. Furthermore, we use 2-loop renormalization group equations to compare the following cases: when the added singlet is a complex field, when it is real, and the pure standard model with neutrino mixing. We show that while running RG equations from unification scale toward current experimental energies, the model with added complex singlet behaves slightly better than the other two cases. Yet, like the standard model itself, one can only attain the experimentally observed gauge couplings at low energies within some percent of accuracy. This agrees with the separations of the standard model gauge couplings at the unification scale when we start from experimental values and run them upward. Subsequently we also discuss the effects of three-loop corrections.

Since the discovery of the Higgs particle in 2012, researchers started to study the instability problem of the standard model effective potential more seriously (For example [23]). Although this instability cannot make the standard model unreliable, even at high energies, because of the long lifetime of the tunneling process, it still could have dramatic consequences during the inflation period [24, 25]. It is interesting to check the effect of any modification of the model on this situation. Therefore the vacuum stability of the models coming from noncommutative geometry will be addressed and compared with the pure standard model.

We will show in this chapter that even though a few extra terms are added to the RG equations due to the complex singlet field, yet their effect on the negativity of the Higgs self-coupling at high energies can be substantial. The reason we cannot predict what exactly happens for the coupling is that the experimentally unknown right-handed neutrino Yukawa coupling contributes in the RG equations as well. This coupling also plays a role in determining the Higgs and top quark masses at low energies. What we can do is to follow its effect by following RG equations down and looking at the particle masses. The proper value of right-handed neutrino Yukawa coupling - turns out to be between 0.411 and 0.455 at unification scale as we will see in section 2.3. The resulting value for this coupling at Z-boson mass region is also between 0.517 and 0.530, while Yukawa coupling of the top quark is about 0.995. Besides, the values of scalar sector couplings are derivable in this scale from RG equations. We argue that in this acceptable range of the couplings, although vacuum instability is not cured, the situation is improved by the presence of the complex scalar field. We use two-loop equations and near to the leading order three-loop equations to assess the loop correction effects in presence of a complex or real singlet field.

We stress that the above results are not merely derivable from the standard model plus a complex singlet. The reason is that in our considerations, we use the initial conditions predicted by the spectral action approach [26]. Moreover here a neutrino coupling is present in RG equations and contributes to the values of particle masses. The form of potential is also restricted and is different from extended standard model cases with complex singlet described in the literature. In our case, the results for stability are slightly better (e.g. compare with [22, 27, 28, 29]).

2.2 The model

After years of investigations by mathematicians to expand the geometrical notions to the spaces with fewer constraints than metric spaces, which led to many developments in various areas of mathematics, finally Alain Connes was able in 1980 to find an applicable set of axioms and definitions to generalize geometrical concepts to a much broader range of spaces [30]. He also used the new geometry to define a noncommutative torus and studied its geometrical properties. Later in 1996, Ali Chamseddine and Alain Connes found an application of this new geometry in physics [6]. They assumed the spacetime is a direct product of $4D$ Riemannian manifold with a noncommutative space. They also introduced the spectral action which is based on the spectrum of the Dirac operator and were able to show that the standard model arises naturally and almost uniquely from these assumptions.

Geometrical structures in noncommutative geometry are defined based on three concepts; a Hilbert space, an algebra of a given set of operators with its faithful representation on the Hilbert space, and a special operator called *Dirac Operator*. These are shown to be enough to define a rich geometry and can yield features of Riemannian manifolds in expected limits[1]. We have therefore what is called spectral triple which is shown by

$$(\mathcal{A}, \mathcal{H}, D).$$

As an example, for a $4D$ spinorial spacetime one can consider Dirac operator to be the familiar 4 by 4 matrix $D = i\gamma^\mu \partial_\mu$. The Hilbert space is then the space of 1 by 4 spinors. In this case $\mathcal{A}$ is the algebra of 4 by 4 complex matrices which is a noncommutative algebra. Now one way to see the geometrical invariants such as curvature is to look at the spectrum of the Dirac operator. One can for example use heat kernel method to asymptotically expand the trace of Dirac operator [14, 5]. This expansion is controlled by a scale called Λ. Doing so, the first term of the expansion turns out to be the cosmological constant and the second term gives the total curvature of spacetime. Higher orders are higher powers of the geometrical invariants such as curvature and Ricci tensor.

Unlike Kaluza-Klein type theories which enlarge geometry by assuming extra dimensions, here the added structure is a finite noncommutative space which possesses no spacetime dimensions. In early models, finding noncommutative structures leading to the standard model was the matter of trial and error. Eventually, in [6], the authors discovered that a noncommutative space with the algebra

$$\mathcal{A}_{\mathcal{F}} = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}) \tag{2.1}$$

is able to produce the standard model when it is tensored with the $4D$ spacetime. $M_3(\mathbb{C})$ is the algebra of 3 by 3 matrices on complex numbers, $\mathbb{H}$ is the algebra

[1]For precise definitions refer to [9, 31].

of quaternions which are represented using 2 by 2 matrices, and $\mathbb{C}$ is the algebra of complex numbers. Later on, the same authors showed that the classification of finite spaces consistent with the noncommutative geometry requirements leads almost uniquely to the same algebra [32]. They also observed that by letting the Dirac operator to have nonlinear fluctuations, the consistent algebra is

$$\mathcal{A}_{\mathcal{F}} = \mathbb{H} \oplus \mathbb{H} \oplus M_4\left(\mathbb{C}\right), \tag{2.2}$$

which leads to the Pati-Salam unified model [33]. As an interesting breakthrough in 2014 it was discovered in [34] that this algebra is dictated by a generalized version of Heisenberg commutation relations. In this letter, we consider the model based on the algebra (2.1), which is a special case of (2.2) that happens when the perturbations of Dirac operator is required to be linear. This is called *first order condition* and we assume its validity in the current work.

Members of $\mathcal{A}_{\mathcal{F}}$ are $2 \times 2 \times 4 = 16$ by 16 matrices and members of the Hilbert space consist of 16 spinors, which means they possess 64 elements. Algebra of the whole space can be written as direct product of $\mathcal{A}_{\mathcal{F}}$ with the algebra of functions on the $4D$ spin manifold. The latter is the commutative algebra of smooth functions on the spin manifold

$$\mathcal{A} = C^{\infty}\left(M\right) \otimes \left(\mathbb{C} \oplus \mathbb{H} \oplus M_3\left(\mathbb{C}\right)\right). \tag{2.3}$$

We have then 16 spinors and it turns out later that they have exactly the same interactions as fermions in one generation including four right and left-handed leptons and 12 colored right and left-handed quarks. Next, one can introduce the chirality operator called γ to enrich the algebra by grading mechanism and add antiparticles to the Hilbert space. Therefore members of the Hilbert space are now 1 by 128 matrices. Next, we can triple this space by hand to take into account the three generations of fermions. Dirac operator of the whole space is then a 384 by

384 matrix which acts on the Hilbert space and is defined as the tensorial sum of the operators on different parts:

$$D = D_M \otimes 1_{96\times 96} + \gamma_5 \otimes D_F. \tag{2.4}$$

The particle content of the model is therefore coming from the above settings of the noncommutative geometry. Then Dirac action provides dynamic to this fermionic part of the model[2]. The vector and scalar parts of the model are described by the spectral action which is the trace of Dirac operator and depends only on the sum of its eigenvalues. The action is:

$$S = \mathrm{Tr}\,(f(D/\Lambda)) + \langle\psi, D\psi\rangle. \tag{2.5}$$

Lambda is an energy cutoff needed to make dimensionless term out of D. Function f is a source to generate physical constants such as G_N and is required to be positive and even.

To start, first we need to make the fermionic part covariant under inner automorphisms of the Hilbert space by adding inner fluctuations of the Dirac operator under such automorphisms. The fluctuations associated with the noncommutative space are responsible for the existence of gauge fields and the Higgs. Inner fluctuations associated with the automorphisms of the continuous $4D$ manifold form Riemannian aspects of the curved $4D$ spacetime. The Dirac action then contains all the fermionic interactions, just like the standard model when all the vector fields are added to the Dirac operator in form of connections. On top of that, here we get the Yukawa terms and the Higgs as parts of the spectrum of Dirac operator.

Next, one can use heat kernel asymptotic expansion to compute the trace. The existence of Λ in the action is crucial so one can rely on the expansion[3]. The trace

[2]To be able to introduce an inner product and define this part of action consistently, another operator called reality operator is needed. For exact definitions refer to [32]

[3]For the special case of Robertson-Walker metric it is shown that the expansion is valid up to energies close to the Planck order [35]

is then reduced to a series with coefficients known as *Seeley de Witt coefficients* [4]:

$$\mathrm{Tr}\left(f\left(D/\Lambda\right)\right) = \mathrm{Tr}\left(F\left(\left(D/\Lambda\right)^2\right)\right) = \sum_{n=0}^{\infty} \Lambda^{4-n} F_{4-n} a_n. \tag{2.6}$$

The function f is supposed to be positive. The odd terms in the expansion vanish for manifolds without boundaries. It is equivalent to saying the square of the Dirac operator has important geometrical information in its spectrum and use a function F such that $F(\alpha^2) = f(\alpha)$. The coefficients a_n depend only on the geometrical invariants such as curvature and therefore reveal the geometrical information embedded in the Dirac operator up to the order defined by powers of Λ. Taylor coefficients F_{4-n} are the spectral function derivatives at zero for $4 - n < 0$ and momenta of spectral function for $4 - n > 0$,

$$F_0 = F(0), \qquad F_2 = \int_0^\infty F(u)du, \qquad F_4 = \int_0^\infty F(u)udu. \tag{2.7}$$

These coefficients along with Yukawa couplings make the physical constants. For example the first one, F_4, is the source of Hubble constant and the third one, F_0, appears in the Higgs kinetic term. Normalization of this term causes F_0 to show up in the mass term of fermions as well as all the coupling constants which is the root of unification in this model [31]. Therefore we trust the model on high energies where the approximation of expansion 2.6 is expected to work well. The unification of the couplings will be then what is expected from GUT theories. Writing the renormalization group equations and running them down to experimental energies is also feasible.

The sum in (2.6) is over even numbers, therefore the fourth term is suppressed by Λ^2. We expect Λ to be right below plank energy which is much higher than any mass in the model. Therefore it is logical to assume higher terms are irrelevant for our purposes. In addition, F is expected to be a cutoff function which can control expansion of the trace.

The Dirac operator for the noncommutative space defined by algebra in (2.3) is [[31]]:

$$D_{AB} = \gamma^{\mu} \otimes \tag{2.8}$$

$$\begin{pmatrix} D_\mu & & & & & \\ & D_\mu + ig_1 B_\mu & & & 0 & \\ & & (D_\mu + \frac{i}{2} g_1 B_\mu) I_{2\times 2} - \frac{i}{2} g_2 W_\mu^i \sigma_i & & & \\ & & & (D_\mu - \frac{2i}{3} g_1 B_\mu) I_{3\times 3} - \frac{i}{2} g_3 V_\mu^a \lambda_a & & \\ & 0 & & & (D_\mu + \frac{i}{3} g_1 B_\mu) I_{3\times 3} - \frac{i}{2} g_3 V_\mu^a \lambda_a & \\ & & & & & (D_\mu - \frac{i}{6} g_1 B_\mu) I_{6\times 6} - \frac{i}{2} g_3 V_\mu^a \lambda_a I_{2\times 2} - \frac{i}{2} g_2 W_\mu^i \sigma_i I_{3\times 3} \end{pmatrix} \otimes 1_3$$

$$+ \gamma^5 \otimes$$

$$\begin{pmatrix} 0_3 & 0 & (\epsilon^{ab} H_b \otimes k^{*\nu})_{6\times 3} & 0 & 0 & 0 \\ 0 & 0_3 & (\bar{H}^a \otimes k^{*e})_{6\times 3} & 0 & 0 & 0 \\ (\epsilon_{ab} \bar{H}^b \otimes k^{\nu})_{3\times 6} & (H_a \otimes k^e)_{3\times 6} & 0_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0_9 & 0 & (\epsilon^{ab} H_b \delta_i^j \otimes k^{*u})_{18\times 9} \\ 0 & 0 & 0 & 0 & 0_9 & (\bar{H}^a \delta_i^j \otimes k^{*d})_{18\times 9} \\ 0 & 0 & 0 & (\epsilon_{ab} \bar{H}^b \delta_j^i \otimes k^u)_{9\times 18} & (H_a \delta_j^i \otimes k^d)_{9\times 18} & 0_{18} \end{pmatrix}$$

The forms of these matrices come from very few axioms, listed in [32], and are not arbitrary. The zeros appear automatically and are necessary to exclude interactions not experimentally observed. Nonzero components are named after their coincidences with the fields and constants in the standard model. The first matrix is block diagonal and contains all the vector bosons. The second matrix contains Higgs terms. D is a 192 by 192 matrix and acts on all 48 known fermions.

The Fermionic part at 2.5 justifies chosen names of fields and their coefficients as for nonzero components of D_{AB}. The first part of D contains gauge fields as it does in the standard model; B, V, and W stand for the $U(1)$, $SU(2)$, and $SU(3)$ gauge fields respectively [4]. The second part is responsible for all the other fermion-fermion interactions which justifies the choice of names, Yukawa couplings k^i and Higgs scalar fields $H^{a,b}$. In the trace part of action 2.5 on the other hand, there is no fermionic field and the spectrum generates bosonic and scalar potentials which have the exact same form of standard model potential terms. In equation 2.11 and what follows, we will study the scalar sector of the action.

To include antiparticles, we can double the algebra, and consequently the Hilbert space, by assuming the existence of a reality operator J for the geometry as an axiom. This operator[5] causes all the other operators to be the direct sum of two dependent parts which can be exchanged by the act of J.

The Dirac operator is however not simply the direct sum of fermionic and anti-fermionic parts. It is shown in [31] that only one off-diagonal element can be nonzero. This element therefore indicates a singlet that gives mass to a right-handed fermion which is coinciding with a right-handed neutrino in the standard model. Dirac operator of the whole space is therefore a 384 by 384 matrix as we noted before

[4] what we see here is a special case of a general theme, starting with a matrix algebra in the noncommutative geometry, the spectral action principle leads to a counterpart gauge theory.

[5] It is evident that J has the role here as the charge conjugate has in the standard model.

$$D = \begin{pmatrix} D_{AB} & D_{AB'} \\ D_{A'B} & D_{A'B'} \end{pmatrix}, \qquad D_{A'B} = \overline{D}_{AB'}, \quad D_{A'B'} = \overline{D}_{AB} \tag{2.9}$$

$$D_{AB'} = \begin{pmatrix} \sigma & 0.. \\ 0.. & 0.. \end{pmatrix}$$

Having the above operator, both parts of the action (2.5) are well defined. The fermionic part of action is containing fermion-gauge and fermion-Higgs interactions, plus terms coming from off-diagonal elements of D, which presents scalar-fermionic interactions absent in the standard model. Since $D_{AB'}$ has only one nonzero element, only one of the fermions is involved with this new sigma-interaction and it is natural to call it right-handed neutrino [20].

$$\langle \psi, D\psi \rangle = c\,\overline{\nu}_R \nu_R + C.C. + \text{fermionic and Yukawa interactions} \tag{2.10}$$

Physically important geometrical information is also derivable from this operator and we need only to find coefficients introduced in (2.6) to identify the bosonic part of the action (2.5). Calculations up to first three terms yield Einstein-Hilbert action along with Gauss-Bonnet terms, plus Higgs potential, σ self-interaction, and $\sigma - H$ interaction. After proper redefinition of the fields, the scalar potential sector of (2.5) is [31]:

$$V = \frac{1}{2}m_h^2 H^2 + \frac{1}{2}m_\sigma^2 |\sigma|^2 + \frac{1}{4}\lambda_\sigma |\sigma|^4 + \frac{1}{4}\lambda_h H^4 + \frac{1}{2}\lambda_{h\sigma}|\sigma|^2 H^2. \tag{2.11}$$

We take σ to be a complex singlet with two degrees of freedom. Although H is a complex doublet with four degrees of freedom, the gauge symmetry allows us to gauge away three of them. The potential has local minimum which occurs when

$$\lambda_\sigma |\sigma|^2 + \lambda_{h\sigma} H^2 + m_\sigma^2 = 0, \quad \lambda_h H^2 + \lambda_{h\sigma}|\sigma|^2 + m_H^2 = 0 \tag{2.12}$$

and proposes the symmetry breaking, which we formulate with the following choices of the vacuum expectation values:

$$H = \begin{pmatrix} 0 \\ h+v \end{pmatrix}, \quad v = \langle h \rangle_0 \tag{2.13}$$

$$\sigma = w + \sigma_1 + i\sigma_2, \quad w = \langle \sigma_1 \rangle_0 .$$

It is obvious from the above setting that the three scalars now mix and due to the $\sigma_1 - \sigma_2$ symmetry, one massless pseudo-Goldstone boson is expected to appear. This massless field does not remain massless at higher loop orders and can be a dark matter candidate [28]. After substituting (2.13) into potential (2.11), and diagonalizing the mass matrix of the square terms, the other two scalar masses turn out to be

$$m_\pm^2 = \left(v^2\lambda_h + \frac{w^2}{4}\lambda_\sigma\right)\left(1 \pm \left(1 - \frac{v^2w^2\lambda_h\lambda_\sigma - v^2w^2(\lambda_{h\sigma})^2}{(v^2\lambda_h + \frac{w^2}{4}\lambda_\sigma)^2}\right)^{\frac{1}{2}}\right). \tag{2.14}$$

It is believed that a highly massive right-handed neutrino can be fitted in the standard model to explain neutrino oscillations. Such a neutral particle is only able to gain mass from a singlet scalar field. In the model described above this mechanism is appearing naturally. The price of this treatment is of course to have a new scalar which is supposed to be highly massive. Here we have another massless field added to the picture which appears since σ is a complex field. We therefore suppose w to be much greater than v and we get

$$M = w\sqrt{\frac{\lambda_\sigma}{2}}, \qquad m_h = v\sqrt{2\lambda_h}\sqrt{1 - \frac{\lambda_{h\sigma}^2}{\lambda_h\lambda_\sigma}}. \tag{2.15}$$

The smaller one is responsible for the Higgs mass and is modified by the factor of $\sqrt{1 - \frac{\lambda_{h\sigma}^2}{\lambda_h\lambda_\sigma}}$ due to the presence of the scalar field. It is remarkable that non-commutative geometry not only predicts the singlet field and its potential terms, but also relates, in the unification scale, the scalar couplings to other parameters such as Yukawa couplings and the unified gauge coupling [31]. Having those relations, we will start from unification and vary all the free parameters to probe the implications of this formula for the Higgs mass.

