AMERICAN UNIVERSITY OF BEIRUT

Towards a Systematic and Unified Method to Solving Nonholonomic Systems

by
Karen Tatarian

A thesis
submitted in partial fulfillment of the requirements
for the degree of Master of Engineering
to the Department of Mechanical Engineering
of the Maroun Semaan Faculty of Engineering and Architecture
at the American University of Beirut

Beirut, Lebanon
July 2018

AMERICAN UNIVERSITY OF BEIRUT

Towards a Systematic and Unified Method to Solving Nonholonomic Systems

by

Karen Tatarian

Approved by:

Dr. Elie Shammas, Associate Professor Mechanical Engineering	Advisor
Dr. Daniel Asmar, Associate Professor Mechanical Engineering	Member of Committee
Dr. Imad Elhajj, Professor Electrical and Computer Engineering	Member of Committee ON BEHALF

Date of thesis defense: 28 July 2018

AMERICAN UNIVERSITY OF BEIRUT

THESIS, DISSERTATION, PROJECT RELEASE FORM

Student Name: TATARIAN KAREN PAUL
Last First Middle

☑ Master's Thesis ◯ Master's Project ◯ Doctoral Dissertation

☑ I authorize the American University of Beirut to: (a) reproduce hard or electronic copies of my thesis, dissertation, or project; (b) include such copies in the archives and digital repositories of the University; and (c) make freely available such copies to third parties for research or educational purposes.

☐ I authorize the American University of Beirut, to: (a) reproduce hard or electronic copies of it; (b) include such copies in the archives and digital repositories of the University; and (c) make freely available such copies to third parties for research or educational purposes after: **One ___ year from the date of submission of my thesis, dissertation or project.**
Two ___ years from the date of submission of my thesis , dissertation or project.
Three ___ years from the date of submission of my thesis , dissertation or project.

Karen T. 31 July 2018
Signature Date

This form is signed when submitting the thesis, dissertation, or project to the University Libraries

Acknowledgements

To Dr. Elie Shammas, my advisor and mentor, thank you for the ever present advice and guidance to help me proceed with my work. It was your continuous support that paved the way for this work. Thank you for the learning and growing process and the journey it led me on and for always knowing the right questions to ask.

To Dr.Daniel Asmar and Dr.Imad El Hajj, my thesis committee members, thank you for your patience and time and most of all for your support and counsel.

To my family and friends, thank you for the continuous love and support no matter where this journey took me and despite all the ups and downs that came along with it.

To my parents, thank you for believing in me and for never making any dream I had, big or small, feel out of reach.

"Deep in the sea, all molecules repeat the patterns of one another till complex new ones are formed. They make others like themselves ... and a new dance starts. Growing in size and complexity ... living things, masses of atoms, DNA, protein ... dancing a pattern ever more intricate. Out of the cradle onto the dry land ... here it is standing ... atoms with consciousness ... matter with curiosity. Stands at the sea ... wonders at wondering ... I ... a universe of atoms ... an atom in the universe.
The same thrill, the same awe and mystery, come again and again when we look at any problem deeply enough. With more knowledge comes deeper, more wonderful mystery, luring one on to penetrate deeper still. Never concerned that the answer may prove disappointing, but with pleasure and confidence we turn over each new stone to find unimagined strangeness leading on to more wonderful questions and mysteries – certainly a grand adventure!" -Richard Feynman

An Abstract of the Thesis of

Karen Tatarian for Master of Engineering
Major: Mechanical Engineering

Title: Towards a Systematic and Unified Method to Solving Nonholonomic Systems

Understanding locomotion is one of the most ongoing quests in the world of mechanics and robotics; ranging widely from legged locomotion to any biomechanically inspired system. While traditional frameworks, such as the Lagrangian and general Hamiltonian formulations, have served to provide a base platform for having an idea about the equations of motion of a system, it has equipped the user with no knowledge on the geometry of the mechanical systems with the nonholonomic constraints and no clarification on its implications. In addition, the resulting equations are of high index Differential-Algebraic equations (DAE's) with the constraints added at the force level as Lagrangian multipliers λ.

In this work, the formulation of reduced constraint Hamiltonian is utilized to model the dynamics of mechanical systems with nonholonomic constraints. Rather than enforcing the constraints by introducing Lagrange multipliers – extra variables and dimensions – to the equations of motion, the constraints are integrated at the geometry level. This allows the definition of a reduced constrained Hamiltonian and a Poisson structure which in turn are utilized to express the full dynamics of the system in a series of first order differential equations. A major contribution of this work it to make the formulation of such equations of motion accessible.

A 20-Step method is designed which requires the knowledge only of: the coordinates of the system q, the Lagrangian L, the constraint equations $\omega(q)$, the generalized forces τ, and the parameters of the variables $\mathbb{P}$. As a first step, the method finds the full symmetry group of the system on which a map is built transforming the dynamics from a general manifold to a reduced constraint submanifold.

Contents

Acknowledgements **v**

Abstract **vi**

1 Introduction **1**

2 Background **5**

2.1 Mathematical Preliminaries 5

2.1.1 Groups & Matrix Groups 6

2.1.2 Lie Groups and Group Actions 7

2.1.3 Tangent Space, Lie Algebras, and Lifted Actions 9

2.1.4 Fiber Bundles 11

2.2 Mechanics 13

2.2.1 Lagrangian Dynamics 13

2.2.2 Hamiltonian Dynamics 16

2.2.3 Reconstruction Equation 22

3 Systematic Method for Dynamic Modeling of Nonholonomic Mechanical Systems **26**

3.1 Overview Comparison between the Frameworks Presented 27

3.2 The 20-Step Method 30

3.2.1 Set-Up 30

3.2.2 Equivalence of Poisson and Lagrangian Reduction 32

3.2.3 Legendre Transformation & Reduced Constraint Hamiltonian 33

3.2.4 Poisson Reduction 33

3.2.5 Reduced Constraint Hamiltonian Equations 34

3.2.6 Summary of the 20-Step Method and Comparison to General Hamiltonian Formulation 35

3.3 Examples 40

4 Conclusion **64**

List of Figures

2.1 Sketch for Elroy Beanie example 8
2.2 The exponential map and the vector field generation on Q along the fiber G [1] 12
2.3 Sketch for unicycle example 15

3.1 Comparison flow chart of the three different formulations: Lagrangian Formulation, the Hamiltonian Formulation, and the Reduced Constraint Hamiltonian Formulation 28
3.2 Continuation of the Comparison flow chart of the three different formulations: Lagrangian Formulation, the Hamiltonian Formulation, and the Reduced Constraint Hamiltonian Formulation 29
3.3 Over-view of Systematic Method to Solving Nonholonomic Mechanical Systems (part 1/4) 31
3.4 General Flow-Chart of General Hamiltonian Formulation and Reduced Constraint Hamiltonian Formulation, where the color orange depicts being within a reduced constraint submanifold and the color blue represents being within the general manifold 35
3.5 Over-view of Systematic Method to Solving Nonholonomic Mechanical Systems (part 2/4) 37
3.6 Over-view of Systematic Method to Solving Nonholonomic Mechanical Systems (part 3/4) 38
3.7 Over-view of Systematic Method to Solving Nonholonomic Mechanical Systems (part 4/4) 39
3.8 The solution for $(x(t), y(t))$ of the Unicycle for a 2π cycle for all three sets of formulation, where C Hamiltonian refers to the Constrained Hamiltonian 50
3.9 The solution for $(\theta(t))$ of the Unicycle for a 2π cycle for all three sets of formulation 51
3.10 The solution for $(\phi(t))$ of the Unicycle for a 2π cycle for all three sets of formulation 51
3.11 The solution for λ_1 and λ_2 of the Unicycle for a 2π cycle for the Hamiltonian formulation and Lagrangian formulation 52
3.12 Sketch for Snakeboard example 52

3.13 The solution for $(x(t), y(t))$ of the Snakeboard for a 10π cycle for all three sets of formulation, where C Hamiltonian refers to the Constrained Hamiltonian 61
3.14 The solution for $(\theta(t))$ of the Snakeboard for a 10π for all three sets of formulation 62
3.15 The solution for $(\psi(t))$of the Snakeboard for a 10π cycle for all three sets of formulation 62
3.16 The solution for $(\phi(t))$ of the Snakeboard for a 10π cycle for all three sets of formulation 63
3.17 The solution for λ_1 and λ_2 of the Snakeboard for a 10π cycle for the Hamiltonian formulation and Lagrangian formulation 63

List of Tables

2.1 Various types of mechanical systems and their respective reconstruction equations 24

3.1 Selecting the full symmetry group G for the Unicycle 42
3.2 Initial conditions of the Unicycle for all three sets of formulations, where C-Hamiltonian refers to the Constrained Hamiltonian . . . 50
3.3 Initial conditions for all three sets of formulations of the Snakeboard, where C-Hamiltonian refers to the Constrained Hamiltonian 60

Chapter 1

Introduction

In the world of robotics, while machine intelligence is flourishing, the mechanical systems governing robots seem to be moving at a much slower pace. The understanding of locomotion is still an ongoing quest for many researchers from modeling legged locomotion to locomoting snake-like robots. In many cases there seems to lack a unified and consistent approach applicable to such mechanical systems specifically arbitrary nonholonomic mechanical systems.

Problems of nonholonomic mechanics, which have played an important role in robotics, wheeled vehicular dynamics and motion generation, have attracted considerable attention since they are intimately connected with fundamental engineering issues, including but not limited to path planning, dynamic stability, and control. Despite the history of nonholonomic mechanism, there has still not been established productive links with corresponding problems in the geometric mechanics of systems with configuration space constraints. In addition, traditional analysis of locomotion, including Lagrangian and Hamiltonian formulations, do not provide the complete examination of the dynamics of the system and the mechanism of the locomotion. This in turn renders issues of controllability and choosing gaits impossible to solve [2]. Unlike traditional frameworks, the work presented in this thesis builds on the geometry of mechanical systems with nonholonomic constraints in the attempt of better highlighting the structure of the equations of motion promoting the analysis and isolation of the crucial geometric objects that govern this motion. This in turn clarifies the basic mechanics underlying locomotion.

Nonholonomic systems are divided into two categories. First, constraints, which are not imposed externally but rather are a consequence of the equation of motions, are known as *dynamic nonholonomic constraints*. Such constraints are conserved by the basic Euler-Lagrangian and Hamiltonian equations and are sometimes regarded as conservation laws instead of constraints. Second, the constraints imposed by kinematics, known as *kinematic nonholonomic constraints*, are linear in velocity, such as rolling constraints[2].

With a long standing history, the theory of mechanical systems with kinematic constraints dates back to two centuries ago during which it had important contributions by Ferrers [1871], Neumann [1888], Vierkandt [1892], Hertz (1894), and Chaplygin [1897]. With ever growing importance of solving the problem, the theory was picked up again by several scientist, each introducing different methods. With the rise of modern differential geometry, various methods were used to obtain the equations of motion under invariant form using: connections([3],[4]), almost product structures ([5],[6]), and jet bundles ([7]) [8]. Many of these papers used reduction by symmetry groups ([9], [10],[11],[2], and [3]) making a promising tool for solving nonholonomic constraint problems. The method introduced in this thesis is a systematic "black box" approach, inspired by the Poisson Geometry within a Hamiltonian framework, that results in the equations of the motion of the nonholonomic mechanical system by only knowing the configuration space, Lagrangian, and constraint equations.

In the vast literature in classical mechanics, i.e Edelen [12] and Arnold [13], nonholonomic mechanical systems are described within the variational framework by Euler-Lagrange equations in addition to extra terms corresponding to the constraint forces. It was Bates & Śniatycki [9] that showed that the dynamics of mechanical systems with nonholonomic constraint may be described within the Hamiltonian framework. However, using the standard Hamiltonian equations were not permissible for such systems since the two-form with respect to which the Hamiltonian equations of motion (on a reduced state space, and without constraint forces) are defined result in equations of motions that do not admit canonical coordinates.

Influenced by Bates & Śniatycki , Van Der Schaft and Maschke [14] used the dual object of a "Poisson" bracket not necessarily satisfying the *Jacobi identity* to introduce a generalized bracket with respect to which the dynamics of the mechanical systems with nonholonomic constraint are portrayed in the Hamiltonian framework. The produced "Poisson" bracket satisfies the Jacobi Identity if and only if the constraints are holonomic. In order to find the dynamical equations of motion on the constrained state space and eliminating the Lagrangian multipliers, new coordinates (generally not canonical) were introduced by establishing new momenta. The final result was a reduced Poisson-like bracket, which allowed expressing the equations of motion in a pseudo-Hamiltonian format. This finding motivated further study of brackets not satisfying the Jacobi identity and their Hamiltonian equations of motion.

Building on the work of Van Der Schaft and Maschke [14], many scholars further developed the theory of nonholonomic Poisson reduction and tied it to other work in the area. Koon and Marsden [15] established the link between the symplectic geometry on the Hamiltonian side of nonholonmic systems, developed by Bates and Śniatycki [9], and the Lagrangian side. Specifically, they described the link between the momentum equation, the reduced Lagrange-d'Alembert equations, and the reconstruction equation corresponding to both sides. Moreover, this was proven to be equivalent to the Lagrangian reduction methods demonstrated by Bloch, Krishnaprasad, Marsden, and Murray [16]. Bloch [17] tied the previously mentioned work together and use the reduction procedure of Van Der Schaft and Maschke [14] to organize nonholonomic dynamics into a reconstruction equation, a nonholonomic momentum equation, and the reduced Lagrange-d'Alembert equations in Hamiltonian form.

In addition to providing the needed links between the symplectic geometry on the Hamiltonian side of nonholonomic systems and the Lagrangian reduction, Koon and Marsden [15] demonstrated where the the momentum lies on the Hamiltonian side and the way it relates to the dynamics of nonholonomic systems with symmetry into three parts:

1. a reconstruction equation for the group element g

2. an equation for the nonholonomic momentum p

3. the reduced Hamilton equations for the *shape variables* r and p_r

Moreover, traditional frameworks, including the Lagrangian and Hamiltonian formulations, lead to Differential-Algebraic equations (DAE) of high indexes. This is due to the fact that in such formulations the constraints are simply added at the last step as add-ons at the force level in the form of Lagrangian multipliers λ's. No special attribute is done concerning the fiber bundle of the system and the effect the constraints have on the geometry of the system. In contrast, the Hamiltonian formulation defines a physical coordinate, which is the momenta related to the motion. This leads to a set of DAE equations of index 1 in comparison to the Lagrangian Formulation, which results in a set of DAE equations of index 2. However, using the constraints along with the geometry of the configuration space results in the ability to built a reduced constraint submanifold in which the equations of motion can be retrieved using a reduced constraint Hamiltonian. This would eliminate the problem of the Lagrangian multipliers since the constraints were added at the initial level. Moreover, the result is a set of Ordinary Differential equations (ODE) of motion.

While using geometry to recover the equations of motion of the system has been approached by influential scientists in the field, all the work seem to diverge and thus making it inaccessible to users. Based on reduced constraint Hamiltonian formulation, this thesis presents a 20-step method that results in a set of $(2n - r)$ ODE's describing the equations of motions of any system with non-holonomic constraints with n and r being the number of coordinates q and the number of constraint equations, respectively. The 20-step method finds the Full Symmetry group of the system and on it builds a map from the general manifold to the reduced constraint submanifold. The method does not require the user to have any background knowledge on any of the formulations listed above nor on differential geometry. This is because, the flow of the steps from one to another take care of any needed changes in the submanifolds.

Chapter 2 is dedicated to introducing all mathematical and mechanical preliminaries needed for the thesis. This includes Lagrangian and Hamiltonian dynamics as well as the reconstruction equation. Moreover, Chapter 3 compares the three formulations: Lagrangian, Hamiltonian, and reduced constraint Hamiltonian. In addition, all the steps of the method and their effects on the flow leading to the final set of equations are highlighted. As a first step, the method finds the full symmetry group of the system and this gives q a structure and allows the mapping of the dynamics. In addition, the method makes use of the reconstruction equation as an intermediate step for simplification purposes and shows that whether one chooses to build the reconstruction equation using geometry or simply using an identity matrix, both lead to the same result. Furthermore, all three frameworks are applied to the Unicycle and Snakeboard problems and are simulated to show that the 20-Step method does indeed render the correct description of the dynamics of the system.

Chapter 2

Background

The purpose of this chapter is to introduce several terms and definitions that will be used throughout this thesis. In order to make the reader more comfortable with the material discussed in the rest of the thesis, this chapter is to refresh the reader about some terms with clarification examples and does not constitute either a reference or a tutorial.

The goal of this chapter is to highlight both mathematical and mechanics preliminary topics needed. First, the aim of the *mathematical preliminaries* section is to point out the special structure of the *configuration space (Q)* of simple mechanical systems. The systems configuration space is the set of configurations the system can assume. In addition, it has as many dimensions as the system has degrees of freedom, and a structure that encodes how these degrees of freedom are coupled or connect back on themselves [18]. Second, the *mechanics* section is to present the Lagrangian and Hamiltonian dynamics along with the *reconstruction equation*, which is generally made of both the conservation of momentum along the allowable directions and the nonholonomic constraints acting on the systems.

2.1 Mathematical Preliminaries

The aim of this chapter is to highlight the special structure of the *configuration space*, which is the space that represents the degrees of freedom of mechanical systems. In addition, it is important to shed a light on *principal fiber bundle* structure, which plays a vital role in the rest of the thesis. This will be done through a brief introduction to the *action* and *lifted action* maps by defining and employing the textitLie group structure of the **fiber space**, which is a sub-space of the configuration space, to map configurations and velocities, respectively. In conclusion, all terms will be defined in this section and the needed tools for each respectively will be introduced.

2.1.1 Groups & Matrix Groups

First, starting with a definition of groups and their properties:

Definition 2.1 : **Group** A group $(G, *)$ is a combination of a set G and an operation $*$, which acts as a map taking $G \times G \to G$. A group has the following four properties:

1. **Closure**: the product of any element belonging to G acting on another by the group operation $(*)$ is in concequence an element of G

$$g_1, g_2 \epsilon G \ , \qquad g_1 * g_2 \epsilon G$$

2. **Associativity**: the order in which a sequence of group operations are assessed does not change their net product

$$(g_1 * g_2) * g_3 \ = \ g_1 * (g_2 * g_3)$$

3. **Identity**: there exists an identity element e in the set, that does not affect other elements when interacting with them

$$e * g \ = \ g * e \ = \ g$$

4. **Inverse**: the inverse, with respect to the group operation, of each group element is an element that belongs to group. When operating on each other, they produce the identity element

$$\begin{gathered} g \ \epsilon \ G \ , \ g^{-1} \ \epsilon \ G \\ g^{-1} * g \ = \ g * g^{-1} \ = \ e \end{gathered}$$

Second, important for the rest of thesis is to introduce the groups that represent rigid body motions. Groups of rigid rotation $SO(n)$ are *special orthogonal groups*. The set of elements for $SO(n)$ is given by

$$SO(n) \ = \ \{R \ \epsilon \ \mathbb{R}^{n \times n} : RR^T \ = \ I^{n \times n}, detR = +1\}$$

where $I^{n \times n}$ is a $n \times n$ identity matrix. For example, to represent all rotations in the plane or the set of all displacements that can be generated by a single revolute joint, taking $n = 2$ results in $SO(2)$, which is the group of rotations in two dimensions. The elements of $SO(2)$ have the following form:

$$SO(2) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} : \theta \ \epsilon \ \mathbb{R} \right\} \tag{2.1}$$

Another significant group is the *special Euclidean group* $SE(n)$, which is the group of rigid translations and rotations in an n-dimensional space. The set of elements for $SE(n)$ is given by:

$$SE(n) = \left\{ \begin{pmatrix} R^{n\times n}| & p^{n\times 1} \\ \hline 0^{1\times n}| & 1 \end{pmatrix} : p \in \mathbb{R}^{n\times n} \in SO(n) \right\} = \mathbb{R}^n \times SO(n) \qquad (2.2)$$

$SE(3)$ is the special Euclidean group of rigid body displacements in three-dimensions. A subgroup of $SE(3)$ is $SE(2)$, which is used to represent the configuration of a rigid body in the plane. For more on rigid body motion groups, the reader is referred to [19] and [20].

2.1.2 Lie Groups and Group Actions

Before describing Lie groups, a few definitions need to be introduced including manifolds. In a nutshell, a manifold is a topological space that locally looks like $\mathbb{R}^n$. A good start would be defining topological space.

Definition 2.2 : **Topological Space** A set X in combination with a collection of open subsets T form a topological space if the following conditions and properties apply:

1. The empty set $\emptyset$ is in T
2. The entire set X is in T
3. The intersection of a finite number of sets in T is also in T
4. The union of an arbitrary number of sets in T is also in T

Definition 2.3 : **Manifold** A manifold $\mathcal{M}$ is a topological space on which the following properties apply:

1. $\mathcal{M}$ is Hausdorff
2. $\mathcal{M}$ is locally Euclidean
3. $\mathcal{M}$ has a countable basis of open sets

A **submanifold** is a subset of a manifold that is itself a manifold, but has a smaller dimension. For more information, the reader is referred to [21] and [22].

Definition 2.4 : **Homeomorphism** A homeomorphism is a function f mapping between two spaces and it has the following properties:

1. **Bijective**, that is invertible. Bijective functions are *surjective (onto)* and *injective (one-to-one)* that is:

(a) f^{-1} is a true function whose domain is the entire range of f

(b) $f * f^{-1}$ and $f^{-1} * f$ are both identity maps

2. **Continuous** with a continuous inverse.

Diffeomorphism is homeomorphism for which both { and $\{^{-1}$ are additionally differentiable; implying that they are not only continuous, but their derivatives are also continuous.

Thus, a n-dimensional manifold is a topological space $\mathcal{M}$ for which every point $q \epsilon \mathcal{M}$ has a neighborhood homeomorphic to Euclidean space $\mathbb{R}^n$.

