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RANDOM WALKS ON PROJECTIVE SPACES

by
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An Abstract of the Thesis of

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Our goal in this thesis is to understand a recent result of Benoist-Quint [BQ14] about the classification of stationary measures on the real projective space $\mathbb{P}(\mathbb{R})^d$. More precisely, consider a probability measure μ on the general linear group $\mathrm{GL}_d(\mathbb{R})$ such that Γ_μ, the semi-group generated by the support of μ, is strongly irreducible. We aim to construct μ-stationary measures on $\mathbb{P}(\mathbb{R}^d)$ using random walks, by proving the existence of limits for the empirical measures. For this purpose, and inspired by previous work of Raugi [Rau94], we introduce a Markov Feller operator P_μ and prove it to be equicontinuous using the theory of random matrix products [BL85].

Contents

Chapter 1

Introduction

The study of products of random matrices and random walks on groups in general has been going on since the beginning of the twentieth century and has enormous applications in mathematical physics, expander graphs, cryptography, geometric group theory $\cdots$. One of the main interests is the question of the asymptotic behavior of the random walks $\{L_n = X_n \cdots X_1\}_{n\geq 1}$ and $\{R_n = X_1 \cdots X_n\}_{n\geq 1}$, where $\{X_n\}_{n\geq 1}$ is a sequence of independent and identically distributed random variables in a group G. When G is a group of $d \times d$ invertible matrices, this problem was addressed by Benoist and Quint [BQ16], Bougerol [BL85], Furstenberg [Fur63], Kifer [FK83], Goldsheid [GG96], Guivarc'h [Gui90], Kesten [FK60], Le Page [LP82], Margulis [GM89], Raugi [Rau94], Tutubalin [Tut65], Viscer [Vir70] and others. The question of the asymptotic behavior of $||X_n \cdots X_1 v||$ for any v in $\mathbb{R}^d$ arises, for example, in solutions of differential equations with random coefficients ([BL85]). Also, it appeared as a fundamental tool in studying properties of subgroups of the linear group; for example in proving the existence of free non-abelian subgroups of the general linear group (the probabilistic proof of the Tits alternative by Guivarc'h [Gui90]) and in homogenous dynamics (we refer for instance to the powerful results of Benoist-Quint [BQ13]). According to Furstenberg [Fur63], one way to understand the behavior of a random matrix product is to consider stationary measures on the $(d-1)$ projective space.

In this thesis, our goal is to describe how one can construct probability measures on the projective space that are invariant with respect to a probability measure on the general linear group $GL_d(\mathbb{R})$ (we refer to Definition to 2.4.3) using random walks. Such a study was conducted previously by Guivarc'h-Raugi [GR07], and Benoist-Quint [BQ14]. We follow in this thesis the approach of Benoist-Quint. Consider a probability measure μ on $GL_d(\mathbb{R})$. Let Γ_μ denote the smallest closed semigroup of $GL_d(\mathbb{R})$ containing the support of μ and consider the canonical action of Γ_μ on $X = \mathbb{P}(\mathbb{R}^d)$. When the action of Γ_μ on $\mathbb{R}^d$ is strongly irreducible (see Definition 2.2.2), we prove the following results that we summarize in the following theorem:

Theorem 1.0.1. *[BQ14, Theorem 1.1 and Theorem 1.3]*
For every $x \in X$, for $\beta := \mu^{\otimes \mathbb{N}}$-almost every $b = (b_1, \cdots) \in GL_d(\mathbb{R})^{\mathbb{N}}$, the limit of the empirical probability measures

$$\nu_{x,b} := \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \delta_{b_k \cdots b_1 x}$$

exists and is a μ-ergodic μ-stationary probability measure on X. Moreover, if ν_x is defined by $\nu_x := \int \nu_{x,b} \, d\beta(b)$, then ν_x is μ-stationary, depends continuously on x and

$$\nu_x = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \mu^{*k} * \delta_x.$$

Theorem 1.0.1 describes the asymptotic behavior of the random walk at time n starting at x in $\mathbb{P}(\mathbb{R}^d)$. These results extend previous work of Guivarc'h and Raugi [GR85] who proved that when the action of Γ_μ on $\mathbb{R}^d$ is strongly irreducible and proximal, there exists a unique μ-invariant measure on X, called the "Furstenberg measure".

To prove Theorem 1.0.1, we take a functional analysis point of view. The basic idea is to use an averaging operator P_μ on X defined by

$$P_\mu : C^0(X) \longrightarrow C^0(X); \quad f \longmapsto P_\mu(f) = \int_{\Gamma_\mu} f(gx) d\mu(g).$$

This Markov-Feller operator explicitly describes the random walk $\{L_n\}_n$ on $\mathbb{P}(\mathbb{R}^d)$. In our context, we will think of random walks on a set X as continuous functions that assign to each element of X a probability measure on X. The strong irreducibility of Γ_μ will play an important role in proving that P_μ is equicontinuous. This is the content of Chapter 3. From this, Theorem 1.0.1 follow as special cases of results of Raugi [Rau94] concerning equicontinuous operators on compact metric spaces (Chapter 4 and 5).

Here is the structure of this thesis:

- In Chapter 2, we state the relevant definitions and we prove all the apparatus needed in the next chapters. This chapter will be divided into a linear algebra part and a measure theoretical one. In the first, we give some properties of subgroups of general linear groups: irreducibility, contraction $\cdots$ and relate them to the action on the projective space. In the second, we define and understand stationary measures on X with respect to a probability measure μ on G.

- In Chapter 3, we prove that, when the action of Γ_μ is strongly irreducible, the averaging operator P_μ is equicontinuous. The proof relies on previous results of Furstenberg, Guivarc'h-Raugi on the theory of random matrix products. The main goal is to study the behavior of the random walk suitably normalized.

- In Chapter 4, we study the decomposition of the space of measures on X under the action of an equicontinuous Markov-Feller operator. This is a purely functional analytic section and is valid for any compact metric space X and for any Markov-Feller operator on it.

- In Chapter 5, we prove the main results of this thesis. Since the proof requires some knowledge of the theory of martingales, we dedicate the first part of this chapter to recall the basic notions needed.

Chapter 2

Background

In this chapter, we will state some definitions and properties that will be used throughout this thesis. We focus on the real projective space, subsemigroups of general linear group, definitions of irreducibility and proximality, convergence of measures and μ-stationary measures.

2.1 The Real Projective Space

We recall in this section the notion of the real projective space along with some of its important properties. All of these results, along with their proofs, can be found for instance in [Tu11, Chapter 2].

Proposition/Definition 2.1.1. *Consider the binary relation $\sim$ on $\mathbb{R}^d \backslash \{0\}$ defined by*

$$x \sim y \iff \text{there exists } \lambda \in \mathbb{R}\backslash\{0\} \text{ such that } y = \lambda x.$$

Then $\sim$ is an equivalence relation on $\mathbb{R}^d \backslash \{0\}$ and the quotient space $\mathbb{R}^d/_\sim$ is called the ***real projective space of dimension d-1****, and is denoted by $\mathbb{P}(\mathbb{R}^d)$ (or sometimes $\mathbb{P}^{d-1}$). Moreover, for any $x \in \mathbb{R}^d \backslash \{0\}$, we denote by $\overline{x}$ the equivalence class of x in $\mathbb{P}(\mathbb{R}^d)$.*

Therefore, the real projective space $\mathbb{P}(\mathbb{R}^d)$ is the space of lines in $\mathbb{R}^d$ passing through the origin.

The $(d-1)$-dimensional real projective space can also be seen as the quotient of the unit $(d-1)$-sphere, S^{d-1}, with the antipodal points identified. The resulting projection $p : S^{d-1} \longrightarrow \mathbb{P}(\mathbb{R}^d)$, which is continuous and surjective, provides important topological structures on $\mathbb{P}(\mathbb{R}^d)$, such as the following:

Proposition 2.1.2. *The real projective space $\mathbb{P}(\mathbb{R}^d)$ is a compact and connected topological space.*

Remark 2.1.1. *It is also important to note that the projection mentioned above implies that S^{d-1} is a double covering space of $\mathbb{P}(\mathbb{R}^d)$.*

Proposition 2.1.3. *The real projective space $\mathbb{P}(\mathbb{R}^d)$ is a separable Hausdorff space.*

The real projective space is a metrizable space and we will be using the following distance on $\mathbb{P}(\mathbb{R}^d)$. We refer for instance to [BG06, Proposition 2.8.18].

Proposition/Definition 2.1.4. ***Fubini-Study metric***
Let $||\cdot||$ be the Euclidean norm on $V = \mathbb{R}^d$. Endow $\wedge^2 V$ with a compatible norm. For every $\overline{x} = \mathbb{R}x$ and $\overline{y} = \mathbb{R}y$ in $\mathbb{P}(\mathbb{R}^d)$, define the map $d : \mathbb{P}(\mathbb{R}^d) \times \mathbb{P}(\mathbb{R}^d) \longrightarrow [0; \infty)$ by

$$d(\overline{x}, \overline{y}) = \frac{||x \wedge y||}{||x|| \, ||y||}.$$

Then d is a metric on $\mathbb{P}(\mathbb{R}^d)$ that measures the absolute value of the sine of the angle between the two lines $\mathbb{R}x$ and $\mathbb{R}y$.

Remark 2.1.2. *Since $\mathbb{P}(\mathbb{R}^d)$ is compact and metrizable, this proves Proposition 2.1.3.*

2.2 Subsemigroups of Linear Groups

Definition 2.2.1. *Let G be a group acting on a set X and let Y be a subset of X. We say that Y is* ***stabilized*** *by G if $\forall g \in G$ and $\forall y \in Y$, $g \cdot y \in Y$.*

Definition 2.2.2. *Let G be a group acting on a vector space V.*

1. *G is said to be* ***irreducible*** *(or equivalently, the action of G is irreducible) if no non-trivial subspace of V is stabilized by all the elements of G. We say that G is* ***reducible*** *if it is not irreducible.*

2. *We say that G is* ***strongly irreducible*** *if there does not exist a finite union of nontrivial subspaces stabilized by all the elements of G.*

Example 2.2.1. *Consider $GL_2(\mathbb{R})$ acting naturally on $\mathbb{R}^2$.*

(i) *Let* $G = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in G, a, b \neq 0 \right\}$.

