AMERICAN UNIVERSITY OF BEIRUT

ADMISSIBLE REPRESENTATIONS OF $GL(2, \mathbb{Q}_p)$, THE KIRILLOV MODEL, AND APPLICATION TO LOCAL NEW VECTORS

by
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To my Mom and Dad for their endless sacrifice, to my sister Rayan and brother Hadi, who stood by my side during the hard times, to my little angel Batoul for the love she brought into our home, I dedicate this work.

An Abstract of the Thesis of

Janane Moussa Krayem for Master of Science
Major: Mathematics

Title: Admissible Representations of $GL(2, \mathbb{Q}_p)$, The Kirillov Model, and Application to Local New Vectors

[Jacquet and Langlands, 1970] showed that every infinite dimensional, irreducible, admissible representation of $GL(2, \mathbb{Q}_p)$ has a unique Kirillov model, i.e. is isomorphic to a representation $(\pi, \mathcal{K})$, where the space $\mathcal{K}$ consists of locally constant functions on $\mathbb{Q}_p^\times$ on which π operates in some special way. The thesis will cover the proof of the above result after constructing the convenient framework. For this purpose I will start by introducing the notion of admissible representations of $GL(2, \mathbb{Q}_p)$ and do some topology on $\mathbb{Q}_p$. After that I will prove the existence of the Kirillov model and prove some properties of the Bruhat-Schwartz space. Then I will prove the uniqueness part, which requires the construction of commutative operators; this part is the heart of the thesis. I shall then study a special example: the principal series representation. The last part of the thesis is dedicated to an application to the local new vectors of a representation.

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Chapter 1

Introduction

The Kirillov model, studied by Kirillov (1963), is a realization of a representation of GL_2 over a local field on a space of functions on the local field. Let F is a non-Archimedean local field, χ_F a fixed nontrivial character of the additive group of F, and π an irreducible representation of $GL(2, F)$, then the Kirillov model for π is a representation π' on a space of locally constant functions f on $F^\times$ with compact support in F such that

$$\pi'\left(\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\right) \cdot f(y) = \chi_F(by)f(ay) \quad (\forall a, y \in F^\times,\ b \in F).$$

[Jacquet and Langlands, 1970] showed that an infinite dimensional admissible irreducible representation of $GL(2, F)$ has an essentially unique Kirillov model. [Casselman, 1973] then extended the theory of newforms, which was known in the sense of modular forms, to the representation theory of $GL(2, F)$.

In this thesis, we limit ourselves to $F = \mathbb{Q}_p$, the p-adic field. However, generalizations to any non-Archimedean local fields F follow easily in most of the theorems.

The second chapter gives a quick tour on the p-adic Fourier analysis that we will be using later. We define the p-adic field, and do some topology on $\mathbb{Q}_p$. We also define the p-adic Schwartz space, and prove the Fourier inversion formula on $\mathbb{Q}_p$.

We define the notion of admissible irreducible representations in the third chapter. We also define the Kirillov model of a representation. We present the proof of the existence and uniqueness of such a model.

In the fourth chapter, we study the application of the Kirillov model on the theory of new forms. We present the theory that allows us to define the local new vectors and the conductor of a representation.

The fifth chapter presents explicitly the Kirillov model for a special kind of representations, called the principal series representations. We prove admissibility of such representations, and study the asymptotic behaviour of functions in the Kirillov space associated to principal series representations.

Chapter 2

Background Theory

In this chapter we establish the basic theoretical tools we will need to generate representations of $GL(2, \mathbb{Q}_p)$. We first outline the definition and properties of p-adic fields, introduce some notions of topology on $\mathbb{Q}_p$, and give some background on the Fourier transform over local fields. The material of this chapter is from [Neukirch, 2013] chapter 2 and [Goldfeld and Hundley, 2011] chapter 1.

2.1 Basic Definitions

Definition 2.1.1. A (multiplicative) absolute value of a field K is a function

$$| \, | : K \to \mathbb{R}^+$$

satisfying the following properties:

1. $|x| \geq 0, \; and \; |x| = 0 \iff x = 0$

2. $|xy| = |x||y|$

3. $|x+y| \leq |x| + |y|$ (triangle inequality)

If the absolute value satisfies the **strong triangle inequality**

$$|x+y| \leq max\{\,|x|, |y|\,\}$$

then it is called **non-Archimedean**, otherwise it is called **Archimedean**.

Let x be a an integer, and let p be a prime number. Then $x = p^m u$, where m is a non-negative integer and $(p,u) = 1$. The exponent m is denoted by $v_p(x)$.

Definition 2.1.2. The p-adic valuation of a non-zero integer x with respect to a prime p, denoted by $v_p(x)$, is defined to be the value of the highest power of p which divides x. For $x = 0$, $v_p(x) = \infty$.

This definition is extended to the rational numbers by defining $v_p(\frac{x}{y}) = v_p(x) - v_p(y)$.

Definition 2.1.3. For $x \in \mathbb{Q}$, the p-adic absolute value of x is defined by:

$$|x|_p = \begin{cases} p^{-v_p(x)} & \text{if } x \neq 0. \\ 0 & \text{if } x = 0. \end{cases}$$

It is easily checked that $|\ |_p$ is a non-Archimedean absolute value over $\mathbb{Q}$.

Definition 2.1.4. The ring of p-adic numbers, denoted by $\mathbb{Q}_p$, is the completion of $\mathbb{Q}$ with respect to $|\ |_p$.

Definition 2.1.5. The ring of integers $\mathbb{Z}_p$ is defined as $\mathbb{Z}_p = \{x \in \mathbb{Q}_p \mid |x|_p \leq 1\}$. Elements of $\mathbb{Z}_p$ are called the p-adic integers.

Proposition 2.1.1. *We have the following properties:*

1. *The set* $p\mathbb{Z}_p = \{x \in \mathbb{Z}_p \mid |x|_p < 1\}$ *is the unique maximal ideal in* $\mathbb{Z}_p$*.*

2. *The set of units in* $\mathbb{Z}_p$ *are:* $\mathbb{Z}_p^\times = \{x \in \mathbb{Z}_p \mid |x|_p = 1\}$*.*

3. $\mathbb{Z}_p = \mathbb{Z}_p^\times \cup p\mathbb{Z}_p$*.*

In order to understand the structure of $\mathbb{Q}_p$ and $\mathbb{Z}_p$, much more properties can be proven. Actually, every p-adic integer x can be represented as a formal power series:

$$\begin{aligned} x &= a_0 + a_1 p + a_2 p^2 + \cdots \\ &= \sum_{n\geq 0} a_n p^n, \end{aligned}$$

where $0 \leq a_n \leq p-1$, called the p-adic digit. This representation is unique, and it is very similar to the decimal fraction representation:

$$a_0 + a_1(\frac{1}{10}) + a_2(\frac{1}{10})^2 + \cdots, \quad 0 \leq a_i < 10.$$

Moreover, every p-adic number $x \in \mathbb{Q}_p$ can be written as:

$$\begin{aligned} x &= a_{-n_0} p^{-n_0} + \cdots + a_1 p + a_2 p^2 + \cdots \\ &= \sum_{n\geq -n_0} a_n p^n, \end{aligned}$$

with $0 \leq a_n \leq p-1$ and $n_0 = v_p(x)$. This representation is unique as well (up to adding terms with zero coefficients). It is conventionally written as

$$(\cdots a_0 \cdot a_{-1} \cdots a_{-n_0+1}\, a_{-n_0})_p.$$

In order to get a more satisfying definition of p-adic numbers, we will define $\mathbb{Z}_p$ using another approach. The material is taken from [Neukirch, 2013].

One can define addition and multiplication of p-adic numbers which turn $\mathbb{Z}_p$ into a ring, and $\mathbb{Q}_p$ into its field of fractions. However, the direct approach, defining sum and product via the usual carry-over rules for digits, as one does it when dealing with real numbers as decimal fractions, leads into complications. They disappear once we use another representation of the p-adic numbers $f = \sum_{v=0}^{\infty} a_v p^v$, viewing them not as sequences of sums of integers

$$s_n = \sum_{v=0}^{n-1} a_v p^v \in \mathbb{Z},$$

but rather as sequences of **residue classes**.

$$\overline{s}_n = s_n \bmod p^n \in \mathbb{Z}/p^n\mathbb{Z}.$$

The terms of such a sequence lie in different rings $\mathbb{Z}/p^n\mathbb{Z}$, but these are related by the canonical projections

$$\mathbb{Z}/p\mathbb{Z} \xleftarrow{\lambda_1} \mathbb{Z}/p^2\mathbb{Z} \xleftarrow{\lambda_2} \mathbb{Z}/p^3\mathbb{Z} \xleftarrow{\lambda_3} \cdots$$

and we find

$$\lambda_n(\overline{s}_{n+1}) = \overline{s}_n.$$

In the direct product

$$\prod_{n=1}^{\infty} \mathbb{Z}/p^n\mathbb{Z} = \{(x_n)_{n\in\mathbb{N}} \mid x_n \in \mathbb{Z}/p^n\mathbb{Z}\},$$

we now consider all elements $(x_n)_{n\in\mathbb{N}}$ with the property that

$$\lambda_n(x_{n+1}) = x_n \quad \text{for all } n = 1, 2, \ldots$$

This set is called the **projective limit** of the rings $\mathbb{Z}/p^n\mathbb{Z}$, and is usually denoted by $\varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$. In other words, we have

$$\varprojlim_n \mathbb{Z}/p^n\mathbb{Z} = \Big\{ (x_n)_{n\in\mathbb{N}} \in \prod_{n=1}^{\infty} \mathbb{Z}/p^n\mathbb{Z} \mid \lambda_n(x_{n+1}) = x_n, \quad n = 1, 2, \dots \Big\}.$$

Proposition 2.1.2. *There is an isomorphism* $\mathbb{Z}_p \xrightarrow{\sim} \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$.

Proof. See [Neukirch, 2013], chapter II proposition 2.5. □

Addition and multiplication extend from $\mathbb{Z}_p$ to $\mathbb{Q}_p$, and $\mathbb{Q}_p$ becomes the field of fractions of $\mathbb{Z}_p$.

Proposition 2.1.3. $\mathbb{Z}_p$ *is compact and* $\mathbb{Q}_p$ *is locally compact.*

Proof. See [Neukirch, 2013], chapter II proposition 5.1. □

2.2 Some Topology on $\mathbb{Q}_p$

Every field with an absolute value defined on it forms a metric space. So it is convenient to discuss the topology of $\mathbb{Q}_p$. The open balls in $\mathbb{Q}_p$ are defined as:

$$B(a,r) = \{x \in \mathbb{Q}_p \mid |x-a|_p < r\}.$$

Similarly, the closed balls are defined as :

$$\overline{B(a,r)} = \{x \in \mathbb{Q}_p \mid |x-a|_p \leq r\}.$$

Let $x, y \in \mathbb{Q}_p$, then $|x-y|_p = p^k$ for some $k \in \mathbb{Z}$; hence it is enough to consider balls of the form $B(a, p^k)$, for $k \in \mathbb{Z}, a \in \mathbb{Q}_p$.

Lemma 2.2.1. *Let* $a, b \in \mathbb{Q}_p$ *:*

1. *If* $b \in B(a, p^k)$, *then* $B(a, p^k) = B(b, p^k)$.

2. *Two balls in* $\mathbb{Q}_p$ *have a non empty intersection if and only if one of them contains the other, i.e.*

$$B(a, p^k) \cap B(b, p^r) \neq \varnothing \iff B(a, p^k) \subset B(b, p^r) \text{ or } B(b, p^r) \subset B(a, p^k).$$

Proof. 1. If $b \in B(a, p^k)$, then $|a-b|_p < p^k$. Let $x \in B(a, p^k)$, then $|x-a|_p < p^k$. Using triangle inequality and the non-Archimedean property of the p-adic absolute value, we get

$$|x-b|_p = |x-a+a-b|_p \leq max\{|x-a|_p, |a-b|_p\} < p^k,$$

so $x \in B(b, p^k)$; it follows that $B(a, p^k) \subset B(b, p^k)$. For the other inclusion, let $x \in B(b, p^k)$ so that $|x-b|_p < p^k$. This implies that

$$|x-a|_p = |x-b+b-a|_p \leq max\{|x-b|_p, |b-a|_p\} < p^k,$$

so $x \in B(a, p^k)$, and $B(b, p^k) \subset B(a, p^k)$.

2. One direction is trivial ($\Leftarrow$).

For the other direction ($\Rightarrow$), we know that there exists $x \in B(a, p^k) \cap B(b, p^r)$. It follows from (1) that

$$B(x, p^k) = B(a, p^k) \quad \text{and} \quad B(x, p^r) = B(b, p^r).$$

If $k \leq r$, then $B(a, p^k) = B(x, p^k) \subset B(x, p^r) = B(b, p^r)$.

If $r \leq k$, then $B(b, p^r) = B(x, p^r) \subset B(x, p^k) = B(a, p^k)$.

□

Now we need to specify our basis of the topology. It follows from the topology of the metric defined on $\mathbb{Q}_p$ that

$$\mathcal{T} = \{B(a, p^{-n}) \text{ such that } x \in \mathbb{Q}_p \text{ and } n \in \mathbb{Z}\}$$

forms a basis of the topology of $\mathbb{Q}_p$. This implies that any open set U in $\mathbb{Q}_p$ can be written as a union of elements of $\mathcal{T}$.

Proposition 2.2.1. $B(a, p^{1-n}) = a + p^n\mathbb{Z}_p, \quad \forall a \in \mathbb{Q}_p,\ n \in \mathbb{Z}.$

Proof.

$$\begin{aligned} x \in a + p^n\mathbb{Z}_p &\iff x - a \in p^n\mathbb{Z}_p \\ &\iff |x-a|_p \leq p^{-n} \\ &\iff |x-a|_p < p^{-n+1} \\ &\iff x \in B(a, p^{1-n}). \end{aligned}$$

□

Therefore, the open sets $a + p^n\mathbb{Z}_p$ form a basis of open sets for $\mathbb{Q}_p$, where $a \in \mathbb{Q}_p$ and n an arbitrary integer. As $p^n\mathbb{Z}_p$ are compact sets, and the function defined by translation is continuous, then $a + p^n\mathbb{Z}_p$ are compact sets as well.

Let us now consider the example of $\mathbb{Z}_7$.

Example 2.2.1. Using figures 2.1 and 2.2 on the top of next page, we can observe that the sphere

$$S(0,1) = \{x \in \mathbb{Z}_7 \mid |x|_7 = 1\} = \bigcup_{x \in \{1,\dots,6\}} B(x, 1),$$

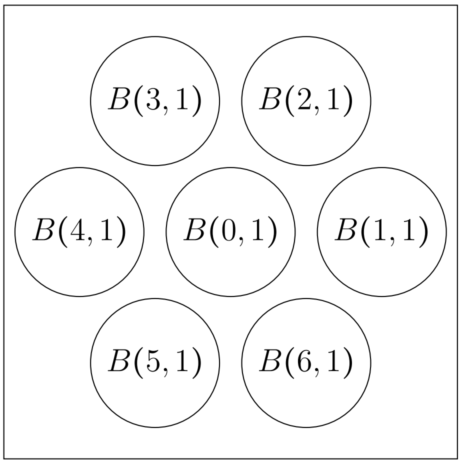

Figure 2.1: $\mathbb{Z}_7$

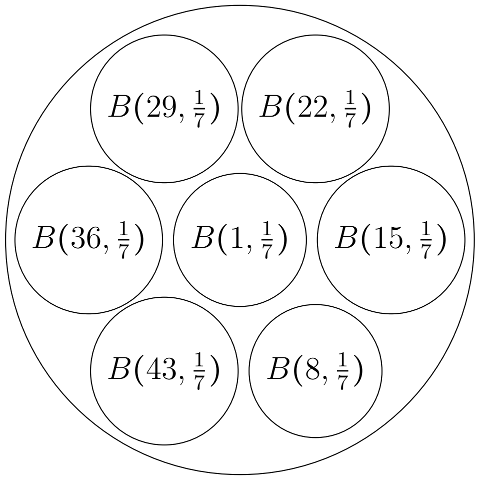

Figure 2.2: A zoom into $B(0,1)$

and that

$$\forall x, y \in \mathbb{Z}_7 \text{ such that } x \in y + 7^k\mathbb{Z}_k, \text{ we have } B(x, 7^{-(k-1)}) = B(y, 7^{-(k-1)}).$$

Moreover, for $x \in \mathbb{Z}_7$ and $k \in \mathbb{N}$,

$$B(x, 7^{-k}) = \bigcup_{j \in \{0, \ldots 6\}} B(x + j \times 7^{k+1}, 7^{-(k+1)}).$$

We generalize this result for $x \in \mathbb{Z}_p$ and $k \in \mathbb{N}$,

$$B(x, p^{-k}) = \bigcup_{j \in \{0, \ldots p-1\}} B(x + j \times p^{k+1}, p^{-(k+1)}).$$

Definition 2.2.1 (p-adic Schwartz function)**.** A function $f : \mathbb{Q}_p \to \mathbb{C}$ is said to be a p-adic Schwartz function if it is locally constant and compactly supported on $\mathbb{Q}_p$, i.e.

1. For every $x \in \mathbb{Q}_p$, there exists an open set $U \subset \mathbb{Q}_p$ containing x such that f is constant on U, i.e. $f(x) = f(u)$ for all $u \in U$.

2. $\text{Supp}(f)$ is compact.

Every locally constant compactly supported function $f : \mathbb{Q}_p \to \mathbb{C}$ can be written as finite linear combination

$$f(x) = \sum_{i=1}^{n} c_i \cdot 1_{U_i}(x),$$

where $c_i \in \mathbb{C}$, $(U_i)_{i=1,\ldots n}$ collection of disjoint open sets in $\mathbb{Q}_p$, and 1_{U_i} is the characteristic function of U_i for $i = 1, 2, \ldots n$.

The space $\mathcal{S}(\mathbb{Q}_p)$ consisting of complex valued p-adic Schwartz functions is called the **p-adic Schwartz space** of $\mathbb{Q}_p$.

2.3 p-Adic Fourier Analysis

We would like to define Haar measure on $\mathbb{Q}_p$. Actually, the Haar measure is defined on locally compact groups in general, so it can be defined on $\mathbb{Q}_p$ as it is a local field. We mean by a local field a field which is equipped by a non trivial absolute value and is locally compact in the corresponding topology. A Haar measure on a locally compact group is a regular Borel measure that is translation invariant. For more details about Haar measure over $\mathbb{Q}_p$, see [Ramakrishnan and Valenza, 2013] Chapter 1 section 2.

Definition 2.3.1 (Haar measure on $\mathbb{Q}_p$)**.** Let $a \in \mathbb{Q}_p$, $n \in \mathbb{Z}$. The Haar measure on $\mathbb{Q}_p$ is the measure satisfying

$$\mu(a + p^n\mathbb{Z}_p) = p^{-n}.$$

We will also use the following notation: $d\mu(x) = dx$.

We can notice that μ is translation invariant and that $\mu(\mathbb{Z}_p) = 1$.

