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Monomial ideals, multigraded resolutions and the subadditivity problem

by
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Let $R = k[x_1, \ldots, x_n]$ be the polynomial ring in n variables, and I a monomial ideal in S. Let $\mathbb{F}$ be a minimal graded free resolution of $S = R/I$. We study properties of multigraded resolutions to establish results on the subadditivity condition for maximal shifts in the minimal graded free resolution $\mathbb{F}$.

Contents

Chapter 1

Introduction

Let $R = k[x_1, x_2, \ldots x_n]$ be the polynomial ring in n variables and I a monomial ideal in R. Let $(\mathbb{F}, \delta)$ be a graded minimal free resolution of R/I:

$$\mathbb{F} : 0 \to \bigoplus_j R(-j)^{\beta_{sj}} \xrightarrow{\partial_s} \ldots \to \bigoplus_j R(-j)^{\beta_{ij}} \xrightarrow{\partial_i} \ldots \to \bigoplus_j R(-j)^{\beta_{1j}} \xrightarrow{\partial_1} R \to R/I \to 0.$$

We denote by $t_i = max\{j, \beta_{ij} \neq 0\}$. We say that the complex satisfies the subadditivity condition if for all a and b, we have

$$t_{a+b} \leq t_a + t_b$$

The structure of the resolution of monomial ideals can be quite complex. We grade R in a refined way, namely by multigrading it, then we study properties of multigraded free resolutions in [10], [11] and [14]. We prove that the minimal resolution of S/I contains as sub-complexes the minimal free resolutions of some smaller monomial ideals. We apply these properties to study the results in [11] on the subadditivity condition for maximal shifts which have been of interest to many authors, even when the ideal I is not a monomial ideal, see [2], [3], [6], [7], [8], [11] and [13] for instance. It is known that the minimal graded free resolution may not satisfy the subadditivity condition for maximal shifts in general, but no counter examples have been known for monomial ideals or Gorenstein algebras. The problem is still open in these cases.

Chapter 2

Preliminaries

2.1 Rings and Algebras

Definition 2.1. Let R be a polynomial ring on n variables $R = [x_1, \ldots, x_n]$. A monomial ideal in R is an ideal generated by monomials in R.

Example 2.2. $I = (x^2, y^5, xz)$ is a monomial ideal in $R = k[x, y, z]$

Definition 2.3. A ring R with exactly one maximal ideal $\mathfrak{m}$ is said to be a ***local ring***.

Example 2.4. Any field is a local ring since the only maximal ideal is $\mathfrak{m} = 0$.

Definition 2.5. A ring R is said to be ***Noetherian*** if it satisfies one of the following equivalent conditions:

i- Every non-empty set of ideals in R has a maximal element.

ii- Every ascending chain of ideals in R is stationary.

iii- Every ideal in R is finitely generated.

Theorem 2.6. *(Hilbert's Basis Theorem)*

If R is a Noetherian ring, Then the polynomial ring $R[x]$ is also Noetherian.

Proof. Let $\mathcal{A}$ be an ideal in $R[x]$. We want to show that $\mathcal{A}$ is finitely generated. Let $I = \{$leading coefficients of polynomials in $\mathcal{A}$ $\}$, which is an ideal in R. Since R is Noetherian, then I is finitely generated. Denote by $\{a_1, \ldots, a_n\}$ the set of generators of I. i.e $I = (a_1, \ldots, a_n)$. So for all t with $q \leq t \leq n$, there exists $f_t \in R[x]$ such that $f_t = a_t x^{r_t} + l$ where l consists of a polynomial of lower terms. Let $\mathcal{B}$ be the ideal generated by the f_t so $\mathcal{B} = (f_1, \ldots, f_n)$. It is easy to show that $\mathcal{B}$ is an ideal of $R[x]$.

We next set r to be as follows: $r = \max\{r_t, 1 \leq t \leq n\}$, and we take a polynomial $f = ax^m + l$ in $\mathcal{A}$. So $a \in I$. If $m \geq r$, we write a as $a = \sum\limits_{1 \leq 1 \leq n} u_i a_i$ where $u \in R$. Hence, $f - \sum\limits_{1 \leq 1 \leq n} u_i f_i x^{m-r}$ is in $\mathcal{A}$ with degree strictly less than m. We proceed in this fashion, until we get a polynomial g of degree strictly less than r. So there exists, $h \in \mathcal{B}$ such that $f = g + h$.

Now, Let M be an R-module generated by $1, x, \ldots, x^{r-1}$. Then, $\mathcal{A} = (\mathcal{A} \cap M) + \mathcal{B}$ Notice that $\mathcal{A} \cap M$ is finitely generated since M is a finitely generated R-module, say generated by $\{g_1, \ldots, g_m\}$. And $\mathcal{B}$ is also finitely generated by $\{f_1, \ldots, f_n\}$ Therefore, $\mathcal{A}$ is finitely generated by $\{f_1, \ldots, f_n, g_1, \ldots, g_m\}$. Hence $R[x]$ is Noetherian.

$\square$

Theorem 2.7. *(Nakayama's Lemma)*

1. *Let R be a commutative ring (not necessarily Noetherian). Let I be an ideal of R which is contained in every maximal ideal of R. Let M be a finitely generated R module. Suppose that $IM = M$, then $M = 0$.*

2. *Let R be a local ring. Let M be a finitely generated R-module and N a submodule of M. If $M = N + \mathfrak{m}M$, where $\mathfrak{m}$ is the maximal Ideal in R, then $M = N$.*

3. *Let R be a local ring. Let M be a finitely generated R-module. If $m_1, \ldots, m_n$ are generators for M/IM, then they are generators of M as well.*

Proof. 1. Let $\{m_1, \ldots, m_s\}$ be generators of M. Since $IM = M$, then there exists $a_1, \ldots, a_s \in I$ such that $m_s = a_1m_1 + \cdots + a_sm_s$. So there exists $a \in I$ such that $(1 + a)m_s \in \mathcal{B}$, where $\mathcal{B}$ is the module generated by the first $s - 1$ generators $\{m_1, \ldots, m_{s-1}\}$. Therefore, $(1 + a)$ is a unit in I. Otherwise, $(a + 1)$ belongs to some maximal ideal and not in I, and a belongs to all maximal ideals. So $1 \in$ some maximal ideal which is impossible. Hence, $m_s \in \mathcal{B}$. Proceeding by induction, we will get the desired result.

2. Applying 1 to M/N, will get directly the result.

3. Apply 2 by taking N to be the module generated by $\{m_1, \ldots, m_n\}$.

$\square$

We next define the notion of an algebra.

Definition 2.8. Let $f : A \to B$ be a ring homomorphism. If $a \in A$ and $b \in B$, define a product

$$ab = f(a)b$$

This definition of scalar multiplication makes the ring B into an A-module. Thus B has an A-module structure as well as a ring structure. The ring B, equipped with this A-module structure, is said to be an A-algebra.

Definition 2.9. The ***tensor algebra*** of the R-module M is the graded, non-commutative algebra

$$T_R(M) := R \oplus M \oplus (M \otimes_R M) \oplus \cdots ,$$

where the product of $x_1 \otimes \cdots \otimes x_m$ and $y_1 \otimes \cdots \otimes y_n$ is $x_1 \otimes \cdots \otimes x_m \otimes y_1 \otimes \cdots \otimes y_n$.

In the most interesting case, where M is a free R-module in the x_i, this is the free (non-commutative) algebra on the x_i. $T_R(M)$ is sometimes denoted by $T(M)$

Definition 2.10. The ***exterior algebra*** of M is the algebra $\wedge_R(M)$ obtained from $T_R(M)$ by imposing the skew-commutativity, that is by factoring out the two-sided ideal generated by the elements $x^2 = x \otimes x = 0$ for all $x \in M$. (From the formula $(x + y) \otimes (x + y) = x \otimes x + x \otimes y + y \otimes x + y \otimes y$ we see that $x \otimes y + y \otimes x$ goes to 0 in $\wedge_R M$ for all $x, y \in M$, so that $\wedge_R M$ really is skew-commutative.

Sometimes we replace $\wedge_R M$ by $\wedge M$.

Remark 2.11. If $x_i = x_j$ for some $i \neq j$ then $x_i \wedge x_j \wedge \ldots \wedge x_p = 0$

Definition 2.12. (Basis and dimension) If the dimension of V is n and $\{e_1, ..., e_n\}$ is a basis of V, then the set $\{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k} / 1 \leq i_1 < i_2 < \cdots < i_k \leq n\}$ is a basis for $\bigwedge^k(V)$. The dimension of $\bigwedge^k(V)$ is $\binom{n}{k}$.

2.2 Standard grading of a polynomial ring

In this section, we introduce the standard grading of a polynomial ring. We define the notion of homogeneous (or graded) ideals and homogeneous homomorphisms, along with useful tools that are used later on.