Clearly, G stabilizes $E_1 = \mathbb{R}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ *and* $E_2 = \mathbb{R}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. *Therefore G is reducible.*

(ii) *Let* $H = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, a, b \neq 0 \right\} \bigcup \left\{ \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}, a, b \neq 0 \right\}$. *This is clearly a subgroup of $GL_2(\mathbb{R})$. Let us check that H is irreducible but not strongly irreducible. Indeed, suppose that there exists a proper subspace E of $\mathbb{R}^2$ stabilized by H. Necessarily E is a one-dimensional subspace, i.e.* $E = \mathbb{R}\begin{pmatrix} c \\ d \end{pmatrix}$ *for some $c, d \in \mathbb{R}$.*

In particular, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} d \\ c \end{pmatrix} = k\begin{pmatrix} c \\ d \end{pmatrix}$ *for some $k \in \mathbb{R}$. So $d = kc$ and $c = kd$ which implies that $d = k^2 d$, i.e $d = \pm c$. Hence* $E = \mathbb{R}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ *or* $E = \mathbb{R}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$. *However, both previous subspaces are not stabilized by H as one can check by looking, for instance, at the action of the element* $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \in H$. *Therefore, H is irreducible.*

However, H is not strongly irreducible: consider $E_1 = \mathbb{R}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $E_2 = \mathbb{R}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ *and let $F = E_1 \cup E_2$. Notice that for all $a, b \in \mathbb{R}$,* $\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ b \end{pmatrix} \in E_2 \subseteq F$

and $\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix} \in E_1 \subseteq F$. *Therefore, H stabilizes F.*

Definition 2.2.3. *We define the* ***general linear group of degree d****, $GL_d(\mathbb{R})$, to be the subgroup of $M_d(\mathbb{R})$ consisting of all $d \times d$ invertible matrices.*

Proposition 2.2.1. *The general linear group acts naturally on the real projective space by the map*

$$GL_d(\mathbb{R}) \times \mathbb{P}(\mathbb{R}^d) \longrightarrow \mathbb{P}(\mathbb{R}^d)$$

$$(g, \overline{x}) \longrightarrow g \cdot \overline{x} = \overline{gx}$$

Definition 2.2.4. *Endow $\mathbb{R}^d$ with the Euclidean norm $||\cdot||$. For simplicity of notation, the operator norm induced on $M_d(\mathbb{R})$ will be denoted also by $||\cdot||$.*
Given a subset Γ of $GL_d(\mathbb{R})$ we define the ***proximal dimension*** *(or* ***index****) of Γ as the smallest integer $r \geq 1$ such that there exists a sequence $(g_n)_n$ in Γ for which the sequence $(||g_n||^{-1}g_n)_n$ converges to a rank r matrix.*

Remark 2.2.1. *The previous definition remains unchanged if we replace the Euclidean norm with any other norm as $M_d(\mathbb{R})$ is a finite dimensional vector space.*

We focus on on giving several examples illustrating the proximal dimension (index).

Example 2.2.2. *(Example with full index) Consider the orthogonal group $O(d) = \{A \in GL_d(\mathbb{R}) : \forall x \in \mathbb{R}^d, ||Ax|| = ||x||\}$. Let G be any subgroup of $O(d)$. We claim that the proximality index of G is equal to d. Indeed, $O(d)$ is a compact topological space as it is the unit ball in the finite dimensional vector space $(M_d(\mathbb{R}), ||\cdot||)$ (where $||\cdot||$ is here the operator norm). Thus if $(g_n)_n$ is any sequence in $O(d)$, then any subsequential limit A of $g_n/||g_n|| = g_n$ remains in $O(d) \subset GL_d(\mathbb{R})$, in particular A has full rank. This proves our claim.*

Example 2.2.3. *(Example with index equal to one) Let $g_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $g_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and consider the group Γ generated by g_1 and g_2 (actually one can prove that $\Gamma = SL_2(\mathbb{Z})$). It is easily seen that Γ is strongly irreducible. Moreover, it has proximal dimension one.*

Indeed, $g := \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix} = {g_1}^3 {g_2}^{-3} {g_1}^2 \in \Gamma$. *This is a diagonalizable matrix with* $\lambda_1 = \frac{5+\sqrt{21}}{2}$ *and* $\lambda_2 = \frac{5-\sqrt{21}}{2}$ *as eigenvalues. Write* $g = P \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} P^{-1}$ *for some invertible matrix* P. *Since* $\lambda_1 > \lambda_2$, *we deduce that* $\frac{g^n}{\lambda_1^n}$ *converges to the matrix* $A := P \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$; *which is a rank one matrix (it is the projection onto* Pe_1 *parallel to* Pe_2*). But, for every* $n \in \mathbb{N}$, $\frac{1}{||P||||P^{-1}||} \leq \frac{||g^n||}{\lambda_1^n} \leq ||P||||P^{-1}||$. *Hence we can extract a convergent subsequence of* $\frac{||g^n||}{\lambda_1^n}$, *say to* α. *Clearly,* $\alpha > 0$. *Writing* $\frac{g^n}{||g^n||} = \frac{g^n}{\lambda_1^n} \times \frac{\lambda_1^n}{||g^n||}$, *we deduce that there exists a subsequence, say* $\frac{g^{n_k}}{||g^{n_k}||}$ *of* $\frac{g^n}{||g^n||}$, *that converges to* $\alpha^{-1}A$ *which is still a rank one matrix. Since* $g^{n_k} \in \Gamma$ *for every* $k \in \mathbb{N}$, *we deduce that the proximal dimension of* Γ *is indeed equal to one. Hence,* Γ *is an example of a strongly irreducible group with proximal dimension equal to one.*

Motivated by the previous example, we give the following definition:

Definition 2.2.5. *Let* A *be an element of* $GL_d(\mathbb{R})$. *We say that* A *is* ***proximal*** *(or* ***contracting****) if it has a simple dominating eigenvalue, i.e it has a unique eigenvalue of maximum modulus.*

A subset Γ *of* $GL_d(\mathbb{R})$ *is said to be* ***proximal*** *if it contains a proximal element.*

As suggested by the proof in Example 2.2.3 , proximality of a subsemigroup of $GL_d(\mathbb{R})$ implies that its index is equal to one. The converse is true provided an irreducibility assumption is imposed. More precisely,

Proposition 2.2.2. *Let* Γ *be a subsemigroup of* $GL_d(\mathbb{R})$. *If* Γ *is proximal, then its proximal dimension is equal to one. Moreover, if* Γ *is irreducible, then the converse is true.*

Proof. For the forward direction, let M be an element of Γ with a simple dominating eigenvalue. By the Jordan decomposition, M is similar to a matrix M' of the form $M' = \begin{pmatrix} \lambda & 0 \\ 0 & N \end{pmatrix}$ where λ is the dominating eigenvalue and N is a square matrix of size $d-1$ and with spectral radius $\rho(N) < |\lambda|$. But by Gelfand's spectral radius formula, $||N^n||^{\frac{1}{n}} \underset{n \to +\infty}{\longrightarrow}$

$\rho(N) < \lambda$. Thus $\frac{M'^{2n}}{||M'^{2n}||}$ converges to a rank one matrix. Using an argument similar to the one in Example 2.2.3, we deduce that there exits a subsequence, say $M^{n_k}/||M_{n_k}||$, of $M^n/||M^n||$ that converges to a rank one matrix. Thus Γ has proximal dimension one.

Conversely, suppose that Γ has proximal dimension one. Let $(g_n)_n$ be a sequence in Γ such that $\dfrac{g_n}{||g_n||}$ converges to a matrix A of rank 1. Then, clearly, A has at most one non-zero eigenvalue with algebraic multiplicity 1. Suppose first that A has such a nonzero eigenvalue. In this case A is proximal. Since the set of proximal matrices is open in $M_n(\mathbb{R})$ (by continuity of the map $A \in M_d(\mathbb{R}) \longmapsto (\lambda_1(A), \cdots, \lambda_d(A)) \in \mathbb{R}^d$), and since $\dfrac{g_n}{||g_n||}$ converges to A, there exists an $N \in \mathbb{N}$ such that for all $n > N$, $\dfrac{g_n}{||g_n||}$ is proximal. In particular, $\dfrac{g_{N+1}}{||g_{N+1}||}$ is proximal, and so is $g_{N+1} \in \Gamma$.

Suppose finally that all the eigenvalues of A are zero. This happens exactly when $Im(A) \subset Ker(A)$ (i.e. $A^2 = 0$). Since Γ is irreducible, there exists $\gamma \in \Gamma$ such that $\gamma \text{Im}(A) \not\subseteq \ker(A)$ (otherwise $\{\gamma x; \gamma \in \Gamma, x \in \text{Im}(A)\}$ would be a non trivial Γ-stable subspace of V). Hence $\text{Im}(\gamma A) \not\subseteq \ker(\gamma A)$, so that γA is proximal. Since $\dfrac{\gamma g_n}{||g_n||}$ converges to γA, from the first case, we get that $\gamma g_n \in \Gamma$ is proximal for all $n > N$. Therefore Γ is proximal.

$\square$

Remark 2.2.2. *The importance of proximality is highlighted in the action of $GL_d(\mathbb{R})$ on $\mathbb{P}(\mathbb{R}^d)$. In fact, if $g \in GL_d(\mathbb{R})$ is proximal, then there exists ${v_g}^+ \in \mathbb{P}(\mathbb{R}^d)$ (namely the line in the direction of the eigenvector associated to the dominating eigenvalue), and a projective hyperplane ${H_g}^-$ such that for all $\overline{x} \notin {H_g}^-$, $g^n \cdot \overline{x} \underset{n\to\infty}{\longrightarrow} {v_g}^+$. Moreover, $\mathbb{R}^d = {v_g}^+ \oplus {H_g}^-$.*

Example 2.2.4. *Here is an example of a strongly irreducible subsemigroup of $GL_4(\mathbb{R})$ with proximal dimension 2. Let $K = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, a, b \in \mathbb{R} \right\}$. This is a field isomorphic to $\mathbb{C}$*

via $\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \mapsto a + ib$. *Consider now the following subset of* $GL_4(\mathbb{R})$*:*

$$\Gamma = \left\{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}; A, B, C, D \in K \text{ and } det(M) \neq 0 \right\} \subset GL_4(\mathbb{R}).$$

This is a subgroup of $GL_4(\mathbb{R})$ *isomorphic to* $GL_2(\mathbb{C})$. *We can show that* Γ *is strongly irreducible by noticing that, for any rotation* R_θ *by an angle* θ, *the block matrix* $\begin{pmatrix} R_\theta & O \\ O & I \end{pmatrix}$ *belongs to* Γ.

Let us check that Γ *has proximality index equal to* 2. *Indeed, exploiting the isomorphism* $\Gamma \simeq GL_2(\mathbb{C})$, *we see that for any matrix* $M \in \Gamma$, *the spectrum of* M *consists of four eigenvalues of the form* $\{\lambda, \overline{\lambda}, \mu, \overline{\mu}\}$. *It follows then that* Γ *is not proximal and so, by irreducibility of* Γ *and Lemma 2.2.2, the index of* Γ *is strictly greater than one. Moreover, let* $M = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \Gamma$. *Clearly,* $M^n/||M^n||$ *converges to a rank two matrix. Hence, the index of* Γ *is* 2.

We end this section by a general result that will be used later.

Lemma 2.2.1. *Let* Γ *be a subsemigroup of* $GL_d(\mathbb{R})$ *acting naturally on* $\mathbb{R}^d$, *and let* $\Gamma^t := \{g^t : g \in \Gamma\}$, *where* g^t *is the transpose matrix of* g. *Then,*

(i) the index of Γ^t *is equal to the index of* Γ*;*

(ii) if Γ *is strongly irreducible, then so is* Γ^t.

Proof. (i) Follows immediately from the fact that $rank(A) = rank(A^t)$ and $||A|| = ||A^t||$ for any matrix A.

