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On The Generalization of The Riemann Mapping Theorem in The Theory of Several Complex Variables

by

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An Abstract of the Thesis of

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One of the most important results in complex analysis is The Riemann Mapping Theorem which states that every non-empty simply connected domain in the complex plane $\mathbb{C}$ which is not the entire $\mathbb{C}$ is biholomorphically equivalent to the open unit disc. However, this theorem does not hold in higher dimensions. For instance, the open unit ball and the open polydisc are not biholomorphic in $\mathbb{C}^n$ for $n > 1$. Generalizations of the Riemann Mapping Theorem in the theory of several complex variables rely on additional characterizations of the complex structure of the domain. For instance, Stanton built his generalization on specific conditions on the Carathéodory and Kobayashi metrics defined on a complex manifold. Whereas Wong-Rosay theorem mainly relies on the group of automorphisms of a domain. In this work, our basic aim is to study Stanton and Wong-Rosay theorems and their proofs. We will also approach the proof of Wong-Rosay theorem using the scaling method of Pinckuk.

Contents

Chapter 1

Introduction

We know from the Riemann Mapping Theorem that any non-empty simply connected domain in the complex plane $\mathbb{C}$ which is not the entire $\mathbb{C}$ is biholomorphically equivalent to the open unit disc. In 1907, Poincaré proved that the open unit ball and the open polydisc in $\mathbb{C}^n$ are not biholomorphic to each other, leaving the door open to questions concerning the characterization of domains in $\mathbb{C}^n$ for $n > 1$. Naturally, the geometry of domains in $\mathbb{C}^n$ for $n > 1$ is much more complicated than in the complex plane and requires the introduction of new invariant objects. This essential question has been addressed extensively since then by many authors. A remarkable result due to B. Wong [11] gives a characterization of domains with curvature in terms of their group of automorphisms. Later on J.-P. Rosay [8] gave a local version of Wong's result with a complete different proof. Another important characterization, also due to B. Wong [12], relates the geometry of the domain with the behaviour of the curvature of two important invariant metrics, both generalizations of the Poincaré metric. C.M. Stanton [10] also obtained a result in that vein.

In this thesis we aim to deeply study these results and their proofs due to Rosay in [8] and to Stanton in [10]. Finally, we will also follow [1] and [4] to present a different and very geometric proof of Wong-Rosay theorem based on S. Pinchuk's scaling method .

In what follows, Chapter 2 will present a brief look at concepts and results that will be used later in the work. In this chapter, we introduce strictly pseudoconvex domains and describe them locally. We then define Kobayashi and Carathédory metrics/volumes and prove some of their basic properties and we give a quick tour on complex manifolds.

Chapter 3 will present the proof of Wong-Rosay theorem by J.-P. Rosay where two main components of the proof will be highlighted: the characterization of the unit ball by volumes and the localization of Kobayashi and Carathéodory volumes near a point of strict pseudoconvexity.

Chapter 4 will then introduce Pinchuk's scaling sequence that will be applied to present a distinct proof of Wong-Rosay theorem.

Lastly, Chapter 5 will present Stanton's characterization of the unit ball by its metrics.

Chapter 2

Preliminaries

2.1 Strict Pseudoconvexity and Local Change of Coordinates

2.1.1 Strictly Pseudoconvex Domains

Definition 2.1.1. *Let D be an open set in $\mathbb{C}^n$ and let u be a C^2 function on D. The* ***Levi Form*** *of u at $z \in D$ is the complex Hessian $L_{z,u}$ of u at z, i.e. the Hermitian form:*

$$L_{z,u}(\zeta) = \sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial z_i \partial \overline{z_j}}(z)\zeta_i\overline{\zeta_j}$$

Definition 2.1.2. *Let Ω be a domain in $\mathbb{C}^n$ with C^2 boundary and let ρ be a real-valued C^2 function defined on a neighborhood $U_{\partial\Omega}$ of the boundary of Ω such that $U_{\partial\Omega} \cap \Omega = \{z \in U_{\partial\Omega} : \rho(z) < 0\}$ and $\nabla\rho(z) \neq 0$ for every $z \in \partial\Omega$. We say* *that* Ω *is* ***strictly pseudoconvex*** *if and only if*

$$L_{z,\rho}(\omega) > 0$$

for every $z \in \partial\Omega$ *and* $\omega \in T_z(\partial\Omega)$*, where* $L_{z,\rho}(\omega)$ *is the Levi form of* ρ *at* z *and* $T_z(\partial\Omega) = \{\omega \in \mathbb{C}^n : \sum_{i=1}^{n} \frac{\partial\rho}{\partial z_i}(z)\omega_i = 0\}$ *is the complex tangent space at* z*.*

Note that ρ *is called a* **defining function** *of* Ω*.*

2.1.2 Local Change of Coordinates

Proposition 2.1.1. *Let* Ω *be a bounded domain of* $\mathbb{C}^n$ *with* C^2 *boundary. Let* $\zeta_0 \in \partial\Omega$ *be a point of strict pseudoconvexity. Then there exists a neighborhood of* ζ_0 *where - up to change of coordinates -* Ω *is defined by:*

$$\rho(\omega) = 2\Re\omega_1 + L_{\zeta_0,\rho}(\omega) + o(\|\omega\|^2)$$

with $\omega = (\omega_1, \cdots, \omega_n)$*.*

Proof. Since Ω is bounded there exists a positive constant K such that $L_{\zeta_0,\rho}(\zeta) \geq K\|\zeta\|^2$, for all $\zeta \in \mathbb{C}^n$. By rotation and translation of coordinates, we can assume that $\zeta_0 = 0$ and $n = (1, 0, \cdots, 0)$ is the unit outward normal to $\partial\Omega$ at ζ_0.

Consider the following second-order Taylor expansion of ρ about $\zeta_0 = 0$:

$$\begin{aligned}
\rho(\zeta) &= \rho(0) + \sum_{i=1}^{n} \frac{\partial \rho}{\partial z_j}(\zeta_0)\zeta_i + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 \rho}{\partial z_i \partial z_j}(\zeta_0)\zeta_i\zeta_j \\
&\quad + \sum_{i=1}^{n} \frac{\partial \rho}{\partial \overline{z}_j}(\zeta_0)\overline{\zeta}_i + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 \rho}{\partial \overline{z}_i \partial \overline{z}_j}(\zeta_0)\overline{\zeta}_i\overline{\zeta}_j \\
&\quad + \sum_{i,j=1}^{n} \frac{\partial^2 \rho}{\partial z_i \partial \overline{z}_j}(\zeta_0)\zeta_i\overline{\zeta}_j + o(\|\zeta\|^2) \\
&= 2\Re\left\{ \sum_{i=1}^{n} \frac{\partial \rho}{\partial z_j}(\zeta_0)\zeta_i + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 \rho}{\partial z_i \partial z_j}(\zeta_0)\zeta_i\zeta_j \right\} \\
&\quad + \sum_{i,j=1}^{n} \frac{\partial^2 \rho}{\partial z_i \partial \overline{z}_j}(\zeta_0)\zeta_i\overline{\zeta}_j + o(\|\zeta\|^2) \\
&= 2\Re\left\{ \zeta_1 + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 \rho}{\partial z_i \partial z_j}(\zeta_0)\zeta_i\zeta_j \right\} \\
&\quad + \sum_{i,j=1}^{n} \frac{\partial^2 \rho}{\partial z_i \partial \overline{z}_j}(\zeta_0)\zeta_i\overline{\zeta}_j + o(\|\zeta\|^2)
\end{aligned}$$

Where $\rho(0) = 0$ since $\zeta_0 = 0$ is a boundary point and $\sum_{i=1}^{n} \frac{\partial \rho}{\partial z_j}(\zeta_0) = (1, 0, \cdots, 0)$ by assumption.

We define next the map $\phi : \zeta = (\zeta_1, \cdots, \zeta_n) \mapsto \omega = (\omega_1, \cdots, \omega_n)$ as follows:

$$\omega_1 = \phi_1(\zeta) = \zeta_1 + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 \rho}{\partial z_i \partial z_j}(\zeta_0)\zeta_i\zeta_j$$

$$\omega_k = \phi_k(\zeta) = \zeta_k \quad \text{for } 2 \leq k \leq n$$

By the *Implicit Function Theorem*, we see that ϕ is a well defined invertible holomorphic mapping on some neighborhood of $\zeta_0 = 0$. Then after a local change of coordinates $\zeta = (\zeta_1, \cdots, \zeta_n) \mapsto \omega = (\omega_1, \cdots, \omega_n)$ the defining function becomes:

$$\rho(\omega) = 2\Re\omega_1 + L_{\zeta_0,\rho}(\omega) + o(\|\omega\|^2)$$

$\square$

2.2 Complex Manifolds

2.2.1 Complex Manifolds, Tangent Space and Tangent Bundle

Definition 2.2.1. *A topological space M is called an n-dimensional **complex manifold** if there exist an open cover $\{U_i\}_{i\in I}$ of M and a family $\{\phi_i\}_{i\in I}$ of homeomorphisms of U_i onto an open set of $\mathbb{C}^n$ such that if $U_i \cap U_j \neq \emptyset$, the mapping $\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$ is biholomorphic.*

Example 2.2.1. *Complex projective spaces $\mathbb{C}P^n$ and complex Lie groups such as $GL(n, \mathbb{C})$ are examples of complex manifolds.*

A complex valued function f defined on an open set $U \subset M$ is said to be holomorphic if for any $i \in I$ the function $f \circ \phi_i^{-1}$ is holomorphic on $\phi_i(U \cap U_i) \subset \mathbb{C}^n$. For every point $p \in U_i$, the mapping ϕ_i is expressed as $\phi_i(p) = (z_1(p), \cdots, z_n(p))$ in terms of the coordinates in $\mathbb{C}^n$ where each z_i is a holomorphic function on U_i. We call $(z_1. \cdots, z_n)$ a **holomorphic local coordinates system** on U_i.

Definition 2.2.2. *Let M and N be two complex manifolds and let $F : M \to N$ be a smooth map. The* ***pull-back*** *of a smooth function $f \in C^\infty(N)$ is given by*

$$F^*(f) = f \circ F \in C^\infty(M)$$

One can play a similar game for Hermitian forms. In particular the pull-back of the Levi form $i\partial\overline{\partial}$ is given by

$$F^*(i\partial\overline{\partial}f) = i\partial\overline{\partial}(f \circ F).$$

Definition 2.2.3. *Let M and N be two n-dimensional oriented manifolds and let $f : M \to N$ be a smooth map. If M is compact and N is connected, and if $y \in N$ is any regular value of f then the* **degree** *of f is defined as follows*

$$deg\, f = deg\,(f, y) = \sum_{x \in f^{-1}(y)} sign\; d_x f$$

where $sign\, df$ is equal to 1 if df preserves orientation and -1 if it reverses orientation.

If f is a diffeomorphism then the $deg\, f = 1$ if f is an orientation preserving map and $deg\, f = -1$ otherwise (see [6]).

We will introduce next the notion of a tangent space on a manifold.

Definition 2.2.4. *Let M be a complex manifold and $p \in M$. A tangent vector*

*v_p to M at p is a derivation on $C^\infty(M)$. The space of all such vectors is called the **tangent space** of M at p and it is denoted by T_pM. The disjoint union of all tangent spaces of M is called the **tangent bundle** of M and it is denoted by TM.*

Note that the tangent bundle is seen as $TM = \bigcup\limits_{p\in M}\{p\} \times T_pM = \{(p, v_p) : p \in M, v_p \in T_pM\}$. Thus one can naturally define the canonical projection $\pi : TM \to M$ where $\pi(p, v_p) = p$.

Definition 2.2.5. *A **vector field** $X : M \to TM$ on a complex manifold M is a section of the tangent bundle TM i.e. it is a right inverse of the projection map π. The space of all vector fields on M is denoted by $\Gamma(TM)$.*

2.2.2 Affine Connections, Geodesics and The Exponential Map

In this section we will generalize the notion of a directional derivative by introducing affine connections and covariant derivatives on a manifold. We will introduce the notion of geodesics (with initial point and initial velocity) as well which played a pivotal role in differential geometry.

Definition 2.2.6. *An **affine connection** on the tangent bundle TM of a manifold M is a map*

$$\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$$

$$(X,Y) \mapsto \nabla_X Y$$

such that for all $X, Y \in \Gamma(TM)$,

1. *∇ is $C^\infty(M)$-linear in the first variable and $\mathbb{C}$-linear in the second.*

2. *∇ satisfies Leibniz rule in the second variable i.e. if $f \in C^\infty(M)$ then,*

$$\nabla_X(fY) = (df)(X)Y + f\nabla_X Y$$

Let M be a complex manifold and $\gamma : [a,b] \to M$ be a smooth parametrized curve. Recall that a vector field along the curve γ is a map

$$V : [a,b] \to \bigsqcup_{t\in[a,b]} T_{\gamma(t)}M$$

$$t \mapsto v_{\gamma(t)}$$

where $\bigsqcup$ stands for the disjoint union. We denote the space of all such vector fields by $\Gamma(TM|_{\gamma(t)})$.