Definition 2.13. Let R be the polynomial ring $k[x_1, \ldots, x_n]$ over a field k. Set $\deg(x_i) = 1$ for each i. A monomial $x_1^{c_1} \cdot \ldots \cdot x_n^{c_n}$ has ***degree*** $c_1 + ... + c_n$. For $i \in \mathbb{N}$, we denote by R_i the k-vector space spanned by all monomials of degree i. In particular, $R_0 = k$.

Definition 2.14. A polynomial u in R is called ***homogeneous*** if $u \in R_i$ for some i. In this case, we say that u has degree i (or that u is a form of degree i) and write $deg(u) = i$. Note that 0 is a homogeneous element with arbitrary degree. We get the following two equivalent properties:

1. $R_i R_j \subseteq R_{i+j}$ for all $i, j \in \mathbb{N}$.

2. $deg(uv) = deg(u) + deg(v)$ for every two homogeneous elements $u, v \in R$.

Every polynomial $f \in R$ can be written uniquely as a finite sum $f = \sum_{i \in \mathbb{N}} f_i$ of non-zero elements $f_i \in R_i$. In this case, f_i is called the ***homogeneous component*** of f of degree i. Thus, we have a direct sum decomposition $\bigoplus_{i \in \mathbb{N}} R_i$ of R as a k-vector space such that $R_i R_j \subseteq R_{i+j}$ for all $i, j \in \mathbb{N}$. We say that R is ***standard graded.***

Example 2.15. Let $R = k[x, y]$. In this case, $R_0 = k$, R_1 in the k-space of all linear forms, R_2 is the k-space of all quadratics, etc. The polynomial $f = x^2y^3 - 2xy^2 + 3x^3$ is not homogeneous and has homogeneous components x^2y^3, $-2xy^2 + 3x^3$

Definition 2.16. A proper ideal J in R is called a ***graded or homogeneous ideal*** if it satisfies the following equivalent conditions:

1. If $f \in J$, then every homogeneous component of f is in J.

2. $J = \bigoplus_{i \in \mathbb{N}} J_i$, where $J_i = R_i \cap J$.

3. If I is the ideal generated by all homogeneous elements in J, then $J = I$.

4. J has a system of homogeneous generators.

In this case, the k-spaces J_i are called the ***homogeneous components*** of J. An element $m \in J$ is called ***homogeneous*** if $m \in J_i$ for some i. We say that m is homogeneous of degree 1 and $deg(m) = i$. Thus, every element $m \in J$ can be written uniquely as a sum $\sum_i m_i$, where each $m_i \in J_i$; m_i is called the ***homogeneous components of*** m ***of degree*** i

Definition 2.17. Let I be a graded ideal in R. Note that $R_i I_j \subseteq I_{i+j}$ for all $i, j \in \mathbb{N}$. The quotient ring $S = R/I$ get the grading from R by $S_i = R_i/I_i$ for every $i \in \mathbb{N}$.

Remark 2.18. We consider the polynomial ring $R = k[x_1, \ldots, x_n]$ to be a local ring with the maximal ideal $\mathfrak{m} = (x_1, \ldots, x_n)$. In a matter of fact, a maximal ideal $\mathfrak{m}$ in R is generated by $(x_1 + c_1, \ldots, x_n + c_n)$ since $R/\mathfrak{m} \cong k$. We only consider homogeneous generators, which forces the c_i's to be zeros.

2.3 Shifts on graded modules and homogeneous homomorphisms

In this section, we define shifts in the graded modules that lead us to define homogeneous homomorphisms.

Definition 2.19. An R-module M is called ***a graded module*** if it has a direct sum decomposition $M = \bigoplus_{i \in \mathbb{Z}} M_i$ as a k-vector space and $R_i M_j \subseteq M_{i+j}$ for all $i, j \in \mathbb{Z}$. The k-spaces M_i are called ***homogeneous components*** of M. An element $m \in M$ is called ***homogeneous*** if $m \in M_i$ for some i. We say that m is homogeneous of degree 1 and $deg(m) = i$.

Definition 2.20. For $p \in \mathbb{Z}$, denote by $M(-p)$ the graded free module of R such that $M(-p)_i = M_{i-p}$ for all i. we say the ring M is ***shifted*** p ***degrees***, and p is ***the shift***. The same thing holds for the quotient ring S.

Example 2.21. Let $R = k[x, y]$, 1 has degree 0 in R, but has degree 1 in $R(-1)$. Similarly, xy has degree 2 in R and degree 4 in $R(-2)$.

Proposition 2.22. *The module $R(-p)$ is the free R module generated by one element in degree p.*

Proof. $R(-p)_p = R_0$. □

Definition 2.23. Let M and N be graded modules in R. We say that a homomorphism $\phi : M \longrightarrow N$ has ***degree*** i if $deg(\phi(m)) = i + deg(m)$ for each homogeneous element $m \in M$.

Example 2.24. Let $R = k[x, y]$, and ϕ be the homomorphism defined below:

$$R(-3) \oplus R(-4) \xrightarrow{\begin{pmatrix} x^3 & y^4 \end{pmatrix}} R$$

is graded and has degree 0. Since the homomorphism $R \xrightarrow{x^3} R$ maps $1 \mapsto x^3$, 1 has degree 0 and x^3 has a degree 3 in R. The homomorphism

$$R \oplus R(-2) \xrightarrow{\begin{pmatrix} x^2 & y^4 \end{pmatrix}} R$$

is graded and has degree 2.

Chapter 3

Graded Resolutions

In this chapter, we consider $R = k[x_1, \dots, x_n]$ to be the graded local polynomial ring in n variables with maximal ideal $\mathfrak{m} = (x_1, \dots, x_n)$. We let I to be the homogeneous ideal over R and $S = R/I$.

3.1 Open and exact sequences

Definition 3.1. A ***complex*** $\mathbb{F}$ ***over*** R is a sequence of homomorphisms of R-modules

$$\mathbf{F} : \cdots \longrightarrow F_i \xrightarrow{d_i} F_{i-1} \longrightarrow \cdots \longrightarrow F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \longrightarrow \dots$$

such that $d_i \circ d_{i+1} = 0$ for all $i \in \mathbb{Z}$. The collection of maps $d = \{d_i\}_i$ is called the ***differential*** of $\mathbb{F}$.

$\mathbb{F}$ is called a ***left complex*** if $F_i = 0$ for all $i < 0$, so

$$\mathbb{F} : \cdots \longrightarrow F_i \xrightarrow{d_i} F_{i-1} \longrightarrow \cdots \longrightarrow F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \longrightarrow 0$$

Definition 3.2. A sequence of homomorphisms of R-modules, M_i's is said to be ***exact*** at M_i if $Im(d_i) = Ker(d_{i-1})$. where Im represents the image of a map and Ker represents the kernel of the map. In particular: if M, M' and M" are R-modules, then:

1. $0 \longrightarrow M' \xrightarrow{f} M$ is exact, $\iff$ f is injective;

2. $M \xrightarrow{g} M" \longrightarrow 0$ is exact, $\iff$ g is surjective;

3. $0 \longrightarrow M" \xrightarrow{f} M \xrightarrow{g} M" \longrightarrow 0$ is exact $\iff$ f is injective and g is surjective; g induces an isomorphism of $CoKer(f) = M/f(M')$ onto M".

A sequence of type 3 is called ***a short exact sequence***. The complex is called ***graded*** if the modules M_i are graded and each d_i is a homomorphism of degree 0.

3.2 Minimal free resolutions

In this section, we introduce free resolutions.

Definition 3.3. Let M be an R-module. A ***free resolution*** of M is a complex

$$\mathbb{F} : \quad \cdots \longrightarrow F_i \xrightarrow{d_i} F_{i-1} \longrightarrow \cdots \longrightarrow F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \longrightarrow 0$$

of free R modules such that $M \cong F_0/Im(d_1)$ and $\mathbb{F}$ is exact. Or, for simplicity, we write it as

$$\mathbb{F}: \quad \cdots \longrightarrow F_i \xrightarrow{d_i} F_{i-1} \longrightarrow \cdots \longrightarrow F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0$$

Definition 3.4. A free resolution of M is called minimal if

$$d_{i+1}(F_{i+1}) \subset \mathfrak{m}F_i \quad \text{for all} \quad i \geq 0.$$

In other words, the maps in the resolutions are represented by matrices with entries in the maximal ideal $\mathfrak{m} = (x_1, \ldots, x_n)$

Construction 3.2.1. *Given an R-module, M generated by a minimal set of generators $\{m_i\}_i$. We write the step-by-step construction of a minimal free resolution of M*

Step1: *Map a graded free module F_0 onto M by sending a basis for F_0 to the set of $\{m_i\}$.*

$$F_0 \xrightarrow{d_0} M$$

Step 2: *Let $M_1 = Ker(d_0)$ which is finitely generated. Choose a minimal set of generators of M_1, then set F_1 to be the free R module with $rank(M_1) = \#\{minimal\ set\ of\ generators\ of\ M_1\}$. Each element of the basis of F_1 will be mapped to an element in the minimal set of generators of M_1. So in the following diagram, that commutes: s_1 is a surjective map from*

$F_1 \to M_1$ and i_1 is an injective map from $M_1 \to F_0$

$$\begin{tikzcd} F_1 \arrow[r, "d_1"] \arrow[dr, "s_1"] & F_0 \arrow[r, "d_0"] & M \\ & M_1 \arrow[u, "i_1"] & \end{tikzcd}$$

Notice that this step guarantees exactness at F_0.