(ii) Suppose that there exist proper subspaces $V_1, \cdots V_n$ of $\mathbb{R}^d$ such that their union is stabilized by all the elements of Γ^t, i.e

$$g^t\left(\bigcup_{i=1}^{n} V_i\right) = \bigcup_{i=1}^{n} V_i \quad \text{for all } g \in \Gamma.$$

For all $i \in \{1, \cdots, n\}$, let $W_i = {V_i}^{\perp}$. Let $x \in \bigcup_{i=1}^{n} W_i$, i.e $x \in W_j$ for some j. Hence,

$$\langle x, v \rangle = 0 \;\forall v \in V_j. \tag{2.1}$$

Now, let V_k be the subspace such that $g^t(V_k) = V_j$. For any $v \in V_k$, we have

$$\langle gx, v \rangle = \langle x, g^t v \rangle = 0$$

by (2.1). Hence, $gx \in W_k \subset \bigcup_{i=1}^{n} W_i$. Since g is invertible, we get $g\left(\bigcup_{i=1}^{n} W_i\right) = \bigcup_{i=1}^{n} W_i$ which contradicts the fact that Γ is strongly irreducible.

□

2.3 Convergence of Measures

In this section, X denotes a compact metric space. We are interested in studying convergence of measures on X. By "measure", we mean a regular Borel complex measure. We denote the set of regular Borel complex measures as $\mathcal{M}(X)$, and the set probability measures on a space X by $\mathcal{P}(X)$.

Definition 2.3.1. *A sequence $\{\nu_n\}$ in $\mathcal{M}(X)$ is said to* ***converges $*$-weakly*** *towards $\nu \in \mathcal{M}(X)$ if, for any continuous real function f on X, $\lim_{n\to\infty} \int_X f \, d\nu_n = \int_X f \, d\nu$.*

Definition 2.3.2. *A sequence $\{\nu_n\}$ in $\mathcal{M}(X)$ is said to* ***converges weakly*** *towards $\nu \in \mathcal{M}(X)$ if, for any bounded continuous real function f on X, $\lim_{n\to\infty} \int_X f \, d\nu_n = \int_X f \, d\nu$.*

Notice that, since X is compact, any continuous function on X is bounded. Therefore $*$-weak convergence and weak convergence are equivalent in our case. We will use both terms interchangeably.

Remark 2.3.1. *In view of Remark 2.2.2, when g is proximal, we get that $g^n \cdot \delta_{\overline{x}}$ converges $*$-weakly to δ_{v_g+} as $n \to \infty$.*

The following is a standard result in measure theory:

Proposition 2.3.1. *If X is a compact metric space, then for any sequence $\{\nu_n\} \in \mathcal{P}(X)$, there is a $*$-weakly convergent subsequence of ν_n, and its limit is a probability measure on X. In other words, if X is compact, then $(\mathcal{P}(X), *)$ is compact.*

One important notion that we will be using a lot in this thesis is that any measure ν on a compact space X can be seen as a vector in $C^0(X)^*$ defined by $\nu(f) = \int f d\nu$ for any $f \in C^0(X)$. The fact that $\nu \in C^0(X)^*$ is continuous follows from the fact that this operator is bounded as X is compact.

Remark 2.3.2. *In fact, in our setting, the converse is also true: any vector in $C^0(X)^*$ corresponds to a measure on X. This follows from* ***The Riesz Representation theorem*** *[Rud91, Theorem 6.19]: Let X be a locally compact Hausdorff space. Then, for any bounded linear functional ψ on $C_c(X)$ (the set of continuous compactly supported function on X), there is a unique regular Borel measure μ on X such that $\psi(f) = \int_X f(x)\, d\mu(x) = \mu(f)$ for all f in $C_c(X)$.*

In our case, since X is compact, $C_c(X) = C^0(X)$. Hence, we get that $C^0(X)^* = \mathcal{M}(X)$.

2.4 Stationary Measures

In this section, X denotes a compact metric space.

Definition 2.4.1. *Let ν be a Borel measure on X, $f : X \to X$ a transformation. We say ν is* ***f-invariant*** *if for any $h \in C^0(X)$,*

$$\int_X h d\nu = \int_X h(fx) d\nu(x),$$

or, equivalently, if for all $A \in \mathfrak{M}$, $\nu(f^{-1}(A)) = \nu(A)$. We denote this property by $f * \nu = \nu$.

Remark 2.4.1. *If ν is invariant under a transformation f, then it is invariant under all compositions of f.*

Example 2.4.1. *Let $X = \mathbb{R}$, $\mathfrak{M}$ the Borel σ-algebra of the Lebesgue measure λ, and let f be the translation by a scalar $u \in \mathbb{R}$, i.e $f(x) = x + u$ for $x \in \mathbb{R}$. Then λ is f-invariant because for any open interval (a, b), $\lambda((a, b)) = b - a$ and*

$$\lambda(f^{-1}((a, b))) = \lambda((a - u, b - u)) = b - u - (a - u) = b - a = \lambda((a, b)).$$

Example 2.4.2. *For every $\theta \in [0, 2\pi]$, let $R_\theta : S^1 \longrightarrow S^1$ be the rotation on the circle S^1 by an angle $\theta \in [0, 2\pi]$. Let $\widetilde{\lambda}$ be the pushforward measure of the Lebesgue measure λ under $q : [0, 1] \to S^1$ defined by $q(x) = e^{2i\pi x}$.*

1. *For any $\theta \in [0, 2\pi]$, the probability measure λ is R_θ-invariant. Indeed, for any $\theta \in [0, 2\pi]$, and for any A in the Borel σ-algebra of S^1, we have*

$$\begin{aligned} \widetilde{\lambda}(R_{-\theta}A) &= \lambda(\{x \in [0, 1]; e^{2\pi ix} \in R_{-\theta}A\}) \\ &= \lambda(\{x \in [0, 1]; e^{2\pi i(x+\theta)} \in A\}) \\ &= \lambda(\{x - \theta \in [0, 1]; e^{2\pi ix} \in A\}) \\ &= \lambda(\{x \in [0, 1]; e^{2\pi ix} \in A\}) = \widetilde{\lambda}(A) \end{aligned}$$

   by translation invariance of the Lebesgue measure.

2. *Suppose that $\theta \notin 2\pi\mathbb{Q}$, i.e. R_θ has infinite order. We claim that the Lebesgue measure $\widetilde{\lambda}$ is the unique R_θ-invariant measure on S^1. Indeed, let μ be such a measure. it is enough to prove that $\int f d\mu = \int f d\widetilde{\lambda}$ for any trigonometric function f (since the set of trigonometric functions form a dense subset of $C^0(S^1)$.) Indeed, let $e_k(t) = e^{2\pi ikt}$ for*

any $k \in \mathbb{Z}$. Notice that $e^{2\pi ik\theta} = 1$ only if $k = 0$ since θ is irrational. Now,

$$\frac{1}{N}\sum_{n=0}^{N-1} e_k({R_\theta}^n(t)) = \frac{1}{N}\sum_{n=0}^{N-1} e^{2\pi ik(t+n\theta)} = \begin{cases} 1 & \text{if } k = 0 \\ \frac{1}{N}e^{2\pi ikt}\frac{e^{2\pi iNk\theta}-1}{e^{2\pi ik\theta}-1} & \text{if } k \neq 0 \end{cases}$$

$$\underset{n\to\infty}{\longrightarrow} \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k \neq 0 \end{cases} = \int e_k(t)d\widetilde{\lambda}(t).$$

But also, $\int e_k(t)d\mu = \frac{1}{N}\sum_{n=0}^{N-1}\int e_k(t)d\mu = \int \frac{1}{N}\sum_{n=0}^{N-1} e_k(t)d\mu \underset{N\to\infty}{\longrightarrow} \int\int e_k \; d\lambda \; d\mu = \int e_k \; d\widetilde{\lambda}.$

Definition 2.4.2. *Let G be a set of transformations on X, and let ν be a Borel measure on X. We say that ν is **G-invariant** if it is invariant under all elements of G, i.e $\forall f \in G$, ν is f-invariant.*

Unfortunately, in a lot of cases, such a G-invariant measure fails to exist. Let's look at an example.

Example 2.4.3. *Consider the unit circle S^1 and the group G of its homeomorphisms onto itself such that G contains at least one rotation of infinite order (i.e a rotation by an irrational angle mod 2π) denoted by R_θ, and it contains at least one transformation which is not an isometry of S^1, denoted by T. Suppose μ is a G-invariant measure on S^1. The existence of R_θ in G forces μ to be the Lebesgue measure $\widetilde{\lambda}$ (as in Example 2.4.2). However, $\widetilde{\lambda}$ is not invariant under T.*
Therefore there exists no G-invariant probability measure on S^1.

Now we need to look for a property that substitutes G-invariance.
From now on, we let G be a topological semigroup acting continuously on a topological space X.

Definition 2.4.3. *Let μ be a Borel measure on G and ν a Borel measure on X. Denote by $\mu * \nu$ the **pushforward measure** of $\mu \otimes \nu$ under the map $G \times X \to X$: $(g, x) \to g \cdot x$. In* *other words, for A in the Borel σ-algebra of X,*

$$(\mu * \nu)(A) = (\mu \otimes \nu)(\{(g,x) \in G \times X : g \cdot x \in A\}),$$

or, equivalently, for any continuous function f on X:

$$\int_X f(x)d(\mu * \nu)(x) = \int_X \int_G f(g.x)d\mu(g)d\nu(x).$$

*We say ν **is μ-invariant** (or **μ-stationary**) if $\mu * \nu = \nu$.*

Remark 2.4.2. *When $X = G$ and G acts on itself by translation, we denote $\mu * \mu$ by μ^{*^2} and we call it the second convolution power of μ. Similarly we define the n-th convolution power of μ, denoted by μ^{*^n}.*

The efficiency of this stationary property is highlighted by the following proposition.

