AMERICAN UNIVERSITY OF BEIRUT

Signal Estimation and Reconstruction at Sub-Nyquist Rates

by
Mohamad Hassan Saab

A thesis
submitted in partial fulfillment of the requirements
for the degree of Master of Engineering
to the Department of Electrical and Computer Engineering
of the Maroun Semaan Faculty of Engineering and Architecture
at the American University of Beirut

Beirut, Lebanon
April 2019

AMERICAN UNIVERSITY OF BEIRUT

Signal Estimation and Reconstruction at Sub-Nyquist Rates

by

Mohamad Hassan Saab

Approved by:

Dr. Karim Kabalan, Professor	Advisor
Electrical and Computer Engineering	
Dr. Youssef Nasser, Senior Lecturer	Co-Advisor
Electrical and Computer Engineering	
Dr. Ali Chehab, Professor	Member of Committee
Electrical and Computer Engineering	
Dr. Mohammad Mansour, Professor	Member of Committee
Electrical and Computer Engineering	

Date of thesis defense: April 23, 2019

AMERICAN UNIVERSITY OF BEIRUT

THESIS, DISSERTATION, PROJECT RELEASE FORM

Student Name: Saab Mohamad Hassan
Last First Middle

☑ Master's Thesis ◯ Master's Project ◯ Doctoral Dissertation

☐ I authorize the American University of Beirut to: (a) reproduce hard or electronic copies of my thesis, dissertation, or project; (b) include such copies in the archives and digital repositories of the University; and (c) make freely available such copies to third parties for research or educational purposes.

☑ I authorize the American University of Beirut, to: (a) reproduce hard or electronic copies of it; (b) include such copies in the archives and digital repositories of the University; and (c) make freely available such copies to third parties for research or educational purposes after: One ___ year from the date of submission of my thesis, dissertation or project.
Two ___ years from the date of submission of my thesis, dissertation or project.
Three ✓ years from the date of submission of my thesis, dissertation or project.

Mohamad May/16th/2019
Signature Date

Acknowledgements

First and foremost, I would like to thank God Almighty for giving me the strength, knowledge, ability and opportunity to undertake this research study and to persevere and complete it satisfactorily. Without his blessings, this achievement would not have been possible.

I would like also to express my gratitude to my beautiful family, especially my parents, for their continuous and unconditional care and support throughout my life, and mainly during my thesis journey. They have all made a tremendous contribution in helping me reach this stage in my life. This would not have been possible without their unwavering and unselfish love and support given to me at all times.

I would like to acknowledge my advisor Prof. Karim Kabalan, and my co-advisor Prof. Youssef Nasser, for their guidance and motivation. My acknowledgement goes also to my respected committee, Prof. Ali Chehab and Prof. Mohammad Mansour for their comments, and questions that were an added value to this work.
In my journey towards this degree, I have found a teacher, a friend, an inspiration, a role model and a pillar of support in my guide, Prof. Youssef Nasser. It was a great opportunity to work with him. I am grateful to his continuous support, patience, understanding, and motivation on the technical and non-technical sides.

My acknowledgement would be incomplete without thanking the professors who taught me the course I've taken during my residence. Taking their courses and being their student was a great experience and helped me in my thesis progress and grew my personality to become more mature. It would be inappropriate if I omit to mention every single person with whom I've worked as a graduate assistantship, especially lab managers and instructors. They gave the opportunity to live the valuable lab assistant experience.

My final thanks goes to all my friends, colleagues throughout my Masters journey. Thank you all for adding to the AUB experience the amazing memories and the precious moments that I will hold through my life.

An Abstract of the Thesis of

Mohamad Hassan Saab for Master of Engineering
Major: Electrical and Computer Engineering

Title: Signal Estimation and Reconstruction at Sub-Nyquist Rates

Frequency estimation is a very important step to correctly detect a signal components. Nowadays, frequency estimation is required in many applications such biomedical signals, spectrum sensing, and military systems. However, as most of these applications require wide bands signals, the implementation of conventional sampling schemes at the Nyquist rate becomes very challenging. Hence, it is primordial to propose advanced frequency estimation methods at subNyquist sampling rates. In literature, Chinese remainder theorem (CRT) has been proposed to estimate the components of a single frequency signal. However, its extension to multiple components has not been addressed due to the complexity of the estimation algorithm. In this proposal, we extend the CRT further by proposing a new approach for frequency estimation of a signal with multiple components as long as they have a particular pattern. The results have been validated by Monte-Carlo simulations and compared with the well-known MUSIC algorithm.

Contents

List of Figures

List of Tables

Chapter 1

Introduction

Frequency estimation and sensing has recently attracted many researchers due to its large number of applications. Among others, signal sensing in communication systems is currently an integral part for spectrum sharing. Also, other applications such as audio, and electric systems highly rely on frequency estimation [1, 2, 3].
In the literature, three different approaches have been adopted for frequency estimation. The first one is based on the classical discrete Fourier transform (DFT) [4, 5]. The second approach, called multiple signal classification (MUSIC), requires an advanced processing as it converts the signal into another subspace and then uses the eigen values of the converted signal to separate between the signal and the noise. MUSIC has been implemented in different forms and applications but we cite here the pioneering work of [6]. The third approach is also an advanced technique, based on the estimation of the signal parameter via rotational invariance techniques (ESPRIT) [7]. Other approaches exist in literature; the reader may refer to [8, 9].
In 5G technologies, the millimeter wave band has gained great research interest as it offers a large Bandwidth (1-2 GHz) that would heavily increase the bit rate to many Gb/s [10]. However, the main problem of these bands resides in very advanced Analog to Digital Converters (ADCs) whose sampling rate is at least to 2-4 Gsamples/s [11]. ADCs with high speed sampling rate are either unavailable, or too costly and power hungry [11, 12]. Therefore, there is a need to find alternate ways for frequency estimation that are based on sub-Nyquist sampling.
Many methods are proposed in the literature for sub-Nyquist sampling. In [13], time delay method between the sampling channels is used to avoid the ambiguity.
In [14], the ambiguity in the frequency estimation is resolved with low rate ADCs by choosing proper delay times and by using sparse linear prediction. Moreover, the Chinese remainder theorem (CRT) is useful to correctly estimate only one frequency [15]. Other methods benefited from the compressed sensing theory to design sub-Nyquist sensing algorithms to estimate the power spectrum of wide band signal [16, 17, 18]. In [19], three co-prime undersampling ratios are used to