Proceed in the same manner to get the full resolution inductively.

Definition 3.5. Let M be an R-module and F a minimal free resolution of M.

Define the i'th ***Betti number of*** M ***over*** R/M by

$$b_i^R(M) = rank(F_i).$$

Example 3.6. Let $R = \mathbb{Q}[x, y]$ and $I = (x^4, x^3y, y^3, x^2y^2)$. We construct the minimal free resolution of R/I. The first map would be the canonical map $R \to R/I$, call it d_0. So $F_0 = R$. We let $M_1 = Ker(d_0) = I$ that is generated by $\{x^4, x^3y, y^3, x^2y^2\}$ which is a minimal set of generators.

Next, the free module $F_1 = R^4$ since $\#\{x^4, x^3y, y^3, x^2y^2\} = 4$, and the matrix representation of the map $d_1 : R^4 \to R$ is (x^4, x^3y, y^3, x^2y^2).

Let $M_2 = Ker(d_1)$ wich is generated by $(a \quad b \quad c \quad d)^T$ such that $ax^4 + bx^3y + cy^3 + dx^2y^2 = 0$. Removing the dependent vectors, we get that $(a \quad b \quad c \quad d)^T = (0 \quad 0 \quad x^2 \quad -y)$ or $(0 \quad y \quad 0 \quad -x)$ or $(-y \quad x \quad 0 \quad 0)$. Thus, the matrix representation of d_2 is $\begin{pmatrix} -y & 0 & 0 \\ x & y & 0 \\ 0 & 0 & x^2 \\ 0 & -x & -y \end{pmatrix}$. The next free module in the resolution has rank 3.

the minimal free resolution of I in R module would be :

$$0 \to R^3 \xrightarrow{\begin{pmatrix} -y & 0 & 0 \\ x & y & 0 \\ 0 & 0 & x^2 \\ 0 & -x & -y \end{pmatrix}} R^4 \xrightarrow{\begin{pmatrix} x^4 & x^3y & y^3 & x^2y^2 \end{pmatrix}} R \to R/I \to 0.$$

We next exhibit an example of the Koszul complex.

The Koszul complex resolves algebras R/I where I is generated by a regular sequence. A regular sequence is a sequence of elements which are as independent as possible. Hence, if $I = (x_1, ..., x_n) \subset R$ then $x_1 \dots x_n$ is a regular sequence if for all $i = 1, ..., n, x_i$ is a non-zero divisor on $R/(x_1, ..., x_{i-1})$.

Let $f_1, \dots, f_r$ be elements in R. Let E be the exterior algebra over k on basis elements $e_1, \dots, e_r$. In other words, E is the following algebra:

$$E = k\langle e_1, \dots, e_r\rangle / (\{e_i^2 | 1 \leq i \leq r\}, \{e_ie_j + e_je_i | 1 \leq i \leq r\})$$

Denote by f the sequence $f_1, \dots, f_r$ and by $\mathbf{K(f)}$ the complex equipped with the differential

$$d(e_{j_1} \wedge \dots \wedge e_{j_i}) = \sum_{1 \leq p \leq i} (-1)^{p+1} f_{j_p} e_{j_1} \wedge \dots \wedge \hat{e_{j_p}} \wedge \dots \wedge e_{j_i}$$

where $\hat{e_{j_p}}$ means that e_{j_p} is omitted in the product. Notice that $d^2 = 0$, this can be shown by computation:

$$d^2(e_{j_1} \wedge \ldots \wedge e_{j_i}) = \sum_{1 \leq p < s \leq i} \gamma_{p,s} e_{j_1} \wedge \ldots \wedge \hat{e_{j_p}} \wedge \ldots \wedge e_{j_i}$$

where the coefficient $\gamma_{p,s}$ is obtained in two steps:

(1) Start by removing e_{j_s} and then remove e_{j_p} from the product to get the coefficient $(-1)^{s+1} f_{j,s} (-1)^{p+1} f_{j,p}$.

(2) Start by removing e_{j_p} and then remove e_{j_s} from the product to get the coefficient $(-1)^{p+1} f_{j,p} (-1)^s f_{j,s}$.

Therefore, $\gamma_{p,s} = (-1)^{s+1} f_{j,s} (-1)^{p+1} f_{j,p} + (-1)^{p+1} f_{j,p} (-1)^s f_{j,s} = 0$.

The complex $\mathbf{K(f)}$ is called the ***Koszul complex*** of $I = (f_1, \ldots, f_r)$, written as follows:

$$\mathbf{K(f)}: \quad 0 \to K_r \to \ldots \to K_1 \to K_0 \to 0.$$

Note that $\{e_{j_1} \wedge \ldots \wedge e_{j_i} | 1 \leq j_1 < \ldots < j_i \leq r\}$ form a basis of the R-module K_i.

Example 3.7. Let $R = k[x, y, z]$ and $f_1 = x^2$ and $f_2 = y^2$. Then, K_0 has basis 1, K_1 has basis e_1, e_2 and K_2 has basis $e_1 \wedge e_2$. For the differential:

$$d(e_1) = x^2 \quad \text{and} \quad d(e_2) = y^2$$

$$d(e_1 \wedge e_2) = d(e_1)e_2 - d(e_2)e_1 = x^2 e_2 - y^2 e_1$$

The Koszul complex $\mathbf{K(x^2, y^2)}$ would be:

$$\mathbf{K(x^2, y^2)}: \quad 0 \to K_2 \xrightarrow{\begin{pmatrix} -y^2 \\ x^2 \end{pmatrix}} K_1 \xrightarrow{\begin{pmatrix} y^2 & x^2 \end{pmatrix}} K_0.$$

The software Macaulay 2 exhibits a minimal free resolution of any module along with the differentials in the resolution.

```
i1 : R=QQ[x,y,z]

o1 = R

o1 : PolynomialRing

i2 : I = ideal (x^2*y^3, y^5*z^2,z*x^4, x^2*y^2*z, x^5, y^6 )

                   2 3   5 2   4    2 2    5   6

o2 = ideal (x y , y z , x z, x y z, x , y )

o2 : Ideal of R

i3 : res o2

         1      6      7      2

o3 = R  <-- R  <-- R  <-- R  <-- 0



     0      1      2      3      4

o3 : ChainComplex

i4 : o3.dd_1

o4 = | x5 x2y3 x4z x2y2z y6 y5z2 |

                   1       6

o4 : Matrix R  <--- R

i5 : o3.dd_2

o5 = {5} | -z 0  0   -y3 0   0   0    |
```

```
     {5} | 0  -z 0   x3  -y3 0   0    |
     {5} | x  0  -y2 0   0   0   0    |
     {5} | 0  y  x2  0   0   0   -y3z |
     {6} | 0  0  0   0   x2  -z2 0    |
     {7} | 0  0  0   0   0   y   x2   |

              6       7
o5 : Matrix R  <--- R

i6 : o3.dd_3

o6 = {6} | -y3 0   |
     {6} | x3  y3z |
     {7} | -xy 0   |
     {8} | z   0   |
     {8} | 0   -z2 |
     {8} | 0   -x2 |
     {9} | 0   y   |

              7       2
o6 : Matrix R  <--- R
```

Next we show that two minimal free resolutions of any module M over R are isomorphic. In order to do so, we need to state Nakayama's lemma in the graded case and a lemma that follows.