Definition 2.2.7. *Let M be a complex manifold with an affine connection ∇ and a smooth curve $\gamma : [a,b] \to M$. The* ***covariant derivative*** *DV/dt (associated to ∇) of the vector field V along the curve $\gamma(t)$ in M is given by the map*

$$\frac{D}{dt} : \Gamma(TM|_{\gamma(t)}) \to \Gamma(TM|_{\gamma(t)})$$

such that for all $V \in \Gamma(TM|_{\gamma(t)})$,

1. *($\mathbb{C}$-linearity)* DV/dt *is* $\mathbb{C}$*-linear in* V.

2. *(Leibniz Rule) for any* $f \in C^{\infty}[a,b]$

$$\frac{D(fV)}{dt} = \frac{df}{dt}V + f\frac{DV}{dt}$$

3. *(Compatibility with* ∇*) if* V *is induced from a vector field* V' *on* M*, in the sense that* $V(t) = V'_{\gamma(t)}$*, then*

$$\frac{DV}{dt}(t) = \nabla_{\gamma'(t)}V'$$

Definition 2.2.8. *Let* M *be a complex manifold with a connection* ∇*. A parametrized curve* $\gamma : I \subset \mathbb{R} \to M$ *is called a* ***geodesic*** *if the covariant derivative of its velocity vector field* $\gamma'(t)$ *is zero i.e.* $\frac{D\gamma'}{dt}(t) = 0$*. The geodesic is said to be maximal if* I *can not be extended to a larger interval.*

Remark 2.2.1. *Given any* $p \in M$ *and* $v_p \in T_pM$ *there exists a unique maximal geodesic denoted by* $\gamma_{v_p}(t)$ *such that* $\gamma_{v_p}(0) = p$ *and* $\gamma'_{v_p}(0) = v_p$*. For simplicity of the notation, we will omit* p *from* $\gamma_{v_p}(t)$ *and* v_p*.*

We are now in the position to make the following definition.

Definition 2.2.9. *Let* M *be a complex manifold with an affine connection. Let* $p \in M$ *and* $v \in T_pM$*, if* $\gamma_v(1)$ *is defined, the* ***exponential map*** *is set to be the* *following*

$$exp_p(v) = \gamma_v(1)$$

One can show that for each t in the domain of γ_v, $exp_p(tv) = \gamma_v(t)$.

Proposition 2.2.1. *The differential of the exponential map at the origin, $d_0(exp_p)$ is the identity map on T_pM.*

Proof. Let $v \in T_pM$. Note that the curve $\alpha_v(t) = tv \subset T_pM$ satisfies $\alpha_v(0) = 0$ and $\alpha_v'(0) = v$. We will use this curve to compute the differential as follows

$$\begin{aligned} d_0(exp_p)(v) &= d_0(exp_p)\left(\frac{d}{dt}\bigg|_{t=0} \alpha_v(t)\right) \\ &= \frac{d}{dt}\bigg|_{t=0} exp_p(\alpha_v(t)) \\ &= \frac{d}{dt}\bigg|_{t=0} exp_p(tv) \\ &= \frac{d}{dt}\bigg|_{t=0} \gamma_v(t) \\ &= \gamma_v'(0) \\ &= v \end{aligned}$$

where γ_v is the unique geodesic for v through p. □

It follows by the inverse function theorem that the exponential map is a local diffeomorphism at the origin of T_pM.

2.2.3 Hermitian Metrics on Complex Tangent Bundles

Definition 2.2.10. *Let M be a $\mathbb{C}^\infty$ complex manifold. A* ***Hermitian metric*** *on the tangent bundle of M assigns smoothly to each $p \in M$ a complex inner product $\langle\ ,\ \rangle_p$ on T_pM. A complex tangent bundle on which there is a Hermitian metric is called a* ***Hermitian bundle****.*

A connection on a Hermitian bundle is said to ***compatible with the Hermitian metric*** if for all $X, Y, Z \in \Gamma(TM)$,

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$$

Remark 2.2.2. *The notion of geodesics defined previously depends only on a connection and does not require the manifold to be equipped with a metric. However, on a Hermitian bundle we will consider the connection that is compatible with the Hermitian metric.*

Proposition 2.2.2. *Let $\gamma(t)$ be a geodesic on a Hermitian manifold M. Then $\|\gamma'(t)\|$ is constant, where $\| . \|= \sqrt{\langle\ .\ ,\ .\ \rangle}$ is the norm associated to the Hermitian metric (if indicated, this norm might also be denoted by $|\ .\ |$ in this work).*

Proof. Let $f(t) = \langle \gamma'(t), \gamma'(t)\rangle$. We want to show that $f(t)$ is constant. Note that

$$\frac{d}{dt}f(t) = \frac{d}{dt}\langle \gamma'(t), \gamma'(t)\rangle = \langle \frac{D}{dt}\gamma'(t), \gamma'(t)\rangle + \langle \gamma'(t), \frac{D}{dt}\gamma'(t)\rangle = 0$$

where the second equality holds by Remark 2.2.2 and the last one holds since

$\gamma(t)$ is a geodesic. Hence, we conclude that $f(t)$ is constant. $\square$

Proposition 2.2.3. *Let M be a Hermitian manifold and let $\gamma : [0,1] \to M$ be the geodesic with $\gamma(0) = p, \gamma(1) = q$ and $\gamma'(0) = v$. Then, the length of γ is*

$$L(\gamma) = \|v\|$$

Proof. By Proposition 2.2.2 we see that

$$L(\gamma) = \int_0^1 \|\gamma'(t)\| dt = \|\gamma'(t)\| = \|v\|$$

where the the last equality holds because $\gamma'(0) = v$.

$\square$

Remark 2.2.3. *Geometrically, $exp_p(v)$ is the point on the geodesic γ_v passing through p and tangent to v that is obtained by going out a distance equal to $\|v\|$ along γ_v starting from p.*

More generally, if $r > 0$ is such that $\gamma_v(t) = exp_p(tv)$ is defined on $[0, r]$ then the length of the geodesic arc from p to $exp_p(rv)$ is

$$L(\gamma_v|_{[0,r]}) = \int_0^r \|\gamma'(t)\| dt = r\|\gamma'(t)\| = r\|v\|$$

In particular, if v is a unit vector, then $L(\gamma_v|_{[0,r]}) = r$.

2.3 Invariant Metrics

Historically, the first invariant metric under biholomorphic mappings was defined by Poincaré on the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\} \subset \mathbb{C}$. In higher dimensions, there are many generalizations of the Poincaré metric; the most classical such metrics are due to Carathéodory in 1926, Bergman in 1950 and Kobayashi in 1967. These metrics turn out to be fundamental tools in the theory of several complex variables, providing important information on the geometry of domains and their boundaries.

2.3.1 Poincaré, Kobayashi and Carathéodory Metrics

For a vector v tangent to the open unit disc $\mathbb{D}$ at a point ζ, the **Poincaré metric** is defined as

$$K_{\mathbb{D}}(\zeta, v) = \frac{|v|}{1 - |\zeta|^2}$$

where $|.|$ is the Euclidean norm.

We will introduce now the Kobayashi and Carathéodory metrics and prove that they coincide with the Poincaré metric on $\mathbb{D}$.

Definition 2.3.1. *Let M be a complex manifold of dimension n and let $p \in M$. For every $v \in T_pM$ we define the* **Kobayashi and Carathéodory pseudomet-**

rics *of v at p respectively as follows*

$$K_M(p,v) = \inf\left\{\frac{1}{r} > 0 : f \in H(\mathbb{D}, M),\ f(0) = p,\ d_0 f(\partial/\partial x) = rv\right\}$$
$$C_M(p,v) = \sup\left\{\ |d_p g(v)| : g \in H(M, \mathbb{D}),\ g(p) = 0\right\}$$

where $H(\mathbb{D}, M)$ ($H(M, \mathbb{D})$ resp.) is the space of holomorphic mappings from $\mathbb{D}$ to M (from M to $\mathbb{D}$ resp.) and $d_p g$ is the differential of the function g at the point p.

Note that these pseudometrics may be degenerate; for instance due to Liouville theorem the Carathéodory pseudometric of $\mathbb{C}$ is identically equal to zero; the Kobayashi pseudometric of $\mathbb{C}$ is also identically equal to zero since the complex plane $\mathbb{C}$ contains discs of arbitrary large size. Naturally, in the case of the unit disc $\mathbb{C}$, on has:

Theorem 2.3.1. *The Kobayashi, Carathéodory and Poincaré metrics coincide on the open unit disc.*

Proof. Note that in this proof we see $\mathbb{D}$ as a domain in $\mathbb{C}$ and we will use the usual notion of derivatives in $\mathbb{C}$. We will start by proving that the Kobayashi metric is equal to the Poincaré metric on $\mathbb{D}$. Let $\zeta \in \mathbb{D}$ and $v \in T_\zeta\mathbb{D}$. If $v = 0$ then the result is trivial. Otherwise, for $v \neq 0$, we consider a candidate mapping f for the Kobayashi metric at ζ on $\mathbb{D}$ with $f'(0) = rv$ for some $r > 0$. Note that the Mobius transformation

$$\varphi_\zeta(z) = \frac{z-\zeta}{1-\bar{\zeta}z}$$

is an automorphism of D satisfying $\varphi_\zeta(\zeta) = 0$ and $\varphi'_\zeta(\zeta) = \dfrac{1}{1-|\zeta|^2}$.

Clearly, $\varphi_\zeta \circ f \in H(\mathbb{D}, \mathbb{D})$ and $\varphi_\zeta \circ f(0) = 0$. Thus, by Schwarz lemma we get

$$|(\varphi_\zeta \circ f)'(0)| \leq 1$$

This implies that

$$\frac{r\,|v|}{1-|\zeta|^2} \leq 1$$

Hence,

$$\frac{1}{r} \geq \frac{|v|}{1-|\zeta|^2}$$

Now, consider the function $f : \mathbb{D} \to \mathbb{D}$ defined by

$$f(z) = \frac{\frac{v}{|v|}z + \zeta}{1 + \frac{v}{|v|}\bar{\zeta}z}$$

We can see that f is a candidate mapping for the Kobayashi metric at ζ on $\mathbb{D}$ with $f'(0) = \dfrac{1-|\zeta|^2}{|v|}v$. Therefore,

$$K_{\mathbb{D}}(\zeta, v) = \frac{|v|}{1-|\zeta|^2}.$$

We will show next that the Carathéodory metric coincides with the Poincaré metric on $\mathbb{D}$. Let $\zeta \in \mathbb{D}$ and $v \in T_\zeta\mathbb{D}$. Let g be a candidate mapping for the Carathéodory metric at ζ on $\mathbb{D}$. By Schwarz-Pick lemma we have

$$|g'(\zeta)| \leq \frac{1}{1-|\zeta|^2}$$

Hence

$$|g'(\zeta)v| \leq \frac{|v|}{1-|\zeta|^2}$$

Since the Mobius transformation $\varphi_\zeta(z)$ considered previously is clearly a candidate mapping for the Carathéodory metric at ζ on $\mathbb{D}$ with $\left|\varphi'_\zeta(\zeta)v\right| = \dfrac{|v|}{1-|\zeta|^2}$ we conclude that

$$C_{\mathbb{D}}(\zeta, v) = \frac{|v|}{1-|\zeta|^2}.$$

□

Given two points p and q in a complex manifold M, the integrated **Kobayashi pseudodistance** d_M is defined as follows

$$d_M(p,q) = \inf\left\{ \int_0^1 K_M(\gamma(t), \gamma'(t))dt, \gamma : [0,1] \to M, \gamma(0) = p, \gamma(1) = q \right\}$$

Similarly, the integrated **Carathéodory pseudodistance** d_M is given by

$$d_M(p,q) = \inf\left\{ \int_0^1 C_M(\gamma(t), \gamma'(t))dt, \gamma : [0,1] \to M, \gamma(0) = p, \gamma(1) = q \right\}.$$

Definition 2.3.2. *We say that a manifold M is Kobayashi **hyperbolic** if the Kobayashi pseudodistance is a distance. We say that M is **complete hyperbolic** if it is complete for the Kobayashi distance d_M.*

For instance, the unit disc $\mathbb{D}$ endowed with the Poincaré metric is complete hyperbolic.

2.3.2 Properties

We will state and prove in this section some of the most important properties of the Kobayashi and Carathéodory metrics.