correctly estimate the frequencies of a signal of three components. The estimation is done using MUSIC algorithm.

All these sub-Nyquist approaches are either limited to one frequency or complex to be implemented [19, 15].

In this thesis, we propose a novel signal estimation and reconstruction technique by dividing the signal band into multiple bands. In each of which, the components follow the following fixed pattern: the separation between two consecutive frequencies is fixed. Moreover, this separation must not be a priori known as in [20]. The proposed method is validated on different number of components. A study on the probability of detection is also included and the results are compared with those obtained by the conventional MUSIC algorithm in [19].

The rest of the thesis is organized as follows. Chapter 2 represents a literature review of the Residue Number System (RNS) and different techniques of treating the residues of a signal. Thereafter, chapter 3 is a review on the Chinese Remainders Theorem (CRT) and its recovery techniques. Then in chapter 4, the proposed technique is explained. Then, the simulation scenarios and results are discussed. Finally, conclusions are drawn in chapter 5.

Chapter 2

Residue Number System (RNS)

2.1 Definition

Residue Number System (RNS) represents an integer in form of a set of residues obtained as the remainders of the division over a set of moduli. It can provide carry-free operations between the residues, fault-tolerance, and parallelism [21]. RNS can solve the problem of many applications that demand high speed computations and increasing complexities as the range of numbers increases, such as cryptography and analog to digital converters [22].

2.2 Problem Formulation

Let $(m_1, m_2, \ldots, m_L)$ L co-prime moduli. The residues $(x_1, x_2, \ldots, x_L)$ of a number X over these moduli can be given by:

$$\begin{aligned} x_1 &= X \ mod \ m_1, \\ x_2 &= X \ mod \ m_2, \\ &\vdots \\ x_L &= X \ mod \ m_L, \\ 0 &\leq x_i \leq m_i - 1. \end{aligned}$$

The residue set $(x_1, x_2, \ldots, x_L)$ is a unique representation of X over the range $[0, M-1]$, where $M = \prod_{i=1}^{L} m_i$ is the dynamic range [23, 24].
To solve this problem, two main methods are presented in the literature: Mixed radix conversion (MRC) technique [25, 26, 27, 28], and Chinese remainder theorem (CRT) and its derivations [29, 23, 15].

2.3 Mixed Radix Conversion (MRC)

Mixed Radix Conversion (MRC) is a sequential process to solve for X given its residues $(x_1, x_2, \ldots, x_N)$ and moduli $(m_1, m_2, \ldots, m_N)$. In MRC, X can be written as [28]:

$$X = a_1 + a_2 m_1 + a_3 m_1 m_2 + \ldots + a_n m_1 m_2 \ldots m_{N-1} \quad (2.1)$$

Where the set $(a_1, a_2, \ldots, a_N)$ are the mixed radix digits (MRD). They can be computed through the following sequence [27]:

$$\begin{aligned}
a_1 &= x_1 \\
a_2 &= |(x_2 - a_1)\ |m_1^{-1}|_{m_2}|_{m_2} \\
a_3 &= |\ \big((x_3 - a_1)\ |m_1^{-1}|_{m_3} - a_2\big)\ |\ |m_2^{-1}|_{m_3}|_{m_3} \\
&\vdots \\
a_N &= |\ \big(((x_N - a_1)\ |m_1^{-1}|_{m_N} - a_2)|\ |m_2^{-1}|_{m_N} - \ldots - a_{N-1}\big)\ |m_{N-1}^{-1}|_{m_N}|_{m_N}
\end{aligned}$$

$|m_i^{-1}|_{m_j}$ is the modular multiplicative inverse of m_i, and can be calculated by:

$$|m_i^{-1}|_{m_j} = N_i \mod m_j, \qquad s.t. \qquad N_i m_i = 1 \mod m_j \quad (2.2)$$

2.4 Chinese Remainder Theorem

A second method to recover X is the chinese remainder theorem (CRT). It is a parallel computation method, where each residue is treated separately. In CRT, X can be written as [29]:

$$X = |M_1|x_1 M_1^{-1}|_{m_1} + |M_2|x_2 M_2^{-1}|_{m_2} + \ldots + |M_N|x_N M_N^{-1}|_{m_N}|_M \quad (2.3)$$

$$X = |\sum_{i=1}^{N} M_i |x_i M_i^{-1}|_{m_i}|_M \quad (2.4)$$

where $M_i = M/m_i$ and M_i^{-1} is the modular multiplication inverse of M_i calculated using Extended Euclidean algorithm [30].
CRT is more desirable over MRC because of the ability of performing parallel computational process.

2.5 New CRT

To enhance the performance of the CRT, three new theorems were proposed in [29].