Lemma 3.8. *(Nakayama). Suppose M is a finitely generated graded R-module* *and* $m_1, \ldots, m_n \in M$ *generate* $M/\mathfrak{m}M$*, then* $m_1, \ldots m_n$ *generate* M*.*

Proof. Let $\bar{M} = M/\sum Rm_i$. If the m_i generate $M/\mathfrak{m}M$ then $\bar{M}/\mathfrak{m}\bar{M} = 0$ so $\mathfrak{m}\bar{M} = \bar{M}$. If $\bar{M} \neq 0$, since $\bar{M}$ is finitely generated, there would be a nonzero element of least degree in $\bar{M}$; this element could not be in $\mathfrak{m}\bar{M}$. Thus $\bar{M} = 0$, so M is generated by the m_i. $\square$

Lemma 3.9. *Let* $R = k[x_1, \ldots, x_n]$ *be a polynomial ring over a field* k *of* n *variables. Let* $\mathbb{F}$ *be a graded free resolution of* R*-module as follows:*

$$\mathbb{F}: \qquad \ldots \to F_i \xrightarrow{\partial_i} F_{i-1} \to \ldots \to F_0$$

$\mathbb{F}$ *is minimal* $\iff$ *for all* i*,* ∂_i *takes a basis of* F_i *to a minimal set of generators of the image of* ∂_i

Proof. Consider the right exact sequence

$$F_{i+1} \xrightarrow{\partial_{i+1}} F_i \xrightarrow{\partial_i} Im(\partial_{i-1}) \to 0.$$

Note that R is a local ring, let $\mathfrak{m}$ be the maximal ideal of R.

$$
\begin{aligned}
\mathbb{F} \text{ is minimal} &\iff \forall i, \quad \partial_{i+1}(F_{i+1}) \subseteq \mathfrak{m}F_i \\
&\iff F_{i+1} \xrightarrow{\overline{\partial_{i+1}}} F_i/\mathfrak{m}f_i \quad \text{is the zero map} \\
&\iff F_{i+1}/\mathfrak{m}F_{i+1} \xrightarrow{\overline{\partial_{i+1}}} F_i/\mathfrak{m}F_i \quad \text{is the zero map} \\
&\iff F_i/\mathfrak{m}F_i \xrightarrow{\phi} Im(\partial_i)/\mathfrak{m}Im(\partial_i) \quad \text{is an isomorphism} \\
&\qquad \text{because } \overline{\partial_{i+1}} \text{ is the zero map in the exact sequence} \\
&\qquad F_{i+1}/\mathfrak{m}F_{i+1} \xrightarrow{\overline{\partial_{i+1}}} F_i/\mathfrak{m}F_i \xrightarrow{\phi} Im(\partial_i)/\mathfrak{m}Im(\partial_i) \\
&\qquad \text{so } Ker(\phi) = Im(\overline{\partial_{i+1}}) = 0 \text{ and } \phi \text{ is surjective, by exactness of } \mathbb{F}.
\end{aligned} \tag{3.1}
$$

We show $\Rightarrow$

Let $\{f_1, \ldots, f_n\}$ be a basis of F_i, it is a minimal set of generators. This implies that $\{\overline{f_1}, \ldots, \overline{f_n}\}$ is a minimal set of generators of $F_i/\mathfrak{m}F_i$ by Nakayama's lemma 2.7. From the above isomorphism of k-vector spaces, $m_i = \phi(\overline{f_i})$ is a minimal set of generators for $Im(\partial_i)/\mathfrak{m}Im(\partial_i)$. By Nakayama's lemma $\{m_i\}$ is a minimal set of generators for $Im(\partial_{i+1})$.

We next show $\Leftarrow$

For every M an R-module, $M/\mathfrak{m}M$ is an $R/\mathfrak{m} = k$ vector space.

Since every ∂_i sends a basis of F_i to a minimal set of generators of the image of ∂_i, then the basis $\{f_1, \ldots, f_n\}$ of F_i is sent to a minimal generating set $\{m_1, \ldots, m_n\}$

of $Im(\partial_i)$. Hence, we get

$$\begin{array}{ccc} \{f_i\} & \longrightarrow & \{\overline{f_i}\} \\ \downarrow{\scriptstyle \partial_i} & & \downarrow{\scriptstyle \phi} \\ \{m_i\} & \longrightarrow & \{\overline{m_i}\} \end{array}$$

where $\{f_1, \ldots, f_n\}$ is a basis of F_i, $\{\overline{f_1}, \ldots, \overline{f_n}\}$ is a basis for $F_i/\mathfrak{m}F_i$, $\{m_1, \ldots, m_n\}$ is a minimal set of generators of $Im(\partial_i)$ and $\{\overline{m_1}, \ldots, \overline{m_n}\}$ is a minimal set of generators of $Im(\partial_i)/\mathfrak{m}Im(\partial_i)$. It follows that $F_i/\mathfrak{m}F_i \xrightarrow{\phi} Im(\partial_i)/\mathfrak{m}Im(\partial_i)$ is an isomorphism. □

Proposition 3.10. *Let M be an R-module and $\mathbb{F}$ and $\mathbb{G}$ be two minimal free resolutions of M. Then, there exists a map: $f : \mathbb{F} \to \mathbb{G}$ such that $f_i : F_i \to G_i$ is an isomorphism and the following diagram commutes.*

$$\begin{array}{cccccccccc} \mathbb{F}: & \cdots & & F_1 & \longrightarrow & F_0 & \xrightarrow{d_0} & M & \longrightarrow & 0 \\ & & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle id_M} & & \\ \mathbb{G}: & \cdots & & G_1 & \longrightarrow & G_0 & \xrightarrow{\delta_0} & M & \longrightarrow & 0 \end{array}$$

Proof. We construct the isomorphic maps inductively.

We consider the following diagram:

$$\begin{array}{cccccccccc} \mathbb{F}: & \cdots & & F_1 & \longrightarrow & F_0 & \xrightarrow{d_0} & M & \longrightarrow & 0 \\ & & & & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle id_M} & & \\ \mathbb{G}: & \cdots & & G_1 & \longrightarrow & G_0 & \xrightarrow{\delta_0} & M & \longrightarrow & 0 \end{array}$$

Since δ' is surjective, F_0 is free and every free module is a projective module, then, there exists $f_0 : F_0 \to G_0$:

$$\begin{tikzcd} F_0 \arrow[d, "f_0"] & \\ G_0 \arrow[r, "\delta_0"] & M \end{tikzcd}$$

such that the above diagram commutes. We need to show that f_0 is an isomorphism. To do so, we tensor both $\mathbb{F}$ and $\mathbb{G}$ with $k = R/m$ and we show that $f_0 \otimes id$ is an isomorphism.

We just note that $F_i \otimes k \cong F_i/MF_i$ and $G_i \otimes k \cong G_i/MG_i$ for all i.

$$\begin{tikzcd} \mathbb{F} : \cdots & F_1 \otimes k \arrow[r] & F_0 \otimes k \arrow[r, "d_0 \otimes id"] \arrow[d, "f_0 \otimes id"] & M \otimes k \arrow[r] \arrow[d, "id_M \otimes k"] & 0 \\ \mathbb{G} : \cdots & G_1 \otimes k \arrow[r] & G_0 \otimes k \arrow[r, "\delta_0 \otimes id"] & M \otimes k \arrow[r] & 0 \end{tikzcd}$$

Since $\mathbb{F}$ and $\mathbb{G}$ are minimal, we have $d_0 \otimes id$, $\delta_0 \otimes id$ are isomorphisms. Since the above diagram commutes, then $f_0 \otimes k$ is an isomorphism.

Let (a_{ij}) be the matrix representation of f_0.

$\Rightarrow$ the matrix representation of $f_0 \otimes k$, $(a_{ij} \otimes 1) = (\overline{a_{ij}})$ is an invertible matrix.

$\Rightarrow det(\overline{a_{ij}})$ is a unit in $k = R/\mathfrak{m}$

$\Rightarrow det(a_{ij}) \notin \mathfrak{m}$

$\Rightarrow det(a_{ij})$ is a unit in R

$\Rightarrow f_0$ in isomorphism.

We proceed in the same manner to show that f_i's are isomorphisms for all $i \geq 1$.

Hence the result. $\square$

Definition 3.11. The ***projective dimension*** of M is

$$pd_R(M) = max\{i|b_i^R(M) \neq 0\}.$$

The following theorem by Hilbert states that in a polynomial ring in n variables the projective dimension of a finitely generated module is less than n.

Theorem 3.12. *(Hilbert syzygy theorem) If $R = k[x_1, \ldots, x_n]$, then every finitely generated R-module has a finitely free resolution of length $\leq n$, by finitely generated free modules.*

3.3 Graded minimal free resolutions

Definition 3.13. A resolution $\mathbb{F}$ is called a ***graded free resolution*** if R is a graded ring, the F_i's are graded free modules and the maps d_i's are homogeneous maps of degree 0.

Construction 3.3.1. *Given a graded R-module, M, we write the step-by-step construction of a graded minimal free resolution of M inductively. For the sake of completeness, we add the steps to construct a minimal free resolution along with the grading.*

Step 0: Set F_0 to be equal to R and d to be the canonical map between R and R/M.

Step 1: Let $\{m_1, \ldots, m_n\}$ be the generators of M with degrees $a_1, \ldots, a_m$ respectively. Now set $F_1 = R(-a_1) \oplus \ldots \oplus R(-a_m)$. For all j with $1 \leq j \leq n$, let

f_j to be the generator of $R(-a_j)$, so $deg(f_j) = a_j$. Now define $d_0 : F_1 \to F_0$, such that $d_0(f_j) = m_j$, which is a homogeneous homomorphism of degree 0.