Proposition 2.3.1 (Distance Decreasing Property)**.** *Let M and N be complex manifolds, $p, q \in M$ and $v \in T_pM$. Suppose that $\varphi : M \to N$ is holomorphic, then*

$$K_N(\varphi(p), d_p\varphi(v)) \leq K_M(p, v) \text{ and } C_N(\varphi(p), d_p\varphi(v)) \leq C_M(p, v)$$

Consequently, $d_N(\varphi(p), \varphi(q)) \leq d_M(p, q)$ where d_M, d_N are the Kobayashi or Carathéodory pseudodistances.

Proof. We will start by the proof of the first inequality. Let f be a candidate mapping for the Kobayashi metric at the point p on M *i.e.* $f \in H(\mathbb{D}, M)$, $f(0) = p$ and $d_0f(\partial/\partial x) = rv$ for some $r > 0$. Clearly, $\varphi \circ f$ is a candidate mapping for the Kobayashi metric at $\varphi(p)$ on N. Also, $d_0(\varphi \circ f)(\partial/\partial x) = d_p\varphi(d_0f(\partial/\partial x)) =$

$d_p\varphi(rv) = rd_p\varphi(v)$. Thus

$$K_N(\varphi(p), d_p\varphi(v)) \leq \frac{1}{r}$$

Now, taking the infimum over all the candidate mappings f we obtain

$$K_N(\varphi(p), d_p\varphi(v)) \leq K_M(p, v)$$

We move now to the proof of the second inequality. Let g be a candidate map for the Carathéodory metric at $\varphi(p)$ on N *i.e.* $g \in H(N, \mathbb{D})$ and $g(\varphi(p)) = 0$. Then, $g \circ \varphi$ is a candidate map for the Carathéodory metric at p on M. Thus

$$C_M(p, v) \geq |d_p(g \circ \varphi)(v)| = \left|d_{\varphi(p)}g(d_p\varphi(v))\right|$$

Taking the supremum over all the candidate mappings g we get

$$C_N(\varphi(p), d_p\varphi(v)) \leq C_M(p, v)$$

Finally, the proof of the last inequality follows from definition of the pseudodistance and the previous inequalities. □

Corollary 2.3.1 (Invariance Under Biholomorphism)**.** *Let* $\varphi : M \to N$ *be a* *biholomorphic map. Then, for all $p \in M$, $v \in T_pM$*

$$K_N(\varphi(p), d_p\varphi(v)) = K_M(p, v) \text{ and } C_N(\varphi(p), d_p\varphi(v)) = C_M(p, v)$$

Proof. Let $p \in M$ and let $v \in T_pM$. By applying Propostion 2.3.1 twice we get

$$K_N(\varphi(p), d_p\varphi(v)) \leq K_M(p, v) = K_M(\varphi^{-1}(\varphi(p)), d_p\varphi^{-1}(\varphi(p)) \leq K_N(\varphi(p), d_p\varphi(v))$$

Therefore, $K_N(\varphi(p), d_p\varphi(v)) = K_M(p, v)$. The proof of $C_N(\varphi(p), d_p\varphi(v)) = C_M(p, v)$ follows by the same procedure. □

2.4 Kobayashi and Carathéodory Volumes

Definition 2.4.1. *Let D be a bounded domain in $\mathbb{C}^n$ and let $z \in D$, we define the* **Kobayashi and Carathéodory volumes** *respectively as follows:*

- $K_D(z) = \inf\left\{\frac{1}{|det\, d_0 f|} : \ f \in H(\mathbb{B}^n, D), f(0) = z\right\}$
- $C_D(z) = \sup\left\{|det\, d_z g| : \ g \in H(D, \mathbb{B}^n), g(z) = 0\right\}$

where $\mathbb{B}^n$ is the Euclidean open unit ball in $\mathbb{C}^n$.

Properties of the Kobayashi and Carathéodory Volumes

In what follows, we will state and prove some properties of the Kobayashi and Carathéodory volumes.

Let D be a bounded domain in $\mathbb{C}^n$ and let $z \in D$, then:

Property 1. $C_D(z) \leq K_D(z)$

Property 2. $C_{D'}(z) \leq C_D(z)$ and $K_{D'}(z) \leq K_D(z)$, $\forall D \subseteq D'$

Property 3. $\dfrac{K_{T(D)}(T(z))}{K_D(z)} = \dfrac{C_{T(D)}(T(z))}{C_D(z)} = |det d_z T|^{-1}$, with T being an injective holomorphic map.

Proof. Property 1. Let f and g be two candidate maps of the Kobayashi and Carathéodory volumes respectively. Consider the map $f \circ g : D \longrightarrow D$. Clearly, this map is holomorphic and $f \circ g\ (z) = f(g(z)) = f(0) = z$. Hence, by Theorem 3.2.1 (see section 3.2), $|det\ d_z(f \circ g)| \leq 1$.

Using the Chain rule we get:

$$\begin{aligned} |det\ d_z(f \circ g)| &= |det\ (d_{g(z)}f\ .\ d_z g)| \\ &= |det\ (d_0 f\ .\ d_z g)| \\ &= |det\ d_0 f|.|det\ d_z g| \end{aligned}$$

Hence, $|det\ d_z g| \leq \dfrac{1}{|det\ d_0 f|}$.

Using the following property:

$$(\forall\, a \in A,\, b \in B,\ \ a \leq b) \implies \sup A \leq \inf B$$

we conclude that $C_D(z) \leq K_D(z)$.

Property 2. Let $D \subset D'$. The following inclusions hold:

$$\left\{\frac{1}{|det\ d_0 f|} : f \in H(\mathbb{B}^n, D), f(0) = z\right\} \subset \left\{\frac{1}{|det\ d_0 g|} : g \in H(\mathbb{B}^n, D'), g(0) = z\right\}$$

$$\left\{|det d_z f| : f \in H(D', \mathbb{B}^n), f(z) = 0\right\} \subset \left\{|det d_z g| : g \in H(D, \mathbb{B}^n), g(z) = 0\right\}$$

The first inclusion holds by composing f with the inclusion map $i : D \to D'$ and the second one holds by restricting the domain of f to D.

Using properties of inf and sup of a set we reach the desired conclusion.

Property 3. First note that:

$$K_{T(D)}(T(z)) = \inf\left\{\frac{1}{|det\ d_0 f|}\ :\ f \in H(\mathbb{B}^n, T(D)), f(0) = T(z)\right\}$$

$$C_{T(D)}(T(z)) = \sup\left\{|det\ d_{T(z)} f|\ :\ f \in H(T(D), \mathbb{B}^n), f(T(z)) = 0\right\}$$

Now, we start by proving that $\dfrac{K_{T(D)}(T(z))}{K_D(z)} = |det\ d_z T|^{-1}$.

Let $f : \mathbb{B}^n \to T(D)$ be a holomorphic map with $f(0) = T(z)$. Since T is injective then $T : D \to T(D)$ is a bijection. Hence, there exists a holomorphic map $g : \mathbb{B}^n \to D$ such that $g(z) = T^{-1}(f(z))$ and $g(0) = T^{-1}(f(0)) = T^{-1}(T(z)) = z$. Consequently, for each map f defined as above corresponds a map $g : \mathbb{B}^n \to D$ with $f = T \circ g$ and $g(0) = z$.

Hence, the following holds:

$$\begin{aligned} \left| \frac{1}{det\ d_0 f} \right| &= \left| \frac{1}{det\ d_0 (T \circ g)} \right| \\ &= \left| \frac{1}{det\ (d_{g(0)} T\ .\ d_0 g)} \right| \\ &= \left| \frac{1}{det\ (d_z T\ .\ d_0 g)} \right| \\ &= \left| \frac{1}{det\ d_z T} \right|\ .\ \left| \frac{1}{det\ d_0 g} \right| \end{aligned}$$

Therefore, $\dfrac{K_{T(D)}(T(z))}{K_D(z)} = |det\ d_z T|^{-1}$.

We next move to the proof of $\dfrac{C_{T(D)}(T(z))}{C_D(z)} = |det\ d_z T|^{-1}$.

Let $f : T(D) \to \mathbb{B}^n$ be a holomorphic map with $f(T(z)) = 0$. Applying the same logic as above, there exists a map $g : D \to \mathbb{B}^n$ such that $g = f \circ T$ and $g(z) = f \circ T(z) = 0$. Hence,

$$\begin{aligned} |det\ d_z g| &= |det\ d_z (f \circ T)| \\ &= \left| det\ d_{T(z)} f \right|\ .\ |det\ d_z T| \end{aligned}$$

Therefore, $\dfrac{C_{T(D)}(T(z))}{C_D(z)} = |det\ d_z T|^{-1}$.

□

2.5 Theory of Normal Families

Definition 2.5.1. *A family $\mathcal{F}$ of analytic functions on a domain $\Omega \subseteq \mathbb{C}$ is called* ***normal*** *if every sequence of functions $\{f_n\} \subseteq \mathcal{F}$ has a subsequence that converges uniformly on compact subsets of Ω.*

Theorem 2.5.1 (Montel's Theorem)**.** *If $\mathcal{F}$ is a family of analytic functions on a domain $\Omega \subseteq \mathbb{C}$, uniformly bounded on compact subsets of Ω, then $\mathcal{F}$ is normal.*

Corollary 2.5.1. *Let D be a domain in $\mathbb{C}$. For $p \in D$ and $v \in T_p(D)$, the supremum in the definition of the Carathéodory metric $C_D(p,v)$ (Carathéodory volume $C_D(z)$ resp.) is assumed by some function $f \in H(D,\mathbb{D})$ ($f \in H(D,\mathbb{B}^n)$ resp.).*

Proof. Let $\mathcal{F}$ be the family of holomorphic functions $f : D \to \mathbb{D}$ such that $f(p) = 0$. By property of the supremum, for all $n \in \mathbb{N}$, there exists $f_n \in \mathcal{F}$ such that:

$$C_D(p,v) - \frac{1}{n} < |d_p f_n(v)| \leq C_D(p,v) \tag{2.1}$$

Let $(f_n)_{n\in\mathbb{N}}$ be the sequence of such functions. Since $f(D) \subset \mathbb{D}$, $\mathcal{F}$ is uniformly bounded on D. By Theorem 2.5.1 we see that - up to subsequence extraction - (f_n) converges uniformly on compact subsets of D to some function f. Clearly, $f \in \mathcal{F}$ and $|d_p f_n(v)|$ converges to $|d_p f(v)|$ as $n \to \infty$. Letting $n \to \infty$ in (2.1) we get the desired result: $C_D(p,v) = |d_p f(v)|$. Note that f is called an **extremal function** for v. The same proof applies for the Carathéodory volume and can be generalized to domains in $\mathbb{C}^n$. $\square$

A complex manifold M is said to be **Montel** if every sequence $(f_n) \subset H(\mathbb{D}, M)$ has a subsequence (f_{n_k}) which either converges uniformly on compact subsets of $\mathbb{D}$ or compactly diverges (*i.e.* for arbitrary compact subsets k, k' of $\mathbb{D}$ and M respectively there exits k_0 such that $f_{n_k}(K) \cap K' = \emptyset$ for all $k \geq k_0$). An important class of Montel manifolds is

Theorem 2.5.2. *[3] Every complete hyperbolic manifold is Montel.*

In that case, we obtain an important result regarding the the Kobayashi metric.

Corollary 2.5.2. *Let M be a complete hyperbolic manifold. Then the infimum in the definition of the Kobayashi metric $K_M(p,v)$ is attained.*

Proof. Let $\mathcal{F}$ be the family of candidate mappings f for the Kobayashi metric at a point p on M such that $d_0 f(\partial/\partial x) = rv$ for some $r > 0$. By property of the infimum, for all $n \in \mathbb{N}$, there exists $f_n \in \mathcal{F}$ with $d_0 f_n(\partial/\partial x) = r_n v$ such that:

$$K_M(p,v) \leq \frac{1}{r_n} < K_M(p,v) + \frac{1}{n} \tag{2.2}$$

Let $(f_n)_{n \in \mathbb{N}}$ be the sequence of such functions. We aim to show that (f_n) has a uniformly convergent subsequence (f_{n_k}) on compact subsets of $\mathbb{D}$, then the result follows in the same way as in the proof of Corollary 2.5.1. Since M is complete hyperbolic, by Theorem 2.5.2, excluding the case of compact divergence of the subsequence (f_{n_k}) will complete the proof. However, this can be easily done by taking the following compact subsets

$$k = \{0\} \subset \mathbb{D}, \quad k' = \{p\} \subset M$$

and noting that $f_{n_k}(0) = p$ for all $n_k \in \mathbb{N}$. □

We call the map that realizes this infimum an **extremal** for v or an **extremal disc** for v; Lempert theory [5] provides an important account on the behaviour of extremal discs for convex domains of $\mathbb{C}^n$.