Proposition 2.4.1. *Let G be a semigroup acting on a compact metric space X. Then, for any probability measure μ of G, there exists a μ-invariant probability measure on X.*

Proof. Let μ be a probability measure on G, $x_0 \in X$, and consider the probability measure δ_{x_0} on X.
Let $\nu_n := \frac{1}{n}\sum_{i=1}^{n} \mu^{*i} * \delta_{x_0}$. Since X is compact, there is a $*$-weakly convergent subsequence of ν_n (that we will denote again by ν_n for simplicity), and its limit, ν is a probability measure on X (Proposition 2.3.1). Now,

$$\begin{aligned} \mu * \nu_n &= \frac{1}{n}\sum_{i=1}^{n} \mu^{*(i+1)} * \delta_{x_0} \\ &= \frac{1}{n}\sum_{i=1}^{n} \mu^{*i} * \delta_{x_0} + \frac{1}{n}\{\mu^{*(n+1)} * \delta_{x_0} - \mu * \delta_{x_0}\} \\ &= \nu_n + \frac{1}{n}\{\mu^{*(n+1)} * \delta_{x_0} - \mu * \delta_{x_0}\}. \end{aligned} \tag{2.2}$$

Let $\eta_n := \frac{1}{n}\{\mu^{*(n+1)} * \delta_{x_0} - \mu * \delta_{x_0}\}$, and let $f \in C^0(X)$. Then

$$\begin{aligned}\left|\int f d\eta_n\right| &= \frac{1}{n}\left|\int f(x) d\mu^{*(n+1)} * \delta_{x_0}(x) - \int f(x) d\mu * \delta_{x_0}(x)\right| \\ &= \frac{1}{n}\left|\int f(gx_0) d\mu^{*(n+1)}(g) - \int f(gx_0) d\mu(g)\right| \\ &\leq \frac{1}{n}\int |f(gx_0)| d\mu^{*(n+1)}(g) + \int |f(gx_0)| d\mu(g).\end{aligned}$$

Since f is continuous over X (compact), there exists $M > 0$ such that $f(x) \leq M$ for all $x \in X$. Therefore

$$\left|\int f d\eta_n\right| \leq \frac{1}{n}\{M \int d\mu^{*(n+1)}(g) + M \int d\mu(g)\} = \frac{2M}{n} \underset{n\to\infty}{\longrightarrow} 0.$$

Hence, η_n converges to the zero measure. It follows that, letting $n \to \infty$ in (2.2), we get $\mu * \nu = \nu$.

□

Remark 2.4.3. *If ν is G-invariant, then it is μ-invariant for any measure μ on G.*

Definition 2.4.4. *A probability measure ν on $\mathbb{P}(\mathbb{R}^d)$ is said to be **non-degenerate** (or **proper**) if for any hyperplane H in $\mathbb{R}^d$,*

$$\nu(\{\overline{x} \in \mathbb{P}(\mathbb{R}^d) : x \in H \backslash 0\}) = 0.$$

We state the following important result due to Furstenburg. For a complete proof, see [BL85, Page 49].

Proposition 2.4.2. *Let μ be a probability measure on $GL_d(\mathbb{R})$. Denote by Γ_μ the smallest closed subsemigroup of $GL_d(\mathbb{R})$ generated by the support of μ. Suppose that Γ_μ is strongly irreducible. Then any μ-invariant measure ν on $\mathbb{P}(\mathbb{R}^d)$ is non-degenerate.*

Chapter 3

Random Walks on $\mathbb{P}(\mathbb{R}^d)$

For a measure μ on $GL_d(\mathbb{R})$, we recall that Γ_μ denotes the smallest closed subsemigroup of $GL_d(\mathbb{R})$ generated by the support of μ and that the averaging operator P_μ is defined on $X = \mathbb{P}(\mathbb{R}^d)$ by

$$P_\mu : C^0(X) \longrightarrow C^0(X) \ ; \ f \longmapsto P_\mu(f) = \int_{\Gamma_\mu} f(gx) d\mu(g). \tag{3.1}$$

Our goal in this Chapter is to prove the following theorem of Benoist-Quint [BQ14, Proposition 3.1]:

Theorem 3.0.1. *When Γ_μ is strongly irreducible, the averaging operator P_μ is equicontinuous.*

We will see in Chapter 4 that the equicontinuity of P_μ is an important condition for the construction of μ-stationary probability measures on $\mathbb{P}(\mathbb{R}^d)$. To prove the above theorem, we need first to get introduced to Markov-Feller operators and equicontinuity.

3.1 Markov-Feller operators

Since this section contains only definitions, we let (X, d) denote here any compact metric space. We endow $C^0(X)$ with the sup norm.

Definition 3.1.1. *An operator $P : C^0(X) \longrightarrow C^0(X)$ is said to be a* ***Markov-Feller operator*** *if it is a bounded operator such that $||P|| \leq 1$, $P1 = 1$ and such that $Pf \geq 0$ for all functions $f \geq 0$.*

It is very straightforward to check that our averaging operator (3.1) is a Markov-Feller operator.

Definition 3.1.2. *A family $F \subset C^0(X)$ is said to be* **equicontinuous** *if for all $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ for all $f \in F$ and for all $x, y \in X$ such that $d(x, y) < \delta$.*

Definition 3.1.3. *We say a Markov-Feller operator P is* **equicontinuous** *if, for every f in $C^0(X)$, the family of functions $(P^n f)_{n \geq 1}$ is equicontinuous.*

Proposition 3.1.1. *If a Markov-Feller operator P on $C^0(X)$ is equicontinuous, then for any $f \in C^0(X)$, $\{P^n f; n \geq 1\}$ is relatively compact (i.e. its closure is compact in $C^0(X)$). We also say that P* ***spans a strongly compact semigroup****.*

Proof. Let $f \in C^0(X)$. Since P is equicontinuous, the $\{P^n f\}_n$ is equicontinuous. Moreover, for all $n \in \mathbb{N}$, for all $x \in X$, $|P^n f(x)| \leq ||P^n f|| \leq ||P^n||\ ||f|| \leq ||f||$ since P is Markov-Feller. Therefore $\{P^n f : n \geq 1\}$ is uniformly bounded in $C^0(X)$, and it follows from Arzela-Ascoli theorem (see for instance [Rud91, A5]) that there is a subsequence of $\{P^n f\}_n$ that converges uniformly to some function in $C^0(X)$, i.e $\{P^n f; n \geq 1\}$ is relatively compact. □

3.2 Equicontinuity on the Projective Spaces

We go back to $X = \mathbb{P}(\mathbb{R}^d)$ on which $GL_d(\mathbb{R})$ is acting. Let μ be a probability measure on $GL_d(\mathbb{R})$ and let Γ_μ be the smallest closed semigroup of $GL_d(\mathbb{R})$ which contains the support of μ.

We denote by B be the set of sequences $b = (b_1, b_2, \cdots, b_n, \cdots)$ with $b_n \in GL_d(\mathbb{R})$, $\mathcal{B}$ its Borel σ-algebra, β the product probability measure $\beta = \mu^{\otimes \mathbb{N}^*}$, and θ the **Bernouilli shift map** defined by

$$\theta : B \longrightarrow B; (b_1, b_2, b_3, \cdots) \longrightarrow (b_2, b_3, b_4 \cdots).$$

The following result of Furstenberg and Guivarc'h-Raugi will be crucial for us. For a detailed proof, check [BL85, Theorem 3.1].

Theorem 3.2.1. *Suppose that Γ_μ is strongly irreducible and write r for its index (see Definition 2.2.4). Then, for β-almost every $b = (b_i)_{i\in\mathbb{N}^*} \in B$, there exists an r-dimensional subspace V_b of $\mathbb{R}^d$ such that any limit point of $\{||b_1 \cdots b_n||^{-1} b_1 \cdots b_n : n \geq 1\}$ is a rank r matrix with range V_b.*

Corollary 3.2.1. *Suppose that Γ_μ is strongly irreducible and write r for its index. Then, if $L_n(b) = b_n \cdots b_2 b_1$, for β-almost all $b = (b_i)_{i\in\mathbb{N}^*} \in B$ there exists a $(d-r)$-dimensional subspace W_b of $\mathbb{R}^d$ such that any limit point of $\{||L_n(b)||^{-1} L_n(b), n \geq 1\}$ is a rank r matrix with kernel W_b.*

We write

$$W_b = \left\{ v \in \mathbb{R}^d : \exists (n_k)_k, \exists \pi : \frac{b_{n_k} \cdots b_1}{||b_{n_k} \cdots b_1||} \underset{k\to\infty}{\longrightarrow} \pi \text{ and } \pi v = 0 \right\}. \tag{3.2}$$

Proof. The result follows easily from Theorem 3.2.1 and the following identity, which is true for every square matrix A: $\ker(A)^\perp = \mathrm{Im}(A^t)$. □

Lemma 3.2.2. *: Suppose that Γ_μ is strongly irreducible with index r, and let W_b be as in (3.2). Then for all $\overline{x} \in \mathbb{P}(\mathbb{R}^d)$, $\beta(\{b \in B : \overline{x} \in \mathbb{P}(W_b \backslash \{0\})\}) = 0$.*

Proof. Define the measure ν on $\mathbb{P}(\mathbb{R}^d)$ by

$$\nu(A) = \beta(\{b \in B : \mathbb{P}({W_b}^\perp) \subset A\}).$$

We claim that ν is μ^t- invariant, where $\mu^t(E) = \mu(\{g \in \Gamma_\mu : g^t \in E\})$ for any E in the Borel σ-algebra of Γ_μ^t. Indeed, let A $\subset \mathbb{P}(\mathbb{R}^d)$, then

$$\begin{aligned}(\mu^t * \nu)(A) &= (\mu^t \otimes \nu)(\{(g,x) \in G^t \times X : g \cdot x \in A\}) \\ &= (\mu \otimes \nu)(\{(g,x) \in G \times X : g^t \cdot x \in A\}) \\ &= (\mu \otimes \beta)(\{(g,b) \in G \times B : g^t \cdot \mathbb{P}({W_b}^{\perp}) \subset A\}) \\ &= \beta(\{b = (b_1, b_2, b_3, \cdots) \in B : {b_1}^t \cdot \mathbb{P}({W_{\theta(b)}}^{\perp}) \subset A)\}\end{aligned}$$

where $\theta(b)$ is the shift map. Now it is enough to show that

$$\forall b \in B,\ {b_1}^t \cdot \mathbb{P}({W_{\theta(b)}}^{\perp}) = \mathbb{P}({W_b}^{\perp}),$$

since our right hand side would be equal to $\nu(A)$.

Then, let $\overline{y} \in {b_1}^t \cdot \mathbb{P}(W_\theta(b)^{\perp})$, i.e $\overline{y} = {b_1}^t \cdot \overline{x}$ such that $x \in {W_{\theta(b)}}^{\perp}$; i.e

$$\forall x' \in W_{\theta(b)},\ \langle x, x' \rangle = 0. \tag{3.3}$$

Let $x'' \in W_b$, then, by (3.2), there exists $(n_k)_k$ and $\pi \in End(V)$ such that

$$\frac{b_{n_k} \cdots b_1}{||b_{n_k} \cdots b_1||} \underset{k \to \infty}{\longrightarrow} \pi \text{ and } \pi x'' = 0$$

so

$$\frac{b_{n_k} \cdots b_1 {b_1}^{-1}}{||b_{n_k} \cdots b_1||} \underset{k \to \infty}{\longrightarrow} \pi {b_1}^{-1},$$

i.e

$$\frac{b_{n_k} \cdots b_2}{||b_{n_k} \cdots b_2||} \underbrace{\frac{||b_{n_k} \cdots b_2||}{||b_{n_k} \cdots b_1||}}_{C_{bnk}} \underset{k \to \infty}{\longrightarrow} \pi {b_1}^{-1}$$

Now since C_{bnk} is bounded ($1/||b_1|| \leq C_{bnk} \leq ||{b_1}^{-1}||$), then it has a subsequence that converges to some $c_b > 0$. To simplify notation, we will denote it also by C_{bn_k}. So,

$$\frac{b_{n_k} \cdots b_2}{||b_{n_k} \cdots b_2||} \underset{k \to \infty}{\longrightarrow} {c_b}^{-1} \pi {b_1}^{-1} := \pi'.$$

Let $z := b_1 x'' \in \mathbb{R}^d$. Then we found $\pi' \in End(V)$ and $(n_k)_k$ as above such that

$$\pi' z = {c_b}^{-1} \pi {b_1}^{-1} b_1 x'' = {c_b}^{-1} \pi x'' = 0.$$

So $z \in W_{\theta(b)}$ and

$$\langle y, x'' \rangle = \langle {b_1}^t \cdot x, x'' \rangle = \langle x, b_1 x'' \rangle = \langle x, z \rangle = 0$$

by (3.3). Thus $\overline{y} \in \mathbb{P}({W_b}^{\perp})$ as desired.