Step i+1: Set $U_{i+1} = Ker(d_{i-1})$ which is a finitely generated module. Let $l_1, \ldots l_s$ be generators of U_{i+1} with degrees $c_1, \ldots, c_s$ respectively. Set $F_i = R(-c_1) \oplus \ldots \oplus R(-c_s)$. For $1 \leq j \leq s$, denote by g_j the generator of $R(-c_j)$. Then, $deg(g_j) = c_j$. Now define

$$d_i : F_{i+1} \to U_{i+1} \subset F_i$$

$$g_i \mapsto l_i \quad \text{for} \quad 1 \leq j \leq s.$$

Notice that this is a surjective homomorphism of degree 0. The complex is exact since $Ker(d_i) = Im(d_{i+1})$

Example 3.14. Let $R = k[x, y]$ and $I = (x^3, xy, y^5)$ an ideal in R.

Step 0: Set $F_0 = R$ and d to be the canonical map between R and R/I.

Step 1: Notice that x^3, xy, y^5 are the homogeneous generators of I. Set $F_1 = R(-3) \oplus R(-2) \oplus R(-5)$. Denote by f_1, f_2, and f_3 the generators of $R(-3), R(-2)$ and $R(-5)$ with degrees 3, 2 and 5 respectively. Define

$$d_1 : F_1 \to R$$

$$f_1 \mapsto x^3$$

$$f_2 \mapsto xy$$

$$f_3 \mapsto y^5.$$

Step 2: To find generators of $Ker(d_1)$, we have to find α, β and γ such that

$$\alpha f_1 + \beta f_2 + \gamma f_3 = 0$$

After some computations, one can find that the relations are $(\alpha, \beta, \gamma) = (y, -x^2, 0)$ or $(0, -y^4, x)$. Thus, $yf_1 - x^2 f_2$ and $-y^4 f_2 + x f_3$ are homogeneous generators of $Ker(d_1)$ with degrees 4 and 6 respectively.

Now set $F_2 = R(-4) \oplus R(-6)$ and repeat the same process to get the following minimal free resolution of I:

$$0 \to R(-4)\oplus R(-6) \xrightarrow{\begin{pmatrix} y & 0 \\ -x^2 & -y^4 \\ 0 & x \end{pmatrix}} R(-3)\oplus R(-2)\oplus R(-5) \xrightarrow{\begin{pmatrix} x^3 & xy & y^5 \end{pmatrix}} R \to R/I \to 0$$

Note that in the above example, we constructed the maps in the resolution along with the grading at the same time. Please note that one can construct all the maps first then grade the resolution next.

3.4 Betti Diagrams and the projective dimension

Definition 3.15. Let F be a minimal graded free resolution of M. Define the ***graded Betti numbers*** of M by:

$$\beta_{i,p}^{R} = \text{number of summands in } F_i \text{ of the form } R(-p).$$

The Betti numbers can be given in a table that we call the ***Betti diagram***. The entry in the i'th column and p'th row is $b_{i,i+p}$

	0	1	...	s
i	$\beta_{0,i}$	$\beta_{1,i+1}$	...	$\beta_{s,i+s}$
i+1	$\beta_{0,i+1}$	$\beta_{1,i+2}$	...	$\beta_{s,i+s+1}$
$\vdots$	...	...	...	...
n	$\beta_{0,n}$	$\beta_{1,n+1}$	...	$\beta_{s,n+s}$

where $F_i = R(-a)^{\beta_{i,a}}$, that is F_i requires $\beta_{i,a}$ minimal generators of degree a.

Example 3.16. The Betti diagram corresponding to the resolution in example 3.14 is:

	0	1	2
0	1	-	-
1	-	1	-
2	-	1	1
3	-	-	-
4	-	1	1

For instance, In the second column of the Betti diagram we can check that $b_{2,4} = 1$ and $b_{2,6} = 1$. Wich shows that $F_2 = R(-4) \oplus R(-6)$.

The software Macaulay 2 can be used to find the graded betti numbers as we can see in what follows:

Example 3.17.

```
i1 : R=QQ[x,y,z]

o1 = R

o1 : PolynomialRing

i2 : I = ideal (x^2*y^3, y^5*z^2,z*x^4, x^2*y^2*z, x^5, y^6 )

             2 3   5 2   4    2 2    5   6
o2 = ideal (x y , y z , x z, x y z, x , y )

o2 : Ideal of R

i3 : betti res o2

            0 1 2 3
o3 = total: 1 6 7 2
         0: 1 . . .
         1: . . . .
         2: . . . .
         3: . . . .
         4: . 4 2 .
         5: . 1 1 .
         6: . 1 3 1
         7: . . 1 1

o3 : BettiTally
```

Chapter 4

Multigraded resolutions and their properties

The structure of a minimal free resolution of monomial ideals can be quite complex, and it turned out to be very hard to describe these resolutions. Even when the monomial ideal is generated by quadratics, the complexity of the resolution made it almost impossible to give an explicit description. For that, in this chapter, we introduce beautiful and useful ideas on monomial minimal free resolutions. These ideas are applied in the next section to prove a result on the subadditivity of monomial ideals.

4.1 Multigraded resolutions

In this section, we consider a refined way to grade the polynomial ring, namely we multi-grade it.

Definition 4.1. let R be the polynomial ring of n variables $R = k[x_1, \dots x_n]$ defined over the field k. For every x_i in R define the $\mathbb{N}^n$ degree or the multidegree of x_i by:

$$mdeg(x_i) = \text{ the i'th standard vector in } \mathbb{N}^n = (0, 0, \dots, 1, \dots, 0).$$

For every $d = (d_1, \dots, d_n) \in \mathbb{N}$, there exists a monomial $m \in R$ of multidegree d. Basically, m would be equal to $x_1^{d_1} \dots x_n^{d_n}$. We call d the ***exponent vector*** of x. We can also say that m is of multidegree x^d.

Definition 4.2. We define R_d the k-vector space spanned by the monomial of multidegree d. Alternatively, we consider m to be the monomial in R of multidegree d. One can define R_m to be the k-vector space spanned by m

Now, R has a direct sum decomposition $R = \bigoplus_d R_d$ where d is an exponent vector. It would be more convenient to replace d by a monomial m of multidegree d in R as defined above.

Definition 4.3. An R-module M in R is multigraded if it can be written as a direct sum decomposition $M = \bigoplus_d M_d$ and $R_d M_{d'} \subseteq M_{dd'}$

Denote by $R(x^d)$ the module in R generated by the monomial of multidegree x^d.

In the next example, we construct a minimal multigraded resolution. The construction is similar to the one of minimal graded resolution.

Example 4.4. Let $R = k[x, y]$ be the polynomial ring over the field k with two variables x and y. Let I be the ideal in R generated by x^2, xy and y^3, so $I = (x^2, xy, y^3)$.

Computing the minimal free resolution of R/I would lead to the following resolution call it $\mathbb{F}$

$$\mathbb{F} : 0 \longrightarrow R^2 \xrightarrow{\begin{pmatrix} -y & 0 \\ x & -y^2 \\ 0 & x \end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix} x^2 & xy & y^3 \end{pmatrix}} R \longrightarrow R/I \longrightarrow 0.$$

Denote by d_0, d_1 and d_2 the maps between R and R/I, R^3 and R, and R^2 and R^3 respectively that appear in the above resolution. Let h be the basis element of R of degree 0. Let f_1, f_2, f_3 be basis elements of R^3 and g_1, g_2 basis elements of R^2. Note that all differentials d_0, d_1 and d_2 in the resolution are homogeneous of degree 0.

Since h has multidegree 1 in R, and $d_1(f_1) = x^2$, and x^2 has multidegree x^2 (or $(2, 0)$) in R, then f_1 must be of multidegree x^2. similarly, f_2 and f_3 have multidegrees xy and y^3 respectively. So we replace R^3 by $R(x^2) \oplus R(xy) \oplus R(y^3)$. Similarly since $d_2(g_1) = -yf_1 + xf_2$, then g_1 has multidegree x^2y. In the same manner, we can conclude that g_2 has multidegree xy^3. We then replace R^2 by

$R(x^2y) \oplus R(xy^3)$. Hence, the multigraded resolution will be written as follows:

$$0 \longrightarrow R(x^2y) \oplus R(xy^3) \longrightarrow R(x^2) \oplus R(xy) \oplus R(y^3) \longrightarrow R \longrightarrow R/I \longrightarrow 0$$

Definition 4.5. Let $m \in R$. The component $\mathbb{F}_m$ of $\mathbb{F}$ in multidegree m is the exact sequence of k-vector spaces, with basis

$\left\{ \frac{m}{mdeg(f)} f; f \text{ is in the fixed basis of } \mathbb{F}, \text{ and } mdeg(f) \text{ divides } m \right\}$

Example 4.6. For example, the component $R(x^2y)$ in multidegree x^2y^2 in example 4.4 is a 1-dimensional k-vector space with basis yg_1, write it as $R(x^2y)_{x^2y^2} = k\{yg_1\}$. Similarly, $R(xy^3)_{x^2y^2} = 0$, $R(x^2)_{x^2y^2} = k\{y^2f_1\}$, $R(xy)_{x^2y^2} = k\{xyf_2\}$, $R(y^3)_{x^2y^2} = 0$, $R_{x^2y^2} = k\{x^2y^2\}$, where $\{f_1, f_2, f_3\}$ is a fixed basis for $R(x^2) \oplus R(xy) \oplus R(y^3)$ and $\{g_1, g_2\}$ is a fixed basis for $R(x^2y) \oplus R(xy^3)$. We note that $(R/I)_{x^2y^2} = 0$ since $x^2y^2 \in I$.