Example 2.3.1. We have that

$$\mathbb{Z}_p^\times = (1 + p\mathbb{Z}_p) \cup (2 + p\mathbb{Z}_p) \cup \cdots \cup (p - 1 + p\mathbb{Z}_p),$$

where the sets on the right hand side are pairwise disjoint open compact sets; this implies that

$$\mu(\mathbb{Z}_p^\times) = \sum_{j=1}^{p-1} \mu(j + p\mathbb{Z}_p) = (p - 1)\mu(p\mathbb{Z}_p) = \frac{p-1}{p}.$$

Definition 2.3.2. (Additive character on $\mathbb{Q}_p$) An additive character $\psi : \mathbb{Q}_p \to \mathbb{C}$ on $\mathbb{Q}_p$ is a continuous function such that

$$\psi(x + y) = \psi(x)\psi(y), \quad |\psi(x)| = 1, \quad x, y \in \mathbb{Q}_p.$$

The minimal n such that ψ is trivial on $p^n\mathbb{Z}_p$ is called the conductor of the additive character ψ.

Definition 2.3.3. (Standard additive character on $\mathbb{Q}_p$) Let $e_p(x)$ be the continuous complex valued homomorphism on $\mathbb{Q}_p$ defined by

$$e_p(x) = e^{-2\pi i \{x\}_p},$$

where

$$\{x\}_p = \begin{cases} \sum_{i=-k}^{-1} a_i p^i, & \text{where } x = \sum_{i=-k}^{\infty} a_i p^i \in \mathbb{Q}_p \text{ with } k > 0,\ 0 \le a_i \le p - 1, \\ 0, & \text{otherwise.} \end{cases}$$

Remark. $e_p(x)$ satisfies the conditions of an additive character on $\mathbb{Q}_p$, and the conductor of e_p is zero. $\{x\}_p$ is intended to be the fractional part (or the tail) of x.

One can prove that any other character on $\mathbb{Q}_p$ is of the form $e_p(ax)$ for some fixed $a \in \mathbb{Q}_p$ (check [Vladimirov et al., 1994] section III).

In classical Fourier analysis, the space of Schwartz functions from $\mathbb{R}$ to $\mathbb{C}$ is preserved by the Fourier transform, and we have the Fourier inversion formula $\hat{\hat{f}}(x) = f(-x)$. We want to get a similar result on $\mathbb{Q}_p$, so it is convenient to consider the space of p-adic Schwartz function i.e. the space of locally constant compactly supported functions on $\mathbb{Q}_p$. We will follow the proof of [Goldfeld and Hundley, 2011] section 1.6.

Definition 2.3.4 (Fourier transform on $\mathbb{Q}_p$)**.** Let $y \in \mathbb{Q}_p$ and let $f : \mathbb{Q}_p \to \mathbb{C}$ be a locally constant compactly supported function. The Fourier transform of f is defined as

$$\hat{f}(x) = (\mathcal{F}f)(x) = \int_{\mathbb{Q}_p} f(y) e_p(-xy)\, dy.$$

Lemma 2.3.1. *For $n \in \mathbb{Z}$,*

$$\int_{p^n\mathbb{Z}_p} e_p(x)\, dx = \begin{cases} p^{-n} & \text{if } n \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $x \in p^n\mathbb{Z}_p$.

If $n \geq 0$, then $\{x\}_p = 0$, so

$$\int_{p^n\mathbb{Z}_p} e_p(x)\, dx = \mu(p^n\mathbb{Z}_p) = p^{-n}.$$

If $n < 0$, let $y = p^n \in p^n\mathbb{Z}_p$ so that $\{y\}_p = p^n$. Define $T : \mathbb{Q}_p \to \mathbb{Q}_p$ by $T(x) = x + y$. Then as μ is translation invariant and as $x + y \in p^n\mathbb{Z}_p$ if and only if $x \in p^n\mathbb{Z}_p$,

$$\begin{aligned}
\int_{p^n\mathbb{Z}_p} e_p(x)\, dx &= \int_{\mathbb{Q}_p} 1_{p^n\mathbb{Z}_p}(x) e_p(x)\, dx \\
&= \int_{\mathbb{Q}_p} 1_{p^n\mathbb{Z}_p}(T(x)) e_p(T(x))\, dx \\
&= \int_{\mathbb{Q}_p} 1_{p^n\mathbb{Z}_p}(x+y) e_p(x+y)\, dx \\
&= e_p(y) \int_{\mathbb{Q}_p} 1_{p^n\mathbb{Z}_p}(x+y) e_p(x)\, dx
\end{aligned}$$

$$= e_p(y) \int_{\mathbb{Q}_p} 1_{p^n\mathbb{Z}_p}(x) e_p(x)\, dx$$
$$= e_p(y) \int_{p^n\mathbb{Z}_p} e_p(x)\, dx.$$

As $e_p(y) = e^{-2\pi i p^n} \neq 1$, we get that $\int_{p^n\mathbb{Z}_p} e_p(x)\, dx = 0$. □

Theorem 2.3.2. *The space of locally constant compactly supported functions on $\mathbb{Q}_p$ is preserved by the Fourier transform.*

Proof. Every locally constant compactly supported function can be written as a finite linear combination of characteristic functions of the compact open sets $a+p^n\mathbb{Z}_p$ with$a \in \mathbb{Q}_p$ and $n \in \mathbb{Z}$. Therefore it suffices to prove that the Fourier transform of $1_{a+p^n\mathbb{Z}_p}$ is a locally constant compactly supported function, as integration is a linear transformation.

$$\begin{aligned}
\widehat{1}_{a+p^n\mathbb{Z}_p}(y) &= \int_{\mathbb{Q}_p} 1_{a+p^n\mathbb{Z}_p}(x) e_p(-xy)\, dx \\
&= \int_{a+p^n\mathbb{Z}_p} e_p(-xy)\, dx \\
&= \int_{p^n\mathbb{Z}_p} e_p(-(a+x)y)\, dx \\
&= e_p(-ay) \int_{p^n\mathbb{Z}_p} e_p(-xy)\, dx \\
&= e_p(-ay) p^{-n} 1_{p^{-n}\mathbb{Z}_p}(y), \qquad (2.1)
\end{aligned}$$

where the last step follows from the previous lemma. Now the function in the last step is supported on $p^{-n}\mathbb{Z}_p$, and it is locally constant since it is the product of locally constant functions. □

Lemma 2.3.3. *For $n \in \mathbb{Z}$ and $y \in \mathbb{Q}_p$,*

$$\int_{\mathbb{Q}_p} 1_{p^n\mathbb{Z}_p}(x) e_p(-xy)\, dx = p^{-n} 1_{p^{-n}\mathbb{Z}_p}(y).$$

Proof. If $y \in p^{-n}\mathbb{Z}_p$ then $xy \in \mathbb{Z}_p$ for all $x \in p^n\mathbb{Z}_p$, so the additive character is trivial and the integral is just $\mu(p^n\mathbb{Z}_p) = p^{-n}$.

If $y \notin p^{-n}\mathbb{Z}_p$, then there is a unique $m \in \mathbb{Z}$ and $u \in \mathbb{Z}_p^\times$ such that $y = p^m u$. Now,

$$y \notin p^{-n}\mathbb{Z}_p \iff p^n y \notin \mathbb{Z}_p \iff p^{m+n}u \notin \mathbb{Z}_p \iff m + n < 0.$$

Applying the change of the variable $t = xy = p^m ux$, and $dt = |p^m u|_p dx = p^{-m} dx$, we get

$$\int_{\mathbb{Q}_p} 1_{p^n\mathbb{Z}_p}(x) e_p(-xy)\, dx = p^m \int_{p^{m+n}\mathbb{Z}_p} e_p(-t)\, dt = 0,$$

where the last step follows from lemma (2.3.1) as $m + n < 0$. □

Theorem 2.3.4. *If $f : \mathbb{Q}_p \to \mathbb{C}$ is locally constant compactly supported function, then $\hat{\hat{f}}(x) = f(-x)$.*

Proof. It suffices to check this for $f = 1_{a+p^n\mathbb{Z}_p}$, $a \in \mathbb{Q}_p$ and $n \in \mathbb{Z}$.

$$\begin{aligned}
\hat{\hat{1}}_{a+p^n\mathbb{Z}_p}(y) &= \int_{\mathbb{Q}_p} \hat{1}_{a+p^n\mathbb{Z}_p}(x) e_p(-xy)\, dx \\
&= \int_{\mathbb{Q}_p} e_p(-ax) p^{-n} 1_{p^{-n}\mathbb{Z}_p}(x) e_p(-xy)\, dx && \text{by equation (2.1)} \\
&= p^{-n} \int_{\mathbb{Q}_p} 1_{p^{-n}\mathbb{Z}_p}(x) e_p(-(a+y)x)\, dx \\
&= 1_{p^n\mathbb{Z}_p}(a+y) && \text{by lemma (2.3.3)} \\
&= \begin{cases} 1 & \text{if } a + y \in p^n\mathbb{Z}_p, \\ 0 & \text{otherwise.} \end{cases} \\
&= \begin{cases} 1 & \text{if } -y \in a - p^n\mathbb{Z}_p = a + p^n\mathbb{Z}_p, \\ 0 & \text{otherwise.} \end{cases} \\
&= 1_{p^n\mathbb{Z}_p}(-y).
\end{aligned}$$

□

Chapter 3

The Kirillov Model

Most of this chapter is taken from [Godement, 1974], chapter 1 and [Goldfeld and Hundley, 2011], chapter 6.

3.1 Admissible Representations of $GL(2, \mathbb{Q}_p)$

In this section, we define the key notions that we need to construct the Kirillov model.

Definition 3.1.1 (Representation of a group on a vector space)**.** Let G be a group and let V be a vector space. A representation of G on V is a group homomorphism $\pi : G \to GL(V)$ where $GL(V)$ is the group of all invertible linear maps $V \to V$.

In other words, a representation is a rule, how to assign a linear transformation of V to each group element in a way that is compatible with the group operation. For $g \in G$ and $v \in V$, one often writes $\pi(g) \cdot v$ to denote the action of $\pi(g)$ on v, where $\pi(g'g'') = \pi(g') \cdot \pi(g'')$ for all $g', g'' \in G$. The ordered pair (π, V) is referred to as a representation, and V is referred to as the space of π.

Remark. If the group G and the vector space V are equipped with topologies, then we shall also require the map $G \times V \to V$, given by $(g, v) \to \pi(g) \cdot v$, to be continuous.

Definition 3.1.2 (Irreducible representation)**.** A representation is said to be irreducible if it has no non-trivial invariant subspaces, i.e. the only G-invariant subspaces are 0 and V.

Definition 3.1.3 (Intertwining maps and isomorphic representations)**.** Let

$$\pi_1 : G \to GL(V_1) \qquad \pi_2 : G \to GL(V_2)$$

be two representations. An intertwining operator is a linear map $L : V_1 \to V_2$ such that

$$L \cdot (\pi_1(g) \cdot v) = \pi_2(g) \cdot (L \cdot v)$$

for all $g \in G, v \in V_1$. If there is an intertwining operator $L : V_1 \to V_2$ which is an isomorphism of vector spaces, then the two representations are said to be isomorphic.

From now on, we shall consider representations (π, V) where $G = GL(2, \mathbb{Q}_p)$.

Lemma 3.1.1. *Let (π, V) be a representation of $GL(2, \mathbb{Q}_p)$ and let $v \in V$. The following are equivalent:*

1. *The function*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \pi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \cdot v, \quad \left(\textit{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{Q}_p)\right),$$

   is a locally constant function.

2. *The stabilizer $Stab_{GL(2,\mathbb{Q}_p)}(v)$ of v in G is open.*

3. There exists a compact open subgroup $K \subset GL(2, \mathbb{Q}_p)$ such that

$$v \in V^K := \{v \in V \mid \pi(x) \cdot v = v \text{ for all } x \in K\}.$$

V^K is called the set of K-fixed vectors in V.

Definition 3.1.4 (Smooth representation)**.** Let (π, V) be a representation of $GL(2, \mathbb{Q}_p)$. We say (π, V) is smooth if every $v \in V$ satisfies one of the above conditions of lemma 3.1.1.

Definition 3.1.5 (Admissible representation)**.** Let (π, V) be a representation of $GL(2, \mathbb{Q}_p)$. We say (π, V) is admissible if it is smooth and, in addition, for every compact open subgroup $K \subset GL(2, \mathbb{Q}_p)$ the dimension of the space V^K is finite.

Lemma 3.1.2. *$GL(2, \mathbb{Z}_p)$ is compact.*

Proof. See [Bump, 1998] Proposition 4.5.2. □

Lemma 3.1.3. *For each $n \geq 1$, define the subgroup K_n of $GL(2, \mathbb{Q}_p)$ by*

$$K_n = \{k \in GL(2, \mathbb{Z}_p) \mid k \equiv I_2 \, (mod \; p^n \mathbb{Z}_p)\}.$$

Then K_n is a compact open subgroup of $GL(2, \mathbb{Q}_p)$, and the collection $\{K_n\}_{n \geq 1}$ constitutes a basis of open neighborhoods of the identity.

Proof. This follows from the topology on $GL(2, \mathbb{Q}_p)$, which is induced from that on $M_2(\mathbb{Q}_p)$. For more details, see [Ramakrishnan and Valenza, 2013] chapter 1. □

Remark. If G is a topological group, then for any $g \in G$, the map given by left multiplication by g is a homeomorphism. Consequently, to determine a basis of open sets for G, it suffices to determine a basis of open neighborhoods around the identity. The collection $\{K_n\}_{n \geq 1}$ mentioned in the lemma 3.1.2 gives us the required basis for $GL(2,\mathbb{Q}_p)$. Now, every compact open subset of $GL(2,\mathbb{Q}_p)$ is a finite disjoint union of cosets of the subgroup

$$K_n = \{k \in GL(2,\mathbb{Z}_p) \,|\, k \equiv I_2 \,(\text{mod } p^n\mathbb{Z}_p)\}$$

for some n non-negative integer. Therefore one gets that a representation (π, V) is smooth if and only if for every $v \in V$, there exists some $n \in \mathbb{N}$ such that $\pi(k) \cdot v = v$ for all $k \in K_n$.

Definition 3.1.6. Let G be a topological group. Then:

1. A *quasi character* is a continuous homomorphism from G to $\mathbb{C}^\times$.
2. A *character* is a continuous homomorphism from G to $S^1 = \{z \in \mathbb{C} \,|\, |z| = 1\}$.

Lemma 3.1.4. *Let $\mu : \mathbb{Q}_p^\times \to \mathbb{C}$ be a quasi character, then there exists $n \in \mathbb{N}$ such that $\mu(1 + p^n\mathbb{Z}_p) = 1$.*

Proof. We have $\mu(1) = \mu(1 \cdot 1) = \mu(1) \cdot \mu(1)$, so that $\mu(1) = 1$. Let V be a small open neighborhood of 1 in $\mathbb{C}^\times$, then the only subgroup of $\mathbb{C}^\times$ contained in V is $\{1\}$. □

Now it is reasonable to have the following definition.

Definition 3.1.7. Let $\mu : \mathbb{Q}_p^\times \to \mathbb{C}$ be a quasi character. Then the conductor of μ is defined to be zero if $\mu|_{\mathbb{Z}_p^\times}$ is trivial. Otherwise, it is defined to be the least $n \in \mathbb{N}$ such that $\mu(1 + p^n\mathbb{Z}_p) = 1$.

Lemma 3.1.5. *$SL(2,\mathbb{Q}_p)$ is generated by $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}$ for $x, y \in \mathbb{Q}_p$.*

Proof. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Q}_p)$, so that $ad - bc = 1$.

If $c = 0$, then $ad = 1$ and we have

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix}$$

Otherwise, if $c \in \mathbb{Q}_p^\times$, then we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & c^{-1}a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -c & 0 \\ 0 & -c^{-1} \end{pmatrix} \begin{pmatrix} 1 & c^{-1}d \\ 0 & 1 \end{pmatrix}.$$

We get that $SL(2, \mathbb{Q}_p)$ is generated by the matrices $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}$, and $\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}$, with $x, y \in \mathbb{Q}_p$ and $z \in \mathbb{Q}_p^\times$. But we have

$$\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ z^{-1} - 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ z - 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -z^{-1} \\ 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

for all $z \in \mathbb{Q}_p^\times$, and so we get our result. □

Lemma 3.1.6. *$SL(2, \mathbb{Q}_p)$ is the commutator subgroup of $GL(2, \mathbb{Q}_p)$.*

Proof. We usually denote the commutator subgroup of a group G by $[G : G]$.
Let $G = GL(2, \mathbb{Q}_p)$, and let $[g, h] \in [G : G]$, then $\det(ghg^{-1}h^{-1}) = 1$ and we get the first inclusion $[G : G] \subset SL(2, \mathbb{Q}_p)$. For the other inclusion, let $x, y \in \mathbb{Q}_p$, we have

the following identities:

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x/2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & x/2 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{-1}$$

$$\begin{pmatrix} 1 & 0 \\ y & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ y/2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ y/2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{-1}.$$

This gives us that $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}$ are commutators. But these generate $SL(2, \mathbb{Q}_p)$, hence we get that $SL(2, \mathbb{Q}_p) \subset [G : G]$. □

Lemma 3.1.7 (Schur's lemma for irreducible smooth representations)**.** *Let (π, V) be an irreducible smooth representation of $GL(2, \mathbb{Q}_p)$. Let $T : V \to V$ be an intertwining map, then there exists a constant $c \in \mathbb{C}$ such that $T \cdot v = c \cdot v$ for all $v \in V$.*

Proof. See [Goldfeld and Hundley, 2011] Chapter 6, lemma 6.1.8. □

It turns out that smooth representations of G on finite dimensional vector spaces are not very interesting. In such a case, admissibility will be automatic. The key assumption then is smoothness. Actually we will prove that a finite dimensional smooth irreducible representation of $GL(2, \mathbb{Q}_p)$ factors through the determinant and some character of $\mathbb{Q}_p^\times$. This is expected because characters are actually one dimensional representations. The difficulty arises for infinite dimensional representations, and this will be discussed in detail in the next section. We first consider the following theorem where the dimension of V is finite.

Theorem 3.1.8. *Let (π, V) be a finite dimensional irreducible smooth representation of $GL(2, \mathbb{Q}_p)$. Then V is one dimensional and there is a quasi-character ω of $\mathbb{Q}_p^\times$ such that $\pi(g) = \omega(det(g))$.*

Proof. Choose a basis for V and for each basis vector choose a compact open subgroup stabilizing it. The intersection of these subgroups is still compact and open, and fixes everything in V. So the kernel of the representation, say H, contains a compact open subgroup, and so it is open. Thus there exists $\epsilon > 0$ such that the matrix $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ belongs to H for all $|x|_p < \epsilon$. Now let $x \in \mathbb{Q}_p$, then there exists an element $a \in \mathbb{Q}_p^\times$ such that $|ax|_p < \epsilon$. So we get

$$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & ax \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

belongs to H for all $x \in \mathbb{Q}_p$, as H is a normal subgroup of $GL(2, \mathbb{Q}_p)$. Similarly $\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}$ is in the kernel of π for all $y \in \mathbb{Q}_p$. It follows that $SL(2, \mathbb{Q}_p) \subset H$ as these two types of matrices generate $SL(2, \mathbb{Q}_p)$. But $SL(2, \mathbb{Q}_p)$ is the commutator subgroup of $GL(2, \mathbb{Q}_p)$, so it acts trivially, and $\pi(g_1)\pi(g_2) = \pi(g_2)\pi(g_1)$ for all $g_1, g_2 \in GL(2, \mathbb{Q}_p)$. It follows by Schur's lemma that each $\pi(g)$ acts by a scalar on all of V, and there exists some constant $c(g) \in \mathbb{C}$ such that $\pi(g) \cdot v = c(g) \cdot v$ for all $v \in V$. This is true for all $g \in GL(2, \mathbb{Q}_p)$, so using the irreducibilty of (π, V), we conclude that the dimension of V is one.