1. **New CRT-I:**
   New Chinese Remainder Theorem I is a parallel MRC. Its advantage over the conventional CRT is in the usage of smaller weights.
   In new CRT-I, X can be computed as:

$$X = |x_1 + k_1 m_1 (x_2 - x_1) + k_2 m_1 m_2 (x_3 - x_2) + \dots \\ + k_{N-1} m_1 m_2 \dots m_{N-1} (x_N - x_{N-1})|_{m_1 m_2 \dots m_N}$$

   where

$$\begin{aligned} k_1 m_1 &= 1 \quad \text{mod} \quad m_2 m_3 \dots m_N, \\ k_2 m_1 m_2 &= 1 \quad \text{mod} \quad m_3 \dots m_N, \\ &\vdots \\ k_{N-1} m_1 m_2 \dots m_{N-1} &= 1 \quad \text{mod} \quad m_N, \end{aligned}$$

   This theorem does not require sequential process like MRC. Therefore, the MRDs $(k_1, k_2, \dots, k_{N-1})$ can be found separately in parallel.

2. **New CRT-II:**
   The main objective of the new Chinese Remainder Theorem II is to compute X using modulo operations of smaller size.
   Let the moduli set $(m_1, m_2, \dots, m_N)$ with $m_1 < m_2 < \dots < m_N$. The algorithm of the new CRT-II is the following:

   - ***translate*** $((x_1, x_2, \dots, x_N), X)$
     (a) if $N > 2$, t $= \lfloor N/2 \rfloor$, then
     translate $((x_1, x_2, \dots, x_t), N_1)$, $M_1 = m_1 m_2 \dots m_t$
     translate $((x_{t+1}, x_2, \dots, x_N), N_2)$, $M_2 = m_{t+1} \dots m_N$,
     findno (N_1, N_2, M_1, M_2, X).
     (b) if $N = 2$, then ***findno*** (x_1, x_2, m_1, m_2, X).
   - ***findno*** (x_1, x_2, m_1, m_2, X)
     (a) find k_0 s. t. $K_0 m_2 = 1 \ mod \ m_1$
     (b) $X = x_2 + |k_0(x_1 - x_2)|_{m_1} m_2$.

   In New CRT-II, the modulo multipliers are bounded by $\sqrt{M}$.

3. **New CRT-III:**
   The New Chinese Remainder Theorem III is an extension of the New CRT-II to the case where the moduli set $(m_1, m_2, \dots, m_N)$ are not pair-wisely co-prime.

The same algorithm of New CRT-II is used in CRT-III with minor modifications as follows:

$$
\begin{aligned}
&N \geq 2, \quad t = \lfloor N/2 \rfloor, \quad M_1 = lcm(m_1, \ldots, m_t),\ M_2 = lcm(m_{t+1}, \ldots, m_N).\\
&GCD(M_1, M_2) = d \implies M_1 = dc_1, \text{ and } M_2 = dc_2,\\
&GCD(c_1, c_2) = 1, \quad k_1 c_2 = 1 \ mod \ c_1\\
&N_1 = y_i \ mod \ m_i, \quad 1 \leq i \leq t,\\
&N_2 = y_i \ mod \ m_i, \quad t+1 \leq i \leq N,\\
&N_1 \in [0, M_1 - 1], \text{ and } N_2 \in [0, M_2 - 1].
\end{aligned}
$$

Therefore,

$$X = N_2 + |k_1(N_1 - N_2)/d|_{c_1} M_2.$$

In literature, the focus is on special moduli sets where the modulo operation is not needed for the conversion. Several moduli sets that have been used are listed in [29]. In the last decade, other moduli sets have been studied, such as $(2^n - 1, 2^n, 2^n + 1, 2^{n+1} - 1, 2^{n-1} - 1)$ [31], $(2n + 2, 2n + 1, 2n)$ using MRC [28], $(2^n - 1, 2^n, 2^n + 1)$ [32]. Most recently, Hiasat has studied additional moduli sets for RNS , i.e. $(2^{2n+p}, 2^n - 1, 2^n + 1, 2^n - 2^{\frac{n+1}{2}} + 1, 2^n - 2^{\frac{n+1}{2}} - 1)$ with $p \leq \frac{n-5}{2}$ [33], $(2^n - 1, 2^n + 1, 2^{2n} + 1, 2^{2n+p})$ with $p \leq n - 2$ [34], and $(2^{n+1} - 1, 2^n, 2^n - 1)$ [24]

2.6 RNS and ADC

The main objective of using RNS ADCs is to reduce the hardware complexity and to increase the speed of the process, by reducing the number of comparators in the circuit [35]. With the increase of the number of output bits, the number of comparators increases exponentially [36]. Therefore, the amount of consumed power and the chip area increases significantly, and high speed flash ADCs have practical limits to meet these factors [36].
RNS ADCs, also known as Folding ADCs, are based on number theory (NT). NT can increase the resolution of the output signals and simplify the conversion process [37]. The concept of these ADCs is to use the residues obtained from input analog signals after sub-Nyquist sampling in order to get the binary output.
The main advantage of RNS of other techniques is that a set of residues can form a unique representation of an integer within a large dynamic range [35].

There are several systems that make use of NT: Symmetrical Number System (SNS) [38], Optimum Symmetrical Number System (OSNS) [39], Robust Symmetrical Number System [40], and Folding Number System (FNS) [37]. SNS and OSNS processes are summarized and their performance for ADCs is compared in [37].