Thus $\mathbb{F}_{x^2y^2}$ is the exact sequence of k-vector spaces

$$0 \longrightarrow k\{yg_1\} \longrightarrow k\{y^2f_1\} \oplus k\{xyf_2\} \longrightarrow k\{x^2y^2\} \longrightarrow 0 \longrightarrow 0.$$

4.2 Homogenization and dehomogenization

Definition 4.7. A ***frame*** $\mathbf{L}$ is a complex of finite k-vector spaces with a differential ∂ and a fixed basis satisfying:

1. $L_i = 0$ for $i < 0$ and $i >> 0$ very large.
2. $L_0 = k$

3. $L_1 = k^r$ for some integer r.

4. $\partial(w_i) = 1$ for each basis vector w_i in $L_1 = k^r$

Definition 4.8. For a monomial ideal M in R that is generated by $\{m_1, \ldots, m_r\}$ for some r, we define L_M to be the set of the least common multiples of $\{m_1, \ldots, m_r\}$.

Definition 4.9. An M***-complex*** $\mathbf{C}$ is a multigraded complex of finitely generated free multigraded R-modules with differentials d and fixed basis with multidegrees in L_M, which satisfies the following conditions:

1. $C_i = 0$ for $i < 0$ and $i >> 0$ very large.

2. $C_0 = R$

3. $C_1 = R(m_1) \oplus \cdots \oplus R(m_r)$

4. $d(f_i) = m_i$ for each basis element f_i in C_1

We seek to find a correspondence between a frame that is a complex of k-vector spaces and a complex of finitely generated free multigraded R-modules.

Construction 4.2.1. *Let $\boldsymbol{L}$ be an r-frame. We aim to get an M-complex $\boldsymbol{C}$ of free R-modules with differential d, where M is an R module generated by $\{m_1, \ldots, m_r\}$. Following the definition, set*

$$C_0 = R \text{ and } C_1 = R(m_1) \oplus \cdots \oplus R(m_r)$$

Let $\{w_{i,1},\ldots,w_{i,p}\ \}$ *and* $\{w_{i-1,1},\ldots,w_{i-1,q}\}$ *be the given basis for* L_i *and* L_{i-1} *respectively; and let* $\{f_{i,1},\ldots,f_{i,p}\}$ *and* $\{f_{i-1,1},\ldots,f_{i-1,q}\}$ *be the basis of* C_i *and* C_{i-1} *respectively. Suppose*

$$\partial(w_{i,j}) = \sum_{1\le s\le q} \alpha_{sj} w_{i,s}$$

with coefficients $\alpha_{sj} \in k$*. Then, we consider*

$$mdeg(f_{i,j}) = lcm\,(mdeg(f_{i-1,s}) \mid \alpha_{sj} \neq 0)$$

$$C_i = \bigoplus_{1\le j\le p} R(mdeg(f_{i,j}))$$

$$d(f_{i,j}) = \sum_{1\le s\le q} \alpha_{sj} \frac{mdeg(f_{i,j})}{mdeg(f_{i-1,s})} f_{i-1,s}$$

Before we exhibit an example we show that C is a complex.

Theorem 4.10. ***C*** *in construction 4.2.1 is a complex.*

Proof. Fix L_i, L_{i-1} and L_{i-2} components of the frame L with basis $\{w_{i,1},\ldots,w_{i,p}\}$, $\{w_{i-1,1},\ldots,w_{i-1,q}\}$, $\{w_{i-2,1},\ldots,w_{i-2,t}\}$ respectively. The corresponding components of **C** that are C_i, C_{i-1} and C_{i-2} with basis elements $\{f_{i,1},\ldots,f_{i,p}\}, \{f_{i-1,1}\ldots,f_{i-1,q}\}$ and $\{f_{i-2,1},\ldots,f_{i-2,t}\}$ respectively. For a fixed j with $1\le j\le p$, we have from the construction of the frame that:

$$\partial^2(w_{i,j}) = 0.$$

Then it follows that:

$$\partial(\sum_{1\le s\le q}\alpha_{sj}w_{i-1,s}) = \sum_{1\le s\le q}\alpha_{sj}\Big(\sum_{1\le l\le t}\beta_{ls}w_{i-2,l}\Big) = \sum_{1\le l\le t}\Big(\sum_{1\le s\le q}\alpha_{sj}\beta(ls)\Big)w_{i-2,l} = 0.$$

with α_{sj} and $\beta_{ls} \in K$. Hence, $\sum\limits_{1 \le s \le q} \alpha_{sj}\beta_{ls} = 0$ for each $1 \le l \le t$.

Doing the analogy with $\mathbf{C}$ we get:

$$\begin{aligned}
d^2(f_{i,j}) &= d\left(\sum_{1\le s\le q} \alpha_{sj} \frac{mdeg(f_{i,j})}{mdeg(f_{i-1,s})} f_{i-1,s}\right) \\
&= \sum_{1\le s\le q} \alpha_{sj} \frac{mdeg(f_{i,j})}{mdeg(f_{i-1,s})} f_{i-1,s} \left(\sum_{1\le l\le t} \beta_{ls} \frac{mdeg(f_{i-1,s})}{mdeg(f_{i-2,l})} f_{i-2,l}\right) \\
&= \sum_{1\le l\le t} \left(\sum_{1\le s\le q} \alpha_{sj}\beta_{ls} \frac{mdeg(f_{i,j})}{mdeg(f_{i-1,s})} f_{i-1,s} \frac{mdeg(f_{i-1,s})}{mdeg(f_{i-2,l})} f_{i-2,l}\right) f_{i-2,l} \qquad (4.1) \\
&= \sum_{1\le l\le t} \left(\sum_{1\le s\le q} \alpha_{sj}\beta_{ls}\right) \frac{mdeg(f_{i,j})}{mdeg(f_{i-2,l})} f_{i-2,l} \\
&= 0.
\end{aligned}$$

Thus, it follows that $\mathbf{C}$ is a complex. $\square$

Note that, we say the complex $\mathbf{C}$ is the M-***homogenization*** of the frame L.

Example 4.11. Consider the following frame

$$0 \longrightarrow k \xrightarrow{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}} k^3 \xrightarrow{\begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}} k^3 \xrightarrow{\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}} k.$$

Let $I = (x^2, xy, y^3)$. The homogenization of this frame is:

$$0 \longrightarrow R(x^2y^3) \xrightarrow{\begin{pmatrix} y^2 \\ x \\ 1 \end{pmatrix}} R(x^2y)\oplus R(xy^3)\oplus R(x^2y^3) \xrightarrow{\begin{pmatrix} -y & 0 & y^3 \\ x & -y^2 & 0 \\ 0 & x & -x^2 \end{pmatrix}} R(x^2)\oplus R(xy)\oplus R(y^3)$$

$$\xrightarrow{\begin{pmatrix} x^2 & xy & y^3 \end{pmatrix}} R.$$

We explain how we obtained the first column in the matrix representation of d_1:

Let $\{w_{1,1}, w_{1,2}, w_{1,3}\}$ be the basis of $L_1 = k^3$ and $\{w_{2,1}, w_{2,2}, w_{2,3}\}$ be the basis of $L_2 = k^3$. Let $\{f_{1,1}, f_{1,2}, f_{1,3}\}$ be the basis of C_1 and $\{f_{2,1}, f_{2,2}, f_{2,3}\}$ be the basis of C_2. We know from the frame that $d(w_{2,1}) = -w_{1,1} + w_{1,2}$.

Thus,

$$\begin{aligned} \partial(f_{2,1}) &= -\frac{mdeg(f_{2,1})}{mdeg(f_{1,1})} f_{1,1} + \frac{mdeg(f_{2,1})}{mdeg(f_{1,2})} f_{1,2} + 0 \\ &= -\frac{-x^2y}{x^2} f_{1,1} + \frac{x^2y}{xy} f_{1,2} + 0 \\ &= -y f_{1,1} + x f_{1,2} + 0 \end{aligned} \tag{4.2}$$

So the first column in d_1 would be $\begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}$.