Chapter 3

Characterization of The Unit Ball by Rosay

The group of automorphisms of a given strictly pseudoconvex domain in $\mathbb{C}^n$ is an important invariant object in complex analysis. It allows to understand dynamic and geometric properties of the domain and its boundary. It is known that the group of automorphisms of a bounded domain is a real Lie group. Moreover the group of automorphisms of the unit ball in $\mathbb{C}^n$, which can be described precisely (see [9]), is non-compact. It seems natural to wonder whether or not the ball is the only strictly pseudoconvex domain with a non-compact automorphism group. In 1977, B. Wong proved in [11] that the ball is indeed the only such domain. In this chapter, we will consider the approach of J.-P. Rosay [8] for B. Wong result.

3.1 Main Theorem

Theorem 3.1.1 (Wong-Rosay Theorem)**.** *let Ω be a bounded domain of $\mathbb{C}^n$ and let $\zeta_0 \in \partial\Omega$ be a point of strict pseudoconvexity. Assume that there exists a compact subset $K \subset \Omega$, a sequence (x_k) of points in K and a sequence (T_k) of automorphisms of Ω such that $T_k(x_k)$ converges to ζ_0. Then Ω is biholomorphically equivalent to the open unit ball.*

Remark 3.1.1. *The existence of points of strict pseudoconvexity is guaranteed in any bounded domain of $\mathbb{C}^n$ of class C^2. For instance, points of the boundary of maximal norm are an example of such points.*
Note that Ω could also be taken to be a domain in a hyperbolic manifold and not in $\mathbb{C}^n$ without changing the proof.

3.2 A First Characterization of The Unit Ball by Volumes

In this section we will state a theorem due to H. Cartan [2], which generalizes the classical Schwarz lemma, and prove a first result on the characterization of the unit ball by volumes (Theorem 3.2.2) that will be useful in the proof of Wong-Rosay Theorem.

Theorem 3.2.1 ([2])**.** *let D be a domain of $\mathbb{C}^n$ and let $z \in D$. If $f : D \longrightarrow D$ is holomorphic such that $f(z) = z$ then:*

1. $|det\, d_z f| \leq 1$ *and if equality holds then* f *is an automorphism of* D.

2. *the eigenvalues of* $d_z f$ *are of norm* ≤ 1 *and if all of them have norm 1 then* f *is the identity map.*

Theorem 3.2.2. *Let* D *be a bounded domain of* $\mathbb{C}^n$*. If there exists* $z \in D$ *such that* $K_D(z) = C_D(z)$*, then* D *is biholomorphically equivalent to the open unit ball* $\mathbb{B}^n$.

Proof. Our aim in this proof is to construct a biholomorphic map from D to $\mathbb{B}^n$. By Corollary 2.5.1 there exists a holomorphic map $g : D \to \mathbb{B}^n$ such that $g(z) = 0$ and $C_D(z) = |det\, d_z g|$. However, the existence of such extremal map is not guaranteed for the Kobayashi volume. Hence, let $(f_k)_{k\in\mathbb{N}}$ be a sequence of holomorphic maps from $\mathbb{B}^n$ to D such that $f_k(0) = z$ and $\frac{1}{|det\, d_0(f_k)|}$ converges to $K_D(z)$ when $k \to \infty$.

We start by proving that g is surjective. For this purpose, we construct the sequence of holomorphic maps $H_k = g \circ f_k : \mathbb{B}^n \to \mathbb{B}^n$. Note that $H_k(0) = g \circ f(0) = g(f_k(0)) = g(z) = 0$ and $|det\, d_0 H_k| = \left|det\, d_{f_k(0)} g\right| \times |det\, d_0 f_k| = |det\, d_z g| \times |det\, d_0 f_k|$. By Theorem 2.5.1, we can extract a subsequence of (H_k) that converges uniformly on compact subsets of $\mathbb{B}^n$ to a map H. Clearly, $H : \mathbb{B}^n \to \mathbb{B}^n$ is holomorphic and $H(0) = 0$. Moreover, $|det\, d_0 H| = C_D(z)\dfrac{1}{K_D(z)} = 1$. Where the last equality holds by the given of the theorem.By Theorem 3.2.1, we conclude that H is an automorphism of $\mathbb{B}^n$.

Remark 3.2.1. *The fact that H is onto (being an automorphism) does not necessarily imply that H_K is onto for all $K \geq 0$ and that g is surjective consequently. For instance, consider the sequence of holomorphic maps $\phi_k : \mathbb{B}^n \to \mathbb{B}^n$ such that $\phi_k(z) = (1 - \frac{1}{k})z$. Clearly, ϕ_k is not onto $\mathbb{B}^n$ for $k \geq 0$, however the sequence converges uniformly to the identity map which is surjective.*

To prove that g is surjective, we start by taking $t \in \mathbb{B}^n$. For k large enough there exists $z \in \mathbb{B}^n$ such that $H_k(z) = t$. If this was not the case, we will reach a contradiction since H is onto. Hence, $t = H_k(z) = g \circ f_k(z) = g(f_k(z))$ and g is surjective.

Next, we show that g is injective. For this purpose, we define the sequence $(F_k)_{k\in\mathbb{N}}$ of holomorphic maps $F_k = f_k \circ g : D \to D$. Note that $F_k(0) = f_k(g(0)) = f_k(z) = 0$.

The following claims will be useful for the proof of injectivity of g.

Claim 1: The modulus of the eigenvalues of $d_z(F_k)$ converges to 1 when $k \to \infty$.

Proof. Let $\lambda_{1,k} \cdots \lambda_{n,k}$ be the eigenvalues of $d_z(F_k)$. By Theorem 3.2.1:

$$|\lambda_{j,k}| \leq 1 \text{ for } 1 \leq j \leq n \tag{3.1}$$

By *Schur decomposition*, $d_z(F_k)$ is similar to a triangular matrix U having $\lambda_{1,k} \cdots \lambda_{n,k}$ as diagonal entries. Hence, $|det\, d_z(F_k)| = |det\, U| = |\lambda_{1,k} \cdots \lambda_{n,k}|$.Thus,

$$\lim_{k\to\infty} |\lambda_{1,k} \cdots \lambda_{n,k}| = \lim_{k\to\infty} |det\, d_z(F_k)| = \lim_{k\to\infty} |det\, d_0(f_k)|\ .\ |det\, d_z g| = 1 \qquad (3.2)$$

The desired result follows from (3.1) and (3.2). □

Proposition 3.2.1. *Let $\theta \in (0, 2\pi]$ be a fixed angle. There exists a sequence of positive integers $(\mu_k)_{k\in\mathbb{N}}$ such that $e^{i\mu_k\theta} \longrightarrow 1$ as $k \to \infty$.*

Claim 2: There exists a subsequence $(F_{\varphi(k)})$ of $(F_k)_{k\in\mathbb{N}}$ and a sequence $(\mu_k)_{k\in\mathbb{N}}$, $\mu_k \in \mathbb{N}^*$, such that the eigenvalues of $(d_z F^{\mu_k}_{\varphi(k)})$ converge to 1 when $k \to \infty$.

Proof. Note that for $k \geq 0$, $d_z F^{\mu_k}_{\varphi(k)} = d_z(F_{\varphi(k)} \circ F_{\varphi(k)} \cdots \circ F_{\varphi(k)})$, where the composition is applied μ_k times. Using the chain rule and the fact that $F_{\varphi(k)}(z) = f_{\varphi(k)}(g(z)) = f_{\varphi(k)}(0) = z$ we get:

$$\begin{aligned} d_z F^{\mu_k}_{\varphi(k)} &= d_{F_{\varphi(k)}(z)}(F^{\mu_k-1}_{\varphi(k)}) \times d_z F_{\varphi(k)} \\ &= d_z(F^{\mu_k-1}_{\varphi(k)}) \times d_z F_{\varphi(k)} \\ &= d_{F_{\varphi(k)}(z)}(F^{\mu_k-2}_{\varphi(k)}) \times d_z F_{\varphi(k)} \times d_z F_{\varphi(k)} \\ &\vdots \\ &= (d_z F_{\varphi(k)})^{\mu_k} \end{aligned}$$

Hence, the eigenvalues of $(d_z F^{\mu_k}_{\varphi(k)})$ are $\lambda^{\mu_k}_{j,k}$, $1 \leq j \leq n$.

First, let us write the eigenvalues of $d_z(F_k)$ as $\lambda_{j,k} = r_{j,k} e^{i\,\theta_{j,k}}$ for $1 \leq j \leq n$ and for some $r \geq 0$ and $\theta \in (0, 2\pi]$. Then, proceed as folloows:

<u>*Step 1:*</u> Consider the sequence $(\lambda_{1,k})_{k\in\mathbb{N}}$. Since $|\lambda_{1,k}| \leq 1$ for $k \geq 0$, by *Bolzano-Weierstrass* theorem, we can extract a subsequence $(\lambda_{1,\varphi_1(k)})$ of $(\lambda_{1,k})$ such that:

$$\lambda_{1,\varphi_1(k)} = r_{1,\varphi_1(k)} e^{i\,\theta_{1,\varphi_1(k)}} \to \lambda_1 \text{ as } k \to \infty, \text{ for some } \lambda_1 = r_1 e^{i\,\theta_1} \in \mathbb{C}$$

Note that $r_1 = 1$ by Claim 1. By Proposition 3.2.1, we can find a sequence of positive integers $(\mu_{1,k})$ such that: $\lambda_1^{\mu_{1,k}} = e^{i\mu_{1,k}\theta_1} \to 1$ when $k \to \infty$. Hence, $\lambda_{1,\varphi_1(k)}^{\mu_{1,k}} \to 1$ when $k \to \infty$.

<u>*Step 2:*</u> We will now consider the subsequence $(\lambda_{2,\varphi_1(k)})$ of the sequence $(\lambda_{2,k})_{k\in\mathbb{N}}$ instead of the sequence itself. Clearly, $|\lambda_{2,\varphi_1(k)}| \leq 1$ and consequently, $|\lambda_{2,\varphi_1(k)}^{\mu_{1,k}}| \leq 1$ for $k \geq 0$.

Applying *Bolzano-Weierstrass* theorem and Proposition3.2.1 as in *Step 1*, we can find a subsequence $(\lambda_{2,\varphi_2\circ\varphi_1(k)})$ of $(\lambda_{2,\varphi_1(k)})$ and a sequence of positive integers $(\mu_{2,k})$ such that:

$$\lambda_{2,\varphi_2\circ\varphi_1(k)}^{(\mu_{2,k}\,\cdot\,\mu_{1,k})} \longrightarrow 1 \text{ when } k \to \infty$$

Note that $(\lambda_{1,\varphi_2\circ\varphi_1(k)}^{(\mu_{2,k}\,\cdot\,\mu_{1,k})})$ corresponds to a subsequence of $(\lambda_{1,\varphi_1(k)}^{\mu_{1,k}})$ raised to the power $(\mu_{2,k})$. Hence, it will also converge to 1 as $k \to \infty$.

Proceeding in an inductive fashion, we reach:

<u>*Step n:*</u> We extract from the sequence $(\lambda_{n,\varphi_{n-1}\circ\varphi_{n-2}\cdots\varphi_1(k)})$ the following subsequence $(\lambda_{n,\varphi_n\circ\varphi_{n-1}\circ\varphi_{n-2}\cdots\varphi_1(k)})$ and we find a sequence of positive integers $(\mu_{n,k})$ such that:

$$\lambda_{n,\varphi_n\circ\varphi_{n-1}\circ\varphi_{n-2}\cdots\varphi_1(k)}^{(\mu_{n,k}\ .\ \mu_{n-1,k}\cdots\mu_{1,k})} \longrightarrow 1 \text{ when } k \to \infty$$

Note that this construction guarantees that $\lambda_{j,\varphi_n\circ\varphi_{n-1}\circ\varphi_{n-2}\cdots\varphi_1(k)}^{(\mu_{n,k}\ .\ \mu_{n-1,k}\cdots\mu_{1,k})}$ converges to 1 as $k \to \infty$ for all $1 \leq j \leq n$. Hence, taking $\varphi(k) = \varphi_n \circ \varphi_{n-1} \circ \varphi_{n-2} \cdots \varphi_1(k)$ and $\mu_k = \mu_{n,k}\ .\ \mu_{n-1,k}\ \cdots\ \mu_{1,k}$ finishes the proof of Claim 2. □

We will next proceed by the proof of injectivity of g as follows: let $\varepsilon > 0$ such that D_ε, the set of points in D such that their Kobayashi distant to z is less than ε, is relatively compact. By the *distance decreasing property of the Kobayashi metric*, (Proposition 2.3.1) $F_{\varphi(k)}^{\mu_k}(D_\varepsilon) \subset D_\varepsilon$ for $k \geq 0$. Hence, up to subsequence extraction, we can assume by Theorem 2.5.1 that the sequence $(F_{\varphi(k)}^{\mu_k})_{k\in\mathbb{N}}$ converges uniformly on compact subsets of D_ε to a map $F : D_\varepsilon \to D_\varepsilon$.

