AMERICAN UNIVERSITY OF BEIRUT

AN INTEGRATED SINGLE-VENDOR MULTI-BUYER PRODUCTION INVENTORY MODEL BASED ON THE CONSIGNMENT STOCK CASE WITH CROSS-SHIPMENTS BETWEEN BUYERS

by
MARIO ABDO KARAM

A thesis
submitted in partial fulfillment of the requirements
for the degree of Master of Engineering Management
to the Department of Industrial Engineering and Management
of the Faculty of Engineering and Architecture
at the American University of Beirut

Beirut, Lebanon
January 2019

AMERICAN UNIVERSITY OF BEIRUT

AN INTEGRATED SINGLE-VENDOR MULTI-BUYER PRODUCTION INVENTORY MODEL BASED ON THE CONSIGNMENT STOCK CASE WITH CROSS-SHIPMENTS BETWEEN BUYERS

by

MARIO ABDO KARAM

Approved by:

Dr. Hussein Tarhini, Assistant Professor — Advisor
Department of Industrial Engineering and Management

Prof. Bacel Maddah, Professor — Member of Committee
Department of Industrial Engineering and Management

Dr. Walid Nasr, Associate Professor — Member of Committee
Department of Industrial Engineering and Management

Date of thesis defense: January 04, 2019

AMERICAN UNIVERSITY OF BEIRUT

THESIS, DISSERTATION, PROJECT RELEASE FORM

Student Name: ___Karam_____________Mario______________Abdo_______
Last First Middle

☑ Master's Thesis ◯ Master's Project ◯ Doctoral Dissertation

☑ I authorize the American University of Beirut to: (a) reproduce hard or electronic copies of my thesis, dissertation, or project; (b) include such copies in the archives and digital repositories of the University; and (c) make freely available such copies to third parties for research or educational purposes.

☐ I authorize the American University of Beirut, to: (a) reproduce hard or electronic copies of it; (b) include such copies in the archives and digital repositories of the University; and (c) make freely available such copies to third parties for research or educational purposes after:

One ____ year from the date of submission of my thesis, dissertation, or project.

Two ____ years from the date of submission of my thesis, dissertation, or project.

Three ____ years from the date of submission of my thesis, dissertation, or project.

Mario Abdo Karam January 17, 2019

Signature Date

ACKNOWLEDGEMENTS

I would like to express my genuine gratitude to my advisor Dr. Hussein Tarhini for his continuous support throughout my graduate study. He showed me great motivation and enthusiasm.

I would also like to thank Dr. Mohammad Jaber and the Natural Sciences and Engineering Research Council of Canada (NSERC) for supporting the research.

I want to thank the rest of my thesis committee: Prof. Bacel Maddah and Dr. Walid Nasr for their help.

I want to express my sincere gratitude to my family and all friends who encouraged me and stand by me through my days.

AN ABSTRACT OF THE THESIS OF

Mario Abdo Karam for Master of Engineering
Major: Engineering Management

Title: An integrated single-vendor multi-buyer production inventory model based on the consignment stock case with cross-shipments between buyers

Vendor managed inventory (VMI) is an approach used by several companies to monitor and control stocks of products. In this paper, we investigate how the Consignment Stock (CS), a particular VMI policy, may exemplify a successful strategy for both buyers and supplier. CS suppresses the vendor's inventory and uses the buyers' warehouses to stock its products. This paper considers the problem of one vendor with multiple buyers collaborating under VMI with a consignment stock policy with the possibility of cross-shipments between buyers and considering the inventory and shipment storage capacities. We consider transshipment between buyers as a tool to decrease the total cost faced by the vendor and buyers. Numerical results are also presented to illustrate it and discuss its importance. The cost function is shown to be jointly convex in the shipment sizes between the vendors and between the supplier and the vendors. The number of shipments between suppliers and vendors can then be found via a proposed genetic algorithm.

CONTENTS

ILLUSTRATIONS

Table Page

CHAPTER I

INTRODUCTION

Companies typically engage in certain supply agreements that would enable them to strengthen their inventory policies. A vertical relationship between a company and its suppliers is key to a profitable supply chain. Collaboration, preferably a strong one, between them is fundamental in a competitive market. Researchers have been focusing on different inventory models, some of which include considering different coordination mechanisms and supply chain structures. Gümüş, Jewkes, and Bookbinder (2008) inspected three business scenarios: retailer-managed inventory (RMI), consignment inventory, and vendor-managed inventory with consignment stock (VMI-CS), where the supplier makes several deliveries in a production cycle. They recommend that VMI-CS would generate higher cost savings than the other two situations. The VMI-CS policy has been shown to be a successful coordination strategy. The supplier (thereby referred to as vendor) will continually supply the buyers with stock between a minimum and maximum level, ensuring that no stock-outs occur. The vendor stocks its products at the buyers' warehouses. The companies (thereby referred to as buyers) may use from these stocks to satisfy their needs. The buyers pay the vendor only for the items withdrawn from stock, and hence the vendor has information on the trends in buyers' needs. For a review of different coordination mechanisms in supply chains with one or more entities at each level, we suggest the work of Glock (2012) and Govindan (2013).

The single-vendor multi-buyer problem has been targeted before. The work of Lal and Staelin (1984) who considered a quantity discount schedule for the case of one vendor facing groups of similar purchasers has some shortcomings, one of which is that customers belong to a homogeneous group and increase their order size simultaneously. Joglekar (1988) argued that, in reality, if this happens, the vendor's production may have to be increased substantially, which will not only impact the revenue stream, as Lal and Staelin (1984) claimed, but also the manufacturing cost stream. The net effect, in Joglekar (1988)'s opinion, is likely to be far smaller than that estimated by Lal and Staelin (1984). Banerjee and Banerjee (1994) considered the case of stochastic demand in the development of the model for coordinated inventory control between the single vendor and multiple buyers. An electronic data interchange (EDI) was considered to be the real-life link between the parties which allows the supplier to monitor the consumption and send materials to the buyers according to the pre-determined policy. Lu (1995) claimed that it is difficult to estimate the buyers' holding and ordering costs. The objective of the model he developed is to minimize the vendor's annual total cost subject to constraints that limit buyers incurred costs. Viswanathan and Piplani (2001) considered a supply chain consisting of a vendor and multiple buyers. They showed that coordination using common replenishment periods is profitable. Nevertheless, their model was missing the inventory cost of the vendor. Woo, Hsu, and Wu (2001) revised the work of Banerjee (1986a, 1986b) and considered an integrated inventory with a single shipment, where a cycle approach was adopted to minimize the joint cost of the vendor and the buyers, while Boyacı and Gallego (2002) assumed a deterministic and price-dependent demand and showed that the optimal policy occurs for a consignment stock agreement. Siajadi, Ibrahim, and Lochert (2006) modified the work of Banerjee and Banerjee (1994) for multiple shipments and showed it to be profitable. Kim, Hong, and Chang (2006) extended the structure and complexity of the models discussed here.

They added a supplier to the chain that supplies the manufacturer (vendor) with raw materials to produce multiple products. The vendor uses lumpy deliveries when shipping products to buyers. They developed a heuristic to solve the problem. The results the heuristic produced contained a high error for large lots, long production setup times, and high variation in products' setup times.

Zavanella and Zanoni (2009) developed an analytical model for a single-vendor multi-buyer integrated production under the consignment case. They considered one product with deterministic demand and equal batches. They used the joint optimum and sequential solution to get the required replenishment policy to minimize the total cost per cycle for the system. They compared both approaches and found that using the joint optimum and not the sequential solution gives a positive economic impact for the buyers and a negative one for the vendor. Hoque (2008) extended the work of R. M. Hill (1999) and R. Hill and Omar (2006) that targeted single-vendor single-buyer problem. Hoque developed three models for a single-vendor multi-buyer case that can also solve the previously mentioned problem, with a close relationship between the vendor and buyers. The first and second models consider equal batch sizes and unequal in the third. Srinivas and Rao (2010) investigated a chain of a vendor and multiple buyers with a consignment stock agreement and controllable lead time. They proposed four models and used the genetic algorithm to determine the optimal decision variables that would minimize the joint total expected cost under stochastic environment. Hoque (2011) developed two models for a single-vendor multi-buyer production-inventory supply chain. They considered two batch transfer policies: (1) once produced, and (2) once depleted. Unlike the work in the earlier literature, Hoque (2011b) considered transportation time and cost, limited truck capacity, buyers’ storage capacity, setup time, non-zero lead time, and batch size. Hariga, Gumus, Ben-Daya, and Hassini (2013) tackled the single-vendor multi-buyer problem under the consignment case. They proposed a nonlinear mixed integer NP-hard model to schedule the deliveries to the buyers and specify the production lot sizes. They used a heuristic approach to compute the near-optimal delivery schedule. Unequal shipment sizes are allowed. Hariga, Hassini, and Ben-Daya (2014) criticized the paper of Hoque (2011) noting that he has used different standards in the ordering cost as that used in Zavanella and Zanoni (2009), and thus the comparison is not applicable. They also showed that the model proposed by Hoque (2011) is not that of good results. Rad, Razmi, Sangari, and Ebrahimi (2014) studied an integrated vendor-managed inventory system for a single-vendor two-buyer system where the supplier incurs both the cost of ordering raw materials and the cost of setting up production. Diabat (2014) developed a hybrid genetic/simulated annealing algorithm to solve a profit maximization problem in a vendor-managed inventory system with a nonlinear, non-concave objective function.