The following theorem was shown by I. Peeva, M. Velasco in [15]

Theorem 4.12. *The M-homogenization of any frame of the minimal multigraded free resolution F of R/M is F.*

Construction 4.2.2. *From a complex* $\boldsymbol{C}$ *as described above, we can get the frame* $\boldsymbol{L}$ *to be the* ***dehomogenization*** *of* $\boldsymbol{C}$*. Where* $\boldsymbol{L}$ *in a complex of finite k-vector spaces with fixed basis. The differentials of* $\boldsymbol{L}$ *can be obtained by setting the variables* $x_1, \ldots, x_n$ *to be all equal to 1 in the differentials of* $\boldsymbol{C}$*.*

4.3 Properties of multigraded resolutions

Definition 4.13. For an M-complex C. Let $C(\leq m)$ be the sub-complex of C that is generated by the homogeneous basis elements of multidegree that divides m.

Example 4.14. In the above example,the complex **C** is

$$0 \longrightarrow R(x^2y^3) \xrightarrow{\begin{pmatrix} y^2 \\ x \\ 1 \end{pmatrix}} R(x^2y) \oplus R(xy^3) \oplus R(x^2y^3) \xrightarrow{\begin{pmatrix} -y & 0 & y^3 \\ x & -y^2 & 0 \\ 0 & x & -x^2 \end{pmatrix}} R(x^2) \oplus R(xy) \oplus R(y^3) \xrightarrow{\begin{pmatrix} x^2 & xy & y^3 \end{pmatrix}} R.$$

For $m = x^2y^2$, then $C(\leq m)$ would be as follows:

$$0 \longrightarrow R(x^2y) \xrightarrow{\begin{pmatrix} -y \\ x \end{pmatrix}} R(x^2) \oplus R(xy) \xrightarrow{\begin{pmatrix} x^2 & xy \end{pmatrix}} R.$$

Proposition 4.15. *Let M be a set of monomials in R generated by $\{m_1, \ldots, m_r\}$ as above. Let $m \in M$ be a monomial and*

$$m' = lcm(m_i | m_i \text{ divides } m)$$

Then $C(\leq m) = C(\leq m')$

Proof. The basis elements of $\mathbf{C}$ have multidegree in L_M. Now, since $m' = lcm(m_i|m_i$ divides $m)$, then $m' \in M$. Also, the components of $C(\leq m)$ are $C(q)$ such that q divides m. Then, $q \in L_M$, so q divides m' as well.

On the other hand if q divides m', then it directly follows that q divides m.

Therefore, $C(\leq m) = C(\leq m')$. □

Proposition 4.16. *Let $\boldsymbol{C}$ be an M-complex as described above, and $m \in M$ a monomial. The component of $\mathbf{C}$ of multidegree m is isomorphic to the frame of the complex $C(\leq m)$.*

Proof. Notice that C_m, the component of C of multidegree m has as basis elements:

$$\left\{ \frac{m}{mdeg(f)} f; f \text{ is the fixed basis of } \mathbf{C}, \text{ and } mdeg(f) \text{ divides } m \right\}.$$

Thus, it follows by construction, that the component of C of multidegree m is isomorphic to the frame of the complex $C(\leq m)$. □

Theorem 4.17. *An M-complex C us a free multigraded resolution of R/M, if and only if, the frame of the complex $C(\leq m)$ is exact for all monomials $m \in L_M$.*

Proof. Since C is a multigraded complex then for any monomial m in R that is not in M, we get all components of $C(\leq m)$ to be zero. Thus it suffices to check the theorem for all monomials $m \in M$.

By proposition 4.16, it suffices to check exactness of the frame of $C(\leq m)$ for all monomials $m \in M$.

Let m be a monomial in M and $m' \in R$ defined as in proposition 4.15, we get $C(\leq m) = C(\leq m')$. Therefore it suffices to check exactness of the monomials $m \in L_M$. $\square$

Proposition 4.18. *(Gasharov-Hibi-Peeva, Miller)*

Let $m \in M$ be a monomial, and consider the monomial ideal $(M_{\leq m})$ generated by the monomial $\{m_i | m_i$ divides $m\}$. Fix a homogeneous basis of a multigraded free resolution F_M of R/M .

1. *The sub-complex $F_M(\leq m)$ is a multigraded free resolution of $R/(M_{\leq m})$.*
2. *If F_M is minimal multigraded free resolution of R/M, then $F_M(\leq m)$ is independent of the choice of basis.*
3. *If F_M is a minimal multigraded free resolution of R/M, then the resolution $F_M(\leq m)$ is minimal as well.*

Proof. 1. Let $v = \operatorname{lcm}(m_i | m_i \text{ divides } m)$. By proposition 4.15, we get that $F_M(\leq m) = F_M(\leq v)$ and hence $M_{\leq m} = M_{\leq v}$. By theorem 4.17, we have to show that for all monomials $u \in L_{M(\leq v)}$, the frame of the complex $F_M(\leq v)(\leq u)$ is exact.

We have $F_M(\leq v)(\leq u) = F_M(\leq w)$ for w being the maximum monomial in L_M that divides v and u. Since F_M is exact, then again by theorem 4.17 the frame of $F_M(\leq w)$ is exact for all $w \in L_M$. Hence the frame of $F_M(\leq v)(\leq u)$ is exact. As a result, the frame of $F_M(\leq m)$ is exact. Therefore, the first statement holds.

2. $F_M(\leq m)$ is the sub-resolution of F_M with multidegree that divides m. Thus the construction of $F_M(\leq m)$ is independent of the basis of F_M. Therefore, the desired result in (2) holds.

3. by the construction of $F_M(\leq m)$, The result in (3) holds.

□

Chapter 5

Subadditivity of monomial ideals

5.1 Introduction

Let $R = k[x_1, \ldots, x_n]$ and I a homogeneous ideal in R. Denote by $(\mathbb{F}, \partial)$ a minimal graded free resolution of $S = R/I$ with $\mathbb{F}_a = \oplus_j R(-j)^{\beta_{aj}}$. For each a, denote by $t_a(\mathbb{F})$ the maximal shift in the resolution $\mathbb{F}$. In other words,

$$t_a(\mathbb{F}) = max\{j : \beta_{aj} \neq 0\}.$$

$\mathbb{F}$ is said to satisfy the *subadditivity* condition for maximal shifts if

$t_{a+b}(\mathbb{F}) \leq t_a(\mathbb{F}) + t_b(\mathbb{F})$, for all a and b.

There is history of looking for bounds of maximal shifts. The subadditivity problem for maximal shifts has been studied by many authors [3], [6], [7], [9], [11], [13]. It was shown that $t_p \leq t_1 + t_{p-1}$ for all graded algebras where p =projdim$S$ [11, Corollary 3], and that $t_p \leq t_a + t_{p-a}$ in some cases of S see [6, Corollary 4.1]. In [3], Avramov, Conca and Iyengar consider the situation when $S = R/I$ is Koszul and show that $t_{a+1}(I) \leq t_a + t_1 = t_a + 2$ for $a \leq$ height (I).

It is known that the minimal graded free resolution of graded algebras may not satisfy the subadditivity for maximal shifts as shown in the following example where $t_2 > 2t_1$:

```
  i1 : R=QQ[x,y,z]

o1 = R

o1 : PolynomialRing

i2 : ideal (x^12, y^12, z^12, -x^6*y^6+x^5*y^5*z^2+x^6*z^6-y^6*z^6)

             12   12   12     6 6    5 5 2    6 6    6 6
o2 = ideal (x  , y  , z  , - x y  + x y z  + x z  - y z )

o2 : Ideal of R

i3 : betti res o2

            0 1  2 3
o3 = total: 1 4 10 7
         0: 1 .  . .
         1: . .  . .
         2: . .  . .
         3: . .  . .
         4: . .  . .
         5: . .  . .
```

```
         6: . .  . .
         7: . .  . .
         8: . .  . .
         9: . .  . .
        10: . .  . .
        11: . 4  . .
        12: . .  . .
        13: . .  . .
        14: . .  . .
        15: . .  . .
        16: . .  . .
        17: . .  . .
        18: . .  1 .
        19: . .  2 1
        20: . .  1 .
        21: . .  2 .
        22: . .  3 2
        23: . .  . 1
        24: . .  1 2
        25: . .  . 1

o3 : BettiTally
```

However, no counter examples are known for monomial ideals or Gorenstein algebras. In the next section, we focus on exhibiting a result by Herzog and Srinivasan that show that $t_{a+1} \leq t_a + t_1$, for all $a \geq 1$ [11, Corollary 4].