The other inclusion can be proven in the same way, since it is equivalent to ${b_1}^{-t} \cdot \mathbb{P}(W_b^{\perp}) \subset \mathbb{P}(W_{\theta(b)}^{\perp})$.

Now that ν is μ^t-invariant, and since ${\Gamma_\mu}^t$ is also strongly irreducible (by Lemma 2.2.1), then ν is non-degenerate, by Proposition 2.4.2. Hence for all $\overline{x} \in \mathbb{P}(\mathbb{R}^d)$, $H = (\mathbb{R}x)^{\perp}$ is a hyperplane and $\nu(\{\mathbb{P}(H)\}) = 0$, i.e for all $\overline{x} \in \mathbb{P}(\mathbb{R}^d)$

$$\begin{aligned} 0 &= \beta(\{b \in B : \mathbb{P}({W_b}^{\perp}) \subset \mathbb{P}(H)\}) \\ &= \beta(\{b \in B : (W_b)^{\perp} \subset \mathbb{R}x^{\perp}\}) \\ &= \beta(\{b \in B : x \in W_b\}) \\ &= \beta(\{b \in B : \overline{x} \in \mathbb{P}(W_b)\}) \end{aligned}$$

□

Before we continue, we recall the following lemma:

Lemma 3.2.3. *Let X be a metric space, $K > 0$ and $(f_n)_n$ a sequence of K-Lipschitz real valued functions on X. Assume that the pointwise infinimum f of the $(f_n)_{n \geq \mathbb{N}}$ is finite (i.e.*

different than $-\infty$), then f is also K-Lipshitz.

Proof. Let $x, y \in X$, let n be arbitrary. Then,

$$f(x) \leq f_n(x) \leq |f_n(x) - f_n(y)| + f_n(y) \leq Kd(x,y) + f(y)$$

which implies that

$$f(x) - f(y) \leq Kd(x,y).$$

Similarly, we prove that $f(y) - f(x) \leq Kd(x,y)$ and hence

$$|f(x) - f(y)| \leq Kd(x,y).$$

□

Lemma 3.2.4. *Suppose Γ_μ is strongly irreducible with index r and let W_b be as in (3.2). Then for all $\alpha > 0$, for β a.e $b \in B$, there exists $c_{\alpha,b} > 0$ such that for all $v \in \mathbb{R}^d \backslash \{0\}$ with $d(\mathbb{R}v, \mathbb{P}(W_b)) \geq \alpha$ (see Proposition/Definition 2.1.4), one has*

$$\inf_{n \geq 1} \frac{||b_n \cdots b_1 v||}{||b_n \cdots b_1||\ ||v||} \geq c_{\alpha,b}.$$

Proof. : First we note that it is enough to prove the result for all $v \in \mathbb{R}^d \backslash \{0\}$ with $||v|| = 1$. Let

$$K_{b,\alpha} = \{v \in \mathbb{R}^d : ||v|| = 1 \text{ and } d(\mathbb{R}v, \mathbb{P}(W_b)) \geq \alpha\}.$$

Clearly, $K_{b,\alpha}$ is compact and $K_{b,\alpha} \subset {W_b}^c$.
For every $n \in \mathbb{N}$, let $(f_n)_n$ be the family of functions defined by $f_n(v) = \dfrac{||b_n \cdots b_1 v||}{||b_n \cdots b_1||}$. Clearly each f_n is 1-Lipshitz, therefore $f(v) = \inf_{n \geq 1} f_n(v)$ is 1-Lipshitz and hence continuous over $K_{b,\alpha}$, which is compact, and thus it attains its minimum, i.e there exists a point $v_0 \in K_{b,\alpha}$ such that $f(v) \geq f(v_0)$ for all $v \in K_{b,\alpha}$. It remains to prove that $f(v_0) > 0$.
Suppose $f(v_0) = 0$. By the definition of infimum, and since $f_n(v) \neq 0$ for all $n > 1$, there

exists a subsequence $(f_{nk})_k$ such that

$$f_{nk}(v_0) = \frac{||b_{nk} \cdots b_1 v_0||}{||b_{nk} \cdots b_1||} \underset{k\to\infty}{\longrightarrow} f(v_0).$$

Consider the sequence

$$(M_{n_k})_k = \frac{b_{nk} \cdots b_1}{||b_{nk} \cdots b_1||},$$

then $(M_{n_k})_k$ has a convergent subsequence, that we will also denote by $(M_{n_k})_k$, for simplicity. Thus there exists $\pi \in End(V)$ such that

$$M_{n_k} \underset{k\to\infty}{\longrightarrow} \pi,$$

so that

$$f_{nk}(v_0) = ||M_{n_k} v_0|| \underset{k\to\infty}{\longrightarrow} ||\pi v_0||.$$

By the uniqueness of limits, it follows that $||\pi v_0|| = f(v_0) = 0$, and so $v_0 \in W_b$ (by (3.2)), which contradicts the fact that $v_0 \in K_{b,\alpha}$. Therefore $f(v_0) > 0$. The proof follows by taking $c_{\alpha,b} = f(v_0)$. □

Lemma 3.2.5. *Let $V = \mathbb{R}^d$ and μ be a probability measure on $GL(V)$ such that the action of Γ_μ on V is strongly irreducible. Then, for all $\epsilon > 0$,*

(a) there exists $c_\epsilon > 0$ such that, for all v in $V \setminus \{0\}$, one has

$$\beta(\{b \in B : \inf_{n\geq 1} \frac{||b_n \cdots b_1 v||}{||b_n \cdots b_1||\ ||v||} \geq c_\epsilon\}) \geq 1 - \epsilon,$$

(b) there exists $M_\epsilon > 0$ such that, for all x, y in $\mathbb{P}(V)$, one has

$$\beta(\{b \in B : \sup_{n\geq 1} d(b_n \cdots b_1 x, b_n \cdots b_1 y) \leq M_\epsilon d(x, y)\}) \geq 1 - \epsilon$$

Proof. (a) Let r be the proximal dimension of Γ_μ. Since Γ_μ is strongly irreducible with index r, then by Corollary 3.2.1, for β a.e $b \in B$, there exists a $(d - r)-$dimensional subspace W_b

of $\mathbb{R}^d$ such that

$$W_b = \left\{ v \in \mathbb{R}^d : \exists (n_k)_k, \exists \pi : \frac{b_{n_k} \cdots b_1}{||b_{n_k} \cdots b_1||} \underset{k\to\infty}{\longrightarrow} \pi \text{ and } \pi v = 0 \right\}.$$

Let $A_x = \{b \in B : \overline{x} \in \mathbb{P}(W_b)\} = \{b \in B : d(\overline{x}, \mathbb{P}(W_b)) = 0\}$ (since $\mathbb{P}(W_b)$ is closed). Then for all $\overline{x} \in \mathbb{P}(\mathbb{R}^d)$, $\beta(A_x) = 0$ (by Lemma 3.2.2). Since $\mathbb{P}(\mathbb{R}^d)$ is a seperable topological space, consider a countable dense subset $D \subset \mathbb{P}(\mathbb{R}^d)$, so that $\bigcap_{\overline{x}\in D} A_x$ is measurable and

$$\beta(\bigcap_{\overline{x}\in D} A_x) = 0.$$

But

$$\bigcap_{\overline{x}\in D} A_x = \bigcap_{n=1}^{\infty} \underbrace{\bigcap_{\overline{x}\in D} \{b \in B : d(\overline{x}, \mathbb{P}(W_b)) < 1/n\}}_{M_n}.$$

Since $(M_n)_n$ is a decreasing sequence of measurable sets and β is a finite measure,

$$0 = \beta(\bigcap_{\overline{x}\in D} A_x) = \beta(\bigcap_{n=1}^{\infty} M_n) = \lim_{n\to\infty} \searrow \beta(M_n).$$

Let $\epsilon > 0$, $\exists\ N_\epsilon \in \mathbb{N}$ such that for all $n \geq N_\epsilon$, $\beta(M_n) < \epsilon/2$.

In particular, $\beta(M_{N_\epsilon+1}) < \epsilon/2$. Therefore $\beta(\{b \in B : \forall \overline{x} \in D, d(\overline{x}, \mathbb{P}(W_b)) < 1/(N_\epsilon + 1)\}) < \epsilon/2$.

Let $\alpha'_\epsilon = 1/(N_\epsilon + 1)$ and let $\alpha_\epsilon = \alpha'_\epsilon/2$. Then

$$\beta(\{b \in B : \forall \overline{x} \in D, d(\overline{x}, \mathbb{P}(W_b)) \geq \alpha'_\epsilon\} \geq 1 - \epsilon/2.$$

A fortiori, for all $\overline{x} \in D, \beta(\{b \in B : d(\overline{x}, \mathbb{P}(W_b)) \geq \alpha'_\epsilon\} \geq 1 - \epsilon/2$, and

$$\forall \overline{x} \in D,\ \beta(\{b \in B : d(\overline{x}, \mathbb{P}(W_b)) \geq \alpha_\epsilon\} \geq 1 - \epsilon/2.$$

Now, since D is dense in $\mathbb{P}(\mathbb{R}^d)$, for all $\overline{x} \in \mathbb{P}(\mathbb{R}^d) \setminus D$, there exists $\overline{x_0} \in D$ such that $d(\overline{x}, \overline{x_0}) < \alpha_\epsilon$.

Notice that if $d(\overline{x_0}, \mathbb{P}(W_b)) \geq \alpha'_\epsilon$ then

$$d(\overline{x}, \mathbb{P}(W_b)) > d(\overline{x_0}, \mathbb{P}(W_b)) - d(\overline{x}, \overline{x_0}) > \alpha'_\epsilon - \alpha_\epsilon = \alpha_\epsilon.$$

It follows that for all $\overline{x} \in \mathbb{P}(\mathbb{R}^d) \setminus D,\ \beta(\{b \in B : d(\overline{x}, \mathbb{P}(W_b)) \geq \alpha_\epsilon\} \geq 1 - \epsilon/2$.