2.2.1 Running of the renormalization group equations

Having the model described in section 2.2, one can find the effective potential and renormalization group equations in some loop order and run them to explore high energy scales. There are however two free parameters here. The neutrino Yukawa coupling and the Higgs self-coupling. Knowing the Higgs mass now, the value of Higgs self-coupling is determined in the pure standard model as

$$\lambda_h(M_z) = \frac{(125.5)^2}{2(246.2)^2} = 0.1299.$$

In models with extra scalars though, there is a seesaw mechanism which determines the Higgs mass and the value of this coupling is not determined even when the mass is measured.

We used SARAH which is a Mathematica package to derive two-loop RG equations ([36]) for this model and presented the results in appendix 2.7. It is clear from RG equations that the extra field cannot correct the gauge couplings evolutions and therefore is not going to help the couplings to meet in exactly one point (Figure 2.1). That is because the scalar field potential terms are quadratic and their couplings appear only in two-loop corrections of Yukawa couplings evolutions, which themselves enter just in two-loop corrections of the gauge couplings. The latter is due to the Yukawa interaction of the particles with square of the singlet. Figure 2.1 also shows that the added singlet field, no matter is it complex or real, does not cause meaningful changes in evolution of Yukawa couplings. However, two-loop corrections shift them for about ten percent if we follow their evolutions to very different energy scales.

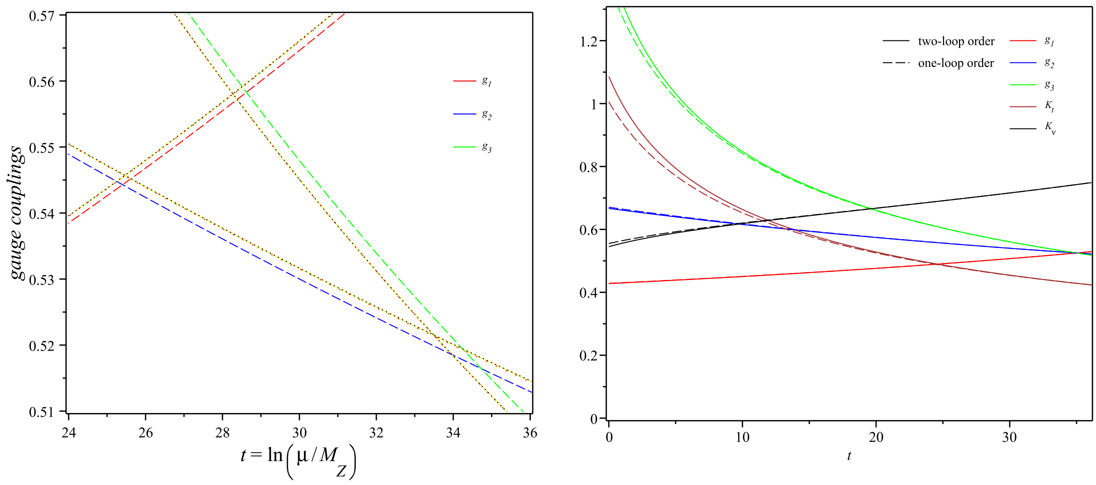


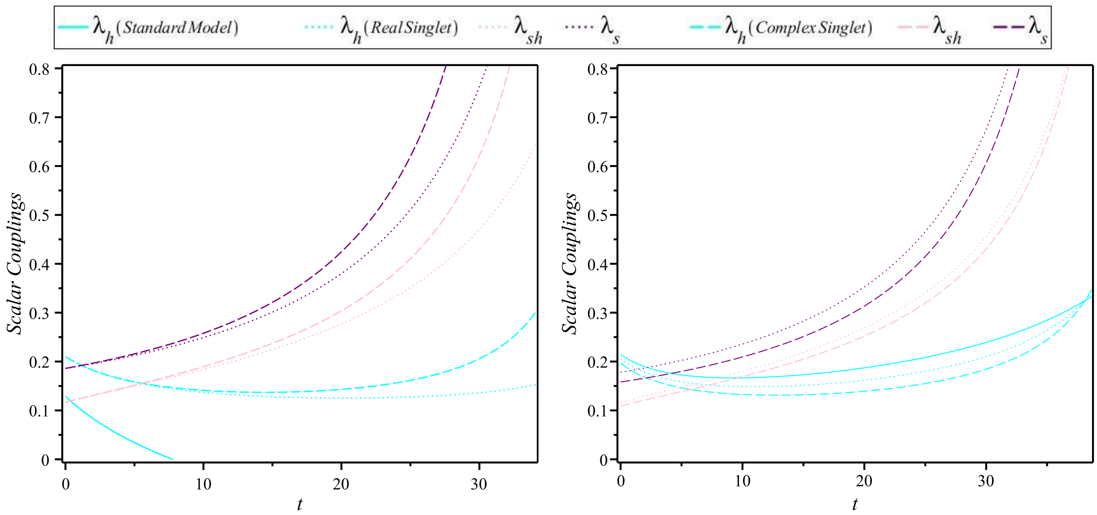
Figure 2.1: The behavior of gauge couplings at the unification scale. Dashed lines are indicating the evolution of standard model couplings up to one-loop corrections. Yellow solid lines show the situation is slightly better when two-loop corrections are also taken into account. The black dotted lines are for RG equations up to three loops for the standard model. The black and yellow lines are so close that their separation cannot be distinguished in this diagram. This difference is from the same order of errors that experimental uncertainties create when we run the equations upward. In all cases, the corrections coming from a real or complex scalar field added to the standard model is negligible. Red dotted lines have two-loop corrections of the complex scalar field, in the model described in section 2.2, and include three-loop corrections of all the other parameters. Yet again it matches with two-loop corrections suggesting that higher orders are not going to make the situation any better. The graph on the left compares one-loop RGEs with two-loop equations for gauge and Yukawa couplings when we start at the same points at high energies and follow them toward experimental values. Again adding a singlet does not create meaningful changes.

Though replacing the real scalar field with a complex field has no remarkable implications on the gauge couplings evolutions, it can cause noticeable consequences for the field couplings as there are new Feynman diagrams between them when we add the imaginary component. Choosing acceptable initial values and running RG equations, including two-loop effects, show that this difference is meaningful. As it is clear from Figure 2.2a, starting from the same points, the couplings behave differently at very high energies. These couplings indicate Higgs mass through the relation (2.15) which can be sensitive to small variations of the couplings.

2.3 Top quark and Higgs masses at low energies

In our approach, we run RG equations downward from the unification scale. The advantage is that the spectral model predicts initial conditions at high energies, and relates all the Yukawa couplings to the unified gauge coupling g. Interestingly, the scalar couplings are also not free parameters at the unification energy, instead they are determined by both g and the ratio of neutrino and top quark Yukawa couplings [26]. We choose the approach of [20] and, for simplicity, define the ratio $n = (\frac{k^\nu}{k^t})^{\frac{1}{2}}$ at the unification scale. n is one of the free parameters of the model which can be fixed, then running this along with other parameters causes predictions for the physical quantities at the experimental arena. As discussed before however, the unification scale itself and the value of gauge couplings at this scale is not predicted. Figure 2.2b shows the evolution of all the parameters in different scenarios. There is about ten percent difference in the values of the couplings at low energies between real and complex models. Since the effects of higher orders of loop corrections are negligible, the difference we see here does worth investigating. Another encouraging fact is that in [20] the effects of the scalar field couplings were shown to be able to save the model after the Higgs small mass discovery.

The other observation which justifies our consideration reveals itself when we compare two-loop and one-loop equations. Whether complex or real singlet is added to the Lagrangian, the scalar couplings get modified for about ten percent at low energies and as noted before this can in principle dramatically modify results of equation (2.15).



(a) (b)

Figure 2.2: Running of the couplings incorporating two-loop corrections, toward unification (a) or from unification (b). Solid lines are for the SM, dots are for when the real scalar singlet is present, and dashed lines are for the case that the singlet field is complex. In case of SM, except for the neutrino Yukawa coupling, the initial values are coming from experiments. In all the cases, the initial values for experimentally unknown couplings are discussed in the next chapter; when we run from unification and look for the best fits.

It is notable that if we use initial values coming from the spectral action, it is not possible to run the minimal standard model from unification scale and find the experimentally acceptable mass of the Higgs particle at low energies. Trivially the reason is that these initial conditions imply unification of the gauge couplings which does not happen for minimal standard model. In [20] however, the authors showed initial conditions and RG equations are consistent with the low Higgs mass for when the added scalar field coming from spectral model is real. Nevertheless, spectral action imposes no restrictions on the singlet field. In this section, we consider both complex and real scalars and use two-loop RG equations to incorporate higher order corrections and asses the importance of loop corrections. We also study prediction of the theory for top quark mass.

Our main result in this section is that the assumption of existence of a new scalar field and the predictions of the spectral model at unification are consistent with the known masses of top quark and Higgs. For simplicity, we neglect lighter particles. As it was noted in section 2.2 however, the neutrino is assumed to play a significant role since it has a Yukawa coupling and its mass comes from a seesaw mechanism. The method is straightforward; We assume the initial conditions predicted in [26]. Then we run the equations supposing n, g, and U are free parameters. It hands us couplings values at low energies. Then it is possible to find the best values for these three parameters by minimizing the errors between the result masses and experimentally known values at low energies. The fact that these errors exist and are more than experimental uncertainties is very important and we will discuss it in the next section.

The other important aspect of the situation is to compare two-loop and one-loop corrections, as we are comparing real scalar and complex scalar fields. Up to one-loop, there is no remarkable change in top quark mass if we replace the real scalar with a complex one. However, the two-loop corrections differentiate top quark mass in these two cases. This differentiation is still one order of magnitude smaller than the current observational uncertainties. The situation is different for the Higgs mass as it depends directly to the scalar couplings (eq. 2.15).

Our considerations show that there is a rather short range for n and g that everything fits together. This happens for a U, unification scale, varying between $2 \times 10^{16} GeV$ and $5 \times 10^{18} GeV$. In figure 2.3 the lines indicate what initial values are acceptable to meet the correct particle masses at low energies. It turns out that for a reasonable g, the correct choices for n and U always exist to fit the Higgs and top quark masses simultaneously in low energies within the experimentally acceptable values.

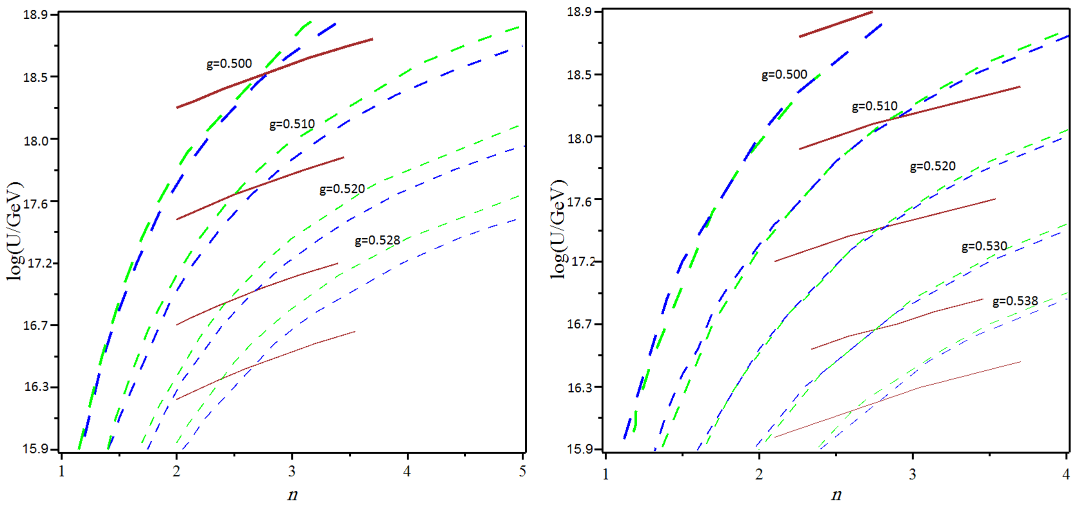


Figure 2.3: Each line shows suitable choices of unification scale and n value at this scale, in order to revive experimental values of particle masses at low energies. Each set of three lines are for a specific g value and are illustrated with a particular thickness. The left hand side diagram incorporates two-loop corrections while the diagram on the right has only one-loop corrections. It can be inferred from diagrams that within a reasonable range of g, the lines associated with top quark, solid brown lines, and Higgs, dashed lines, always have a collision point. Therefore suitable n and U can be always found to assure the low energy values for the Higgs and top quark masses. This is true for both real and complex cases which are distinguished by blue and green lines respectively.

To illustrate even more, we show possible choices for g and n at unification energy in figure 2.4. The colored strips in two diagrams show all the choices which lead to retrieving particle masses at low energies, incorporating one-loop or two-loop corrections. As we noted before however, the correct choice of unification scale is depending on g and n. To give some examples, the small window of correct choices of g and n for three different unification energies are indicated by lighter colors on the strips.

Up to two loop corrections, the suitable n is obtained to be around 2.7 for the real scalar and around 2.5 for the complex scalar case which means that at the unification scale, Yukawa coupling of neutrino is around 6 times bigger than the top quark coupling.

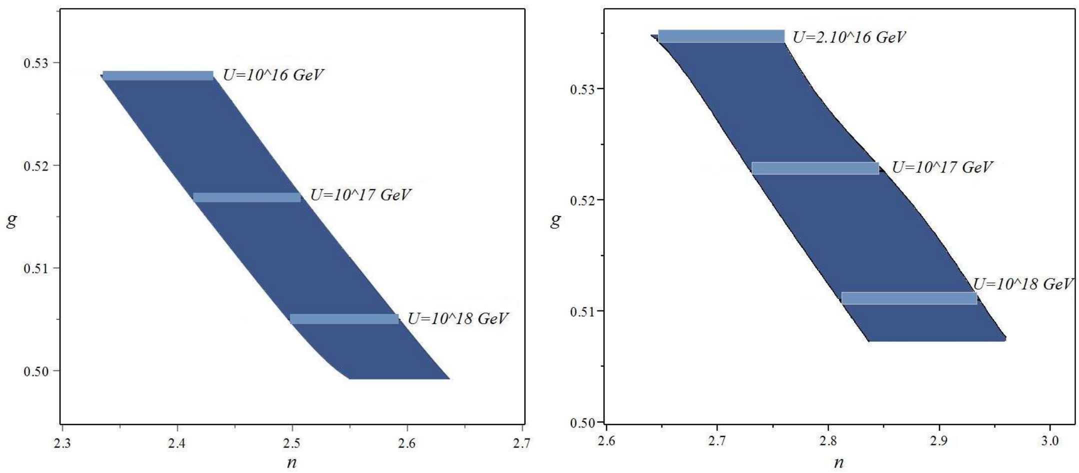


Figure 2.4: At any unification scale, there is a small window of choices for unified gauge coupling, g, and the root of neutrino and top quark Yukawa couplings ratio, n, which lead to consistent low energy particle masses with experimental values. Two-loop corrections, left diagram, make the choices a little more restricted.