Transshipment has proven to be an efficient way to enhance the performance of supply chains. The first to study this practice was Krishnan and Rao (1965) who used an order-up-to policy, same costs at different locations, and a single order period. Robinson (1990) extended their work to a multi-period problem. He showed that the introduction of transshipment would reduce the holding and shortage costs. Many researchers study centralized (e.g., Diks and De Kok (1996); Salameh and Jaber (1997)) and decentralized (e.g., Rudi, Kapur, and Pyke (2001)) decision-making in supply chains or networks. Jemai, Rekik, and Kalaï (2013) considered a vendor-managed inventory system with consignment stock policy and transshipment. They showed that transshipment is useful especially when vehicle capacity is not enough to replenish the inventories of the buyers. Lee and Park (2016) considered a network of one supplier and two retailers. The supplier has an uncertain capacity, and the retailers have uncertain demand. With uncertain supplier capacity, the retailers tend to inflate their orders (rationing game), and hence cooperation between the retailers is needed. They proposed the transshipment between the two retailers to pass the surplus of one retailer to the other who is out of stock. Ji, Sun, and Wang (2017) considered demand disruption in a two-stage supply chain from the manufacturer to the retailer and then the consumer, with a transshipment-before-buyback contract. The contract was also investigated for a supply chain of two retailers and a manufacturer and showed that it is beneficial for all to enter this contract. Dehghani and Abbasi (2018) considered the lateral shipment policy for the case of perishable items. They targeted the transshipment of blood units between hospitals. Villa and Castañeda (2018) investigated a system of two retailers and one supplier with transshipment between the retailers. They focused on the behavioral study to determine the effect of coordination of actors through transshipment.

Inspired by the work of Zavanella and Zanoni (2009), the focus of this paper is the consignment stock case and a two-level supply chain with a single vendor on one level and multiple buyers on the second level. It determines the sizes of shipment to buyers and transshipments between the buyers and their frequencies. It proposes a single-vendor multi-buyer production inventory model under the consignment policy (CS). A CS policy ensures the vendor guarantees the availability of items in stock when needed by a buyer. When a vendor replenishes the stock at the buyer's side, the buyer does not pay on replenishment. It pays only for the items it depletes. The vendor benefits by moving its inventory to a cheaper location. The buyer saves by only incurring storage costs. The environment considered is a deterministic one, and the optimal strategy that minimizes the joint total cost per unit time is determined. A genetic algorithm (GA) approach is used to solve the problem; i.e., to find the sizes of shipments from the vendor to the buyers, between the buyers and their frequencies. Transshipments are known as the monitored movement of material among multiple locations at the same echelon level (Herer, Tzur, & Yücesan, 2006). Transshipments or cross-shipments between buyers happen as needed between the buyers to ensure that no stock-out or shortages occur. Cross-shipments could be better than direct shipments from the vendor in cases of high transportation costs for example. Having a limited capacity on the size of shipments affect the optimal strategy, and so it is studied in this paper. The constraint that can face the buyers is the capacity of their warehouses, which also affects the optimal strategy. The problem description and the approach are different from those in Jemai et al. (2013). In this paper, on the contrary to the latter one, there is no specific distribution hub with a possibility of using the products at different warehouses to satisfy demand. This paper allows multiple distributions hubs whose number is determined.

CHAPTER II
MODEL

The model is summarized as follows. There are *y* buyers and one vendor that will collaborate to minimize their total cost per unit time. The constraints that are described later respect the set assumptions. The number and sizes of shipments sent by the vendor to the buyers, and the number and sizes of cross-shipments sent between the buyers (e.g., from *i* to *j* or *j* to *i*), and the cycle length that minimizes the total cost define the optimal policy. The model proposed in this paper is a continuation of the work of Zavanella and Zanoni (2009) and fixes some of its assumptions. In their model, the number of shipments from the vendor to the buyers is a real number, whereas in ours it is an integer number to reflect reality. This assumption by them might result in an infeasible solution. They also required that at least one batch be sent per cycle from the vendor to each buyer. This paper; however, allows for cross-shipments between buyers, where the vendor ships all to one (or more) buyer location and uses it as a distribution hub, result in reducing total cost at optimality. Zavanella and Zanoni (2009) ignored the capacity of a buyer's warehouse. This paper assumes the buyer's stocking capacity to be limited. The other assumptions are the same as in Zavanella and Zanoni (2009), which are deterministic and constant demand, zero lead time, no variable transportation costs, and equal batch sizes (check assumptions in section 2.3.1).

The following subsections will illustrate the parameters and decision variables required for the construction of the model.

A. Parameters

- A vendor setup cost ($)
- A_i order cost by buyer i for a shipment from the vendor ($/setup)
- A_{ij} order cost by buyer j for a shipment from buyer i ($/setup)
- h_1 vendor holding cost per unit per time-unit ($/unit/time-unit)
- h_2 buyer's holding cost per unit per time-unit ($/unit/time-unit)
- P vendor continuous production rate (unit/time unit)
- d_i buyer's i demand rate (unit/time unit)
- y number of buyers
- C_i storage capacity of buyer i (unit)
- C_v storage capacity of the vendor (unit)
- s_i shipment capacity to buyer i (unit/shipment)
- s_{ij} shipment capacity from buyer i to buyer j (unit/shipment)
- TC average total cost of the system per unit time ($/time-unit)

B. Decision Variables

- q_i quantity transported per delivery to buyer i (unit/shipment)
- q_{ij} i_{th} buyer quantity transported per delivery to buyer j (unit/shipment)
- T replenishment cycle length (time-unit)
- n_i number of deliveries from the vendor to buyer i in a cycle
- n_{ij} number of deliveries from buyer i to buyer j

C. The Analytical Model

1. Assumptions

The following assumptions will be made:

- A cycle is defined as the period in which the vendor experiences one setup cost and sends the required amounts to the buyers to satisfy their needs in this same cycle. During this cycle also, the cross-shipments between buyers happen without any violation of the previous requirement.
- The buyers' holding costs are equal. However, this assumption can be extended into groups of neighborhoods having the same holding costs each with no cross-shipments between neighborhoods.
- Production rate is higher than the sum of all demand rates ($P > \sum_i^y d_i$).
- There is no lead time for shipments from the vendor to the buyers and cross-shipments between buyers.
- Ordering cost includes all fixed costs such as transportation, order preparation, loading/unloading, and others as was considered by Jemai et al. (2013).

2. Vendor's Total Cost

The vendor's total cost per unit-time is the sum of its setup cost, SC_v, and holding cost, HC_v, divided by the cycle time, T, and is given as:

$$TC_v = TC_v(T, n_i, q_i | i = 1, \cdots, y) = \frac{SC_v}{T} + \frac{HC_v}{T} = \frac{A}{T} + \frac{h_1}{2PT} \sum_{i=1}^{y} n_i q_i^2 \quad (1)$$

3. Buyers' Total Cost

The total cost of the buyers, TC_b, is computed as the total of all the buyers, $TC_b = \sum_{i=1}^{y} TC_b^i$, since the cross-shipments make it harder to estimate the total cost of each buyer alone and all buyers have the same holding cost. The buyers' total cost per unit-time is the sum of the buyers ordering, OC_b, and holding, HC_b, costs and it is given as:

$$TC_b = TC_b(T, n_i, q_i | i = 1, \cdots, y) = \frac{OC_b}{T} + \frac{HC_b}{T}$$
$$= \frac{1}{T}\left[\sum_{i=1}^{y}\left(n_i A_i + \sum_{\substack{i=1 \\ j\neq i}}^{y} n_{ij} A_{ij}\right)\right] \tag{2}$$
$$+ \frac{h_2}{2}\left[\sum_{i=1}^{y} n_i q_i (1 - \frac{d_i}{P}) + \frac{1}{P}\sum_{i=1}^{y} q_i d_i\right]$$

4. Total Cost and Constraints

The total supply chain cost is the sum of Eqs. (1) and (2) and is given as:

$$TC = TC_v + TC_b = \frac{A}{T} + \frac{h_1}{2PT}\sum_{i=1}^{y} n_i q_i^2 + \frac{1}{T}\left[\sum_{i=1}^{y}\left(n_i A_i + \sum_{\substack{i=1 \\ j\neq i}}^{y} n_{ij} A_{ij}\right)\right]$$
$$+ \frac{h_2}{2}\left[\sum_{i=1}^{y} n_i q_i \left(1 - \frac{d_i}{P}\right) + \frac{1}{P}\sum_{i=1}^{y} q_i d_i\right] \tag{3}$$