Now we need to check that π factors through the determinant. For each $g \in GL(2, \mathbb{Q}_p)$, define the function

$$\begin{aligned} \lambda : GL(2, \mathbb{Q}_p) &\to \mathbb{C}^\times \\ g &\to c(g) \end{aligned}$$

such that $\pi(g) \cdot v = c(g) \cdot v$ for all $v \in V$. It is easily proven that this function is a homomorphism. Observe that $det : GL(2, \mathbb{Q}_p) \to \mathbb{Q}_p^\times$ is surjective with kernel $SL(2, \mathbb{Q}_p)$; hence $GL(2, \mathbb{Q}_p)/SL(2, \mathbb{Q}_p) \cong \mathbb{Q}_p^\times$. But as $SL(2, \mathbb{Q}_p) \subset H$, then $SL(2, \mathbb{Q}_p)$

is in the kernel of λ as well, and so λ must factor through the determinant. We get a homomorphism $\omega : \mathbb{Q}_p^\times \to \mathbb{C}^\times$ such that $c(g) = \omega(det(g))$. It follows from the smoothness of π that H contains a compact open subgroup K_n for some non negative integer n, which shows that $\omega(det(K_n)) = \omega(1 + p^n\mathbb{Z}_p) = 1$. Hence ω is continuous. □

Because of this result, we can confine our attention to infinite dimensional representations.

Proposition 3.1.1. *$GL(2, \mathbb{Q}_p)$ is generated by the matrices* $\begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}, \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ *and* $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Proof. Observe that

$$\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Lemma 3.1.4 shows that we can generate $SL(2, \mathbb{Q}_p)$. Now, if $g \in GL(2, \mathbb{Q}_p)$ has determinant a, then $g = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g'$ for some $g' \in SL(2, \mathbb{Q}_p)$. □

From the above proposition, it is enough to know the action of generator matrices on V in order to determine the representation π completely. It turns out that diagonal matrices act by some special character called the central character. The proof of this fact is a direct result of **Schur's lemma**.

Proposition 3.1.2. *(Central Character) Let V be a complex vector space and let (π, V) be an irreducible smooth representation of $GL(2, \mathbb{Q}_p)$. Then there exists a* *unique multiplicative character* $\omega_\pi : \mathbb{Q}_p^\times \to \mathbb{C}^\times$ *such that*

$$\pi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) \cdot v = \omega_\pi(a) \cdot v, \quad (\forall a \in \mathbb{Q}_p^\times,\ v \in V).$$

The character ω_π *is called the central character associated to* (π, V)*.*

Proof. Let $a \in \mathbb{Q}_p^\times$ and let $z = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ be in the center of $GL(2, \mathbb{Q}_p)$. Then $\pi(z) \cdot (\pi(g) \cdot v) = \pi(g) \cdot (\pi(z) \cdot v)$ for all $g \in GL(2, \mathbb{Q}_p)$, $v \in V$. Thus $\pi(z) : V \to V$ is an intertwining map. By Schur's lemma, there exists a constant $c(a) \in \mathbb{C}$ such that $\pi(z) \cdot v = c(a) \cdot v$ for all $v \in V$. Let $\omega_\pi(z) \coloneqq c(a)$. Now let us check that ω_π is a multiplicative character. Let $z = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$, $z' = \begin{pmatrix} a' & 0 \\ 0 & a' \end{pmatrix}$ be any two elements in the center of $GL(2, \mathbb{Q}_p)$. As $\pi(zz') \cdot v = \pi(z) \cdot (\pi(z') \cdot v)$, we get that $c(aa') = c(a)c(a')$ for all $a, a' \in \mathbb{Q}_p^\times$. □

3.2 Existence of the Kirillov Model

In this section, we present the main theorem which treats with local representations of $GL(2)$ over non-Archimedean places. This requires the following definition.

Definition 3.2.1 (Kirillov Representation)**.** Fix a prime p and let $\mathcal{K}$, X be two non-trivial complex vector spaces. Let $(\pi, \mathcal{K})$ be a representation of $GL(2, \mathbb{Q}_p)$ such that $\mathcal{K}$ consists of locally constant functions $f : \mathbb{Q}_p^\times \to X$ where π operates as:

$$\pi\left(\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\right) \cdot f(y) = e_p(by) f(ay) \quad (\forall f \in \mathcal{K},\ a, y \in \mathbb{Q}_p^\times,\ b \in \mathbb{Q}_p)$$

Then $(\pi', \mathcal{K})$ is called a Kirillov representation and the vector space $\mathcal{K}$ is called a

Kirillov space.

Theorem 3.2.1 (**Main Theorem**). *Let p be a prime number and let V be complex infinite dimensional vector space. Assume that (π, V) is an admissible irreducible representation of $GL(2, \mathbb{Q}_p)$. Then (π, V) is isomorphic to one and only one Kirillov representation $(\pi', \mathcal{K})$ whose space of functions is complex valued. $(\pi', \mathcal{K})$ is called the Kirillov model of the representation (π, V).*

Remark. The terminology for $\mathcal{K}$ is apt because it was Kirillov who first proved that (over any local field) every irreducible *unitary* representation (that is, representations where the group action is also required to respect the inner product of the vector space) remains irreducible upon restriction to the Borel subgroup $\left\{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\right\}$. (This paragraph is mentioned in [Gelbart, 1975] section 4).

The proof of this theorem needs some work, and is done over several steps. In this section, our goal is to prove the existence of such a model. This requires the following lemma.

Lemma 3.2.2. *Let (π, V) be an infinite dimensional irreducible admissible representation of $GL(2, \mathbb{Q}_p)$. Then there is no nonzero vector invariant by all the matrices $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, $x \in \mathbb{Q}_p$.*

Proof. Assume that there is some $v \in V$ such that $\pi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) \cdot v = v$ for all $x \in \mathbb{Q}_p$. Let H be the stabilizer of the space $\mathbb{C}v$, i.e.

$$H = \{g \in GL(2, \mathbb{Q}_p) \mid \pi(g) \cdot v = \lambda \cdot v \text{ for some } \lambda \in \mathbb{C}\}$$

First, the subgroup of matrices $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ that fix v is contained H. It follows from the smoothness of π that $\pi\left(\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}\right) \cdot v = v$ for those $c \in \mathbb{Q}_p$ such that $|c|_p$ is sufficiently small. Fix such a matrix $\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \in H$. Then

$$\begin{pmatrix} 0 & -c^{-1} \\ c & 0 \end{pmatrix} = \begin{pmatrix} 1 & -c^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}\begin{pmatrix} 1 & -c^{-1} \\ 0 & 1 \end{pmatrix} \in H.$$

This implies that for any $x \in \mathbb{Q}_p$,

$$\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} = \begin{pmatrix} 0 & -c^{-1} \\ c & 0 \end{pmatrix}\begin{pmatrix} 1 & -xc^{-2} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -c^{-1} \\ c & 0 \end{pmatrix}^{-1} \in H.$$

Thus, $SL(2, \mathbb{Q}_p) \in H$. On the other hand, we have that $\pi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) \cdot v = \omega(a) \cdot v$ for all $a \in \mathbb{Q}_p^\times$, where ω is the central character associated to π. Therefore, H also contains the center Z of $GL(2, \mathbb{Q}_p)$. So H contains the subgroup of $GL(2, \mathbb{Q}_p)$ with elements of square determinant. The index of such group is finite. As $Z \cdot SL(2, \mathbb{Q}_p)$ stabilizes $\mathbb{C}v$, hence, $\{\pi(g) \cdot v \mid g \in GL(2, \mathbb{Q}_p)\}$ spans a finite dimensional invariant subspace. But (π, V) is both infinite dimensional and irreducible, so the only such subspace is the zero subspace. Thus v must be zero. □

Now we restate the existence of the Kirillov model and we prove it.

Theorem 3.2.3. *Let V be a complex infinite dimensional vector space. Then every admissible irreducible representation (π, V) of $GL(2, \mathbb{Q}_p)$ is isomorphic to some Kirillov representation $(\pi', \mathcal{K})$. Moreover, every function $f \in \mathcal{K}$ vanishes outside a compact subset of $\mathbb{Q}_p$.*

Proof. Following [Jacquet and Langlands, 1970], [Godement, 1974] and

[Goldfeld and Hundley, 2011], we complete the proof of this theorem in several steps. Let $V_0 \subset V$ be the subset defined by

$$V_0 := \left\{ v \in V \mid \int_{p^{-n}\mathbb{Z}_p} e_p(-u)\pi\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right) \cdot v\, du = 0, \quad \text{for all large } n \right\}.$$

Step 1: V_0 is a subspace of V.

Subproof. Let $v, v' \in V_0$, then there is some $n, m \in \mathbb{N}$ large enough such that

$$\int_{p^{-n}\mathbb{Z}_p} e_p(-u)\pi\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right) \cdot v\, du = 0 \quad \text{and} \quad \int_{p^{-m}\mathbb{Z}_p} e_p(-u)\pi\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right) \cdot v'\, du = 0$$

If $m \geq n$, then $p^{-n}\mathbb{Z}_p \subset p^{-m}\mathbb{Z}_p$. It follows that

$$\begin{aligned} &\int_{p^{-m}\mathbb{Z}_p} e_p(-u)\pi\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right) \cdot v\, du \\ &= \sum_{y \in p^{-m}\mathbb{Z}_p / p^{-n}\mathbb{Z}_p} e_p(-y)\pi\left(\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}\right) \int_{p^{-n}\mathbb{Z}_p} e_p(-u)\pi\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right) \cdot v\, du \\ &= 0. \end{aligned}$$

Thus $\int_{p^{-m}\mathbb{Z}_p} e_p(-u)\pi\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right) \cdot (v + v')\, du = 0$, and so $v + v' \in V_0$. Moreover, V_0 is closed under the multiplication by scalars in $\mathbb{C}$. It follows that V_0 is a subspace of V. ∎

For each $v \in V$, define $f_v : \mathbb{Q}_p^\times \to V/V_0$ by $f_v(y) := \pi\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) \cdot v \pmod{V_0}$, for all

$y \in \mathbb{Q}_p^\times$. Define the space of functions $\mathcal{K}$ as

$$\mathcal{K} = \left\{ f_v : \mathbb{Q}_p^\times \to V/V_0 \mid v \in V \right\}.$$

Define the map L by

$$V \to \mathcal{K}$$
$$v \to L(v) \coloneqq f_v$$

where f_v is defined above on $\mathbb{Q}_p^\times$. We claim that L is an isomorphism of vector spaces.

Step 2: The map $L : V \to K$ is an isomorphism of vector spaces.

Subproof. Notice that L is surjective by definition. We need only to prove that it is injective. Assume there exists some $v \in V$ such that $L(v) = 0$, i.e. $f_v(y) = 0$ for all $y \in \mathbb{Q}_p^\times$. Define the function $\phi : \mathbb{Q}_p \to V$ by $\phi(u) \coloneqq \pi\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right) \cdot v$, for $u \in \mathbb{Q}_p$. If we can prove that $\phi(u)$ is constant, then it follows from lemma 3.2.2 that v must be zero, and consequently L is injective. In particular, $V_0 \neq V$ and $X \neq 0$.

It only remains to prove that $\phi(u)$ is constant for all $u \in \mathbb{Q}_p$. As $L(v) = 0$, then the function

$$\phi_n(y) \coloneqq \int_{p^{-n}\mathbb{Z}_p} e_p(-uy)\pi\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right) \cdot v \, du = \int_{p^{-n}\mathbb{Z}_p} e_p(-uy)\phi(u) \, du$$

vanishes for some large n (where n depends on y). Furthermore, for any compact set $K \subset \mathbb{Q}_p^\times$, a sufficiently large n can be chosen so that $\phi_n(y) = 0$ for all $y \in K$ (i.e. n can be made independent of y on compact sets). Let $\psi : \mathbb{Q}_p \to \mathbb{C}$ be a locally constant compactly supported function such that $\int_{\mathbb{Q}_p} \psi(u) \, du = 0$, or equivalently, $\hat{\psi}(0) = 0$

(this is also equivalent to saying that ψ is orthogonal to the constant function 1). We also know that the Fourier transform $\hat{\psi}(y) = \int_{\mathbb{Q}_p} e_p(-uy)\psi(y)\,du$ vanishes outside a compact subset K of $\mathbb{Q}_p^\times$ by Theorem 2.3.2.

Now using the Fourier inversion formula given in Theorem 2.3.4, we get that

$$\begin{aligned}
\int_{\mathbb{Q}_p} \psi(u)\phi(u)\,du &= \int_{p^{-n}\mathbb{Z}_p} \psi(u)\phi(u)\,du && \text{(as } \psi \text{ is compactly supported)}\\
&= \int_{p^{-n}\mathbb{Z}_p} \phi(u)\Big(\int_{\mathbb{Q}_p} \hat{\psi}(t)e_p(tu)\,dt\Big)\,du && \text{(by Fourier inversion formula)}\\
&= \int_{p^{-n}\mathbb{Z}_p} \phi(u)\Big(\int_{K} \hat{\psi}(t)e_p(tu)\,dt\Big)\,du && \text{(where } K \text{ is the support of } \hat{\psi})\\
&= \int_{K} \phi_n(-t)\hat{\psi}(t)\,dt = 0. && (3.1)
\end{aligned}$$

Here we choose n such that the support of ψ lies in $p^{-n}\mathbb{Z}_p$, and $\phi_n(y) = 0$ for all $y \in K$, where K is the support of $\hat{\psi}$. We have shown that the integral (3.1) vanishes. This implies that $\phi : \mathbb{Q}_p \to V$ is orthogonal to every locally constant compactly supported function $\psi : \mathbb{Q}_p \to \mathbb{C}$ which is orthogonal to the constant function 1. So we get that

$$\int_{\mathbb{Q}_p} \psi(u)\Big(\phi(u) - 1\Big)\,du = 0.$$

It follows then that ϕ itself must be the constant function, i.e.

$$\pi\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right) \cdot v = v \qquad \text{for all } u \in \mathbb{Q}_p.$$

■

This makes it possible to identify each element v of V with a V/V_0-valued function f_v on $\mathbb{Q}_p^\times$, so that $GL(2,\mathbb{Q}_p)$ operates on $\mathcal{K}$ through π' in a way such that

$$\pi'(g) \cdot f_v(y) \coloneqq f_{\pi(g)\cdot v}$$

$$= \pi\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \cdot (\pi(g)\cdot v)\right) \quad (\text{mod } V_0).$$

We will show that $(\pi', \mathcal{K})$ is a Kirillov representation which is isomorphic to (π, V).

Step 3: Each $f_v \in \mathcal{K}$ satisfies $\pi'\left(\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\right) \cdot f_v(y) = e_p(by) f_v(ay)$.

Subproof. Let $a, y \in \mathbb{Q}_p^\times$ and $b \in \mathbb{Q}_p$. On one side, we have

$$\begin{aligned}
\pi'\left(\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\right) \cdot f_v(y) &= \pi\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \cdot \left(\pi\left(\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\right) \cdot v\right)\right) \\
&= \pi\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\right) \cdot v \\
&= \pi\left(\begin{pmatrix} ya & yb \\ 0 & 1 \end{pmatrix}\right) \cdot v \qquad \text{mod } (V_0).
\end{aligned}$$

On the other side, we have: $e_p(by) f_v(ay) = e_p(by)\pi\left(\begin{pmatrix} ay & 0 \\ 0 & 1 \end{pmatrix}\right) \cdot v \quad \text{mod } (V_0).$

We must show that

$$\int_{p^{-n}Z_p} e_p(-u)\pi\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right) \cdot \left(\left[\pi\left(\begin{pmatrix} ya & yb \\ 0 & 1 \end{pmatrix}\right) - e_p(by)\pi\left(\begin{pmatrix} ya & 0 \\ 0 & 1 \end{pmatrix}\right)\right] \cdot v\right) du = 0$$

for n large enough. Choose n_0 sufficiently large so that $|by|_p < p^{n_0}$. Let $n \geq n_0$, then using the change of variable $u \to u - by$, we get that

$$\begin{aligned}
&\int_{p^{-n}\mathbb{Z}_p} e_p(-u)\pi\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right) \cdot \left(\pi\left(\begin{pmatrix} ya & yb \\ 0 & 1 \end{pmatrix}\right) \cdot v\right) du \\
&= \int_{p^{-n}\mathbb{Z}_p} e_p(-u + by)\pi\left(\begin{pmatrix} 1 & u - by \\ 0 & 1 \end{pmatrix}\right) \cdot \left(\pi\left(\begin{pmatrix} ya & yb \\ 0 & 1 \end{pmatrix}\right) \cdot v\right) du
\end{aligned}$$

$$= \int_{p^{-n}\mathbb{Z}_p} e_p(-u+by)\pi\left(\begin{pmatrix} ya & u \\ 0 & 1 \end{pmatrix}\right)\cdot v\,du.$$

The difference of the right and left hand sides is then zero, which is exactly what we want. ■

Step 4: Each f_v is locally constant and vanishes outside a compact subset of $\mathbb{Q}_p$.

Subproof. The action of π on V is smooth, then for all $v \in V$, if $a \in \mathbb{Q}_p^\times$ with $|a-1|_p$ sufficiently small, then $\pi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)\cdot v = v$. This implies that

$$f_v(ay) = \pi\left(\begin{pmatrix} ya & 0 \\ 0 & 1 \end{pmatrix}\right)\cdot v = \pi\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right)\pi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)\cdot v = \pi\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right)\cdot v = f_v(y)$$

for all $y \in \mathbb{Q}_p^\times$, provided that $a \in \mathbb{Q}_p^\times$ with $|a-1|_p$ sufficiently small. So f_v is locally constant. Similarly, by smoothness of π on V there exists a sufficiently large positive integer m such that $\pi\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right)\cdot v = v$ for $|b|_p < p^{-m}$. Then by step 2, it follows that

$$\begin{aligned} e_p(by)f_v(y) = \pi'\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right)\cdot f_v(y) &= \pi\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\cdot\left(\pi\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right)\cdot v\right)\right) \\ &= \pi\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\cdot v\right) = f_v(y) \end{aligned}$$

for all $y \in \mathbb{Q}_p^\times$ and $|b|_p < p^{-m}$. Thus f_v has a compact support in $\mathbb{Q}_p$. ■

This completes the proof of the stated theorem. □

3.3 The Bruhat-Schwartz Space

Let V_0 be the subspace defined in the proof of Theorem 3.2.1. The material of this section is from [Godement, 1974] and [Goldfeld and Hundley, 2011].