For $k \geq 0$ we have that $F_{\varphi(k)}(z) = z$, then $F_{\varphi(k)}^{\mu_k}(z) = z$ and consequently $F(z) = z$. By Claim 2, we see that the eigenvalues of $d_z F_{\varphi(k)}^{\mu_k}$ are equal to 1. Hence, we conclude by Theorem 3.2.1 that F is the identity map. It follows that $(F_{\varphi(k)}^{\mu_k})_{k\in\mathbb{N}}$ converges uniformly to the identity map on compact subsets of D.

Now, let $z_1, z_2 \in D$ such that $g(z_1) = g(z_2)$. Since each $F_{\varphi(k)}^{\mu_k}$ is of the form $h_k \circ g$

for some map $h_k : \mathbb{B}^n \to D$ we see that $F^{\mu_k}_{\varphi(k)}(z_1) = F^{\mu_k}_{\varphi(k)}(z_2)$ for all $k \geq 0$ and consequently $F(z_1) = F(z_2)$. Thus, $z_1 = z_2$ and g is injective.

Therefore g is a biholomorphic map from D to $\mathbb{B}^n$. □

3.3 Localization of The Kobayashi and Carathéodory Volumes

Proposition 3.3.1. *Let Ω and D be two bounded domains of $\mathbb{C}^n$. Let $y \in D$ and $\zeta_0 \in \partial\Omega$ be a point of strict pseudoconvexity. Suppose that (φ_j) is a sequence of holomorphic maps from D to Ω such that $\varphi_j(y) \to \zeta_0$ when $j \to \infty$, then φ_j converges to ζ_0 uniformly on compact subsets of D.*

Proof. Since Ω is bounded then by Theorem 2.5.1 every subsequence of (φ_j) has a uniformly convergent subsequence on compact subsets of D. Let φ be an accumulation point of a subsequence (φ_{j_k}) of (φ_j) in the topology of uniform convergence on compact sets. We aim to show that (φ_{j_k}) has only one such point. For this purpose, we consider a holomorphic map P on $\overline{\Omega}$ defined as:

$$P(\zeta) = exp\left(p_{\zeta_0}(\zeta)\right)$$

where:

$$p_{\zeta_0}(\zeta) = \sum_{i=1}^{n} \frac{\partial \rho(\zeta_0)}{\partial z_i}(\zeta - \zeta_0)_i + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 \rho(\zeta_0)}{\partial z_i \partial z_j}(\zeta - \zeta_0)_i(\zeta - \zeta_0)_j$$

with ρ being the defining function of Ω.

Clearly, $P(\zeta_0) = 1$ and $|P(\zeta)| = exp\,(\Re(p_{\zeta_0}(\zeta)))$. By Proposition 2.1.1 the expansion of ρ about the boundary point ζ_0 is:

$$\rho(\zeta) = 2\Re(p_{\zeta_0}(\zeta)) + L_{\zeta_0,\rho}(\zeta - \zeta_0) + o(\|\zeta - \zeta_0\|^2)$$

Since $L_{\zeta_0,\rho}(\zeta - \zeta_0) > 0$ (because ζ_0 is a point of strict pseudoconvexity) and $\rho(\zeta) < 0$ in $\Omega \cap U$, where U is a neighborhood of $\partial\Omega$ near ζ_0, there exists a neighborhood V of ζ_0 such that $\Re(p_{\zeta_0}(\zeta)) < 0$ in $V \cap \Omega$. Thus, $|P(\zeta)| < 1$ for all $\zeta \in V \cap \overline{\Omega}$.

Now, consider the holomorphic map $P \circ \varphi$. Since φ is an accumulation point of (φ_{j_k}) we see that $P \circ \varphi(y) = P(\varphi(y)) = P(\zeta_0) = 1$ and $|P \circ \varphi(x)| = |P(\varphi(x))| = |P(\zeta)| < 1$ for all $x \in D$ close enough to y. Hence, $P \circ \varphi$ has a local maximum at y and therefore constant by the *Maximum modulus Principle*. Consequently, $p_{\zeta_0}(\varphi(\zeta))$ is constant and therefore so is $\varphi(\zeta)$ (precisely, $\varphi(\zeta) = \zeta_0$ is the only accumulation point). We conclude then that (φ_j) converges to ζ_0 uniformly on compact subsets of D. $\square$

Theorem 3.3.1. *Let Ω be a bounded domain of $\mathbb{C}^n$ and let $\zeta_0 \in \partial\Omega$ be a point of strict pseudoconvexity. Let $A > 0$ and let $B(\zeta_0, A)$ be the ball of center ζ_0 and radius A. If $(z_k)_{k\in\mathbb{N}}$ is a sequence of points in Ω converging to ζ_0, then:*

(i) $\dfrac{K_\Omega(z_k)}{K_{\Omega\cap B(\zeta_0,A)}(z_k)} \to 1$

(ii) If there exists $x \in \Omega$ such that for all $k \in \mathbb{N}$ there exists T_k, an automorphism of Ω, satisfying $T_k(x) = z_k$ then:

$$\frac{C_\Omega(z_k)}{C_{\Omega\cap B(\zeta_0,A)}(z_k)} \to 1$$

Proof. We will start by the proof of *(i)*.

Let $(f_k)_{k\in\mathbb{N}}$ be a sequence of holomorphic maps from $\mathbb{B}^n$ to Ω such that $f_k(0) = z_k$. Clearly, $f_k(0)$ converges to ζ_0 as $k \to \infty$. Let $A > 0$, by Proposition 3.3.1 we see that f_k converges uniformly on compact subsets of $\mathbb{B}^n$ to ζ_0. Hence, for all $\eta > 0$ there exists $k_0 \geq 0$ such that for all $k \geq k_0$, $\|f_k(t) - \zeta_0\| < A$, for all t satisfying $\|t\| < 1 - \eta$ (note that η here corresponds to the choice of compact subsets of $\mathbb{B}^n$). Next, we define on $\mathbb{B}^n$ the holomorphic map $\psi_k(t) = f_k((1-\eta)t)$. Clearly, $\|(1-\eta)t\| \leq 1 - \eta$ and consequently, $\psi_k(\mathbb{B}^n) \subset \Omega \cap B(\zeta_0, A)$. Also, $\psi_k(0) = f_k((1-\eta) \times 0) = f_k(0) = z_k$. We conclude that:

$$K_{\Omega\cap B(\zeta_0,A)}(z_k) \leq \frac{1}{|det\ d_0\psi_k|}$$

To compute $|det\, d_0\psi_k|$ we notice that $\psi_k = f_k \circ g$ where $g : \mathbb{B}^n \to \mathbb{B}^n$ is given by $g(t) = (1-\eta)t$. Therefore,

$$\begin{aligned}|det\, d_0\psi_k| &= |det\, d_0(f_k \circ g)| \\ &= \left|det\,(d_{g(0)}f_k \times d_0 g)\right| \\ &= |det\,(d_0 f_k \times d_0 g)| \\ &= |det\, d_0 f_k| \times |det\, d_0 g|\end{aligned}$$

Since $g(t) = g(t_1, t_2, \cdots, t_n) = ((1-\eta)t_1, (1-\eta)t_2, \cdots, (1-\eta)t_n)$ we can easily see that $d_0 g = (1-\eta)I_n$, where I_n is the $n \times n$ identity matrix. Hence, $|det\, d_0 g| = (1-\eta)^n$ and consequently,

$$\begin{aligned}K_{\Omega\cap B(\zeta_0,A)}(z_k) &\leq \frac{1}{|det\, d_0 f_k|} \times \frac{1}{(1-\eta)^n} \\ (1-\eta)^n \times K_{\Omega\cap B(\zeta_0,A)}(z_k) &\leq \frac{1}{|det\, d_0 f_k|}\end{aligned}$$

Thus,

$$(1-\eta)^n \times K_{\Omega\cap B(\zeta_0,A)}(z_k) \leq K_\Omega(z_k) \tag{3.3}$$

Since $\Omega \cap B(\zeta_0, A) \subset \Omega$ we conclude by Property 2 (section 2.4) that:

$$k_\Omega(z_k) \leq K_{\Omega\cap B(\zeta_0,A)}(z_k) \tag{3.4}$$

Finally, by (3.3) and (3.4) we see that:

$$1 \le \frac{K_\Omega(z_k)}{K_{\Omega\cap B(\zeta_0,A)}(z_k)} \le \frac{1}{(1-\eta)^n} \tag{3.5}$$

letting $\eta \longrightarrow 0$ in (3.5) we obtain the desired result.

Next, we will prove *(ii)*.

Let $x \in \Omega$ with the property of the given and let $\eta > 0$. We claim that there exists a domain Ω' relatively compact in Ω containing x such that:

$$C_{\Omega'}(x) \le (1+\eta)C_\Omega(x) \tag{3.6}$$

Assume by contradiction that there exists $\eta > 0$ such that for all relatively compact domains Ω' in Ω containing x we have $C_{\Omega'}(x) > (1+\eta)C_\Omega(x)$. For $k > 0$, take Ω'_k to be $B(x,k)\cap\Omega$, where $B(x,k)$ is the ball of center x and radius k. Clearly, $x \in \Omega'_k$ and Ω'_k is relatively compact in Ω thus $C_{\Omega'_k}(x) > (1+\eta)C_\Omega(x)$. Note that by Corollary 2.5.1, $C_{\Omega'_k}(x) = |det\, d_x g_k|$ for some holomorphic map $g_k : \Omega'_k \to \mathbb{B}^n$ with $g_k(x) = 0$. Construct then the sequence (g_k) of such extremal maps. For k large enough the sequence (g_k) converges to the holomorphic map $g : \Omega \to \mathbb{B}^n$ such that $g(x) = 0$ and $C_\Omega(x) = |det\, d_x g|$. Hence, $C_{\Omega'_k}(x)$ converges to $C_\Omega(x)$ as $k \to \infty$. However, we previously showed that for some $\eta > 0$, $C_{\Omega'_k}(x) > (1+\eta)C_\Omega(x)$ for all $k > 0$. Thus we reached a contradiction and the claim is then proved.

Now, consider the following series of inequalities:

$$1 \leq \frac{C_{\Omega \cap B(\zeta_0, A)}(z_k)}{C_\Omega(z_k)} \leq \frac{C_{T_k(\Omega')}(z_k)}{C_\Omega(z_k)} = \frac{C_{\Omega'}(x)}{C_\Omega(x)} \leq 1 + \eta \tag{3.7}$$

where the first inequality holds by Property 2 (section 2.4) and the fact that $\Omega \cap B(\zeta_0, A) \subset \Omega$, the second inequality also holds by the same property and the fact that for k large enough, $T_k(\Omega') \subset \Omega \cap B(\zeta_0, A)$ and the last inequality holds by (3.6). To show the equality in (3.7), it is enough to see that by Property 3 (section 2.4) we have the following:

$$\frac{C_{T_k(\Omega')}(z_k)}{C_{\Omega'}(x)} = \frac{C_{T_k(\Omega')}(T_k(x))}{C_{\Omega'}(x)} = |det\ d_x T|^{-1}$$

and

$$\frac{C_\Omega(z_k)}{C_\Omega(x)} = \frac{C_{T_K(\Omega)}(T_k(x))}{C_\Omega(x)} = |det\ d_x T|^{-1}$$

Therefore, by letting $\eta \to 0$ in (3.7) we finish the proof of *(ii)*. □

3.4 Proof of The Main Theorem

In this section we will prove Theorem 3.1.1 stated in section 3.1. Hence we assume that the assumptions of the theorem are satisfied throughout this section.

We start by proving the following modification of Proposition 3.3.1.

Proposition 3.4.1. *let $x \in \Omega$, then $T_k(x)$ converges to ζ_0 as $k \to \infty$.*

Proof. Since K is compact, up to subsequence extraction, the sequence (x_k) converges in K. Let y be the limit of (x_k) in K. We aim to show that $T_k(y)$ converges to ζ_0.

We know that $T_k(x_k) \to \zeta_0$ as $k \to \infty$ *i.e.* for all $\epsilon > 0$ there exists $k_1 \geq 0$ such that for all $k \geq k_1$, $\|T_k(x_k) - \zeta_0\| < \frac{\epsilon}{2}$. Also, by continuity of each T_k at y, there exists $k_2 \geq 0$ such that for all $k \geq k_2$, $\|T_l(x_k) - T_l(y)\| < \frac{\epsilon}{2}$ for all $l \geq 0$.