Hence for the given ϵ, we found $\alpha_\epsilon > 0$ such that

$$\forall \overline{x} \in \mathbb{P}(\mathbb{R}^d),\ \beta(\{b \in B : d(\overline{x}, \mathbb{P}(W_b)) \geq \alpha_\epsilon\} \geq 1 - \epsilon/2.$$

From Lemma 3.2.4, there exists $c_{\alpha_\epsilon,b} > 0$ such that for all $v \in \mathbb{R}^d \setminus \{0\}$ with $d(\mathbb{R}v, \mathbb{P}(W_b)) \geq \alpha_\epsilon$, one has

$$\inf \frac{||b_n \cdots b_1 v||}{||b_n \cdots b_1||\ ||v||} \geq c_{\alpha_\epsilon,b}.$$

Since $\beta\{b \in B; c_{\alpha_\epsilon,b} > 0\} = 1$, then by writing $\{b \in B; c_{\alpha_\epsilon,b} > 0\} = \bigcup_{n\in\mathbb{N}}\{b \in B; c_{\alpha_\epsilon,b} > 1/n\}$, we see that there exists $c_\epsilon > 0$ such that

$$\beta(\{b \in B : c_{\alpha_\epsilon,b} \geq c_\epsilon\}) \geq 1 - \epsilon/2.$$

We proved that for all $v \in V \setminus \{0\}$, if $b \in B$ such that $d(\mathbb{R}v, \mathbb{P}(W_b)) \geq \alpha_\epsilon$, then

$$\inf_{n\geq 1} \frac{||b_n \cdots b_1 v||}{||b_n \cdots b_1||\ ||v||} \geq c_{\alpha_\epsilon,b}.$$

Therefore,

$$\inf_{n\geq 1} \frac{||b_n \cdots b_1 v||}{||b_n \cdots b_1||\ ||v||} \geq c_\epsilon \text{ or } c_{\alpha_\epsilon,b} < c_\epsilon.$$

Hence,

$$\{b \in B : d(\mathbb{R}v, \mathbb{P}(W_b)) \geq \alpha_\epsilon\} \subset \{b \in B : \inf_{n\geq 1} \frac{||b_n \cdots b_1 v||}{||b_n \cdots b_1||\ ||v||} \geq c_\epsilon\} \cup \{b \in B : c_{\alpha_\epsilon,b} < c_\epsilon\},$$

which implies that

$$\begin{aligned}\beta(\{b \in B : \inf_{n\geq 1} \frac{||b_n \cdots b_1 v||}{||b_n \cdots b_1||\;||v||} \geq c_\epsilon\}) &\geq \beta(\{b \in B : d(\mathbb{R}v, \mathbb{P}(W_b)) \geq \alpha_\epsilon\}) - \beta(\{b \in B : c_{\alpha_\epsilon, b} < c_\epsilon\}) \\ &\geq 1 - \epsilon/2 - \epsilon/2 = 1 - \epsilon\end{aligned}$$

as required.

(b) Let $\epsilon > 0$, there exists $M_\epsilon = (c_{\epsilon/2})^{-2}$ such that for all $x, y \in \mathbb{P}(V)$, if we let $p_n = b_n \cdots b_1$, $x = \mathbb{R}v$ and $y = \mathbb{R}w$, then we have

$$\frac{d(p_n x, p_n y)}{d(x,y)} = \frac{||p_n v \wedge p_n w||}{||v \wedge w||} \frac{||v||}{||p_n v||} \frac{||w||}{||p_n w||} \leq \frac{||p_n||\;||v||}{||p_n v||} \frac{||p_n||\;||w||}{||p_n w||},$$

which implies that

$$\{b \in B : \inf_{n\geq 1} \frac{||p_n v||}{||p_n||\;||v||} \geq c_{\epsilon/2}\}$$

is a subset of

$$\{b \in B : \inf_{n\geq 1} \frac{||p_n v||}{||p_n||\;||v||} \frac{||p_n w||}{||p_n||\;||w||} \geq (c_{\epsilon/2})^2\} \cup \{b \in B : \inf_{n\geq 1} \frac{||p_n w||}{||p_n||\;||w||} < c_{\epsilon/2}\}.$$

But,

$$\begin{aligned}\{b \in B : \inf_{n\geq 1} \frac{||p_n v||}{||p_n||\;||v||} \frac{||p_n w||}{||p_n||\;||w||} \geq (c_{\epsilon/2})^2\} &= \{b \in B : \sup_{n\geq 1} \frac{||p_n||\;||v||}{||p_n v} \frac{||p_n||\;w||}{||p_n w||} \leq 1/(c_{\epsilon/2})^2\} \\ &\subset \{b \in B : \sup_{n\geq 1} \frac{d(p_n x, p_n y)}{d(x,y)} \leq M_\epsilon\} \\ &= \{b \in B : \sup_{n\geq 1} d(p_n x, p_n y) \leq M_\epsilon d(x,y)\} := B_{\epsilon,x,y}.\end{aligned}$$

Hence,

$$\begin{aligned} \beta(B_{\epsilon,x,y}) &\geq \beta(\{b \in B : \inf_{n\geq 1} \frac{||p_n v||}{||p_n||\ ||v||} \geq c_{\epsilon/2}\}) - \beta(\{b \in B : \inf_{n\geq 1} \frac{||p_n w||}{||p_n||\ ||w||} < c_{\epsilon/2}\}) \\ &\geq 1 - \epsilon/2 - \epsilon/2 = 1 - \epsilon. \end{aligned}$$

□

We can now finally prove the equicontinuity of our operator.

Theorem 3.2.6. *When Γ_μ is strongly irreducible, the averaging operator on $X = \mathbb{P}(\mathbb{R}^d)$*

$$P_\mu : C^0(X) \longrightarrow C^0(X); f \longrightarrow P_\mu(f) = \int_{\Gamma_\mu} f(gx)d\mu(g)$$

is equicontinuous.

Proof. Let f be a continuous function on X. We want to prove that the family of functions $({P_\mu}^n f)_{n\geq 1}$ is equicontinuous. Without loss of generality, we can assume that $||f||_\infty \leq 1$, because, otherwise consider $g = \dfrac{f}{||f||_\infty}$ satisfies the condition, i.e for all $x_0 \in X$ for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $n \in \mathbb{N}$ and for all $x \in X$ with $d(x, x_0) < \delta$,

$$|({P_\mu}^n g)(x) - ({P_\mu}^n g)(x_0)| < \frac{\epsilon}{||f||_\infty},$$

then

$$|({P_\mu}^n f)(x) - ({P_\mu}^n f)(x_0)| = ||f||_\infty\ |({P_\mu}^n g)(x) - ({P_\mu}^n g)(x_0)| < \epsilon.$$

Now, fix $\epsilon > 0$. Since f is continuous over the compact set X, it is uniformly continuous. Hence, there exists $\eta_\epsilon > 0$ such that for all $x', y' \in X$ with $d(x', y') < \eta_\epsilon$, we have $|f(x') - f(y')| < \epsilon$.

Let $x, y \in X$. From the previous lemma, there exists $M_\epsilon > 0$ such that the set

$$B_{\epsilon,x,y} := \{b \in B : \sup_{n \geq 1} d(b_n \cdots b_1 x, b_n \cdots b_1 y) \leq M_\epsilon d(x, y)\}$$

satisfies $\beta(B^c_{\epsilon,x,y}) \leq \epsilon$.

Let $\delta = \eta_\epsilon / M_\epsilon$, then for all $n \in \mathbb{N}$ for all $x, y \in X$ with $d(x, y) < \delta$, we have

$$\begin{aligned} |(P_\mu{}^n f)(x) - (P_\mu{}^n f)(y)| &= \left| \int_{\Gamma_\mu} (f(bx) - f(by)) d\mu^{*n}(b) \right| \\ &\leq \int_B |f(b_n \cdots b_1 x) - f(b_n \cdots b_1 y)| d\beta(b) \\ &= \int_{B_{\epsilon,x,y}} |f(b_n \cdots b_1 x) - f(b_n \cdots b_1 y)| d\beta(b) \\ &\quad + \int_{B^c_{\epsilon,x,y}} |f(b_n \cdots b_1 x) - f(b_n \cdots b_1 y)| d\beta(b) \\ &\leq \int_{B_{\epsilon,x,y}} \epsilon d\beta(b) + \int_{B^c_{\epsilon,x,y}} (1+1) d\beta(b) \\ &\leq \epsilon\beta(B_{\epsilon,x,y}) + 2\beta(B^c_{\epsilon,x,y}) \\ &\leq \epsilon + 2\epsilon = 3\epsilon. \end{aligned}$$

Therefore, the family $(P_\mu{}^n f)_{n \geq 1}$ is equicontinuous, and hence so is the operator P_μ. □

Chapter 4

Decomposition theorems and Empirical Measures for Markov-Feller Operators

As mentioned in the introduction, our method of constructing probability measures on $X = \mathbb{P}(\mathbb{R}^d)$ is a functional one. We will see probability measures as vectors of $C^0(X)^*$ (see the end of section 2.3). For this reason, we dedicate Subsection 4.1 to all the general notions of functional analysis needed to prove the results in Chapter 5. This part of the chapter is based on Raugi's work [Rau94].

The space X can be any compact metric space and P an equicontinuous Markov-Feller operator on X. One of the main results of this section is Theorem 4.1.2 which shows that the space of measures on X can be decomposed into a suitable direct sum of P -stable subspaces: the space of P -invariant vectors and a space on which P contracts any vector to zero in Cesaro mean. We then explain the link between this functional analytic chapter and our main study in this thesis.

The last part of this chapter is the powerful Breiman's law of Large number Theorem 4.2.2 that describes the subsequential limits of the empirical measures for Markov-Feller operators on a compact metric space.

4.1 Decomposition Theorems

In this section, X is a compact metric space and E is the Banach space $E = C^0(X)$ with $||\cdot||$ the sup norm. We endow the dual Banach space E^* with the $*$-weak topology defined in the same way as in Definition 2.3.1.

Definition 4.1.1. *For any operator $P : E \longrightarrow E$, we define the* ***adjoint operator of*** *P in E^* by $\nu \longrightarrow \nu P$ where $(\nu P)(f) = \nu(P(f))$ for all $f \in E$.*

Lemma 4.1.1. *If $\{P^n f; n \geq 1\}$ is strongly relatively compact for any $f \in E$, then for any $\nu \in E^*$, $\{\nu P^n; n \geq 1\}$ is $*$-weakly relatively compact.*

Proof. Let $\nu \in E^*$. For any $f \in E$, $\{P^n f\}_n$ has a subsequence $\{P^{n_k} f\}_k$ that converges to some $y_f \in E$. Let $\eta \in E^*$ be defined by $\eta(f) = \nu(y_f)$ for any $f \in E$. Then, for all $f \in E$, by continuity of ν, we have that $\lim_{k\to\infty} \nu P^{n_k}(f) = \nu(y_f) = \eta(f)$. Hence, νP^{n_k} is a subsequence of $\{\nu P^n; n \geq 1\}$ that converges $*$-weakly to η. $\square$

For any operator P, we will introduce the following sets:

$$E^P := \{f \in E : Pf = f\}, \quad (E^*)^P := \{\nu \in E^* : \nu P = \nu\}$$

$$E_P := \{f \in E : \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} P^k f = 0 \text{ strongly}\}$$

and

$$(E^*)_P := \{\nu \in E^* : \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \nu P^k = 0 \ * -\text{weakly}\}$$

An element in E^P will be called a P-invariant function and an element in $(E^*)^P$ will be called a P-invariant measure.