Definition 3.3.1 (normalized unitary character)**.** A normalized unitary character of $\mathbb{Q}_p^\times$ is a continuous function $\omega : \mathbb{Q}_p^\times \to \mathbb{C}$ which satisfies

- $\omega(yy') = \omega(y)\omega(y'), \qquad (\forall\, y, y' \in \mathbb{Q}_p^\times)$
- $|\omega(y)|_{\mathbb{C}} = 1, \qquad (\forall\, y \in \mathbb{Q}_p^\times)$
- $\omega(p) = 1.$

Definition 3.3.2. We define the Bruhat-Schwartz spaces as:

$$S_X(\mathbb{Q}_p^\times) := \Big\{ f : \mathbb{Q}_p^\times \to X \mid f \text{ is locally constant, and } \exists N_f > \epsilon_f > 0$$
$$\text{such that } f(y) = 0 \text{ if } |y|_p < \epsilon_f \text{ or } |y|_p > N_f \Big\},$$
$$S(\mathbb{Q}_p^\times) := \Big\{ f : \mathbb{Q}_p^\times \to \mathbb{C} \mid f \text{ is locally constant, and } \exists N_f > \epsilon_f > 0$$
$$\text{such that } f(y) = 0 \text{ if } |y|_p < \epsilon_f \text{ or } |y|_p > N_f \Big\}.$$

For the rest of this chapter, let $X = V/V_0$.

Proposition 3.3.1. *The Bruhat-Schwartz subspace $S(\mathbb{Q}_p^\times)$ is irreducible under the action of the subgroup* $B(\mathbb{Q}_p) = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a \in \mathbb{Q}_p^\times,\, b \in \mathbb{Q}_p \right\}$.

Proof. Consider the representation $\pi : B(\mathbb{Q}_p) \to GL(S(\mathbb{Q}_p^\times))$, with the action

$$\pi\left(\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\right) \cdot f(y) = e_p(by)f(ay), \quad (\forall f \in S(\mathbb{Q}_p^\times),\, a, y \in \mathbb{Q}_p^\times,\, b \in \mathbb{Q}_p).$$

Notice that $S(\mathbb{Q}_p^\times)$ is stable under the action above.

Every element of $S(\mathbb{Q}_p^\times)$ can be written as a finite linear combination of characteristic functions of cosets $a + p^n\mathbb{Z}_p$, $n \in \mathbb{Z}$, $a \in \mathbb{Q}_p$, $a \notin p^n\mathbb{Z}_p$. We will prove that, for all a and n, the characteristic function $1_{a+p^n\mathbb{Z}_p}$ is a linear combination of translates of $1_{1+p^n\mathbb{Z}_p}$, by the action of $B(\mathbb{Q}_p)$ given above.

First, suppose that $a = 1$ and $n = 2$. For $1 \le i \le p$ and $y = 1 + y_1 p + y_2 p^2 + \dots$ with $y_j \in \{0, 1, \dots, p-1\}$ for $j = 1, 2, \dots$ we have

$$\pi\left(\begin{pmatrix} 1 & i/p^2 \\ 0 & 1 \end{pmatrix}\right) \cdot 1_{1+p\mathbb{Z}_p}(y) = e_p(\frac{i}{p^2} + \frac{i \cdot y_1}{p}) \cdot 1_{1+p\mathbb{Z}_p}(y).$$

It follows that

$$\begin{aligned}
&\frac{1}{p}\sum_{i=1}^{p} e_p(-\frac{i}{p^2}) \cdot \pi\left(\begin{pmatrix} 1 & i/p^2 \\ 0 & 1 \end{pmatrix}\right) \cdot 1_{1+p\mathbb{Z}_p}(y) \\
&= \frac{1}{p}\sum_{i=1}^{p} e_p(-\frac{i}{p^2}) e_p(\frac{i}{p^2} + \frac{i \cdot y_1}{p}) \cdot 1_{1+p\mathbb{Z}_p}(y) \\
&= \frac{1}{p}\sum_{i=1}^{p} e_p(\frac{i \cdot y_1}{p}) \cdot 1_{1+p\mathbb{Z}_p}(y) \\
&= \begin{cases} \dfrac{1}{p}\displaystyle\sum_{i=1}^{p} 1 & \text{if } y_1 = 0,\ y = 1 + y_2 p^2 + \dots \in 1 + p^2\mathbb{Z}_p, \\ \dfrac{1}{p}\displaystyle\sum_{i=1}^{p} e_p(\frac{i \cdot y_1}{p}) & \text{if } y_1 \ne 0,\ y \in 1 + p^2\mathbb{Z}_p, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

$$= \begin{cases} 1 & \text{if } y \in 1 + p^2\mathbb{Z}_p, \\ 0 & \text{otherwise.} \end{cases}$$

$$= 1_{1+p^2\mathbb{Z}_p},$$

where here we have used the fact that

$$\sum_{i=1}^{p} e_p(\frac{i \cdot y_1}{p}) = \sum_{i=1}^{p-1} \left(e_p(\frac{y_1}{p})\right)^i + 1 = \frac{1 - \left(e_p(\frac{y_1}{p})\right)^p}{1 - e_p(\frac{y_1}{p})} = 0.$$

Equation (5.3) also holds for $y \notin 1 + p\mathbb{Z}_p$, where we get in this case $1_{1+p\mathbb{Z}_p}(y) = 0$ on both sides. Repeating this trick as needed, we can obtain $1_{1+p^n\mathbb{Z}_p}$ for any $n \geq 0$.

Now let a and n be arbitrary, subject to the condition $a \notin p^n\mathbb{Z}_p$. Write $a = a_0 p^m$ with $a_0 \in \mathbb{Z}_p^\times$. Then $m < n$, and $a + p^n\mathbb{Z}_p = a(1 + p^{n-m}\mathbb{Z}_p)$. Since $y \in a + p^n\mathbb{Z}_p$ if and only if $a^{-1}y \in 1 + p^{n-m}\mathbb{Z}_p$, it follows that $\pi\left(\begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right) \cdot 1_{1+p^{n-m}\mathbb{Z}_p} = 1_{1+p^n\mathbb{Z}_p}$. This shows that any arbitrary element of $S(\mathbb{Q}_p^\times)$ can be obtained, via the action of $B_1(\mathbb{Q}_p) = \left\{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a \in \mathbb{Q}_p^\times, b \in \mathbb{Q}_p\right\}$, from the single element $1_{1+p^n\mathbb{Z}_p}$, i.e. any subrepresentation of $B_1(\mathbb{Q}_p)$ on the Bruhat-Schwartz space $S(\mathbb{Q}_p^\times)$ that contains $1_{a+p\mathbb{Z}_p}$ is in fact equal to $S(\mathbb{Q}_p^\times)$.

Using a similar trick, one can prove that given any arbitrary linear combination of characteristic functions $1_{a+p^n\mathbb{Z}_p}$, we can recover $1_{1+p\mathbb{Z}_p}$ via the action of $B_1(\mathbb{Q}_p)$, i.e. if $W \in S(\mathbb{Q}_p^\times)$ is a non-zero subrepresentation, then W contains $1_{a+p\mathbb{Z}_p}$. This completes the proof that $S(\mathbb{Q}_p)$ is irreducible. □

Proposition 3.3.2. *The Bruhat-Schwartz space* $S_X(\mathbb{Q}_p^\times) = S(\mathbb{Q}_p^\times) \cdot X \subset \mathcal{K}$.

Proof. Let $x \in X$. Let $v \in V$ be any element whose image in $X = V/V_0$ is x. Then $f_v(1) = x$.

Set $f_1 = f_v$ and define

$$f_2 := \frac{1}{1 - e_p(p^{-1})} \left(f_1 - \pi' \left(\begin{pmatrix} 1 & p^{-1} \\ 0 & 1 \end{pmatrix} \right) \cdot f_1 \right).$$

Then $f_2(1) = x$ and f_2 is supported on $\{y \in \mathbb{Q}_p^\times \mid |y|_p \geq 1\}$.

Next, take n so that the support of f_1 (and hence also f_2) is contained in $p^{-n}\mathbb{Z}_p$, and define

$$f_3(y) := \left(\frac{1}{p^n} \sum_{j=0}^{p^n-1} \pi' \left(\begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix} \right) \cdot f_2 \right)(y) = \left(\frac{1}{p^n} \sum_{j=0}^{p^n-1} e_p(jy) \right) f_2(y).$$

Then $f_3(1) = x$ and f_3 is supported on $\mathbb{Z}_p^\times$. Now, let N be the smallest positive integer such that f_3 is constant on cosets of $1 + p^N\mathbb{Z}_p$.

Let $\omega : \mathbb{Z}_p^\times \to \mathbb{C}^\times$ be a normalized unitary character of conductor p^N. Then ω is constant on cosets of $1 + p^N\mathbb{Z}_p \in \mathbb{Z}_p^\times$. By the previous lemma, there are exactly $\phi(p^N) = (p-1)p^{N-1}$ such characters $\omega \pmod{p^N}$ and they satisfy the orthogonality relation

$$\frac{1}{\phi(p^N)} \sum_{\omega (\mathrm{mod}\, p^N)} = \omega(j) = \begin{cases} 1 & \text{if } j - 1 \in p^N\mathbb{Z}_p, \\ 0 & \text{otherwise.} \end{cases}$$

Now, for each character $\omega \pmod{p^N}$, define

$$\begin{aligned} f_w(y) &:= \int_{\mathbb{Z}_p^\times} \omega(-u)^{-1} \cdot \pi' \left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \right) \cdot f_3(y) \, d^\times u \\ &= \frac{1}{(p-1)p^{N-1}} \sum_{\substack{j=1 \\ (j,p)=1}}^{p^N} \omega(j)^{-1} \cdot \pi' \left(\begin{pmatrix} j & 0 \\ 0 & 1 \end{pmatrix} \right) \cdot f_3(y). \end{aligned}$$

Notice that

- $f_\omega \in \mathcal{K}$ and is supported on $\mathbb{Z}_p^\times$.

- $f_\omega(a \cdot y) = \omega(a) f_\omega(y), \quad (\forall a \in \mathbb{Z}_p^\times,\ y \in \mathbb{Q}_p^\times)$.

Thus $f_\omega \in S(\mathbb{Q}_p^\times) \cdot f_\omega(1)$. It follows from the previous proposition that the whole space $S(\mathbb{Q}_p^\times) \cdot f_\omega(1)$ is contained in $\mathcal{K}$. On the other hand, by orthogonality relation,

$$\begin{aligned}
\sum_{\omega(\mathrm{mod}\, p^N)} f_\omega(1) &= \sum_{\substack{j=1 \\ (j,p)=1}}^{p^N} \left[\frac{1}{\phi p^N} \sum_{\omega(\mathrm{mod}\, p^N)} \omega(j)^{-1} \right] \cdot \pi' \left(\begin{pmatrix} j & 0 \\ 0 & 1 \end{pmatrix} \right) \cdot f_3(1) \\
&= \pi' \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \cdot f_3(1) \\
&= x.
\end{aligned}$$

It follows from the previous proposition that for every $x \in X$, the whole space $S(\mathbb{Q}_p^\times) \cdot x$ is contained in $\mathcal{K}$. So we get our desired result. □

Theorem 3.3.1. *$\mathcal{K} = S_X(\mathbb{Q}_p^\times) + \pi'(w) \cdot S_X(\mathbb{Q}_p^\times)$, where $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.*

Proof. Let $f \in S_X(\mathbb{Q}_p^\times) = S(\mathbb{Q}_p^\times) \cdot X$. $\mathcal{K}$ is spanned by $\{\pi'(g) \cdot f \mid g \in GL(2, \mathbb{Q}_p)\}$ because $(\pi', \mathcal{K})$ is irreducible. Let $g \in GL(2, \mathbb{Q}_p) - B(\mathbb{Q}_p)$, then we have the identity $g = hwh'$ where

$$h = \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix}, \quad w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad h' = \begin{pmatrix} -c & -d \\ 0 & b - adc^{-1} \end{pmatrix}.$$

Now, let $F \coloneqq \pi'(h') \cdot f$. We proved that the space $S_X(\mathbb{Q}_p^\times)$ is invariant under the action of $B_1(\mathbb{Q}_p)$ (proposition 3.3.1) and under the action of diagonal matrices (proposition 3.1.2). It follows then that $F \in S_X(\mathbb{Q}_p^\times)$. Then we have

$$\pi'(g) \cdot f = \pi'(h) \cdot (\pi'(w) \cdot F)$$

$$= \Big(\pi'(h)\cdot(\pi'(w)\cdot F) - \pi'(w)\cdot F\Big) + \pi'(w)\cdot F.$$

The function

$$\begin{aligned}&\Big(\pi'(h)\cdot(\pi'(w)\cdot F) - \pi'(w)\cdot F\Big)(y)\\ &\quad = \Big(e_p(ac^{-1}y) - 1\Big)(\pi'(w)\cdot F)(y)\end{aligned}$$

vanishes for all values of y such that $e_p(ac^{-1}y) = 1$. It also vanishes outside a compact subset of $\mathbb{Q}_p$. We get then that this function is in $S_X(\mathbb{Q}_p^\times)$. Therefore $\pi'(g)\cdot f \in S_X(\mathbb{Q}_p^\times) + \pi'(w)\cdot S_X(\mathbb{Q}_p^\times)$. □

3.4 Uniqueness of the Kirillov Model

For the rest of this section, $X = V/V_0$ as defined previously, and $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. We need to prove the uniqueness of the Kirillov model. For this aim, we shall prove that X is one dimensional. This requires the construction of some operators in order to describe the action of w on $\mathcal{K}$. We consider then the following definition.

Definition 3.4.1. Fix $\chi : \mathbb{Z}_p^\times \to \mathbb{C}^\times$ a unitary multiplicative character. Let $t \in \mathbb{Q}_p^\times$. We define the linear operator $J(t,\chi) : X \to X$ by the action

$$J(t,\chi)\cdot x = [\pi'(w)\cdot \psi_{t,\chi}(y')x]\Big|_{y'=1}$$

for all $x \in X$, where $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and for fixed $x \in X$,

$$\psi_{t,\chi} : \mathbb{Q}_p^\times \to \mathbb{Q}_p^\times$$

$$y' \to \psi_{t,\chi}(y')x = \chi(t^{-1}y') \cdot 1_{\mathbb{Z}_p^\times}(t^{-1}y') \cdot x$$

Remark. The function $y' \to \psi_{t,\chi}(y')x \in S_X(\mathbb{Q}_p^\times)$, which is in turn contained in $\mathcal{K}$. Therefore, the action of $\pi'(w)$ is well defined on this function. The operator $J_\pi(t,\chi)$ is defined by applying $\pi'(w)$ to $y' \to \psi_{t,\chi}(y')x$ and then setting "y'"= 1 afterwards, where this "y'" is not the same as y' in $\psi_{t,\chi}(y')x$.

Remark. Note that for a fixed $x \in X$, $J(t,\chi) \cdot x$ vanishes for large $|t|_p$.

Lemma 3.4.1. *Let $t \in \mathbb{Q}_p^\times$, $f \in S_X(\mathbb{Q}_p^\times)$, and $\omega_{\pi'}$ be the central character of π'. Let p^N be the conductor of f. Then for each $y \in \mathbb{Q}_p^\times$, we have*

$$[\pi'(w) \cdot f](y) = \omega_{\pi'}(y) \sum_{\chi \text{ mod } p^N} \int_{\mathbb{Q}_p^\times} J(ty,\chi) \cdot f(t)\, d^\times t.$$

Proof. We recall the following: If χ is a normalized unitary character of $\mathbb{Z}_p^\times$ which has conductor p^N, then χ is constant on cosets of $1 + p^N\mathbb{Z}_p$ in $\mathbb{Z}_p$, and there are exactly $\phi(p^N) = (p-1)p^N$ such characters χ such that

$$\frac{1}{\phi(p^N)} \sum_{\chi \text{ mod } p^N} \chi(j) = \begin{cases} 1, & \text{if } j \in 1 + p^N\mathbb{Z}_p, \\ 0, & \text{otherwise.} \end{cases}$$

Now, let $y \in \mathbb{Q}_p^\times$. Then

$$\begin{aligned}
&\omega_{\pi'}(y) \sum_{\chi \text{ mod } p^N} \int_{t \in \mathbb{Q}_p^\times} J(ty,\chi) \cdot f(t)\, d^\times t \\
&= \omega_{\pi'}(y) \sum_{\chi \text{ mod } p^N} \int_{t \in \mathbb{Q}_p^\times} [\pi'(w) \cdot \psi_{ty,\chi}(y') f(t)]_{y'=1}\, d^\times t && (\text{for } y' \in \mathbb{Q}_p^\times) \\
&= \omega_{\pi'}(y) \left[\int_{t \in \mathbb{Q}_p^\times} \sum_{\chi \text{ mod } p^N} \pi'(w) \cdot \psi_{ty,\chi}(y') f(t) \right]_{y'=1} d^\times t && (\text{finite sum}) \\
&= \omega_{\pi'}(y) \left[\int_{t \in \mathbb{Q}_p^\times} \pi'(w) \sum_{\chi \text{ mod } p^N} \chi((ty)^{-1}y') \cdot 1_{\mathbb{Z}_p^\times}((ty)^{-1}y') f(t) \right]_{y'=1} d^\times t \\
&= \omega_{\pi'}(y) \left[\int_{t \in \mathbb{Q}_p^\times} \pi'(w) \phi(p^N) \cdot 1_{1+p^N\mathbb{Z}_p}((ty)^{-1}y') f(t) \right]_{y'=1} d^\times t
\end{aligned}$$

$$= \omega_{\pi'}(y) \left[\pi'(w)\phi(p^N) \cdot \int_{t \in \mathbb{Q}_p^\times} 1_{1+p^N\mathbb{Z}_p}(t^{-1}) f(ty^{-1}y')\, d^\times t \right]_{y'=1}$$

$$= \omega_{\pi'}(y) \left[\pi'(w)\phi(p^N) \cdot \frac{1}{\phi(p^N)} f(y^{-1}y') \right]_{y'=1},$$

where the last step follows from the fact that the volume of $1 + p^N\mathbb{Z}_p$ is $\dfrac{1}{\phi(p^N)}$, and that f is invariant under multiplication by $1 + p^N\mathbb{Z}_p$, while the next to last step is a substitution, replacing t by $ty^{-1}y'$. Now,

$$f(y^{-1}y') = \pi'\left(\begin{pmatrix} y^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right) \cdot f(y'),$$

and

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} y^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} y^{-1} & 0 \\ 0 & y^{-1} \end{pmatrix}\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Using these two identities, we get that

$$\omega_{\pi'}(y) \left[\pi'\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) \cdot f(y^{-1}y') \right]_{y'=1}$$

$$= \omega_{\pi'}(y) \left[\pi'\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) \cdot \pi'\left(\begin{pmatrix} y^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right) \cdot f(y') \right]_{y'=1}$$

$$= \omega_{\pi'}(y) \left[\pi'\left(\begin{pmatrix} y^{-1} & 0 \\ 0 & y^{-1} \end{pmatrix}\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) \cdot f(y') \right]_{y'=1}$$

$$= \omega_{\pi'}(y)\, \omega_{\pi'}(y^{-1}) \left[\pi'\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) \cdot f(y') \right]_{y'=1}$$

$$= \left[\pi'\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) \cdot f(y') \right]_{y'=1}$$

Now suppose that $h : y' \to h(y') = \pi'\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) \cdot f(y')$. Then

$$h(yy') : y' \to h(yy') = \left(\pi'\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) h\right)(y') = \left(\pi'\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) \cdot f\right)(y')$$

It follows then that

$$\begin{aligned}
&\left[\pi'\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) \cdot f(y')\right]_{y'=1} \\
&= [h(yy')]_{y'=1} \\
&= h(y) \\
&= \left(\pi'\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) \cdot f\right)(y).
\end{aligned}$$

This completes the proof. □

Lemma 3.4.2. *If $T \in End\ (X)$ commutes with all the operators $J(t,\chi)$, where $t \in \mathbb{Q}_p^\times$ and $\chi : \mathbb{Z}_p^\times \to \mathbb{C}^\times$ (unitary multiplicative character), then T acts by a scalar on all of X.*

Proof. Let $\mathcal{F}_X(\mathbb{Q}_p^\times)$ denote the space of all functions $f : \mathbb{Q}_p^\times \to X$. Any linear operator $T : X \to X$ induces an operator

$$\begin{aligned}
T' : \mathcal{F}_X(\mathbb{Q}_p^\times) &\to \mathcal{F}_X(\mathbb{Q}_p^\times) \\
f &\to T' \cdot f,
\end{aligned}$$

provided that $(T' \cdot f)(y) = T \cdot f(y)$ for all $y \in \mathbb{Q}_p^\times$. Here $T \cdot x$ denotes the action of T on $x \in X$. Notice that T' maps $S_X(\mathbb{Q}_p^\times)$ to $S_X(\mathbb{Q}_p^\times)$ as T is a linear operator.