To solve this problem, we formulate it as a mathematical programming problem with Eq. (3) being the objective function to be minimized subject to the following constraints:

$$n_i q_i + \sum_j n_{ji} q_{ji} - \sum_j n_{ij} q_{ij} = d_i T \text{ , } \forall \ i, j \tag{4}$$

$$n_i q_i - (n_i - 1)\frac{q_i d_i}{P} \leq C_i \text{ , } \forall \ i \tag{5}$$

$$\sum_j q_{ji} - \sum_j q_{ij} \leq C_i \text{ , } \forall \ i, j \tag{6}$$

$$q_i \leq C_v \text{ , } \forall \ i \tag{7}$$

$q_i \leq s_i, \forall\ i$ (8)

$q_{ij} \leq s_{ij}\ , \forall\ i, j$ (9)

$n_{ij}\ , \forall\ i, j$ non-negative integer (10)

$n_i\ , \forall\ i$ non-negative integer (11)

$q_i\ \geq 0, \forall\ i$ (12)

$q_{ij}\ \geq 0, \forall\ i, j$ (13)

$T > 0$ (14)

The first constraint, Eq. (4), is a set of equations for different i and j ensuring that the quantities flowing into a buyer's node minus those flowing out is equal to the total demand for a buyer in T. The second constraint, Eq. (5), is a set of equations for different i ensuring that the inventory level at a buyer does not exceed its warehouse capacity. The third constraint, Eq. (6), is a set of equations ensuring that the net cross-shipped quantities sent from buyer i to j and received by buyer i from j do not exceed its warehouse capacity. The fourth constraint, Eq. (7), is a set of equations ensuring the vendor's production quantity for buyer i does not exceed the vendor's warehouse. The fifth and sixth constraints, Eqs. (8) and (9), are sets of equations ensuring that the shipped quantities do not exceed shipping capacity. It is worth to be noted that the shipment capacity considered is the effective capacity when its utilization is taken into consideration. The seventh and eight constraints, Eqs. (10) and (11), are two sets of equations setting the numbers of shipments to non-negative integers. The ninth and tenth constraints, Eqs. (12) and (13), ensure the shipped quantities are non-negative numbers. The eleventh and last constraint, Eq. (14), guarantee that the cycle length is strictly positive.

The following theorem illustrates a structural property useful in finding the optimal solution to the problem described in Eqs. (3)-(14).

Theorem: The objective function TC, Eq. (3), and the constraints, Eq. (4)-(14) are jointly **convex** in T and q_i's for any given value of n_i and n_{ij} ,$\forall$ i, j, for any number of buyers y.

Proof: see appendix

CHAPTER III
METHODOLOGY: GENETIC ALGORITHM

In this paper, we introduce a genetic algorithm (GA) approach to solve the model presented earlier. A GA is a randomized search technique that is based on ideas from the natural selection process (Goldberg, 1989). Starting from an initial set of solutions, generations of new solutions are obtained through applying neighborhood search operators (crossover and mutation). These operators are applied to randomly selected solutions from the current set of solutions, where the selection probability is proportional to the solutions objective function value. GAs have been successfully implemented to a wide range of combinatorial optimization problems (Gen & Cheng, 1997).

A. Chromosome Representation

A chromosome is represented by a one-dimensional matrix. The matrix is formed of one row, which is divided into two parts. The first part has the number of genes equal to the number of buyers y. They represent the number of batches sent per cycle by each buyer in the order 1, 2, …, y. For example, for y= 3, the three genes will represent n_1, n_2, and n_3, respectively. The second part indicates if a shipment will be sent from buyer i to buyer j or not. The order of the variables is such that the first buyer is combined with other buyers in order, then the second buyer, up until the y^{th} buyer. So, the total number of variables in the second part is ($y^2 - y$). For example, for y= 3, the six genes will start from the fourth element of the row vector and will be n_{12}, n_{13}, n_{21}, n_{23}, n_{31} and n_{32} respectively.

$\mathbf{n_1}$	$\mathbf{n_2}$	$\mathbf{n_3}$	$\mathbf{n_{12}}$	$\mathbf{n_{13}}$	$\mathbf{n_{21}}$	$\mathbf{n_{23}}$	$\mathbf{n_{31}}$	$\mathbf{n_{32}}$

Figure 1. Illustration of the structure of a chromosome for the case of 3 buyers

B. Generation of Initial Population

The genetic algorithm function creates a random initial population unless specified otherwise. The population size and the initial population can be determined. A large population size will enable a wider search in the search space and thus reduce the probability of returning a local minimum instead of a global one. Nevertheless, the time for computation will increase in this case. Subpopulations can be also created with each having a specified size. The initial population will take the values of the specified values up to the population size. If the specified initial population has a size smaller than the population size, then the rest of the values will be specified randomly using a Creation function.

For this paper, the solution of the problem solved as a minimum spanning tree (MST) problem will resemble the optimal flow of products in several case studies as will be discussed later. Thus, it is a good approach to use the ordering costs as the distances between nodes (vendor and buyers) in order to find the MST for the problem. Thus, include the MST as follows:

1. Draw the tree of the problem showing the distance between the vendor and buyers and between the buyers themselves.

2. Determine the MST of the problem.
3. Include the MST in the initial population by using the same flow of products while replacing different number of shipments.

Another good initial solution to be included is to include the solution of the problem using the approach of Zavanella and Zanoni (2009) by considering that no cross-shipments between the buyers will occur.

C. Fitness Function

The fitness function of the genetic algorithm is the total cost TC of the model that was defined previously. At this point, the decision variables which are the genes of the chromosome as shown above should take specific values. The number of batches n_i sent to buyer i and the number of batches cross-shipped between the buyers n_{ij} must be positive integers. Once a chromosome is selected, the fitness function will be convex in T and q_i's for any given values of n_i's and n_{ij}'s. To find the value of the fitness corresponding to this chromosome, the values of T and q_i's can be obtained by a simple line search. The value of the objective function will be returned to the GA and the process will continue as will be discussed.

D. Mutation

In the sake of searching for a close solution to the current solution, mutation is needed. A local better solution can be found by slightly altering the present solution. As a result, it can be realized that no point in the search space will have a zero probability of being tested (Srinivas & Rao, 2010). The genes of the new generation are subjected to a low probability of being changed or mutated. The resulting chromosomes form the new population and will be evaluated to check their feasibility and optimality. The algorithm keeps going from one generation to the other until the stopping criteria are met.

E. Solution Procedure

Following are the steps for the evaluation of a chromosome:

Step 1. Start with chromosomes from an initial population in the evaluation of the fitness function.

Step 2. For each chromosome, replace its values in the objective function. The latter is now a function of T and $q_i, \forall\ i$.

Step 3. For each chromosome, solve the optimization problem having the function of Step 2 as the objective function and subject to constraints (4) to (14) to get the optimal values of T and $q_i, \forall\ i$.

Step 4. Return the optimal value of the objective function from Step 3 to the fitness function. Each chromosome from Step 1 corresponds now to a strategy with the total cost equal to the value of the fitness function from Step 3 relative to that chromosome.

Step 5. Select chromosomes from the previous population to apply mutation on them.

Step 6. Repeat Steps 2 to 5 until a stopping criterion is reached (the change in the optimal fitness value is negligible). When this criterion is reached, stop the process and return the optimal fitness value and optimal chromosome.

CHAPTER IV
SPECIAL CASE: CASE OF LARGE STORAGE CAPACITY

A 'special case' of the model developed is when the storage capacities at any location and shipping capacities between locations are very large. Constraints (5)-(9) become nonbinding since the right-hand-side values tend, theoretically, to infinity. As a result, the model proposed in this section can never result in an optimal solution and produce a higher unit-time total cost than that of Zavanella and Zanoni (2009). The rationale behind this is simple. Both models are similar in every aspect except that now the model proposed in this paper gives the possibility of a cross-shipment between the buyers. Considering the model proposed in this section where the optimal strategy is that no cross-shipments between buyers occur, both papers produce the same results. Then, n_{ij} is zero for all *i* and *j*, and the objective function, Eq. (3), reduces to:

$$TC = \frac{1}{T}\left(A + \sum_{i=1}^{y} n_i A_i\right) + \frac{h_1}{2} T \sum_{i=1}^{y} \frac{d_i^2}{P n_i} + \frac{h_2 T}{2} \sum_{i=1}^{y} d_i (1 - \frac{d_i}{P} + \frac{d_i}{n_i P}) \qquad (15)$$

Eq. (15) above is the total cost as given by Zavanella and Zanoni (2009). Thus, the unit-time total cost from Zavanella and Zanoni (2009) is an upper bound to that of the model of this paper.