- **Symmetrical Number System (SNS):**
SNS consists of using Pairwise Relatively Prime (PRP) moduli set. The signal is converted into set of residues obtained by folding it over the modulus. The output if this operation is called Symmetrical Residues (SRs)[37]. Let $0 \leq h \leq m$, the SNS's SRs are of the form $x_h = min\{h, m-h\}$.
For m odd,
$$\{x_m\} = \left[0, 1, \ldots, \lfloor\frac{m}{2}\rfloor, \lfloor\frac{m}{2}\rfloor, \ldots, 2, 1\right]$$
For m even,
$$\{x_m\} = \left[0, 1, \ldots, \frac{m}{2} - 1, \frac{m}{2}, \ldots, 2, 1\right]$$
The dynamic range (DR) of SNS, which is the maximum number of distinct vectors for N moduli is:
$$M_{SNS} = min\left\{\frac{m_1}{2}\prod_{l=2}^{j} m_{i_l} + \prod_{l=j+1}^{N} m_{i_l}\right\}$$
where $j \in [2, N-1]$.
If the moduli set is all odd co-prime, then DR becomes:
$$M_{SNS} = min\left\{\frac{1}{2}\prod_{l=1}^{j} m_{i_l} + \frac{1}{2}\prod_{l=j+1}^{N} m_{i_l}\right\}$$
where $j \in [1, N-1]$.

- **Optimum Symmetrical Number System (OSNS):**
The main difference between SNS and OSNS is that the folding period in OSNS is twice the modulus ($T = 2m_i$). The SRs formula of OSNS becomes:
$$\{x_m\} = [0, 1, \ldots, m-1, m-1, \ldots, 0]$$
The OSNS's DR is:
$$M_{OSNS} = \prod_{i=1}^{N} m_i$$

- **Folding Number System (FNS):**
Folding Number System (FNS) is proposed in [37]. It is a modified version of OSNS, that reduces the complexity. An additional bit is introduced at the output. It is called Folding Bit (FB). Because of that bit, the DR of FNS is twice that of OSNS. This also provides a better resolution of ADC. The folding bit is used to eliminate the ambiguities coming from the folding process. The SRs of the FNS can be calculated as below:
$$\begin{cases} s = d, & f = 0 \quad \text{if } d \leq m_i - 1 \\ s = 2m_i - d - 1, & f = 1 \quad \text{if } m_i \leq d \leq T - 1 \end{cases}$$

d is the decimal offset within the folding period $T = 2m_i$, and f is the folding bit.

Chapter 3

Frequency Estimation Techniques Using CRT

One of the most famous applications of the Chinese Remainder Theorem (CRT) is the frequency estimation from undersampled waveforms. The estimation process started by recovering only one frequency component. Nowadays, latest CRT algorithms are able to detect and estimation multiple components even from noisy signals. In this section, we are going to present a review of different kinds estimation algorithms in the literature.

3.1 Estimation of One Component

The early stages of using CRT in frequency estimation started by applying the basic CRT on a error-free signal composed from one frequency component. Then, it was extended to cover the case of noisy signal.

3.1.1 Conventional CRT

The error free case of frequency estimation is first discussed in [41]. The CRT is applied as following:

1. The moduli of the CRT $m_l(l = 1, \dots, L)$ are considered as the low-rate undersampling frequencies. These frequencies should be pairwisely coprime.

2. The remainders are the indexes corresponding to the peaks of the DFT output of each undersampled waveforms.

3. The true frequency value of the input signal is calculated using the conventional CRT in Equation (2.4).

3.1.2 Robust CRT

Real-life signals are transmitted and received in noisy environment. Therefore, more realistic approach should be made in the estimation process in order to deal with noisy received signals.

Several approaches are presented in the literature to deal with this kind of problems. All of them require that the maximum allowed error is upper bounded by certain value τ.

In [42], the undersampling frequencies $m_1, m_2, \ldots, m_L$ assumed to have a common GCD between them Γ, where $m_l = \Gamma M_l$, and M_l are pairwisely coprime. Therefore, the upper error bound becomes $\tau < \Gamma/4$. Then, the relation between exact and erroneous remainders is:

$$|r_l - \tilde{r}_l| \leq \tau \tag{3.1}$$

Thereafter, several techniques are introduced, such as: single [42, 43] and multi-stage [44] minimum distance, maximum likelihood [45], and normalized remainders [46].

All of them aim to recover n_l, such that:

$$\hat{N} = \left[\frac{1}{L}\sum_{l=1}^{L}(n_l m_l + \tilde{r}_l)\right] = N + \left[\frac{1}{L}\sum_{l=1}^{L}(\tilde{r}_l - r_l)\right] \tag{3.2}$$

3.2 Estimation of Multiple Components

Signals with more than one frequency component is the most widely used in practice. Therefore, estimation of signals with one component is not so useful unless if the band is divided into narrower sub-bands, where each of them contain one component. However, it is very expensive in terms of number and complexity of filters needed. Therefore, the main question became: How can we correctly detect more than one component using CRT? The main challenge of this problem is the ambiguity in the association of the remainders from different undersampled signals corresponding to the same component.

The following reviews the evolution of estimating multiple components using CRT and the assumptions taken into consideration. It started by simply estimating two components, then it was expanded to higher number of components under predefined assumptions. Moreover, these new approaches covered both error-free and noisy signals.

3.2.1 Estimation of Two Components

Let N_1, N_2 the two frequency components to be recovered. The first step is to determine the largest possible value that those two components could take. In

[47], the dynamic range of $\{N_1, N_2\}$ where they can be uniquely determined is:

$$\max\{N_1, N_2\} < \max\left\{\frac{1+\sqrt{1+4\cdot lcm(M)}}{2}, m_L\right\} \tag{3.3}$$

where $M = \{m_1, m_2, \ldots, m_L\}$ is the set of the undersampling frequencies in the ascending order. The only condition on M is $M \geq 5$
Thereafter, the exact components can be recovered following these steps:

1. Let $k_{1,l}$ and $k_{2,l}$ the indices of the two peaks at sampling frequency m_l. Let $\langle k_{1,l} + k_{2,l}\rangle_{m_l}$ and $\langle k_{1,l}k_{2,l}\rangle_{m_l}$, the sum S and the product P of remainders at ℓ-th sampling frequency modulo m_l.