Let $f \in S_X(\mathbb{Q}_p^\times)$. As $S_X(\mathbb{Q}_p^\times) \subset \mathcal{K}$, it follows from the previous lemma that for every $y \in \mathbb{Q}_p^\times$, we have

$$\begin{aligned}
&\pi'\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) \cdot (T' \cdot f)(y) \\
&= \omega_{\pi'}(y) \sum_{\chi \bmod p^N} \int_{\mathbb{Q}_p^\times} J(ty, \chi) \cdot (T \cdot f(t))\, d^\times t \\
&= T \cdot \left[\omega_{\pi'}(y) \sum_{\chi \bmod p^N} \int_{\mathbb{Q}_p^\times} J(ty, \chi) \cdot f(t)\, d^\times t\right] \\
&= T \cdot \left[\pi'\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) \cdot f(y)\right] \\
&= T' \cdot \left(\pi'\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) \cdot f\right)(y).
\end{aligned} \tag{3.2}$$

We have thus proved that T' commutes with $\pi'\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)$ when T' is considered as an operator on $S_X(\mathbb{Q}_p^\times)$. We need to prove that $\mathcal{K}$ is an invariant subspace of T'. Indeed, if $f \in \mathcal{K}$, then we can write

$$f = f_1 + \pi'\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) \cdot f_2$$

with $f_1, f_2 \in S_X(\mathbb{Q}_p^\times)$. Then

$$\begin{aligned}
T' \cdot f = T' \cdot f_1 + T' \cdot \pi'\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) \cdot f_2 &= T' \cdot f_1 + \pi'\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) \cdot T' \cdot f_2 \\
&\in S_X(\mathbb{Q}_p^\times) + \pi'\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) \cdot S_X(\mathbb{Q}_p^\times) \subset \mathcal{K}.
\end{aligned}$$

Thus $\mathcal{K}$ is an invariant subspace of T'.

Now we want to show that $T' : \mathcal{K} \to \mathcal{K}$ commutes with any $\pi'(g)$ where $g \in GL(2, \mathbb{Q}_p)$. T' commutes with $\pi'\left(\begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix}\right)$ for any $d \in \mathbb{Q}_p^\times$ because $\pi'\left(\begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix}\right)$ acts by a scalar (central character). Moreover, we have already shown this for $g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ just when T' is considered to be acting on $S_X(\mathbb{Q}_p^\times)$. Now for any $f \in \mathcal{K}$, $f = f_1 + \pi'(w) \cdot f_2$ with $f_1, f_2 \in S_X(\mathbb{Q}_p^\times)$. Let $y \in \mathbb{Q}_p$, then

$$\begin{aligned}
T' \cdot \pi'(w) \cdot f(y) &= T' \cdot \pi'(w) \cdot [f_1(y) + \pi'(w) \cdot f_2(y)] \\
&= T' \cdot \pi'(w) \cdot f_1(y) + T' \cdot \pi'(-I_2) \cdot f_2(y) \\
&= \pi'(w) \cdot (T' \cdot f_1)(y) + \pi'(-I_2) \cdot T' \cdot f_2(y) && \text{(by (3.2))} \\
&= \pi'(w) \cdot (T' \cdot f_1)(y) + \pi'(w) \cdot [\pi'(w) \cdot T' \cdot f_2(y)] \\
&= \pi'(w) \cdot (T' \cdot f_1)(y) + \pi'(w) \cdot [T' \cdot \pi'(w) \cdot f_2(y)] && \text{(by (3.2))} \\
&= \pi'(w) \cdot (T' \cdot f_1 + T' \cdot \pi'(w) \cdot f_2)(y) \\
&= \pi'(w) \cdot (T' \cdot f)(y).
\end{aligned}$$

So T' commutes with $\pi'(w)$. Consider now $g = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ ($a \in \mathbb{Q}_p^\times$, $b \in \mathbb{Q}_p$). Then for any $f \in \mathcal{K}$ and $y \in \mathbb{Q}_p$, we have

$$T' \cdot \pi'(g) \cdot f(y) = T' \cdot [e_p(by) f(ay)] = e_p(by)(T' \cdot f)(ay) = \pi'(g) \cdot (T' \cdot f)(y).$$

Consequently, T' commutes with $\pi'\left(\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\right)$. The matrices $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ with $a, d \in \mathbb{Q}_p^\times$ and $b \in \mathbb{Q}_p$ generate $GL(2, \mathbb{Q}_p)$. Thus T' commutes with all $\pi'(g)$, with $g \in GL(2, \mathbb{Q}_p)$, and so T' acts by a scalar. It follows that T also acts by a scalar. □

Theorem 3.4.3 (Commutativity of the operators)**.** *For all $t, t' \in \mathbb{Q}_p^\times$, and all w, w' :*

$\mathbb{Z}_p^\times \to \mathbb{C}^\times$ *(unitary multiplicative characters), we have*

$$J(t,w)\cdot J(t',w') = J(t',w')\cdot J(t,w).$$

Proof. We begin with the matrix identity:

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -t^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} t^{-1} & 0 \\ 0 & t^{-1} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} t^2 & -t \\ 0 & 1 \end{pmatrix}$$

Let $f \in S_X(\mathbb{Q}_p^\times)$. For $t, y \in \mathbb{Q}_p^\times$ set

$$f_t(y) = \pi'\left(\begin{pmatrix} t^2 & -t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} t^{-1} & 0 \\ 0 & t^{-1} \end{pmatrix}\right)\cdot f(y) = e_p(-ty)\omega_{\pi'}(t^{-1})\cdot f(t^2 y).$$

And so $f_t \in S_X(\mathbb{Q}_p^\times)$. We apply π' on both sides.

On the right hand side: Let $y \in \mathbb{Q}_p^\times$ and let N_t be the conductor of f_t:

$$\begin{aligned}
y \to &\left[\pi'\left(\begin{pmatrix} 1 & -t^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} t^{-1} & 0 \\ 0 & t^{-1} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} t^2 & -t \\ 0 & 1 \end{pmatrix}\right)\cdot f\right](y) \\
&= \pi'\left(\begin{pmatrix} 1 & -t^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} t^{-1} & 0 \\ 0 & t^{-1} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} t^2 & -t \\ 0 & 1 \end{pmatrix}\right)\cdot f(y) \\
&= \pi'\left(\begin{pmatrix} 1 & -t^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)\cdot f_t(y) \\
&= e_p(-t^{-1}y)\,\pi'\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)\cdot f_t(y) \\
&= e_p(-t^{-1}y)\,\omega_{\pi'}(y) \sum_{\chi (mod\, p^{N_t})} \int_{a\in\mathbb{Q}_p^\times} J(ay,\chi) f_t(a)\, d^\times a \\
&= e_p(-t^{-1}y)\,\omega_{\pi'}(y) \sum_{\chi (mod\, p^{N_t})} \int_{a\in\mathbb{Q}_p^\times} J(ay,\chi)[e_p(-ta)\omega_{\pi'}(t^{-1})\cdot f(t^2 a)]\, d^\times a \\
&= \omega_{\pi'}(t^{-1}y) \sum_{\chi (mod\, p^{N_t})} \int_{a\in\mathbb{Q}_p^\times} e_p(-t^{-1}y - ta) J(ay,\chi) f(t^2 a)\, d^\times a
\end{aligned}$$

$$= \omega_{\pi'}(t^{-1}y) \sum_{\chi (mod\, p^{N_t})} \int_{a \in \mathbb{Q}_p^\times} e_p(-\frac{a+y}{t}) J(\frac{ay}{t^2}, \chi) f(a)\, d^\times a.$$

Now we apply π^* to the left hand side.

Let $y \in \mathbb{Q}_p^\times$, then

$$y \to \left[\pi' \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right) \cdot f \right] (y)$$

$$= \pi' \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right) \cdot f(y)$$

$$= \pi' \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) \Bigg[\underbrace{\pi' \left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right)}_{\text{is not in general in } S_X(\mathbb{Q}_p^\times)} \cdot f(y) \Bigg]$$

$$= \pi' \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) \Bigg[\underbrace{\pi' \left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right) \cdot f(y) - \pi' \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right) \cdot f(y)}_{\text{this function now is in } S_X(\mathbb{Q}_p^\times)} \Bigg] + f(y)$$

Let p^N be the conductor of f, define

$$f_t'(y) := \pi' \left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right) \cdot f(y) - \pi' \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right) \cdot f(y)$$

$$= \pi' \left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \right) \Bigg[\omega_{\pi'}(-y) \sum_{\chi'' (mod\, p^N)} \int_{a \in \mathbb{Q}_p^\times} J(ay, \chi'') f(a)\, d^\times a \Bigg]$$

$$- \omega_{\pi'}(-y) \sum_{\chi'' (mod\, p^N)} \int_{a \in \mathbb{Q}_p^\times} J(ay, \chi'') f(a)\, d^\times a$$

$$= (e_p(ty) - 1) \omega_{\pi'}(-y) \sum_{\chi'' (mod\, p^N)} \int_{a \in \mathbb{Q}_p^\times} J(ay, \chi'') f(a)\, d^\times a \qquad (3.3)$$

Let $p^{N'_t}$ be the conductor of f'_t, so that the left hand side becomes:

$$\begin{aligned}
&\pi'\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right)\cdot f(y) \\
&= \pi'\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)\cdot f'_t(y) + f(y) \\
&= \omega_{\pi'}(y) \sum_{\chi'(mod\, p^{N'_t})} \int_{z\in\mathbb{Q}_p^\times} J(zy,\chi') f'_t(y)\, d^\times z + f(y) \\
&= \omega_{\pi'}(y) \sum_{\chi'(mod\, p^{N'_t})} \int_{z\in\mathbb{Q}_p^\times} J(zy,\chi') \Bigg[(e_p(tz)-1)\omega_{\pi'}(-z) \\
&\qquad \cdot \sum_{\chi''(mod\, p^N)} \int_{a\in\mathbb{Q}_p^\times} J(az,\chi'') f(a)\, d^\times a \Bigg] d^\times z + f(y) \\
&= \omega_{\pi'}(y) \sum_{\chi'(mod\, p^{N'_t})} \sum_{\chi''(mod\, p^N)} \int_{z\in\mathbb{Q}_p^\times} J(zy,\chi')(e_p(tz)-1)\omega_{\pi'}(-z) \\
&\qquad \cdot \underbrace{\left[\int_{a\in\mathbb{Q}_p^\times} J(az,\chi'') f(a)\, d^\times a \right]}_{\substack{\text{This is a function of z. Basically, } |az|_p < \text{bdd, but } |a|_p > m > 0, \text{ so } |z|_p < bdd/m. \\ \text{However, problem of } z \text{ remains near zero; but this is fixed by the} \\ (e_p(tz)-1) \text{ factor, where } t \text{ here is being fixed.}}} d^\times z + f(y) \qquad (3.4)
\end{aligned}$$

We note that:

- t is temporarily fixed, so we can limit ourselves to compact subsets of $\mathbb{Q}_p^\times$, so that $|t|_p$ is bounded.
- N_t and N'_t also can be made bounded for a couple of t's.
- We need to bound z when a is fixed ($m_f < |a|_p < M_f$), so that we can justify interchanging the order of integration.

Let K_f be compact subset containing all a's in the support of f, and let K' be

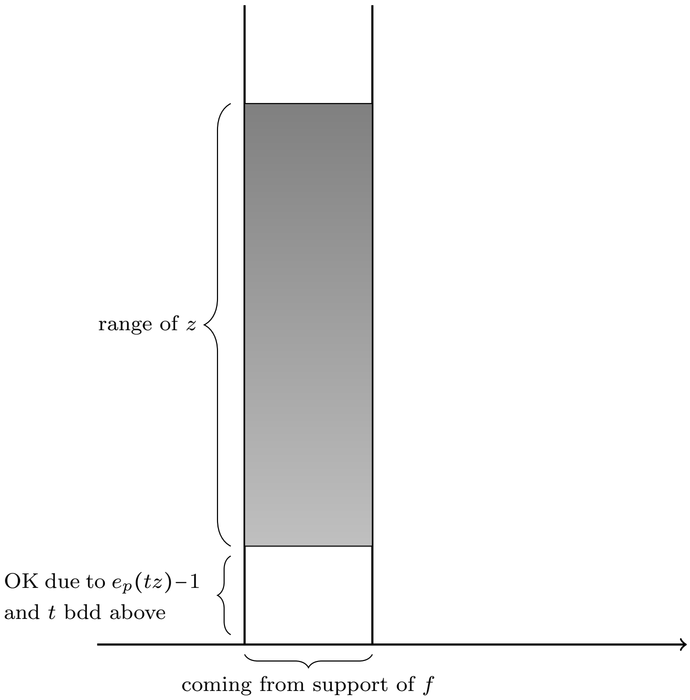

Figure 3.1: A sketch of the case

the compact set containing all the z's for z bounded as mentioned in 3.4. Then

$$\int_{z\in\mathbb{Q}_p^\times}\int_{a\in\mathbb{Q}_p^\times}\cdots = \int_{z\in K'} \underbrace{\cdots}_{\substack{\text{vanishes for}\\ |z|_p \text{ small}}} \left[\underbrace{\int_{a\in K_f}\cdots}_{\substack{\text{quantity which van-}\\ \text{ishes for } |z|_p \text{ large}}} \right]$$

$$= \int_{z\in K'}\int_{a\in K_f}\cdots$$

$$= \int_{a\in K_f}\int_{z\in K'}\cdots$$

Now define the function

$$K_1 : \mathbb{Q}_p^\times \times \mathbb{Q}_p^\times \times \mathbb{Q}_p^\times \to \mathrm{End}(X)$$

$$(a,y,t) \to K_1(a,y,t) \coloneqq \omega_{\pi'}(t^{-1}y) \sum_{\chi(mod\, p^{N_t})} e_p(-\frac{a+y}{t}) J(\frac{ay}{t^2},\chi).$$

So we get RHS $= \int_{a\in K_f} K_1(a,y,t) f(a)\, d^\times a$. Define also

$$\begin{aligned}
K_2 : \mathbb{Q}_p^\times \times \mathbb{Q}_p^\times \times \mathbb{Q}_p^\times &\to \mathrm{End}(X)\\
&(a,y,t) \to K_2(a,y,t) \coloneqq\\
&\omega_{\pi'}(y) \sum_{\chi'(mod\, p^{N'_t})} \sum_{\chi''(mod\, p^N)} \int_{z\in K'} \Big[e_p(tz) - 1\Big] \omega_{\pi'}(-z) J(zy,\chi') J(az,\chi'')\, d^\times z.
\end{aligned}$$

So we get

$$\begin{aligned}
\text{LHS} &= \int_{z\in K'} \int_{a\in K_f} K_2(a,y,t)\, f(a)\, d^\times a\, d^\times z + f(y)\\
&= \int_{a\in K_f} \int_{z\in K'} K_2(a,y,t)\, f(a)\, d^\times z\, d^\times a + f(y).
\end{aligned}$$

Note that K' is chosen sufficiently large so that the above equation holds true.

In order to get rid of the term $+f(y)$, we will take two different t_1 and t_2, subtract the term with t_2 from the term with t_1 in the RHS (respectively LHS), and then equate the two resulting equations. It follows then that

$$\int_{a\in K_f} \big(K_1(a,y,t_1) - K_1(a,y,t_2)\big) f(a)\, d^\times a = \int_{a\in K_f} \big(K_2(a,y,t_1) - K_2(a,y,t_2)\big) f(a)\, d^\times a$$

for all $f \in S_X(\mathbb{Q}_p^\times)$. For $a, y \in \mathbb{Q}_p^\times$, define the two functions R and Q to be

$$R(a,y) = K_1(a,y,t_1) - K_1(a,y,t_2) \qquad Q(a,y) = K_2(a,y,t_1) - K_2(a,y,t_2).$$

Now we try to make $z \in K'$ uniformly in f (where f is restricted to conductor p^N). Note that

- K_f is fixed.
- We are only going to consider functions f whose support is in K_f and f is invariant under $1 + p^N$.

- The domain of such functions f has finitely many points (because finitely many cosets of $K_f/(1+p^N)$), but the values also can be in X. So let us limit ourselves to values in some finite dimensional subspace $X' \subset X$.

Thus the functions f concerned can only be in a finite dimensional space. Thus we can find a uniform upper bound for the support. So $z \in K' \subset p^{-M}\mathbb{Z}_p$ for some M, is a uniform function of K_f and N.

Our **goal** now is to prove that $\forall v \in X'$, $Q(a,y)v = Q(y,a)v$.