Twenty numerical examples were generated and solved to test this case. The results showed that the TC values were lower than (16 of 20) or equal to (4/20) the TC of the model of Zavanella and Zanoni (2009). Accounting for cross-shipments between the buyers broadens the feasible solution space allowing to achieve lower TC values. It was also observed that out of 20, the solutions of 16 examples were the same as those produced by using a Minimum Spanning Tree (MST) when the ordering costs in our model represent the distances between nodes in the MST. However, the number of shipments required and their sizes, as well as the cycle length are not provided by MST, and hence, the latter only provides the shipment map with no further details. The numerical examples presented in the next section will provide more insights and examples in this case.

CHAPTER V
NUMERICAL EXAMPLE

A. Large Capacity for the Buyers and Vendor

Consider the case of a vendor and three buyers. Assume the warehouses' capacities for the vendor and buyers are extremely large. The shipping capacity is also unlimited. These assumptions are in line with Zavanella and Zanoni (2009). The values of the input parameters for the vendor and the buyers are:

P = 5,000 unit/year	A = 100 $
d_1 = 1,500 unit/year	A_1 = 60 $
d_2 = 1,300 unit/year	A_2 = 60 $
d_3 = 1,000 unit/year	A_3 = 10 $
h_1 = 5 $/unit/year	$A_{12} = A_{21}$ = 55 $
h_2 = 4 $/unit/year	$A_{13} = A_{31}$ = 70 $
	$A_{23} = A_{32}$ = 40 $

The initial population fed to the genetic algorithm contains the MST solution of the problem, which is to send a batch to buyer 3 and two cross-shipments from buyer 3 to buyer 2 and from buyer 2 to buyer 1, respectively. It also contains another solution consisting of sending batches to all buyers with no cross-shipments. The MST for this problem is the optimal solution where n_3 = 5 as shown in Table 1. The optimal solution corresponding to the n_i, n_j, and n_{ij} values is given in Table 1. The solution is that the vendor produces 675.25 units and ships to buyer 3 in 5 equal shipments of 134.45 each, 495.35 units from the warehouse of buyer 3 to that of 2, and ship 265.37 units from the warehouse of buyer 2 to that of 1 at a total cost of 2,769.7 $/year. The schedule of shipments and the behavior of inventory for the vendor and the buyers are shown in Figure 2. The model of Zavanella and Zanoni (2009) produced the following solution: T = 0.151, n_1= 1, n_2= 1, n_3 = 2, q_1= 236.92, q_2= 205.33, and q_3= 78.97 at a total cost of TC = 3,038.95 $/year, which is about 10% ((3,038.95 – 2,769.70)/ 2,769.70) higher than total cost of the model in this paper.

Table 1. Optimal strategy for case A

Optimal chromosome								
n_1	n_2	n_3	n_{12}	n_{13}	n_{21}	n_{23}	n_{31}	n_{32}
0	0	5	0	0	1	0	0	1
Optimal shipment sizes								
q_1	q_2	q_3	q_{12}	q_{13}	q_{21}	q_{23}	q_{31}	q_{32}
0	0	134.45	0	0	265.37	0	0	495.35
T = 0.1769 year				TC = 2,769.7 $/year				

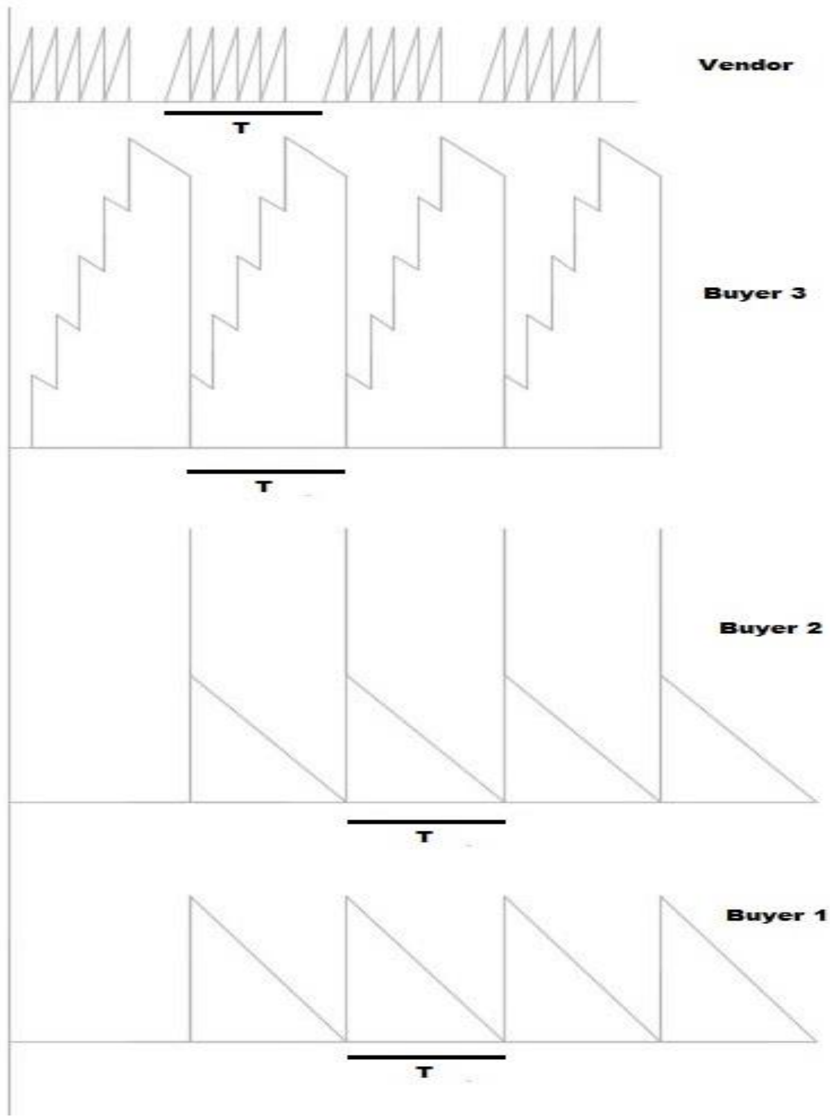

Figure 2. Inventory level at vendor and buyers with time for the case of large warehouse and no cross-shipment capacity

Sensitivity analysis was performed on the case presented above. Figure 3 shows the results obtained from performing a one-way sensitivity analysis on A_i's and A_{ij}'s. The results show that the total cost is most sensitive to the order cost of buyer 3. Thus, more effort must be put on reducing the order cost of buyer 3 than those of the other buyers in the case where order costs are high for the three buyers. In all cases, the curve reaches a plateau and becomes constant indicating that the optimal strategy at that point is independent of the considered ordering cost. For example, when A_1 is equal to

43$ with everything held constant, the optimal solution occurs when: T = 0.154, n_1= 1, n_2= 0, n_3 = 3, q_1= 230.8, q_2= 0, q_3= 118, and q_{32}= 200.1 at a total cost of TC = 2,768.1 $/year. Increasing A_1 to 44$ gives the following solution: T = 0.177, n_1= 0, n_2= 0, n_3 = 5, q_1= 0, q_2=0, q_3= 134.5, q_{21}= 265.4, and q_{32}= 495.4 at a total cost of TC = 2,769.7 $/year. The latter case is seen to have no shipment between the vendor and buyer 1, and thus the optimal strategy remains the same if A_1 is increased further.