2. Using conventional CRT, calculate the sum and the product of the true frequencies.

3. N_1 and N_2 are the roots of the following second degree equation:

$$N^2 - SN + P = 0 \tag{3.4}$$

The dynamic range is extended in [48]. It becomes:

$$max\{N_1, N_2\} < 2\sqrt{lcm(M)} \tag{3.5}$$

Then, the new estimation algorithm proposed in [48] is:

1. For each residue set, calculate

$$s_l = \langle k_{1,l} + k_{2,l}\rangle_{m_l} \tag{3.6}$$

2. Using s_l $(l = 1, \ldots, L)$, calculate S the sum of the two frequencies.

3. Let $\zeta = \max[0, \lceil S - 2\sqrt{lcm(M)}\rceil]$. Calculate P by applying conventional CRT on remainders P_l, where P_l:

$$P_l = \langle (k_{1,l} - \zeta)(k_{2,l} - \zeta)\rangle_{m_l} \tag{3.7}$$

4. N_1 and N_2 are the roots of the following equation:

$$(N - \zeta)^2 - (S - 2\zeta)(N - \zeta) + P = 0 \tag{3.8}$$

- **Estimation of Noisy Signals**
  The estimation of a noisy signal with two frequency components is similar to the robust CRT with one component described above. In [49], the first assumption is on the GCD of the undersampling frequencies, $\Gamma > 1$. The second one is on the upper error bound $\tau < \Gamma/8$ [49].
  Noisy signal components are recovered in [49] following these next steps:

1. Calculate $\tilde{r}^c_{k,l} = \langle \tilde{r}_{k,l} \rangle_{m_l}$.
2. Sort $\tilde{r}^c_{k,l}$ is the ascending order. Then, compute D_k.

$$
\begin{aligned}
D_l &= \tilde{r}^c_{\zeta(l+1)} - \tilde{r}^c_{\zeta(l)}, \qquad \text{if } l = 1, \ldots, 2L-1. \\
D_l &= \tilde{r}^c_{\zeta(1)} - \tilde{r}^c_{\zeta(2K)}, \qquad \text{if } l = 2L.
\end{aligned}
$$

3. Find l_0 as:

$$
l_0 = \arg\max_{l \in \{1,\ldots,L\}} \{D_l + D_{l+L}\} \tag{3.9}
$$

4. Get Ω_1 and Ω_2 such as:

$$
\begin{aligned}
\Omega_1 &\triangleq \{\omega_1, \ldots, \omega_L\} = \{\tilde{r}^c_{\zeta(l_0+1)}, \ldots, \tilde{r}^c_{\zeta(l_0+L)}\} \\
\Omega_2 &\triangleq \{v_1, \ldots, v_L\} \\
&= \{\tilde{r}^c_{\zeta(1)}, \ldots, \tilde{r}^c_{\zeta(L)}\} \qquad \text{if } l_0 = L \\
&= \{\tilde{r}^c_{\zeta(l_0+1+L)} - \Gamma, \ldots, \tilde{r}^c_{\zeta(2L)} - \Gamma, \tilde{r}^c_{\zeta(1)}, \ldots, \tilde{r}^c_{\zeta(l_0)}\} \qquad \text{if } l_0 \neq L
\end{aligned}
$$

5. Get $\omega_1^{'}$ and $\omega_2^{'}$ such that:

$$
\begin{aligned}
\omega_l^{'} &= \omega_l \qquad \text{if } \omega_L - v_1 \leq \Gamma/2 \\
\omega_l^{'} &= \omega_l - \Gamma \qquad \text{if } \omega_L - v_1 > \Gamma/2
\end{aligned}
$$

6. Calculate $\bar{\omega_1}$ and $\bar{\omega_2}$ as:

$$
\begin{aligned}
\bar{\omega_1} &\triangleq \frac{\omega_1^{'} + \cdots + \omega_L^{'}}{L} \\
\bar{\omega_2} &\triangleq \frac{v_1 + \cdots + v_L}{L}
\end{aligned}
$$

7. Determine $\hat{q}_{i,l}$:

$$
\begin{aligned}
\hat{q}_{i,l} &= \left\lceil \frac{\tilde{r}_{i,l} - \bar{\omega}_t}{\Gamma} \right\rfloor, \; i = 1, 2; \; l = 1, \ldots, L. \\
t &= 1, \text{if } d_\Gamma(\tilde{r}^c_{i,l}, \omega_{l_1}) = 0 \qquad \text{for some } l_1 \\
t &= 2, \text{if } d_\Gamma(\tilde{r}^c_{i,l}, v_{l_2}) = 0 \qquad \text{for some } l_2 \\
\text{with } d_C(x, y) &\triangleq x - y + \left\lceil \frac{x-y}{C} \right\rfloor C
\end{aligned}
$$

   d_C is the circular distance.
8. Determine residue sets R_l as $R_l(\hat{Q}_1, \hat{Q}_2) = \{\hat{q}_{1,l}, \hat{q}_{2,l}\}$
9. Reconstruct $\{\hat{Q}_1, \hat{Q}_2\}$ using the generalized CRT algorithm in [48].
10. Recover $\{\hat{N}_1, \hat{N}_2\}$ using the following equation:

$$
\{\hat{N}_1, \hat{N}_2\} = \{\Gamma\hat{Q}_1 + \hat{r}^c_1, \{\Gamma\hat{Q}_2 + \hat{r}^c_2\}
$$

3.2.2 Generalization of Multiple Components Estimation

Recently, the estimation of a multiple components signal using the Chinese Remainders Theorem (CRT) is widely explored in the research to extract the best possible designs in terms of performance, complexity, and dynamic range. There are two main tracks: the first one focuses on the estimation of noise free signal, while the second aims to find a robust estimation in a noisy environment.