2.3.1 comparing the complex and the real cases

We saw that for both scenarios (complex or real singlets), it is possible to find acceptable initial conditions. On the other hand, again in both cases, gauge values deviations at low energies do not fit within the experimental uncertainties. Yet, the situation is slightly better in the complex case for g_3 and g_2. For any g at unification, U is about 0.2, and n is about 0.28 higher in the real case compared with the complex one.

For the standard model alone, best quantities are: $g \sim 0.49$ and $u \sim 39$ When the scalar (complex or real) is added, up to one-loop, $g \sim 0.52$ and $u \sim 35$ and up to two-loop, $g \sim 0.53$ and $u \sim 32$ end to the best results.

2.4 Implications on the vacuum instability

In the standard model, the observed masses of Higgs and top quark imply effective potential of the Higgs field to become unstable at high energies. This can be seen, in the tree level, by the fact that the Higgs self-coupling changes its sign at some energy scale below the unification. For the standard model itself, one can use the renormalization group equations up to some order and find the point at which λ_h changes its sign. It turns out that this happens at the energy scale of order $10^6 GeV$ [6] which is much smaller than the unification scale. Figure 2.5 shows two-loop corrections have an effective role to make the situation better while three-loop corrections are too small to have any significance. Thus, we do not expect higher order corrections to resolve this issue.

With an additional scalar field, it is interesting to see what happens for the effective potential. There are two new couplings associated with the scalar quadratic term and its interaction with the Higgs in the model described earlier. These two couplings along with the Higgs self-coupling are only constrained by the masses of the Higgs and the supposedly heavily massive singlet. Therefore there are not enough known initial conditions and one cannot run the renormalization group equations from low energies. It is however useful to investigate whether this additional field could in principle modify the equations as much as needed in order to cure instability. A straightforward investigation shows that the addition of a complex field could in principle cure the equations (Figure 2.6a). As noted before,

[6] To find this result, in addition of all the known couplings of SM, we take into account the Yukawa of neutrino which is around 0.5. It shifts the instability to lower energies, however later when we add the singlet and all the parameters of the model, the instability goes to much higher energies.

it is especially important due to the fact that higher loop corrections are not being expected to save the potential.

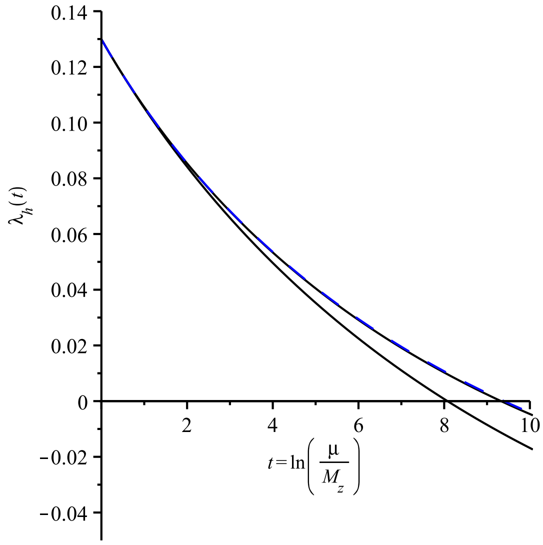
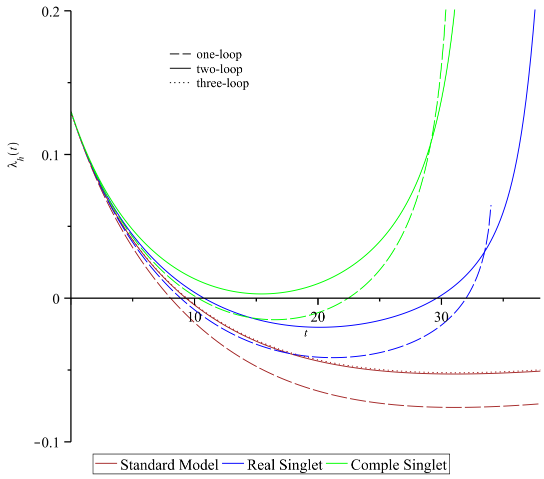
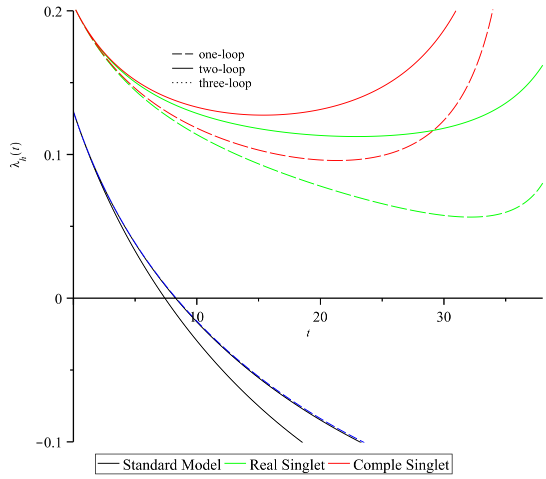


Figure 2.5: After the Higgs discovery, all of the initial values are known for the standard model parameters and one can follow the evolution of Higgs self-coupling. The lower line has only one-loop corrections. The line shifts to the right when two-loop effects are added to RG equations, and the tunneling time increases consequently. The blue dashed line includes three-loop corrections and suggests that going to higher orders will not improve the situation.

As we saw in the previous sections, there are a number of predictions at high energy scales in spectral approach which suggest to start from the unification and run RG equations downward. Doing this gives ideas about the acceptable range of values for couplings; particularly this is useful for the extra couplings which we have no clue about their magnitudes as they are not constrained with the current experimental data. One result is that the Higgs self-coupling is stronger than in the pure standard model, and this pushes the instability of the effective potential to higher energies. It does not affect the Higgs mass because of the seesaw mechanism between Higgs and new scalar. Figure 2.6b illustrates what happens when we use such initial conditions. All the couplings in the potential are now positive all along the way up to unification scale and the potential is expected to be stable. In this respect both real and complex scalar models behave desirably.



(a) The initial values used to draw this diagram are not realistic. However it shows that addition of new fields can in principle have effects more than higher loop corrections. The errors of these lines at high energies due to the experimental uncertainties of the initial values are less than ten percent of loop effects.



(b) Comparison between the behavior of Higgs self-coupling in different scenarios. In the scalar extended standard model, the coupling is not determined with Higgs mass and could have a greater initial value which might save the potential from being instable.

Figure 2.6

2.5 Conclusion

The noncommutative model introduced in section 2.2 adds some familiar extra features to the standard model, for example a new singlet field with quadratic potential, higher powers of geometrical invariants, and prediction of gauge couplings unification at high energies. Our considerations in this paper show that starting from the unification point predicted by the theory, it is possible to revive both top quark and Higgs masses. We however witnessed that there is a deviation for the gauge coupling values at experimental arena which is of the same order of deviation of the gauge couplings in the standard model at unification scale. Comparing these errors we conclude that the complex singlet field makes the theory slightly better than the pure standard model or when a real scalar is added; however, the full treatment is not possible.

Comparing the results of two-loop corrections and near to leading order, for three-loop, shows that there is no hope for loop corrections to contribute in a significant way. We believe the root of all of such inconsistencies goes back to the issue of gauge couplings not meeting at one point and therefore lack of a true unification. Equivalently in noncommutative geometry approach there is unification, but the price in the simple version that we considered here was that one could not fully revive the gauge couplings at low energies. Yet, the little change toward better results with this minimal change in the settings of the standard model might urge us to investigate other more generalized models derived from noncommutative geometry principles.

The spectral action approach coming from the noncommutative geometry point of view, however, does not uniquely lead to the model we considered here. Further investigations showed in 2014 that imposing generalized versions of Heisenberg uncertainty relations leads to Pati-Salam model as the most general possible outcome of this approach [33]. The model we considered here is the simplest special case of that general theory. The Pati-Salam model has a rich content of beyond SM fields that might help the situation and will be the subject of our further investigations.

2.6 Further Discussions

Despite the experimental success of the standard model, there are some fundamental questions left unanswered. For example, there are no compelling reasons for the choice of gauge groups and representations of the fermions. In addition, the origin of Higgs field and its particular potential is not understood which also leads to ambiguity in the Yukawa sector.

Along with other benefits such as some predictions for the unification of gauge couplings, noncommutative geometry is able to provide concrete geometrical understanding for the Higgs and gauge fields by generalizing the geometrical concepts to noncommutative spaces and assuming a richer structure for spacetime ([6],[31]). It turns out that the Higgs and the gauge fields are rooted in the fluctuations of the generalized Dirac operator. This operator has a part associated with the $4D$ spacetime and another part associated with the additional finite noncommutative structure. Potentials and interactions of gauge fields, as well as gravitational potential and its minimal coupling with matter, are parts of geometrical invariants of the space and appear in the spectrum of Dirac operator. Gauge transformations have also geometrical origins and are rooted in inner automorphisms of the finite noncommutative structure. All in all, this theory provides an elegant platform for a *geometrical unification* of all the known forces.

Having established that noncommutative geometry approach is promising, the important task is to find some new phenomena with experimental implications which could be examined with the current experimental facilities. This is a tough test for any unified model which is trying to replace the standard model. The reason is the glorious experimental success of the standard model. It is also extremely difficult to go beyond the present experimental energy scales. In this thesis we study two available models based on noncommutative geometry.

In this chapter, we saw that the spectral singlet extended standard model has reasonable implications. Yet the theory lacks complexity to provide unification consistent with the experimental values achieved for gauge couplings at low energies. Yet, there is no ambiguity for any maneuver at this level except to take the field to be complex and to go to higher loops. These, of course, are not expected to solve the problem fully.

The reason behind all of these difficulties might be that the vast desert between the current experimental scope and unification scale might not be as empty as predicted by the standard model. The platform of noncommutative geometry to go beyond the standard model is also richer than the model described in this chapter. It is shown that imposing few axioms of noncommutative geometry along with a generalized form of Dirac uncertainty principles leads uniquely to Pati-Salam model [33]. We try to investigate this model in the next chapter to find some phenomenological ground in it.

The feature of the models that we study is that the form of the scalar sector potential is not ambiguous and derived from the spectral action principle. Parameters of the models are also not totally independent and there are relations between them. All of this, along with the fact that simpler outcomes of noncommutative geometry, like the singlet model described above, have been revealing some good results, convince us to look at Pati-Salam model in noncommutative geometry as a promising model.

There is a rich content of scalar fields in this model which are distributing in the desert and in principle are able to modify the way parameters of the model evolve with energy. As we will see however, there come other problems which lead us to believe the model needs a jump to include yet further scalars. In the next chapter, it will be clarified that, to go beyond the standard model with the noncommutative approach, scalars in the adjoint representation of the higher symmetries are needed that are not coupled to the fermions. These live naturally on the diagonal of the

Dirac operator but the mathematical mechanism to enter them to the picture is not yet developed. In general, it seems a hard task to build a unified theory based on noncommutative geometry approach because of the tight restrictions that the spectral action principle imposes on the scalar potential.

2.7 2-loop RGEs for complex singlet extended standard model

Here we present 2-loop renormalization group equations of the complex singlet extended standard model with the right-handed neutrino. These equations are derived using SARAH package for Mathematica [36]. The equations are consistent with the literature [22, 27, 28].

$$\frac{dg_1}{dt} = \frac{41\,{g_1}^3}{160\,\pi^2} + \frac{{g_1}^3}{12800\,\pi^4}\Big(- 15\,{K_\nu}^2 - 85\,{K_t}^2 + 199\,{g_1}^2 + 135\,{g_2}^2 + 440\,{g_3}^2\Big), \tag{2.16}$$

$$\frac{dg_2}{dt} = -\frac{19\,{g_2}^3}{96\,\pi^2} + \frac{{g_2}^3}{7680\,\pi^4}\Big(- 15\,{K_\nu}^2 - 45\,{K_t}^2 + 27\,{g_1}^2 + 175\,{g_2}^2 + 360\,{g_3}^2\Big),$$

$$\frac{dg_3}{dt} = -\frac{7\,{g_3}^3}{16\,\pi^2} + \frac{{g_3}^3}{2560\,\pi^4}\Big(- 20\,{K_t}^2 + 11\,{g_1}^2 + 45\,{g_2}^2 - 260\,{g_3}^2\Big),$$

$$
\begin{aligned}
\frac{dK_\nu}{dt} &= \frac{K_\nu}{16\pi^2}\Big(- 9/20\, g_1^2 - 9/4\, {g_2}^2 + 3\, {K_t}^2 + 5/2\, {K_\nu}^2\Big) \qquad (2.17)\\
&+ \frac{1}{256\,\pi^4}\Bigg(1/40\,\Big(21\,{g_1}^4 - 54\,{g_1}^2{g_2}^2 - 230\,{g_2}^4 + 240\,{\lambda_h}^2 + 80\,{\lambda_{sh}}^2\\
&+ 5\,\left(17\,{g_1}^2 + 45\,{g_2}^2 + 160\,{g_3}^2\right){K_t}^2 + 15\,\left({g_1}^2 + 5\,{g_2}^2\right){K_\nu}^2 - 270\,{K_t}^4 - 90\,{K_\nu}^4\Big)K_\nu\\
&+ \frac{{K_\nu}^3}{80}\Big(- 60\,{K_\nu}^2 - 540\,{K_t}^2 + 279\,{g_1}^2 + 675\,{g_2}^2 - 960\,\lambda_h\Big)\Bigg),\\
\frac{dK_t}{dt} &= \frac{K_t}{16\pi^2}\left(-\frac{17\,{g_1}^2}{20} - 9/4\,{g_2}^2 - 8\,{g_3}^2 + 9/2\,{K_t}^2 + {K_\nu}^2\right)\\
&+ \frac{1}{256\,\pi^4}\Bigg(\frac{K_t}{600}\Big(1187\,{g_1}^4 - 270\,{g_1}^2{g_2}^2 - 3450\,{g_2}^4 + 760\,{g_1}^2{g_3}^2\\
&\qquad + 5400\,{g_2}^2{g_3}^2 - 64800\,{g_3}^4 + 3600\,{\lambda_h}^2 + 1200\,{\lambda_{sh}}^2\\
&\qquad + 75\,\left(17\,{g_1}^2 + 45\,{g_2}^2 + 160\,{g_3}^2\right){K_t}^2 + 225\,\left({g_1}^2 + 5\,{g_2}^2\right){K_\nu}^2\\
&\qquad - 4050\,{K_t}^4 - 1350\,{K_\nu}^4\Big)\\
&+ \left(\frac{223\,{g_1}^2}{80} + \frac{135\,{g_2}^2}{16} + 16\,{g_3}^2 - 12\,\lambda_h - \frac{27\,{K_t}^2}{4} - 9/4\,{K_\nu}^2\right){K_t}^3 + 3/2\,{K_t}^5\Bigg),
\end{aligned}
$$

$$
\begin{aligned}
\frac{d\lambda_h}{dt} = \frac{1}{16\pi^2}\Bigg(&\frac{27\,{g_1}^4}{200} + \frac{9\,{g_1}^2{g_2}^2}{20} + \frac{9\,{g_2}^4}{8} - 9/5\,{g_1}^2\lambda_h - 9\,{g_2}^2\lambda_h \\
&+ 24\,{\lambda_h}^2 + 4\,{\lambda_{sh}}^2 + 12\,\lambda_h\,{K_t}^2 + 4\,\lambda_h\,{K_\nu}^2 - 6\,{K_t}^4 - 2\,{K_\nu}^4\Bigg) \\
+ \frac{1}{256\,\pi^4}\Bigg(&- \frac{3411\,{g_1}^6}{2000} - \frac{1677\,{g_1}^4{g_2}^2}{400} - \frac{289\,{g_1}^2{g_2}^4}{80} + \frac{305\,{g_2}^6}{16} + \frac{1887\,{g_1}^4\lambda_h}{200} \\
&+ \frac{117\,{g_1}^2{g_2}^2\lambda_h}{20} - \frac{73\,{g_2}^4\lambda_h}{8} + \frac{108\,{g_1}^2{\lambda_h}^2}{5} + 108\,{g_2}^2{\lambda_h}^2 - 312\,{\lambda_h}^3 \\
&- 40\,\lambda_h\,{\lambda_{sh}}^2 - 32\,{\lambda_{sh}}^3 + \Big(- \frac{171\,{g_1}^4}{100} - 9/4\,{g_2}^4 + \frac{45\,{g_2}^2\lambda_h}{2} \\
&+ 80\,{g_3}^2\lambda_h - 144\,{\lambda_h}^2 + 1/10\,{g_1}^2\left(63\,{g_2}^2 + 85\,\lambda_h\right)\Big){K_t}^2 \\
&- \frac{{K_\nu}^2}{200}\Big(18\,{g_1}^4 + 15\,{g_1}^2\left(4\,{g_2}^2 - 20\,\lambda_h\right) + 150\,{g_2}^4 - 300\,{g_2}^2\lambda_h \\
&+ 9600\,{\lambda_h}^2\Big) - 8/5\,{g_1}^2{K_t}^4 - 32\,{g_3}^2{K_t}^4 - 3\,\lambda_h\,{K_t}^4 - \lambda_h\,{K_\nu}^4 \\
&+ 30\,{K_t}^6 + 10\,{K_\nu}^6\Bigg),
\end{aligned}
\tag{2.18}
$$