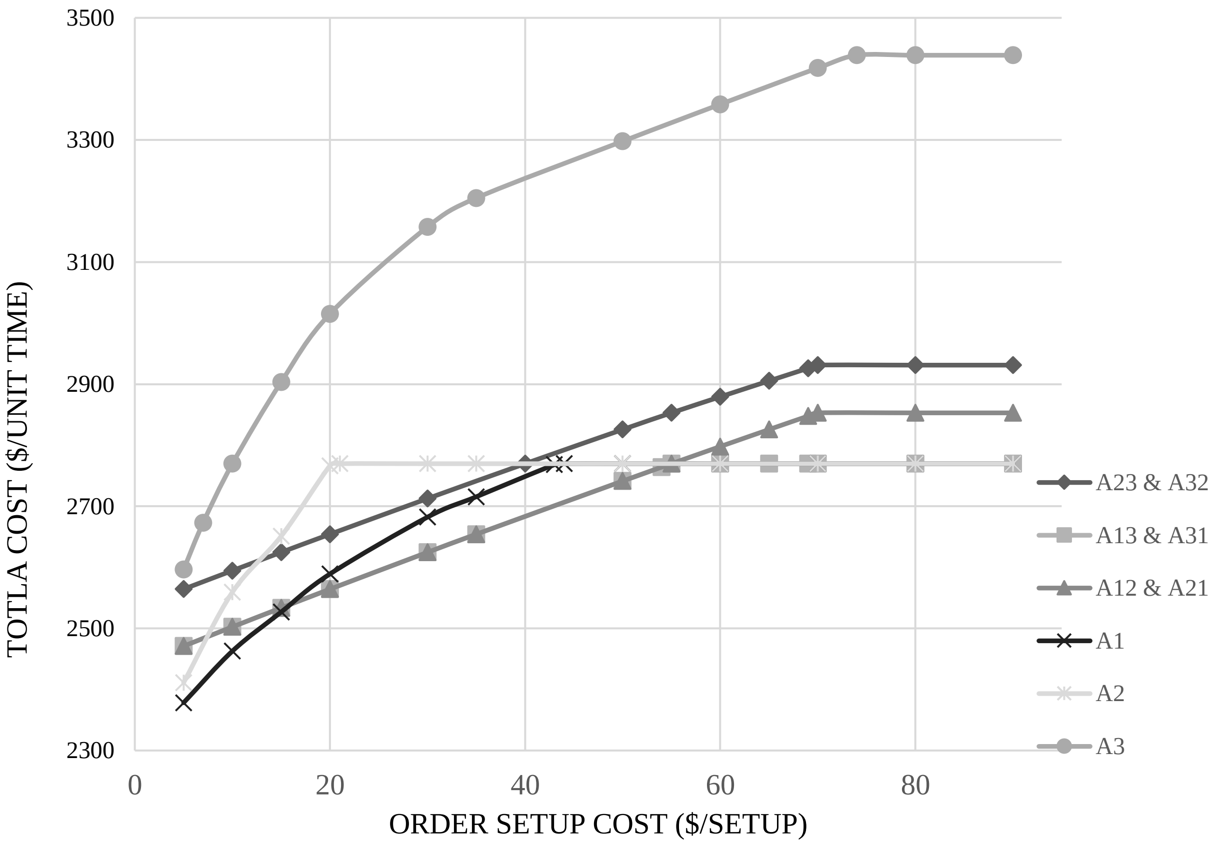

Figure 3. One-way sensitivity analysis for order costs

A closer look at the one-way sensitivity analysis performed on the order cost of buyer 3 leads to the results shown in Table 2. It is observed that the optimal strategy includes cross-shipments for A_3 equal up to 34$. The policy is the same as in Zavanella and Zanoni (2009) for 34 < $A_3 \leq$ 73$. Beyond this cost, the optimal strategy will no more have a direct shipment between the vendor and buyer 3. It is also observed that for A_3 = 30$ or more, the optimal strategy is the same as using MST when the ordering costs in our model represent the distances between nodes in the MST. For A_3 =A_1 and A_2 = 60$, the optimal strategy turned out to be the same as that of Zavanella and Zanoni (2009).

Table 2. One-way sensitivity analysis on buyer 3 order cost

A_3	5	10	20	30	35	60	70	74	90
Proposed Model	2596.6	2769.7	3015.2	3157.7	3204.9	3358.3	3417.8	3438.7	3438.7
Zavanella and Zanoni	2975.0	3038.9	3109.2	3173.3	3204.9	3358.3	3417.8	3441.3	3533.7
n_1	0	0	0	1	1	1	1	1	1
n_2	0	0	0	0	1	1	1	1	1
n_3	8	5	4	2	1	1	1	0	0
n_{12}	0	0	0	0	0	0	0	0	0
n_{21}	1	1	1	0	0	0	0	0	0
n_{13}	0	0	0	0	0	0	0	0	0
n_{31}	0	0	0	0	0	0	0	0	0
n_{23}	0	0	0	0	0	0	0	1	1
n_{32}	1	1	1	1	0	0	0	0	0
T	0.181	0.177	0.182	0.165	0.159	0.167	0.170	0.151	0.151
	MST	MST	MST	Not MST	Not MST	Not MST	Not MST	Not MST	Not MST

To have a further look at what happens when A_1 = A_2 = A_3, several cases were considered as shown in Table 3. It can be realized that when the order cost for the buyers is low, the vendor sends its shipments directly to the three buyers without cross-shipments, and the strategy is that of MST. The same strategy, which is no longer MST, continues. This result shows that having low order costs is not what drives having direct shipments from the vendor to the buyers. On the contrary, the optimal strategy is dependent on the ordering cost and the holding cost that is another major player in determining the optimal strategy. When A_1 = A_2 = A_3 ≥ \$80, it becomes more efficient to send a cross-shipment from buyer 3 to buyer 2 with a direct shipment from the vendor to buyers 1 and 3. An alternative to that is to switch buyers 2 and 3 giving the same unit-time total cost.

Columns 6-10 in Table 3 show that the ordering cost does not drive the optimal solution. On the contrary, when the order cost increased to 60 and then to 120, the optimal solution was not MST anymore, and hence the holding cost was affecting greatly the optimal solution. So, it was better to send a batch from the vendor to buyer 1 rather than sending a larger shipment to buyer 3 to be cross-shipped to buyer 2 and then from buyer 2 to buyer 1. In fact, increasing the shipment means a higher holding cost at the vendor is realized with a larger impact than the order cost of buyer 1.

Table 3. Optimal strategies for cases with equal order costs for buyers

A_1 = A_2 = A_3	10	35	40	60	75	80	100	120
n_1	2	1	1	1	1	1	1	1
n_2	2	1	1	1	1	0	0	0
n_3	1	1	1	1	1	1	1	1
n_{12}	0	0	0	0	0	0	0	0
n_{21}	0	0	0	0	0	0	0	0
n_{13}	0	0	0	0	0	0	0	0
n_{31}	0	0	0	0	0	0	0	0
n_{23}	0	0	0	0	0	0	0	0
n_{32}	0	0	0	0	0	1	1	1
T	0.134	0.143	0.148	0.167	0.180	0.162	0.173	0.218
TC	2231.2	2873.6	2976.8	3358.3	3618.1	3693.8	3932.3	4955.7
	MST	MST	MST	Not MST	Not MST	Not MST	Not MST	Not MST

Therefore, when capacity is not a concern, the results show the proposed model produces either similar results to those of Zavanella and Zanoni (2009) or a better.

B. Limited Capacity

1. Limited Warehouse Capacity

Consider now the same example as in section 3.1 but with limited warehouses' storage capacities (in units) for the vendor and the three buyers as C_v = 550, C_1 =200, C_2 = 250, and C_3 = 500, respectively. The optimal solution, Table 4, is the vendor produces 506.68 units ($n_3 q_3$) and ships to buyer 3 in 3 equal shipments of 126.67 each, 373.33 units from the warehouse of buyer 3 to that of 2, and ship 200 units from the warehouse of buyer 2 to that of 1 at a total cost of 2,864.5 $/year.

Table 4. Optimal chromosome for case B.1

Optimal chromosome								
n_1	n_2	n_3	n_{12}	n_{13}	n_{21}	n_{23}	n_{31}	n_{32}
0	0	4	0	0	1	0	0	1
Optimal shipment sizes								
q_1	q_2	q_3	q_{12}	q_{13}	q_{21}	q_{23}	q_{31}	q_{32}
0	0	126.67	0	0	200	0	0	373.33
T = 0.1333 year				TC = 2,864.5 $/year				

The flow of the items from vendor to buyers is still like that of the MST. The graphs for the vendor and buyers inventory level still have the same shape as the previous case (Figure 2) with a modification in values. The limited capacity of warehouses resulted in a narrower feasible region for the optimal solution leading to an increase in the optimal objective function value. The maximum inventory levels reached by the vendor and the buyers when there are no capacity concerns represents a lower capacity bound. This constraint, limit, is binding.

2. Limited Warehouse and Shipment Capacity

Now consider the same example in section B.1 by adding shipment capacities between locations as:

$s_1 = 80$	items/shipment	$s_{12} = s_{13} = 60$	items/shipment
$s_2 = 80$	items/shipment	$s_{21} = s_{23} = 50$	items/shipment
$s_3 = 80$	items/shipment	$s_{31} = s_{32} = 50$	items/shipment

The optimal solution is to ship directly from the vendor to the buyers with one cross-shipments from buyer 3 to 2 at a total cost of TC = 4,105.4 $/year, with the results shown in Table 5. The flow of the products from vendor to buyers is no longer the as in the MST problem solution.

Table 5. Optimal chromosome for case B.2

Optimal chromosome								
n_1	n_2	n_3	n_{12}	n_{13}	n_{21}	n_{23}	n_{31}	n_{32}
3	2	3	0	0	0	0	0	1
Optimal shipment sizes								
q_1	q_2	q_3	q_{12}	q_{13}	q_{21}	q_{23}	q_{31}	q_{32}
80	79	70	0	0	0	0	0	50
T = 0.16 year				TC = 4,105.4 $/year				

Figure 3 shows the change in inventory levels with time for the vendor and three buyers. Imposing a limit on the shipment capacity between locations is binding in this example. Buyers 1, 2 and 3 receive close to full capacity shipments. On the other hand, buyer 3 sends a maximum size shipment to buyer 2 to satisfy demand. Buyer 3 could consider investing to increase its shipping capacity to decrease the total cost. However, in this case, buyer 3 weighs the reduction in cost against the invested amount. If some investment on the shipment capabilities of buyer 3 is done, then increasing the items/shipment to 127 would enable a decrease in the total cost per unit time to 2,864.5 $/year. This is equivalent to a reduction of 30.23%. If this investment is offset by this reduction, then it is worth to be done.