1. **Noise-Free estimation:**
   One of the first estimation approaches of multiple components signal was made in [50], where the main assumption is the following:

   - Let $W = \max_{1 \leq i \leq j \leq K} |f_j - f_i|$ $(f_1 < f_2 < \ldots < f_L)$ the range of the frequencies. Then, the undersampling frequencies $\min\{m_1, m_2, \ldots, m_L\} > 2W$. This ensures that there is no overlapping in the distinct remainders after undersampling.
   - Let $\alpha_{i,l}$ are the peaks at the DFT-output after undersampling at m_l, where $\alpha_{1,l} < \alpha_{2,l} < \ldots < \alpha_{K,l}$. Then, the difference between $\alpha_{i,l}$ is checked. If there exists i_0 such that $\alpha_{i_0+1,l} - \alpha_{i_0,l} > W$, then the remianders will have the following order:

     $$r_{1,l} = \alpha_{i_0+1,l} \ \ldots, r_{K-i_0} = \alpha_{K,l}, r_{K-i_0+1,l} = \alpha 1, l, \ldots, r_{K,l} = \alpha_{i_0,l} \tag{3.10}$$

     This process is repeated at each undersampled signal. each of the frequency components are recovered using the CRT formula in 2.4 following the obtained order of the remainders.

   The main drawback of this approach is that the allowed band is narrow with respect to the maximum possible dynamic range $DR = lcm\{m_1, m_2, \ldots, m_L\}$. However, a new method is proposed in [15] where no limitations have been issued on the frequency band, but a smaller dynamic range:

   $$DR = \max\{N_1, \ldots, N_K\} < \max\{m_{total}, m_1, \ldots m_L\} \tag{3.11}$$

   where $m_{total} = \prod m_1 m_2 \ldots m_K$, such that $m_1 < m_2 < \ldots < m_L$ Frequency estimation is done following these steps:

   (a) Let S_l the residue set at each m_l. $S_l \triangleq \{r_{1,l}, \ldots, r_{K,l}\}$.

   (b) Let $S \triangleq S_1 \times S_2 \times \ldots \times S_L$ the product set of all residues. Therefore S includes all possible combinations between residues in sets S_l. each entry of S is a L-dimensional vector.

   (c) Take a random vector $(d_1, \ldots, d_L)$ from S.

   (d) Let $N_l = \{d_l + nm_l : d_l \leq d_l + nm_l \leq DR, n \in \mathbb{N}\}$.

(e) Form the set $I \triangleq N_1 \cap \ldots \cap N_L$. Following *Lemma 1* in [15], if $I \neq \emptyset$, then I has only one element $I = \{\bar{N}\}$.

(f) Calculate the residue vector of $\bar{N} = \{\bar{d}_1, \ldots, \bar{d}_L\}$. If this vector belongs to S, then $\bar{N}$ is a valid frequency. If not, find another vector from S and repeat the above process.

(g) If $\bar{N}$ is a valid frequency, remove from sets S_l all its corresponding remainders. Then reform the set S and repeat the process until all frequencies are determined.

A recent approach has been proposed in [51], where two estimation algorithms were introduced. In this approach, the dynamic range is the largest possible ($DR = \prod_{l=1}^{L} = m_l$).The assumptions in the first algorithms are based on theorem 1 in [51]:

- The allowed band is $|\max_k(N_k) - \min_k(N_k)| < \min_l\{m_1, \ldots, m_L\} = m_1$ to avoid the duplication of the residues.
- $GCD(K, m_j) = 1$.
- $L \geq K$.

Therefore, the first estimation algorithm is as following:

(a) Let $B_j = \{r_{1,j}, \ldots, r_{K,j}\}$ sorted in ascending order. Calculate:

$$S_{i,j} \triangleq \sum_{\eta=1}^{K} \langle r_{\eta,j} - r_{i,j} \rangle_{m_j} \tag{3.12}$$

(b) Let $S_j = \{S_{i,j} : i = 1, \ldots, K\}$ for $1 \leq j \leq L$. Obtain $\cap_{j=1}^{L} S_j = \{C\}$

(c) Obtain the indices $i_1, i_2 \ldots, i_L$ such that $S_{i_1,1} = S_{i_2,2} = \cdots = S_{i_L,L} = C$. these indices should be unique in each residue set B_j.

(d) Let $d_{1,j} = r_{i_j,j}$. Therefore, N_1 can be reconstructed from the residues $r_{i_j,j}$ via the conventional CRT for a single integer.

(e) Calculate:

$$C_i = N_1 + \langle r_{i,1} - r_{i_1,1} \rangle_{m_1} \tag{3.13}$$

Then arrange C_i for $1 \leq i \neq i_1 \leq K$ in the ascending order. If $i_1 = 1$, then the remaining frequencies N_i can be obtained as $N_i = C_i$

In the second algorithm, the last assumption in theorem 1 ($L \geq K$) has changed to:

$$K^2 - K(m_1 + k) + (k-1)m_L > 0, \qquad \text{for } 2 \leq k \leq K \tag{3.14}$$

Hence, the second algorithm in [51] is:

(a) Calculate $S_{i,j}$ and obtain S_j as in the first algorithm.