$$
\begin{aligned}
\frac{d\lambda_{sh}}{dt} =& \frac{\lambda_{sh}}{160\,\pi^2}\Big(60\,{K_t}^2 + 20\,{K_\nu}^2 - 9\,{g_1}^2 - 45\,{g_2}^2 + 120\,\lambda_h + 80\,\lambda_{sh} + 80\,\lambda_s\Big) \\
&- \frac{1}{102400\,\pi^4}\lambda_{sh}\Bigg(- 1671\,{g_1}^4 - 450\,{g_1}^2{g_2}^2 + 3625\,{g_2}^4 - 5760\,{g_1}^2\lambda_h - 28800\,{g_2}^2\lambda_h \\
&\quad + 24000\,{\lambda_h}^2 - 480\,{g_1}^2\lambda_{sh} - 2400\,{g_2}^2\lambda_{sh} + 57600\,\lambda_h\,\lambda_{sh} + 17600\,{\lambda_{sh}}^2 \\
&\quad + 38400\,\lambda_{sh}\,\lambda_s\,(t) + 16000\,\left(\lambda_s\,(t)\right)^2 - 100\,\Big(17\,{g_1}^2 + 45\,{g_2}^2 + 160\,{g_3}^2 \\
&\quad - 288\,\lambda_h - 96\,\lambda_{sh}\Big){K_t}^2 - 100\,\Big(3\,{g_1}^2 + 15\,{g_2}^2 - 96\,\lambda_h - 32\,\lambda_{sh}\Big){K_\nu}^2 \\
&\quad + 5400\,{K_t}^4 + 1800\,{K_\nu}^4\Bigg),
\end{aligned} \tag{2.19}
$$

$$
\begin{aligned}
\frac{d\lambda_s}{dt} =& \frac{1}{16\,\pi^2}\left(8\,{\lambda_{sh}}^2 + 20\,\left(\lambda_s\,(t)\right)^2\right) \\
&+ \frac{1}{256\,\pi^4}\Bigg(\frac{48\,{g_1}^2{\lambda_{sh}}^2}{5} + 48\,{g_2}^2{\lambda_{sh}}^2 - 64\,{\lambda_{sh}}^3 \\
&\quad - 80\,{\lambda_{sh}}^2\lambda_s\,(t) - 60\,\left(\lambda_s\,(t)\right)^3 - 48\,{\lambda_{sh}}^2{K_t}^2 - 16\,{\lambda_{sh}}^2{K_\nu}^2\Bigg).
\end{aligned}
$$

Chapter 3

Spectral Pati-Salam Model

3.1 Introduction

Noncommutative geometry has shown interesting results in physics when it is applied to a hyperspace which is the direct product of four dimensional Riemannian spacetime and a noncommutative space characterized by the spectral triple: algebra of operators which are spacetime functions on the matrices of the form $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$, the relevant Hilbert space of this algebra, and a special operator which generalizes the notion of conventional Dirac operator. The inverse of this operator is looked at as the ultimate propagator between fermions which contains all the useful information of generalized geometry that is the source of all the forces. Therefore this operator gives dynamic to the fermions in a usual Dirac action way. Hence, all the fluctuations of the Dirac operator in principle connect fermions together. These (classical) fluctuations make the operator invariant under unitary automorphisms of the hyperspace algebra and show themselves as spacetime connections, scalars, and gauge fields. When we try to go beyond the standard model, this property of the Dirac operator puts serious constraints on the scalars in theory. Moreover, these scalars and vector fields along with the spacetime geometry gain dynamic by the spectral action principle which states that their kinetic and potential terms are in the spectrum of the Dirac operator [6]. This principle therefore dictates the scalar and gauge sectors. For the gauge sector, with the correct choice of algebra, favorable Yang-Mills type action appears [31, 33]. For the scalar sector, the constraints coming from these two different parts of the action are typically conflicting. In the case of spectral standard model these are consistent and reduce the number of free parameters. For a generalized model however we need to make sure of their implications.

One specific property of the Higgs sector in noncommutative geometry is that Higgses which originate from offdiagonal elements of the self-adjoint Dirac operator of the noncommutative part of hyperspace, connect particles with antiparticles and violate flavor. This is due to the settings of the theory which incorporate antiparticles and particles and treat them quite equally. What is called the first order condition can restrict this phenomenon to happen only for the neutrinos [26]. This is then the source of their Majorana masses.

In 2013, there were two attempts in [37] and [33] to use noncommutative geometry methods to explore beyond the standard model. In the former, a higher algebra is chosen and the first order condition is imposed. The result is a model with higher symmetry and with nice features such as having no extra fermions and having the scalar fields needed to get the right mass for the Higgs field. In the latter, the algebra is the same as (2.2), but the first order condition is relaxed. This path is what we follow here to explore implications of the spectral action for its scalar sector.

There are spaces which do not admit the first order condition like quantum spheres [38, 39]. Authors of [40] have shown that with adding the correct nonlinear terms to the fluctuations of the Dirac operator, $A_{(2)}$ bellow, noncommutative geometry approach can be used for these spaces as well. Invariant physical Dirac operator of the whole space is then (refer to [31, 33, 40] for exact definitions)

$$D_A = D + A_{(1)} + JA_{(1)}J^{-1} + A_{(2)}. \tag{3.1}$$

They also have shown that if Dirac operator can admit first order condition for a subalgebra, starting with it will cause fields in $A_{(2)}$ to be products of fields in $A_{(1)}$ and eat them as their vacuum remnants. The same authors have subsequently used these facts to build a Pati-Salam model based on the algebra $\mathbb{H} \oplus \mathbb{H} \oplus M_4(\mathbb{C})$ in [33]. We will discuss the physical content of this model and also argue that what is observed for scalars originating from $A_{(1)}$ and $A_{(2)}$ is only a special case of a general phenomenon by reasoning that for a generic Dirac operator $A_{(2)}$ will always eat $A_{(1)}$. This means in all the physical scalar fields resulting from a generic Dirac operator, there is no trace of linear fluctuations left. We also argue that preconditions on elements of the discrete Dirac operator might lead to unrealistic Higgs fields.

Relaxing the first order condition and writing the spectral action based on a broader algebra can cause two main challenges. Firstly, the nonzero offdiagonal sectors of the Dirac operator cause particles other than neutrinos to connect to their antiparticles and produce diquarks which were previously absent in the model. These along with leptoquark scalars on offdiagonal blocks of the diagonal sectors might lead to proton or nucleon decay. Secondly, since the spectral action dictates the scalar potential, it should be checked that the potential can break the symmetry to the standard model in an acceptable way. This is especially important in noncommutative geometry approach because here one does not have the free hand in manipulating the scalar sector which is the case in grand unified or effective field theories. In those cases, if there is a Higgs with correct representation like a Higgs in the adjoint representation, one may often safely assume that there is a potential for it which provides just enough needed Goldstone bosons. Here however one needs to make sure of that by considering the vacuum of the given potential.

3.1.1 Spectral action without first order condition

The fluctuations in (3.1) are written as

$$A_{(1)} = \sum_{a,b} a[D, b], \quad A_{(2)} = \sum_{\hat{a},\hat{b}} \hat{a}[A_{(1)}, \hat{b}], \quad \hat{a} = JaJ^{-1}, \tag{3.2}$$

to have the correct transformational properties that make Dirac operator invariant under unitary automorphisms of the algebra [40]. J is called the reality operator and constructs the opposite algebra $\mathcal{A}^0 = J\mathcal{A}^* J^{-1}$ in such a way to have $[a, b^0] = 0$ for any $a \in \mathcal{A}$ and $a^0 \in \hat{\mathcal{A}}$. In physics language, J is charge conjugate operator which exchanges particles and antiparticles of the same chirality in the Hilbert space. Elements of algebra act on right and left isospin indices of particles and four-color indices of antiparticles. Effect of J on algebra is to make operators which do the opposite.

The sum in (3.2) is over any favorite set of the elements of algebra and opposite algebra. The only conditions the chosen elements in these sums have to satisfy is normalization, $\sum ab = 1$, and to make $A_{(1)}$ and $A_{(2)}$ self-adjoint [1]. We emphasize that elements of opposite algebra used to calculate $A_{(2)}$ need not be associated with the elements of algebra which are used in $A_{(1)}$. Any set of elements will ensure the invariance of the Dirac operator as far that they come from the above recipes. We also notice that these fluctuations are spacetime connections plus function-valued and vector-valued matrices. Nothing is still a physical field. This notion only appears when the spectral action principle gives dynamic to the components of the final invariant Dirac operator which has accumulated all the fluctuations. Total Dirac operator is a linear combination of the operators associated with each

[1]Even a milder condition might be needed since only the final invariant Dirac operator which includes all the fluctuations is required to be self-adjoint.

one of the continuum and discrete subspaces, that is

$$D = \gamma^\mu D_\mu \otimes 1 + \gamma^5 \otimes D_F. \tag{3.3}$$

Two sets of fields emerge which are spacetime tensors and vector fields from the first part and scalar fields from the second part.

To make the Dirac operator of the discrete space invariant, intuitively one might expect that any nonzero component of the original operator needs to admit fluctuations. This however also depends on the degrees of freedom in the algebra which might identify some of these fluctuations, especially at the linear order.

The Dirac operator is divided into distinct blocks determined by the structure of the algebra and nonlinear fluctuations assure of the maximum degrees of freedom for any block of the original Dirac operator with no zeros. Linear fluctuations of $\mathcal{A}$ and $\hat{\mathcal{A}}$ create all the degrees of freedom for chiral and four-color indices respectively. In general, these cannot coexist if the nonlinear fluctuations are to vanish. This is manageable for some blocks, but is not legitimate since the result fields have nonrealistic representations.

We start with the generic Dirac operator of the discrete space [31]

$$D_F = \begin{pmatrix} D_B^{\ A} & D_B^{\ A'} \\ D_{B'}^{\ A} & D_{B'}^{\ A'} \end{pmatrix} \tag{3.4}$$

$$D_{BA} \equiv D_B^{\ A'} = \bar{D}_{B'}^{\ A} \equiv \bar{D}^{AB}, \quad D_B^{\ A} = \bar{D}^B_{\ A},$$

$$D^B_{\ A} \equiv D_{B'}^{\ A'} = \bar{D}^{B'}_{\ A'} \equiv \bar{D}_B^{\ A}.$$

These relations assure self-adjointness. Further, we need $J_F D_F J_F^{-1} = D_F$ to be sure antiparticles are associated to their particle counterparts [41, 26]. It means

$$D_B^{\ A'} \xrightarrow{J} D_{B'}^{\ A*} = D_B^{\ A'T} = D^{A'}_{\ \ B} \implies D_{AB} = D_{BA}. \tag{3.5}$$

This symmetry is understandable because these two blocks connect particles and antiparticles of different chiralities [42]. Next is to impose [43]

$$\gamma_F D_F = -D_F \gamma_F, \tag{3.6}$$

which eliminates diagonal blocks of diagonal sectors and offdiagonal blocks of offdiagonal sectors in 3.4. So the original Dirac operator of the discrete noncommutative space can be shown as

$$D_F = \begin{pmatrix} \begin{pmatrix} 0 & D_{2}{}^{\dot{a}I}_{bJ} \\ \bar{D}_{2}{}^{aI}_{\dot{b}J} & 0 \end{pmatrix} & \begin{pmatrix} D_{4dLcK} & 0 \\ 0 & D_{6\dot{d}L\dot{c}K} \end{pmatrix} \\ \\ \begin{pmatrix} \bar{D}_{4}{}^{dLcK} & 0 \\ 0 & \bar{D}_{6}{}^{\dot{d}L\dot{c}K} \end{pmatrix} & \begin{pmatrix} 0 & D_{2}{}^{\dot{a}I}_{bJ} \\ \bar{D}_{2}{}^{aI}_{\dot{b}J} & 0 \end{pmatrix} \end{pmatrix}. \tag{3.7}$$

Small letters, dotted letters, and capital letters are $\mathbb{H}_L$, $\mathbb{H}_R$, and $M_4(\mathbb{C})$ indices. Properties in (3.4) and (3.5) hold for the invariant Dirac operator in (3.1) because $A_{(2)} = JA_{(2)}J^{-1}$ and because fluctuations are hermitian by construction [40].

Further, fluctuations of each block are totally independent of other blocks. Moreover, $A_{(2)}$ assures that if a block has no zeros, the most general *field* sits on that block inheriting all the indices and having all the possible degrees of freedom.

Operators of Pati-Salam algebra are in the form of:

$$a = \begin{pmatrix} X^a_b \delta^{\dot{a}}_{\dot{b}} \delta^I_J & & \\ & \delta^a_b Y^{\dot{a}}_{\dot{b}} \delta^I_J & \\ & & \delta^{a'}_{b'} \delta^{\dot{a}'}_{\dot{b}'} W^{I'}_{J'} \end{pmatrix} \tag{3.8}$$

Elements of quaternions X and Y are complex functions of spacetime. W is in the algebra of 4 by 4 complex function-valued matrices. The offdiagonal block on the diagonal sector of the invariant Dirac operator is:

$$D_{A}{}^{\dot{a}I}_{bJ} = \cancelto{0}{\delta^{\dot{a}}_{b}} \delta^I_J \gamma^\mu \partial_\mu + D^{\dot{a}I}_{bJ} + A_{(1)}{}^{\dot{a}I}_{bJ} + JA_{(1)}J^{-1}{}^{\dot{a}I}_{bJ} + A_{(2)}{}^{\dot{a}I}_{bJ}. \tag{3.9}$$

Due to the block diagonal form of elements of algebra, the fluctuations are:

$$\begin{aligned} A_{(1)}{}^{\dot{a}I}_{bJ} &= \sum \left(a^{A}_{bJ} D^{B}_{A} b^{\dot{a}I}_{B} - a^{A}_{bJ} b^{B}_{A} D^{\dot{a}I}_{B} \right) \\ &= \sum \left(a^{cK}_{bJ} D^{\dot{b}L}_{cK} b^{\dot{a}I}_{\dot{b}L} - a^{cK}_{bJ} b^{dL}_{cK} D^{\dot{a}I}_{dL} \right) \\ &= \sum \left(X^{c}_{b} Y^{\dot{a}}_{\dot{b}} D^{\dot{b}I}_{cJ} - X^{c}_{b} Z^{d}_{c} D^{\dot{a}I}_{dJ} \right) \\ &= \sum \left(X^{d}_{b} Y^{\dot{a}}_{\dot{b}} - X^{c}_{b} Z^{d}_{c} \delta^{\dot{a}}_{\dot{b}} \right) D^{\dot{b}I}_{dJ} = \sum_{f} \left(f^{\dot{a}}_{b} \right)^{d}_{\dot{b}} D^{\dot{b}I}_{dJ}. \end{aligned} \tag{3.10}$$

If we reorganization constant elements of the Dirac operator like $D^{\dot{a}I}_{bJ} = M^{\dot{b}}_{d} N^{I}_{J}$, then

$$\sum_{f} \left(f^{\dot{a}}_{b} \right)^{d}_{\dot{b}} M^{\dot{b}}_{d} N^{I}_{J} = \sum_{f} f^{\dot{a}}_{b} N^{I}_{J}. \tag{3.11}$$

Because of the sum over the elements of algebra, f is an arbitrary function with the two indices which show its transformational properties.