Let $\epsilon > 0$, take $k' = max\{k_1, k_2\}$ then for all $k \geq k'$ the following holds:

$$\|T_k(y) - \zeta_0\| \leq \|T_k(y) - T_k(x_k)\| + \|T_k(x_k) - \zeta_0\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Hence, $T_k(y)$ converges to ζ_0 as $k \to \infty$.

By Proposition 3.3.1, T_k converges to ζ_0 uniformly on compact subsets of Ω. In particular, fixing $x \in \Omega$, $T_k(x) \to \zeta_0$ as $k \to \infty$. □

Let $z_k = T_k(x)$. By Property 3 (section 2.4), for $k \in \mathbb{N}$:

$$\frac{K_\Omega(x)}{C_\Omega(x)} = \frac{K_{T_k(\Omega)}(T_k(x))}{C_{T_k(\Omega)}(T_k(x))} = \frac{K_\Omega(z_k)}{C_\Omega(z_k)}$$

Remark 3.4.1. *By showing that $\dfrac{K_\Omega(z_k)}{C_\Omega(z_k)}$ converges to 1 and then applying Theorem 3.2.2 we finish the proof of Theorem 3.1.1.*

By Proposition 2.1.1 we see that after a change of coordinates in a neighborhood of ζ_0, Ω is defined by a function $\rho < 0$ of the form:

$$\rho(\omega) = 2\Re\omega_1 + L_{\zeta_0,\rho}(\omega) + o(\|\omega\|^2)$$

By the same machinery done in Section 2.1.2, we see that for all $z \in \Omega$ close enough to ζ_0, we can locally change the coordinates so that the new coordinates of z are of the form $(a, 0, \cdots, 0)$ for some $a < 0$.

We introduce next the notion of ellipsoids that will be used in the proof as local domains of comparison.

Definition 3.4.1. *For every Hermitian positive definite form Q we associate the ellipsoid defined as follows:*

$$E = \{\omega \in \mathbb{C}^n : \rho = 2\Re\omega_1 + Q(\omega) < 0\}$$

Let $\varepsilon > 0$. Let $E_\varepsilon^+(z_k)$ and $E_\varepsilon^-(z_k)$ be the ellipsoids defined respectively by the forms

$$Q^+ = L_{z_k,\rho}(\omega) - \varepsilon\|\omega\|^2 \text{ and } Q^- = L_{z_k,\rho}(\omega) + \varepsilon\|\omega\|^2$$

Let $A > 0$ and let Ω_1 be the image of $\Omega \cap B(\zeta_0, A)$ in the ω-coordinates. For $r > 0$ small enough the following holds:

$$E_\varepsilon^-(z_k) \cap B(0, r) \subset \Omega_1 \subset E_\varepsilon^+(z_k) \tag{3.8}$$

We aim next to show that:

$$\lim_{k\to\infty}\frac{K_\Omega(z_k)}{C_\Omega(z_k)} \leq \lim_{k\to\infty}\frac{K_{E_\varepsilon^-(z_k)}(a_k,0,\cdots,0)}{C_{E_\varepsilon^+(z_k)}(a_k,0,\cdots,0)}$$

for some $a_k < 0$.

By Proposition 3.4.1 and Theorem 3.3.1 we see that as $k \to \infty$

$$\frac{K_\Omega(z_k)}{K_{\Omega\cap B(\zeta_0,A)}(z_k)} \to 1 \text{ and } \frac{C_\Omega(z_k)}{C_{\Omega\cap B(\zeta_0,A)}(z_k)} \to 1.$$

Hence,

$$\lim_{k\to\infty}\frac{K_\Omega(z_k)}{C_\Omega(z_k)} = \lim_{k\to\infty}\frac{K_{\Omega\cap B(\zeta_0,A)}(z_k)}{C_{\Omega\cap B(\zeta_0,A)}(z_k)} \tag{3.9}$$

Since the change of coordinates was done locally through a biholomorphic mapping we have the following equality:

$$\lim_{k\to\infty}\frac{K_{\Omega\cap B(\zeta_0,A)}(z_k)}{C_{\Omega\cap B(\zeta_0,A)}(z_k)} = \lim_{k\to\infty}\frac{K_{\Omega_1}(a_k,0,\cdots,0)}{C_{\Omega_1}(a_k,0,\cdots,0)} \tag{3.10}$$

for some $a_k < 0$. Now, by (3.8) and Property 2 (section 2.4) the following holds:

$$\lim_{k\to\infty}\frac{K_{\Omega_1}(a_k,0,\cdots,0)}{C_{\Omega_1}(a_k,0,\cdots,0)} \leq \lim_{k\to\infty}\frac{K_{E_\varepsilon^-(z_k)\cap B(0,r)}(a_k,0,\cdots,0)}{C_{E_\varepsilon^+(z_k)}(a_k,0,\cdots,0)} \tag{3.11}$$

Finally, applying Theorem 3.3.1 again on $E_\varepsilon^-(z_k)$ for the sequence $(a_k, 0, \cdots, 0)$ converging to 0, we get that:

$$\lim_{k\to\infty} K_{E_\varepsilon^-(z_k)\cap B(0,r)}(a_k, 0, \cdots, 0) = \lim_{k\to\infty} K_{E_\varepsilon^-(z_k)}(a_k, 0, \cdots, 0) \tag{3.12}$$

Therefore, by (3.9), (3.10), (3.11) and (3.12) we get the desired result:

$$\lim_{k\to\infty} \frac{K_\Omega(z_k)}{C_\Omega(z_k)} \leq \lim_{k\to\infty} \frac{K_{E_\varepsilon^-(z_k)}(a_k, 0, \cdots, 0)}{C_{E_\varepsilon^+(z_k)}(a_k, 0, \cdots, 0)}$$

Clearly, $1 \leq \dfrac{K_\Omega(z_k)}{C_\Omega(z_k)}$ for $k \geq 0$ (Property 1 in section 2.4). Since the ellipsoids are biholomorphically equivalent to the open unit ball $\mathbb{B}^n$ and given the fact that $K_{\mathbb{B}^n}(0) = C_{\mathbb{B}^n}(0) = 1$ we see that:

$$\lim_{k\to\infty} \frac{K_{E_\varepsilon^-(z_k)}(a_k, 0, \cdots, 0)}{C_{E_\varepsilon^+(z_k)}(a_k, 0, \cdots, 0)} = 1$$

Hence,

$$\lim_{k\to\infty} \frac{K_\Omega(z_k)}{C_\Omega(z_k)} = 1$$

By Remark 3.4.1 the proof of Theorem 3.1.1 is now complete.

Chapter 4

Proof of Wong-Rosay Theorem By Pinchuk's Scaling Method

In 1991, S. Pinchuk [7] introduced a powerful technique in complex analysis known as the scaling method. The scaling method can be seen as a standard tool in differential geometry which consists in flattening and localizing a domain Ω near a given boundary point. In case $\Omega = \{\Re(z_0) > |z_1|^2 + \cdots + |z_n|^2 +$ higher order terms$\}$ the main idea of this method is to make use of the group of automorphisms of the unbounded ball $\{\Re(z_0) > |z_1|^2 + \cdots + |z_n|^2\}$, and more precisely the set of anisotropic dilatations $(z_0, z_1, \cdots, z_n) \to (\lambda z_0, \sqrt{\lambda} z_1, \cdots, \sqrt{\lambda} z_n)$, where $\lambda \in \mathbb{R}^+ \setminus \{0\}$. In this chapter, we will present Pinchuk's scaling method and, following K.-T. Kim and S.G. Krantz [4] and F. Berteloot [1] we discuss a new proof of Wong-Rosay theorem.

We will restate Wong-Rosay theorem as follows:

Theorem 4.0.1 (Wong-Rosay Theorem)**.** *let Ω be a bounded domain of $\mathbb{C}^{n+1}$ with C^2 boundary and let $\zeta_0 \in \partial\Omega$ be a point of strict pseudoconvexity. Assume that there exist a point $x \in \Omega$ and a sequence (T_k) of automorphisms of Ω such that $T_k(x)$ converges to ζ_0. Then Ω is biholomorphically equivalent to the open unit ball.*

proof by Pinchuk's scaling method. Before proceeding by the scaling method we shall get everything set for the implementation of the method. Hence, the proof of the theorem is divided into four main steps: (1) preparation, (2) localization, (3) dilatation and (4) synthesis.

Step 1. Preparation. Clearly, we may assume that ζ_0 is in the origin of $\mathbb{C}^{n+1}$ and $n = (-1, 0, \cdots, 0)$ is the unit outward normal at ζ_0. Thus, by the proof of Proposition 2.1.1, the Taylor expansion of the defining function ρ about $\zeta_0 = 0$ is

$$\rho(\zeta) = 2\Re\Big\{ -\zeta_0 + \frac{1}{2}\sum_{i,j=0}^{n} \frac{\partial^2\rho}{\partial z_i \partial z_j}(\zeta_0)\zeta_i\zeta_j \Big\} + \sum_{i,j=0}^{n} \frac{\partial^2\rho}{\partial z_i \partial \overline{z}_j}(\zeta_0)\zeta_i\overline{\zeta}_j + o(\|\zeta\|^2)$$

Note that the matrix $A = \left(\frac{\partial^2\rho}{\partial z_i \partial \overline{z}_j}(\zeta_0)\right)_{i,j}$ is the complex Hessian of ρ at ζ_0. Consequently A is positive definite. By Cholesky decomposition, we have that $A = L\overline{L}^{\intercal}$ where L is a lower triangular matrix with strictly positive diagonal entries. Thus

$$\sum_{i,j=0}^{n} \frac{\partial^2 \rho}{\partial z_i \partial \overline{z}_j}(\zeta_0)\zeta_i\overline{\zeta}_j = \zeta A \overline{\zeta}^{\intercal} = \zeta L \overline{L}^{\intercal}\overline{\zeta}^{\intercal} = \big(\zeta L\big)\big(\overline{\zeta L}\big)^{\intercal}$$

Now, we perform the holomorphic coordinate change at the origin $\zeta = (\zeta_0, \cdots, \zeta_n) \mapsto z = (z_0, \cdots, z_n)$ given by

$$z_0 = -\frac{1}{2}\Big\{ -\zeta_0 + \frac{1}{2}\sum_{i,j=0}^{n} \frac{\partial^2 \rho}{\partial z_i \partial z_j}(\zeta_0)\zeta_i\zeta_j \Big\}$$

$$z_k = \big(\zeta L\big)_k \qquad \text{for } 1 \leq k \leq n$$

Hence, for some $r > 0$ we may assume that the set $\Omega \cap B(0, 10r)$, where $B(0, 10r)$ is the open Euclidean ball of radius $10r$ centered at the origin, is defined by

$$\rho(z) < 0$$

where

$$\rho(z) = -\Re z_0 + |z_1|^2 + \cdots + |z_n|^2 + R(z)$$

with

$$R(z) = O\big(\|(z_1, \cdots, z_n)\|^3\big) + O\big(\,|\Im z_0|\,\|(z_1, \cdots, z_n)\|\big) + O\big(\,|\Im z_0|^2\,\big)$$

Note that for $r > 0$ small enough the boundary $\partial\Omega$ is strictly convex in $B(0, 2r)$.

For simplicity of the notation we will use the following coordinates $z = (z_0, z_\alpha)$ in $\mathbb{C}^{n+1}$ with

$$z_\alpha = (z_1, \cdots, z_n).$$

Step 2. Localization. Under the assumptions of the previous step we see that there exists a function $P : B(0, 2r) \to \mathbb{C}$ such that

$$P(0) = 1 \text{ and } |P(z)| < 1 \text{ for every } z \in \overline{\Omega} \cap B(0, 2r) \setminus \{0\}$$

Note that the existence of such a map is given in the proof of Proposition 3.3.1.

Now, since Ω is bounded, Theorem 2.5.1 guarantees that every subsequence (T_{k_l}) of (T_k) has a uniformly convergent subsequence on compact subsets of Ω. Let T be a subsequential limit of (T_{k_l}). Then $T : \Omega \to \overline{\Omega}$ is holomorphic and $T(x) = 0$. Hence, there exists an open neighborhood U (relatively compact) of x such that $T(U) \subset B(0, 2r) \cap \Omega$. By uniform convergence of (T_{k_l}) - up to subsequence extraction - to T on compact subsets we may assume that $T_{k_l}(U) \subset B(0, 2r) \cap \Omega$ for k_l sufficiently large. Consider now the holomorphic map $P \circ T_{k_l}|_U : U \to \mathbb{D}$ where $\mathbb{D}$ is the open unit disc. Clearly, $P \circ T(x) = 1$ and $|P \circ T(z)| < 1$ for all $z \in U$. Hence, by the *Maximum Modulus Principle* $T(z)$ is constant for all $z \in U$, more precisely $T(z) = 0$ on U. Since U contains a non-empty open set, the *Identity Theorem* implies that $T \equiv 0$ on Ω. Therefore, T_k converges uniformly to $T \equiv 0$ on compact subsets of Ω *i.e.* for every compact subset K of Ω there exists

$n \in \mathbb{N}^*$ such that

$$T_k(K) \subset B(0, r) \cap \Omega \tag{4.1}$$

for every $k \geq n$.