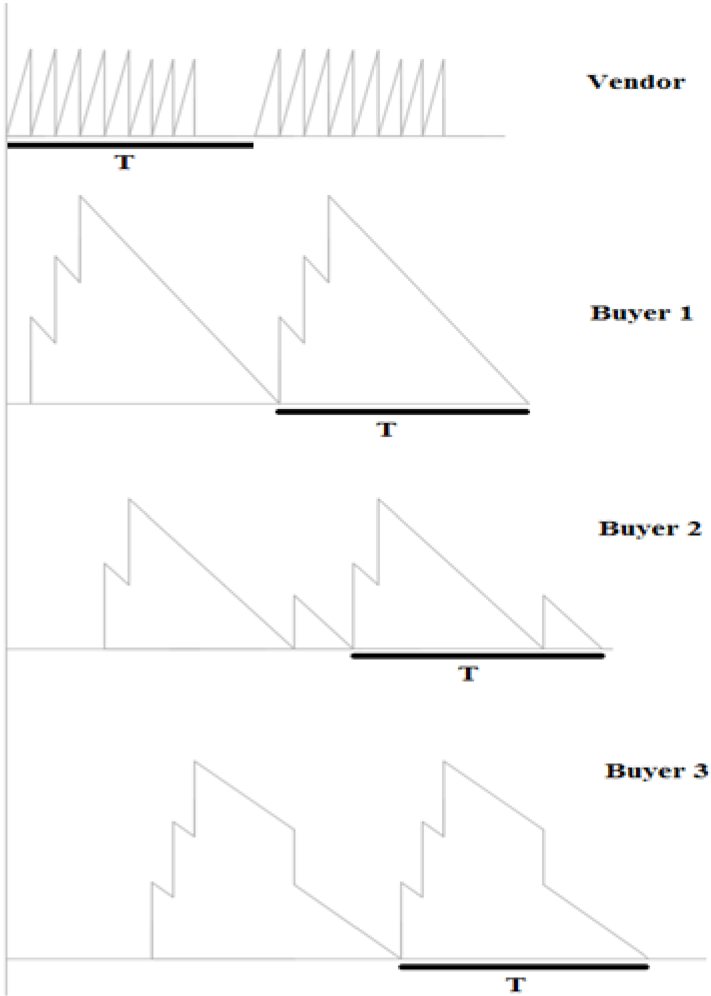

Figure 4. Inventory level at vendor and buyers with time for the case of limited warehouse and cross-shipment capacity

After obtaining the optimal strategy that would minimize the total cost per year, some actions can be taken by managers that allow a further reduction in that cost. For instance, it can be observed that the vendor shipment capacity constraint is binding since the optimal strategy shows that the shipment quantity from the vendor to buyer 1 is 80 items, which is also the capacity of the truck. Investing to increase storage capacities results in a lager feasible solution space and more solution points allowing for further reductions in cost. The graph of Figure 4 shows this reduction as a function of the capacity of the shipment from the vendor to buyer i. The vendor is assumed to use the same truck type to send independent shipments to buyers. Thus, increasing the shipment capacity up to 200 items/shipment would reduce the total cost from 4,105.4 $/year to 3067.667 $/year, a reduction of 25.28%. The amount saved should exceed the amount invested. If not, then no investment is recommended. The total cost will not decrease further for an increase in capacity exceeding 200 items/shipment. Investing to increase the capacity beyond 200 will be a waste of money.

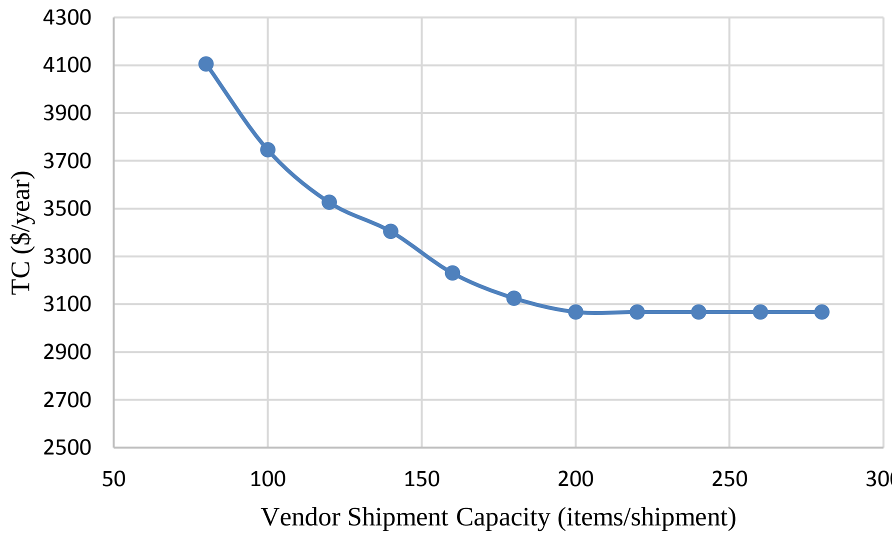

Figure 5. Total cost per year versus vendor shipment capacity

In general, it is clear that the model provides benefits to both the vendor and the buyers by decreasing the total cost per unit time. The model suggests that cross-shipment could be advantageous to the vendor and the buyers as it reduces, including holding and storage, costs and frees space.

3. Limited Shipment Capacity

By referring to the example of section B.2, the shipment capacities were governing the solution. Thus, by considering that the warehouse capacities were very high, the same solution would have been returned. This unlimited capacity is a waste of space and money unless was used for another objective.

CHAPTER VI
CONCLUSIONS

This paper developed a model for a single-vendor multi-buyer system under the consignment case with a reduction in holding costs while descending the chain. The model extended the results of Zavanella and Zanoni (2009) and offered the possibility of cross-shipments among buyers. It also took into consideration the capacity of the vendor and buyers' warehouses and that of the shipments. The results showed that the introduction of the transshipment possibility between the buyers allows the reduction of the joint optimum total cost per unit time. For the case of unlimited capacity, the developed model performed better than the model of Zavanella and Zanoni (2009). The total cost was either the same or lower. The first represents the upper bound on the total cost. For the case of limited capacity, the total cost was higher. The same applies to the capacity of shipments, which influence the solution especially when that capacity is small compared to that of demand and warehouse. Investing to increase warehouse and shipment capacities reduced the total cost, given that savings are more than the invested amount.

The model presented in this paper assumed that the holding costs for all buyers are the same not reflecting reality where the holding costs are different. However, this assumption could be extended into groups of neighborhoods having same holding costs each with no cross-shipments between neighborhoods. It also did not take into consideration the variable transportation cost as should be, but it only considered the fixed transportation cost as part of the ordering cost.

Future work could study batches of different shipments, truck capacity, and environmental issues (e.g., Bazan, Jaber, and Zanoni (2015)). Having different holding costs at different buyer locations is also interesting to consider. Investigating a more complex supply chain structure than the one considered in this paper with tier-1 and/or tier-2 suppliers is an interesting and challenging problem (e.g., Jaber and Goyal (2009); Zahran, Jaber, and Zanoni (2016); Zahran and Jaber (2017)). Accounting for product returns with cross-shipment is also a problem to consider (e.g., Batarfi, Jaber, and Aljazzar (2017)).

REFERENCES

Banerjee, A., & Banerjee, S. (1994). A coordinated order-up-to inventory control policy for a single supplier and multiple buyers using electronic data interchange. *International Journal of Production Economics, 35*(1-3), 85-91.

Batarfi, R., Jaber, M. Y., & Aljazzar, S. M. (2017). A profit maximization for a reverse logistics dual-channel supply chain with a return policy. *Computers & Industrial Engineering, 106*, 58-82. doi:https://doi.org/10.1016/j.cie.2017.01.024

Bazan, E., Jaber, M. Y., & Zanoni, S. J. A. M. M. (2015). Supply chain models with greenhouse gases emissions, energy usage and different coordination decisions. *39*(17), 5131-5151.

Boyacı, T., & Gallego, G. (2002). Coordinating pricing and inventory replenishment policies for one wholesaler and one or more geographically dispersed retailers. *International Journal of Production Economics, 77*(2), 95-111. doi:https://doi.org/10.1016/S0925-5273(01)00229-8

Dehghani, M., & Abbasi, B. (2018). An age-based lateral-transshipment policy for perishable items. *International Journal of Production Economics, 198*, 93-103.