Theorem 4.1.2. *Recall that $E = C^0(X)$. Let $P : E \longrightarrow E$ be an an equicontinuous Markov-Feller operator (see Definitions 3.1.1 and 3.1.3). Then:*

(i) One has the isomorphism $(E^*)^P \simeq (E^P)^*$

(ii) E can be decomposed as $E = E^P \oplus E_P$

(iii) E^* can be decomposed as $E^* = (E^*)^P \oplus (E^*)_P$

Proof. (i) Let $\psi : (E^*)^P \longrightarrow (E^P)^*$ be defined by $\psi(\nu) = \nu|_{E^P}$ for all $\nu \in (E^*)^P$.
ψ is injective: Let $\nu \in (E^*)^P$ be such that $\nu(g) = 0,\ \forall g \in E^P$. Let $f \in E$ and let $K = \overline{\{P^n f : n \geq 1\}}$. Since P is equicontinuous, by Proposition 3.1.1, K is compact in E. So by [Rud91, Theorem 3.20.c], the convex hull $co(K)$ is relatively compact, i.e the sequence $\{a_{f,n} = \frac{1}{n}\sum_{k=1}^{n} P^k f\} \in K$ has a subsequence converging to some $y_\infty \in \overline{co}(K)$. For simplicity, we will denote this subsequence again by $\{a_{f,n}\}$. Notice that

$$P(a_{f,n}) = a_{f,n} + \frac{1}{n}(P^{n+1}f - Pf).$$

But since $P^{n+1}f, Pf \in K$, which is compact, then they are bounded and we get that, as n goes to ∞, $P(y_\infty) = y_\infty$ (by continuity of P). Now,

$$\nu(f) = \lim_{n\to\infty} \frac{1}{n}\sum_{k-1}^{n} \nu(f) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \nu(P^k f) = \nu(y_\infty) = 0.$$

Therefore ν is the zero operator.
ψ is surjective: Let $\eta \in (E^P)^*$. By the Hahn Banach theorem, there exists $\nu \in E^*$ such that $\nu|_{E^P} = \eta$. By Lemma 4.1.1, the sequence $\{b_n = \frac{1}{n}\sum_{k-1}^{n} \nu P^k\}$ is $*$-weakly relatively compact, so it contains a subsequence (denoted by $\{b_n\}$ for simplicity) that converges $*$-weakly to some ν_∞. As before, notice that, for any $f \in E$

$$\nu_\infty P(f) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \nu P^{k+1}(f) = \lim_{n\to\infty} \left(\frac{1}{n}\sum_{k=1}^{n} \nu P^k(f) + \frac{\nu P^{k+1}(f) - \nu P(f)}{n} \right) = \nu_\infty(f).$$

Therefore, $\nu_\infty \in (E^*)^P$ and for any $f \in E^P$, $\nu_\infty(f) = \lim_{n\to\infty} \sum_{k=1}^{n} \nu P^k(f) = \lim_{n\to\infty} \sum_{k=1}^{n} \nu(f) = \nu(f)$. Hence we found $\nu_\infty \in (E^*)^P$ such that $\psi(\nu_\infty) = \nu_\infty|_{E^P} = \nu|_{E^P} = \eta$.

(ii) Let $f \in E$. As before, any cluster point y_∞ of the sequence $\{\frac{1}{n}\sum_{k=1}^{n} P^k f\}$ satisfies

$y_\infty \in E^P$ and $\forall \nu \in (E^*)^P$, $\nu(y_\infty) = \nu(f)$ (which is a constant). Using the isomorphism (i) and the canonical injection $E^P \hookrightarrow (E^P)^{**}$, we deduce that $\{\frac{1}{n}\sum_{k=1}^n P^k f\}$ admits a unique cluster point and so it converges to $y_\infty \in E^P$. So now let $\pi_P : E \longrightarrow E$ be the map $f \longmapsto \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^n P^k f$. Then, π_P is a well defined function and is clearly a projection with $\mathrm{Im}(\pi_P) = E^P$ and $\ker(\pi_P) = E_P$. Hence, $E = E^P \oplus E_P$

(iii) Let $\Pi_P : E^* \longrightarrow E^*$ be the map $\nu \longmapsto \nu\pi_P$ where π_P is defined above. It is easy to see that Π_P is a projection whose image is $(E^*)^P$ and kernel is $(E^*)_P$. Thus, $E^* = (E^*)^P \oplus (E^*)_P$.

□

Proposition 4.1.1. *If $\nu \in E^*$ is P-invariant, then for all $f \in E$, $\int_X f d\nu = \int_X \pi(f) d\nu$ where π is the projection of E onto E^P parallel to E_P.*

Proof. Let $f \in E$, $f = \pi(f) + f_2$ where $f_2 \in E_p$ (Theorem 4.1.2), $\int_X f d\mu = \nu(f) = \nu(\pi(f)) + \nu(f_2)$. Since ν is P-invariant, $\nu(f_2) = \nu(P^k f_2)$ for all k. But $\nu(Pf_2) = \frac{1}{n}\sum_{k-1}^n \nu(Pf_2) = \frac{1}{n}\sum_{k-1}^n \nu(P^k f_2) = \nu(\frac{1}{n}\sum_{k-1}^n P^k f_2) \underset{n\to\infty}{\longrightarrow} 0$ (by definition of E_P and continuity of ν). Therefore, $\displaystyle\int f d\nu = \nu(\pi(f)) = \int \pi(f) d\nu$. □

We end this section by linking the notions covered in this chapter with the ones discussed in Chapter 2.4 concerning μ-stationary measures.

Remark 4.1.1. *(Relation between μ-invariant and P_μ-invariant) Markov Feller operators occur in our thesis as the averaging operator P_μ where μ is a probability measure on $GL_d(\mathbb{R})$. It is important to note that for any probability measure ν on X, one has*

$$\mu * \nu = \nu \iff \nu P_\mu = \nu. \tag{4.1}$$

Indeed, for any $f \in C^0(X)$,

$$(\mu*\nu)(f) = \int f(x) d(\mu*\nu) = \int \left(\int f(gx) d\mu(g) \right) d\nu(x) = \int P_\mu(f(x)) d\nu(x) = \nu(P_\mu(f)) = \nu P_\mu(f).$$

So $\mu * \nu = \nu P_\mu = \nu$.

4.2 Empirical Measures

In this section, we follow the notation of Section 3.2.

Definition 4.2.1. *We define the* ***empirical measures*** *for any* $x \in X$, $b = (b_i)_{i\geq 1} \in B$ *and* $n \geq 1$ *to be the probability measures* $\nu_{x,b,n} := \frac{1}{n}\sum_{k=1}^{n} \delta_{b_k \cdots b_1 x}$.

Theorem 4.2.1. ***Breiman's law of large numbers*** *[Bre60]*

Assume that there exists only one μ*-stationary probability measure* ν *on* X*. Then, for all* x *in* X*, for* β*-almost every* b *in* B*, the sequence of empirical measures* $\nu_{x,b,n}$ *converges* $*$*-weakly to* ν*.*

In fact, we are interested in a stronger version of the statement above that arises from its proof and that is, as observed by Benoist and Quint [BQ12, page 17], very useful even if there are more than one μ-stationary probability measure. We will state it as follows:

Theorem 4.2.2. ***Breiman's law of large numbers bis***

Let X *be a compact metrizable space. Then, for every* x *in* X*, for* β*-almost every trajectory* $b \in B$*, every* $*$*-weak cluster point of the sequence of empirical measures is* μ*-invariant.*

Here is a proposition that we will need in Chapter 5

Proposition 4.2.1. *Suppose that* $\nu_{x,b,n}$ *converges to a probability measure* $\nu_{x.b}$*. Then, for all* $y \in supp(\nu_{x,b})$*,* y *is a limit point (subsequential limit) of the sequence* $\{b_n \cdots b_1 x\}_{n\geq 1}$*.*

Proof. Let $y \in \mathrm{supp}(\nu_{x,b})$, i.e $\forall \epsilon > 0$, $\nu_{x,b}(B(y,\epsilon)) > 0$. Suppose y is not a limit point of $\{b_n \cdots b_1 x\}_{n\geq 1}$. Then, there exists $\epsilon_0 > 0$ and $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $b_n \cdots b_1 x \notin B(y,\epsilon_0)$. Therefore for all $n \geq n_0$, $\nu_{x,b,n}(B(y,\epsilon_0)) \leq \frac{n_0}{n}$. By porte-manteau theorem, we deduce that $\nu_{x,b}(B(y,\epsilon_0)) \leq 0$, i.e. $\nu_{x,b}(B(y,\epsilon_0)) = 0$. This is a contradiction with $y \in \mathrm{supp}(\nu_{x,b})$. □

Chapter 5

Limit Laws on Projective Spaces

In this chapter, we will prove Theorem 1.0.1 at two times: Theorem 5.2.1 and Theorem 5.2.2 below. We recall that our space $X = \mathbb{P}(\mathbb{R}^d)$ is a compact metric space and for any probability measure μ on $GL_d(\mathbb{R})$, the averaging operator P_μ is a Markov-Feller equicontinuous operator by Chapter 3, which allows us to use the results of Chapter 4. Theorem 5.2.1 will follow from the equicontinuity of P_μ (Theorem 3.0.1) and the Decompositions theorems of Chapter 4 (Theorem 4.1.2) while Theorem 5.2.2 will be the result of the equicontinuity of P_μ, Breiman's law of large number (Theorem 4.2.2) and Doob's martingale theorem we recall herebelow.

5.1 Basic Notion of Martingales

As observed by Furstenberg, the theory of Martingales is very useful to describe stationary measures ([Fur63]). In this section, we will just recall the notions that we need for martingale theory. We refer for instance to [Bil12, Chapter 6].

Definition 5.1.1. *A **martingale** is a sequence of random variables $X_1, X_2, \cdots$ such that for all $n \in \mathbb{N}$, $E(|X_n|) < \infty$ and $E(X_{n+1}|X_1, \cdots, X_n) = X_n$*

Definition 5.1.2. *A sequence of random variables $X_1, X_2, \cdots$ satisfying $E(X_{n+1}|X_1, \cdots, X_n) \leq X_n$ is said to be a **supermartingale**.*

Remark 5.1.1. *Any martingale is a supermartingale.*

We now state the following important theorem due to Doob.

Theorem 5.1.1. ***Doob's Martingale Theorem***

Suppose $\{X_n\}$ *is a supermartingale satisfying* $\sup_n E(|X_n|) < \infty$*, then, almost surely, the limit* $X_\infty = \lim_{n\to\infty} X_n$ *exists. Moreover,* $X_\infty \in L^1$*.*

5.2 Construction of Stationary Measures

In this section, we let $X = \mathbb{P}(\mathbb{R}^d)$ and $E = C^0(X)$, the Banach space of continuous function over X, endowed with the supremum norm $||\cdot||$. In this section, we prove the following result:

Theorem 5.2.1. *Let* μ *be a probability measure on* $GL_d(\mathbb{R})$ *such that the action of* Γ_μ *on* $\mathbb{R}^d$ *is strongly irreducible. Then, for every* x *in* X*, the limit probability measure*

$$\nu_x := \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mu^{*k} * \delta_x$$

exists, is μ*-stationary, and depends continuously on* x*.*

Recall from (4.1) that $\mu^{*k} * \delta_x = \delta_x P_\mu^k$ and ν is $\mu - stationary$ if and only if it is P_μ-invariant. Therefore, proving Theorem 5.2.1 is equivalent to proving, under the same hypothesis, the following statement: for every x in X, the limit probability measure

$$\nu_x := \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \delta_x P_\mu^k$$

exists, is P_μ-invariant, and depends continuously on x.