Step 3. Dilatation. Consider now the sequence $T_k(x)$ in Ω and let $T_k(x) = q^k$ for each k. Choose next the boundary point p^k that is the nearest to q^k in the normal direction of $\partial\Omega$ *i.e.* $p^k = (p_0^k, p_\alpha^k) \in \partial\Omega$ such that

(i) $q_0^k - p_0^k \in \mathbb{R}^+ \setminus \{0\}$

(ii) $p_\alpha^k = q_\alpha^k$

for every $k \geq 1$. We set $\lambda_0^k = q_0^k - p_0^k > 0$ (for convenience we took $\Im(p_0^k) = \Im(q_0^k)$).

Pinchuk's scaling method consists of two main procedures: centering and dilatation processes. The centering process is given by the sequence of maps $A_k : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ defined in local coordinates by $A_k(z) = w$ with $w = (w_0, w_\alpha)$ given by

$$w_0 = (z_0 - p_0^k) - c^k(z_\alpha - p_\alpha^k)$$
$$w_\alpha = z_\alpha - p_\alpha^k$$

where the complex constant c^k is chosen such that $c^k \to 0$ as $k \to \infty$ and such that the domain $\Omega_k = A_k(\Omega)$ is represented in the neighborhood of the origin by

$$\Re w_0 > \Psi_k(\Im(w_0), w_\alpha) \tag{4.2}$$

satisfying

$$\Psi_k(0,0) = 0 \tag{4.3}$$

and

$$\nabla\Psi_k|_0 = (0,0) \tag{4.4}$$

We see that (A_k) is a sequence of translations and complex rotations mapping p^k to the origin and q^k to the point $(\lambda_0^k, 0)$ for every k. Let us examine now how the domain Ω_k looks locally in the new coordinates. We aim then to compute the new defining function $\rho_k(w) = \rho(A_k^{-1}(w))$. Note that A_k^{-1} is given by $A_k^{-1}(w) = z$ with

$$z_0 = w_0 + c^k w_\alpha + p_0^k$$

$$z_\alpha = w_\alpha + p_\alpha^k$$

Hence,

$$
\begin{aligned}
\rho_k(w) &= \rho(A_k^{-1}(w)) \\
&= -\Re\Big\{ w_0 + c^k w_\alpha + p_0^k \Big\} + \left|w_\alpha + p_\alpha^k\right|^2 + R\big(A_k^{-1}(w)\big) \\
&= -\Re(w_0) - \Re(c^k w_\alpha) - \Re(p_0^k) + \left|w_\alpha + p_\alpha^k\right|^2 + R\big(A_k^{-1}(w)\big) \\
&= -\Re(w_0) - \Re(c^k w_\alpha) - \Re(p_0^k) + |w_\alpha|^2 + \left|p_\alpha^k\right|^2 + 2\Re(\overline{p_\alpha^k} w_\alpha) + R\big(A_k^{-1}(w)\big)
\end{aligned}
$$

In order to determine c^k we note that Ψ_k in (4.2) is given here by

$$\Psi_k(\Im(w_0), w_\alpha) = -\Re(c^k w_\alpha) - \Re(p_0^k) + \left|w_\alpha + p_\alpha^k\right|^2 + R\big(A_k^{-1}(w)\big)$$

Therefore

$$(4.3) \Rightarrow \Re(p_0^k) = \left|p_\alpha^k\right|^2 + R(p^k)$$

and

$$(4.4) \Rightarrow c^k = 2\overline{p_\alpha^k} + r^k$$

with $r^k \to 0$ as $k \to \infty$. Substituting these results in the expression of $\rho_k(w)$ yields to the following result

$$\rho_k(w) = -\Re(w_0) + |w_\alpha|^2 + R(\beta^k w)$$

where $\beta^k \to 1$ as $k \to \infty$.

Next, we consider the dilatation sequence given by the linear maps $L_k : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ defined by

$$L_k(w_0, w_\alpha) = (\tilde{w}_0, \tilde{w}_\alpha) = \left(\frac{w_0}{\lambda_0^k}, \frac{w_\alpha}{\sqrt{\lambda_0^k}} \right)$$

for all $k \geq 1$.

Note that for each k, L_k stretches the coordinates and sends $A_k(q^k)$ to the point $(1, 0)$. We may examine now Pinchuk's dilatation sequence $\Lambda_k = L_k \circ A_k$. For some $r > 0$, the sequence of domains $\Lambda_k(\Omega \cap B(0, r))$ is defined by

$$\begin{aligned}
\frac{1}{\lambda_0^k}\tilde{\rho}_k(\tilde{w}) &= \frac{1}{\lambda_0^k}\rho(A_k^{-1} \circ L_k^{-1})(\tilde{w}) \\
&= \frac{1}{\lambda_0^k}\rho_k(L_k^{-1}(\tilde{w})) \\
&= \frac{1}{\lambda_0^k}\rho_k\left((\lambda_0^k \tilde{w}_0, \sqrt{\lambda_0^k}\tilde{w}_\alpha) \right) \\
&= -\Re(\tilde{w}_0) + |\tilde{w}_\alpha|^2 + R(\beta^k \sqrt{\lambda_0^k}\tilde{w})
\end{aligned}$$

After dropping the $\sim$'s and substituting $\rho_k = \frac{1}{\lambda_0^k}\tilde{\rho}_k$ we have the form

$$\rho_k(w) = -\Re(w_0) + |w_\alpha|^2 + R(\beta^k \sqrt{\lambda_0^k} w)$$

Clearly as $k \to \infty$, $\lambda_0^k \to 0$. Hence ρ_k converges uniformly on compact subsets to $\rho(w) = -\Re(w_0) + |w_\alpha|^2$ as $k \to \infty$. Consequently, the sequence of domains

$\Lambda_k(\Omega \cap B(0,r))$ converges to

$$E = \{(z_0, z_1, \cdots, z_n) \in \mathbb{C}^n + 1 : \Re(z_0) \geq |z_1|^2 + \cdots + |z_n|^2\}$$

Moreover, we may assume that for k large enough we have

$$\Lambda_k(\Omega \cap B(0,r)) \subset \tilde{E}$$

with

$$\tilde{E} = \{(z_0, z_1, \cdots, z_n) \in \mathbb{C}^n + 1 : \Re(z_0) \geq \frac{1}{2}(|z_1|^2 + \cdots + |z_n|^2)\}.$$

Step 4. Synthesis. Consider a sequence of relatively compact subdomains W_k of Ω such that

$$\overline{W_l} \subset W_{l+1} \;\; l \geq 1$$

and

$$\bigcup_{l=1}^{\infty} W_l = \Omega$$

Let (σ_k) be the scaling sequence given by $\sigma_k = \Lambda_k \circ T_k : \Omega \to \Lambda_k(\Omega)$. By (4.1) we may assume that for k large enough

$$T_k(W_l) \subset \Omega \cap B(0,r)$$

for every $l \geq 1$. Note that by the preceding step we see that for every l

$$\sigma_k(W_l) \subset \Lambda_k(\Omega \cap B(0,r)) \subset \tilde{E}$$

as $k \to \infty$. Hence for every l, (σ_k) forms a normal family for k large enough and we may assume that σ_k converges uniformly on compact subsets of Ω to a holomorphic mapping $\sigma : \Omega \to E$.

It remains to show that σ is actually a biholomorphism between Ω and V. Let $\psi_k : \Lambda_k \to \Omega$ be the inverse of σ_k for each k. Since Ω is bounded, Theorem 2.5.1 implies that - up to subsequence extraction - ψ_k converges uniformly to the holomorphic mapping $\psi : E \to \overline{\Omega}$ on compact subset. Consequently, ψ is the inverse of σ and σ is a biholomorphism from Ω to V. Since E is biholomorphic to the open unit ball, the theorem is now proved. □

Chapter 5

Characterization of The Unit Ball by Stanton

In this part of the thesis we care about other important invariants of domains, namely invariant metrics. Carathéodory and Kobayashi both introduced infinitesimal invariant metrics in order to generalize the Poincaré metric of the unit disc in higher dimensions. Although the Carathéodory and Kobayashi metrics coincide on the unit ball, this is not the case in general. Hence, one might question whether the unit ball is the only domain with such characterization. Using a negative curvature argument, B. Wong [12] obtained a metric characterization of the unit ball. In 1983, C.M. Stanton [10] improved significantly the previous result. In this chapter we will present Stanton's characterization of the unit ball.

5.1 Main Theorem

Theorem 5.1.1 (Stanton Theorem)**.** *Let M be a connected complete hyperbolic complex manifold of complex dimension n. Assume that there exists a point p of M at which the Carathéodory and Kobayashi metrics are equal and that one of these metrics is hermitian and of class C^∞. Then M is biholomorphically equivalent to the open unit ball in $\mathbb{C}^n$.*

5.2 Proof of The Main Theorem

Throughout this section we assume that the given of Theorem 5.1.1 holds and we take the Kobayashi metric to be the Hermitian metric of class C^∞ on M. We proceed by proving some results that are useful for the proof of the theorem.

Proposition 5.2.1. *Let M be a complete hyperbolic complex manifold such that its Kobayashi and Carathéodory metrics coincide at some point $p \in M$. Let $v \in T_pM$ and let f be an extremal disc for v (f exists by Corollary 2.5.2), then f is a distance preserving isometry relative the Poincaré metric in $\mathbb{D}$ and the Kobayashi/Carathéodory metric in M.*

Proof. Assume that $d_0f(\partial/\partial x) = rv$ for some $r > 0$. Since f is an extremal disc for v, $K_M(p, v) = \frac{1}{r}$. Knowing that the Kobayashi and Carathéodory metrics coincide at p we conclude that $C_M(p, v) = \frac{1}{r}$. Next, we define $g : M \to \mathbb{D}$ to be an extremal function for v (g exists by Corollary 2.5.1). Thus $C_M(p, v) = |d_pg(v)|$

and consequently $\frac{1}{r} = |d_p g(v)|$. By the chain rule we get

$$\frac{1}{r} = |d_p g(v)| = |d_p g(d_0 f((\partial/r\partial x))| = \left|d_{f(0)} g(d_0 f(\partial/r\partial x))\right| = \frac{1}{r} |d_0(g \circ f)| \quad (5.1)$$

Note that $g \circ f : \mathbb{D} \to \mathbb{D}$ is a holomorphic map with $(g \circ f)(0) = 0$, then by Schwarz lemma $|d_0(g \circ f)| \leq 1$ with equality holds if $g \circ f$ is an isometry. By (5.1) we conclude that $g \circ f$ is a distance preserving isometry of the open unit disc. However, the *distance decreasing property* of the Kobayashi and the Carathéodory metrics (Proposition 2.3.1) tells us that f and g can not increase distances. Therefore, f is a distance preserving isometry relative the Poincaré metric in $\mathbb{D}$ and the Kobayashi/Carathéodory metric in M. □

Remark 5.2.1. *Since the kobayahsi metric defined on M in Theorem 5.1.1 is assumed to be Hermitian, the tangent space T_pM of M at p has the structure of an n-dimensional complex inner product space. Moreover, for all $v \in T_pM$ the Kobayashi metric is given by $K_M(p, v) = \overline{v}^{\intercal} A_p v$, where A_p is a Hermitian matrix.*

Denoting by $\mathbf{B} = \{v \in T_pM : \overline{v}^{\intercal} A_p v < 1\}$ the open unit ball relative to the Kobayashi metric in T_pM, We will prove Theorem 5.1.1 for $\mathbf{B}$ instead of the open unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$ *i.e.* we aim to show that under the assumptions of the theorem M is is biholomorphically equivalent to $\mathbf{B}$. For this purpose we will prove the following proposition.

Proposition 5.2.2. *The open unit ball $\boldsymbol{B}$ in T_pM is biholomorphically equivalent* *to the open unit ball* $\mathbb{B}^n$ *in* $\mathbb{C}^n$.

Proof. First recall that $\mathbf{B}$ and $\mathbb{B}^n$ are defined as follows

$$\mathbb{B}^n = \{v \in \mathbb{C}^n : \|v\|^2 < 1\} \text{ and } \mathbf{B} = \{v \in T_pM : \overline{v}^\intercal A_p v < 1\}$$

where $\|.\|$ is the Euclidean norm. Since A_p is a Hermitian matrix it follows that $A_p = U^{-1}D_pU$, where U is a unitary matrix and

$$D_p = \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{pmatrix}$$

is a diagonal matrix with real diagonal entries. Furthermore $d_i > 0$ for all $1 \leq i \leq n$. This can be seen by noting that the d_i's are the eigenvalues of A_p, hence for all $1 \leq i \leq n$ we have

$$A_p v = d_i v$$

$$\overline{v}^\intercal A_p v = \overline{v}^\intercal d_i v$$

$$\overline{v}^\intercal A_p v = d_i \|v\|^2$$

$$K_M(p, v) = d_i \|v\|^2$$

Thus $d_i > 0$ (for $v \neq 0$).