Integrate both sides against the specific function f, which is defined as follows: f is supported on coset of $1+p^N$, say $a_0 + (1+p^N)$ and of value $v \in X'$ (i.e. *one* operator is applied to *one* vector). Then we get that

$$\begin{aligned} Q(a_0,y)v &= R(a_0,y)v \\ &= R(y,a_0)v \qquad \text{(as the RHS is symmetric)} \\ &= Q(y,a_0)v \qquad \text{(here we need to apply to another function near } y_0 \text{ whose value on } V \text{ is } y_0(1+p^N)) \end{aligned}$$

Note that Q depends on K_f, N and the finite dimensional space X'. Moreover, note that we can write Q as

$$Q(a,y)v = \sum_{\chi',\chi''} G(a,y,\chi',\chi'')v$$

$$Q(y,a)v = \sum_{\chi',\chi''} G(y,a,\chi'',\chi')v$$

where $G(a,y,\chi',\chi'') = \int_{z\in K'} \big(e_p(t_1 z) - e_p(t_2 z)\big)\omega_{\pi'}(-z)J(zy,\chi')J(az,\chi'')\, d^\times z$.

From the definition of $J(t,\chi)$, notice that $J(tu,\chi) = \overline{\chi}(u)\, J(t,\chi)$ for all $u \in \mathbb{Z}_p^\times$. It follows that

$$G(au,yw,\chi'.\chi'')v = \overline{\chi''}(u)\,\overline{\chi'}(w)\, G(a,y,\chi',\chi'')$$

$$G(yw, au, \chi''.\chi')v = \overline{\chi'}(w)\,\overline{\chi''}(u)\,G(y,a,\chi'',\chi')$$

for all $u, w \in \mathbb{Z}_p^\times$. We get then that

$$Q(au, yw)v = \sum_{\chi',\chi''} \overline{\chi''}(u)\,\overline{\chi'}(w)\,G(a,y,\chi',\chi'')v$$
$$Q(yw, au)v = \sum_{\chi',\chi''} \overline{\chi'}(w)\,\overline{\chi''}(u)\,G(y,a,\chi'',\chi')v.$$

Now, for ψ_1, ψ_2 unitary characters of $\mathbb{Z}_p^\times$, we have

$$\begin{aligned}
&\int_{u,w\in\mathbb{Z}_p^\times} (\text{LHS})\cdot\psi_1(u)\,\psi_2(w)\,d^\times u\,d^\times w \\
&= \sum_{\chi',\chi''} \int_{u,w\in\mathbb{Z}_p^\times} \underbrace{\overline{\chi''}(u)\,\psi_1(u)\,\overline{\chi'}(w)\,\psi_2(w)}_{\substack{=0 \text{ unless } \chi''=\psi_1 \text{ and } \chi'= \\ \psi_2, \text{ in which we get } 1.}} G(a,y,\chi',\chi'')\,d^\times u\,d^\times w \\
&= G(a,y,\chi',\chi'').
\end{aligned}$$

Similarly, $\int_{u,w\in\mathbb{Z}_p^\times} (\text{RHS})\cdot\psi_1(u)\,\psi_2(w)\,d^\times u\,d^\times w = G(y,a,\chi'',\chi')$.
But RHS=LHS, then $G(a,y,\chi',\chi'') = G(y,a,\chi'',\chi')$. Thus

$$\begin{aligned}
&\int_{z\in K'} \big(e_p(t_1z) - e_p(t_2z)\big)\omega_{\pi'}(-z)J(zy,\chi')J(az,\chi'')v\,d^\times z \\
&= \int_{z\in K'} \big(e_p(t_1z) - e_p(t_2z)\big)\omega_{\pi'}(-z)J(az,\chi'')J(zy,\chi')v\,d^\times z
\end{aligned}$$

for all $v \in X'$, where K' depends on N, K_f, t_1, t_2. We want to **check** that K' is uniform for all t_i such that $|t_i|_p \le B$, $t_i \ne 0$. Notice that

$$\begin{aligned}
t_1, t_2 \in p^{-B}\mathbb{Z}_p &\implies e_p(t_iz) = 1 && \text{for } z \in p^B\mathbb{Z}_p \\
&\implies e_p(t_1z) - e_p(t_2z) = 0 && \text{for } z \in p^B\mathbb{Z}_p.
\end{aligned}$$

So we can assume that $K' \cap p^B\mathbb{Z}_p = \varnothing$, so that as $|t_i|_p$'s are bounded above, we get that a part of K' is bounded below. Now we need to check if the $|t_i|_p$'s are bounded below. We look at the functions that span $e_p(t_1 z) - e_p(t_2 z)$ for $z \in p^{-M}\mathbb{Z}_p - p^B\mathbb{Z}_p$. Such functions are constant on the cosets of $p^B\mathbb{Z}_p$. So we can see t_i's as elements of $p^{-B}\mathbb{Z}_p/p^M\mathbb{Z}_p$.

Start with a locally constant compactly supported function on $\mathbb{Q}_p$. Every such function can be written as finite linear combination of characteristic functions compact open sets of the form $a+p^n\mathbb{Z}_p$, $a \in \mathbb{Q}_p$, $n \in \mathbb{Z}$. So it suffices to consider $g = 1_{1+p^n\mathbb{Z}_p}$, where g vanishes at zero. Now, using Fourier inversion formula, we have

$$g(z) = \int_{y\in\mathbb{Q}_p} \widehat{g}(y) e_p(yz)\, dy,$$

where by lemma 2.3.2, we have

$$\widehat{g}(y) = \widehat{1}_{1+p^n\mathbb{Z}_p}(y) = e_p(-y)p^{-n}1_{p^{-n}\mathbb{Z}_p}(y.)$$

We also know that $\int_{\mathbb{Q}_p} \widehat{g}(y)\, dy = g(0) = 0$. Choose $\mathcal{B}$, such that $p^M\mathbb{Z}_p \subset \mathcal{B}$ and $\mathcal{B}$ contains the support of $\widehat{g}$. If $S = t_i U$ is the set of representatives of $\mathcal{B}/p^M\mathbb{Z}_p$, and if c is the volume of $p^M\mathbb{Z}_p$, then

$$g(z) = c\sum_{\xi\in S} e_p(\xi z)\widehat{g}(\xi) = \sum_{t_i U} c e_p(t_i z)\widehat{g}(t_i) = \sum_{i=1}^{l} c e_p(t_i z)\widehat{g}(t_i).$$

Now, we fix one t, say t_1, and we write g as differences

$$\begin{aligned} g(z) &= \sum_{i=1}^{l} c_i \left(e_p(t_i z) - e_p(t_1 z)\right) \\ &= \begin{cases} 1 & \text{if } z \equiv 1 \bmod (p^N\mathbb{Z}_p), \\ 0 & \text{if } z \in K',\ z \not\equiv 1 \bmod (p^N\mathbb{Z}_p). \end{cases} \end{aligned}$$

This proves that the function

$$z \to \omega_{\pi'}(-z) \cdot (J(zy, \chi')J(za, \chi'') - J(za, \chi'')J(zy, \chi'))$$

is orthogonal to all functions in $S(\mathbb{Q}_p^\times)$, and hence must vanish. Therefore

$$J(zy, \chi')J(za, \chi'') = J(za, \chi'')J(zy, \chi'),$$

which establishes the commutativity of the operators. □

Theorem 3.4.4. *The dimension of X is one.*

Proof. By Theorem 3.4.3, the family of operators $J(t, \chi)$ is commutative. It immediately follows from Lemma 3.4.2 that each $J(t, \chi)$ is a scalar operator. Consequently, any $T \in \mathrm{End}(X)$ commutes with every $J(t, \chi)$. It follows again from Lemma 3.4.2 that every linear operator $T \in \mathrm{End}(X)$ must be a scalar operator. This forces the space X to be one dimensional. □

Chapter 4

Application to Local New Vectors

In this chapter, we present one application of the Kirillov model: the theory of local new vectors. This is similar to the theory of new forms in the sense of modular forms. The classical theory is due to *Atkin & Lehner*. We will present the p-adic analogue of this theory, which is due to [Casselman, 1973]. However, we will follow the proof of [Deligne, 1973].

Lemma 4.1. *Let H be the subgroup of $SL(2,\mathbb{Q}_p)$ generated by the subgroups* $\begin{pmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix}$ *and* $\begin{pmatrix} 1 & 0 \\ p^n\mathbb{Z}_p & 1 \end{pmatrix}$*, then*

i. For $n > 0$, H is the subgoup of SL(2,$\mathbb{Z}_p$) of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $a \equiv d \equiv 1\ (mod\ p^n\mathbb{Z}_p)$ and $c \equiv 0\ (mod\ p^n\mathbb{Z}_p)$.

ii. For $n = 0$, $H = SL(2,\mathbb{Z}_p)$.

iii. For $n < 0$, $H = SL(2,\mathbb{Q}_p)$.

Proof. :

i. We want to prove that

$$\left\langle \begin{pmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ p^n\mathbb{Z}_p & 1 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} \equiv 1 \,(\mathrm{mod}\, p^n\mathbb{Z}_p) & \mathbb{Z}_p \\ \equiv 0 \,(\mathrm{mod}\, p^n\mathbb{Z}_p) & \equiv 1 \,(\mathrm{mod}\, p^n\mathbb{Z}_p) \end{pmatrix} \right\rangle.$$

Let G be the subgroup of $SL(2,\mathbb{Z}_p)$ defined by

$$G = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}_p) \text{ such that } a \equiv d \equiv 1 \;\&\; c \equiv 0 \;(\mathrm{mod}\; p^n\mathbb{Z}_p) \right\}.$$

As $\begin{pmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix} \subset H$ & $\begin{pmatrix} 1 & 0 \\ p^n\mathbb{Z}_p & 1 \end{pmatrix} \subset H$, it is enough to prove that $G \subset H$. Let $\begin{pmatrix} 1+p^n\alpha & \beta \\ p^n\gamma & 1+p^n\delta \end{pmatrix} \in G$. Now we reduce this into a diagonal matrix using column and row operations:

$$\begin{pmatrix} 1+p^n\alpha & \beta \\ p^n\gamma & 1+p^n\delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\gamma(1+p^n\alpha)^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1+p^n\alpha & 0 \\ 0 & 1+p^n\delta' \end{pmatrix} \begin{pmatrix} 1 & -\beta(1+p^n\alpha)^{-1} \\ 0 & 1 \end{pmatrix}.$$

Notice that

$$\begin{pmatrix} 1+p^n\alpha & 0 \\ 0 & 1+p^n\alpha \end{pmatrix} = \begin{pmatrix} 1 & -\alpha(1+p^n\alpha)^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ p^n & 1 \end{pmatrix} \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -p^n(1+p^n\alpha)^{-1} & 1 \end{pmatrix}.$$

It follows that $\begin{pmatrix} 1+p^n\alpha & 0 \\ 0 & 1+p^n\alpha \end{pmatrix} \in G$.

We can deduce that $\begin{pmatrix} 1+p^n\alpha & \beta \\ p^n\gamma & 1+p^n\delta \end{pmatrix} \in H$ as it can be written as a product of generators of H, and so we get our result.

ii. We need to prove that $SL(2,\mathbb{Z}_p)$ is generated by $\begin{pmatrix} 1 & 0 \\ \mathbb{Z}_p & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix}$.

Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}_p)$, then $ad - bc = 1$.

Case 1: If $b \in \mathbb{Z}_p^\times$, then $c = b^{-1}(ad-1)$.

Let $y = b$,

$x = b^{-1}(d-1)$,

$z = b^{-1}(a-1)$.

Then we get that

$$1 + yz = 1 + bb^{-1}(a-1) = a,$$

$$y = b,$$

$$x + z + xyz = b^{-1}(d-1) + b^{-1}(a-1) + bb^{-1}(d-1)(a-1) = b^{-1}(ad-1) = c,$$

$$xy + 1 = bb^{-1}(d-1) + 1 = d.$$

So $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1+yz & y \\ x+z+xyz & xy+1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}.$

Case 2: If $b \notin \mathbb{Z}_p^\times$, then $b \in p\mathbb{Z}_p$, so

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -c & -d \\ a & b \end{pmatrix},$$

where $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in H$ (by the above case). Notice that $d \in \mathbb{Z}_p^\times$ because otherwise we get $b = p^k u$ and $d = p^r v$ where $k, r > 0$ and $u, v \in \mathbb{Z}_p^\times$. This implies that $ad - bc = p(avp^{r-1} - cup^{k-1}) = 1$, which is impossible over $\mathbb{Z}_p$. It follows that

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is generated by $\begin{pmatrix} 1 & 0 \\ \mathbb{Z}_p & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix}$.

iii. First, notice that $SL(2,\mathbb{Z}_p) \subset H$. Moreover, we have the following identity

$$\begin{pmatrix} p & 0 \\ 0 & p^{-1} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & -p \\ 0 & 1 \end{pmatrix}}_{\in H} \underbrace{\begin{pmatrix} 1 & 0 \\ p^{-1} & 1 \end{pmatrix}}_{\in H} \underbrace{\begin{pmatrix} p & 1 \\ -1 & 0 \end{pmatrix}}_{\in SL(2,\mathbb{Z}_p)\subset H} \in H$$

Thus $\begin{pmatrix} p^n & 0 \\ 0 & p^{-n} \end{pmatrix} \in H$, for $n > 0$.

We already know that $SL(2,\mathbb{Q}_p)$ is generated by $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$, where $b, c \in \mathbb{Q}_p$. It is enough to to prove that $\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \in H$ because one can get the other matrix by conjugating by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Now for $c \in \mathbb{Q}_p$, we can write $c = up^k$, for some $u \in \mathbb{Z}_p^\times$ and $k \in \mathbb{Z}$. We can assume that $k < 0$ because otherwise the result follows trivially. Notice that

$$\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} = \begin{pmatrix} p^{-k} & 0 \\ 0 & p^k \end{pmatrix} \begin{pmatrix} 1 & 0 \\ p^{-2k}c & 0 \end{pmatrix} \begin{pmatrix} p^k & 0 \\ 0 & p^{-k} \end{pmatrix} \in H,$$

which is exactly what we want.

□

Theorem 4.2. *Let π be an infinite dimensional admissible irreducible representation of $GL(2,\mathbb{Q}_p)$. Let α and β be two characters of $\mathbb{Z}_p^\times$ such that $\alpha\beta = \omega_\pi$ (the central character of π).*

i. There exists a non zero vector $v \in V$ such that

$$\pi\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\right) v = \alpha(a)\beta(d)\cdot v \quad (a, d \in \mathbb{Z}_p^\times,\ b \in \mathbb{Z}_p)$$

ii. Let n be the smallest integer such that there exists v in (i), invariant by $\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$ for $c \in p^n\mathbb{Z}_p$. Then n is at least equal to the conductor of $\alpha\beta^{-1}$, in particular $n \geq 0$.

Proof. :

i. We need to identify v with a function $v(x) \in \mathcal{K}$ that is supported on $\mathbb{Z}_p$ and transforms by α on the cosets of $\mathbb{Z}_p^\times$. So we need to choose a $v(x)$ satisfying the relation $v(ax) = \alpha(a)\, v(x)$ for all $a \in \mathbb{Z}_p^\times$, i.e. $v(x)$ transforms by $\alpha(a)$ on any coset of $\mathbb{Z}_p^\times$. The simplest example here is to choose $v(x) = \alpha(x)\cdot 1_{\mathbb{Z}_p^\times}(x)$, where $1_{\mathbb{Z}_p^\times}(x)$ is the characteristic function of $\mathbb{Z}_p^\times$. This function is a Schwartz function, so it is in $\mathcal{K}$, the Kirillov model of π. Then

$$\begin{aligned}
&\pi\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\right)\cdot v(x) \\
&= \pi\left(\begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix}\begin{pmatrix} ad^{-1} & bd^{-1} \\ 0 & 1 \end{pmatrix}\right)\cdot v(x) \\
&= \omega_\pi(d) e_p(bd^{-1}x) v(ad^{-1}x) \\
&= \alpha(d)\beta(d)\alpha(ad^{-1}) v(x) \\
&= \alpha(a)\beta(d) v(x).
\end{aligned}$$

ii. Take v and n as in the statement of the theorem. Then we have three cases:

For $n = 0$, we have that $\pi\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\right) v = \alpha(a)\beta(d)\cdot v$ and $\pi\left(\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}\right) v = v$.

The stabilizer of v includes $SL(2,\mathbb{Z}_p)$ (by lemma 4.1), so it includes $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$.

This implies that $\alpha\beta^{-1}(a) = 1$ for all $a \in \mathbb{Z}_p^\times$, and so we get that the conductor of $\alpha\beta^{-1}$ is zero.

For $n > 0$, v is fixed by $\begin{pmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ p^n\mathbb{Z}_p & 1 \end{pmatrix}$. Hence it is fixed by

$$\left\{\begin{pmatrix} \equiv 1 \ (\mathrm{mod}\, p^n\mathbb{Z}_p) & \mathbb{Z}_p \\ \equiv 0 \ (\mathrm{mod}\, p^n\mathbb{Z}_p) & \equiv 1 \ (\mathrm{mod}\, p^n\mathbb{Z}_p) \end{pmatrix}, \text{ of determinant } 1\right\}$$

For all such $a \equiv 1$ and $d = a^{-1} \equiv 1 \ (\mathrm{mod}\, p^n\mathbb{Z}_p)$, we have $\alpha(a)\beta(a^{-1})v = v$, which gives us that $\alpha\beta^{-1}$ is trivial on $1 + p^n\mathbb{Z}_p$.

For $n < 0$, we have that v is fixed by $SL(2,\mathbb{Q}_p)$ (by lemma 4.1). But we proved in a previous theorem that there is no such non-zero vector. This is a contradiction. Hence we get that n cannot be strictly negative.

□

Now we introduce the following set. Let X_k be the subgroup of V defined by

$$X_k = \left\{ v \in V, \text{ such that } \pi\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\right) v = \alpha(a)\beta(d)\cdot v \text{ and } \pi\left(\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}\right) v = v \ \forall a,d \in \mathbb{Z}_p^\times,\ b \in \mathbb{Z}_p,\ c \in p^k\mathbb{Z}_p \right\}.$$

Definition 4.1. The smallest integer n defined in Theorem 4.2 (ii) means that $X_n \neq 0$ and $X_k = 0$ for $k < n$. n is called the conductor of the representation with respect to α and β.

If α is the trivial character and $\beta = \omega_\pi$ (the central character of π), then the corresponding n is called the conductor of the representation. In this case, n is at least equal to the conductor of ω_π.

For the rest of this chapter, fix a non zero vector $v_0 \in X_n$.

Lemma 4.3. *For $i \geq 0$, we have* $\pi\left(\begin{pmatrix} p^{-i} & 0 \\ 0 & 1 \end{pmatrix}\right) X_k \subset X_{k+i}$.