Diabat, A. (2014). Hybrid algorithm for a vendor managed inventory system in a two-echelon supply chain. *European Journal of Operational Research, 238*(1), 114-121.

Diks, E. B., & De Kok, A. (1996). Controlling a divergent 2-echelon network with transshipments using the consistent appropriate share rationing policy. *International Journal of Production Economics, 45*(1-3), 369-379.

Gen, M., & Cheng, R. (1997). *Genetic algorithms and engineering design*. New York: Wiley.

Glock, C. H. (2012). The joint economic lot size problem: A review. *International Journal of Production Economics, 135*(2), 671-686.

Goldberg, D. E. (1989). *Genetic algorithms in search, optimization, and machine learning*. Reading, Mass.: Addison-Wesley Pub. Co.

Govindan, K. (2013). Vendor-managed inventory: a review based on dimensions. *International Journal of Production Research, 51*(13), 3808-3835.

Gümüş, M., Jewkes, E. M., & Bookbinder, J. H. (2008). Impact of consignment inventory and vendor-managed inventory for a two-party supply chain. *International Journal of Production Economics, 113*(2), 502-517.

Hariga, M., Gumus, M., Ben-Daya, M., & Hassini, E. (2013). Scheduling and lot sizing models for the single-vendor multi-buyer problem under consignment stock partnership. *Journal of the operational research society, 64*(7), 995-1009.

Hariga, M., Hassini, E., & Ben-Daya, M. (2014). A note on generalized single-vendor multi-buyer integrated inventory supply chain models with better synchronization. *International Journal of Production Economics, 154*, 313-316.

Herer, Y. T., Tzur, M., & Yücesan, E. (2006). The multilocation transshipment problem. *IIE transactions, 38*(3), 185-200.

Hill, R., & Omar, M. (2006). Another look at the single-vendor single-buyer integrated production-inventory problem. *International Journal of Production Research, 44*(4), 791-800.

Hill, R. M. (1999). The optimal production and shipment policy for the single-vendor singlebuyer integrated production-inventory problem. *International Journal of Production Research, 37*(11), 2463-2475.

Hoque, M. (2008). Synchronization in the single-manufacturer multi-buyer integrated inventory supply chain. *European Journal of Operational Research, 188*(3), 811-825.
Hoque, M. (2011). Generalized single-vendor multi-buyer integrated inventory supply chain models with a better synchronization. *International Journal of Production Economics, 131*(2), 463-472.
Jaber, M., & Goyal, S. J. I. J. o. P. R. (2009). A basic model for co-ordinating a four-level supply chain of a product with a vendor, multiple buyers and tier-1 and tier-2 suppliers. *47*(13), 3691-3704.
Jemai, Z., Rekik, Y., & Kalaï, R. (2013). Inventory routing problems in a context of vendor-managed inventory system with consignment stock and transshipment. *Production Planning & Control, 24*(8-9), 671-683.
Ji, X., Sun, J., & Wang, Z. (2017). Turn bad into good: Using transshipment-before-buyback for disruptions of stochastic demand. *International Journal of Production Economics, 185*, 150-161.
Joglekar, P. N. (1988). Note—Comments on "A quantity discount pricing model to increase vendor profits". *Management Science, 34*(11), 1391-1398.
Kim, T., Hong, Y., & Chang, S. Y. (2006). Joint economic procurement—production–delivery policy for multiple items in a single-manufacturer, multiple-retailer system. *International Journal of Production Economics, 103*(1), 199-208.
Krishnan, K., & Rao, V. (1965). Inventory control in N warehouses. In (Vol. 16, pp. 212-&).
Lal, R., & Staelin, R. (1984). An approach for developing an optimal discount pricing policy. *Management Science, 30*(12), 1524-1539.
Lee, C., & Park, K. S. (2016). Inventory and transshipment decisions in the rationing game under capacity uncertainty. *Omega, 65*, 82-97.
Lu, L. (1995). A one-vendor multi-buyer integrated inventory model. *European Journal of Operational Research, 81*(2), 312-323.
Rad, R. H., Razmi, J., Sangari, M. S., & Ebrahimi, Z. F. (2014). Optimizing an integrated vendor-managed inventory system for a single-vendor two-buyer supply chain with determining weighting factor for vendor′ s ordering cost. *International Journal of Production Economics, 153*, 295-308.
Robinson, L. W. (1990). Optimal and approximate policies in multiperiod, multilocation inventory models with transshipments. *Operations research, 38*(2), 278-295.
Rudi, N., Kapur, S., & Pyke, D. F. (2001). A two-location inventory model with transshipment and local decision making. *Management Science, 47*(12), 1668-1680.
Salameh, M., & Jaber, M. (1997). Reserve stock and transhipment in hierarchical inventory systems. *Production Planning & Control, 8*(5), 469-474.
Siajadi, H., Ibrahim, R. N., & Lochert, P. B. (2006). Joint economic lot size in distribution system with multiple shipment policy. *International Journal of Production Economics, 102*(2), 302-316.
Srinivas, C., & Rao, C. (2010). Optimization of supply chains for single-vendor–multibuyer consignment stock policy with genetic algorithm. *The International Journal of Advanced Manufacturing Technology, 48*(1-4), 407-420.
Villa, S., & Castañeda, J. A. (2018). Transshipments in supply chains: A behavioral investigation. *European Journal of Operational Research, 269*(2), 715-729.
Viswanathan, S., & Piplani, R. (2001). Coordinating supply chain inventories through common replenishment epochs. *European Journal of Operational Research, 129*(2), 277-286.
Woo, Y. Y., Hsu, S.-L., & Wu, S. (2001). An integrated inventory model for a single vendor and multiple buyers with ordering cost reduction. *International Journal of Production Economics, 73*(3), 203-215.

Zahran, S. K., Jaber, M. Y., & Zanoni, S. J. C. (2016). The consignment stock case for a vendor and a buyer with delay-in-payments. *98*, 333-349.
Zahran, S. K., & Jaber, M. Y. J. A. M. M. (2017). Investigation of a consignment stock and a traditional inventory policy in a three-level supply chain system with multiple-suppliers and multiple-buyers. *44*, 390-408.
Zavanella, L., & Zanoni, S. (2009). A one-vendor multi-buyer integrated production-inventory model: The 'Consignment Stock'case. *International Journal of Production Economics, 118*(1), 225-232.

APPENDIX I
THEOREM PROOF

Theorem: The objective function TC, Eq. (3), and the constraints, Eq. (4)-(14) are jointly **convex** in T and q_i for any given value of n_i and n_{ij} ,$\forall\ i, j$, for any number of buyers y.

Proof: Consider the objective function TC that was introduced in the previous model. For any given n_i and n_{ij}, the objective function TC is a function of the cycle length T and the amount shipped from vendor to buyers, qi's. It is twice differentiable and continuous over **R**. Hence, its Hessian Matrix can be computed with a size (y×y) having the following form:

$$\begin{bmatrix} \frac{\partial^2 TC}{\partial T^2} & \frac{\partial^2 TC}{\partial T \partial q_1} & \cdots & \frac{\partial^2 TC}{\partial T \partial q_y} \\ \frac{\partial^2 TC}{\partial q_1 \partial T} & \frac{\partial^2 TC}{\partial q_1^2} & \cdots & \frac{\partial^2 TC}{\partial q_1 \partial q_y} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 TC}{\partial q_y \partial T} & \frac{\partial^2 TC}{\partial q_y \partial q_1} & \cdots & \frac{\partial^2 TC}{\partial q_y^2} \end{bmatrix}$$

Let $A_t = A + \sum_i n_i A_i + \sum_i \sum_j n_{ij} A_{ij}$

Then, the partial derivatives of TC with respect to T and q_i's are given by the following:

$$\frac{\boldsymbol{\partial TC}}{\boldsymbol{\partial T}} = -\frac{1}{T^2} A_t - \frac{1}{T^2}\frac{h_1}{2P} \sum_i n_i q_i^2$$

$$\frac{\boldsymbol{\partial TC}}{\boldsymbol{\partial q_1}} = \frac{h_1}{TP} n_1 q_1 + \frac{h_2}{2}\left(1 - \frac{d_1}{P}\right) + \frac{h_2}{2P} d_1$$

$\vdots$

$$\frac{\partial TC}{\partial q_y} = \frac{h_1}{TP} n_y q_y + \frac{h_2}{2}\left(1 - \frac{d_y}{P}\right) + \frac{h_2}{2P} d_y$$