5.2 Subadditivity

In this section, we use properties of multigraded resolutions to exhibit a result by [11] on the subadditivity for monomial ideals. Let $R = k[x_1, \ldots, x_n]$ and I a homogeneous ideal in R. Let $(\mathbb{F}, \delta)$ be a graded minimal free resolution of R/I:

$$\mathbb{F} : 0 \to \bigoplus_j R(-j)^{\beta_{sj}} \xrightarrow{\partial_s} \ldots \to \bigoplus_j R(-j)^{\beta_{ij}} \xrightarrow{\partial_i} \ldots \to \bigoplus_j R(-j)^{\beta_{1j}} \xrightarrow{\partial_1} R \to R/I \to 0.$$

Proposition 5.1. *Suppose there exists a homogeneous basis $f_1, f_2, \ldots, f_r$ of F_a such that*

$$\partial(F_{a+1}) \subset \bigoplus_{1 \leq i \leq r-1} Rf_i$$

then $deg f_r \leq t_{a-1} + t_1$.

Before we prove proposition 5.1, we introduce the notion of dual basis.

Definition 5.2. Let $\mathbf{F}$ be a complex of R-modules. Denote by $(\mathbf{F}^*, \partial^*)$ the complex $Hom_R(\mathbf{F}, R)$ which is dual to $\mathbf{F}$:

$$\mathbb{F}^* : \quad \cdots \longrightarrow F^*_{a-1} \xrightarrow{\partial^*_a} F^*_a \xrightarrow{\partial^*_{a+1}} F^*_{a+1} \cdots$$

For any basis $f_1, \ldots, f_r$ of F_a, we denote by f^*_j the basis element of F^*_a with $f^*_j(f_l) = 1$ if $j = l$ and $f^*_j(f_l) = 0$ if $j \neq l$. Hence, $f^*_1, \ldots, f^*_r$ is a basis of F^*_a, the so-called dual basis of $f_1, \ldots, f_r$. The maps in the dual complex are defined by $\partial^*(f_i^*) = f_i^* \circ \partial$ for all i, obtained from the following commutative diagram:

$$\begin{tikzcd} F_{a+1} \arrow[r, "\partial"] \arrow[dr, "*"] & F_a \arrow[d, "f_a^*"] \\ & k \end{tikzcd}$$

Proof. Let $\{f_1^*, \ldots, f_r^*\}$ be a dual basis of $\{f_1, \ldots, f_r\}$. The hypothesis implies that $\partial^*(f_r^*) = 0$. Hence f_r^* is a generator of $H^a(\mathbb{F}^*) = Ker\partial_{a+1}^*/Im\partial_a^*$ which is an R/I-module and hence $If_r^* = 0$ in $H^a(\mathbb{F}^*)$.

On the other hand, if $g_1, \ldots, g_m$ is a basis of F_{a-1} and $\partial(f_r) = c_1g_1 + \ldots + c_mg_m$, then $\partial^*(g_i^*) = c_if_r^* + m_i$ where each m_i is a linear combination of the remaining basis elements of F_a^*.

We denote by $c \in I$ to be a generator of maximal degree i.e $deg(c) = t_1(I)$. Since $If_r^* = 0$ in $H^a(\mathbb{F}^*)$, then it implies that $cf_r^* = 0$ in $H^a(\mathbb{F}^*)$. This means that cf_r^* belongs to $Im\partial_a^*$. For that, there exist homogeneous elements $s_i \in R$ such that

$$cf_r^* = \sum_{1 \leq i \leq m} s_i \partial^*(g_i^*) = \sum_{1 \leq i \leq m} s_i(c_if_r^* + m_i).$$

The above equation is possible only if $t_1(I) = deg(c_i) + deg(s_i)$ for some i. In particular, $deg(c_i) \leq t_1(I)$. It follows that

$$deg(f_r) = deg(c_i) + deg(g_i) \leq t_1(I) + t_{a-1}(I)$$

. $\square$

We get to prove the main result:

Theorem 5.3. *Let I be a monomial ideal. Then $t_a(I) \leq t_{a-1}(I) + t_1(I)$ for all $a \geq 1$.*

Proof. Let $\mathbb{F}$ be a minimal multigraded free R-resolution of R/I and let $f \in F_a$ be a homogeneous element of multidegree $\alpha \in \mathbb{N}^n$ whose total degree is equal to the maximal shift $t_a(I)$. We apply the result of proposition 4.18, and consider the restricted complex $F(\leq \alpha)$. Let $f_1, \ldots, f_r$ be a homogeneous basis of $(F(\leq \alpha))_a$ with $f_r = f$. Since there is no basis element of $(F(\leq \alpha))_{a+1}$ of multidegree bigger than α, and since the resolution of $(F(\leq \alpha))_a$ is minimal, it follows that $\partial(F(\leq \alpha)_{a+1}) \subset \bigoplus_{1 \leq i \leq r-1} Rf_i$. Thus we apply proposition 5.1 and deduce that $t_a(I(\leq \alpha)) \leq t_{a-1}(I(\leq \alpha)) + t_1(I(\leq \alpha))$. Since $t_a(I) = t_a(I(\leq \alpha))$, $t_{a-1}(I(\leq \alpha)) \leq t_{a-1}(I)$ and $t_1(I(\leq \alpha)) \leq t_1(I)$, and hence we get the result. $\square$

Example 5.4.
```
i48 : R= QQ[x, y, z, w]

o48 = R

o48 : PolynomialRing

i49 : ideal (x^3*y^2, x^4*z^7, w^5, w^3*x*y*z)

                3 2   4 7   5          3
o49 = ideal (x y , x z , w , x*y*z*w )

o49 : Ideal of R

i50 : res o49

       1      4      6      3
```

```
o50 = R  <-- R  <-- R  <-- R  <-- 0

      0      1      2      3      4

o50 : ChainComplex

i51 : ideal (x^3*y^2, x^4*z^7, w^5, w^3*x*y*z, x^2*y^2*z, x*y*z^10*w)

              3 2   4 7   5         3   2 2        10

o51 = ideal (x y , x z , w , x*y*z*w , x y z, x*y*z  w)

o51 : Ideal of R

i52 : res o51

       1      6      10      6      1

o52 = R  <-- R  <-- R   <-- R  <-- R  <-- 0

      0      1      2       3      4      5

o52 : ChainComplex

i53 : o52.dd_1

o53 = | x3y2 x2y2z w5 xyzw3 x4z7 xyz10w |

              1       6

o53 : Matrix R  <--- R

i54 : o52.dd_2

o54 = {5}  | -z 0   0    -w5  0     0    0     0   0     0     |

      {5}  | x  -w3 0    0    -x2z6 -z9w 0     0   0     0     |

      {5}  | 0  0   -xyz x3y2 0     0    0     0   0     -x4z7 |

      {6}  | 0  xy  w2   0    0     0    -x3z6 -z9 0     0     |
```

```
      {11} | 0  0   0    0    y2    0    yw3   0    -yz3w w5     |
      {13} | 0  0   0    0    0     xy   0     w2   x3    0      |

              6        10
o54 : Matrix R  <--- R

i55 : o52.dd_3

o55 = {6}  | w5   0    0   0   0     0   |
      {8}  | xw2  x2z6 0   z9  0     0   |
      {8}  | -x2y 0    0   0   -x3z6 0   |
      {10} | -z   0    0   0   0     0   |
      {13} | 0    -w3  z3w 0   0     0   |
      {15} | 0    0    -x2 -w2 0     0   |
      {15} | 0    y    0   0   -w2   z3  |
      {15} | 0    0    0   xy  0     -x3 |
      {16} | 0    0    y   0   0     w2  |
      {16} | 0    0    0   0   y     0   |

              10       6
o55 : Matrix R   <--- R

i56 : o52.dd_4

o56 = {11} | 0   |
      {16} | -z3 |
      {17} | -w2 |
```

```
      {17} | x2  |

      {17} | 0   |

      {18} | y   |

              6       1

o56 : Matrix R  <--- R

i57 : o52.dd_5

o57 = 0

              1

o57 : Matrix R  <--- 0

i58 : betti res o51

             0 1  2 3 4

o58 = total: 1 6 10 6 1

          0: 1 .  . . .

          1: . .  . . .

          2: . .  . . .

          3: . .  . . .

          4: . 3  1 . .

          5: . 1  . . .

          6: . .  2 . .

          7: . .  . . .

          8: . .  1 1 .
```

```
 9: . .  . . .
10: . 1  . . .
11: . .  1 . .
12: . 1  . . .
13: . .  3 1 .
14: . .  2 3 .
15: . .  . 1 1

o58 : BettiTally
```

In this Example, $R = \mathbb{Q}[x, y, z, w]$ and $I = (x^3y^2, x^4z^7, w^5, w^3xyz)$

From the betti diagram, we can see that

$t_1(I) = 13$

$t_2(I) = 16$

$t_3(I) = 18$

$t_4(I) = 19.$

Clearly, the main result hold for this example since:

$$t_4(I) = 19 \leq t_3(I) + t_1(I) = 13 + 18 = 31$$

$$t_3(I) = 18 \leq t_2(I) + t_1(I) = 16 + 13 = 29$$

$$t_2(I) = 16 \leq t_1(I) + t_1(I) = 13 + 13 = 26$$

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