Similarly,

$$\begin{aligned} \hat{A}_{(1)}{}^{\dot{a}I}_{bJ} &= J A_{(1)} J^{-1}{}^{\dot{a}I}_{bJ} = A_{(1)}{}^{\dot{a}'I'}_{b'J'} \\ &= \sum \left(a^{c'K'}_{b'J'} D^{\dot{b}'L'}_{c'K'} b^{\dot{a}'I'}_{\dot{b}'L'} - a^{c'K'}_{b'J'} b^{d'L'}_{c'K'} D^{\dot{a}'I'}_{d'L'} \right) \\ &= \sum \left(W^{K'}_{J'} V^{I'}_{L'} D^{\dot{a}'L'}_{b'K'} - W^{K'}_{J'} V^{L'}_{K'} D^{\dot{a}'I'}_{b'L'} \right) \\ &= \sum \left(W^{t}{}^{K}_{J} V^{t}{}^{I}_{L} \bar{D}^{\dot{a}L}_{bK} - W^{t}{}^{K}_{J} V^{t}{}^{L}_{K} \bar{D}^{\dot{a}I}_{bL} \right) \\ &= \sum \left(W^{t}{}^{K}_{J} V^{t}{}^{I}_{L} - W^{t}{}^{M}_{J} V^{t}{}^{K}_{M} \delta^{I}_{L} \right) \bar{D}^{\dot{a}L}_{bK} \\ &= \sum_{g} \left(g^{I}_{J} \right)^{L}_{K} \bar{D}^{\dot{a}K}_{bL} = \sum_{g} \left(g^{I}_{J} \right)^{L}_{K} \bar{N}^{K}_{L} \bar{M}^{\dot{a}}_{b} = \sum_{g} g^{I}_{J} \bar{M}^{\dot{a}}_{b}. \end{aligned} \tag{3.12}$$

Nonlinear corrections also do not incorporate other blocks because again they are linear with respect to $A_{(1)}$ and the algebra is block diagonal:

$$\begin{aligned} A_{(2)}{}^{\dot{a}I}_{bJ} &= \sum \left(c^{A}_{bJ}\ \hat{A}_{(1)}{}^{B}_{A}\ d^{\dot{a}I}_{B} - c^{A}_{bJ}\ d^{B}_{A}\ \hat{A}_{(1)}{}^{\dot{a}I}_{B} \right) \\ &= \sum \left(c^{cK}_{bJ}\ \hat{A}_{(1)}{}^{\dot{b}L}_{cK}\ d^{\dot{a}I}_{\dot{b}L} - c^{cK}_{bJ} d^{dL}_{cK}\ \hat{A}_{(1)}{}^{\dot{a}I}_{dL} \right) \\ &= \sum \left(T^{d}_{b} Z^{\dot{a}}_{\dot{b}} - T^{c}_{b} X^{d}_{c} \delta^{\dot{a}}_{\dot{b}} \right) \hat{A}_{(1)}{}^{\dot{b}I}_{dJ} \\ &= \sum_{g,h} \left(h^{\dot{a}}_{b} \right)^{d}_{\dot{b}} \left(g^{I}_{J} \right)^{L}_{K} \bar{D}^{\dot{b}K}_{dL} = \sum_{g,h} h^{\dot{a}}_{b} g^{I}_{J} = \sum_{q} q^{\dot{a}I}_{bJ} = p^{\dot{a}I}_{bJ}. \end{aligned} \tag{3.13}$$

Therefore

$$D_{A}{}^{\dot{a}I}_{bJ} = f^{\dot{a}}_{b} N^{I}_{J} + g^{I}_{J} \bar{M}^{\dot{a}}_{b} + p^{\dot{a}I}_{bJ} = p^{\dot{a}I}_{bJ} \tag{3.14}$$

Which is interesting because arbitrariness of h in (3.13) ruins the marks of the linear fluctuations in the nonlinear terms. Every block is now one tensor which is going to be a physical scalar field when the spectral action principle is applied. So nonlinear fluctuations promote all the degrees of freedom to tensors, while the linear ones do that only partially.

The Higgses in this block are in $(15+1,2,2)_{422}$ representation. However if we choose $\bar{D}^{\dot{a}K}_{bL} = \bar{M}^{\dot{a}}_{b} \delta^{K}_{L}$, then

$$D_{A}{}^{\dot{a}I}_{bJ} = \sum_{f} f^{\dot{a}}_{b} \delta^{I}_{J}.$$

Which leads to a $(1,2,2)$ Higgs. The process could be repeated for $A_{(1)}$ instead of $\hat{A}_{(1)}$ and $D^{\dot{a}K}_{bL} = \delta^{\dot{a}}_{b} N^{K}_{L}$ also will admit only linear corrections. It leads to $(15+1,2,2)_{422}$ Higgses. Elimination of the nonlinear fluctuations for this block is therefore nothing but splitting the Higgses to belong to representations of only one of the gauge groups. This is something however that can only be legitimate if happens through an acceptable spontaneous symmetry breaking process. For this block this would be a forbidden symmetry breaking. Put another way, even if at the classical level we start with the block which has suitable zeros and later on quantum fluctuations, for example, create those entries, there will be no satisfactory way to go from $(15,2,2)$ to $(15,1,1)$.

If now one promotes $f^{\dot{a}}_{b}$ to a field, it seems like a double Higgs which includes the standard model Higgs. However the only part of this block which could do that was its trace of the four-color indices. Instead, starting with $D = D^{\dot{a}}_{b}\delta^{I}_{J}$, we have actually four different fields on diagonal of the general Higgs associated with this block. Two of them are inside $(8, 2, 2)_{322}$ which cannot survive to low energies. This was the only block which could produce standard model Higgs, and killing nonlinear fluctuations has ruined the possibility of having a proper Higgs. Thus, by turning off some components of the Dirac operator in each block we might get Higgses which only carry some quantum numbers of those blocks, but we should be careful to not trust nonrealistic representations. Elimination of the entire of a block, or making the whole Higgs content of it massive, can be managed easier to be backed by a symmetry breaking process and does not lead to unrealistic scalars.

Turning off nonlinear fluctuations of the blocks on the offdiagonal sector is possible only if the entire block is zero. This is because again either $A_{(1)}$ or $\hat{A}_{(1)}$ should vanish, but for example

$$A_{(1)}{}_{aI}^{b'J'} = \sum \left(X^{d}_{a} W^{J'}_{K'} - X^{e}_{a} Y^{d}_{e} \delta^{J'}_{K'} \right) M_{d}{}^{K'} N^{b'}_{I} \tag{3.15}$$

which cannot be zero if this block is not entirely zero. But this block is needed because of the singlet in it which gives Majorana mass to the sterile neutrino and brings the Higgs mass down to the observed value [20, 44, 26, 45]. These do not require the singlet to survive to low energies.

One other way is to find what subalgebra can have a nonzero block in the offdiagonal sector. The maximal subalgebra with this property is [31] the standard model subalgebra introduced in the introduction. We can then require the first order condition, equivalently elimination of the nonlinear fluctuations, and break the symmetry at the algebraic level. This means we have assumed there is a proper Higgs mechanism which keeps renormalizability intact. Also, all the extra Higgses have presumably gained heavy masses. However since the scalar potential is dictated by the spectral principle, as we will see in the next section, such breaking scenario is not for granted and it is preferable to start with the unified theory and study the symmetry breaking possibilities.

3.2 Spectral Pati-Salam Model

According to the previous section, the following is the most general Dirac operator starting with the generic operator in (3.7) for the discrete space with no preconditions imposed on it.

$$D_A = \begin{pmatrix} \begin{pmatrix} \nabla\!\!\!\!/_L & \gamma^5 \Sigma_{bJ}^{\dot{a}I} \\ \gamma^5 \bar{\Sigma}_{\dot{b}J}^{aI} & \nabla\!\!\!\!/_R \end{pmatrix} & \begin{pmatrix} \gamma^5 H_{dLcK} & 0 \\ 0 & \gamma^5 \dot{H}_{\dot{d}L\dot{c}K} \end{pmatrix} \\ \begin{pmatrix} \gamma^5 \bar{H}^{dLcK} & 0 \\ 0 & \gamma^5 \bar{\dot{H}}^{\dot{d}L\dot{c}K} \end{pmatrix} & \begin{pmatrix} \bar{\nabla\!\!\!\!/}_L & \gamma^5 \Sigma_{bJ}^{\dot{a}I} \\ \gamma^5 \bar{\Sigma}_{\dot{b}J}^{aI} & \bar{\nabla\!\!\!\!/}_R \end{pmatrix} \end{pmatrix}. \tag{3.16}$$

Next is to derive the scalar potential from spectral principle to look for possibilities of symmetry breaking. Follow the usual procedure [4, 31, 33], one reads the potential from the second and fourth terms in the expansion of the Heat Kernel of the Dirac operator. This is the expansion of the sum over spectrum of Dirac operator with respect to a cutoff energy. To start, we need to form the square of the Dirac operator and read connections from it

$$D_A^2 = \nabla_\mu \nabla^\mu + A^\mu \nabla_\mu + B. \tag{3.17}$$

Components of B are

$$
\begin{aligned}
B^{aI}_{bJ} &= B^{a'I'}_{b'J'} = \Sigma^2 + H^2 \\
B^{\dot{a}I}_{\dot{b}J} &= B^{\dot{a}'I'}_{\dot{b}'J'} = \Sigma^2 + \dot{H}^2 \\
B^{\dot{a}I}_{bJ} &= -\gamma^5 \nabla\!\!\!/ \Sigma, \quad \nabla\!\!\!/ \Sigma = \nabla\!\!\!/_L \Sigma - \Sigma \nabla\!\!\!/_R, \quad B^{bJ}_{\dot{a}I} = \bar{B}^{\dot{a}I}_{bJ}, \\
B^{\dot{a}'J'}_{b'I'} &= -\gamma^5 \bar{\nabla\!\!\!/} \Sigma, \quad B^{b'I'}_{\dot{a}'J'} = \bar{B}^{\dot{a}'J'}_{b'I'} \\
B_{aIbJ} &= -\gamma^5 \nabla\!\!\!/ H, \quad \nabla\!\!\!/ H = \nabla\!\!\!/_L H - H \bar{\nabla\!\!\!/}_L, \\
B_{\dot{a}I\dot{b}J} &= -\gamma^5 \nabla\!\!\!/ \dot{H}, \quad \nabla\!\!\!/ \dot{H} = \nabla\!\!\!/_R \dot{H} - \dot{H} \bar{\nabla\!\!\!/}_R, \\
B_{\dot{a}IbJ} &= \bar{\Sigma} H + \dot{H} \bar{\Sigma}, \\
B_{aI\dot{b}J} &= \Sigma \dot{H} + H \Sigma \\
B^{aI\dot{b}J} &= \bar{B}_{aI\dot{b}J}, \quad B^{\dot{a}I\dot{b}J} = \bar{B}_{\dot{a}I\dot{b}J}, \quad B^{aIbJ} = \bar{B}_{aIbJ}, \quad B^{\dot{a}IbJ} = \bar{B}_{\dot{a}IbJ}.
\end{aligned}
\tag{3.18}
$$

Now, trace of B shows the invariants of the second order in D spectrum and trace of B^2 presents terms of the fourth order.

$$
\begin{aligned}
Tr(B) &= 2Tr\Big(2\Sigma^2 + H^2 + \dot{H}^2\Big) \\
Tr(B^2) &= 2Tr\left(2\Sigma^4 + H^4 + \dot{H}^4\right) \\
&\quad + 8Tr\left(\Sigma^2\left(H^2 + \dot{H}^2\right)\right) \\
&\quad + 4Tr\Big(2\bar{H}\Sigma\dot{H}\bar{\Sigma} + h.c.\Big) \\
&\quad + 2Tr\Big(2\nabla_\mu\bar{\Sigma}\nabla^\mu\Sigma + \nabla_\mu\bar{H}\nabla^\mu H + \nabla_\mu\bar{\dot{H}}\nabla^\mu\dot{H}\Big).
\end{aligned}
\tag{3.19}
$$

After normalizing the kinetic terms, the potential is

$$
\begin{aligned}
V &= -\frac{1}{2}M^2\left(\Sigma^2 + H^2 + \dot{H}^2\right) \\
&\quad + \frac{1}{2}g^2 Tr\left(2\Sigma^4 + H^4 + \dot{H}^4\right) \\
&\quad + 2g^2 Tr\left(\Sigma^2 H^2 + \Sigma^2\dot{H}^2 + \left(\bar{H}\Sigma\dot{H}\bar{\Sigma} + h.c.\right)\right).
\end{aligned}
\tag{3.20}
$$

This is the model introduced in [33]. Irreducible representations of these fields with respect to $(4C, 2L, 2R)$ are

$$\begin{aligned} \Sigma &= (15, 2, 2) + (1, 2, 2), \\ H &= (10, 3, 1) + (6, 1, 1), \quad \dot{H} = (10, 1, 3) + (6, 1, 1). \end{aligned} \tag{3.21}$$

The $(1, 2, 2)$ which is the trace of Σ is the only field which can contain the standard model Higgs. After breaking of the four-color, other scalars will pop up with the same quantum numbers, however they just couple with either quarks or leptons or do not couple with fermions at all. It is also notable that there is not any mark left from quaternionic nature of the right and left algebras in the Higgs sector for a generic original Dirac operator because there are always enough degrees of freedom left to build the generic fields. Therefore this is also the Higgs content of the algebra $M_2(\mathbb{C}) \oplus M_2(\mathbb{C}) \oplus M_4(\mathbb{C})$.

There are only two constants here which have originated from the cutoff function in the trace of Dirac operator. one can add Yukawa couplings to the picture by the direct product of our Dirac operator with a three dimensional discrete space of families, yet we can always choose the gauge eigenstates rather than mass eigenstates to explore symmetry breaking. In addition, adding Yukawa couplings in noncommutative geometry approach is a way to produce coupling constants in the scalar sector. Yukawa couplings get absorbed into the fields in kinetic terms and, therefore, reappear in the quartic terms. However these couplings are not free parameters and at unification scale there are relations between them and with the unified gauge coupling g [20, 45]. Therefore we don't expect them to change the picture.

Since the potential in (3.20) is not the most general quartic potential and lacks terms such as $(Tr(\Phi^2))^2$ and $Tr(\Phi^2)Tr(\chi^2)$, there is an appealing need to check whether the scalar potential is actually able to break symmetry properly. In general, the spectral action passes the first test by providing opposite signs for the quadratic and quartic terms [6]. Still one should be careful that this is the accidental symmetry of the Higgs sector and its vacuum which has to offer the correct number of massless bosons. Another aspect of (3.20), which is quite general due to the Gilkey's formula for heat kernel expansion ([4]), is that all the Higgses have the same negative square term at high energies prior to the breaking, yet the quartic part does not have all the possible terms. Therefore one needs to make sure of the signature of the massive Higgses after the breaking.

3.2.1 Symmetry Breaking

In noncommutative approach, unlike usual grand unified models, our hands are not open to choose scalars and customize symmetry breaking along with fermionic mass spectrum. The Higgses and their representations are prescribed by the algebra in the spectral triple and the form of scalar-scalar interactions is dictated by the spectral action principle. This is thought of as the privilege of the theory and is also the exact reason that it is highly restrictive and predictive.

With the above scalars, the only possible scenario is to break $SU(4)$ and $SU_R(2)$ groups at once using the neutral element of $\dot{H}$. After the breaking, the 36 degrees of freedom in $\dot{H}$ appear as the following scalars.

$$\begin{aligned}(10,3,1)_{422} &= 6^{-\frac{2}{3}} + 6^{\frac{1}{3}} + 6^{\frac{4}{3}} + 3^{\frac{2}{3}} + 3^{-\frac{1}{3}} + 3^{-\frac{4}{3}} + 1^{0} + 1^{-} + 1^{--}, \\ (6,1,1)_{422} &= 3_A^{\frac{1}{3}} + \bar{3}_A^{-\frac{1}{3}}.\end{aligned} \tag{3.22}$$

On the right hand sides, 3 stands for color triplet, when $\bar{3}$ and 6 are for anti-symmetric and symmetric 3×3 representations respectively. Superscripts are the electric charges. The 3 fields are leptoquarks leading to $\Delta L = \Delta B = 1$ processes, while the $\bar{3}_A$ and 6 fields are diquarks leading to $\Delta B = 2$ (figure 3.1). In [42], the role of $\bar{3}_A$ as the diquark candidate responsible for the observed B-decay anomalies is discussed and it is noted that this field does not couple with diquarks and does not lead to proton decay processes. Unification of the gauge couplings and the intermediate scale are also addressed in [43] and [46]. Here we want to carefully study the potential and its breaking.