(b) Find a unique minimum in S_L. Assume that $\min S_L = S_{i_L,L}$

(c) In each S_u for $1 \leq u \leq L-1$, find a unique $S_{i_u,u}$ such that $S_{i_u,u} = S_{i_L,L}$ and obtain i_u.

(d) Apply steps (d) and (e) in the first algorithm to recover the desired frequencies.

In all techniques above, the main condition so that all signal components can be recovered is the distinction of the residues at each undersampling frequency. In other words, if a signal has two components resulting the same residue at the output of a such DFT, the reconstruction cannot be done perfectly. To solve this issue, a novel estimation technique is proposed in [52].
The estimation algorithm in [52] benefits from **Viete Theorem** where:
Any N-degree polynomial $P(x) = a_N x^N + a_{N1} x^{N1} + \cdots + a_1 x + a_0$ is known to have N roots $\{X_1, X_2, ..., X_N\}$ by the fundamental theorem of algebra and relations between roots and coefficients are:

$$\sum_{i=1}^{N} X_i = -\frac{a_{N-1}}{a_N} = c_1 \tag{3.15}$$

$$\sum_{1\leq i<j\leq N} X_i X_j = \frac{a_{N-2}}{a_N} = c_2 \tag{3.16}$$

$$\vdots \tag{3.17}$$

$$\prod_{i=1}^{N} X_i = (-1)^N = \frac{a_0}{a_N} = c_N \tag{3.18}$$

The estimation algorithm in [52] works as following:

(a) Let $R_l = \{r_{1,l}, \ldots, r_{K,l}\}$ the residue set at sampling frequency m_l. Calculate its corresponding vector moduli m_l, $c_l = \{c_{1,l}, \ldots, c_{K,l}\}$ as following:

$$c_{1,l} = \langle \sum_{i=1}^{K} r_{i,l} \rangle_{m_l} \tag{3.19}$$

$$c_{2,l} = \langle \sum_{1\leq i<j\leq K} r_{i,l} r_{j,l} \rangle_{m_l} \tag{3.20}$$

$$\vdots \tag{3.21}$$

$$c_{K,l} = \langle \prod_{i=1}^{K} r_{i,l} \rangle_{m_l} \tag{3.22}$$

(b) Recover each elements of $c = \{c_1, \ldots, c_K\}$ using conventional CRT for one integer from its corresponding residue vector.

(c) Let $c_0 = 1$. Construct the polynomial $P(N) = \sum_{i=0}^{K}(-1)^i c_i N^{K-i}$

(d) Solve $P(N) = 0$. The solution gives K distinct integers which represent the desired frequency components.

Using this technique, frequency components with repeated residues can be recovered. However, the major drawback in this technique is its small dynamic range. The latter can be retrieved from the following equation:

$$\max_{i\in\{1,2\ldots,K\}} C_K^i G^i < \prod_{l=1}^{L} m_l \tag{3.23}$$

where G is the upper bound of frequencies N_i.
The last approach of using the Chinese Remainders Theorem in the estimation of unknown signal components in an error free environment is our preliminary proposal in [20].
In our paper, the estimation approach is based in the following assumptions:

(a) Consecutive signal components are separated by a known interval Δ such that:

$$N_{k+1} = N_k + \Delta, \qquad 1 \leq k \leq K-1. \tag{3.24}$$

(b) Δ is bounded by an upper value to avoid the overlap between the remainders, this bound is:

$$\Delta \leq \frac{min(m_i)}{K-1} \tag{3.25}$$

To solve the estimation problem with these conditions, it can be easily shown:

$$N_{k+1} \mod m_i = N_k \mod m_i + \Delta \mod m_i \tag{3.26}$$

$$r_{k+1,i} = (r_{k,i} + \Delta) \mod m_i \tag{3.27}$$

Hence, (3.26) can be used to develop the proposed estimation algorithm:

(a) At the output of each DFT after undersmapling by m_ℓ, detect the peaks and compute the difference between each two consecutive residues.

(b) If this difference between consecutive peaks is equal to Δ, then these two residues correspond to two consecutive frequencies. If not, then the latter represents the remainder of the first component in the signal, i.e. we should swap the remainders.

(c) The order of residues is the same of the signal components. Therefore, the recovery of each components could be then easy using (2.4).

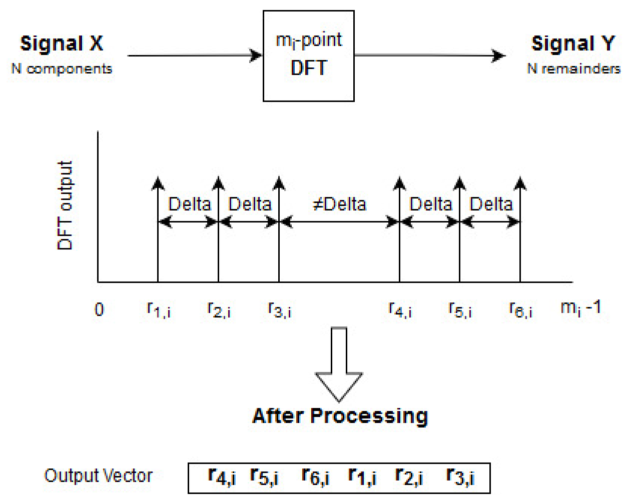

Figure 3.1: The proposed algorithm to solve CRT with multiple components.