Calculation of first column of Hessian Matrix is given by the following:

$$\frac{\partial^2 TC}{\partial T^2} = \frac{2}{T^3} A_t + \frac{1}{T^3}\frac{h_1}{P} \sum_i n_i q_i^2$$

$$\frac{\partial^2 TC}{\partial q_1 \partial T} = -\frac{h_1}{PT^2} n_1 q_1$$

$\vdots$

$$\frac{\partial^2 TC}{\partial q_y \partial T} = -\frac{h_1}{PT^2} n_y q_y$$

Calculation of second column of Hessian Matrix is given by the following:

$$\frac{\partial^2 TC}{\partial T \partial q_1} = -\frac{h_1}{PT^2} n_1 q_1$$

$$\frac{\partial^2 TC}{\partial q_1^2} = \frac{h_1}{TP} n_1$$

$$\frac{\partial^2 TC}{\partial q_2 \partial q_1} = 0$$

$\vdots$

$$\frac{\partial^2 TC}{\partial q_y \partial q_1} = 0$$

Calculation of last column of Hessian Matrix is given by the following:

$$\frac{\partial^2 TC}{\partial T \partial q_y} = -\frac{h_1}{PT^2} n_y q_y$$

$$\frac{\partial^2 \boldsymbol{TC}}{\partial \boldsymbol{q_1} \partial \boldsymbol{q_y}} = 0$$

$\vdots$

$$\frac{\partial^2 \boldsymbol{TC}}{\partial \boldsymbol{q}_{(y-1)} \partial \boldsymbol{q_y}} = 0$$

$$\frac{\partial^2 \boldsymbol{TC}}{\partial \boldsymbol{q_y^2}} = \frac{h_1}{TP} n_y$$

As a result, the Hessian Matrix H of the objective function TC will be as follows:

$$\begin{bmatrix} \frac{2}{T^3}A_t + \frac{1}{T^3}\frac{h_1}{P}\sum_i n_i q_i^2 & -\frac{h_1}{PT^2}n_1 q_1 & -\frac{h_1}{PT^2}n_2 q_2 & \cdots & \cdots & -\frac{h_1}{PT^2}n_y q_y \\ -\frac{h_1}{PT^2}n_1 q_1 & \frac{h_1}{TP}n_1 & 0 & 0 & \cdots & 0 \\ -\frac{h_1}{PT^2}n_2 q_2 & 0 & \frac{h_1}{TP}n_1 & 0 & \cdots & 0 \\ \vdots & 0 & 0 & \ddots & 0 & \vdots \\ \vdots & \vdots & \vdots & 0 & \ddots & 0 \\ -\frac{h_1}{PT^2}n_y q_y & 0 & 0 & \cdots & 0 & \frac{h_1}{TP}n_y \end{bmatrix}$$

The leading principal minor method is used to prove convexity of TC. Consider the following leading principal minors H_k of order k = 1 to y of the Hessian matrix:

1) $\mathrm{H}_1 = |\mathrm{H}(1,1)| = \boldsymbol{\frac{2}{T^3} A_t + \frac{1}{T^3}\frac{h_1}{P} \sum_i n_i q_i^2 > 0}$

2) $H_2 = \begin{vmatrix} \frac{2}{T^3}A_t + \frac{1}{T^3}\frac{h_1}{P}\sum_i n_i q_i^2 & -\frac{h_1}{PT^2}n_1 q_1 \\ -\frac{h_1}{PT^2}n_1 q_1 & \frac{h_1}{TP}n_1 \end{vmatrix}$

$$H_2 = \left(\frac{2}{T^3}A_t + \frac{1}{T^3}\frac{h_1}{P}\sum_i n_i q_i^2\right)\left(\frac{h_1}{TP}n_1\right) - (-\frac{h_1}{PT^2}n_1q_1)(-\frac{h_1}{PT^2}n_1q_1)$$

$$H_2 = \frac{2A_t n_1 h_1}{T^4 P} + \frac{n_1 h_1^2}{T^4P^2}\sum_i n_i q_i^2 - \frac{h_1^2 n_1^2 q_1^2}{T^4P^2}$$

where $\frac{n_1 h_1^2}{T^4P^2}\sum_i n_i q_i^2 - \frac{h_1^2 n_1^2 q_1^2}{T^4P^2} = \frac{h_1^2}{T^4P^2}\left(n_1 \sum_{\substack{i \\ i\neq 1}} (n_i q_i^2) + n_1^2 q_1^2 - n_1 q_1^2\right) =$

$\frac{h_1^2 n_1}{T^4P^2}\sum_{\substack{i \\ i\neq 1}} n_i q_i^2$

$$H_2 = \frac{2A_t n_1 h_1}{T^4 P} + \frac{h_1^2 n_1}{T^4P^2}\sum_{\substack{i \\ i\neq 1}} n_i q_i^2 > 0$$

$$\boldsymbol{H_2 = \frac{n_1 h_1}{TP} H_{\substack{1 \\ i\neq 1}} > 0}$$ where $H_{\substack{1 \\ i\neq 1}} = \frac{2}{T^3}A_t + \frac{1}{T^3}\frac{h_1}{P}\sum_{\substack{i \\ i\neq 1}} n_i q_i^2$

3) $$H_3 = \begin{vmatrix} \frac{2}{T^3}A_t + \frac{1}{T^3}\frac{h_1}{P}\sum_i n_i q_i^2 & -\frac{h_1}{PT^2}n_1 q_1 & -\frac{h_1}{PT^2}n_2 q_2 \\ -\frac{h_1}{PT^2}n_1 q_1 & \frac{h_1}{TP}n_1 & 0 \\ -\frac{h_1}{PT^2}n_2 q_2 & 0 & \frac{h_1}{TP}n_2 \end{vmatrix}$$

$$H_3 = \frac{2n_1 n_2 h_1^2}{P^2T^5}\left(A_t + \frac{h_1}{2P}\sum_{\substack{i \\ i\neq 1 \\ i\neq 2}} n_i q_i^2\right)$$

$$\boldsymbol{H_3 = \frac{n_2 h_1}{PT} H_{\substack{2 \\ i\neq 2}} > 0}$$ where $H_{\substack{2 \\ i\neq 2}} = \frac{n_1 h_1}{TP} H_{\substack{1 \\ i\neq 1 \\ i\neq 2}}$

$$H_{\substack{1 \\ i\neq 1 \\ i\neq 2}} = \frac{2}{T^3}A_t + \frac{1}{T^3}\frac{h_1}{P}\sum_{\substack{i \\ i\neq 1 \\ i\neq 2}} n_i q_i^2$$

4) $$H_4 = \begin{vmatrix} \frac{2}{T^3}A_t + \frac{1}{T^3}\frac{h_1}{P}\sum_i n_i q_i^2 & -\frac{h_1}{PT^2}n_1 q_1 & -\frac{h_1}{PT^2}n_2 q_2 & -\frac{h_1}{PT^2}n_3 q_3 \\ -\frac{h_1}{PT^2}n_1 q_1 & \frac{h_1}{TP}n_1 & 0 & 0 \\ -\frac{h_1}{PT^2}n_2 q_2 & 0 & \frac{h_1}{TP}n_2 & 0 \\ -\frac{h_1}{PT^2}n_3 q_3 & 0 & 0 & \frac{h_1}{TP}n_3 \end{vmatrix}$$

$$H_4 = \frac{2n_1n_2n_3h_1^3}{P^3T^6}\left(A_t + \frac{h_1}{2P}\sum_{\substack{i \\ i\neq 1 \\ i\neq 2 \\ i\neq 3}} n_i q_i^2 \right)$$

$$\boldsymbol{H_4 = \frac{n_3 h_1}{PT} H_{\substack{3 \\ i\neq 3}} > 0}$$

where

$$H_{\substack{3 \\ i\neq 3}} = \frac{n_2 h_1}{PT} H_{\substack{2 \\ i\neq 2 \\ i\neq 3}}$$

$$H_{\substack{2 \\ i\neq 2 \\ i\neq 3}} = \frac{n_1 h_1}{TP} H_{\substack{1 \\ i\neq 1 \\ i\neq 2 \\ i\neq 3}}$$

$$H_{\substack{1 \\ i\neq 1 \\ i\neq 2 \\ i\neq 3}} = \frac{2}{T^3}A_t + \frac{1}{T^3}\frac{h_1}{P}\sum_{\substack{i \\ i\neq 1 \\ i\neq 2 \\ i\neq 3}} n_i q_i^2$$

5) In general, Hessian Matrix H has the same structure as the number of buyers increases.

Let H_{y+1} be the determinant of the whole Hessian Matrix, then, following the above calculations, H_{y+1} will have the following form:

$$\boldsymbol{H_{y+1} = \frac{n_y h_1}{PT} H_{\substack{y \\ i\neq y}} > 0}$$

where $H_{\substack{k \\ i\neq m}} = H_k$ *with all terms with subscript m removed*

Therefore, since the **leading principal minors** are positive for any number of buyers y, then Hessian Matric is positive definite for any given value of n_i's and n_{ij}'s. Thus, the objective function TC is **convex** in T and q_i's for any given value of n_i's and n_{ij}'s.

Furthermore, the constraints are all linear and convex for any given value of n_i's and n_{ij}'s, and hence the problem becomes a convex in T and q_i's.