Definition 2.5 : **Lie group** A Lie group is a finite dimensional smooth manifold G together with a group structure on G, such that both

1. product map: $G \times G \rightarrow G$

2. inverse map $g \rightarrow g^{-1}: \; G \rightarrow G$

are both smooth maps. In addition, Lie groups are continuous groups, implying that the group elements act on each other in a continuous manner. $SO(2)$, $SE(3)$, and $SE(2)$ are also Lie groups

Example 2.1 *:* **Elroy Beanie**
Elroy's Beanie, shown in Figure 2.1, which a robot composed of two rigid bodies allowing rotation with respect to each other. Its variables are $q = (x, y, \theta, \phi)$, where (x, y, θ) identify the robot's location and orientation with respect to an inertial frame and (ϕ) specifies the relative angle between the rigid bodies. Lie

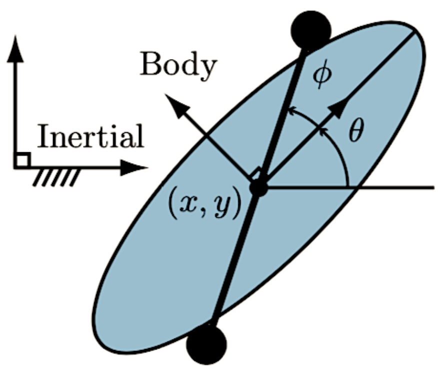

Figure 2.1: Sketch for Elroy Beanie example

group $SE(2)$ represents the location and orientation of the beanie; that is for initial configuration $g = (x, y, \theta) \,\epsilon\, SE(2)$, it can be represented as a matrix using homogeneous coordinates, where:

$$g = \begin{pmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{pmatrix} \tag{2.3}$$

Group actions are maps associated with Lie groups that allow the manifold elements to act on each other.

Definition 2.6 Group action An action of a group G on a set Q is represented by the map $\Phi(g, q) : G \times Q \to Q$ for some $g \,\epsilon G$ and $q \,\epsilon\, Q$. Action Φ has the following properties with $g, h \,\epsilon\, G$, $q \,\epsilon\, Q$ and e being the identity element of G:

1. $\Phi(g, \Phi(h, q)) = \Phi(gh, q)$
2. $\Phi(e, q) = q$

Left and Right actions are defined on matrix groups such that $L(g, h) = gh$ and $R(g, h) = hg$.

2.1.3 Tangent Space, Lie Algebras, and Lifted Actions

The first part of this section is to present the tangent spaces of manifolds used to represent the velocities of the robot. For instance, as seen in Example 2.1, $q = (x, y, \theta, \phi) \,\epsilon\, Q$ was defined as the configuration of the robot and Q as the configuration manifold of the mechanical system. The elements of the tangent space of this manifold would represent the configuration velocity elements, which are $\dot{q} = (\dot{x}, \dot{y}, \dot{\theta}, \dot{\phi})$.

Definition 2.7 : **Tangent Spaces on general differentiable manifolds** Take a curve $\gamma(t)$ on a manifold Q, it is parameterized by time and passes through a point $q \,\epsilon\, Q$ at time $t = 0$. Thus, $\gamma'(t)$ is the velocity vector to the curve $\gamma(t)$ at $t = 0$. There are infinitely many possible curves passing through point q at $t = 0$. As a result, the tangent vectors to these curves span a vector space, generally denoted by T_qQ, which describes the tangent space of Q at point q. In its turn, the tangent bundle, TQ is represented as

$$TQ = \bigcup_{q \epsilon Q} T_qQ$$

Definition 2.8 : **Vector field** A vector field is a smooth map X from a manifold Q to its tangent space T_qQ such that:

$$\begin{aligned} X \quad : \quad & Q \to T_qQ \\ & q \mapsto X(q) \end{aligned}$$

Thus, $X(Q)$ is the set of all smooth vector fields on Q.

Definition 2.9 : Lie algebra A Lie algebra is a vector space $\mathfrak{g}$ with an operation $[.,.] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, which is a Lie bracket, and the following axioms are satisfied:

1. It is **skew symmetric**, $[\xi, \xi] = 0$ for all $\xi \in \mathfrak{g}$
2. It is **bilinear**, $[a\xi, b\eta] = a[\xi, \eta] + b[\xi, \eta]$ and $[\xi, a\eta + b\lambda] = a[\xi, \eta] + b[\xi, \lambda]$ for all $a, b, \in \mathbb{R}$ and $\xi, \eta, \lambda \in \mathfrak{g}$.
3. It satisfies the **Jacobi Identity**, $[[\xi, \eta], \lambda] + [[\lambda, \xi], \eta] + [[\eta, \lambda], \xi] = 0$ for all $\xi, \eta, \lambda \in \mathfrak{g}$

Using the *exponential map*, Lie algebra elements act like generators, which can recover the entire Lie group.

Definition 2.10 : Exponential map For a Lie group G with its associated Lie algebra $\mathfrak{g}$, one can define the exponential map with $\xi \in \mathfrak{g}$ and $t \in \mathbb{R}$ defined by:

$$\begin{aligned} exp \quad : \quad & \mathfrak{g} \to G \\ & \xi \mapsto g = exp(t\xi) \end{aligned}$$

The exponential map generates a configuration fiber variable reached by flowing along the fiber for time t and whose initial velocity was ξ.

Now that the group actions acting on group and configuration elements were defined, it is crucial at this point to define *lifted actions*, which act on tangent vectors of manifolds.

Definition 2.11 : Lifted Action For a manifold Q, the lifted action is defined as a linear map represented as:

$$\begin{aligned} T_q\Phi_g \quad : \quad & T_qQ \to T_{\Phi_g q}Q \\ & \dot{q} \mapsto T_{\Phi_g q}\Phi_g \dot{q} \end{aligned}$$

The above translates to: the action and lifted action map q and $\dot{q}$ to a new configuration,$\Phi_g q$, and velocity, $T_{\Phi_g q}\Phi_g \dot{q}$, respectively.

In addition, for a given group velocity $\dot{g}$:

- The *left* lifted action which maps $\dot{g}$ to the Lie algebra, yields the *body velocity* representation of $\dot{g}$.
- The *right* lifted action, on the other hand, yields the *spatial velocity* representation of $\dot{g}$.

Definition 2.12 Body and spatial velocity representations For a given Lie group G with point $g \in G$ and $v_g \in T_gG$, the body and spatial body representations are defined respectively as:

$$\begin{aligned} \xi^b = T_g L_{g^{-1}} v_g \\ \xi^s = T_g R_{g^{-1}} v_g \end{aligned}$$

2.1.4 Fiber Bundles

A configuration is the minimum number of variables required to specifically represent the location in two or three dimensions of each physical point of the mechanism or robot [17]. While rigid bodies have a fixed shape with the location and orientation of a body-attached reference coordinate frame being sufficient, robots or mechanisms made up of several rigid bodies require additional variables to specify the robot's shape. Thus, the configuration space of multi-bodied robots are made of position and shape variables.

Since the position variables belong to the configuration manifold, the position variables are governed by a Lie group structure. Thus, one can assume that the position variables are elements of a set that has a group structure such as $\mathbb{R}^n$, $SO(n)$, or $SE(n)$. Thereby, for mechanical systems, general configuration manifold is represented by $Q = G \times M$ with G being the *fiber space* indicating the position of the robot and M being the *base space* specifying the internal shape of the robot. In this thesis, the configuration manifolds dealt with have a *principal fiber bundle* structure.

Definition 2.13 : **Fiber bundle** A fiber bundle is a manifold Q with a base subspace M and a projection map $\pi : Q \to M$ with fiber $G(r)$, which is defined by $G = \pi^{-1}(r)$,and it has every neighborhood $U \subset M$ of r satisfying:

The bundle is locally trivial: $\pi^{-1}(U)$ is homeomorphic to $G \times U$

Thus, the fiber bundle is denoted by $Q = (G, M)$.

Important notes to consider:

- Q is a *principal* fiber bundle if the fibers of the bundle are homeomorphic to the structure group, *i.e* fiber G has a group structure.
- Q is a *trivial* fiber bundle if $Q = G \times M$ globally.
- The configuration space of all mechanical systems are trivial principal fiber bundles.

For Example 2.1, $Q = SE(2) \times \mathbb{S}^1$ is a trivial principal bundle with G having a group structure. A lifted map, which is induced by the projection map associated with the fiber bundle manifold, acts on the manifold's tangent space as:

$$\pi : Q \to M \Rightarrow T\pi : TQ \to TM$$

Thus, the lifted action divides the manifold's tangent space into two subspaces.

Definition 2.14 : **Vertical and horizontal spaces** For a given *fiber bundle* Q along with a projection map $\pi : Q \to M$, a vertical space V_q is represented at point $q \in Q$ such that:

$$V_q = ker(T\pi)$$

On the other hand, the horizontal space H_q is simply the complement of V_q so that:

$$T_qQ = V_q \oplus H_q$$

As previously explained, Lie algebras define the entire group using the exponential map. This notion will now be extended to principal fiber bundles.

Definition 2.15 : **Lie group generator** $Q = G \times M$ is a trivial principal fiber bundle with an action $\Phi(g, q) : Q \to Q$ for which a generator on Q is defined as:

$$\begin{aligned} \xi_Q \quad : \quad & \mathfrak{g} \to TQ \\ & \xi \mapsto \tfrac{d}{dt}(\Phi(expt\xi, q))_{t=0} \end{aligned}$$

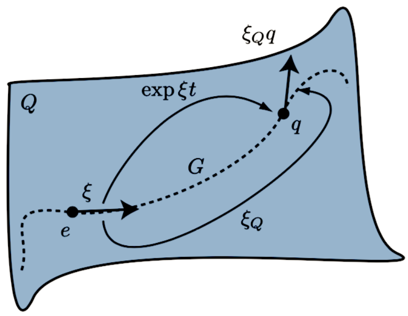

Figure 2.2: The exponential map and the vector field generation on Q along the fiber G [1]

Lemma 2.1 ***: Generator on principal bundles*** *For a given* $Q = G \times M$*, principal fiber bundle, the Lie group generator is equivalent to:*

$$\xi_Q(g, r) = (T_eR_g\xi, \dot{r})$$

For a configuration manifold with a principal fiber bundle, one can now define a *principal connection.*

Definition 2.16 : **Principal Connection**: For a given $Q = G \times M$, trivial principal fiber bundle, associated with a projection map $\pi : Q \to M$, a principal connection, which is a Lie algebra valued map $\mathcal{A} : TQ \to \mathfrak{g}$, is defined such that:

- $\mathcal{A}(\xi_Q(q)) = \xi$, for all $\xi \in \mathfrak{g}$ $q \in Q$, *i.e, for a vector in tangent space of Q was generated by a Lie group generator, the principal connection maps the generated vector back to the Lie algebra elements which generated it.*

- $\mathcal{A}(T_q\Phi_g v_g) = Ad_g\mathcal{A}(v_q)$, for all $q \epsilon Q$, $v_q \epsilon T_qQ$, and $g \epsilon G$, *i.e, for two elements belonging to the tangent space of Q and related by a lifted map $T_q\Phi_g$, the Lie algebra elements correlated to these vectors are connected by the Adjoint map.*

Lemma 2.2 ***: Kernel of the principal connection*** *For a given $Q = G \times M$, trivial principal fiber bundle, associated with a projection map $\pi : Q \to M$, V_q is the vertical space at $q \epsilon Q$ and $H_q = ker(\mathcal{A}(v_q)$ is the horizontal space such that $T_qQ = V_q \oplus H_q$.*

2.2 Mechanics

2.2.1 Lagrangian Dynamics

The first approach to model of mechanical systems and compute the governing equations of motion is the Lagrangian. The result is second order of differential equations of motion of the mechanical system that will be reduced by the end of the thesis using a different procedure.

Given a system with a n-dimensional configuration space Q, then the Lagrangian is a map $L(t, q, \dot{q}) : [a, b] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$. The Lagrangian takes a specific form for mechanical systems composed of multiple rigid bodies. For the following definitions, consider a mechanical system of n rigid bodies and whose configuration space is Q.

Definition 2.17 : Kinetic energy of multi-bodied mechanical systems The kinetic energy of the system is defined by:

$$KE = \sum_{i=1}^{n} \frac{1}{2}(\dot{q}_i^T m_i \dot{q}_i + \dot{\mu}_i^T j_i \dot{\mu}_i) \tag{2.4}$$

where $\dot{q}_i$ is the linear velocity of the center of mass of each rigid body, μ_i is the angular velocity of each of the bodies, m_i and j_i are the mass and inertia of each of the bodies respectively.

Definition 2.18 : Mass Matrix Through the kinetic energy of mechanical systems, a kinetic energy metric, which is on the tangent space of the configuration manifold, utilizing the mass matrix is defined by:

$$KE = \langle\langle v_1, v_2 \rangle\rangle = \frac{1}{2} v_1^T M(q) v_2 \tag{2.5}$$

where v_1,v_2 ϵ T_qQ and $M(q)$ is the mass matrix.

Definition 2.19 : Lagrangian for mechanical systems The Lagrangian for such systems is defined by:

$$L(q, \dot{q}) = \frac{1}{2}\dot{q}^T M \dot{q} - V(q) \tag{2.6}$$

where M is the mass matrix of the mechanical system, $(\frac{1}{2}\dot{q}^T M \dot{q})$ is the kinetic energy, and $V(q)$ is the potential energy.

The principle of *least action*, which is known as Hamiltons principle, states that the mechanical system is characterized by the Lagrangian function $L(q, \dot{q})$ such motion of the system satisfies a certain condition. The condition states that the mechanical system evolves in time such that the *action integral*, $\int_{t_0}^{t_1} L(q, \dot{q})dt$ is minimized. The resultant dynamics equations of motion for mechanical system is equivalent to the first Euler-Lagrange equations. Thus, the configuration of the system will evolve along the curve $q(t)$, which is a minimizer of the integral action.

Euler-Lagrange Equations

First, for isolated mechanical systems with no external forces nor nonholonomic constraints, q(t) must satisfy the first Euler-Lagrange equations which are the dynamic equations of motion:

$$\frac{d}{dt}\Big(\frac{\partial}{\partial \dot{q}_i} L(q, \dot{q})\Big) - \frac{\partial}{\partial q_i} L(q, \dot{q}) = 0, \qquad (i = 1, ..., n) \tag{2.7}$$

Second, for systems with k nonholonomic constraints given in the Pfaffian form $\omega(q)\dot{q} = 0$, the dynamic equations of motion given by:

$$\frac{d}{dt}\Big(\frac{\partial}{\partial \dot{q}_i} L(q, \dot{q})\Big) - \frac{\partial}{\partial q_i} L(q, \dot{q}) = \lambda_j \omega_i^j + \tau_i, \quad (i = 1, ..., n, \quad j = 1, ..., k) \tag{2.8}$$

where τ_i are the generalized external forces and λ_j's are the Lagrange multipliers.

Example 2.2 : The Unicycle
Given a unicycle as represented in figure 2.3. The system is made of a homogeneous disk rolling without slipping (represented as its nonholonomic constraints) on a horizontal plane. The robot's configuration space, which is four-dimensional manifold, is represented as $Q = \mathbb{R} \times \mathbb{R} \times \mathbb{S} \times \mathbb{S}$.The robot is described using the generalized coordinates $q = (x, y, \theta, \phi)$, where (x, y) denote the position of the contact point in the xy-plane, θ represents the orientation of the disk, and ϕ is the rotation angle of the disk.

As a side note, (x, y, θ) may be regarded as an element of the Euclidean group

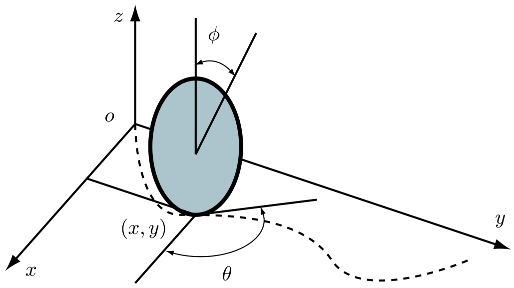

Figure 2.3: Sketch for unicycle example

$SE(2)$, which is the group of rigid motions in the plane, since (x, y, θ) give both the translational position of the disk as well as its rotational position.

Using Equation 2.6, the Lagrangian of the unicycle is equal to its kinetic energy, with the potential energy being equal to zero, and is given by:

$$L(q, \dot{q}) = \frac{1}{2}\big(m(\dot{x}^2 + \dot{y}^2) + J_\theta \dot{\theta}^2 + J_\phi \dot{\phi}^2\big) \tag{2.9}$$

where m is the total mass of the disk, J_θ is the moment of inertia about the z-axis, and J_ϕ is the moment of inertia pf rolling of the wheel. With r being the radius of the unicycle, the two nonholonomic constraints, describing the "rolling without slipping" and "no sideways slipping" conditions, are written as:

$$\begin{aligned} \dot{x} - r(cos\theta)\dot{\phi} = 0 \\ \dot{y} - r(sin\theta)\dot{\phi} = 0 \end{aligned} \tag{2.10}$$

Writing the constraint equations in the Pfaffian form $\omega(q)\dot{(q)} = 0$:

$$\begin{pmatrix} 1 & 0 & 0 & -r(cos\theta) \\ 0 & 1 & 0 & -r(sin\theta) \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.11}$$

Furthermore, having the generalized forces in the steering and rolling directions are τ_θ and τ_ϕ respectively, the equations of motion can be written using Equation 2.8 such that:

$$\begin{gathered} m\ddot{x} = \lambda_1 \\ m\ddot{y} = \lambda_2 \\ m\ddot{\theta} = \tau_\theta \\ m\ddot{\phi} = -r(\cos\theta)\lambda_1 - r(sin\theta)\lambda_2 + \tau_\phi \\ \dot{x} - r(\cos\theta)\dot{\phi} = 0 \\ \dot{y} - r(\sin\theta)\dot{\phi} = 0 \end{gathered} \tag{2.12}$$

As one can notice, the resultant equations of motion are second order differential equations and thus have an index of reduction of two.

An important Theorem to point to and which will be used in Chapter 3 is the following:

Theorem 2.1 ***Euler-Poincaré Equations*** *Given a Lie group G with a left-invariant Lagrangian $L : TG \to \mathbb{R}$ and $l : \mathfrak{g} \to \mathbb{R}$ being its restriction to the tangent space to G at the identity; thus for a curve $g(t) \epsilon G$ let:*

$$\xi(t) = g(t)^{-1}.\dot{g}(t);\ i.r,\ \xi(t) = T_{g(t)}L_{g(t)^{-1}}\xi\dot{g}(t) \text{ as seen in Definition 2.12.}$$

The following are equivalent:

1. *$g(t)$ satisfies the Euler-Lagrange equations for L on G.*
2. *The variational principal $\delta \int L(g(t), \dot{g}(t))dt = 0$ applies for variations with fixed endpoints.*
3. *The* ***Euler-Poincaré equations*** *hold.*

2.2.2 Hamiltonian Dynamics

The second approach is looking at Hamiltonian mechanics. Used commonly in Quantum Mechanics and Statistical Mechanics, the Hamiltonian Formulation is equivalent to Newton's Laws and to the Lagrangian Formulation. The Hamiltonian is described in terms of coordinates and their conjugate momenta rather than the coordinates and their time derivatives as with the Lagrangian.

As previously mentioned, the principal of critical action for a curve $q(t)$ is equivalent to the condition that $q(t)$ satisfies the textitEuler-Lagrange equation, which is 2.7. Let L be the Lagrangian on TQ and let $\mathbb{F}L : TQ \to T^*Q$ be defined in coordinates by:

$$(q^i, \dot{q}^j) \mapsto (q^i, p_j) \tag{2.13}$$

where $\mathbb{F}L$ is the *fiber derivative*, which differentiates L in the fiber direction, and p_j being the generalized momenta for a Lagrangian mechanical system defined as:

$$p_j = \frac{\partial L}{\partial \dot{q}^j} \quad (j = 1, ..., n) \tag{2.14}$$

Definition 2.20 : Legendre Transformation If $\mathbb{F}L$ is a diffeomorphism, then the Lagrangian L on TQ is *hyperrregular*. As a consequence, the corresponding Hamiltonian is defined by:

$$H(q^i, p_j) = p_i\dot{q}^i - L \qquad (i, j = 1, ..., n) \tag{2.15}$$

Thus, the **Legendre Transform** is the change of data from L on TQ to H on T^*Q.

Definition 2.21 : Hamilton's Equations of Motion Lagranges equations of motion imply Hamiltons canonical equations, for $i = 1, ..., n$, being:

$$\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i} \quad (i = 1, ..., n) \end{aligned} \tag{2.16}$$

Thus, this results in $2n$ first order equations of motion.

It is important to note that for N degrees of freedom in a system, the Lagrangian formulation leads to N generalized coordinates $q_i, (i = 1, .., N)$ thus a N dimensional configuration space. On the other hand, in the Hamiltonian formulation, there are N pairs of canonical conjugate pairs, $(q, p)_i, (i = 1, ..., N)$, and as such a 2N dimensional phase space. Moreover, there are as many canonical conjugate pairs as there are degrees of freedom.

Definition 2.22 Poisson Bracket For two functions $u = u(q, p)$ and $v = v(q, p)$, where $q = (q_1, ..., q_n)^T$and $p = (p_1, ..., p_n)^T$, we define their Poisson bracket to be:

$$[u, v] := \sum_{i=1}^{n} \Big(\frac{\partial u}{\partial q_i} \frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i} \frac{\partial v}{\partial q_i} \Big) \tag{2.17}$$

The Poisson bracket have three important properties shown in the following lemma 2.3.

Lemma 2.3 *: **Lie Algebra Properties** The bracket satisfies the following properties for all functions $u = u(q, p)$, $v = v(q, p)$ and $w = w(q, p)$ and scalars a and b:*

1. *Skew-symmetry: $[v, u] = [u, v]$;*
2. *Bilinearity (Leibniz rule): $[au + bv, w] = a[u, w] + b[v, w]$;*
3. *Jacobis identity: $[u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0$.*

These three properties define the Lie algebra, which is a non-associative algebra.