Proof. First of all, notice that the condition of transforming under $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ is unaffected by $\begin{pmatrix} p^{-i} & 0 \\ 0 & 1 \end{pmatrix}$ because

$$\pi\left(\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\begin{pmatrix} p^{-i} & 0 \\ 0 & 1 \end{pmatrix}\right) z = \pi\left(\begin{pmatrix} p^{-i} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\right) z = \alpha(a)\beta(d)z$$

for all $a, d \in \mathbb{Z}_p^\times$. We want to prove that the transformation $\begin{pmatrix} p^{-i} & 0 \\ 0 & 1 \end{pmatrix}$ transforms vectors that are $\begin{pmatrix} 1 & 0 \\ p^k\mathbb{Z}_p & 1 \end{pmatrix}$- invariant into vectors that are invariant under $\begin{pmatrix} 1 & 0 \\ p^{k+i}\mathbb{Z}_p & 1 \end{pmatrix}$. Indeed, let $H = \left\{\begin{pmatrix} 1 & 0 \\ p^k c & 1 \end{pmatrix}, \text{ such that } c \in \mathbb{Z}_p\right\}$ be a subgroup of $GL(2, \mathbb{Q}_p)$. Let $g = \begin{pmatrix} p^{-i} & 0 \\ 0 & 1 \end{pmatrix}$. Note that $gHg^{-1}\begin{pmatrix} 1 & 0 \\ p^{k+i}\mathbb{Z}_p & 1 \end{pmatrix}$. One checks that if v is invariant under H, then gv is invariant under gHg^{-1}. Indeed, let $h \in H$ be arbitrary, so ghg^{-1} is arbitrary element of gHg^{-1}. Then

$$ghg^{-1}(gv) = g(hv) = gv.$$

Lastly, we need to check invariance under $\begin{pmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix}$ for gv. Let $b \in \mathbb{Z}_p$, then

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} gv = gg^{-1}\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} gv = g\begin{pmatrix} 1 & p^i b \\ 0 & 1 \end{pmatrix} v = gv$$

where the last step follows because $i \geq 0$. □

Lemma 4.4. *$\pi\left(\begin{pmatrix} p^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right) X_{k-1} = \left\{z \in X_k \text{ such that } supp(v) \subset p\mathbb{Z}_p\right\}$.*

Proof. $\supseteq$) We know that $\pi\left(\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}\right) z(t) = z(pt)$ for all $t \in \mathbb{Q}_p^\times$. Let $z' = \pi\left(\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}\right) z$. We want to show that $z' \in X_{k-1}$.

- $\pi\left(\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\right) z' = \pi\left(\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}\right) z = \pi\left(\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\right) z = \alpha(a)\beta(d)z'$
- Let $c \in p^{k-1}\mathbb{Z}_p$, so that $c = p^{k-1}\gamma$ for some $\gamma \in \mathbb{Z}_p$. Then using the identity

$$\begin{pmatrix} p^{-1} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ p^{k-1}\gamma & 1 \end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ p^k\gamma & 1 \end{pmatrix},$$

we get that

$$\begin{aligned}\pi\left(\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}\right) z' &= \pi\left(\begin{pmatrix} 1 & 0 \\ p^{k-1}\gamma & 1 \end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}\right) z \\ &= \pi\left(\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ p^k\gamma & 1 \end{pmatrix}\right) z = \pi\left(\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}\right) z = z'.\end{aligned}$$

- We still need to prove that $\pi\left(\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}\right)z' = z'$. However, we will prove a stronger statement:

  claim $\mathrm{supp}(z') \subset \mathbb{Z}_p \iff \pi\left(\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}\right)z' = z' \ \forall b \in \mathbb{Z}_p$.

  Proof of the claim: Assume that $\mathrm{supp}(z') \subset \mathbb{Z}_p$. We know that $\mathrm{supp}(z) \subset p\mathbb{Z}_p$, then

  $$z(x) \neq 0 \implies x \in p\mathbb{Z}_p.$$

  But we also know that $z'(x) = z(px)$, then

  $$z'(x) \neq 0 \implies px \in p\mathbb{Z}_p \implies x \in \mathbb{Z}_p.$$

  This implies that $e_p(bx)z'(x) = z'(x)$. Indeed, if $x \in \mathbb{Z}_p$, we get that $e_p(bx) = 1$. If $x \notin \mathbb{Z}_p$, then $z'(x) = 0$. Therefore, $\pi\left(\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}\right)z' = z'$ for all $b \in \mathbb{Z}_p$.

  For the other implication, assume that $\mathrm{supp}(z') \not\subset \mathbb{Z}_p$, then we can find a point on the support for which $e_p(bx) \neq 1$. But $z'(x)$ is different than zero on its support; this contradicts the fact that $e_p(bx)z'(x) = z'(x)$.

Under all of the above, we get that $z' \in X_{k-1}$.

$\subseteq)$ Conversely, let $z \in \pi\left(\begin{pmatrix}p^{-1} & 0\\ 0 & 1\end{pmatrix}\right)X_{k-1}$, then by lemma 4.3, we get that $z \in X_k$. One can write $z = \pi\left(\begin{pmatrix}p^{-1} & 0\\ 0 & 1\end{pmatrix}\right)v$, where $v \in X_{k-1}$. Identifying the functions with the Kirillov model, we get that $z(pt) = v(t)$. From the above claim, we know that $\mathrm{supp}(v) \subset \mathbb{Z}_p$. Then

$$z(pt) \neq 0 \iff v(t) \neq 0 \iff t \in \mathbb{Z}_p.$$

It follows that

$$z(u) \neq 0 \iff p^{-1}u \in \mathbb{Z}_p \iff u \in p\mathbb{Z}_p.$$

And so $\mathrm{supp}(z) \subset p\mathbb{Z}_p$. □

Lemma 4.5. *We have*

$$dim\left(X_k \Big/ \pi\left(\begin{pmatrix} p^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right) X_{k-1} \right) = 1.$$

Proof. Let $x, y \in X_k$. Identifying our space with the Kirillov model, we can think of x and y as functions supported on $\mathbb{Z}_p$ due to $\pi\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right)$ invariance (because of the claim mentioned in the proof of lemma 4.4), where $x(ut) = \alpha(u)x(t)$ and $y(ut) = \alpha(u)y(t)$ for all $u \in \mathbb{Z}_p^\times$ from $\pi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)$ behaviour. Because of the Kirillov model, the restrictions of x and y to $\mathbb{Z}_p^\times$ must be proportional, i.e. we can find $(\lambda, \mu) \neq (0,0)$ such that $\lambda x + \mu y = 0$. Let $z = \lambda x + \mu y$, then

- $\mathrm{supp}(z) \subset p\mathbb{Z}_p$,
- $z \in X_k$.

It follows from lemma 4.4 that $z \in \pi\left(\begin{pmatrix} p^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right) X_{k-1}$. Therefore, any two elements $x, y \in X_k$ are linearly dependent in $X_k \Big/ \pi\left(\begin{pmatrix} p^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right) X_{k-1}$, so that the latter has dimension 1. The dimension is not zero because we one can write down explicitly an element $\pi\left(\begin{pmatrix} p^{n-k} & 0 \\ 0 & 1 \end{pmatrix}\right) v_0$ where $v_0 \in X_n$. □

Theorem 4.6. *For $k \geq n$, the space X_k is of dimension $k - n + 1$, and has as basis elements $\pi\left(\begin{pmatrix} p^{-i} & 0 \\ 0 & 1 \end{pmatrix}\right) v_0$.*

Proof. Let $k \geq n$. From lemma 4.5, we have $\dim\left(X_k \middle/ \pi\left(\begin{pmatrix} p^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right) X_{k-1} \right) = 1$.

This implies that $\dim(X_k) = \dim\left(\pi\left(\begin{pmatrix} p^{-i} & 0 \\ 0 & 1 \end{pmatrix}\right) X_{k-1}\right) + 1 = \dim(X_{k-1}) + 1$. As X_{n-1} is zero by definition of n, we get that $\dim(X_n)$=1. One can argue by induction to get that $\dim(X_k)$= $k - n + 1$. Now, we want to find the basis vectors for X_k. First, note that $v_0 \in X_n$, which is 1− dimensional, then v_0 spans X_n.

From lemma 4.3, we have that $\pi\left(\begin{pmatrix} p^{-i} & 0 \\ 0 & 1 \end{pmatrix}\right) X_k \subset X_{k+i}$ for all $i \geq 0$. As $v_0 \in X_n$, then $v_0 \in X_{n+1}$, but also $\pi\left(\begin{pmatrix} p^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right) v_0 \in X_{n+1}$, and the latter two elements are linearly independent. Indeed, assume there exists $\alpha, \beta \in \mathbb{C}$ such that

$$\alpha v_0 + \beta\pi\left(\begin{pmatrix} p^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right) v_0 = 0 \qquad \text{in } X_{n+1}.$$

This implies that

$$\alpha v_0 = 0 \qquad \text{in } X_k \middle/ \pi\left(\begin{pmatrix} p^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right) X_{k-1}.$$

But v_0 is a non zero vector in $X_k \middle/ \pi\left(\begin{pmatrix} p^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right) X_{k-1}$, then we get that $\alpha = 0$. Substituting in the above equation, we get then $\beta = 0$. It follows that X_2 is spanned by these two vectors as X_2 is 2-dimensional. We complete the proof by induction on k. □

Remark. The space X_k defined previously has an equivalent definition. It is actually the space of all vectors $v \in V$ such that v is invariant under $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $a \in \mathbb{Z}_p^\times$, $b \in \mathbb{Z}_p$, $c \equiv 0 \bmod(p^n\mathbb{Z}_p)$, $d \equiv 1 \bmod(p^n\mathbb{Z}_p)$.

Now we can define the local new vector of a representation.

Definition 4.2. Let π be an infinite dimensional admissible irreducible representation of $GL(2,\mathbb{Q}_p)$. Let α and β be two characters of $\mathbb{Z}_p^\times$ such that $\alpha\beta = \omega_\pi$. Assume that α is the trivial character and $\beta = \omega_\pi$, and let n be the conductor of π, i.e. the smallest integer such that X_n is non-zero (and in fact one dimensional). A vector v in X_n is called the new vector of the representation.

Remark. The new vector is unique up to multiplying by a scalar.

Actually, for $k \geq n$, the space X_k is a local p-adic analogue of the congruence subgroup

$$\Gamma_{0,1}(p^k\mathbb{Z}_p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mod (p^n\mathbb{Z}_p) \right\} \subset GL(2,\mathbb{Z}_p).$$

Moreover, the space $V^{\Gamma_{0,1}(p^n)}$ of vectors that transform under $\Gamma_{0,1}(p^n)$ by $\beta(d)$ is one dimensional, so that $v \in V^{\Gamma_{0,1}(p^n)}$ is the new form. Moreover

$$V^{\Gamma_{0,1}(p^k)} = \left\langle v, \begin{pmatrix} p^{-1} & 0 \\ 0 & 1 \end{pmatrix} v, \cdots, \begin{pmatrix} p^{-(k-n)} & 0 \\ 0 & 1 \end{pmatrix} v \right\rangle$$

for all $k \geq n$. Therefore, a new vector is analogue to the notion of a new form in the sense of modular forms.

Chapter 5

The Principal Series Representations

One of the main goals of [Jacquet and Langlands, 1970] was to classify the local admissible irreducible representations of $GL(2, \mathbb{Q}_p)$. It turns out that this task would be much easier by using the Kirillov space. In this chapter, we define the principal series representations and find the associated Kirillov model explicitly.

5.1 Admissibility

In order to guarantee the existence of the Kirillov model of principal series representations, we need to check that conditions of Theorem 3.2.1 are satisfied. In this section, we define the principal series representations of $GL(2, \mathbb{Q}_p)$ and prove that they are admissible.

First, we recall the definition of smooth induced representations.

Definition 5.1.1. Let G, H be two groups with $H \in G$. Let $\rho : H \to GL(V)$ be a smooth representation. We define the smooth induction of V to G, denoted by $\mathrm{Ind}_H^G(V)$ or $\mathrm{Ind}_H^G(\rho)$, to be the space

$$\mathrm{Ind}_H^G(V) \coloneqq \left\{ f : G \to V \mid f \text{ is locally constant and } f(hg) = \rho(h)(f(g)) \ \forall h \in H,\, g \in G \right\}.$$

G here acts on the induced space by right translations, as we will see in the formula below.

Let ω_1, ω_2 be two normalized unitary characters of $\mathbb{Q}_p^\times$, and let $s_1, s_2 \in \mathbb{C}$. Then we can consider the characters χ_1, χ_2 of $\mathbb{Q}_p^\times$ given by

$$\chi_1(x) = \omega_1(x)|x|_p^{s_1} \qquad \text{and} \qquad \chi_2(x) = \omega_2(x)|x|_p^{s_2}.$$

$\chi = (\chi_1, \chi_2)$ extends to a character of the diagonal matrices D as $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \to \chi_1(a)\chi_2(d)$. Consequently, $\chi = (\chi_1, \chi_2)$ extends to a character of the Borel subgroup $B(\mathbb{Q}_p)$ by

$$\chi\left[\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right] = \chi_1(a)\chi_2(d) \text{ for all } a, d \in \mathbb{Q}_p^\times,\, x \in \mathbb{Q}_p.$$

Definition 5.1.2. We define the "normalized parabolic induction" of χ to be the space

$$\mathcal{B}_{\chi_1,\chi_2} \coloneqq \Bigg\{ \phi : GL(2, \mathbb{Q}_p) \to \mathbb{C} \text{ such that } \phi \text{ is locally constant and}$$
$$\phi\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} g\right) = \chi_1(a)\chi_2(d) \left|\frac{a}{d}\right|_p^{\frac{1}{2}} \phi(g) \ \forall\, a, d \in \mathbb{Q}_p^\times,\, b \in \mathbb{Q}_p,\, g \in GL(2, \mathbb{Q}_p) \Bigg\},$$

where $GL(2, \mathbb{Q}_p)$ acts on $\mathcal{B}_{\chi_1,\chi_2}$ through right translations, i.e. $g_1 \cdot \phi(g) = \phi(gg_1)$ for all $g, g_1 \in GL(2, \mathbb{Q}_p)$, $\phi \in \mathcal{B}_{\chi_1,\chi_2}$.

Remark. In the above definition, we have $\mathcal{B}_{\chi_1,\chi_2} = \mathrm{Ind}_{B(\mathbb{Q}_p)}^{GL(2,\mathbb{Q}_p)}(\rho_{\chi_1,\chi_2})$, where

$$\rho_{\chi_1,\chi_2}\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} g\right) = \chi_1(a)\chi_2(d)\left|\frac{a}{d}\right|_p^{\frac{1}{2}} \rho_{\chi_1,\chi_2}(g)\ \forall\, a,d \in \mathbb{Q}_p^\times,\ b \in \mathbb{Q}_p,\ g \in GL(2,\mathbb{Q}_p).$$

Definition 5.1.3. If $(\rho_{\chi_1,\chi_2}, \mathcal{B}_{\chi_1,\chi_2})$ is an irreducible representation, then it is called the principal series representation of $GL(2,\mathbb{Q}_p)$ associated to $\chi = (\chi_1,\chi_2)$.

Remark. [Goldfeld and Hundley, 2011] work with non-normalized induction:

$$\mathcal{B}_{\text{non-normalized}}(\chi_1,\chi_2) \coloneqq \Bigg\{\phi : GL(2,\mathbb{Q}_p) \to \mathbb{C} \text{ such that } \phi \text{ is locally constant and}$$
$$\phi\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} g\right) = \chi_1(a)\chi_2(d)\phi(g)\ \forall\, a,d \in \mathbb{Q}_p^\times,\ b \in \mathbb{Q}_p,\ g \in GL(2,\mathbb{Q}_p)\Bigg\}.$$

One can go between the two definitions via:

$$\mathcal{B}_{\text{normalized}}(\chi_1,\chi_2) = \mathcal{B}_{\text{non-normalized}}(\chi_1 \cdot |\ |_p^{\frac{1}{2}}, \chi_2) \cdot |\ |_p^{\frac{-1}{2}}).$$

Lemma 5.1.1 (Iwasawa's decomposition)**.** *$GL(2,\mathbb{Q}_p) = B(\mathbb{Q}_p)K$, where $K = GL(2,\mathbb{Z}_p)$.*

Proof. See [Bump, 1998] Proposition 4.5.2. □

Theorem 5.1.2. *Principal series representations are admissible.*

Proof.
- We first prove that $(\rho_{\chi_1,\chi_2}, \mathcal{B}_{\chi_1,\chi_2})$ is smooth. Let $f \in \mathcal{B}_{\chi_1,\chi_2}$. As f is locally constant, it follows that for each $g \in GL(2,\mathbb{Z}_p)$, there exists n such that $f(gk) = f(g)$ whenever $k \equiv I_2 \bmod (p^n\mathbb{Z}_p)$. What we want to show is that n can be made independent of g.

  Recall from chapter 3 that $K_n = \{k \in GL(2,\mathbb{Z}_p) \mid K \equiv I_2 \bmod p^n\mathbb{Z}_p\}$ is compact open subgroup for all $n \geq 0$. Note that for each $k \in GL(2,\mathbb{Z}_p)$, there exists an integer $n(k) \geq 1$ such that

$$f(kk') = f(k) \quad (\forall k' \in K_{n(k)}).$$

The cosets

$$k \cdot K_{n(k)} \quad (k \in GL(2, \mathbb{Z}_p)$$

form an open cover of $GL(2, \mathbb{Z}_p)$. As $GL(2, \mathbb{Z}_p)$ is compact,then we can choose a finite subcover. Let then n to be the largest value on $n(k)$. As $K_n \subset K_m$ for $n > m$, it follows that

$$f(kk') = f(k) \quad (\forall k \in GL(2, \mathbb{Z}_p),\, k' \in K_n).$$

Now, using Iwasaswa decomposition, we get that

$$f(gk') = f(g) \quad (\forall g \in GL(2, \mathbb{Q}_p),\, k' \in K_n).$$

Therefore we get that $(\rho_{\chi_1,\chi_2}, \mathcal{B}_{\chi_1,\chi_2})$ is smooth.

- We now need to prove that $(\rho_{\chi_1,\chi_2}, \mathcal{B}_{\chi_1,\chi_2})$ is admissible. Fix a positive integer n and consider

$$\mathcal{B}^n_{\chi_1,\chi_2} \coloneqq \left\{ f \in \mathcal{B}_{\chi_1,\chi_2} \mid f(gk) = f(g),\, \forall g \in GL(2, \mathbb{Q}_p),\, k \in K_n \right\}.$$

We must prove that this space is finite dimensional for each n. By Iwasawa's decomposition, we have that $GL(2, \mathbb{Q}_p) = B(\mathbb{Q}_p) \cdot K$. It follows that $B \backslash G / K$ has one element. Thus $B \backslash G / K_n$ has finitely many representatives, using the representatives of K / K_n.

Now, any function in $\mathcal{B}^n_{\chi_1,\chi_2}$ that is fixed by K_n is determined by its values on these representatives, and the latter are just finitely many. It follows that $\mathcal{B}^n_{\chi_1,\chi_2}$ is finite dimensional, and so the principal series representations of

$GL(2, \mathbb{Q}_p)$ are admissible.

□

Remark. [Goldfeld and Hundley, 2011] find the exact set of representatives and computes the exact dimension. For more details, refer to Proposition 6.5.5.

5.2 The Kirillov Model of the Principal Series

Goal: In this section, we are going to find the Kirillov model of the principal series representation $(\rho_{\chi_1,\chi_2}, \mathcal{B}_{\chi_1,\chi_2})$ of $GL(2, \mathbb{Q}_p)$. For the rest of the section, let $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and let $\chi = \chi_1\chi_2^{-1}$ be a character on $\mathbb{Q}_p^\times$. The content of this section is taken from [Godement, 1974] chapter 1, sections 8 and 9.