Proof. To simplify notation, let $P = P_\mu$.

Let $x \in X$. Since P is equicontinuous, and $\delta_x \in \mathcal{M}(X) = E^*$, then, by Theorem 4.1.2, there exists $\eta_1 \in (E^*)^P$ and $\eta_2 \in (E^*)_P$ such that $\delta_x = \eta_1 + \eta_2$, which implies that

$$\frac{1}{n} \sum_{k=1}^{n} \delta_x P_\mu^k = \frac{1}{n} \sum_{k=1}^{n} \eta_1 P_\mu^k + \frac{1}{n} \sum_{k=1}^{n} \eta_2 P_\mu^k.$$

By definition, the second term of the above equation converges $*$-weakly to zero as n goes to ∞, and $\eta_1 P^k = \eta_1$. Hence, $\nu_x = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \eta_1 = \eta_1$ exists and is P-invariant. Moreover, since $\mathcal{P}(X)$ is closed in the $*$-weak topology (Proposition 2.3.1), ν_x is a probability measure.

It remains to prove that ν_x depends continuously on x, i.e that the map $\psi : X \longrightarrow \mathcal{P}(X), x \longmapsto \nu_x$ is continuous with $\mathcal{P}(X)$ endowed with the weak-$*$ convergence. Indeed, let $x_n \in X$ that converges to some x. Fix $f \in C^0(X)$. By equicontinuity of P, we can find $\delta > 0$ such that, for every $k \in \mathbb{N}$, $|(P^k f)(x) - (P^k f)(y)| < \epsilon$ whenever $d(x, y) < \delta$. Let $n_0 = n_0(\epsilon) \in \mathbb{N}$ such that $d(x_n, x) < \delta$ for all $n > n_0$. Fix $n > n_0$. We have then for every $k \in \mathbb{N}$ that $|(P^k f)(x) - (P^k f)(y)| < \epsilon$, i.e. $|(\delta_x P^k)(f) - (\delta_{x_n} P^k(f)| < \epsilon$. In particular, for every $k \in \mathbb{N}$,

$$\left| \frac{1}{k}\sum_{l=1}^{k}(\delta_x P_\mu^l)(f) - \frac{1}{k}\sum_{l=1}^{k}(\delta_x P_\mu^l)(f) \right| < \epsilon.$$

Tending k to $+\infty$ (n being fixed), we deduce that $|\nu_x(f) - \nu_{x_n}(f)| < \epsilon$. Thus $\nu_{x_n}(f) \underset{n\to+\infty}{\longrightarrow} \nu_x(f)$. Thus ν_{x_n} converges to ν_x in the weak-$*$ convergence. Hence ψ is sequentially continuous and hence continuous, by the metrizability of the space X.

□

Now, our second result shows how we can construct the same measures of Theorem 5.2.1 using empirical measures. Here, we use the same notation introduced in Section 3.2.

Before we state the theorem, we give a few definitions.

Definition 5.2.1. *A measure ν is said to be μ-**ergodic** if, when a measurable set A satisfies $g^{-1}A = A$ for all $g \in \Gamma_\mu$, we have $\nu(A) = 0$ or $\nu(A^c) = 0$.*

Definition 5.2.2. *We say that the action of μ on X is **uniquely ergodic** if there is a unique μ-invariant probability measure on X.*

Remark 5.2.1. *In fact, the μ-ergodic measures are the extreme points of the set of μ-invariant measures. In the case where X is uniquely ergodic, this implies that the unique measure on X must be ergodic.*

Definition 5.2.3. *Let $P : C^0(X) \longrightarrow C^0(X)$ be an operator, and let $\nu \in \mathcal{P}(X)$. We say that ν is P-ergodic if it is an extreme point of the compact convex set of P-invariant probability measures on X.*

Now we prove our main theorem.

Theorem 5.2.2. *Let μ be a probability measure on $GL_d(\mathbb{R})$ such that the action of Γ_μ on $\mathbb{R}^d$ is strongly irreducible. Then, for every $x \in X$, for β almost every $b \in B$, the limit of the empirical probability measures*

$$\nu_{x,b} := \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \delta_{b_k \cdots b_1 x}$$

exists and is a μ-ergodic μ-stationary probability measure on X. Moreover,

$$\nu_x = \int \nu_{x,b}\ d\beta(b).$$

As before, we note that proving μ-invariance and μ-ergodicity of ν is equivalent to proving its P-invariance and P-ergodicity.

Proof. Fix $x \in X$. Applying Proposition 4.2.2, we get, for β-almost every $b \in B$, every $*$-weak cluster point $\nu_{b,x}$ of the sequence $\nu_{b,x,n} := \frac{1}{n}\sum_{k=1}^{n} \delta_{b_k \cdots b_1 x}$ is μ- invariant (i.e P_μ- invariant).
Claim: For $b \in B$, knowing that $\nu_{b,x} \in (E^*)^{P_\mu}$, then

$$\int f d\nu_{b,x,n} \longrightarrow \int f d\nu_{b,x} \text{ for all } f \in E,$$

if and only if

$$\int f d\nu_{b,x,n} \longrightarrow \int f d\nu_{b,x} \text{ for all } f \in E^{P_\mu}.$$

Proof: Let $f \in E$, $f = \pi(f)+f_2$ where $f_2 \in E_{P_\mu}$. Then, $\displaystyle\int f d\nu_{b,x,n} = \int \pi(f) d\nu_{b,x,n} + \int f_2 d\nu_{b,x,n}$.
By writing $f = f^+ - f^-$ (decomposition into positive and negative parts), we can assume that f_2 is non-negative. By definition, for every $x \in X$,

$$\frac{1}{n}\sum_{k=1}^{n} P^k f_2(x) = \frac{1}{n}\sum_{k=1}^{n}\int_{\mathrm{GL}_d(\mathbb{R})} f_2(gx)d\mu^{*k}(g) = \int_B \frac{1}{n}\sum_{k=1}^{n} f_2(b_n\cdots b_1 x)\, d\beta(b) \underset{n\to+\infty}{\longrightarrow} 0.$$

Since f_2 is non-negative, we deduce that for β-almost every $b \in B$, $\frac{1}{n}\sum_{k=1}^{n} f_2(b_k \cdots b_1 x) \longrightarrow 0$, i.e. $\int_X f_2(y)\ d\nu_{b,x,n}(y) \longrightarrow 0$. Therefore, for every $f \in E$, for almost every $b \in B$,

$$\lim_{n\to\infty}\int f\ d\nu_{b,x,n} = \lim_{n\to\infty}\int \pi(f)\ d\nu_{b,x,n} = \int \pi(f)\ d\nu_{b,x} = \int f\ d\nu_{b,x}$$

because $\nu_{b,x}$ is P-invariant. This proves our claim by the separability of E. The claim implies that to show that $\nu_{b,x,n}$ converges, it is enough to show that $\nu_{b,x,n}(f)$ converges for all $f \in E^{P_\mu}$, for β- almost every $b \in B$.

Let $f \in E^{P_\mu}$ and consider the map

$$\Phi_n : B \longrightarrow \mathbb{R}$$

$$b = (b_1, b_2, \cdots) \longrightarrow f(b_n \cdots b_1 x).$$

Notice that for any $n \in \mathbb{N}$,

$$E[\Phi_{n+1}(b)] = \int f(g \cdot b_n \cdots b_1 x)\ d\mu(g) = P_\mu f(b_n \cdots b_1 x) = f(b_n \cdots b_1 x) = \Phi_n(b)$$

since f is P_μ-invariant. Therefore $\{\Phi_n\}$ is a martingale, and since f is continuous over X which is compact, then Φ_n is bounded on B. Hence, by Doob's Martingale Theorem, $f(b_n \cdots b_1 x)$ converges for β-a.e $b \in B$, and thus the Cesaro sum, which is $\nu_{b,x,n}(f)$, converges to a measure $\nu_{b,x}(f)$, as desired. By compactness of $\mathcal{P}(X)$, we know that $\nu_{b,x}$ is a probability measure.

So, by separability of E, we proved that for $x \in X$, for β-a.e $b \in B$, the limit of the empirical probability measures $\nu_{b,x} = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \delta_{b_n \cdots b_1 x}$ exists and is μ-invariant. It remains to show that it is μ-ergodic for β-a.e $b \in B$. To do so, it is enough to show that for β-a.e $b \in B$, the action of μ on $S_b := \mathrm{supp}(\nu_{b,x})$ is uniquely ergodic, i.e for β-a.e $b \in B$, there is a unique

μ-stationary measure on S_b (see Remark 5.2.1).

Fix a generic sequence $b \in B$, and let $f \in E^{P_\mu}$. We showed that $f(b_n \cdots b_1 x)$ converges to $\ell_f = \nu_{b,x}(f)$ (which is a constant) as n goes to ∞. Then, any limit point of the sequence $\{b_n \cdots b_1 x\}_n$ in X is in the level set $f^{-1}(\ell_f)$, which implies that $S_b \subset f^{-1}(\ell_f)$ (Proposition 4.2.1).

Therefore, for all $f \in E^{P_\mu}$, for all $y \in S_b$, $f(y) = \ell_f$. Suppose now that there exists a μ-invariant measure η such that $\operatorname{supp}(\eta) \subset S_b$, i.e $\eta(S_b) = 1$. Then, for all $f \in E^{P_\mu}$, $\displaystyle\int f(y)\ d\eta(y) = \int \ell_f\ d\eta(y) = \ell_f$. But, since η and $\nu_{x,b}$ are both P_μ-invariant, then by Proposition 4.1.1, it follows that

$$\forall f \in E,\ \int f(y)\ d\eta(y) = \ell_f = \int f(y)\ d\nu_{x,b}(y)$$

and hence $\eta = \nu_{x,b}$.

Now, for any $f \in E$,

$$\begin{aligned}
\int \nu_{x,b}(f) d\beta(b) &= \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \int_B \delta_{b_k \cdots b_1 x}(f) d\beta(b) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \int_{\Gamma_\mu} \delta_{gx}(f) d\mu^{*k}(g) \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \int_{\Gamma_\mu} f(gx) d\mu^{*k}(g) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \int_{\Gamma_\mu} \int_X f(gy) d\delta_x(y) d\mu^{*k}(g) \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mu^{*k} * \delta_x \\
&= \nu_x(f).
\end{aligned}$$

Hence, $\nu_x = \int \nu_{x,b}\ d\beta(b)$ as desired. □

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