Corollary 2.1 *Knowing that all the coordinates $(q_1, ..., q_n)$ and $(p_1, ..., p_n)$ are independent, by direct substitution into the definition, the following results are deduced for anyu $= u(q, p)$ and all $(i, j = 1, ..., n)$:*

$\frac{\partial u}{\partial q_i} = [u, p_i]$, $\frac{\partial u}{\partial p_i} = -[u, q_i]$,$[q_i, q_j] = 0$, $[p_i, p_j] = 0$, *and* $[q_i, p_j] = \delta_{ij}$.

where δ_{ij} is the Kronecker delta such that:

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Nonholonomic Constraints. The constraints are written as $\omega^a(\dot{q}) = 0$, and the basic equations are given by the Lagrange-d'Alembert principle:

$$\frac{d}{dt}\Big(\frac{\partial}{\partial \dot{q}_i}L(q,\dot{q})\Big) - \frac{\partial}{\partial q_i}L(q,\dot{q}) = \lambda_a \omega_i^a \quad (i = 1, ..., n, \quad a = 1, ..., k) \tag{2.18}$$

The generalized Hamiltonian formulation and the Legendre Transformation, which is defined as the momentum associated with each variable and the Hamiltonian, become:

$$\begin{aligned} H &= \sum_{i=1}^{n} p_i \dot{q}^i - L \\ p_i &= \frac{\partial L}{\partial \dot{q}_i} \quad (i = 1, ..., n) \end{aligned} \tag{2.19}$$

with the moment being $p = \frac{\partial L}{\partial \dot{q}}$. This will result in the equations of motion written in the Hamiltonian form being:

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p_i} \qquad (i = 1, .., n) \\ \dot{p} &= -\frac{\partial H}{\partial q_i} + \lambda_a \omega_i^a + \tau_i \quad (a = 1, ..., k) \end{aligned} \tag{2.20}$$

The cotangent bundle T^*Q contains its canonical Poisson bracket expressed in natural canonical coordinates $(q, p) = (q_l, ..., q_n, P_1, ..., P_n)$ for T^*Q as:

$$\{F, G\}(q, p) = \sum_{i=1}^{n} \Big(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}\Big)(q, p) = \big(\frac{\partial F^T}{\partial q}, \frac{\partial F^T}{\partial p}\big) J \begin{pmatrix} \frac{\partial G}{\partial q} \\ \frac{\partial G}{\partial p} \end{pmatrix} \tag{2.21}$$

where J is the standard Poisson structure matrix:

$$J = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix} \tag{2.22}$$

and it is intrinsically determined by the Poisson bracket $\{,\}$:

$$J = \begin{pmatrix} (\{q^i, q^j)_{i,j}\} & (\{q^i, p_j\})_{i,j} \\ (\{p_i, q^j)_{i,j}\} & (\{p_i, p_j\})_{i,j} \end{pmatrix}, \quad (i, j = 1, .., n) \tag{2.23}$$

Thus, the constrained Hamiltonian equations can be written as

$$\begin{aligned} \begin{pmatrix} \dot{q}^i \\ \dot{p}_i \end{pmatrix} &= J \begin{pmatrix} \frac{\partial H}{\partial q^j} \\ \frac{\partial H}{\partial p_j} \end{pmatrix} + \begin{pmatrix} 0 \\ \lambda_a \omega_i^a \end{pmatrix} + \begin{pmatrix} 0 \\ \tau_i \end{pmatrix} \quad (i = 1, .., n) \\ \omega_i^a \frac{\partial H}{\partial p_i} &= 0 \quad (a = 1, ..., k) \end{aligned}, \tag{2.24}$$

In order to get rid of the Lagrangian multipliers and write the the dynamical equations of motion on the constrained state space, which is defined as

$$\mathcal{M} = \left\{(q,p)\epsilon T^*Q \quad | \quad \omega_i^a \frac{\partial H}{\partial p_i} = 0\right\} \tag{2.25}$$

Van Der Schaft and Maschke [14] introduced a change in coordinates. With the rank(ω_i^a) = k, there exists locally a smooth $n \times (n-k)$ matrix X_i^α of rank $n-k$ such that $\omega_a^i X_i^\alpha = 0$. Thus, the coordinate transformation $(q,p) \mapsto (q, \widetilde{p_\alpha}, \widetilde{p_a})$ is defined by:

$$\begin{aligned} \widetilde{p_\alpha} &= X_\alpha^i p_i \\ \widetilde{p_a} &= \omega_a^i p_i \end{aligned} \tag{2.26}$$

With the new coordinates, the Poisson matrix structure becomes as follows:

$$\tilde{J}(q,\tilde{p}) = \begin{pmatrix} (\{q^i, q^j)_{i,j}\} & (\{q^i, \tilde{p}_j\})_{i,j} \\ (\{\tilde{p}_i, q^j)_{i,j}\} & (\{\tilde{p}_i, \tilde{p}_j\})_{i,j} \end{pmatrix} \tag{2.27}$$

This leads to the following constrained Hamiltonian equations

$$\begin{gathered} \begin{pmatrix} \dot{q}^i \\ \dot{\tilde{p}}_\alpha \\ \dot{\tilde{p}}_a \end{pmatrix} = \tilde{J}(q,\tilde{p}) \begin{pmatrix} \frac{\partial \tilde{H}}{\partial q^j} \\ \frac{\partial \tilde{H}}{\partial \tilde{p}_\beta} \\ \frac{\partial \tilde{H}}{\partial \tilde{p}_b} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \lambda_a \end{pmatrix} \quad (i = 1, .., n) \\ \frac{\partial \tilde{H}}{\partial \tilde{p}_a}(q,\tilde{p}) = 0 \quad (a = 1, ..., k) \end{gathered} \tag{2.28}$$

where $\tilde{H}(q,\tilde{p})$ is the Hamiltonian $H(q,p)$ expressed in the new coordinates. A second change in coordinates is done, transforming $(q, \widetilde{p_\alpha}, \widetilde{p_a}) \mapsto (q, \widetilde{p_\alpha})$. The reduced Hamiltonian is defined by $H_\mathcal{M}$ as $\tilde{H}(q, \widetilde{p_\alpha}, \widetilde{p_a})$ with $(q, \widetilde{p_\alpha}, \widetilde{p_a})$ satisfying the constraint equation $\frac{\partial \tilde{H}}{\partial \tilde{p}_a} = 0$. Restricting the dynamics on the constraint state space $\mathcal{M}$ and having the constraint equation as 2.28, a $(2n-k) \times (2n-k)$ skew-symmetric matrix $J_\mathcal{M}$ is defined by truncating the last k row and k column from the Poisson structure $\tilde{J}$. By disregarding the last equations, the Lagrange multipliers λ are eliminated.

As a note to consider, when first introduced, Van der Schaft and Maschke [14] defined a $J_\mathcal{M}$ as:

$$J_\mathcal{M} = \begin{pmatrix} 0_n & X(q) \\ -X^T & (-p^T[X_i, X_j](q))_{i,j=1,..,n-k} \end{pmatrix} \tag{2.29}$$

where k is the rank of $\omega(q)$ and $[X_i, X_j]$ is a lie bracket, defined in local coordinates q as:

$$[X,Y](q) = \tfrac{\partial Y}{\partial q}(q)X(q) - \tfrac{\partial X}{\partial q}(q)Y(q)$$

with $\frac{\partial Y}{\partial q}$ and $\frac{\partial X}{\partial q}$ are the Jacobian matrices.

Finally, the dynamical equations on $\mathcal{M}$ expressed in the coordinates $(q, \widetilde{p_\alpha})$, which serve as local coordinates for the constrained state space, are given as:

$$\begin{pmatrix} \dot{q}^i \\ \dot{\tilde{p}}_\alpha \end{pmatrix} = J_{\mathcal{M}}(q, \tilde{p}_\alpha) \begin{pmatrix} \frac{\partial H_{\mathcal{M}}}{\partial q^j}(q, \tilde{p}_\alpha) \\ \frac{\partial H_{\mathcal{M}}}{\partial \tilde{p}_\beta}(q, \tilde{p}_\alpha) \end{pmatrix}, \quad \begin{pmatrix} q^i \\ \tilde{p}_\alpha \end{pmatrix} \epsilon \quad \mathcal{M} \tag{2.30}$$

These equations are in pseudo-Hamiltonian format. Furthermore, the matrix $J_{\mathcal{M}}$ defines a bracket $\{,\}_{\mathcal{M}}$ on constrained state space $\mathcal{M}$ as follows:

$$\{F_{\mathcal{M}}, G_{\mathcal{M}}\}_{\mathcal{M}}(q, \tilde{p}_\alpha) := \Big(\frac{\partial F_{\mathcal{M}}^T}{\partial q}, \frac{\partial F_{\mathcal{M}}^T}{\partial \tilde{p}_\alpha}\Big) J_{\mathcal{M}} \begin{pmatrix} \frac{\partial G_{\mathcal{M}}}{\partial q^j} \\ \frac{\partial G_{\mathcal{M}}}{\partial \tilde{p}_\beta} \end{pmatrix} \tag{2.31}$$

for any two smooth functions $F_{\mathcal{M}}, G_{\mathcal{M}}$ on $\mathcal{M}$. Indeed, this bracket satisfies the first two defining properties of a Poisson bracket: the skew-symmetry and the Leibniz rule. In addition, the bracket $\{,\}_{\mathcal{M}}$ on $\mathcal{M}$, satisfies the Jacobi identity and as such becoming a Poisson bracket, if and only if the constraints are holonomic.

Example 2.3 : The Unicycle
Following Example 2.2, which resulted in the equations of motion using the Lagrangian formulation, this example uses the Hamiltonian formultion with nonholonomic constraints, that is the method presented by Van der Schaft and Maschke [14], to obtain the dynamics of the unicycle of Figure 2.3.

After calculating the equations of motion in Equation 2.12 using Euler-Lagrangian, one defines the Hamiltonian by the Legendre transformation using Equation 2.19 such as:

$$p = \frac{\partial L}{\partial \dot{q}_i} = \begin{pmatrix} m\dot{x} \\ m\dot{y} \\ J_\theta \dot{\theta} \\ J_\phi \dot{\phi} \end{pmatrix} \tag{2.32}$$

where the momentum p is also defined as $p = (p_x, p_y, p_\theta, p_\phi)^T$. Thus using Equation 2.19, $H(q,p)$ is defined as:

$$H = \frac{1}{2}\big(m\dot{x}^2 + m\dot{y}^2 + J_\theta \dot{\theta}^2 + J_\phi \dot{\phi}^2\big) \tag{2.33}$$

which is also equivalent to:

$$H(q,p) = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{p_\theta^2}{2J_\theta} + \frac{p_\phi^2}{2J_\phi} \tag{2.34}$$

Recall that $\omega(q)$ was defined in the Pfaffian form in Equation (2.11), using Equations (2.20) the constrained Euler-Lagrangian equations transform into:

$$\dot{q} = \begin{pmatrix} \frac{p_x}{m} & \frac{p_y}{m} & \frac{p_\theta}{J_\theta} & \frac{p_\phi}{J_\phi} \end{pmatrix}^T$$
$$\dot{p} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ 0 \\ -r(\cos\theta)\lambda_1 - r(\sin\theta)\lambda_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \tau_\theta \\ \tau_\phi \end{pmatrix} \tag{2.35}$$

Equations (2.35) are the second set of equations of motion found for the Unicycle problem.

The rest of the example is to highlight the method introduced by Van der Schaft and Maschke [14]. In order to eliminate λ_i in the Euler-Lagrangian equations, new momenta are introduced using Equation 2.26 by defining a new matrix $X(q)$ such that $\omega(q)^T X(q) = 0$ where

$$\omega(q) = \begin{pmatrix} 1 & 0 & 0 & -r(cos\theta) \\ 0 & 1 & 0 & -r(sin\theta) \end{pmatrix} \tag{2.36}$$

as previously defined. As a result,$X(q)$ becomes:

$$X(q) = \begin{pmatrix} r(cos\theta) & 0 \\ r(sin\theta) & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.37}$$

Now introducing the new momenta $\tilde{p} = (\tilde{p}_\alpha, \tilde{p}_a)$ where $\tilde{p}_\alpha = (\tilde{p}_1, \tilde{p}_2)$ and $\tilde{p}_a = (\tilde{p}_3, \tilde{p}_4)$, using Equation 2.26 such that

$$\tilde{p}_\alpha := X^T(q)p = \begin{pmatrix} p_\phi + p_x r(cos\theta) + p_y r(sin\theta) \\ p_\theta \end{pmatrix} \tag{2.38}$$

$$\tilde{p}_a := \omega^T(q)p = \begin{pmatrix} p_x - p_\phi r(cos\theta) \\ p_y - p_\phi r(sin\theta) \end{pmatrix} \tag{2.39}$$

As suggested by Van der Schaft and Maschke [14], $J_{\mathcal{M}}$ was found using Equation 2.29 as

$$J_{\mathcal{M}} = \begin{pmatrix} 0 & 0 & 0 & 0 & r(cos\theta) & 0 \\ 0 & 0 & 0 & 0 & r(sin\theta) & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -r(cos\theta) & -r(sin\theta) & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix} \tag{2.40}$$

Note: If all parameters were taken to be equal to one, then $\tilde{p}_3 = 0$ and $\tilde{p}_4 = 0$.

As shown in Equation 2.28:

$\frac{\partial \tilde{H}}{\partial \tilde{p}_3}(q,\tilde{p}) = 0$ and $\frac{\partial \tilde{H}}{\partial \tilde{p}_4}(q,\tilde{p}) = 0$

Finally, using the Equations 2.38, the reduced Hamiltonian becomes

$$H_{\mathcal{M}}(x,y,\theta,\phi,\tilde{p}_1,\tilde{p}_2) = \frac{1}{2}\Big(\quad \frac{\tilde{p}_2}{J_\theta} + m\dot{x}^2 + \frac{(-\tilde{p}_1 + mr(cos\theta)\dot{x} + mr(sin\theta)\dot{y})}{J_\phi} \quad \Big) \tag{2.41}$$

While using the Van der Schaft and Maschke method did introduce a new way of looking at solving nonholonomic constraint systems without worrying about the Lagrange multipliers, the Hamiltonian formulation still needed some work. For these reasons, the method introdcued in Chapter 3 makes use of other methods such as the Reduced Lagrangian and uses a *reduced constrained* Hamiltonian and a new way of formulating the reduced Poisson structure matrix $J_{\bar{\mathcal{M}}}$.
By focusing on the Poisson structure and combining the reduced Lagrangian methods [16] with Van der Schaft and Maschke's method [14], one can derive the equations of motion without the Lagrangian multipliers in a reduced constraint state space by focusing on the building a new Poisson matrix structure to accommodate the constraints. Instead of keeping the constraints outside the Poisson matrix, a new Poisson-like matrix will be introduced and inside of which the constraints will be integrated resulting in a set of first order equations with no Lagrangian multipliers (λ).

2.2.3 Reconstruction Equation

The reconstruction equation for mechanical systemsplays a major role in the method presented in this thesis. This section simply introduces a partition over the family of simple mechanical systems. In addition, a general type of systems, *generalized mixed systems*, is defined and it represents a superset of all the mechanical systems considered for the method. Finally, the respective reconstruction equations for these mechanical systems are presented. It is important to note that this section merely serves as a summary and introduction of mechanical systems classification. Further study of the topic and its relation to gait generation techniques can be found in E.Shammas [1].

Definition 2.23 : Generalizeed mixed systems A simple mechanical system (as defined by Smale ([23], [24]) with an n-dimensional configuration space Q, which has a trivial principal fiber bundle structure where $Q = G \times M$ with G being an l-dimensional Lie group fiber space and M is an m-dimensional base space, subjected to a k-dimensional set of nonholonomic constraints is consider to be of the *generalized mixed* type if:

1. Its Lagrangian is made of the system's kinetic energy alone:

$$L(q,\dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} \tag{2.42}$$

where $M(q)$ is the $n \times n$ mass amtrix defining the kinetic energy metric as defined in 2.18

2. The nonholonomic constraints, which the system is subjected to, can be written in a Pfaffian form:
$$\omega(q)\dot{q} = 0 \tag{2.43}$$
where $\omega(q)$ is $k \times n$ matrix describing the constraints.

3. Both the Lagrangian and the set of nonholonomic constraints are invariant with respect to the fiber group Lie actions such as:
$$L(q, \dot{q}) = L(\Phi_g q, T_q \Phi_g \dot{q}) \tag{2.44}$$
$$\omega(q)\dot{q} = \omega(\Phi_g q) T_q \Phi_g \dot{q} \tag{2.45}$$
where Φ_g is the Lie group action on Q and $T_q\Phi_g$ is the lifted action on TQ

4. Away from singular configurations of the systems, the nonholonomic are linearly independent, which ensures that none of the constraints is a linear combination of the other velocity constraints, such that:
$$det(\omega(q)) \neq 0 \tag{2.46}$$

In the rest of the thesis, the mechanical systems assumed are considered generalized mixed type. In the following part of the section, the different sub-types of mixed systems are introduced with their respective reconstruction equations. For the derivation of the reconstruction equation for generalized mixed systems, please refer to E.Shammas [1]. In addition, the techniques needed to obtain the reconstruction equation of the systems for the method presented in this thesis will be shown in the next chapter.

In order to define sub-types of mechanical systems, additional conditions are imposed on the dimensions of fiber and base spaces and the number of nonholonomic constraints. Table **??** summarized the conditions to have different types of mechanical systems and their respective reconstruction equations.

Definition 2.24 : Mixed Belonging to the generalized mixed system family, **mixed** systems have at least one nonholonomic constraint and at most one less nonholonomic constraints than the dimension of the fiber space, that is:

$$0 < k < l$$

where l is the dimension of the fiber space G and k is the number of non-holonomic velocity constraints the system is exposed to.

Sub-types of Mixed Systems		
System Type	Condition	Reconstruction Equation
Mixed	$0 < l < k$	$\xi = -\mathbf{A}(r)\dot{r} + \Gamma(r)p^T$
Principally Kinematic	$l = k$	$\xi = -\mathbb{A}(r)\dot{r}$
Purely Mechanical	$k = 0$(no nonholonomic constraints) & $p(t) = 0$	$\xi = -A(r)\dot{r}$
Purely Dynamic	$m = 1$	$\xi = \Gamma(r)p^T$

Table 2.1: Various types of mechanical systems and their respective reconstruction equations

Based on the condition stated, it can be infered that mixed systems do not have enough nonholonomic constraints to fully span the fiber space. As a result, there are directions of motion that are in a sense "orthogonal" to all of the system's nonholonomic constraints. Along these allowable directions, there exists a generalized momentum whose varaibles are instantaneously conserved and governed by a first order differential equation.

Furthermore, the reconstruction equation for mixed systems is as follows

$$\xi = -\mathbf{A}(r)\dot{r} + \Gamma(r)p^T \tag{2.47}$$

where $\mathbf{A}(r)$ is a $l \times m$ matrix representing the local form of the *mixed* connections and $\Gamma(r)$ is a $l \times (l-k)$ matrix multiplying the transpose of the generalized nonholonomic momentum. In addition, ξ is a Lie algebra element or a fiber velocity at the group identity such that $\xi = T_g L_{g^{-1}}\dot{g}$.

Definition 2.25 : Principally Kinematic systems Principally kinematic systems, also known as *Chaplygin*, unlike *mixed systems* have enough nonholonomic constraints to fully span the fiber space since

$$k = l$$

where l is the dimension of the fiber space and k is the number of nonholonomic velocity constraints on the system. Thus, there will be no generalized momentum variables for such systems.

As a result, the reconstruction equation for *principally kinematic systems* reduces to

$$\xi = -\mathbb{A}(r)\dot{r} \tag{2.48}$$

where $\mathbb{A}(r)$ is an $l \times m$ matrix representing the local form of the *principal connection*.

Definition 2.26 : Purely Mechanical systems On the opposite end of the spectrum across from the principally kinematic systems, purely kinematic systems have no nonholonmic constraints on the system such that

$$\text{k=0}$$

where k is the number of nonholonmic velocity constraints acting on the system. As shown in E.Shammas [1], an additional condition is introduced ensuring that there are no external forces acting on system and that the system starts from rest. As a result, this forces all the momentum variables to start and stay at zero at all time. This leads to the following reconstruction equation for purely mechanical systems:

$$\xi = -A(r)\dot{r} \tag{2.49}$$

where $A(r)$ is a $l \times m$ matrix representing the local form of the *mechanical connection*.

Definition 2.27 : Purely Dynamic systems Belonging to the family of mixed type of systems, purely dynamic systems have an one-dimensional base space such that

$$\text{m=1}$$

where m is the dimension of the base space M. Its reconstruction equation reduceds to :

$$\xi = \Gamma(r)p^T \tag{2.50}$$

where $\Gamma(r)$ is an $l \times (l - k)$ matrix multiplying the transpose of the generalized nonholonomic momentum variable p.

Chapter 3

Systematic Method for Dynamic Modeling of Nonholonomic Mechanical Systems

Understanding locomotion is one of the most ongoing quests in the world of mechanics and robotics; ranging widely from legged locomotion to any biomechanically inspired system. While traditional frameworks, such as the Lagrangian and general Hamiltonian formulations, have served to provide a base platform for having an idea about the equations of motion of a system, it has equipped the user with no knowledge on the geometry of the mechanical systems with the nonholonomic constraints and no clarification on its implications. In the context of locomotion's mechanism, controllability and choice of gait are the stepping stones on which successful mechanical systems with nonholonomic constraints are built. Without the special structure, which is built on the geometry of the system, traditional framework makes choosing gaits and similar issues impossible [2]. The method presented in this chapter, builds on the geometry of mechanical systems with nonholonomic constraints in the attempt of better highlighting the structure of the equations of motion promoting the analysis and isolation of the crucial geometric objects that govern this motion. This in turn clarifies the basic mechanics underlying locomotion.

However, in contrast to the Lagrangian and Hamiltonian formulations, nontraditional frameworks, which are based on the geometry of the system, still lack a unified systematic procedure and as such making it inaccessible to all users. This thesis presents a 20-step method that results in a set of $(2n - r)$ Ordinary Differential Equations (ODE) describing the equations of motions of any system with nonholonomic constraints with n and r being the number of coordinates q and the number of constraint equations, respectively. The 20-step method finds the Full Symmetry group of the system and on it builds a map from the general manifold to the reduced constraint submanifold.

The method requires simply and only the knowledge on the following:

1. The local coordinated (q^i)
2. The Lagrangian $(L(q, \dot{q}))$, which can be calculated using 2.42
3. The constraint equations $(\omega(q))$
4. The generalized forces on the system (τ)
5. The parameters of the variables $(\mathbb{P})$

3.1 Overview Comparison between the Frameworks Presented

Before diving into the method, one may take a step back to zoom out and comprehend the impact of the main differences of the three formulations: the Lagrangian Formulation, the Hamiltonian Formulation, and the Reduced Constraint Hamiltonian Formulation. A comparison chart is presented in Figures 3.1 and 3.2.

First regarding the Lagrangian Formulation, while the Euler-Lagrangian equations provide a one-step method to find the equations of motion, the constraints are added at the force level as add-ons to the equations. This results in equations of high index, specifically Differential-Algebraic Equations (DAE) of index 2, which require additional mathematical steps and effort for solver to find the equations of motion. In addition, adding the constraints as such forces the introduction of new variables, the Lagrangian multipliers $\lambda' s$, which are of zero order and act as constraint forces. Furthermore, both Lagrangian and Hamiltonian formulations make no special contribution to the constraints and make no use of the fiber bundle information found in Q.
On the other hand, the Hamiltonian Formulation introduces a physical term, the momenta of the coordinates q , and which is attributed to the motion of the system. This results in first order $2n$, where n is the number of the coordinates q, set of equations $(\dot{q}, \dot{p})$. However the constraints in the Hamiltonian Formulation are still added at the force level using the Lagrangian multipliers $\lambda' s$, which are of zero order. This makes the entire set of equations DAE's of index 1. There is still no association between the equations of motion and the actual geometry of the system and its nonholonomic constraints.