Fields in Σ can be written with respect to the standard model quantum numbers as well:

$$(15,2,2)_{422} = \begin{pmatrix} 8^0 \\ 8^- \end{pmatrix}_{\frac{1}{2}} + \begin{pmatrix} 8^+ \\ 8^0 \end{pmatrix}_{-\frac{1}{2}} + \begin{pmatrix} \chi_1^0 \\ \chi_1^- \end{pmatrix}_{\frac{1}{2}} + \begin{pmatrix} \chi_2^+ \\ \chi_2^0 \end{pmatrix}_{-\frac{1}{2}} + \begin{pmatrix} 3^{\frac{1}{3}} \\ 3^{-\frac{2}{3}} \end{pmatrix}_{\frac{1}{2}} + \begin{pmatrix} 3^{-\frac{2}{3}} \\ 3^{-\frac{5}{3}} \end{pmatrix}_{-\frac{1}{2}} \tag{3.23}$$

$$+ \begin{pmatrix} \bar{3}^{\frac{5}{3}} \\ \bar{3}^{-\frac{2}{3}} \end{pmatrix}_{\frac{1}{2}} + \begin{pmatrix} \bar{3}^{\frac{2}{3}} \\ \bar{3}^{-\frac{1}{3}} \end{pmatrix}_{-\frac{1}{2}},$$

$$(1,2,2)_{422} = \begin{pmatrix} \phi_1^0 \\ \phi_1^- \end{pmatrix}_{\frac{1}{2}} + \begin{pmatrix} \phi_2^+ \\ \phi_2^0 \end{pmatrix}_{-\frac{1}{2}}.$$

These are left doublets with the noted hypercharges. Fine tuning is required since uncolored elements in trace of Σ, presented by ϕ, will have to survive to low energies to make a double Higgs effective action while the colored ones have to gain high masses. We see in the next section that the Higgs sector of the model is proton decay free at the tree level which suggests the tuning might not be severe, yet these scalars couple with fermions and that can put serious restrictions on the intermediate scale. The field χ is just like standard model Higgs, but only interacts with the leptons.

To have a proper breaking to the standard model, the potential in (3.20) and its vacuum need to satisfy the following expectations at the tree level. One of the components of the complex field 1^0 needs to get a VEV and break $(4C, 2R, 2L)$ directly to $(3C, 2L, 1Y)$. At the same time, it will be providing Majorana mass for the sterile neutrino due to its Yukawa interaction. The other component of 1^0 along with two components of 1^- have correct quantum numbers to be Goldstone bosons needed for breaking $SU_R(2)$. The six degrees of freedom in the color triplet $3^{\frac{2}{3}}$ and its conjugate should also come massless to play the role of Goldstone bosons for making the six unwanted Higgs fields in $SU(4)$ massive. Moreover, the potential has to provide positive masses proportional to the unification scale for all the other components of $\dot{H}$ and Σ since they all are colored or charged and also have Yukawa interactions with the fermions. With the above Higgs content, this scenario is the only possibility to break symmetry to the standard model group which must be implemented automatically by the scalar potential. Specifically, here one cannot break the right and color symmetries at different scales.

Since H does not acquire a VEV, to study the symmetry breaking of the potential in (3.20), we concentrate only on $\dot{H}$ and Σ. After normalizing the kinetic terms, in terms of particles in (3.22), quadratic and quartic parts are

$$\begin{aligned}
&-\frac{1}{2}M^2\Big(|6^{-\frac{2}{3}}|^2+|6^{\frac{1}{3}}|^2+|6^{\frac{4}{3}}|^2+|3^{-\frac{4}{3}}|^2+|3^{-\frac{1}{3}}|^2+|3^{\frac{2}{3}}|^2+|1^0|^2+|1^-|^2+|1^{--}|^2 \\
&\qquad + |3_A|^2+|\bar{3}_A|^2+\Sigma^2\Big) \\
&+\frac{1}{4}g^2\Big(2|1^0|^4+2|1^{--}|^4+2|3_A|^4+|1^-|^4+|3^{-\frac{4}{3}}|^4+|3^{\frac{2}{3}}|^4+\frac{1}{2}|3^{-\frac{1}{3}}|^4+\frac{1}{2}|\bar{3}_A|^4 \\
&\qquad +4|1^0|^2(|1^-|^2+|3^{\frac{2}{3}}|^2+|3_A|^2+\frac{1}{2}|3^{-\frac{1}{3}}|^2+\frac{\sqrt{2}}{2}(3_A 3^{\frac{1}{3}}+h.c.)\Big)+...
\end{aligned} \tag{3.24}$$

Handy calculation shows that this potential can provide needed massless Goldstones. However components of 3_A will also appear as six unwanted massless bosons and the rest of scalars will acquire negative masses which means this is a local maximum. To explore a little further, one can easily see that the more general case of

$$-\frac{1}{2}M^2 Tr(\bar{\dot{H}}\dot{H})+\frac{1}{4}\lambda_1 Tr(|\bar{\dot{H}}\dot{H}|^2)+\frac{1}{4}\lambda_2 Tr(|\bar{\dot{H}}\dot{H}|)^2, \tag{3.25}$$

does not work either. The leptoquarks $3^{\frac{2}{3}}$ in the symmetric part with $I_R=1$ and 3_A in the antisymmetric part with $I_R=0$ have always the same masses. This is because the underlying group theory causes them to interact with $|1^0|^2$ in the exact same way. Therefore symmetry of the vacuum is bigger than what is needed and any useful vacuum leads to pseudo-Goldstone bosons. Since it is an effect of the accidental symmetry in the Higgs sector, one can easily accept that these fields acquire masses at higher loop orders. However these should not be high masses and that is problematic since these scalars couple with the fermions plus being charged and colored. If there was a freedom to choose the coefficients, by either $0 < -2\lambda_1 < \lambda_2$ or $0 < 2\lambda_1 < -\lambda_2$ one could get rid of negative masses and arrange a suitable local minimum (refer to [47] for more).

The other scalar, Σ, evidently cannot improve the situation since it is not allowed to have expectation values at high scales. The spectral action mixes Σ with $\dot{H}$ which is favorable to make the unwanted scalars massive, but there is a need for fine tuning which is hard to arrange especially if the first breaking cannot happen safely in reasonably low energies. In the literature, the Higgs sector of Pati-Salam is usually thought of as the remnants of 126 and 120 or 45 of $SO(10)$. The last two ones have a $(15, 1, 1)$ in them and the 126 has a $(10, 3, 1)$. So breaking of the right symmetry and the four-color can happen independently [48, 49, 50, 51, 52, 53].

Hence, lack of terms such as $(Tr(\dot{H}^2))^2$ and $Tr(\dot{H}^2)Tr(\Sigma^2)$ is preventing the potential to have a suitable vacuum. More importantly, current settings of the noncommutative geometry approach cannot offer Higgses in the adjoint representation [54] and that is problematic for symmetry breaking procedure as we saw here and was seen in [55]. As was noted in section 3.1.1, it is possible to eliminate Dirac operator elements on offdiagonal blocks in order to have adjoint Higgses. However, these would be remnants of forbidden symmetry breakings. Even if the breaking itself was legal, it should be backed by spectral action and not be done in the algebraic level. Obviously, once the most general form of Pati-Salam model is obtained, the only reliable scenario to go down to the standard model is a proper symmetry breaking scenarios. Otherwise, the renormalizability of the model in jeopardy and survived Higgses are not trustable.

3.2.2 fermion masses

Dirac operator in (3.16) leads to the following Yukawa interaction terms

$$L_Y = g\bar{\psi_R}\Sigma\psi_L + g\bar{\psi_R^C}\dot{H}\psi_R + g\bar{\psi_L^C}H\psi_L + h.c. \,. \tag{3.26}$$

$\dot{H}$ only transforms right-handed particles to right-handed antiparticles while Σ transforms right (left) handed particles to left (right) handed particles.

Each one of ϕ and χ in (3.23) provides different Dirac masses for fermions with isospin $\frac{1}{2}$ and $-\frac{1}{2}$ by admitting two independent VEVs. However χ only couples with the leptons while χ couples with all the fermions. Neutrino has seesaw mechanism of type one. If H also exists, active neutrino gets a Majorana mass as well and seesaw mechanism of type two happens naturally for the neutrino.

With four independent VEVs and Majorana masses for the neutrinos, there is enough freedom and the model basically does not have predictions for fermion masses. It is also interesting that in the offdiagonal part of generic Dirac operator, naturally a colorless double Higgs arises which gives masses to all the particles. A restriction however emerges when the quartic part of potential is not able to provide high masses for the components of 8 in Σ, which is just like what happens for the 6 in $\dot{H}$. Through Σ^4, only χ can give a mass term to the 8 which will inevitably be from the same order of quarks masses. Therefore Σ will come with an overall negative mass term. The problem is originating from tight restrictions of two parts of the action. The bosonic part which does not possess all the possible quartic terms due to the spectral action principle, and the fermionic part which couples all the scalars to fermions.

This model basically predicts Yukawa and gauge couplings to be equal to the unified gauge coupling at unification. They of course will run differently. Even if we implement a procedure to produce Yukawas as 3 by 3 matrices and find some room for maneuver, the scalar couplings will not be independent of Yukawas and the g. These two originate from constants in the Dirac operator of the discrete space and the cutoff function in the spectral action which are the only sources to produce couplings in noncommutative geometry approach. The scalar couplings therefore are expected to be from the same order of these couplings at unification. It is seen that the relation between these couplings are consistent with particle masses at low energies [20, 45]. Therefore from any aspect, it is inevitable to try to find a way for entering new fields into the picture.

3.3 Proton Decay

Another interesting test for the potential suggested by the spectral action principle for the Pati-Salam model is to look for proton decay diagrams. The importance of this task in general is to see whether we can safely bring the intermediate breaking mass scale down. This could add to this model the privilege of needing a less severe fine tuning and also can have phenomenological implications for the achievable energies at LHC [46]. Another encouraging fact is that noncommutative geometry approach tries to yield an effective theory at the unification scale and proton decay is one of the few probes available to examine a theory which is written at those high energies with our current experimental abilities.

In this section, we first do a general analysis of the possible proton decay diagrams with the Yukawa interactions of (3.26). Then we look for the needed vertexes of Higgs interactions in the potential. It is enough to look for vertices originating from $\dot{H}^2$ and $\dot{H}^4$ because Σ does not lead to diquarks or leptoquarks and H has the same exact Yukawa terms for left-handed fermions as $\dot{H}$ has for

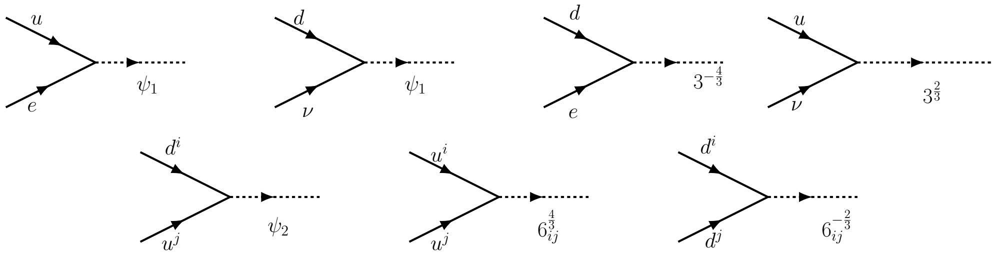


Figure 3.1: Diquark and leptoquark diagrams. ψ_1 is a combination of $3^{-\frac{1}{3}}$ and $3_A^{-\frac{1}{3}}$ and ψ_2 is a combination of $\bar{3}_A$ and $6^{\frac{1}{3}}$.

right-handed ones. This feature originates from grading of the algebra and left-right symmetry because each one of the left and right-handed particles has a Higgs field to produce flavor violating diagrams. In contrast, what happens in the $SU(5)$ unified theory is that the same extra gauge fields couple with different chiralities.

If the proton decay diagrams are allowed by potential at tree level, then one should make sure all the elements which are involved get indeed a mass at high enough scales. Then we should also be worried since H cannot admit VEV and doesn't couple with $\dot{H}$ in a way to gain a high mass from it. However, it becomes clear in the following that the needed order six and order nine operators cannot form by vertexes that the potential provides.

It is evident that the extra gauge fields in this model cannot cause proton decay on their own because diquark vertices do not exist. This roots in the fact that in noncommutative approach all particles sit in the ordinary, rather than ordinary and conjugate, fundamental representations (similar to SO(10) and usual Pati-Salam unified models). Yet the Yukawa terms in (3.26) suggest that diquark scalars exist. We saw in 3.2.1 that $\dot{H}$ alone could not break the symmetry. If that was the case, three fields in $3^{-\frac{2}{3}}$ and their conjugates would have been eaten by gauge bosons and disappeared from Lagrangian. Now however one part of each remains as a Higgs field. We are interested to see whether the Higgs fields can cause proton decay at tree level or not. With the particle content in (3.22) and their dangerous diagrams in figure 3.1 we witness that only following vertexes can cause proton decay diagrams with up to dimension nine operators.

$$\begin{aligned} &\psi_1\psi_2, \quad 3^{-\frac{4}{3}}6^{\frac{4}{3}}, \quad 3^{\frac{2}{3}}6^{-\frac{2}{3}}, \\ &\psi_1\psi_1 3^{\frac{2}{3}}, \quad \psi_2\psi_2 6^{-\frac{2}{3}}, \quad 3^{\frac{2}{3}}3^{\frac{2}{3}}3^{-\frac{4}{3}}, \quad 6^{\frac{4}{3}}6^{-\frac{2}{3}}6^{-\frac{2}{3}}. \end{aligned} \tag{3.27}$$

Here ψ_1 is a combination of 3_A and $3^{-\frac{1}{3}}$. Also ψ_2 is a combination of $6^{\frac{1}{3}}$ and $\bar{3}_A$.

Obviously all the terms in $\dot{H}^4$ are from order four, so we seek for the above terms coupled with one or two 1^0. This field admits a VEV and dimension of diagrams remain intact. However such terms are absent in the potential. All the terms in $\dot{H}$ are presented in appendix 3.6. For the relevant dimension six operators we read from (3.29)

$$|1^0|^2\left(|3^{\frac{2}{3}}|^2+|3_A|^2+\frac{1}{2}|3^{-\frac{1}{3}}|^2+\frac{\sqrt{2}}{2}(3_A 3^{\frac{1}{3}}+h.c.)\right),$$

which can only lead to pion decay. The followings are all of the possible dimension nine vertexes. None of these is in the above form and they do not lead to proton decay.

$$\begin{aligned} 1^0\Big(&3^{-\frac{2}{3}}6^{\frac{4}{3}}3^{-\frac{2}{3}} + 3^{-\frac{2}{3}}6^{\frac{1}{3}}3^{\frac{1}{3}} + 3^{\frac{1}{3}}6^{\frac{1}{3}}3^{-\frac{2}{3}} + 3^{\frac{1}{3}}6^{-\frac{2}{3}}3^{\frac{1}{3}} \\ &+ {3_A}^*6^{-\frac{2}{3}}{3_A}^* + 2(3^{-\frac{2}{3}}\bar{3}_A {3_A}^*) + 3^{-\frac{2}{3}}\bar{3}_A {3_A}^* + h.c.\Big) \end{aligned} \tag{3.28}$$

3.4 Conclusion

In this note we considered relaxation of the first order condition and worked on nonlinear fluctuations found in [40] to show that the nonlinear fluctuations swipe all the marks of the linear fluctuations and no trace of them remains in physical fields. The structure of algebra divides Dirac operator to blocks and nonlinear fluctuations provide all the degrees of freedom for each block and the result is a scalar field with all the indices of that block promoted to tensorial indices. Linear fluctuations do this only partially and promote only two out of four indices to tensorial indices. Neglecting the nonlinear effects produces Higgses which are in nonrealistic representations. It is also discussed that any precondition on the original discrete Dirac operator which kills nonlinear fluctuations, even for one block of the Dirac operator, lead to unphysical results by implying not acceptable symmetry breaking scenarios.

When the spectral action is used to go beyond the standard model, clearly, the challenge is that the spectral potential has to be able to support a suitable symmetry breaking scenario. With the current settings of noncommutative geometry approach it is not possible to arrange an acceptable scenario as it is seen here and in reference [55]. The model especially lacks the presence of Higgses in the adjoint representation and Higgses which decouple with the fermions in the action. Both of these are automatically true for Higgses on diagonal blocks of diagonal sectors in 3.16. However, the algebra is even graded and odd parts of the Dirac operator cannot coexist with its even parts. The only exception is to have a nonzero diagonal singlet. This couples with all the Higgses and might be able to provide a suitable minimum and solve the problem of negative mass terms in section 3.2.1. Another way might be to expand the geometry to include a product space exactly like the even discrete space, but with an odd grading which will be able to provide all the adjoint Higgses on diagonal of its Dirac operator. It is an important fact that spectral potential is in a way that no Higgs can break the symmetry alone even if it is in the adjoint of the four-color and the right symmetries. Therefore all the Higgses on diagonal blocks would be needed to break the symmetry in a complicated form.

In the last section, we saw that the offdiagonal Higgses which violate baryonic and leptonic numbers in their Yukawa interactions do not lead to proton decay at tree level up to nine dimensional operators, despite the presence of the dangerous leptoquark and diquark vertexes. To see this, we indicated the scalar vertexes needed to cause proton decay and witnessed that they are absent in the potential.

3.5 Further discussions

The standard model is an effective model which is established by matching some different sectors delicately. Its scalar sector plays an important role by affecting all the other sectors in a way or another. Although at some points in the past other ideas such as dynamic symmetry breaking were also popular, after the Higgs discovery at LHC in 2012, there is more or less no hope to glue all of these parts in a way other than having such scalars. Still, there is some ambiguity in the Yukawa sector along with the fine tuning problem and the hierarchy of the Higgs mass which seem to be challenging our trust in the standard model.