Next, we define $\phi : T_pM \to \mathbb{C}^n$ to be the holomorphic map $v \mapsto \phi_1 \circ \phi_2(v)$

where

$$\phi_1 : \mathbb{C}^n \to \mathbb{C}^n$$

$$\phi_1(v) = \left(\sqrt{d_1}v_1, \sqrt{d_2}v_2, \cdots, \sqrt{d_n}v_n\right)^{\intercal}$$

with $v = \left(v_1, v_2, \cdots, v_n\right)^{\intercal}$, and

$$\phi_2 : T_pM \to \mathbb{C}^n$$

$$\phi_2(v) = Uv$$

Let us show that ϕ maps bijectively $\mathbf{B}$ to $\mathbb{B}^n$. For any $v \in \mathbf{B}$ the following holds

$$\begin{aligned}
\overline{v}^{\intercal} A_p v &< 1 \\
\overline{v}^{\intercal}\,(U^{-1} D\, U)\, v &< 1 \\
\overline{v}^{\intercal}\,(\overline{U}^{\intercal} D\, U)\, v &< 1 \\
(\overline{Uv})^{\intercal} D\,(Uv) &< 1 \\
(\overline{\phi_2(v)})^{\intercal} D\, \phi_2(v) &< 1 \\
\sum_{i=1}^{n} \|\sqrt{d_i}(\phi_2(v))_i\|^2 &< 1 \\
\|\phi_1 \circ \phi_2(v)\|^2 &< 1 \\
\|\phi(v)\|^2 &< 1
\end{aligned}$$

Thus ϕ maps $\mathbf{B}$ to $\mathbb{B}^n$. Clearly, ϕ_1 and ϕ_2 are invertible and consequently so is

ϕ. Therefore ϕ is biholomorphic. $\square$

We start the proof of Theorem 5.1.1 by defining the following map $\Phi : \mathbf{B} \to M$

$$\Phi(v) = exp_p\left[\frac{\tanh^{-1}|v|}{|v|}v\right],\ \ v \in \mathbf{B}$$

where exp_p is the exponential map at p relative to the Kobayashi metric and $|.| = K_M(p, .)$.

Note that

$$\frac{\tanh^{-1}|v|}{|v|} = \frac{1}{|v|}\sum_{n=1}^{\infty}\frac{|v|^{2n-1}}{2n-1} = \sum_{n=1}^{\infty}\frac{|v|^{2n-2}}{2n-1} = \sum_{k=0}^{\infty}\frac{|v|^{2k}}{2k+1}$$

Hence, $\dfrac{\tanh^{-1}|v|}{|v|}$ is an analytic function of $|v|^2$ and consequently Φ is a C^∞ mapping.

Clearly, showing that Φ is a biholomorphic map will finish the proof of Theorem 5.1.1. We start by showing that it is holomorphic on $\mathbf{B}$.

Lemma 5.2.1. Φ *is holomorphic on each complex line through the origin of* $\mathbf{B}$.

Proof. Let u be a unit vector in T_pM. We define the restriction of Φ to the complex line through the origin of $\mathbf{B}$ determined by u to be the map $G : \mathbb{D} \to M$

such that

$$G(\zeta) = \Phi(\zeta u) = exp_p\left[\frac{\tanh^{-1}|\zeta|}{|\zeta|}\zeta u\right], \ \ \zeta \in \mathbb{D}$$

where $\mathbb{D}$ is the open unit disc in $\mathbb{C}$. By Remark 2.2.3 and the fact that u is a unit vector, we see that $G(\zeta)$ is the point on the geodesic arc through p tangent to ζu such that the length of the geodesic arc from p to $G(\zeta)$ is equal to $\tanh^{-1}|\zeta|$. Let $H : \mathbb{D} \to M$ be an extremal disc for u such that $H(0) = q$. By Proposition 5.2.1 H is a distance preserving isometry from $\mathbb{D}$ into M. Thus, H maps geodesics in $\mathbb{D}$ to geodesics in M and consequently $H(\zeta)$ lies on the geodesic arc through p tangent to ζu. Since geodesics on M are (locally) shortest paths between points, the length of this geodesic from p to $H(\zeta)$ is the same as the distance between p and $H(\zeta)$ which is equal to $\tanh^{-1}|\zeta|$ (since H is an isometry). Therefore, $G(\zeta) = H(\zeta)$ for all $\zeta \in \mathbb{D}$, and consequently G is holomorphic. □

We will show now that Φ is holomorphic on the entire unit ball in T_pM.

Lemma 5.2.2. Φ *is holomorphic on* $\mathbf{B}$.

Proof. Let us define two maps that will be essential in the proof of the Lemma. The first map $\varphi : M \to [0, +\infty)$ is given by

$$\varphi(q) = \frac{1}{1 - \tanh^2 d_M(p, q)}, \ \ q \in M$$

where d_M is the Kobayashi distance, and the second map $\psi : \mathbf{B} \to [0, +\infty)$ is given by

$$\psi(v) = \frac{1}{1 - |v|^2}, \;\; v \in \mathbf{B}.$$

Note that

$$\varphi \circ \Phi(v) = \frac{1}{1 - \tanh^2 d_M(q, \phi(v))} = \frac{1}{1 - \tanh^2\left(\frac{\tanh^{-1}|v|}{|v|}\,|v|\right)} = \frac{1}{1 - |v|^2} = \psi(v)$$

where the second equality holds by Remark 2.2.3.

Claim 5.2.1. *ψ is a strictly plurisubharmonic function in $\boldsymbol{B}$.*

Proof. We showed in Proposition 5.2.2 that $\mathbf{B}$ is biholomorphically equivalent to $\mathbb{B}^n$ with $\phi : \mathbf{B} \to \mathbb{B}^n$ being the biholomorphic mapping. Thus, by invariance of the Levi form $i\partial\overline{\partial}$ under biholomorphism, it is enough to show that $\psi \circ \phi^{-1} : \mathbb{B}^n \to [0, +\infty)$ is strictly plurisubharmonic *i.e.* its complex Hessian is positive definite. By reversing the computation done in the proof of Proposition 5.2.2 one can easily see that for all $v \in \mathbb{B}^n$ we have

$$\psi \circ \phi^{-1}(v) = \frac{1}{1 - \|v\|^2}.$$

A simple computation shows that the complex Hessian of $\psi \circ \phi^{-1}$ is given by

$$H = \frac{1}{(1 - \|v\|^2)^3}\left[I_n + 2V\right]$$

where $V = \overline{v}v^{\intercal}$ is the $n \times n$ matrix $(\overline{v}_i v_j)_{ij}$.

Since $\frac{1}{(1-\|v\|^2)^3} > 0$ for all $v \in \mathbb{B}^n$ we will omit this factor from the next computation. For all $w^\intercal \in \mathbb{C}^n$ the following holds

$$\begin{aligned}
\overline{w} H\, w^\intercal &= \overline{w} \Big[I_n + 2V \Big]\, w^\intercal \\
&= \overline{w} w^\intercal + 2\, \overline{w}\, V\, w^\intercal \\
&= \|w\|^2 + 2\, \overline{w}\, \overline{v} v^\intercal w^\intercal \\
&= \|w\|^2 + 2\, \overline{(wv)}\, (wv)^\intercal \\
&= \|w\|^2 + 2\, \|w.v\|^2 \\
&\geq 0
\end{aligned}$$

with the last equality is strict for all $w \neq 0$. Therefore H is positive definite and the proof of the claim is now complete. □

For each $a > 0$, let

$$B_a = \{v \in \mathbf{B} : \psi(v) < a\}$$

It is easy to see that for all $a_1 < a_2 < a_3 < \cdots$ we have $B_{a_1} \subset B_{a_2} \subset B_{a_3} \subset \cdots \subset \mathbf{B}$. Now, let

$$A = \sup\{a : \Phi \text{ is holomorphic on } B_a\}.$$

Showing that $A = \infty$ finishes the proof of the lemma.

We start by showing that $A > 0$. Let $U \subset M$ be an open neighborhood of p

with a holomorphic coordinate system $(z_1.\cdots, z_n)$ on U. Then, there exists $a > 0$ such that $\Phi(B_a) \subset U$ and consequently each of the functions $z_k \circ \Phi : B_a \to \mathbb{C}$ is holomorphic on complex lines through the origin of B_a and of class C^∞. The following version of the classical Forelli's theorem will allow us to conclude that $z_k \circ \Phi$ is holomorphic on B_a and consequently $A > 0$.

Theorem 5.2.1. *[9] Let $\mathbb{B}^n$ be the open unit ball in $\mathbb{C}^n$, and suppose $f : \mathbb{B}^n \to \mathbb{C}$ statisfies*

1. *$f \in C^\infty(\{0\})$*

2. *all slice functions f_z are holomorphic.*

Thus $f \in H(\mathbb{B}^n)$.

Next, we aim to show that if Φ is holomorphic on B_a for some $a > 0$, then it is holomorphic on $B_{a+\epsilon}$ for some $\epsilon > 0$. Hence, assuming that Φ is holomorphic on B_a (of class C^∞) we have that

$$\Phi^*(i\partial\overline{\partial}\varphi) = i\partial\overline{\partial}(\varphi \circ \Phi) = i\partial\overline{\partial}(\psi)$$

By continuity, the above equality is true on $cl(B_a)$, the closure of B_a. By Claim 5.2.1 and by invariance of the Levi form under pull-back we see that $(i\partial\overline{\partial}\varphi)$ is positive definite on $\Phi(cl(B_a))$. Again, by continuity we have that the previous statement is true in a neighborhood W of $\Phi(cl(B_a))$. Note that

$$B_a = \psi^{-1}(-\infty, a) = (\varphi \circ \Phi)^{-1}(-\infty, a) = \Phi^{-1} \circ \varphi^{-1}(-\infty, a) = \Phi^{-1}(H_a)$$

where $H_a = \{q \in M : \varphi(q) < a\}$. Thus Φ maps B_a into the set $\{q \in M : \varphi(q) < a\}$. Hence, by continuity, $w = \{q \in M : \varphi(q) < a + \varepsilon\}$ for some $\varepsilon > 0$. Since $(i\partial\overline{\partial}\varphi)$ is positive definite on W, we conclude that W is a Stein manifold and consequently W is locally endowed with a holomorphic coordinate system $(z_1, \cdots, z_n)$. Now, for some $\epsilon > 0$, $\Phi(B_{a+\epsilon}) \subset W$. Repeating the same argument done previously to show that $A > 0$, we conclude that $z_k \circ \Phi$ is holomorphic on $B_{a+\epsilon}$ and consequently so is Φ.

The last step in the proof of the lemma is to show that $A = \infty$. Assume by contradiction that $A < \infty$ *i.e.* Φ is holomorphic on B_A and A is the largest such number. However, we showed that if Φ is holomorphic on B_A then it will be holomorphic on $B_{A+\epsilon}$ for some $\epsilon > 0$. Thus $A = \infty$ and Φ is holomorphic on $\mathbf{B}$. □

Lemma 5.2.3. *Φ is bijective.*

Proof. For each $a > 0$ we define the level sets $S_a = \{v \in \mathbf{B} : \psi(v) = a\} \subset \mathbf{B}$ and $M_a = \{q \in M : \varphi(q) = a\} \subset M$. One can easily see that $\mathbf{B} = \bigsqcup_{a>0} S_a$ and $M = \bigsqcup_{a>0} M_a$ where $\bigsqcup$ is the disjoint union.

Since $\psi = \varphi \circ \Phi$, Φ maps S_a into M_a. Hence, it is enough to show that the restriction of Φ to S_a, $\Phi|_{S_a}$ is bijective. We start by observing that Φ - given by the exponential map - is a diffeomorphism from a neighborhood of the origin of $\mathbf{B}$ to a neighborhood of p in M. Hence, for all a close enough to 1, $\Phi|_{S_a}$ maps S_a diffeomorphically to M_a. By Proposition 2.2.1 we see that differential of $\Phi|_{S_a}$ preserves orientations in a neighborhood of the origin of $\mathbf{B}$. Thus, $\Phi|_{S_a}$ is an orientation preserving diffeomorphism for a near 1 and consequently it has degree 1 for a near 1. By connectedness, the degree of $\Phi|_{S_a}$ is 1 for all a. Being also an orientation preserving map, $\Phi|_{S_a}$ is a diffeomorphism for all a. The proof of Theorem 5.1.1 is now complete. $\square$

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