Finally, the Reduced Constraint Hamiltonian Formulation uses the nonholonomic constraints along with the geometry of the configuration space to create a map from the general manifold to a reduced constraint submanifold, where the constraints are integrated, and finds the dynamics within that submanifold. This eliminates the need to introduce and then solve the constraint forces $\lambda' s$ since the constraints were added at an initial level, at the geometry level. The outcome is a set of $(2n - r)$ ODE equations of motion.

The method presented in this chapter is a 20-step procedure that makes the simplifies the Reduced Constraint Formualtion in order to make it more accessible to users. Along the 20-steps, important information on the system is revealed and can be extracted for different purposes. The map, designed in this 20-step method,to move to the reduced constraint submanifold requires finding the Full Symmetry group of the system and thus introducing new momenta coordinates $(\tilde{p}_i, \tilde{p}_\alpha)$. The Full Symmetry group divides the fiber bundle Q and coordinates q by highlighting what is crucial for the dynamics of the system. In addition, the procedure makes use of the Reconstruction Equation to simplify intermediate steps.

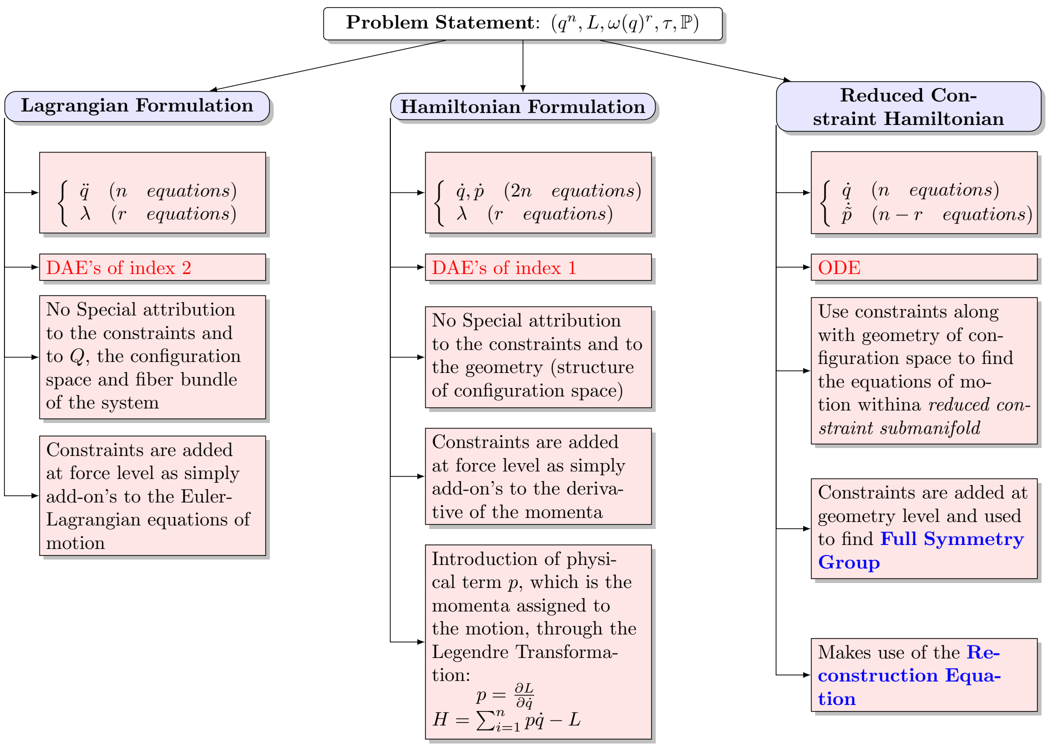

Figure 3.1: Comparison flow chart of the three different formulations: Lagrangian Formulation, the Hamiltonian Formulation, and the Reduced Constraint Hamiltonian Formulation

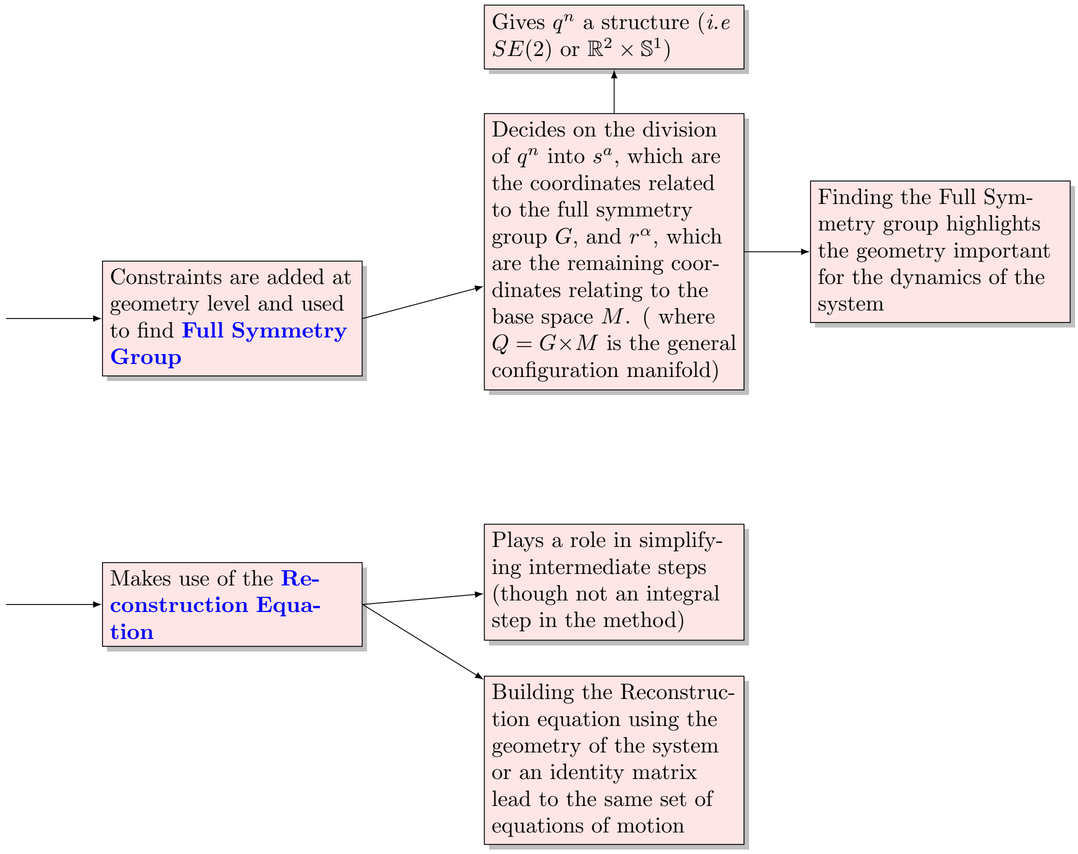

Figure 3.2: Continuation of the Comparison flow chart of the three different formulations: Lagrangian Formulation, the Hamiltonian Formulation, and the Reduced Constraint Hamiltonian Formulation

Comparison between method presented and the Van der Schaft and Maschke method

Inspired by the work of Van der Schaft and Maschke [14], the method presented makes use of the poisson geometry of nonholonmic systems to create a unified systematic approach to solving all mechanical systems with nonholonmic constraints acting on them. Building on the work of Van der Schaft and Maschke [1994], the method also makes use of the work of Bloch, Krishnaprasad, Marsden, and Murray [1996] [2]. In addition, the method encompasses the findings of Marle [8] on the bracket $\{,\}_{\mathcal{M}}$ found in 2.31 can be given an intrinsic interpretation.

While Van der Schaft and Maschke [14] set a stepping stone for Poisson Reduction formulation, the method introduced in this chapter builds on the later and makes use of different formulations and theorems.

1. As seen in Section 2.2.2, the Van der Schaft and Maschke method focuses

on the constraint submanifold ($\mathcal{M}$). On the other hand, with G, the configuration group of the system, being the symmetry group of the system, the Hamiltonian H is G-invariant; thus, the method presented uses reduction by symmtery to build the reduced poisson structural matrix to finally get the equations of motion on the reduced constraint submanifold ($\bar{\mathcal{M}}$).

2. Second, instead of focusing only on the Poisson Reduction formulation alone, the method links and makes use of both the Poisson Reduction formulation and the Lagrangian Reduction procedure.

3. Third, as seen in Equation 2.41, $h_{\mathcal{M}}$ is not fully in terms of the new momenta introduced but rather includes extra terms. Moreover, $h_{\bar{\Updownarrow}}$ solves that problem by employing the reduced lagrangian.

4. Fourth, unlike the method of Van der Schaft and Maschke, the method introduced in this chapter utilizes the reconstruction equation, presented in Section 2.2.3, to recover the full dynamics of the system.

5. Fifth, a big role in the following method is given to the geometry of the system,where the first step is to test to find the full symmetry group of the system.

3.2 The 20-Step Method

The method is over-viewed in Figures 3.3,3.5,3.6, and 3.7 and as shown it is divided into five parts. First is the *Set-Up*, which is simply to describe the local coordinates, the configuration space, and the Lagrangian. Second is the *Equivalence of Poisson and Lagrangian reduction*, where both equations representing ξ, the body representation of the fiber velocity $\dot{g}$, are calculated to finally find the reduced constrained Lagrangian. Third is the *Legendre Transformation & Reduced Hamiltonian constraint*, where the new momenta are introduced leading to the reduced constrained Hamiltonian. Fourth is the *Poisson Reduction*, in which are the steps needed to construct the reduced Poisson structure matrix. Finally, the *Reduced Hamiltonian* results in $(2n-r)$ first order differential equations (ODE's).

3.2.1 Set-Up

As previously mentioned, the method requires merely the knowledge of a maximum of five ingredients: the coordinates q, the Lagrangian L, the constraint matrix $\omega(q)$, if present the generalized forces acting on the system τ, and for simulation purposes the parameters of the variables $\mathbb{P}$. As seen in Figure 3.3,a general configuration manifold denoted by $Q = G \times M$ exists for mechanical

systems.In general, G is the *fiber space* specifying the position and orientation of the robot and M is the *base space* specifying the internal shape of the robot. In the problem statement given throughout the thesis, the configuration manifolds have a *principal fiber bundle* structure.
However, at this point, one does not need to know how Q is divided into G and M since G will be picked by a test, which reveals the full symmetry group of the system. Similarly, the local variables are divided as $q^i = (r^\alpha, s^a)$, where s^a are the coordinates related to the configuration group G and r^α are the remaining coordinates known as the shape or base variables.
Once again, the Lagrangian L is defined as the kinetic energy of the system minus its potential energy. In addition, the matrix $\omega(q)$ is the arrangement of the constraint equations with respect to $\dot{q}$ as portrayed in Figure 3.3.

Set-Up

For mechanical systems, one assumes a general configuration manifold denoted by $Q = G \times M$, where G is the *fiber space* specifying the position of the robot and M is the *base space* specifying the internal shape of the robot. Here, the configuration manifolds have a *principal fiber bundle* structure.

Let $q^i = (r^\alpha, s^a)$ be the local coordinates, where s^a are the coordinates related to the configuration group G and r^α are the remaining coordinates. Typically, s^a are referred to as the fiber variables, and r^α as the base or shape variables.

For mechanical systems with configuration space Q, the Lagrangian is defined by:

$$L(q, \dot{q}) = \frac{1}{2}\dot{q}^T M \dot{q} - V(q) \tag{1}$$

where M is the mass matrix of the mechanical system, $\dot{q}^T M \dot{q}$ is the kinetic energy, and $V(q)$ is the potential energy.
For systems with k nonholonomic constraints c_i given in the Paffian form $\omega(q)\dot{q} = 0$, we define ω^a, which is the matrix representing the constraints, has the form :

$$\omega^a(q) = ds^a + A^a_\alpha(r, s) dr^\alpha \tag{2}$$

Figure 3.3: Over-view of Systematic Method to Solving Nonholonomic Mechanical Systems (part 1/4)

3.2.2 Equivalence of Poisson and Lagrangian Reduction

The first "box" establishes the map from the general manifold to the reduced constraint submanifold $\overline{\mathcal{M}}$ and as such the building ground of the method. The steps are found in Figure 3.5.
The three main purposes are:

1. To find the full symmetry group, which will allow the mapping from the general manifold to $\overline{\mathcal{M}}$ through ξ_Q^q, the vector field of the Lie algebra element found.

2. To retrieve the Reconstruction Equation and rewrite it within a reduced constraint submanifold.

3. To calculate the Lagrangian within the submanifold $\overline{\mathcal{M}}$ giving l_c.

The nonholonomic constraints equations reveal information on the geometry of the system that may be extremely useful to better structure the equations of motion and highlight the way the system's geometry influences its dynamics. One of the crucial information one can obtain from the nonholonomic constraints equations is the full symmetry group of the system. By taking the null space of $\omega(q)$, one can acquire which set of coordinates from q would lead to a non-zero Lie algebra element, which acts as the fixed basis on which the submanifold $\overline{\mathcal{M}}$ is build. The set of coordinates, which achieve that, are referred to in the problem as s^a and thus matching to its configuration space disclose the full symmetry group G. This implies that even without a full understand of the configuration space Q of the system, one can recover the way it is divided into $G \times M$. It is important to note that having a Lie algebra element made of a zero in one of the basis means that no motion is allowed in that direction. For this reason, only the full symmetry group would give a full non-zero lie algebra element ξ.

After having the Set-Up make up the first two steps, the rest of the steps in this section are as follows (which are detailed in Figure 3.5):

3. Find the Full Symmetry group from the Null Space of $\omega(q)$, this leads to finding ξ_Q^q and respectively the Lie algebra element ξ.

4. Calculate the nonholonomic momentum: $p = \frac{\partial L}{\partial \dot{q}}(\xi_Q^q)$

5. Obtain the Reconstruction equation $g^{-1}\dot{s} + A(r)\dot{r} = \Gamma(r)p$ by solving for $g^{-1}\dot{s}$ by setting the constraints to zero and using the nonholonomic momentum.

6. Compute the body fixed axis $e = g^{-1}\xi_Q^q$ and replace in the Reconstruction equation $\Gamma(r)p = \Omega e$, where Ω is the body angular velocity.

7. Find the reduced Lagrangian $l(r, \dot{r}, \xi)$

8. Get the reduced constrained Lagrangian $l_c(r, \dot{r}, \Omega)$

3.2.3 Legendre Transformation & Reduced Constraint Hamiltonian

The motivation for this part of the method is the following:

After having defined the reduced constraint lagrangian l_c, the new momenta are introduced and act as Legendre Transformation allowing the movement from the Lagrangian to Hamiltonian formulation but within the reduced constraint submanifold $\overline{\mathcal{M}}$.

Continuing the steps from the list 3.2.2, the steps of this part are detailed in Figures 3.5 and 3.6 and summarized below:

9. The new momenta and Legendre Transformation are calculated $\widetilde{P}_\alpha = \frac{\partial l_c}{\partial \dot{r^\alpha}}$ and $\widetilde{P}_i = \frac{\partial l_c}{\partial \Omega^i}$

10. Retrieve Ω and $\dot{r^\alpha}$ from the Legendre Transformation

11. Find the reduced Hamiltonian formulation $h_{\overline{M}} = \widetilde{P}_i \Omega^i + \widetilde{P}_\alpha \dot{r^\alpha} - l_c$ by expanding the expressions of Ω and $\dot{r^\alpha}$ calculated previously.

3.2.4 Poisson Reduction

This section of the method is dedicated to building the new Poisson-like structure matrix $J_{\mathcal{M}}$ that can be used within the reduced constraint submanifold $\overline{M}$.

To able to use the Poisson brackets $\{,\}$ on the new momenta introduced, the constraints need to be conserved within a variable μ,which is an element of the dual of the Lie algebra g^*, and μ_a are its coordinates with respect to a fixed dual basis. Once the $\{,\}$ are calculated for $\widetilde{P}_i$ and $\widetilde{P}_\alpha$ written in terms of μ_a, they can be mapped to the reduced constraint submanifold $\overline{\mathcal{M}}$ by expanding μ, where $\mu = \frac{\partial l}{\partial \xi}$. Finally, the new Poisson-like structure can be built. All steps are highlighted in this section's box in Figures 3.6 and 3.7.

Continuing the steps from the list 3.2.3, the steps of this section are as follows:

12. Write $\widetilde{P}_i$ in terms of $p_i = \frac{\partial L}{\partial \dot{q}^i}$

13. Write $\widetilde{P}_\alpha$ as $\widetilde{P}_\alpha = p_\alpha + \mu_\alpha(\frac{\partial \xi^\alpha}{\partial \dot{r^\alpha}})$ with $p_\alpha = \frac{\partial L}{\partial \dot{r}^\alpha}$ and $\mu_a = \frac{\partial l}{\partial \xi^a}$

14. Calculate μ, such that $\mu_d = \frac{\widetilde{P}_i}{e_i^d}$ and write $\widetilde{P}_i$ and $\widetilde{P}_\alpha$ in terms of it.

15. Apply the Poisson bracket $\{,\}$ to $\widetilde{P}_i$ and $\widetilde{P}_\alpha$ to calculate $\{\widetilde{P}_i, \widetilde{P}_j\}$,$\{\widetilde{P}_i, \widetilde{P}_\alpha\}$,and $\{\widetilde{P}_\alpha, \widetilde{P}_\beta\}$ in terms of μ.

16. Expand $\mu = \frac{\partial L}{\partial \xi}$ by replacing Ω and $\dot{r}^\alpha$,by their respective expressions. This will restrict μ to submanifold $\overline{\mathcal{M}}$

17. Construct the new Poisson structure matrix:

$$\begin{bmatrix} 0 & e_j^b & 0 & -A_\beta^b \\ -(e_i^c)^T & \{\widetilde{P}_i, \widetilde{P}_j\}_{\overline{M}} & 0 & \{\widetilde{P}_i, \widetilde{P}_\beta\}_{\overline{M}} \\ 0 & 0 & 0 & \delta_\beta^\alpha \\ (A_\alpha^c)^T & \{\widetilde{P}_\alpha, \widetilde{P}_j\}_{\overline{M}} & -\delta_\alpha^\beta & \{\widetilde{P}_\alpha, \widetilde{P}_\beta\}_{\overline{M}} \end{bmatrix}$$

where $\widetilde{P}_\alpha = P_\alpha - \mu_b A_\alpha^b$ and $\widetilde{P}_\beta = P_\beta - \mu_d A_\beta^d$ and $\left[(A_\alpha^c)^T\right] = \begin{bmatrix} A_\alpha^b \\ A_\beta^d \end{bmatrix}$

3.2.5 Reduced Constraint Hamiltonian Equations

Finally, the purpose of this section is the to retrieve all ODE equations of motion. The steps are pointed out in Figure 3.7 and are as follows:

18. Calculate the derivative of $h_{\overline{\mathcal{M}}}$ with respect to $\widetilde{P}_i$, r, and $\widetilde{P}_\alpha$

19. Find $(\xi, \dot{\widetilde{P}}_i, \dot{r}_\alpha, \dot{\widetilde{P}}_\alpha)$ by expanding the following:

$$\begin{bmatrix} \xi^b \\ \dot{\widetilde{P}}_i \\ \dot{r}^\alpha \\ \dot{\widetilde{P}}_\alpha \end{bmatrix} = \begin{bmatrix} 0 & e_j^b & 0 & -A_\beta^b \\ -(e_i^c)^T & \{\widetilde{P}_i, \widetilde{P}_j\}_{\overline{M}} & 0 & \{\widetilde{P}_i, \widetilde{P}_\beta\}_{\overline{M}} \\ 0 & 0 & 0 & \delta_\beta^\alpha \\ (A_\alpha^c)^T & \{\widetilde{P}_\alpha, \widetilde{P}_j\}_{\overline{M}} & -\delta_\alpha^\beta & \{\widetilde{P}_\alpha, \widetilde{P}_\beta\}_{\overline{M}} \end{bmatrix} \begin{bmatrix} 0 \\ \frac{\partial h_{\overline{M}}}{\partial \widetilde{P}_j} \\ \frac{\partial h_{\overline{M}}}{\partial r^\beta} \\ \frac{\partial h_{\overline{M}}}{\partial \widetilde{P}_\beta} \end{bmatrix}$$

20. Obtain the equations of motion of $\dot{s}_a$ within the reduced constraint submanifold by knowing that $\xi = g^{-1}\dot{s}_a$

3.2.6 Summary of the 20-Step Method and Comparison to General Hamiltonian Formulation

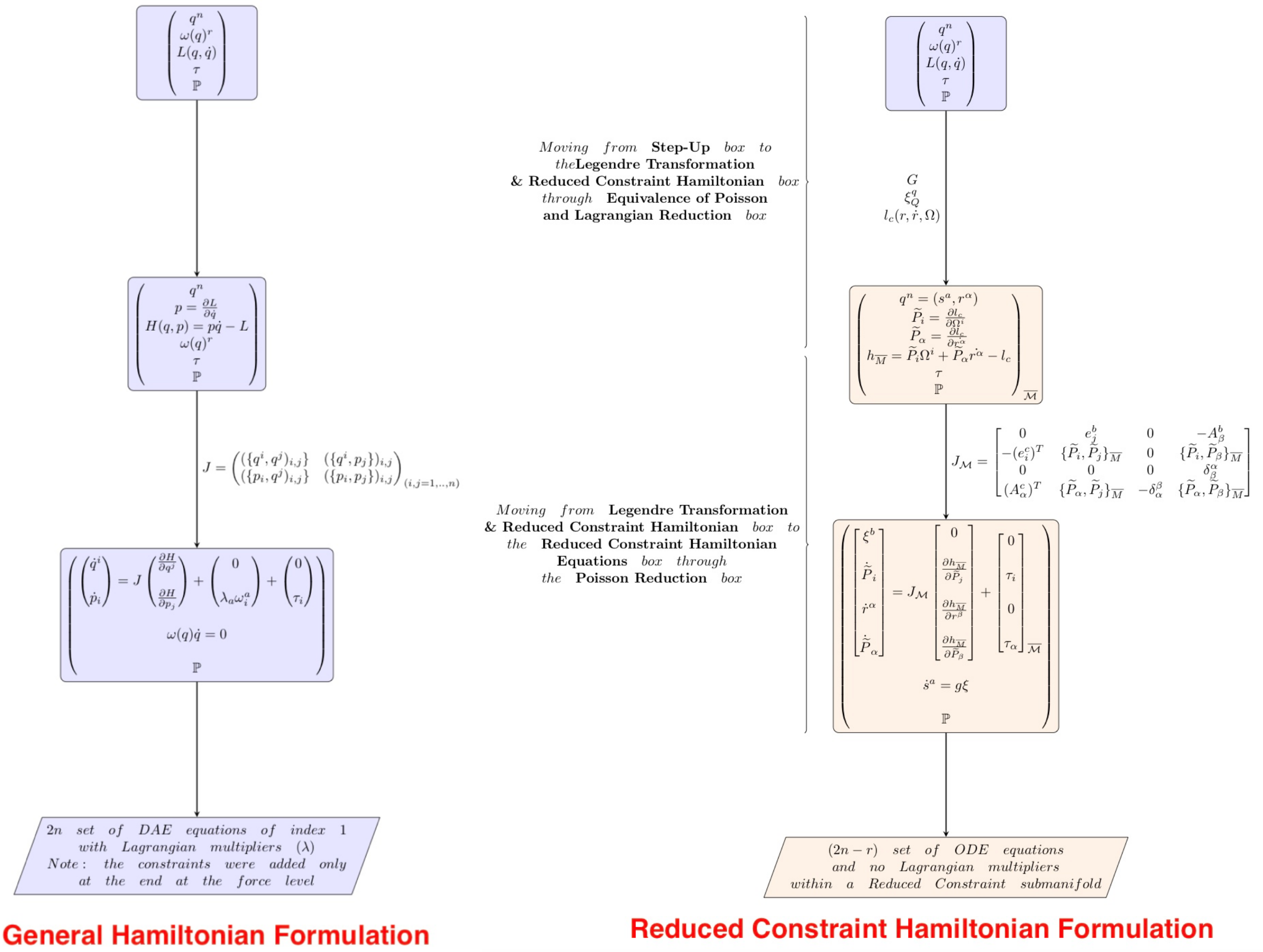

Figure 3.4: General Flow-Chart of General Hamiltonian Formulation and Reduced Constraint Hamiltonian Formulation, where the color orange depicts being within a reduced constraint submanifold and the color blue represents being within the general manifold

At this point, one may zoom out to better understand the effect of the 20-step method and be able to locate the changes in comparison to the general Hamiltonian formulation. Figure 3.4 highlights the consequences of the sections (boxes) of the 20-step method in comparison to the general Hamiltonian Formulation. The color orange depicts being within a reduced constraint submanifold and the color blue represents being within the general manifold. As it is notices, **Equivalence of Poisson and Lagrangian Reduction** section directly transforms the entire system in a general manifold to the submanifold $\overline{\mathcal{M}}$. As a consequence, the use of the constraint matrix is no longer in need since the system now within a submanifold that already includes the constraints. This eliminates the Lagrangian Multipliers, which only appeared in the very last step general Hamiltonian formulation. This is because the constraints were added at the force level. In addition, the constraint forces λ, which are of zero order, cause the final $2n$ set of equations to be DAE's of index 1. On the other hand, entering the constraints at the initial geometry level using a full symmetry group G in the reduced constraint Hamiltonian framework leads to a $(2n - r)$ ODE set of equations of motion.