If one looks at the standard model as an effective model, one needs to build a model which contains it and is more elegant. The real challenge is then to comprehend how a theory can be more elegant and yet complex enough to accommodate different parts of this model with their delicacy and with all the accurate tunings.

The project of noncommutative geometry approach to the standard model is based on the idea to expand the notion of Riemannian space and Dirac operator in order to find geometrical origins for the gauge fields. The potential of the gauge sector appears in the spectrum as the geometrical invariants of the generalized geometry. Amazingly, scalar fields, their mutual and self interactions, and their interactions with fermions come forth as a bonus. This is the best opportunity to try having a unified picture at the classical level.

It is the aim of this dissertation to explore some phenomenological aspects of the scalar sector for currently available models based on noncommutative geometry. We started with the spectral standard model and showed its scalar sector is reliable despite the tight relations between the couplings at the unification scale. These relations imply that scalar couplings are not free parameters at the unification scale. At the same time, the Yukawa coupling of the neutrino exists in the model and gives enough freedom to find a window in the free parameters to match the particle masses at low energies with their experimental values. These free parameters basically are the unified gauge coupling, the Yukawa couplings, and the unification scale. This part of the thesis is based on the paper [45]. The rest of the thesis is based on [56].

There are two problems, however, which make the need to go beyond this model inevitable. First, the model is not consistent with the experimental values for the gauge couplings at low energies. This is totally expected since it is effectively a singlet extended standard model which has no intermediate scale. Second, the algebraic conditions which kill extra Higgses and break the symmetries to reach to the spectral standard model are not backed with proper and clear symmetry breaking processes.

We choose the spectral Pati-Salam model suggested in [33] as the reliable beyond standard model based on noncommutative geometry because the first order condition is not imposed to obtain this model. Therefore, as the second part of this thesis, we explored the implications of the spectral action principle on the scalar sector of this model. It turned out that the model lacks proper Higgses to break the symmetry properly. Since all the Higgses which are consistent with the usual settings of the noncommutative geometry approach exist in this model, one reaches to the conclusion that new ways must be invented to generate new scalars. More specifically, the model lacks fields which are in the adjoint representation of the gauge groups. These Higgses live naturally on the diagonal blocks of the Dirac operator and their very favorable feature is that they naturally decouple from the fermions in the action. This seems should be the next natural step to find a consistent way to enter such Higgses to the picture.

The other fact about the scalar sector is that its potential is very specific and does not contain all the possible quartic terms. This highly restrictive nature of the spectral action principle reveals itself for Higgses which are in higher order representations, instead of the fundamental representations, of the gauge symmetries. We showed in chapter 3 that this restrictive nature works in our favor as it comes to the proton decay possibilities by not allowing for the dangerous vertices to combine.

Yet, the spectral action is in a way that it does not allow any single Higgs field of higher representations to break a higher symmetry on its own. This is correct even if we had adjoint Higgses. The main reason for this phenomenon is that the spectral scalar potential lacks terms which are quadratic with respect to the traces of scalars.

Altogether, we need to find a way to enter new suitable scalar fields into the picture and hope that they can break the symmetry together and their vacuum expectation values give us some room for maneuver to have suitable intermediate scale(s).

3.6 Terms in $\dot{H}$ with respect to standard model representations

Here we present all terms in $\dot{H}^4$ which is the only part of the potential with diquarks and leptoquarks. Similar terms exist in the H^4. Terms are normalized so that the kinetic terms be in canonical form.

$$
\begin{aligned}
\dot{H}^4 = {} & |1^0|^4 + |1^{--}|^4 + \frac{1}{2}|1^-|^4 + 2|1^-|^2\left(|1^0|^2 + |1^{--}|^2\right) \\
& + (1^+)^2 1^0 1^{--} + (1^-)^2 1^{0*} 1^{++} \\
& + 2|1^0|^2\left(|3^{\frac{2}{3}}|^2 + |3_A|^2 + \frac{1}{2}|3^{-\frac{1}{3}}|^2 + \frac{\sqrt{2}}{2}(3_A 3^{\frac{1}{3}} + h.c.)\right) \\
& |1^-|^2\left(|3^{\frac{2}{3}}|^2 + 2|3_A|^2 + |3^{-\frac{1}{3}}|^2 + |3^{-\frac{4}{3}}|^2\right) \\
& + 2|1^{--}|^2\left(|3_A|^2 + \frac{1}{2}|3^{-\frac{1}{3}}|^2 + |3^{-\frac{4}{3}}|^2 - \frac{\sqrt{2}}{2}(3_A 3^{\frac{1}{3}} + h.c.)\right) \\
& + (1^+ 1^0 + 1^{++} 1^-)\left(3^{-\frac{2}{3}} 3^{-\frac{1}{3}} + 3^{\frac{1}{3}} 3^{-\frac{4}{3}} + \sqrt{2}({3_A}^* 3^{-\frac{4}{3}} + 3^{-\frac{2}{3}} 3_A)\right)
\end{aligned}
\tag{3.29}
$$

$$
\begin{aligned}
& + \frac{1}{2}|3^{-\frac{4}{3}}|^4 + \frac{1}{2}|3^{\frac{2}{3}}|^4 + \frac{1}{4}|3^{-\frac{1}{3}}|^4 + |3^{-\frac{1}{3}}|^4 + |3_A|^4 \\
& + \left(|3^{\frac{2}{3}}|^2 + |3^{-\frac{4}{3}}|^2\right) + 2\left((3^{\frac{1}{3}})^2 3^{\frac{2}{3}} 3^{-\frac{4}{3}} + h.c.\right) \\
& + {3_A}^* 3_A\left(|3^{\frac{2}{3}}|^2 + |3^{-\frac{1}{3}}|^2 + |3^{-\frac{4}{3}}|^2\right) \\
& + \frac{1}{2}\left(3^{-\frac{2}{3}} 3_A {3_A}^* 3^{\frac{2}{3}} + 3^{\frac{4}{3}} 3_A {3_A}^* 3^{-\frac{4}{3}} + 3^{\frac{1}{3}} 3_A {3_A}^* 3^{-\frac{1}{3}}\right) \\
& + \left(3^{-\frac{2}{3}} \bar{3}_A {\bar{3}_A}^* 3^{\frac{2}{3}} + 3^{\frac{4}{3}} \bar{3}_A {\bar{3}_A}^* 3^{-\frac{4}{3}} + 3^{\frac{1}{3}} \bar{3}_A {\bar{3}_A}^* 3^{-\frac{1}{3}} + {3_A}^* \bar{3}_A {\bar{3}_A}^* 3_A\right) \\
& + \sqrt{2}\left(3^{-\frac{2}{3}} \bar{3}_A {3_A}^* 1^0 + 3^{\frac{4}{3}} \bar{3}_A {3_A}^* 1^{--} + 3^{\frac{1}{3}} \bar{3}_A {3_A}^* 1^- + h.c.\right)
\end{aligned}
$$

$$
\begin{aligned}
& + 4Tr\Big(3^*.6.6^*.3 + {3_A}^* 6.6^* 3_A + ((6^* 3_A).(\epsilon 3^* \epsilon \bar{3}_A) \\
& \qquad\qquad + (3^*.6){\bar{3}_A}^* 3_A + {\bar{3}_A}^* \epsilon.3.3^*.6 - 3^*.6.6^*.\epsilon 3_A + h.c.)\Big) \\
& + |\bar{3}_A|^4 + Tr\Big(|6^*.6|^2 + 2{\bar{3}_A}^* 6.6^* \bar{3}_A + 2(6^*.6)(\bar{3}_A^* \bar{3}_A) \\
& \qquad\qquad + (6^* \bar{3}_A.\epsilon 6^* \epsilon \bar{3}_A - 26^*.6.6^*.\epsilon + h.c.)\Big) \\
& + \sqrt{2}\left(3^{-\frac{2}{3}} \bar{3}_A {3_A}^* 1^0 + 3^{\frac{1}{3}} \bar{3}_A {3_A}^* 1^- + 3^{\frac{4}{3}} \bar{3}_A {3_A}^* 1^{--} + h.c.\right) \\
& + \left({3_A}^* 6^{-\frac{2}{3}} {3_A}^* 1^0 + {3_A}^* 6^{\frac{1}{3}} {3_A}^* 1^- + {3_A}^* 6^{\frac{4}{3}} {3_A}^* 1^{--} + h.c.\right) \\
& + \left(3^{-\frac{2}{3}} 6^{\frac{4}{3}} 3^{-\frac{2}{3}} + 3^{-\frac{2}{3}} 6^{\frac{1}{3}} 3^{\frac{1}{3}} + 3^{\frac{1}{3}} 6^{\frac{1}{3}} 3^{-\frac{2}{3}} + 3^{\frac{1}{3}} 6^{-\frac{2}{3}} 3^{\frac{1}{3}} + h.c.\right) 1^0
\end{aligned}
$$

Appendix A

Computational Tools

SARAH

SARAH is a package under the software Mathematica. Many of the usual notions of a beyond standard model such as gauge groups, gauge fields, fermionic fields, scalars, vacuum expectation values and so on are known for this package. It also knows many of the representations of various useful groups and is able to, for example, multiply fields with correct rules. Once one learns the simple language of this package, by implementing a model in it one can benefit from many nice features especially because it is able to do some loop calculations.

Once a model is introduced, SARA can calculate two loop renormalization group equations for all the couplings and parameters of the model. It is also able to do straightforward but messy things such as defining all the vertices, mass matrices, and tadpoles equations. Moreover, one-loop corrections for tadpoles and self-energies will be derived.

SARAH also helps a lot by providing input files for many other packages in high energy physics such as FeynArts, CalcHep/CompHep, MicrOmegas, MadGraph 5, WHIZARD, OMEGA, SPheno. We will see in the next section how one can benefit from files that this package creates to explore phenomenological features of the model using Vevacious.

For a large number of beyond standard models, model files for SARAH already exist and one can explore different aspects or manipulate the models to investigate more. Finally, it produces latex and PDF output files which can save time for later use.

Here we illustrate how the complex singlet extended standard model of chapter 2 is implemented in the package and how one can obtain the renormalization group equations that we used in the same chapter. To start we need to define different pieces of the model in a model file with the format ".m". There are few logical steps. First is to define local and global symmetries. In our simple model, there are only standard model gauge symmetries. This can be easily done by

```
Gauge[[1]]={B,   U[1], hypercharge, g1,False};
Gauge[[2]]={WB, SU[2], left,        g2,True};
Gauge[[3]]={G,  SU[3], color,       g3,False};
```

Gauge symmetries and associated coupling constants are introduced. The "true" for left symmetry shows that we explicitly introduce different components of fields under this symmetry and SARAH expands field multiplications with respect to the indices for this group.

Next is to define different fields. This is done by indicating the type of each field and its quantum numbers under defined symmetries. For fermions, the number of generations should also be mentioned. If one needs, Yukawa couplings can be defined later as matrices.

```
(* Matter Fields *)

FermionFields[[1]] = {q, 3, {uL, dL},    1/6, 2,  3};
FermionFields[[2]] = {l, 3, {vL, eL},   -1/2, 2,  1};
FermionFields[[3]] = {d, 3, conj[dR],    1/3, 1, -3};
FermionFields[[4]] = {u, 3, conj[uR],   -2/3, 1, -3};
FermionFields[[5]] = {e, 3, conj[eR],      1, 1,  1};
FermionFields[[6]] = {v, 3, conj[vR],      0, 1,  1};

ScalarFields[[1]] =  {H, 1, {Hp,H0},      1/2, 2,  1};
ScalarFields[[2]] =  {S,  1, ss,            0,1 ,  1};
ScalarFields[[2]] =  {TT, 1, ttt,           0,1 ,  1};

RealScalars = {ss};
RealScalars = {ttt};
```

In the last line, scalar fields are emphasized to be real. Instead of a single complex field in this simple case, we have used two real scalars. As mentioned, fields which are in fundamental representation of left symmetry are defined by their components. Right-handed particles are defined by their left-handed charge conjugate counterparts so to have only left-handed fields. These are therefore in conjugate representation, indicated by -3, instead of fundamental representation. Right-handed neutrino is also defined as the sixth fermion fields. Its Yukawa coupling plays an important role at unification scale as we saw in chapter 2.

Next step is to define different states of the model with respect to symmetry breaking. At the beginning the model is in the gauge state indicated by GaugeES. The Lagrangian is defined for this state in different pieces and it is emphasized whether hermitian conjugate should be added to each part or not.

```
NameOfStates={GaugeES, EWSB};

(* ----- Before EWSB ----- *)

DEFINITION[GaugeES][LagrangianInput]= {
    {LagYuk, {AddHC->True}},
    {Lagsh, {AddHC->False}},
    {LagHmass,{AddHC->False}}
};
```

For this simple model, there is only one more step which is the electroweak symmetry breaking and there is no need to change the default.

While the gauge symmetries and fields with their quantum numbers are introduced in the previous steps, the package is already aware of all the kinetic terms in the Lagrangian. The only pieces which have to be introduced now are scalar potential and Yukawa terms. Implementing a model like this has this pedagogical benefit that one notices how much of a model is arbitrary and must be defined and what parts are automatic in any field theory. Representations, parameters, and scalar sector have to be defined. In noncommutative geometry approach however, the scalar sector is dictated by spectral action and naturally one can expect that a program could take much less impute for models under that category.

```
LagHmass =   ( mH2 conj[H].H + mS2 S.S + mT2 TT.TT + mv v v
Lagsh =     - ( lss S.S.S.S + lss TT.TT.TT.TT + lss S.S.TT.TT
            + lh conj[H].H.conj[H].H + 2 lsh conj[H].H.S.S
            + 2 lsh conj[H].H.TT.TT );
LagYuk =    + Yu H.u.q + Yv H.v.l);
```

The dot between fields is important and SARAH already knows how to multiply these fields based on their representations. The last line is for Yukawa interactions. All the couplings are free parameters of the model and are defined in a separate file called "parameters.m". For noncommutative geometry version, there are relations between these parameters at unification scale which are usually applied as initial conditions when the parameters are renormalized and run with respect to energy scales.

After these, we naturally come to the symmetry breaking and need to indicate which fields admit vacuum expectation values and how the gauge sector changes after spontaneous symmetry breaking happens. Naturally only W and Z bosons appear in our simple extension and there is no need to change the default.

```
DEFINITION[EWSB][GaugeSector] =
{
  {{VB,VWB[3]},{VP,VZ},ZZ},
  {{VWB[1],VWB[2]},{VWp,conj[VWp]},ZW}
};
```

For each scalar in the following lines, the first pair shows the vacuum expectation value and its coefficient (for simplicity), and the other two show names of the real and imaginary parts and their coefficients.

```
DEFINITION[EWSB][VEVs]=
{    {H0,   {VEV, 1/Sqrt[2]}, {Ah, I/Sqrt[2]},{hh, 1/Sqrt[2]}},
     {ss,   {s0,1/Sqrt[2]},    {0,0}, {ssss,1/Sqrt[2]}} ,
     {ttt,  {t0,1/Sqrt[2]},    {0,0}, {tttt,1/Sqrt[2]}}
};   (*I/Sqrt[2]*)
```

The last step is to indicate names for scalars which mix together, and also for fermionic mass eigenstates which are clear and one rarely needs to change them so there is no need to explain them further. There are also two more files which have information about the parameters and particles. One use of these files is for when we want to create an output file by SARAH and use it as an input file in another package for example for numerical purposes.

We have basically implemented our model in SARAH. To get benefits now, one needs to call the package and the model file in a Mathematica notebook file and use SARAH commands. For example, the command "CalcRGEs[ ]" finds renormalization group equations for all the parameters and couplings to two loop order. If the model is complicated or one is not interested in two loop order, "CalcRGEs[TwoLoop -> False]" calculates the equations to one loop order. Another interesting aspect of SARAH is that it also recognizes supersymmetric fields and susy models can be easily implemented in it as well. One can refer to https://sarah.hepforge.org for more about the package and its benefits.

VEVACIOUS

Here we just briefly introduce one more aspect of this package that we have tried for the complex extended model of chapter 2. The goal is to use Vevacious, an under Linux package, to find transitional time from a local minimum to the deeper minimum next to it. SARAH provides three input files to be used for this purpose.

1. Tree level todpole equations which are in the format required by HOM4PS2.

2. A SPheno file which has the parameters of the model in SLHA format. This is a file which of course could be written by user but would take time and effort. SARAH can provide it and this reduces the chance of errors. This is done by the command "MakeSPeno[ ]" in SARAH which produces a folder named SPheno. Copying this folder in SPheno directory and running the command "sodu make model=folder name", produces SPhenoModelName file in the bin directory of the package. Now running this file in any directory will produce "SPeno.spc.ModelName" in that directory.