Lemma 5.2.1 (Bruhat Decomposition)**.** *Fix a prime p. The group $GL(2, \mathbb{Q}_p)$ is the disjoint union of the double cosets*

$$B(\mathbb{Q}_p) \dot{\bigcup} B(\mathbb{Q}_p) w^{-1} N(\mathbb{Q}_p),$$

where

$$B(\mathbb{Q}_p) = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, \quad \text{for } a, d \in \mathbb{Q}_p^\times, b \in \mathbb{Q}_p \right\} \text{ is called the Borel subgroup;}$$

$$N(\mathbb{Q}_p) = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \quad \text{for } b \in \mathbb{Q}_p \right\}.$$

Remark. The two double cosets are called the "Bruhat cells". $B(\mathbb{Q}_p)$ is called the "little cell". $B(\mathbb{Q}_p)w^{-1}N(\mathbb{Q}_p)$ is called the "big cell".

Proof. The little cell "$B(\mathbb{Q}_p)$" consists of matrices over $GL(2, \mathbb{Q}_p)$ with $c = 0$. We must prove that $B(\mathbb{Q}_p)w^{-1}N(\mathbb{Q}_p)$ is exactly the set of matrices:

$$B(\mathbb{Q}_p)w^{-1}N(\mathbb{Q}_p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{Q}_p) \text{ such that } c \neq 0 \right\}.$$

Indeed, let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{Q}_p)$ such that $c \neq 0$. Then

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \underbrace{\begin{pmatrix} c^{-1}\det(g) & a \\ 0 & c \end{pmatrix}}_{\in B(\mathbb{Q}_p)} \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{=w^{-1}} \underbrace{\begin{pmatrix} 1 & dc^{-1} \\ 0 & 1 \end{pmatrix}}_{\in N(\mathbb{Q}_p)}$$

□

Let $x \in \mathbb{Q}_p$, $\phi \in \mathcal{B}_{\chi_1,\chi_2}$. Define $\Phi_\phi(x) := \phi\left(w^{-1}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)$ to be a function on $GL(2,\mathbb{Q}_p)$.

Proposition 5.2.1. *Φ_ϕ is locally constant, and the map $\phi \to \Phi_\phi$ is injective.*

Proof. As ϕ is locally constant, it follows that Φ_ϕ is locally constant as well. We need now to check injectivity. Assume that $\phi \in \mathcal{B}_{\chi_1,\chi_2}$ satisfies

$$\Phi_\phi(x) = 0 \qquad \forall x \in \mathbb{Q}_p.$$

We want to show that this implies $\phi \equiv 0$. Actually, this can be proved using the Bruhat decomposition. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{Q}_p)$.

- If $c \neq 0$, then

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c^{-1}\det(g) & a \\ 0 & c \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & dc^{-1} \\ 0 & 1 \end{pmatrix},$$

which gives us that

$$\begin{aligned}
\phi(g) &= \phi\left(\begin{pmatrix} c^{-1}\det(g) & a \\ 0 & c \end{pmatrix}\right) \phi\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & dc^{-1} \\ 0 & 1 \end{pmatrix}\right) \\
&= \chi_1\Big(c^{-1}\det(g)\Big)\, \chi_2(c) \left|c^{-2}\det(g)\right|_p^{\frac{1}{2}} \cdot \Phi_\phi(dc^{-1}) \\
&= \chi_1\Big(\det(g)\Big)\, \chi^{-1}(c) \left|c\right|_p^{-1} \left|\det(g)\right|_p^{\frac{1}{2}} \cdot \underbrace{\Phi_\phi(dc^{-1})}_{=0} \\
&= 0.
\end{aligned}$$

- We still need to prove that $\phi(g) = 0$ if $c = 0$. Actually, this uses the fact that the "big cell" in the Bruhat decomposition is everywhere dense in $GL(2, \mathbb{Q}_p)$. Let us prove that $B(\mathbb{Q}_p)w^{-1}N(\mathbb{Q}_p)$ is a dense subset of $GL(2, \mathbb{Q}_p)$.

  Now we know that $\det(g) \neq 0$, i.e. $ad \neq 0$ where $a, d \in \mathbb{Q}_p^\times$. Fix a, b, d and take a sequence c_n such that

  - $c_n \to 0$ as $n \to \infty$.
  - c_n are suffieciently small so that $ad - bc_n \neq 0$ for all n.

  Then $g_n = \begin{pmatrix} a & b \\ c_n & d \end{pmatrix} \to g$ as $n \to \infty$, just as required. Using the continuity of ϕ, we get that

$$\phi(g) = \phi(\lim_{n\to\infty} g_n) = \lim_{n\to\infty}(\phi(g_n)) = 0.$$

□

Proposition 5.2.2. *$\Phi_\phi(x) \left|x\right|_p \chi(x)$ is constant for large $\left|x\right|_p$.*

Remark. The space of functions of Φ_ϕ will be denoted $\mathcal{F}_\chi$.

Proof. Let $x \in \mathbb{Q}_p^\times$. Using the identity

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -x^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x^{-1} & 0 \\ 0 & x \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -x^{-1} & 1 \end{pmatrix},$$

we get that

$$\begin{aligned} \phi\left(w^{-1}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) &= \chi_1(x^{-1})\chi_2(x)|x^{-2}|_p^{\frac{1}{2}}\phi\left(\begin{pmatrix} 1 & 0 \\ -x^{-1} & 1 \end{pmatrix}\right) \\ &= \chi(x)^{-1}|x|_p^{-1}\phi\left(\begin{pmatrix} 1 & 0 \\ -x^{-1} & 1 \end{pmatrix}\right). \end{aligned}$$

As ϕ is locally constant, then $\phi\left(\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}\right)$ is constant for y in the neighborhood of zero. Then for $|x|_p$ sufficiently large, we get that $\phi\left(\begin{pmatrix} 1 & 0 \\ -x^{-1} & 1 \end{pmatrix}\right) = \phi(e)$ which is constant, and thus

$$\phi\left(w^{-1}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) = \chi(x)^{-1}|x|_p^{-1}\phi(e).$$

Therefore $\Phi_\phi(x)\chi(x)|x|_p = \phi(e) =$ constant for $|x|_p$ sufficiently large. □

To get the Kirillov model for the representation $(\rho_{\chi_1,\chi_2}, \mathcal{B}_{\chi_1,\chi_2})$, we associate for each $\phi \in \mathcal{B}_{\chi_1,\chi_2}$, the function

$$\begin{aligned} \xi_\phi(x) &= \chi_2(x)\,|x|_p^{\frac{1}{2}} \int \phi\left(w^{-1}\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}\right) e_p(-xy)\,dy \\ &= \chi_2(x)\,|x|_p^{\frac{1}{2}}\,\widehat{\Phi}_\phi(x) \end{aligned} \tag{5.1}$$

For the next proposition, we assume the convergence of the integral given in equation 5.1.

Proposition 5.2.3. *The space* $\left\{\xi_\phi;\ \text{where } \phi \in \mathcal{B}_{\chi_1,\chi_2}\right\}$ *is the Kirillov model of* ρ_{χ_1,χ_2}*. In other words, we have*

$$\rho_{\chi_1,\chi_2}\left(\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\right)\xi(x) = e_p(bx)\xi(ax) \qquad \forall a \in \mathbb{Q}_p^\times,\ b \in \mathbb{Q}_p.$$

Proof. Let $\phi \in \mathcal{B}_{\chi_1,\chi_2}$, then ϕ, and so Φ_ϕ are locally constant functions. Now using the fact that

$$\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}\phi(x) = \phi(x) \qquad \forall u \in 1 + p^N\mathbb{Z}_p \ (N \text{ sufficiently large}),\ x \in \mathbb{Q}_p,$$

and that $\widehat{\Phi}_\phi(ux) = \widehat{\Phi}_\phi(x)$, it follows that $\widehat{\Phi}_\phi$ is also locally constant. This implies that ξ_ϕ is a locally constant function.

Now, using the matrix identity

$$w^{-1}\begin{pmatrix} a & ay \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} w^{-1}\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix},$$

we get that

$$\begin{aligned}
\rho_{\chi_1,\chi_2}\left(\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\right)\xi_\phi(x) &= \chi_2(x)\,|x|_p^{\frac{1}{2}}\left[\widehat{\rho_{\chi_1,\chi_2}\left(\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\right)\Phi_\phi}\right](x) \\
&= \chi_2(x)\,|x|_p^{\frac{1}{2}}\int [\rho_{\chi_1,\chi_2}\phi]\left(w^{-1}\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}\right)e_p(-xy)\,dy \\
&= \chi_2(x)\,|x|_p^{\frac{1}{2}}\int \phi\left(w^{-1}\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\right)e_p(-xy)\,dy
\end{aligned}$$

$$= \chi_2(x)\,|x|_p^{\frac{1}{2}} \int \phi\left(w^{-1}\begin{pmatrix} a & b+y \\ 0 & 1 \end{pmatrix}\right) e_p(-xy)\,dy$$

$$= \chi_2(x)\,|x|_p^{\frac{1}{2}} \int \phi\left(w^{-1}\begin{pmatrix} a & y \\ 0 & 1 \end{pmatrix}\right) e_p(-xy+bx)\,dy \qquad (y \to y-b)$$

$$= e_p(bx)\chi_2(x)\,|x|_p^{\frac{1}{2}} \int \phi\left(w^{-1}\begin{pmatrix} a & y \\ 0 & 1 \end{pmatrix}\right) e_p(-xy)\,dy$$

$$= e_p(bx)\chi_2(x)\,|x|_p^{\frac{1}{2}}\,|a|_p \int \phi\left(w^{-1}\begin{pmatrix} a & ay \\ 0 & 1 \end{pmatrix}\right) e_p(-axy)\,dy \qquad (y \to ay)$$

$$= e_p(bx)\chi_2(x)\,|x|_p^{\frac{1}{2}}\,|a|_p \int \phi\left(\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} w^{-1}\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}\right) e_p(-axy)\,dy$$

$$= e_p(bx)\chi_2(x)\,|x|_p^{\frac{1}{2}}\,|a|_p\chi_2(a)|a|_p^{\frac{-1}{2}} \int \Phi_\phi(y)e_p(-axy)\,dy$$

$$= e_p(bx)\,\chi_2(ax)\,|x|_p^{\frac{1}{2}}\,|a|_p^{\frac{1}{2}}\,\widehat{\Phi}_\phi(ax)$$

$$= e_p(bx)\,\chi_2(ax)\,|ax|_p^{\frac{1}{2}}\,\widehat{\Phi}_\phi(ax)$$

$$= e_p(bx)\,\xi_\phi(ax).$$

Now, the Mirabolic subgroup $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ acts on this space exactly by the same formulas that we got for the action of the Mirabolic subgroup on the Kirillov model. Therefore, by the uniqueness of the Kirillov model, we arrive to the desired result. □

It remains now to study the convergence of the integral in 5.1. For this purpose, we need to distinguish between two types of characters, the ramified and unramified characters. Any $x \in \mathbb{Q}_p^\times$ can be written uniquely as $p^n u$, where $n \in \mathbb{Z}$ and $u \in \mathbb{Z}_p^\times$ is a unit. This gives an isomorphism $\mathbb{Q}_p^\times \equiv \mathbb{Z} \times \mathbb{Z}_p^\times$. The characters of $\mathbb{Z}$ are just given by $n \to e^{ns}$ for some $s \in \mathbb{C}$, which for our purposes we will rewrite in the form $p^{-ns'}$

where $s' = \frac{-s}{ln(p)}$. Hence, for all $x = p^n u$ ($n \in \mathbb{Z}, u \in \mathbb{Z}_p^\times$), we can rewrite any character χ of $\mathbb{Q}_p^\times$ as $\chi(x) = p^{-ns}\omega(u) = |x|_p^{\,s}\omega(u)$ for some $s \in \mathbb{C}$ and ω a character of $\mathbb{Z}_p^\times$. Any character ω of $\mathbb{Z}_p^\times$ is unitary, i.e. has image in S^1.

If the conductor of χ is zero, i.e. $\omega = 1$, we say that χ is **unramified**. Otherwise, χ is **ramified**. This means that the only unramified characters of $\mathbb{Q}_p^\times$ are $|\ |_p^{\,s}$.

Proposition 5.2.4. *For $\Phi \in \mathcal{F}_\chi$, the function $\widehat{\Phi}$ has the following asymptotic behaviour near zero:*

$$\begin{cases} a\chi(x) + b & \textit{if } \chi(x) \neq 1, |x|_p^{\,-1}, \\ av_p(x) + b & \textit{if } \chi(x) = 1, \\ b & \textit{if } \chi(x) = |x|_p^{\,-1}, \end{cases}$$

where $v_p(x)$ is the p-adic valuation of x, and a, b are complex constants.

Proof. The space $\mathcal{F}_\chi$ is the direct sum of the Schwartz space $S(\mathbb{Q}_p)$ and the one-dimensional subspace spanned by the function

$$\Phi_\chi(x) = \begin{cases} \chi(x)^{-1}\,|x|_p^{\,-1} & \text{if } |x|_p \geq 1\ , \\ 0 & \text{if } |x|_p < 1\ . \end{cases}$$

The behaviour of $\hat{\Phi}$ near 0 is clear for all $\Phi \in S(\mathbb{Q}_p)$, so that the main part of the proof will be for Φ_χ. We will do the case where χ is unramified, i.e. $\chi(x) = |x|_p^{\,s}$ for some $s \in \mathbb{C}$. For all $x \in \mathbb{Q}_p^\times$, we write

$$\begin{aligned} \hat{\Phi}_\chi(x) &= \sum_{n\in\mathbb{Z}} \int_{v_p(y)=n} \Phi_\chi(y) e_p(-xy)\, d^\times y \\ &= \sum_{n\in\mathbb{Z}} \int_{v_p(y)=n} e_p(-xy)\, \chi(y)^{-1}\, |y|_p^{\,-1}\, d^\times y \end{aligned}$$

So it is enough to consider

$$\sum_{n\leq 0}\int_{v_p(y)=n} e_p(-xy)\,|y|_p^{-s}\,d^\times y$$

up to some constant due to the choice of Haar measure. Now we have that

$$\begin{aligned}
\int_{v_p(y)=n} e_p(-xy)\,|y|_p^{-s} d^\times y &= p^{ns}\int_{v_p(y)=n} e_p(-xy)\,d^\times y \\
&= p^{ns}\int_{u\in\mathbb{Z}_p^\times} e_p(-xp^n u)\,du \\
&= p^{ns}\left[\int_{u\in\mathbb{Z}_p} e_p(-xp^n u)\,du - \int_{u\in p\mathbb{Z}_p} e_p(-xp^n u)\,du\right] \\
&= p^{ns}\left[\int_{\mathbb{Z}_p} e_p(-xp^n u)\,du - p^{-1}\int_{\mathbb{Z}_p} e_p(-xp^{n+1} u)\,du\right] \\
&= p^{ns}\cdot\left(\begin{cases} 0 & \text{if } v_p(x) < -n \\ 1 & \text{if } v_p(x) \geq -n\end{cases} - p^{-1}\cdot\begin{cases} 0 & \text{if } v_p(x) < -n-1 \\ 1 & \text{if } v_p(x) \geq -n-1\end{cases}\right) \\
&= p^{ns}\begin{cases} 0 & \text{if } v_p(x) < -n-1 \\ -p^{-1} & \text{if } v_p(x) = -n-1 \\ 1-p^{-1} & \text{if } v_p(x) > -n-1\end{cases} \\
&= \begin{cases} 0 & \text{if } v_p(x) < -n-1 \\ -p^{ns-1} & \text{if } v_p(x) = -n-1 \\ p^{ns}(1-p^{-1}) & \text{if } v_p(x) > -n-1\ ,\end{cases}
\end{aligned}$$

where step 4 to 5 follows from Lemma 2.3.3. Thus we get that

$$\begin{aligned}
\hat{\Phi}_\chi(x) &= \sum_{n\leq 0}\int_{v_p(y)=n} e_p(-xy)\,|y|_p^{-s} d^\times y \\
&= \sum_{-v_p(x)-1\leq n\leq 0}\int_{v_p(y)=n} e_p(-xy)\,|y|_p^{-s} d^\times y \\
&= -p^{(-v_p(x)-1)s-1} + (1-p^{-1})\sum_{-v_p(x)\leq n\leq 0} p^{ns}. \qquad (5.2)
\end{aligned}$$

Case 1: If $s = 0$, i.e. $\chi(x) = 1$, then

$$\hat{\Phi}(x) = -p^{-1} + (1 - p^{-1})(v_p(x) + 1) = (1 - p^{-1})v_p(x) + 1.$$

Otherwise, we need to compute the geometric series $\sum_{-v_p(x) \le n \le 0} p^{ns}$. Indeed,

$$\begin{aligned}
\sum_{-v_p(x) \le n \le 0} p^{ns} &= \sum_{0 \le n \le v_p(x)} p^{(n - v_p(x))s} \\
&= p^{-v_p(x)s} \sum_{0 \le n \le v_p(x)} p^{ns} \\
&= p^{-v_p(x)s} \cdot \frac{1 - p^{(v_p(x)+1)s}}{1 - p^s} \\
&= \left(\frac{1}{1 - p^s}\right) p^{-v_p(x)s} - \frac{p^s}{1 - p^s} \\
&= \left(\frac{1}{1 - p^s}\right) |x|_p^{\,s} - \frac{p^s}{1 - p^s}
\end{aligned}$$

Substituting in equation (5.2), we get that

$$\begin{aligned}
\hat{\Phi}_\chi(x) &= -p^{(-v_p(x)-1)s-1} + (1 - p^{-1})\left[\left(\frac{1}{1 - p^s}\right) p^{-v_p(x)s} - \frac{p^s}{1 - p^s}\right] \\
&= \left(-p^{-s-1} + \frac{1 - p^{-1}}{1 - p^s}\right) |x|_p^{\,s} - \frac{p^s(1 - p^{-1})}{1 - p^s} \\
&= a\chi(x) + b,
\end{aligned} \tag{5.3}$$

where $a = \left(-p^{-s-1} + \frac{1 - p^{-1}}{1 - p^s}\right)$ and $b = -\frac{p^s(1 - p^{-1})}{1 - p^s}$. This gives us the following cases where $s \neq 0$:

Case 2: If $s = -1$, i.e. $\chi(x) = |x|_p^{-1}$, then $a = -1 + 1 = 0$ and $b = p^{-1}$. Thus $\hat{\Phi}_\chi(x) = b = p^{-1}$.

Case 3: If $s \neq 0, -1$, i.e. $\chi(x) \neq 1, |x|_p^{-1}$, then $a \neq 0$, and $\hat{\Phi}_\chi(x) = a\chi(x) + b$, where a and b are as given in equation (5.3).

The case where χ is a ramified character assumes some knowledge of Gauss sums. For more details, refer to [Godement, 1974] chapter 1 lemma 9. □

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