Figures 3.3 , 3.5, 3.6, and 3.7 represent all the steps of the 20-Step Method while depicting the transition from one section to another.

In addition, in Section 3.3, all three formulations are applied to the Unicycle and the Snakeboard problems. A simulation of all three results is done for each, highlighting that the 20-Step Method does indeed give the correct physics and equations of motion for the system.

Equivalence of Poisson and Lagrangian Reduction

First step is to test to find the Full Symmetry Group on which the map to the reduced constraint submanifold will be based.

Taking the Null Space of $\omega(q)$ gives the constraint distribution of $\mathcal{D}_q$. $\mathcal{T}_q Orb(q)$, which is the span of the coordinates of the symmetry group G, is tested and chosen such that $\mathcal{S}_q = \mathcal{D}_q \cap \mathcal{T}_q Orb(q)$ has a full span in terms of all the coordinates of $\mathcal{T}_q Orb(q)$.

Accordingly, the full symmetry group G is chosen. ξ_Q^q is the vector field of $\mathcal{S}_q$ and ξ the Lie algebra element.

(Please refer to the Examples below for a demonstration)

Find the nonholonomic momentum:

$$p = \frac{\partial L}{\partial \dot{q}}(\xi_Q^q) \tag{3}$$

Choose g either based on the geometry of the system ($i.e$ $SE(2)$ or $\mathbb{R}^2 \times \mathbb{S}^1$) or as an identity matrix.

Calculate $g^{-1}\dot{s}$ by using the solution of $\dot{s}$ that one gets by equating the constraint equations to zero and using the nonholonomic momentum from equation 3. This will result in the reconstruction equation and accordingly one can find $A(r)$ and $\Gamma(r)$ sucht that:

$$g^{-1}\dot{s} + A(r)\dot{r} = \Gamma(r)p \tag{4}$$

, where $g^{-1}\dot{s} = \xi$ and s are the fiber variables related to G

But $\Gamma(r)p = \Omega e$, where Ω is defined as the body angular velocity and $e = g^{-1}\xi_Q^q$ is the body fixed axis. Write ξ as:

$$\xi = -A(r)\dot{r} + \Omega e \tag{5}$$

From the Lagrangian L, obtain the reduced Lagrangian $l(r, \dot{r}, \xi)$ using $\xi = g^{-1}\dot{s}$

Obtain the reduced constrained Lagrangian $lc(r, \dot{r}, \Omega)$ by plugging ξ using equation 5

Legendre Transformation & Reduced Constraint Hamiltonian

Find the Legendre Transformation:

$$\widetilde{P}_\alpha = \frac{\partial l_c}{\partial \dot{r^\alpha}} \tag{6}$$

$$\widetilde{P}_i = \frac{\partial l_c}{\partial \Omega^i} \tag{7}$$

Figure 3.5: Over-view of Systematic Method to Solving Nonholonomic Mechanical Systems (part 2/4)

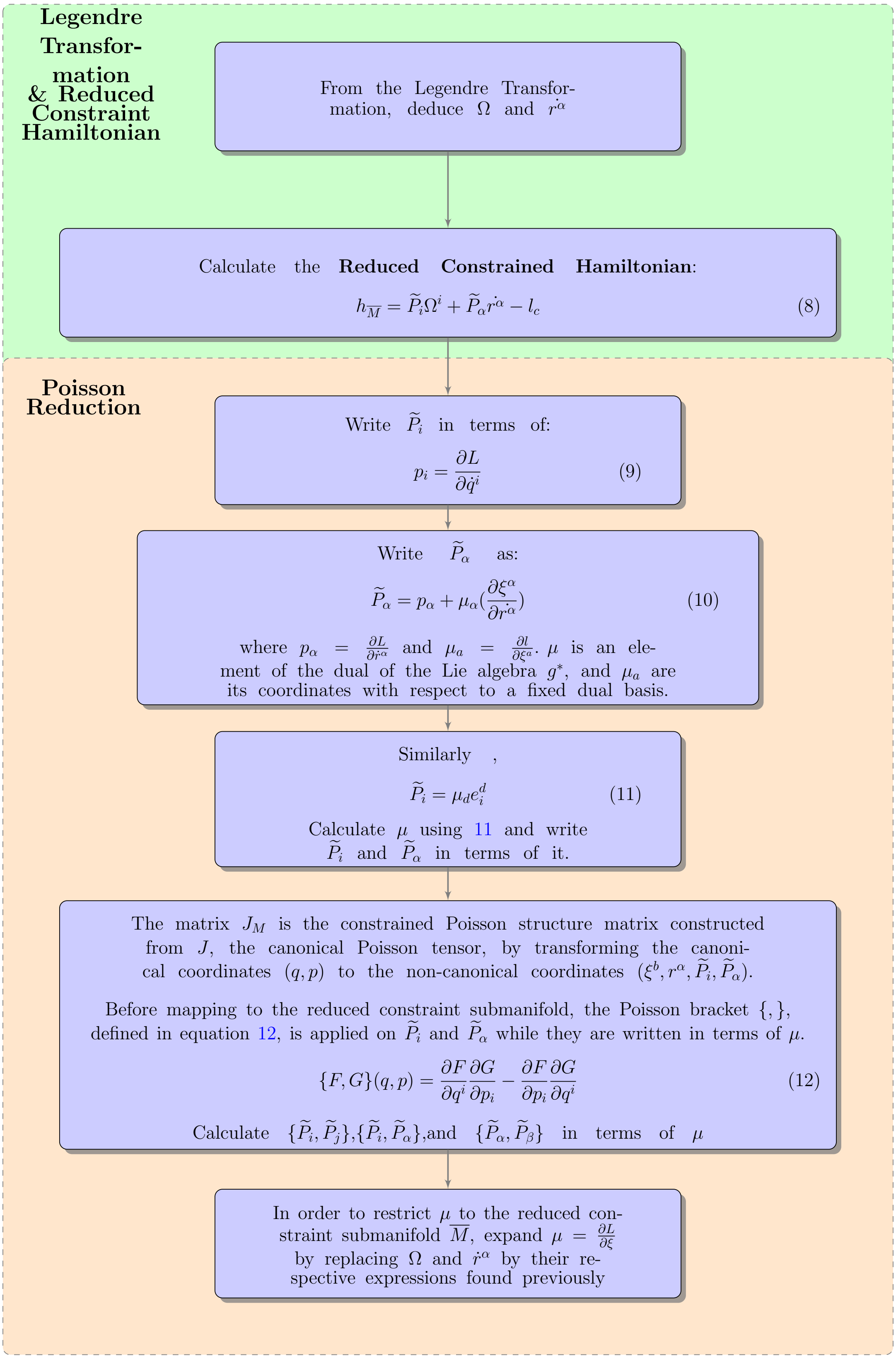

Figure 3.6: Over-view of Systematic Method to Solving Nonholonomic Mechanical Systems (part 3/4)

Expand the μ_i, using the previous step, in the Poisson brackets calculated and as such restricting them to the reduced constraint submanifold $\overline{M}$.

This results in obtaining $\{\widetilde{P}_i, \widetilde{P}_j\}_{\overline{M}}$,$\{\widetilde{P}_i, \widetilde{P}_\alpha\}_{\overline{M}}$,and $\{\widetilde{P}_\alpha, \widetilde{P}_\beta\}_{\overline{M}}$

Construct the reduced Poisson structure matrix:

$$\begin{bmatrix} \xi^b \\ \dot{\widetilde{P}}_i \\ \dot{r}^\alpha \\ \dot{\widetilde{P}}_\alpha \end{bmatrix} = \begin{bmatrix} 0 & e^b_j & 0 & -A^b_\beta \\ -(e^c_i)^T & \{\widetilde{P}_i, \widetilde{P}_j\}_{\overline{M}} & 0 & \{\widetilde{P}_i, \widetilde{P}_\beta\}_{\overline{M}} \\ 0 & 0 & 0 & \delta^\alpha_\beta \\ (A^c_\alpha)^T & \{\widetilde{P}_\alpha, \widetilde{P}_j\}_{\overline{M}} & -\delta^\beta_\alpha & \{\widetilde{P}_\alpha, \widetilde{P}_\beta\}_{\overline{M}} \end{bmatrix} \begin{bmatrix} 0 \\ \frac{\partial h_{\overline{M}}}{\partial \widetilde{P}_j} \\ \frac{\partial h_{\overline{M}}}{\partial r^\beta} \\ \frac{\partial h_{\overline{M}}}{\partial \widetilde{P}_\beta} \end{bmatrix} \tag{13}$$

Where

$$\begin{bmatrix} (A^c_\alpha)^T \end{bmatrix} = \begin{bmatrix} A^b_\alpha \\ A^d_\beta \end{bmatrix} \tag{14}$$

and $\widetilde{P}_\alpha = P_\alpha - \mu_b A^b_\alpha$ and $\widetilde{P}_\beta = P_\beta - \mu_d A^d_\beta$. In addition,$\{\widetilde{P}_i, \widetilde{P}_j\}_{\overline{M}} = -\{\widetilde{P}_j, \widetilde{P}_i\}_{\overline{M}}$ and the same applies for all such brackets and $\{\widetilde{P}_i, \widetilde{P}_i\} = 0$. Furthermore,$\delta^\beta_\alpha$ is the Kronecker delta.

Reduced Constraint Hamiltonian Equations

Calculate the derivative of $h_{\overline{M}}$ with respect to $\widetilde{P}_i$, r, and $\widetilde{P}_\alpha$

Calculate the momentum equation and the reduced constrained Hamilton equations using 13. It is important to note that the generalized forces τ are added to the equations of $\widetilde{P}_i$ and $\widetilde{P}_\alpha$ accordingly.

Knowing that $\xi = g^{-1}\dot{s}$, write $\dot{s}$ in terms of ξ_i.

Replace the ξ_i with the ones found from the previous step, obtaining the reconstruction equations on the Hamiltonian side.

Having both sets of equations gives the dynamics of the full constrained systems but in a more suitable and useful form.

$$\begin{cases} \dot{q} & (n \quad equations) \\ \dot{\widetilde{p}} & (n-r \quad equations) \end{cases}$$

Figure 3.7: Over-view of Systematic Method to Solving Nonholonomic Mechanical Systems (part 4/4)

3.3 Examples

Example 3.1 :The Unicycle

After addressing the Unicycle problem in Example 2.3, where the method introduced by Van der Schaft and Maschke [14] was applied, it was concluded that while the elimination of Lagrangian multipliers was very helpful, the reduced Poisson structure matrix and reduced Hamiltonian needed improvements.

In this example, the Unicycle's equations of motions will be obtained using the method introduced in this chapter.

First, looking at the **first step**, which is the *Set-Up* box, as previously established the local coordinates are $q = (x, y, \theta, \phi)$.The configuration manifold is four-dimensional with $Q = \mathbb{R} \times \mathbb{R} \times \mathbb{S} \times \mathbb{S}$. Since Q has a principle fiber bundle structure as $Q = G \times M$, where $G = SE(2)$ is the fiber Lie group and $M = \mathbb{S}^1$ is the base space. In addition, the Lagrangian was first calculated in Equation 2.9 in Example 2.2 as:

$$L(q, \dot{q}) = \frac{1}{2}\big(m(\dot{x}^2 + \dot{y}^2) + J_\theta \dot{\theta}^2 + J_\phi \dot{\phi}^2\big) \tag{3.1}$$

and the constraint equations were given in Equation 2.10 as:

$$\begin{aligned} \dot{x} - r(\cos\theta)\dot{\phi} = 0 \\ \dot{y} - r(\sin\theta)\dot{\phi} = 0 \end{aligned}$$

Writing the constraint equations in Pfaffian form $\omega(q)\dot{q}$ resulted in Equation 2.11, which was:

$$\begin{pmatrix} 1 & 0 & 0 & -r(\cos\theta) \\ 0 & 1 & 0 & -r(\sin\theta) \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.2}$$

Now that the *Set-Up* has been established, one can translate to the second box *Equivalence of Poisson and Lagrangian Reduction.* After calculating the Lagrangian and writing the constraint equations, making it the first step in the method, the **second step** is taking the Null Space of $\omega(q)$, which results in the following matrix:

$$\mathbf{NullSpace}(\omega(q)) = \begin{pmatrix} r(\cos\theta) & r(\sin\theta) & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{3.3}$$

which translate to the constraint distribution D_q such that

$$\mathcal{D}_q = \mathbf{span}\Big\{\frac{\partial}{\partial\theta}, \quad r(\cos\theta)\frac{\partial}{\partial x} + r(\sin\theta)\frac{\partial}{\partial y} + \frac{\partial}{\partial\phi}\Big\} \tag{3.4}$$

Putting aside the knowledge on the configuration space and choice of G, in order to get a convenient $\mathcal{S}_q$, which is the intersection given by the tangent space of the orbit with the constraint distribution, it is reasonable to choose $\mathcal{T}_q Orb(q)$ as

$$\mathcal{T}_q Orb(q) = \textbf{span}\Big\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial \phi}\Big\} \tag{3.5}$$

Equation 3.5 represents the tangent space to the orbits of the group $G = \mathbb{R}^2 \times \mathbb{S}^1$. The resultant $\mathcal{S}_q$ becomes

$$\mathcal{S}_q = \mathcal{D}_q \cap \mathcal{T}_q Orb(q) = \textbf{span}\Big\{r(\cos\theta)\frac{\partial}{\partial x} + r(\sin\theta)\frac{\partial}{\partial y} + \frac{\partial}{\partial \phi}\Big\} \tag{3.6}$$

Thus, for $G = \mathbb{R}^2 \times \mathbb{S}^1$ the section of $\mathcal{S}_q$, which was calculated in 3.6, is taken to be the vector field:

$$\xi_Q^q = r(\cos\theta)\frac{\partial}{\partial x} + r(\sin\theta)\frac{\partial}{\partial y} + \frac{\partial}{\partial \phi} \tag{3.7}$$

with a corresponding Lie algebra element being:

$$\xi = (r(\cos\theta), r(\sin\theta), 1) \tag{3.8}$$

In addition, since $q = (r^\alpha, s^a)$ are the local coordinates, where s^a are the coordinates related to the configuration group G and r^α are the remaining coordinates, the coordinates here of $s^a = (x, y, \phi)$ and r^α is composed of $r^\alpha = (\theta)$.

Before proceeding it is important to note the following:

Remark 3.1 *The configuration space is a $SE(2) \times \mathbb{S}^1$, which is the group over which the dynamics takes place. While the whole group $SE(2) \times \mathbb{S}^1$ is a symmetry group, using it as is, in the presence of controls, could violate symmetry. In such a case, it is appropriate to consider smaller symmetry groups. Here, the subgroup $G = \mathbb{R}^2 \times \mathbb{S}^1$ is a symmetry group, which is invariant under translation and axial rotation.*

G has to be the full symmetry group of the system at hand and as result have a full lie algebra element. Table 3.1 shows the effect of selecting different G groups on the lie algebra element.
Reminder: Configuration of the Unicycle is $Q = \mathbb{R}^1 \times \mathbb{R}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$ with $q = (x, y, \theta, \phi)$. Since it is a fiber bundle, the configuration space is divided as such $Q = G \times M$ and $q = (s^a, r^\alpha)$ where s^a are the fiber variables thus related to G and r^α are the shape variables related to M. Note that one has no control on $\mathcal{D}_q$; however has control over selecting $\mathcal{T}_q Orb(q)$, based on G, to get the proper $\mathcal{S}_q$ with a full lie algebra element.

Choosing the fully symmetry group G for the Unicycle				
s^a	G	$\mathcal{T}_q Orb(q)$	$\mathcal{S}_q = \mathcal{D}_q \cap \mathcal{T}_q Orb(q)$	ξ
(x)	$\mathbb{R}^1$	**span**$\left\{\frac{\partial}{\partial x}\right\}$	**span**$\left\{0\right\}$	(0)
(x, y)	$\mathbb{R}^1 \times \mathbb{R}^1$	**span**$\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right\}$	**span**$\left\{0\right\}$	$(0, 0)$
(x, y,θ)	$SE(2) = \mathbb{R}^2 \times \mathbb{S}^1$	**span**$\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial \theta}\right\}$	**span**$\left\{\frac{\partial}{\partial \theta}\right\}$	$(0, 0, 1)$
(x, y,ϕ)	$\mathbb{R}^2 \times \mathbb{S}^1$	**span**$\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial \phi}\right\}$	**span**$\left\{r(\cos\theta)\frac{\partial}{\partial x} + r(\sin\theta)\frac{\partial}{\partial y} + \frac{\partial}{\partial \phi}\right\}$	$(r(\cos\theta), r(\sin\theta), 1)$

Table 3.1: Selecting the full symmetry group G for the Unicycle

Remark 3.2 *If one choose G to be $SE(2)$, thus leading to $\mathcal{S}_q$ = **span**$\frac{\partial}{\partial\theta}$, and follows the method, the equations of motion would be zero. In addition, having such an $\mathcal{S}_q$ would lead to a lie algebra element of $\xi = (0, 0, 1)$. Having a Lie algebra element made of a zero in one of the basis means that no motion is allowed in that direction; after all, the lie algebra element acts as the fixed basis on which the constraint space, in which motion is allowed, is being built. In this case, having a lie algebra element of $(0, 0, 1)$ implies that direction in θ is the only allowed direction. It is then no surprise that the result was that all equations of motions were zero except for θ's dynamical equation. The reason is that $G = SE(2)$ does not provide full symmetry in this example.*

Remark 3.3 *As one notices, taking the null space of $\omega(q)$ reveals the appropriate G group to choose and thus leading to a convenient Lie algebra ξ to work with. This will be shown even more with the next Examples.*

The **third step** is to calculate the nonholonmic momentum such that

$$p = \frac{\partial L}{\partial \dot{q}}(\xi_Q^q) = mr(\cos\theta)\dot{x} + mr(\sin\theta)\dot{y} + J_\phi\dot{\phi} \tag{3.9}$$

The **fourth step** is to build the reconstruction equation using Equation 2.47, where $\xi = g^{-1}\dot{s}$ with G being the symmetry group, such that:

$$g^{-1}\dot{s} = -\mathbf{A}(r)\dot{r} + \Gamma(r)p^T \tag{3.10}$$

While one has no longer liberty in selecting G or the fiber variables s^a since they have to represent full symmetry, one has the choice to select g, which can selected to be an Identity matrix or the g of $SE(2)$. Even though $SE(2)$ in this example is not the symmetry group, it still represents the position and orientation of the system. Form this point, this example will be divided into two versions.

The first has g as an Identity matrix (Version A) while the second has g as representing $SE(2)$ (Version B). However, it is important to note that while both lead to the same dynamical equations of the system, Version B results in a simpler representation of the reconstruction equation.