   One needs now to use these two files as inputs for HOM4PS2 to find the tree level extrema and produce an input file for iminuit.

3. SARAH calculates one loop effective potential and stores it in a file with "vin" format which can be used by iminuit. This file is generated by the command "MakeVevacious[ ]" with the name of "ModelName.vin".

Now, iminuit, a strong Python interface, uses these two files as inputs. One is produced by HOM4PS2 which tells were the tree level extrema are, the other is produced by SARAH and shows how the potential is changed by the loop effects. It then finds both one loop corrections for the minima and the global minima.

Next steps are done by VEVACIOUS automatically. It uses the files produced in the previous steps as well as CosmoTransitions package to calculate the transitional time from the local minimum to the global minimum. The will be stored in an output file named as ModelName.vout.

For more information refer to https://vevacious.hepforge.org and reference [57].

Bibliography

[1] K. S. Stelle. "Renormalization of Higher Derivative Quantum Gravity". In: *Phys. Rev.* D16 (1977), pp. 953–969. DOI: 10.1103/PhysRevD.16.953 (cit. on p. 8).

[2] Ali H. Chamseddine and Alain Connes. "Quantum Gravity Boundary Terms from Spectral Action". In: *Phys. Rev. Lett.* 99 (2007), p. 071302. DOI: 10.1103/PhysRevLett.99.071302. arXiv: 0705.1786 [hep-th] (cit. on pp. 8, 11).

[3] Masoud Khalkhali and Matilde Marcolli, eds. *An invitation to noncommutative geometry. Proceedings, International Workshop, NCG 2005, Tehran, Iran, September 11-22, 2005.* 2008 (cit. on p. 10).

[4] Peter B. Gilkey. "The Spectral geometry of a Riemannian manifold". In: *J. Diff. Geom.* 10.4 (1975), pp. 601–618 (cit. on pp. 10, 27, 59, 62).

[5] P. Gilkey. *Invariance theory, the heat equation, and the Atiyah-Singer index theorem.* 2nd Edition. Boca Raton : CRC Press, 1995 (cit. on pp. 10, 19, 24).

[6] Ali H. Chamseddine and Alain Connes. "The Spectral action principle". In: *Commun. Math. Phys.* 186 (1997), pp. 731–750. DOI: 10.1007/s002200050126. arXiv: hep-th/9606001 [hep-th] (cit. on pp. 11, 18, 20, 23, 24, 44, 51, 62).

[7] Farzad Fathizadeh, Asghar Ghorbanpour, and Masoud Khalkhali. "Rationality of Spectral Action for Robertson-Walker Metrics". In: *JHEP* 12 (2014), p. 064. DOI: 10.1007/JHEP12(2014)064. arXiv: 1407.5972 [math-ph] (cit. on p. 11).

[8] Ali H. Chamseddine and Alain Connes. "The Uncanny Precision of the Spectral Action". In: *Commun. Math. Phys.* 293 (2010), pp. 867–897. DOI: 10.1007/s00220-009-0949-3. arXiv: 0812.0165 [hep-th] (cit. on p. 11).

[9] Alain Connes. *Noncommutative geometry.* 1994. ISBN: 9780121858605. URL: http://www.alainconnes.org/docs/book94bigpdf.pdf (cit. on pp. 14, 19, 24).

[10] Paul Langacker. *The standard model and beyond.* 2016. ISBN: 9781420079067. URL: http://www.taylorandfrancis.com/shopping_cart/search/search.asp?search=978-1-4200-7906-7 (cit. on p. 19).

[11] J. Madore. "An introduction to noncommutative differential geometry and its physical applications". In: *Lond. Math. Soc. Lect. Note Ser.* 257 (2000), pp. 1–371 (cit. on p. 19).

[12] T Hungerford. *Algebra (graduate texts in mathematics).* 2000 (cit. on p. 19).

[13] Masoud Khalkhali. "Lectures on noncommutative geometry". In: *An invitation to noncommutative geometry. Proceedings, International Workshop, NCG 2005, Tehran, Iran, September 11-22, 2005.* 2007, pp. 169–273. arXiv: math/0702140 [math-qa] (cit. on p. 19).

[14] D. V. Vassilevich. "Heat kernel expansion: User's manual". In: *Phys. Rept.* 388 (2003), pp. 279–360. DOI: 10.1016/j.physrep.2003.09.002. arXiv: hep-th/0306138 [hep-th] (cit. on pp. 19, 24).

[15] Walter D. van Suijlekom. *Noncommutative geometry and particle physics.* Mathematical Physics Studies. Dordrecht: Springer, 2015. ISBN: 9789401791618, 9789401791625. DOI: 10.1007/978-94-017-9162-5. URL: http://www.springer.com/978-94-017-9161-8 (cit. on p. 19).

[16] Williams; , Dana P. *Lecture Notes on C-algebras.* 2011. URL: https://math.dartmouth.edu/archive/m123s11/public_html/cstar.pdf (cit. on p. 19).

[17] V. S. Sunder. *Functional Analysis: Spectral Theory.* 2000. ISBN: 978-93-80250-90-8. DOI: 10.1007/978-93-80250-90-8 (cit. on p. 19).

[18] M. Nakahara. *Geometry, topology and physics.* 2003 (cit. on p. 19).

[19] Fedele Lizzi. "Noncommutative Geometry and Particle Physics". In: *PoS* CORFU2017 (2018), p. 133. DOI: 10.22323/1.318.0133. arXiv: 1805.00411 [hep-th] (cit. on p. 19).

[20] Ali H. Chamseddine and Alain Connes. "Resilience of the Spectral Standard Model". In: *JHEP* 09 (2012), p. 104. DOI: 10.1007/JHEP09(2012)104. arXiv: 1208.1030 [hep-ph] (cit. on pp. 21, 30, 34, 36, 58, 61, 67).

[21] C. P. Burgess, Maxim Pospelov, and Tonnis ter Veldhuis. "The Minimal model of nonbaryonic dark matter: A Singlet scalar". In: *Nucl. Phys.* B619 (2001), pp. 709–728. DOI: 10.1016/S0550-3213(01)00513-2. arXiv: hep-ph/0011335 [hep-ph] (cit. on p. 21).

[22] Raul Costa et al. "Two-loop stability of a complex singlet extended Standard Model". In: *Phys. Rev.* D92 (2015), p. 025024. DOI: 10.1103/PhysRevD.92.025024. arXiv: 1411.4048 [hep-ph] (cit. on pp. 21, 23, 46).

[23] Giuseppe Degrassi et al. "Higgs mass and vacuum stability in the Standard Model at NNLO". In: *JHEP* 08 (2012), p. 098. DOI: 10.1007/JHEP08(2012)098. arXiv: 1205.6497 [hep-ph] (cit. on p. 22).

[24] Joan Elias-Miro et al. “Higgs mass implications on the stability of the electroweak vacuum”. In: *Phys. Lett.* B709 (2012), pp. 222–228. DOI: 10.1016/j.physletb.2012.02.013. arXiv: 1112.3022 [hep-ph] (cit. on p. 22).

[25] Dario Buttazzo et al. “Investigating the near-criticality of the Higgs boson”. In: *JHEP* 12 (2013), p. 089. DOI: 10.1007/JHEP12(2013)089. arXiv: 1307.3536 [hep-ph] (cit. on p. 22).

[26] Ali H. Chamseddine, Alain Connes, and Matilde Marcolli. “Gravity and the standard model with neutrino mixing”. In: *Adv. Theor. Math. Phys.* 11.6 (2007), pp. 991–1089. DOI: 10.4310/ATMP.2007.v11.n6.a3. arXiv: hep-th/0610241 [hep-th] (cit. on pp. 23, 34, 36, 51, 54, 58).

[27] Chian-Shu Chen and Yong Tang. “Vacuum stability, neutrinos, and dark matter”. In: *JHEP* 04 (2012), p. 019. DOI: 10.1007/JHEP04(2012)019. arXiv: 1202.5717 [hep-ph] (cit. on pp. 23, 46).

[28] Matthew Gonderinger, Hyungjun Lim, and Michael J. Ramsey-Musolf. “Complex Scalar Singlet Dark Matter: Vacuum Stability and Phenomenology”. In: *Phys. Rev.* D86 (2012), p. 043511. DOI: 10.1103/PhysRevD.86.043511. arXiv: 1202.1316 [hep-ph] (cit. on pp. 23, 31, 46).

[29] Joan Elias-Miro et al. “Stabilization of the Electroweak Vacuum by a Scalar Threshold Effect”. In: *JHEP* 06 (2012), p. 031. DOI: 10.1007/JHEP06(2012)031. arXiv: 1203.0237 [hep-ph] (cit. on p. 23).

[30] Alain Connes. “C* algebras and differential geometry”. In: *Compt. Rend. Hebd. Seances Acad. Sci.* A290.13 (1980), pp. 599–604. arXiv: hep-th/0101093 [hep-th] (cit. on p. 23).

[31] Ali H. Chamseddine and Alain Connes. “Noncommutative Geometry as a Framework for Unification of all Fundamental Interactions including Gravity. Part I”. In: *Fortsch. Phys.* 58 (2010), pp. 553–600. DOI: 10.1002/prop.

201000069. arXiv: 1004.0464 [hep-th] (cit. on pp. 24, 27–30, 32, 44, 51, 52, 54, 58, 59).

[32] Ali H. Chamseddine and Alain Connes. "Why the Standard Model". In: *J. Geom. Phys.* 58 (2008), pp. 38–47. DOI: 10.1016/j.geomphys.2007.09.011. arXiv: 0706.3688 [hep-th] (cit. on pp. 25, 26, 28).

[33] Ali H. Chamseddine, Alain Connes, and Walter D. van Suijlekom. "Beyond the Spectral Standard Model: Emergence of Pati-Salam Unification". In: *JHEP* 11 (2013), p. 132. DOI: 10.1007/JHEP11(2013)132. arXiv: 1304.8050 [hep-th] (cit. on pp. 25, 43, 45, 51, 52, 59, 61, 72).

[34] Ali H. Chamseddine, Alain Connes, and Viatcheslav Mukhanov. "Geometry and the Quantum: Basics". In: *JHEP* 12 (2014), p. 098. DOI: 10.1007/JHEP12(2014)098. arXiv: 1411.0977 [hep-th] (cit. on p. 25).

[35] Ali H. Chamseddine and Alain Connes. "Spectral Action for Robertson-Walker metrics". In: *JHEP* 10 (2012), p. 101. DOI: 10.1007/JHEP10(2012)101. arXiv: 1105.4637 [hep-th] (cit. on p. 26).

[36] F. Staub. "SARAH". In: (2008). arXiv: 0806.0538 [hep-ph] (cit. on pp. 32, 46).

[37] Agostino Devastato, Fedele Lizzi, and Pierre Martinetti. "Grand Symmetry, Spectral Action, and the Higgs mass". In: *JHEP* 01 (2014), p. 042. DOI: 10.1007/JHEP01(2014)042. arXiv: 1304.0415 [hep-th] (cit. on p. 51).

[38] Ludwik Dabrowski et al. "The Dirac operator on SU(q)(2)". In: *Commun. Math. Phys.* 259 (2005), pp. 729–759. DOI: 10.1007/s00220-005-1383-9. arXiv: math/0411609 [math-qa] (cit. on p. 51).

[39] Ludwik Dabrowski and Giovanni Landi. "Instanton algebras and quantum four spheres". In: *Differ. Geom. Appl.* 16 (2002), pp. 277–284. DOI: 10.1016/S0926-2245(02)00066-9. arXiv: math/0101177 [math-qa] (cit. on p. 51).

[40] Ali H. Chamseddine, Alain Connes, and Walter D. van Suijlekom. "Inner Fluctuations in Noncommutative Geometry without the first order condition". In: *J. Geom. Phys.* 73 (2013), pp. 222–234. DOI: 10.1016/j.geomphys.2013.06.006. arXiv: 1304.7583 [math-ph] (cit. on pp. 51–53, 55, 69).

[41] John W. Barrett. "A Lorentzian version of the non-commutative geometry of the standard model of particle physics". In: *J. Math. Phys.* 48 (2007), p. 012303. DOI: 10.1063/1.2408400. arXiv: hep-th/0608221 [hep-th] (cit. on p. 54).

[42] Ufuk Aydemir et al. "B-decay anomalies and scalar leptoquarks in unified Pati-Salam models from noncommutative geometry". In: *JHEP* 09 (2018), p. 117. DOI: 10.1007/JHEP09(2018)117. arXiv: 1804.05844 [hep-ph] (cit. on pp. 55, 62).

[43] Ali H. Chamseddine, Alain Connes, and Walter D. van Suijlekom. "Grand Unification in the Spectral Pati-Salam Model". In: *JHEP* 11 (2015), p. 011. DOI: 10.1007/JHEP11(2015)011. arXiv: 1507.08161 [hep-ph] (cit. on pp. 55, 63).

[44] Agostino Devastato and Pierre Martinetti. "Twisted spectral triple for the Standard Model and spontaneous breaking of the Grand Symmetry". In: *Math. Phys. Anal. Geom.* 20.1 (2017), p. 2. DOI: 10.1007/s11040-016-9228-7. arXiv: 1411.1320 [hep-th] (cit. on p. 58).

[45] Hosein Karimi. "Implications of the complex Singlet field for Noncommutative Geometry model". In: *JHEP* 12 (2017), p. 040. DOI: 10.1007/JHEP12(2017)040. arXiv: 1705.01605 [hep-ph] (cit. on pp. 58, 61, 67, 72).

[46] Ufuk Aydemir et al. "Pati–Salam unification from noncommutative geometry and the TeV-scale W_R boson". In: *Int. J. Mod. Phys.* A31.01 (2016),

p. 1550223. DOI: 10.1142/S0217751X15502231. arXiv: 1509.01606 [hep-ph] (cit. on pp. 63, 67).

[47] Ling-Fong Li. “Group Theory of the Spontaneously Broken Gauge Symmetries”. In: *Phys. Rev.* D9 (1974), pp. 1723–1739. DOI: 10.1103/PhysRevD.9.1723 (cit. on p. 65).

[48] Luca Di Luzio. “Aspects of symmetry breaking in Grand Unified Theories”. PhD thesis. SISSA, Trieste, 2011. arXiv: 1110.3210 [hep-ph]. URL: http://www.infn.it/thesis/thesis_dettaglio.php?tid=6024 (cit. on p. 65).

[49] S. Rajpoot. “Symmetry Breaking and Intermediate Mass Scales in the SO(10) Grand Unified Theory”. In: *Phys. Rev.* D22 (1980), pp. 2244–2254. DOI: 10.1103/PhysRevD.22.2244 (cit. on p. 65).

[50] C. S. Aulakh and Rabindra N. Mohapatra. “Implications of Supersymmetric SO(10) Grand Unification”. In: *Phys. Rev.* D28 (1983), p. 217. DOI: 10.1103/PhysRevD.28.217 (cit. on p. 65).

[51] K. S. Babu and R. N. Mohapatra. “B-L Violating Proton Decay Modes and New Baryogenesis Scenario in SO(10)”. In: *Phys. Rev. Lett.* 109 (2012), p. 091803. DOI: 10.1103/PhysRevLett.109.091803. arXiv: 1207.5771 [hep-ph] (cit. on p. 65).

[52] R. N. Mohapatra. *Unification and Supersymmetry. The Frontier od Quark-Lepton Physics.* Berlin: Springer, 1986. ISBN: 9781475719307, 9781475719284. DOI: 10.1007/978-1-4757-1928-4 (cit. on p. 65).

[53] Langacker. “Grand Unified Theories and Proton Decay”. In: *Phys. Rept.* 72 (1981), p. 185. DOI: 10.1016/0370-1573(81)90059-4 (cit. on p. 65).

[54] Thomas Krajewski. “Classification of finite spectral triples”. In: *J. Geom. Phys.* 28 (1998), pp. 1–30. DOI: 10.1016/S0393-0440(97)00068-5. arXiv: hep-th/9701081 [hep-th] (cit. on p. 65).

[55] Matthias Sars. “Noncommutative Geometry and SU(5) Grand Unification”. In: (2011). URL: http://www.waltervansuijlekom.nl/wp-content/uploads/2014/05/thesisMatthias.pdf (cit. on pp. 65, 70).

[56] Hosein Karimi. “Symmetry Breaking and Proton Decay in Spectral Pati-Salam Model”. In: *submitted* (2018) (cit. on p. 72).

[57] J. E. Camargo-Molina et al. “**Vevacious**: A Tool For Finding The Global Minima Of One-Loop Effective Potentials With Many Scalars”. In: *Eur. Phys. J.* C73.10 (2013), p. 2588. DOI: 10.1140/epjc/s10052-013-2588-2. arXiv: 1307.1477 [hep-ph] (cit. on p. 81).