Version A:

Take g to be an Identity matrix:

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.11}$$

Solving for the left side of the equation for $g^{-1}\dot{s}$ gives

$$g^{-1}\dot{s} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\phi} \end{pmatrix} \tag{3.12}$$

Solving for the right side of the equation is done also by calculating $g^{-1}\dot{s}$; however, this time by solving for $\dot{s}^a = (\dot{x}, \dot{y}, \dot{\phi})$ using the constraint equations in Equation 2.10 and the nonholonomic momentum p found in (3.9). The resultant reconstruction equation is the following:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} \frac{r\cos(\theta)}{J_\phi + mr^2}\, p \\ \frac{r\sin(\theta)}{J_\phi + mr^2}\, p \\ \frac{1}{J_\phi + mr^2}\, p \end{pmatrix} \tag{3.13}$$

With $A(r) = 0$, the resultant reconstruction equation shows that system at hand is purely dynamic. Moving to the **fifth step**, $\Gamma(r)p = \Omega e$ where $e = g^{-1}\xi$ is the moving basis and Ω is the body angular velocity. With

$$e = (r\cos(\theta), r\sin(\theta), 1) \tag{3.14}$$

the reconstruction equation becomes:

$$\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} r\Omega\cos(\theta) \\ r\Omega\sin(\theta) \\ \Omega \end{pmatrix} \tag{3.15}$$

Step six is to find the reduced Lagrangian l by replacing $\dot{s} = (\dot{x}, \dot{y}, \dot{\phi})$, knowing that $\xi = g^{-1}\dot{s}$, in Equation 3.1:

$$l(r, \dot{r}, \xi) = \frac{1}{2}\big(m(\xi_1^2 + \xi_2^2) + J_\theta\dot{\theta}^2 + J_\phi\xi_3^2\big) \tag{3.16}$$

This leads to **step seven** to find the *reduced constraint* Lagrangian by replacing ξ in Equation 3.16 by Equation 3.15:

$$l_c(r, \dot{r}, \Omega) = \frac{1}{2}\big((J_\phi + mr^2)\Omega^2 + J_\theta\dot{\theta}^2 \tag{3.17}$$

Moving to the *Legendre Transformation and Reduced Hamiltonian Constraint* box and to **step eight**, in order to find the Legendre Transformation needed with $p = \frac{\partial l_c}{\partial \Omega}$ and $\tilde{p}_\alpha = \frac{\partial l_c}{\partial (\dot{r})^\alpha}$:

$$
\begin{aligned}
p &= \frac{\partial l_c}{\partial \Omega} = (J_\phi + mr^2)\Omega \\
\tilde{p}_\theta &= \frac{\partial l_c}{\partial \dot{\theta}} = J_\theta \dot{\theta}
\end{aligned}
\tag{3.18}
$$

From the Legendre Transformation in Equation 3.18 one can derive Ω and $(\dot{r})^\alpha$ in **step nine**:

$$
\begin{aligned}
\Omega &= \frac{p}{J_\phi + mr^2} \\
\dot{\theta} &= \frac{\tilde{p}_\theta}{J_\theta}
\end{aligned}
\tag{3.19}
$$

To finally reach **step ten** that gives the *reduced constraint* Hamiltonian such that $h_{\bar{\mathcal{M}}} = (\tilde{p}_i \Omega^i + \tilde{p}_\alpha (\dot{r})^\alpha) - l_c$:

$$
h_{\bar{\mathcal{M}}} = \frac{1}{2}\left(\frac{p^2}{J_\phi + mr^2} + \frac{\tilde{p}_\theta^2}{J_\theta}\right) \tag{3.20}
$$

Moving to the *Poisson Reduction* box, where each step is dedicated towards building the reduced Poisson structure matrix, **step eleven** is to write $\tilde{p}_i$, here being p of Equation 3.9 in terms of $p_i = \frac{\partial L}{\partial \dot{q}^i}$ (by replacing $\dot{q}$ in p accordingly):

$$
p_i = (p_x, p_y, p_\theta, p_\phi)^T = (m\dot{x}, m\dot{y}, J_\theta \dot{\theta}, J_\phi \dot{\phi})^T \tag{3.21}
$$

thus p becomes:

$$
p = r(\cos\theta)p_x + r(\sin\theta)p_y + p_\phi \tag{3.22}
$$

Similarly, **step twelve** is to write $\tilde{p}_\alpha$ in terms of $p_\alpha = \frac{\partial L}{\partial (\dot{r})^\alpha}$, the equation $\tilde{p}_\alpha = p_\alpha + \mu_\alpha (\frac{\partial (\xi)^\alpha}{\partial (\dot{r}^\alpha)}$ must be followed such that the ξ is that of Equation 3.15 (*note*: If necessary expand Ω using Equation 3.19).
In addition, μ_α is found by $\mu_\alpha = \frac{\partial l}{\partial \xi}$ and then expanding the result using Equation 3.15 with l being the reduced Lagrangian found in Equation 3.16.

$$
\mu = \frac{\partial l}{\partial \xi} = (m\xi_1, m\xi_2, J_\phi \xi_3)^T = (mr(\cos\theta)\Omega, mr(\sin\theta)\Omega, J_\phi \Omega)^T \tag{3.23}
$$

$$
\tilde{p}_\theta = p_\theta \tag{3.24}
$$

In order to make sure the constraint will be conserved while applying the Poisson brackets, in **step thirteen**, $\tilde{p}_i$ and $\tilde{p}_\alpha$ need to be written in terms of μ, which this time will derived using $\tilde{p}_i = \mu_d e_i^d$ with e being described in Equation 3.14:

$$
p = e_1 p_x + e_2 p_y + e_3 p_\phi \tag{3.25}
$$

this gives :

$$\mu = (p_x, p_y, p_\phi) \tag{3.26}$$

Now rewrite Equations 3.22 and 3.24 in terms of μ:

$$\begin{gathered} p = \mu_1 r\cos(\theta) + \mu_2 r\sin(\theta) + \mu_3 \\ \tilde{p}_\theta = p_\theta \end{gathered} \tag{3.27}$$

Step fourteen is to find $\{\tilde{p}_i, \tilde{p}_\alpha\}$ and $\{\tilde{p}_\alpha, \tilde{p}_\beta\}$ that will go in the reduced Poisson structure. In this case, there is only one $\tilde{p}_\alpha$, so one needs to find $\{\tilde{p}_i, \tilde{p}_\alpha\}(q,p)$ using the equation of the brackets in Figure 3.6:

$$\{p, \tilde{p}_\theta\} = r(\mu_2\cos(\theta) - \mu_1\sin(\theta)) \tag{3.28}$$

As previously mentioned $\tilde{p}_i$ and $\tilde{p}_\alpha$ were written in terms μ in order to conserved the constraint and thus once μ is expanded the brackets $\{,\}(q,p)$ will become restricted to the submanifold $\bar{\mathcal{M}}$. Before expanding μ, it needs to be defined once again by writing μ found in Equation 3.23 and replacing Ω and $\dot{r}^\alpha$, which are calculated in Equation 3.19. Thus, **step fifteen** gives:

$$\mu = (\frac{mr\cos(\theta)}{J_\phi + mr^2}p, \quad \frac{mr\sin(\theta)}{J_\phi + mr^2}p, \quad \frac{J_\phi}{J_\phi + mr^2}p) \tag{3.29}$$

Finally, **step sixteen** restricts the brackets $\{,\}$ to the reduced constraint submanifold $\bar{\mathcal{M}}$ by replacing the μ in Equation 3.28 using Equation 3.29. In this case,

$$\{p, \tilde{p}_\alpha\}_{\bar{\mathcal{M}}} = 0 \tag{3.30}$$

Finally **step seventeen** is the construction of the reduced Poisson structure matrix as shown in Figure 3.7. In this case, it is:

$$\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \dot{p} \\ \dot{\theta} \\ \dot{\tilde{p}}_\theta \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & r\cos(\theta) & 0 & 0 \\ 0 & 0 & 0 & r\sin(\theta) & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -r\cos(\theta) & -r\sin(\theta) & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{\partial h_{\bar{\mathcal{M}}}}{\partial p} \\ \frac{\partial h_{\bar{\mathcal{M}}}}{\partial r} \\ \frac{\partial h_{\bar{\mathcal{M}}}}{\partial \tilde{p}_\theta} \end{pmatrix} \tag{3.31}$$

Finally moving to the last box *the Reduced Hamiltonian*, **step eighteen** is to calculate the derivative of $h_{\bar{\mathcal{M}}}$ with respect to $\tilde{p}_i$, r^α, and $\tilde{p}_\alpha$ to build the matrix on the right seen in Equation 3.31. The matrix has the first three element zero representing the three ξ:

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{p}{J_\phi + mr^2} \\ 0 \\ \frac{\tilde{p}_\theta}{J_\theta} \end{pmatrix} \tag{3.32}$$

Since the left side of reconstruction equation for g being an identity matrix directly has $(\xi_1, \xi_2, \xi_3)^T = (\dot{x}, \dot{y}, \dot{\phi})^T$, steps **step nineteen** and **step twenty** can be combined in one. In **step nineteen**, expanding Equation 3.31 using Equation 3.32 gives the **reduced constraint Hamiltonian equations**: beginequation

$$
\begin{aligned}
\xi_1 = \dot{x} &= \frac{r\cos(\theta)}{J_\phi + mr^2} p \\
\xi_2 = \dot{y} &= \frac{r\sin(\theta)}{J_\phi + mr^2} p \\
\xi_3 = \dot{\phi} &= \frac{p}{J_\phi + mr^2} \\
\dot{p} &= \tau_\phi \\
\dot{\theta} &= \frac{\tilde{p}_\theta}{J_\theta} \\
\dot{\tilde{p}}_\theta &= \tau_\theta
\end{aligned}
\tag{3.33}
$$

It is important to note that the external forces (τ_ϕ and τ_θ) were added to their respective momenta for simulation purposes. As defined in Equation (2.20), $(\dot{p}_x, \dot{p}_y, \dot{p}_\theta, \dot{p}_\phi)$ include their respective external forces in them as seen in Equation (2.20). Since p includes (p_x, p_y, p_ϕ) as seen in Equation (3.22), its derivative would include the external forces related to (x, y, ϕ) (in this case τ_ϕ). The same applies to $\dot{\tilde{p}}_\theta$.

<u>Version B:</u>

While the user does have a choice in using either an identity matrix or the group representing the configuration of the orientation of the system(i.e (x, y, θ) in this case leading to a $SE(2)$ configuration), only the latter would result in simplified middle steps to reach the exact final solution.
Take g to represent the position and orientation of the system $SE(2)$:

$$
g = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.34}
$$

Solving for the left side of the equation for $g^{-1}\dot{s}$ gives

$$
g^{-1}\dot{s} = \begin{pmatrix} (\cos\theta)\dot{x} + (\sin\theta)\dot{y} \\ (-\sin\theta)\dot{x} + (\cos\theta)\dot{y} \\ \dot{\phi} \end{pmatrix} \tag{3.35}
$$

The resultant reconstruction equation here becomes:

$$
\begin{pmatrix} (\cos\theta)\dot{x} + (\sin\theta)\dot{y} \\ (-\sin\theta)\dot{x} + (\cos\theta)\dot{y} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} \frac{r}{J_\phi + mr^2} p \\ 0 \\ \frac{1}{J_\phi + mr^2} p \end{pmatrix} \tag{3.36}
$$

The resultant reconstruction equation system is once again revealed to be purely dynamic with $A(r) = 0$. In **step five**, $\Gamma(r)p = \Omega e$ where $e = g^{-1}\xi$, being the moving basis, and Ω the body angular velocity are found to differ from . With

$$e = (r, 0, 1) \tag{3.37}$$

the reconstruction equation becomes:

$$\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} r\Omega \\ 0 \\ \Omega \end{pmatrix} \tag{3.38}$$

Step six through **step ten** are found to be exactly the same as Version A, where g is an identity matrix, since they are not affected by the choice of representation of the reconstruction equation.
Jumping directly to the *Poisson Reduction* box and specifically to **step eleven**. Writing $\tilde{p}_i$ of Equation 3.9 in terms of $p_i = \frac{\partial L}{\partial \dot{q}^i}$, which were found in Version A, Equation (3.21):

$$p = r(\cos\theta)p_x + r(\sin\theta)p_y + p_\phi \tag{3.39}$$

As for **step twelve**, a simplification pf μ is noticed:

$$\mu = \frac{\partial l}{\partial \xi} = (m\xi_1, m\xi_2, J_\phi\xi_3)^T = (mr\Omega, 0, J_\phi\Omega)^T \tag{3.40}$$

while $p_\alpha = \frac{\partial L}{\partial (\dot{r})^\alpha}$ remains unaffected by μ in this example:

$$\tilde{p}_\theta = p_\theta \tag{3.41}$$

Looking at **step thirteen**, $\tilde{p}_i = \mu_d e_i^d$ with e being described in Equation 3.37, changed accordingly:

$$p = e_1((\cos\theta)p_x + (\sin\theta)p_y) + e_3 p_\phi \tag{3.42}$$

leading to a change in μ:

$$\mu = (\cos\theta p_x + \sin\theta p_y, 0, p_\phi) \tag{3.43}$$

Now rewrite Equations 3.39 and 3.41 in terms of μ:

$$\begin{gathered} p = \mu_1 r + \mu_3 \\ \tilde{p}_\theta = p_\theta \end{gathered} \tag{3.44}$$

In this version, $\{p, \tilde{p}_\theta\}(q, p)$ goes to zero as of **step fourteen**:

$$\{p, \tilde{p}_\theta\} = 0 \tag{3.45}$$

Using Equation 3.40 and replacing Ω and $\dot{r}^\alpha$, which are calculated in Equation **??**,μ redefined in **step fifteen**:

$$\mu = (\frac{mr}{J_\phi + mr^2}p, \quad 0, \quad \frac{J_\phi}{J_\phi + mr^2}p) \tag{3.46}$$

Finally, **step sixteen**, which restricts the brackets $\{,\}$ to the reduced constraint submanifold $\bar{\mathcal{M}}$, is now the same as Version A.

$$\{p, \tilde{p}_\alpha\}_{\bar{\mathcal{M}}} = 0 \tag{3.47}$$

Step seventeen reveals a much a reduced and simplified Poisson structure matrix being:

$$\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \dot{p} \\ \dot{\theta} \\ \dot{\tilde{p}_\theta} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & r & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -r & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{\partial h_{\bar{\mathcal{M}}}}{\partial p} \\ \frac{\partial h_{\bar{\mathcal{M}}}}{\partial r} \\ \frac{\partial h_{\bar{\mathcal{M}}}}{\partial \tilde{p}_\theta} \end{pmatrix} \tag{3.48}$$

Moving to the last box *the Reduced Hamiltonian*, the derivative of $h_{\bar{\mathcal{M}}}$ with respect to $\tilde{p}_i$, r^α, and $\tilde{p}_\alpha$ is calculated **step eighteen**. Reminder, the matrix has the first three element zero representing the three ξ:

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{p}{J_\phi + mr^2} \\ 0 \\ \frac{\tilde{p}_\theta}{J_\theta} \end{pmatrix} \tag{3.49}$$

In **step nineteen**, expanding Equation 3.48 using Equation 3.49 gives the **reduced constraint Hamiltonian equations**:

$$\begin{gathered} \xi_1 = \frac{r}{J_\phi + mr^2}p \\ \xi_2 = 0 \\ \xi_3 = \frac{p}{J_\phi + mr^2} \\ \dot{p} = \tau_\phi \\ \dot{\theta} = \frac{\tilde{p}_\theta}{J_\theta} \\ \dot{\tilde{p}}_\theta = \tau_\theta \end{gathered} \tag{3.50}$$

Finally in the last step,**step twenty**, one obtains the reconstruction equations on the Hamiltonian side:

$$
\begin{aligned}
\dot{x} = \xi_1 \cos(\theta) - \xi_2 \sin(\theta) &= \frac{r\cos(\theta)p}{J_\phi + mr^2} \\
\dot{y} = \xi_1 \sin(\theta) + \xi_2 \cos(\theta) &= \frac{r\sin(\theta)p}{J_\phi + mr^2} \\
\dot{\phi} = \xi_3 &= \frac{p}{J_\phi + mr^2}
\end{aligned}
\tag{3.51}
$$

These equations (3.50) and (3.51) are equivalent to equations (3.1),which is found in Version A.

Simulation

In this section, the four sets of equations representing the unicycle are simulated. Equation (2.12) represents the Euler-Lagrangian formulation and equation (2.35) represents the Hamiltonian formulation with both including the Lagrangian multipliers (λ_1, λ_2). On the other hand, equations (3.1) and (3.50, 3.51), while both being equivalent, represent the constraint Hamiltonian formulation using Poisson geometry with no Lagrangian multipliers.
For all three sets, a Proportional Derivative (PD) controller was designed with specified angles and control parameters. The parameters were taken to be $m = 0.5$, $r = 0.1$, $J_\phi = 0.2$, and $J_\theta = 0.3$. The desired trajectories for θ and ϕ were specified as:

$$
\begin{aligned}
\theta_d &= \sin(t) \\
\phi_d &= t
\end{aligned}
\tag{3.52}
$$

The PD controller for both θ and ϕ were defined as:

$$
\begin{aligned}
\tau_\theta &= k_p(\theta_d - \theta) + k_d(\dot{\theta}_d - \dot{\theta}) \\
\tau_\phi &= k_p(\phi_d - \phi) + k_d(\dot{\phi}_d - \dot{\phi})
\end{aligned}
\tag{3.53}
$$

where the controller gains are $k_p = 20$ and $k_d = 5$. The initial conditions were designed as to start at the origin and move along the $x - axis$.

For the equations of motion, (2.12), which are derived using the Lagrangian formulation , the initial conditions (x, y) were set at zero where as the initial conditions for (θ, ϕ) and their first derivatives, they were set their desired values, respectively.
Second, for the results of the Hamiltonian Formulation with Lagrangian multipliers,the same set of initial conditions were kept, but instead of setting the initial condition of the first order derivatives of (θ, ϕ), the initial conditions of their respective momenta were defined by using the relationship between the moments

<table>
<tr><td>Lagrangian</td><td rowspan="3">$x(0) = 0$
$y(0) = 0$</td><td rowspan="3">$\theta(0) = \theta_d(0)$
$\phi(0) = \phi_d(0)$</td><td>$\dot{\theta}(0) = \dot{\theta}_d(0)$
$\dot{\phi}(0) = \dot{\phi}_d(0)$</td></tr>
<tr><td>Hamiltonian</td><td>$p_\theta(0) = J_\theta \dot{\theta}_d(0)$
$p_\phi(0) = J_\phi \dot{\phi}_d(0)$</td></tr>
<tr><td>Constrained Reduced Hamiltonian</td><td>$\tilde{p}_\theta(0) = J_\theta \dot{\theta}_d(0)$
$p(0) = (J_\phi + mr^2)\dot{\phi}_d(0)$</td></tr>
</table>

Table 3.2: Initial conditions of the Unicycle for all three sets of formulations, where C-Hamiltonian refers to the Constrained Hamiltonian

and $(\dot{\theta}_d, \dot{\phi}_d)$, as shown in equations (3,4) of Equation 2.35 at the origin, respectively.
Third, for the results of the constraint Hamiltonian formulation with no Lagrangian multipliers, the same was done for the initial conditions as set two but this time with respect to the newly introduced set of momenta using equations (3,5) in Equation (3.1). All initial conditions chosen are specified in Table 3.2.

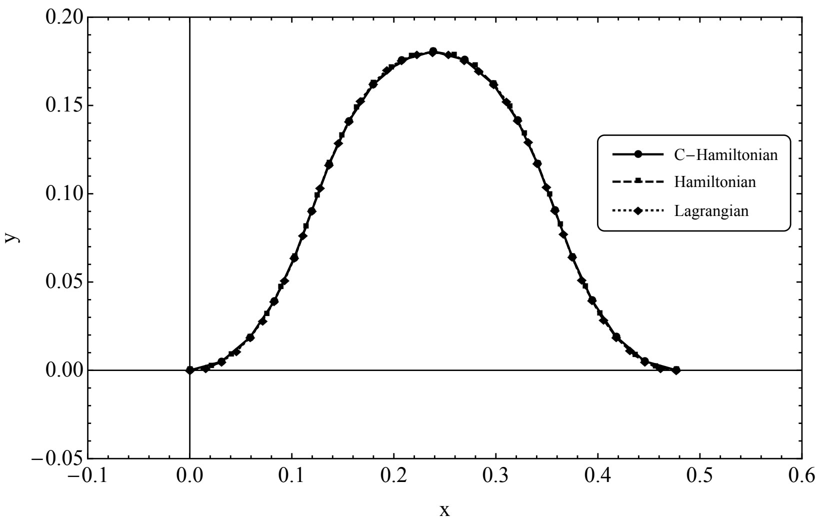

Figure 3.8: The solution for $(x(t), y(t))$ of the Unicycle for a 2π cycle for all three sets of formulation, where C Hamiltonian refers to the Constrained Hamiltonian

Figures 3.8, 3.9, and 3.10 show how all three sets result in solutions that specifically overlap; however, it is only the constraint Hamiltonian formulation that has no Lagrangian multipliers. In addition, Figure 3.17 shows how for both the classical Hamiltonian formulation and the Lagrangian formulation have solutions for the Lagrangian multipliers that overlap.

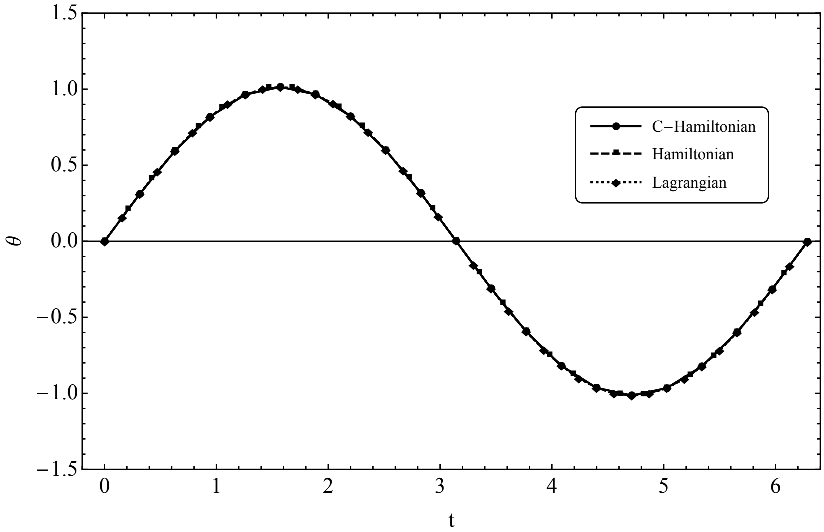

Figure 3.9: The solution for $(\theta(t))$ of the Unicycle for a 2π cycle for all three sets of formulation

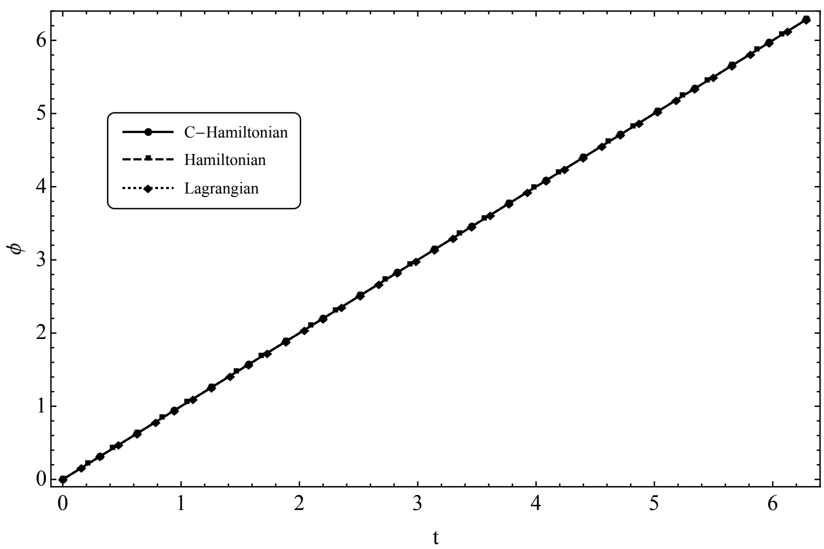

Figure 3.10: The solution for $(\phi(t))$ of the Unicycle for a 2π cycle for all three sets of formulation

Now that one has a clearer idea of how the steps flow, here is another example.

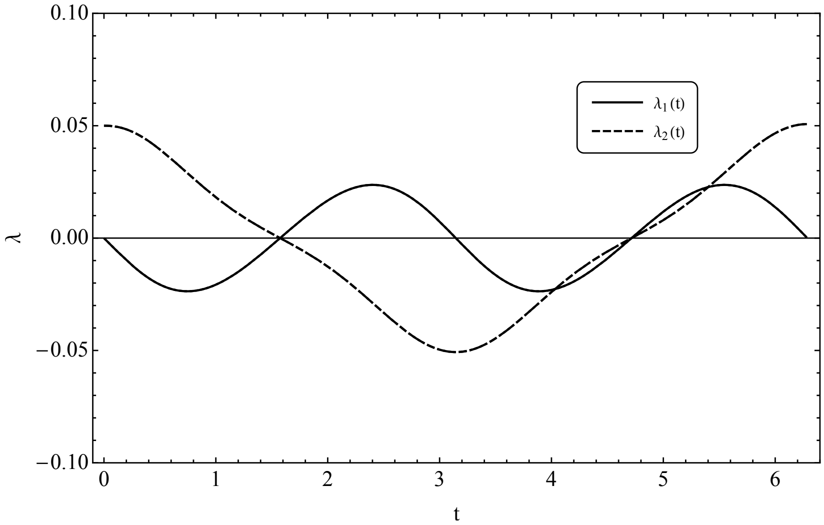

Figure 3.11: The solution for λ_1 and λ_2 of the Unicycle for a 2π cycle for the Hamiltonian formulation and Lagrangian formulation

Example 3.2 : The Snakeboard
Taking the Snakeboard example, which is inspired [2] and [25], and applying the method introduced in this chapter on it. The configuration of the board is given by $q = (x, y, \theta, \psi, \phi)$, where (x, y, θ) represent the position and orientation of the board in the plane, and ψ represents the angle of the momentum wheel relative to the board, and ϕ is set to be $\phi = \phi_f = -\phi_b$, where ϕ_f and ϕ_b are the angles of the back and front wheels relative to the board, respectively. Figure 3.12 shows the the geometry of the snakeboard. The configuration space is $Q = SE(2) \times \mathbb{S}^1 \times \mathbb{S}^1$.

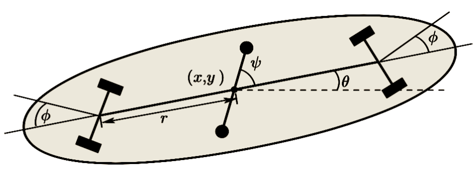

Figure 3.12: Sketch for Snakeboard example

As a first step for the *Set-up*, the Lagrangian $L(q, \dot{q})$ is the total kinetic energy of the system defined using Equation 2.4:

$$L(q, \dot{q}) = \frac{1}{2}m(\dot{x}^2+\dot{y}^2)+\frac{1}{2}J\dot{\theta}^2+\frac{1}{2}J_r(\dot{\theta}+\dot{\psi})^2+1/2J_w(\dot{\theta}-\dot{\phi})^2+1/2J_w(\dot{\theta}+\dot{\phi})^2 \quad (3.54)$$

where m is the total mass of the board, J is the inertia of the board,J_r is the inertia of the rotor, and J_w is the inertia of each of the wheels. A condition is set such that $J + J_r + 2J_w = mr^2$ for simplification purposes. L, being independent of the of the configuration of the system, is invariant to all possible group actions by $SE(2)$. The Lagrangian shortens to:

$$L(q,\dot{q}) = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) + \frac{1}{2}mr^2\dot{\theta}^2 + \frac{1}{2}J_r\dot{\psi}^2 + J_r\dot{\psi}\dot{\theta} + J_w\dot{\phi}^2 \tag{3.55}$$

Second part of the first step of *Set-Up* is to define the constraint equations which are:

$$\begin{aligned} -\sin(\theta-\phi)\dot{x} + \cos(\theta-\phi)\dot{y} - r\cos(\phi)\dot{\theta} = 0 \\ -\sin(\theta+\phi)\dot{x} + \cos(\theta+\phi)\dot{y} + r\cos(\phi)\dot{\theta} = 0 \end{aligned} \tag{3.56}$$

From Equation 3.56, $\omega(q)$, in Pfaffian form $\omega(q)\dot{q} = 0$ is:

$$\begin{pmatrix} -\sin(\theta+\phi) & \cos(\theta+\phi) & -r(\cos\phi) & 0 & 0 \\ -\sin(\theta-\phi) & \cos(\theta-\phi) & r(\cos\phi) & 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\psi} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.57}$$

Lagrangian Formulation:

The first set of equations of motions for the Snakeboard is found through the Euler-Lagrangian equations (2.8) with the generalized forces on the coordinates (ψ, ϕ):

$$\begin{aligned} m\ddot{x} &= -\sin(\theta-\phi)\lambda_1 - \sin(\theta+\phi)\lambda_2 \\ m\ddot{y} &= \cos(\theta-\phi)\lambda_1 + \cos(\theta+\phi)\lambda_2 \\ J_r\ddot{\theta} + J_w\ddot{\psi} &= \tau_\psi \\ 2J_r\ddot{\phi} &= \tau_\phi \\ -\sin(\theta-\phi)\dot{x} + \cos(\theta-\phi)\dot{y} - r\cos(\phi)\dot{\theta} &= 0 \\ -\sin(\theta+\phi)\dot{x} + \cos(\theta+\phi)\dot{y} + r\cos(\phi)\dot{\theta} &= 0 \end{aligned} \tag{3.58}$$

The first set of equation, as seen, yields second-order equations of motion with extra variables, which are the Lagrangian multipliers.

Hamiltonian Formulation:

The second set of equations of motions, which includes the Lagrangian multipliers, is derived using the he generalized Hamiltonian formulation and the Legendre Transformation, which is defined as the momentum associated with each variable and the Hamiltonian, via Equation 2.19. The Legendre Transformation are calculated to be:

$$p = \frac{\partial L}{\partial q} = \begin{pmatrix} m\dot{x} \\ m\dot{y} \\ mr^2\dot{\theta} + J_r\dot{\psi} \\ J_r(\dot{\theta}+\dot{\psi}) \\ 2J_w\dot{\phi} \end{pmatrix} \tag{3.59}$$

where $p = (p_x, p_y, p_\theta, p_\psi, p_\phi)^T$. Using Equation (2.19), this leads to the following Hamiltonian:

$$H(q,p) = \frac{1}{2}\Big(p_x^2 + p_y^2 - \frac{p_\theta^2}{J_r - mr^2} + \frac{p_\phi^2}{J_w} + \frac{2p_\theta p_\psi}{J_r - mr^2} - \frac{mr^2 p_\psi}{J_r(J_r - mr^2)}\Big) \tag{3.60}$$

Using Equation (2.20), the second set of equations of motion are of first-order, yet still include the Lagrangian multipliers:

$$\begin{aligned}
\dot{x} &= \frac{p_x}{m} \\
\dot{y} &= \frac{p_y}{m} \\
\dot{\theta} &= \frac{p_\theta - p_\psi}{J_r - mr^2} \\
\dot{\psi} &= \frac{-J_r p_\theta + mr^2 p_\psi}{J_r(J_r - mr^2} \\
\dot{\phi} &= \frac{p_\phi}{2J_w} \\
\dot{p}_x &= -\sin(\theta - \phi)\lambda_1 - \sin(\theta + \phi)\lambda_2 \\
\dot{p}_y &= \cos\theta - \phi)\lambda_1 + \cos(\theta + \phi)\lambda_2 \\
\dot{p}_\theta &= r(-\lambda_1 + \lambda_2)\cos\phi \\
\dot{p}_\psi &= \tau_\psi \\
\dot{p}_\phi &= \tau_\phi \\
-\sin(\theta - \phi)\dot{x} + \cos(\theta - \phi)\dot{y} - r\cos(\phi)\dot{\theta} &= 0 \\
-\sin(\theta + \phi)\dot{x} + \cos(\theta + \phi)\dot{y} + r\cos(\phi)\dot{\theta} &= 0
\end{aligned} \tag{3.61}$$

While the generalized Hamiltonian formulation decreased the index of reduction of the equations of motion, it increased the equations from second order n equations to first order $2n$ equations and the Lagrangian multipliers are still present.

Reduced Constraint Hamiltonian formulation:

In order to get the third set of equations with no lagrangian multipliers and simplified first order equations of motion, the method introduced in this chapter is implemented on the Snakeboard problem.
Continuing to the *Equivalence of Poisson and Lagrangian Reduction* box, the **second step** is to find ξ_Q^q by taking the Null Space of $\omega(q)$, which gives:

$$\mathbf{NullSpace}(\omega(q)) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ r\cos(\theta)\cot(\phi) & r\sin(\theta)\cot(\phi) & 1 & 0 & 0 \end{pmatrix} \tag{3.62}$$

Thus, the constraint distribution $\mathcal{D}_q$ is:

$$\mathcal{D}_q = \mathbf{span}\Big\{\frac{\partial}{\partial\phi}, \quad \frac{\partial}{\partial\psi}, \quad (r\cos(\theta)\cot(\phi))\frac{\partial}{\partial x} + (r\sin(\theta)\cot(\phi))\frac{\partial}{\partial y} + \frac{\partial}{\partial\theta}\Big\} \tag{3.63}$$

For simplification purposes of trigonometry and in turn simplify further steps down the method, simplify $\mathcal{D}_q$ to:

$$\mathcal{D}_q = \textbf{span}\Big\{\frac{\partial}{\partial\phi}, \quad \frac{\partial}{\partial\psi}, \quad (2r\cos(\theta)\cos^2(\phi))\frac{\partial}{\partial x}+(2r\sin(\theta)\cos^2(\phi))\frac{\partial}{\partial y}+Sin(2\phi)\frac{\partial}{\partial\theta}\Big\} \tag{3.64}$$

Here, one can see by looking at $\mathcal{D}_q$ that $SE(2)$ gives full symmetry and as such is the convenient G group to have. In its turn, $\mathcal{T}_qOrb(q)$ is chosen to be:

$$\mathcal{T}_qOrb(q) = \textbf{span}\Big\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial\theta}\Big\} \tag{3.65}$$

Finally, $\mathcal{S}_q$ is:

$$\mathcal{S}_q = \mathcal{D}_q \cap \mathcal{T}_qOrb(q) = \textbf{span}\Big\{(2r\cos(\theta)\cos^2(\phi))\frac{\partial}{\partial x}+(2r\sin(\theta)\cos^2(\phi))\frac{\partial}{\partial y}+\sin(2\phi)\frac{\partial}{\partial\theta}\Big\} \tag{3.66}$$

All of this is done, to conclude that for $G = SE(2)$ the section of $\mathcal{S}_q$ takes the vector field:

$$\xi_Q^q = (2r\cos(\theta)\cos^2(\phi))\frac{\partial}{\partial x} + (2r\sin(\theta)\cos^2(\phi))\frac{\partial}{\partial y} + \sin(2\phi)\frac{\partial}{\partial\theta} \tag{3.67}$$

with a corresponding Lie algebra element:

$$\xi = (2r\cos(\theta)\cos^2(\phi), 2r\sin(\theta)\cos^2(\phi), \sin(2\phi)) \tag{3.68}$$

Furthermore, with the local coordinates being written as $q = (r^\alpha, s^a)$, here $r^\alpha = (\psi, \phi)$ and $s^a = (x, y, \theta)$. Moving to the **third step**, which is to find the nonholonomic momentum:

$$p = \frac{\partial L}{\partial \dot{q}}(\xi_Q^q) = 2mr\cos(\theta)\cos^2(\phi)\dot{x}2mr\sin(\theta)\cos^2(\phi)\dot{y} + \sin(2\phi)(mr^2\dot{\theta} + J_0\dot{\phi}) \tag{3.69}$$

The **fourth step** is to write the reconstruction equation $g^{-1}g + A(r)\dot{r} = \Gamma(r)p$ with $\xi = g^{-1}g$ by solving Equations 3.56 and 3.69. Due to the simplification in geometry, $G = SE(2)$ was chosen for the reconstruction equation:

$$g = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.70}$$

To finally get:

$$\begin{pmatrix} (\cos\theta)\dot{x} + (\sin\theta)\dot{y} \\ (-\sin\theta)\dot{x} + (\cos\theta)\dot{y} \\ \dot{\theta} \end{pmatrix} + \begin{pmatrix} \frac{J_r\sin(2\phi)}{2mr}\dot{\phi} \\ 0 \\ \frac{2J_r\sin^2(\phi)}{2mr^2}\dot{\phi} \end{pmatrix} = \begin{pmatrix} \frac{1}{2mr}p \\ 0 \\ \frac{\tan(\phi)}{2mr^2}p \end{pmatrix} \tag{3.71}$$

Fifth step is to replace $\Gamma(r)p = \Omega e$, where Ω is the local version of the locked angular velocity and e, the moving basis, is found by $e = g^{-1}.\xi^q$:

$$e = (2r\cos^2(\phi), 0, \sin(2\phi)) \tag{3.72}$$

As a result, Equation 3.71 becomes $\xi = -A(r)\dot{r} + e\Omega$ such that:

$$\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} -\frac{J_r\cos(\phi)\sin(\phi)}{mr}\dot{\phi} + 2r\cos^2(\phi)\Omega \\ 0 \\ -\frac{J_r\sin^2(\phi)}{mr^2}\dot{\phi} + (\sin(2\phi)\Omega \end{pmatrix} \tag{3.73}$$

Step six is to find $l(r, \dot{r}, \xi)$ by solving for $\dot{r}$ in terms of ξ (via $\xi = g^{-1}\dot{s}$) and replacing the found $\dot{r}$ in Lagrangian L calculated in Equation 3.55:

$$l(r, \dot{r}, \xi) = \frac{1}{2}\left(m\xi_1^2 + m\xi_2^2 + mr^2\xi_3^2 + 2J_w\dot{\phi}^2 + 2J_r\xi_3\dot{\phi} + J_r\dot{\phi}^2\right) \tag{3.74}$$

Finally reaching **step seven** to calculate the **reduced constrained Lagrangina** $l_c(r, \dot{r}, \Omega)$ by replacing ξ with the ξ found in Equation 3.73:

$$l_c(r, \dot{r}, \Omega) = 2mr^2\Omega^2\cos^2(\phi) + J_w\dot{\phi}^2 + (\frac{1}{2}J_r + \frac{-J_r\sin^2(\phi)}{2mr^2})\dot{\psi}^2 \tag{3.75}$$

Moving to *Legendre Transformation & Reduced Hamiltonian Constraint*, **step eight** is to find the **Legendre Transformation** to lead to the reduced Poisson:

$$p = \frac{\partial l_c}{\partial \Omega} = 4mr^2\cos^2(\phi)\Omega \tag{3.76}$$

$$\begin{gathered} \tilde{p}_\psi = \frac{\partial l_c}{\partial \dot{\psi}} = (\frac{1}{2}J_r + \frac{-J_r\sin^2(\phi)}{2mr^2})\dot{\psi} \\ \tilde{p}_\phi = \frac{\partial l_c}{\partial \dot{\phi}} = 2J_w\dot{\phi} \end{gathered} \tag{3.77}$$

Step nine, from the Legendre Transformation, deduce $\Omega, \dot{r}^\alpha$:

$$\Omega = \frac{\sec^2(\phi)}{4mr^2}p \tag{3.78}$$

$$\begin{gathered} \dot{\psi} = \frac{2mr^2}{J_r(-J_r + 2mr^2 + J_r\cos(2\phi))}\tilde{p}_\psi \\ \dot{\phi} = \frac{1}{2J_w}\tilde{p}_\phi \end{gathered} \tag{3.79}$$

Finally, **step ten** gives the **Reduced Constrained Hamiltonian** ($h_{\bar{\mathcal{M}}} = p\Omega + \tilde{p}_\alpha\dot{r}^\alpha - l_c$):

$$h_{\bar{\mathcal{M}}} = \frac{1}{4J_w}\tilde{p}_\phi^2 + \frac{mr^2}{J_r(-J_r + 2mr^2 + J_r\cos(2\phi))}\tilde{p}_\psi^2 + \frac{\sec^2(\phi)}{8mr^2}p^2 \tag{3.80}$$

Moving to *Poisson Reduction*, **step eleven** is to write p in terms of $p_i = \frac{\partial L}{\partial q^i}$:

$$p = -2r\cos(\theta)\cos^2(\phi)p_x + -2r\sin(\theta)\cos^2(\phi)p_y + \sin(2\phi)p_\theta \tag{3.81}$$

Step twelve: write $\tilde{p}_\alpha$ such that $\tilde{p}_\alpha = p_\alpha + \mu_\alpha(\frac{\partial \xi^\alpha}{\partial \dot{r}^\alpha})$ with $p_\alpha = \frac{\partial L}{\dot{r}}$ and $\mu_\alpha = \frac{\partial l}{\partial \xi}$:

$$\mu = \begin{pmatrix} 2mr\cos^2(\phi)\Omega - \frac{J_r\cos(\phi)\sin(\phi)}{r}\dot{\psi} \\ 0 \\ \cos(\phi)\left(2mr^2\sin(\phi)\Omega + J_r\cos(\phi)\dot{\psi}\right) \end{pmatrix} \tag{3.82}$$

$$\begin{gathered} \tilde{p}_\psi = p_\psi + \frac{J_r\sin^2(\phi)}{mr^2}p_\theta + \frac{J_r\cos(\theta)\sin(2\phi)}{2mr}p_x + \frac{J_r\sin(\theta)\sin(2\phi)}{2mr}p_y \\ \tilde{p}_\phi = p_\phi \end{gathered} \tag{3.83}$$

Step thirteen is to find $p, \tilde{p}_\alpha$ in terms of μ, which can also be found from $p = \mu_d e_i^d$ using e from Equation 3.72.

$$p = (\cos\theta p_x + \sin\theta p_y)e_1 + p_\theta e_3 \tag{3.84}$$

thus:

$$\mu = \begin{pmatrix} cos(\theta)p_x + sin(\theta)p_y \\ 0 \\ p_\theta \end{pmatrix} \tag{3.85}$$

To finally write $p, \tilde{p}_\alpha$ in terms of μ:

$$p = 2r\cos^2(\phi)\mu_1 + \sin(2\phi)\mu_3 \tag{3.86}$$

$$\begin{gathered} \tilde{p}_\psi = p_\psi - \frac{J_r\sin(2\phi)}{2mr}\mu_1 - \frac{J_r\sin^2(\phi)}{mr^2}\mu_3 \\ \tilde{p}_\phi = p_\phi \end{gathered} \tag{3.87}$$

Using brackets $\{,\}_\mathcal{M}$ defined in Figure 3.6, **step fourteen** is to find $\{p, \tilde{p}_\alpha\}_\mathcal{M}$ and $\{\tilde{p}_\alpha, \tilde{p}_\beta\}_\mathcal{M}$ while they are written in terms of μ:

$$\begin{gathered} \{\tilde{p}_\psi, \tilde{p}_\phi\}_\mathcal{M} = \frac{J_r r\cos(2\phi)}{mr^2}\mu_1 - \frac{J_r\sin(2\phi)}{mr^2}\mu_2 \\ \{p, \tilde{p}_\phi\}_\mathcal{M} = 2r\sin(2\phi)\mu_1 + 2\cos(2\phi)\mu_3 \\ \{p, \tilde{p}_\psi\}_\mathcal{M} = 0 \end{gathered} \tag{3.88}$$

In order to restrict to $\bar{\mathcal{M}}$, in **step fifteen**let the μ found in Equation 3.82 in terms of $p, \tilde{p}_\alpha$ using Equations 3.78 and 3.79:

$$\begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2r}p - \frac{mr\sin(2\phi)}{2(mr^2 - J_r\sin^2\phi)}\tilde{p}_\psi \\ 0 \\ \frac{1}{2}\tan(\phi)p + \frac{mr^2\cos^2(\phi)}{mr^2 - J_r\sin^2\phi}\tilde{p}_\psi \end{pmatrix} \tag{3.89}$$

Now that μ is restricted to $\bar{\mathcal{M}}$, it is time to restrict the brackets to the submanifold $\bar{\mathcal{M}}$ by replacing the mu found in Equation 3.89 in the brackets in **step sixteen**:

$$\begin{aligned} \{\tilde{p}_\psi, \tilde{p}_\phi\}_{\bar{\mathcal{M}}} &= -\frac{J_r}{2mr^2}p - \frac{J_r \sin(2\phi)}{2(mr^2 - J_r \sin^2\phi)}\tilde{p}_\psi \\ \{p, \tilde{p}_\phi\}_{\bar{\mathcal{M}}} &= -\tan(\phi)p + \frac{2mr^2\cos^2(\phi)}{mr^2 - J_r\sin^2(\phi)}\tilde{p}_\psi \\ \{p, \tilde{p}_\psi\}_{\bar{\mathcal{M}}} &= 0 \end{aligned} \tag{3.90}$$

Finally, the **Reduced Poisson structure matrix** is now ready to built using the format found in Figure 3.7 in **step seventeen**:

$$\begin{pmatrix} 0 & 0 & 0 & 2r\cos^2(\phi) & 0 & 0 & -\frac{J_r}{2mr}\sin(2\phi) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sin(2\phi) & 0 & 0 & -\frac{J_r}{mr^2}\sin^2\phi & 0 \\ -2r\cos^2(\phi) & 0 & -\sin(2\phi) & 0 & 0 & 0 & \{p,\tilde{p}_\psi\}_{\bar{\mathcal{M}}} & \{p,\tilde{p}_\phi\}_{\bar{\mathcal{M}}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{J_r}{2mr}\sin(2\phi) & 0 & \frac{J_r}{mr^2}\sin^2\phi & 0 & -1 & 0 & -\{p,\tilde{p}_\psi\}_{\bar{\mathcal{M}}} & \{\tilde{p}_\psi,\tilde{p}_\phi\}_{\bar{\mathcal{M}}} \\ 0 & 0 & 0 & -\{p,\tilde{p}_\phi\}_{\bar{\mathcal{M}}} & 0 & -1 & -\{\tilde{p}_\psi,\tilde{p}_\phi\}_{\bar{\mathcal{M}}} & \{p,\tilde{p}_\psi\}_{\bar{\mathcal{M}}} \end{pmatrix} \tag{3.91}$$

Now ready to move to *The Reduced Hamiltonian*, **step eighteen** is to calculate the derivatives of $h_{\bar{\mathcal{M}}}$ with respect to $p, r, \tilde{p}_\alpha$ in that order while keeping the first three elements zero in reference to the three ξ:

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{\partial h_{\bar{\mathcal{M}}}}{\partial p} \\ \frac{\partial h_{\bar{\mathcal{M}}}}{\partial \psi} \\ \frac{\partial h_{\bar{\mathcal{M}}}}{\partial \phi} \\ \frac{\partial h_{\bar{\mathcal{M}}}}{\partial \tilde{p}_\psi} \\ \frac{\partial h_{\bar{\mathcal{M}}}}{\partial \tilde{p}_\phi} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{\sec^2\phi}{4mr^2}p \\ 0 \\ \frac{mr^2\sin(2\phi)}{2(mr^2 - J_r\sin^2\phi)^2}\tilde{p}_\psi^2 + \frac{\sec(\phi)^2\tan(\phi)}{4mr^2}p^2 \\ \frac{mr^2}{J_r(mr^2 - J_r\sin^2\phi))}\tilde{p}_\psi \\ \frac{1}{2J_w}\tilde{p}_\phi \end{pmatrix} \tag{3.92}$$

Finally, **step nineteen** gives the **Reduced Hamiltonian Equations** by multiplying the reduced Poisson structure matrix found by Equation 3.92:

$$
\begin{aligned}
\xi_1 &= \frac{p}{2mr} - \frac{r\sin(2\phi)}{-J_r + 2mr^2 + J_r\cos(2\phi)}\tilde{p}_\psi \\
\xi_2 &= 0 \\
\xi_3 &= -\frac{2\sin^2(\phi)}{-J_r + 2mr^2 + J_r\cos(2\phi)}\tilde{p}_\psi + \frac{\tan(\phi)}{2mr^2}p \\
\dot{p} &= \frac{\tilde{p}_\phi}{2J_w}\Big(\frac{4mr^2\cos^2(\phi)}{-J_r + 2mr^2 + J_r\cos(2\phi)}\tilde{p}_\psi - \tan(\phi)p\Big) \\
\dot{\psi} &= \frac{2mr^2}{J_r(mr^2 - J_r\sin^2\phi)}\tilde{p}_\psi \\
\dot{\phi} &= \frac{\tilde{p}_\phi}{2J_w} \\
\dot{\tilde{p}}_\psi &= \frac{p_\phi}{2J_w}\Big(\frac{-J_r}{2mr^2}p + \frac{J_r\sin(2\phi)}{J_r - 2mr^2 - J_r\cos(2\phi)}\tilde{p}_\psi\Big) + \tau_\psi \\
\dot{\tilde{p}}_\phi &= \tau_\phi
\end{aligned}
\tag{3.93}
$$

It is important to note that the generalized forces were added to the first order equation of motion of $\dot{\tilde{p}}_\psi$ and $\dot{\tilde{p}}_\phi$ respectively. At last, **step twenty**, which is the last step, gives the reconstruction equations on the Hamiltonian side by using ξ found in Equation 3.93 and replacing it in left side of Equation 3.94 to give:

$$
\begin{aligned}
\dot{x} &= \cos\theta\Big(\frac{p}{2mr} - \frac{r\sin(2\phi)}{-J_r + 2mr^2 + J_r\cos(2\phi)}\tilde{p}_\psi\Big) \\
\dot{y} &= \sin\theta\Big(\frac{p}{2mr} - \frac{r\sin(2\phi)}{-J_r + 2mr^2 + J_r\cos(2\phi)}\tilde{p}_\psi\Big) \\
\dot{\theta} &= \frac{\tan(\phi)}{2mr^2}p - \frac{2\sin^2(\phi)}{-J_r + 2mr^2 + J_r\cos(2\phi)}\tilde{p}_\psi
\end{aligned}
\tag{3.94}
$$

The Reduced Constraint Hamiltonian formulation using Poisson geometry lead to a set of first order equations of motion with no Lagrangian multiplier and less than $2n$ equations.

Simulation

Three sets of equations of motion representing the Snakeboard were calculated. Equations (3.58) represents the Euler-Lagrangian formulation and equation (3.61) represents the Hamiltonian formulation with both including the Lagrangian multipliers (λ_1, λ_2) and both have a High Index of Reduction. However, equations (3.93,3.94), developed using the constraint Hamiltonian formulation via Poisson geometry, make up the only set of equations that has no Lagrangian multipliers and of first order.

<table>
<tr><td>Lagrangian</td><td rowspan="3">$x(0) = 0$
$y(0) = 0$
$\theta(0) = 0$</td><td rowspan="3">$\psi(0) = \psi_d(0)$
$\phi(0) = \phi_d(0)$</td><td>$\dot{\psi}(0) = \dot{\psi}_d(0)$
$\dot{\phi}(0) = \dot{\phi}_d(0))$
$\dot{x}(0) = \dot{y}(0) = \dot{\theta}(0) = 0$</td></tr>
<tr><td>Hamiltonian</td><td>$p_x(0) = p_y(0) = 0$
$p_\theta(0) = p_\psi(0) = J_r\dot{\psi}_d(0)$
$p_\phi(0) = 2J_w\dot{\phi}_d(0)$</td></tr>
<tr><td>C-Hamiltonian</td><td>$p(0) = J_r \sin(2\phi_d(0))\dot{\psi}_d(0)$
$\tilde{p}_\psi(0) = \frac{2J_r}{mr^2}\left(mr^2 - J_r \sin^2\phi_d(0)\right)\dot{\psi}_d(0)$
$\tilde{p}_\phi(0) = 2J_w\dot{\phi}_d(0)$</td></tr>
</table>

Table 3.3: Initial conditions for all three sets of formulations of the Snakeboard, where C-Hamiltonian refers to the Constrained Hamiltonian

For all three sets, a Proportional Derivative (PD) controller was designed with specified angles and control parameters. The parameters were taken to be $m = 0.5$, $J = 0.3$, $J_r = 0.2$,$J_w = 0.1$, and $r = 1.18322$ was chosen as to not violate the condition imposed $J + J_r + 2J_w = mr^2$. The desired trajectories for ψ and ϕ were specified as:

$$\begin{aligned} \psi_d &= \sin(t) \\ \phi_d &= \sin(t + \frac{\pi}{2}) \end{aligned} \tag{3.95}$$

The PD controller for both ψ and ϕ were defined as:

$$\begin{aligned} \tau_\psi &= \ddot{\psi}_d + k_p(\psi_d - \theta) + k_d(\dot{\psi}_d - \dot{\psi}) \\ \tau_\phi &= \ddot{\phi}_d + k_p(\phi_d - \phi) + k_d(\dot{\phi}_d - \dot{\phi}) \end{aligned} \tag{3.96}$$

where the controller gains are $k_p = 10$ and $k_d = 5$. The initial conditions were designed as to start at the origin and move along the $x - axis$. All initial conditions chosen are specified in Table 3.3.

For the first set of equations, (3.58) derived using the Lagrangian formulation , the initial conditions were set similarly to the way it was done in the Unicycle problem. For all three sets, the initial conditions of (x, y, θ) and their first derivatives were set to zero while the initial conditions for (ψ, ϕ) and their first derivatives were set at their desired values, respectively.
Moving to the second set of equations, (3.61) derived using the generalized Hamiltonian Formulation with Lagrangian multipliers, the same set of initial conditions

were kept, but here the initial conditions that were previously set for $\dot{q}(0)$ were used to find the initial conditions of their respective momenta $p(0)$, which were derived in equations (3.59).
Finally, for the third set of equations,(3.93, 3.94) derived using the reduced constraint Hamiltonian formulation with no Lagrangian multipliers, once again while keeping the same initial conditions of $q(0)$, the initial conditions set for $\dot{q}(0)$ were used to get the initial conditions of $(P(0), \tilde{p}_\psi(0), \tilde{p}_\phi(0))$.

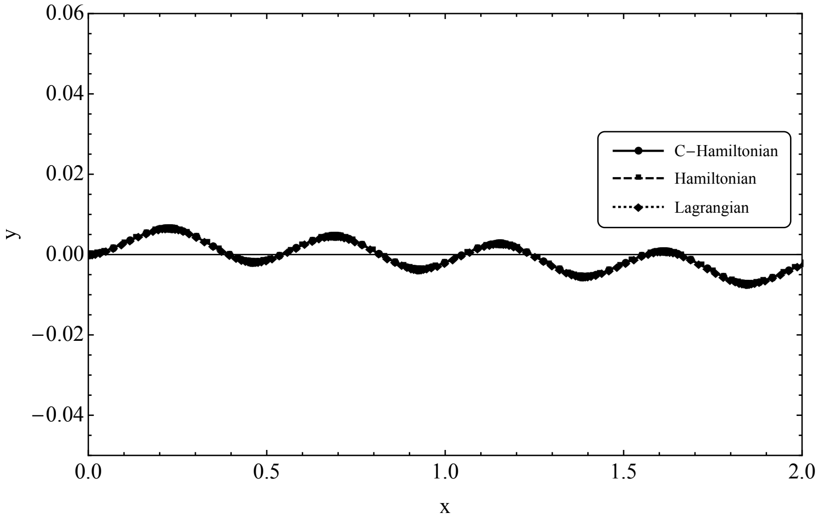

Figure 3.13: The solution for $(x(t), y(t))$ of the Snakeboard for a 10π cycle for all three sets of formulation, where C Hamiltonian refers to the Constrained Hamiltonian

Figures 3.13, 3.14, 3.15, and 3.16 show how all three sets result in solutions that specifically overlap. Once again, it is important to highlight the fact that it is only the constraint Hamiltonian formulation that has no Lagrangian multipliers. In addition, Figure 3.17 shows how for both the classical Hamiltonian formulation and the Lagrangian formulation have solutions for the Lagrangian multipliers that overlap.

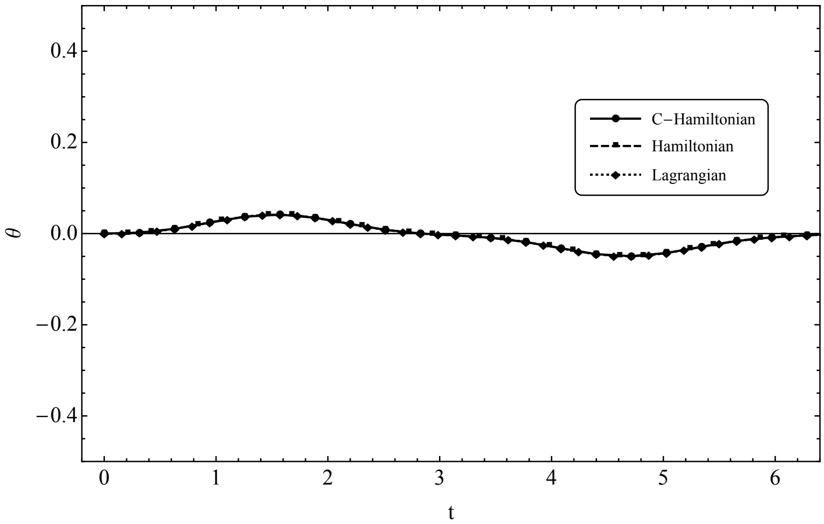

Figure 3.14: The solution for ($\theta(t)$) of the Snakeboard for a 10π for all three sets of formulation

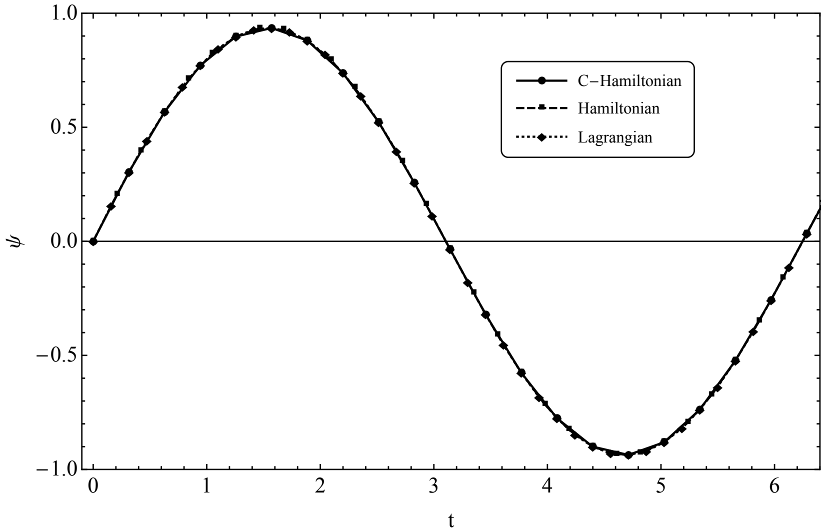

Figure 3.15: The solution for ($\psi(t)$)of the Snakeboard for a 10π cycle for all three sets of formulation

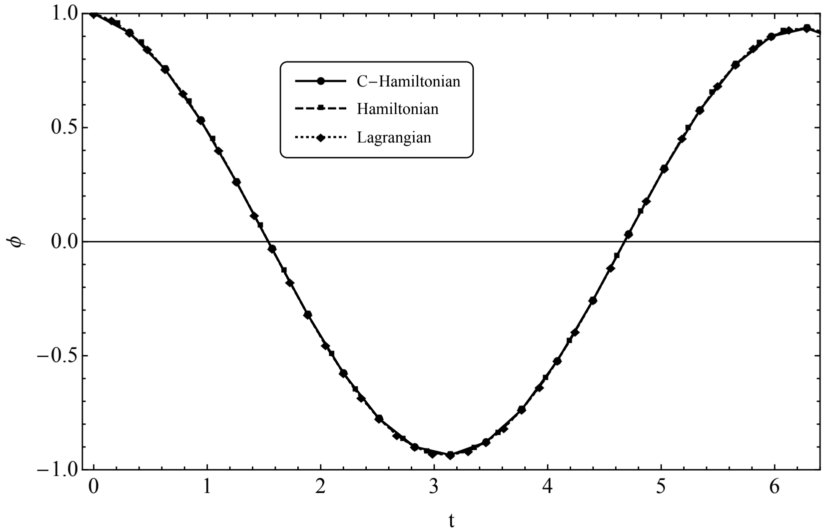

Figure 3.16: The solution for $(\phi(t))$ of the Snakeboard for a 10π cycle for all three sets of formulation

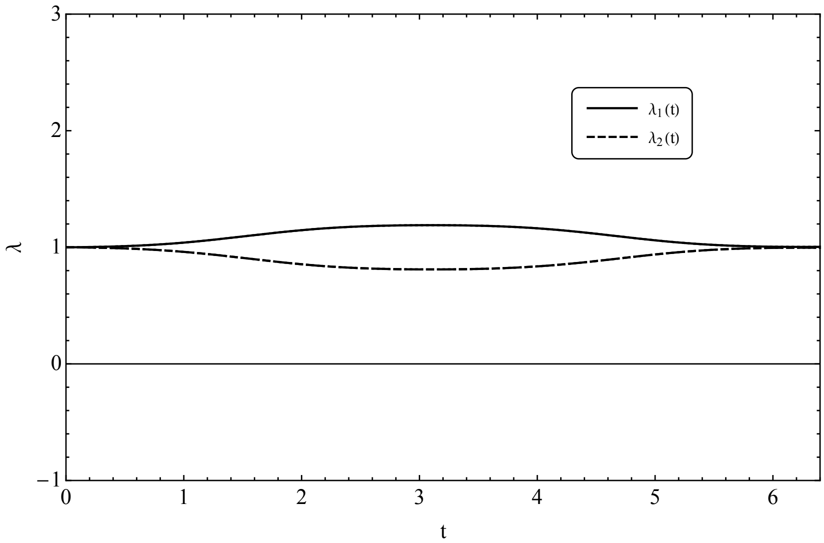

Figure 3.17: The solution for λ_1 and λ_2 of the Snakeboard for a 10π cycle for the Hamiltonian formulation and Lagrangian formulation

Chapter 4

Conclusion

This work presented a study on robotic systems with nonholonomic constraints. A 20-Step method was designed to make reduced constraint Hamiltonian formulation more accessible to the user. The result is a set of $(2n - r)$ set of ODE equations of motion suitable for observability, controllability, and choice of gait studies, with n and r being the number of coordinates q and the number of constraint equations, respectively.

The method integrates the constraints at the initial and geometry level building a reduced constraint submanifold where the dynamics of the system are retrieved. This removes the need to solve for the constraint forces λ, which are of zero order. With only the knowledge of the coordinates q, the Lagrangian L, and the constraint equations $\omega(q)$, first the method finds the full symmetry group of the system. This allows the mapping from the general manifold to the reduced constraint submanifold. In addition, it gives structure to q and decides on its division along with the system's fiber bundle Q. In addition, the reconstruction equation is used as an intermediate step. It shows that choosing to build the reconstruction equation using an identity matrix or based on the system's geometry lead to the same result and set of equations. However, only the latter simplifies all intermediate steps.Moreover, a Poisson like structure is built using the Poisson brackets but within the reduced constraint submanifold.

As part of the work, the 20-Step method was applied to the Unicycle and the Snakeboard problem to clarify and highlight the effects of the steps on the problen and the results. In addition, all three sets of equations from the three frameworks were simulated to show that the result of the 20-Step method coincide with the results of the traditional frameworks. Moreover, a set of initial conditions had to be established for each problem. Furthermore, the new method allows the user to retrieve many valuable information about the system that the tradition frameworks fail to achieve. For instance, focusing on the unicycle problem, inspecting the geometry of the constrained distribution revealed that the group of full symmetry is in fact not the $SE(2)$ but rather $G \times \mathbb{R}^2$. In addition, it also showed that despite being given two choices while building the reconstruction equation, both lead to equivalent equations of motion. This reduced constraint Hamiltonian formulation opens the door to more concrete and solid ways of solving nonholonomic systems.

Bibliography

[1] E. A. Shammas, *Generalized Motion Planning for Underactuated Mechanical Systems*. PhD thesis, Carnegie Mellon University, 2006.

[2] J. E. M. Anthony M. Bloch, P.S Krishnaprasad and R. M. Murray, "Nonholonomic mechanical systems with symmetry," *CDS Technical Report, California Institute of Technology*, vol. 94-013, June 1994.

[3] J. Koiller, "Reduction of some classical non-holonomic systems with symmetry," *Archive for Rational Mechanics and Analysis*, vol. 118, pp. 113–148, Jun 1992.

[4] W. Sarlet, F. Cantrijn, and D. J. Saunders, "A geometrical framework for the study of non-holonomic lagrangian systems," *Journal of Physics A: Mathematical and General*, vol. 28, no. 11, p. 3253, 1995.

[5] J. F. Carinena and M. F. Ranada, "Lagrangian systems with constraints: a geometric approach to the method of lagrange multipliers," *Journal of Physics A: Mathematical and General*, vol. 26, no. 6, p. 1335, 1993.

[6] M. de Len, "Nonholonomic reduction," *Journal of Mathematical Physics*, vol. 37, April 1996.

[7] E. Massa and E. Pagani, "A new look at classical mechanics of constrained systems," *Annales de lI. H. P., section A*, vol. 66, April 1997.

[8] C.-M. Marle, "Various approaches to conservative and nonconservative nonholonomic systems," *Reports on Mathematical Physics*, vol. 42, no. 1, pp. 211 – 229, 1998. Proceedings of the Pacific Institute of Mathematical Sciences Workshop on Nonholonomic Constraints in Dynamics.

[9] L. Bates and J. Śniatycki, "Nonholonomic reduction," *Reports on Mathematical Physics*, vol. 32, 1991.

[10] R. Cushman, D. Kemppainen, J. niatycki, and L. Bates, "Geometry of nonholonomic constraints," vol. 36, pp. 275–286, 12 1995.

[11] R. Cushman, J. Hermans, and D. Kemppainen, "The rolling disc," in *Nonlinear Dynamical Systems and Chaos* (H. W. Broer, S. A. van Gils, I. Hoveijn, and F. Takens, eds.), (Basel), pp. 21–60, Birkhäuser Basel, 1996.

[12] D. Edelen, *Lagrangian Mechanics of Nonconservative Nonholonomic Systems*, vol. 2 of *Mechanics: Dynamical Systems*. Springer Netherlands, 1977.

[13] V. Arnold, *Dynamical Systems III*, vol. 3. Springer (Encyclopaedia of Mathematical Sciences), 1985. translation of the Russian original.

[14] A. V. D. Schaft and B. Maschke, "On the hamiltonian formulation of nonholonomic mechanical systems," *Reports on Mathematical Physics*, vol. 34, no. 2, pp. 225 – 233, 1994.

[15] W. S. Koon and J. E. Marsden, "The hamiltonian and lagrangian approaches to the dynamics of nonholonomic systems," *Reports on Mathematical Physics*, vol. 40, pp. 21 – 62, 1997.

[16] A. Bloch, P. Krishnaprasad, J. Marsden, and R. Murray, "Nonholonomic mechanical systems with symmetry," *Archive for Rational Mechanics and Analysis*, vol. 136, no. 1, pp. 21–99, 1996.

[17] A. Bloch, *Nonholonomic Mechanics and Control*, vol. 24 of *Interdisciplinary Applied Mathematics*. Springer Verlag, 2003.

[18] R. L. Hatton and H. Choset, *An Introduction to Geometric Mechanics and Differential Geometry*. Carnegie Mellon University, 2011.

[19] V. Kumar, "Lectures notes on rigid body motion and the euclidean group," 2002.

[20] H. Choset and R. L. Hatton, "Introduction to geometric mechanics and differential geometry," December 2011.

[21] W. Boothby, *An Introduction to Differentiable Manifolds and Riemannian Geometry*. Pure and Applied Mathematics, Elsevier Science, 1975.

[22] B. Hall, *Lie Groups, Lie Algebras, and Representations: An Elementary Introduction*. Graduate Texts in Mathematics, Springer, 2003.

[23] S. SMALE, "Topology and mechanics. i," *Inventiones mathematicae*, vol. 10, pp. 305–331, 1970.

[24] S. Smale, "Topology and mechanics. ii. the planar n-body problem.," *Inventiones mathematicae*, vol. 11, pp. 45–64, 1970.

[25] J. W. B. Andrew D. Lewis, James P. Ostrowski and R. M. Murray, "Nonholonomic mechanics and locomotion: the snakeboard example," March 1994.