For the sake of clarification, Figure 3.1 shows a simple example on the association between the frequency components and the reminders by the division over m_ℓ. This will lead to the following:

$$\begin{aligned} f_{1,\ell} &= r_{4,\ell} mod\ m_\ell, \\ f_{2,\ell} &= r_{5,\ell} mod\ m_\ell, \\ f_{3,\ell} &= r_{6,\ell} mod\ m_\ell, \\ f_{4,\ell} &= r_{1,\ell} mod\ m_\ell, \\ f_{5,\ell} &= r_{2,\ell} mod\ m_\ell, \\ f_{6,\ell} &= r_{3,\ell} mod\ m_\ell. \end{aligned}$$

In this approach, the dynamic range is:

$$DR = \prod_{\ell=1}^{L} m_\ell - 1 \tag{3.28}$$

This DR is equal to the one in single component estimation which is the maximum possible. However, the main drawback is the limited number of components with respect two the dynamic range and Δ. We will try to enhance it in this thesis.

2. **Noisy Signal Estimation:**
To estimate a noisy signal with multiple components using CRT, a robust estimation technique is needed in order to deal with erroneous remainders. In the literature, several robust techniques are proposed for robust CRT. One of the earliest techniques was proposed in [53]. It was an extension to that proposed in [50] and explained in the above previous section.
The dynamic range in [53] is:

$$DR = \max\{N_1, \ldots, N_K\} < \max\{m(\zeta), m_\gamma\} \tag{3.29}$$

where

$$m(\zeta) \triangleq m_1 m_2 \ldots m_\zeta \tag{3.30}$$

for $0 > \zeta < \eta$, where

$$\gamma = \eta K + \theta \tag{3.31}$$

for $0 < \theta < K$. Therefore, integers $\{N_k\}$ can be uniquely determined if the number of residue sets with errors $\tilde{S}_l$ satisfy

$$e \leq \frac{(\eta - \zeta)K + \theta}{2} \tag{3.32}$$

according to Theorem 1 in [53]. The determination algorithm is as following:

(a) S_l the residue set at each m_l. $S_l \triangleq \{r_{1,l}, \ldots, r_{K,l}\}$.

(b) Let $S \triangleq S_1 \times S_2 \times \ldots \times S_L$ the product set of all residues. Therefore S includes all possible combinations between residues in sets S_l. each entry of S is a L-dimensional vector.

(c) Take a random vector $(d_1, \ldots, d_L)$ from S. There is at least $L - e$ correct remainders in this residue vector.

(d) Let $N_l = \{d_l + nm_l : d_l \leq d_l + nm_l \leq m(\zeta), n \in \mathbb{N}\}$.

(e) From Equation (3.32), the number of correct remainders $L - e \geq \zeta K$. Since there are only K distinct integers, therefore, ζ is the minimum number of correct remainders sharing same integer in S.

(f) If the set $I \triangleq N_{l_1} \cap \ldots \cap N_{l_\zeta} \neq \emptyset$. then I has only one element $I = \{\bar{N}\}$.

(g) Calculate the residue vector of $\bar{N} = \{\bar{d}_1, \ldots, \bar{d}_L\}$. Let $\bar{L}$ the cardinality of the set $R \triangleq \{\bar{d}_l \in \tilde{S}_l : 1 \leq l \leq L\}$.

(h) If

$$\bar{L} \geq L - \frac{(\eta - \zeta)K + \theta}{2} = \frac{L}{2} + \frac{\zeta K}{2} \tag{3.33}$$

then $\bar{N}$ is a valid integer. If not, find another index set of size ζ such that $N_{l_1} \cap \ldots \cap N_{l_\zeta} \neq \emptyset = \{\bar{N}\}$.

(i) If $\bar{N}$ is a valid frequency, remove from sets S_l all its corresponding remainders. Then reform the set S and repeat the process until all frequencies are determined.

Other estimation technique was proposed in [54]. It was assumed that the maximum error bound δ satisfies $\delta < \Gamma/4$ such that the undersampling frequencies $m_l = \Gamma M_l$for $l = 1 \ldots L$. where $\{M_l\}$ are pair-wisely co-prime. The proposed estimation algorithm is:

(a) Calculate $\tilde{r}_l^c = \langle \sum_{k=1}^{K} \tilde{r}_{kl} \rangle_\Gamma \quad (l = 1 \ldots L)$

(b) Let $a = \min_l \tilde{r}_l^c$ and $b = \max_l \tilde{r}_l^c$. Calculate $\tilde{q}_l$ and $\tilde{\omega}_l$ as following:
If $b - a \leq 2\delta$, then:

$$\tilde{q}_l = \lfloor \frac{\tilde{r}_l}{\Gamma} \rfloor; \qquad \omega_l = \tilde{r}_l^c \tag{3.34}$$

Under the condition of $b - a \geq \Gamma - 2\delta$. If $\tilde{r}_l^c \in [0, 2\delta)$, then:

$$\tilde{q}_l = \lfloor \frac{\tilde{r}_l}{\Gamma} \rfloor; \qquad \omega_l = \tilde{r}_l^c \tag{3.35}$$

and if $\tilde{r}_l^c \in [\Gamma - 2\delta, \Gamma)$, then:

$$\tilde{q}_l = \lfloor \frac{\tilde{r}_l}{\Gamma} \rfloor + 1; \qquad \omega_l = \tilde{r}_l^c - \Gamma \tag{3.36}$$

(c) Reconstruct $\tilde{Q}$ with $\tilde{q}_l$ as residues using conventional CRT.

(d) Determine $\tilde{\tilde{X}}$ as:

$$\tilde{\tilde{X}} = \tilde{Q}\Gamma + \left[\frac{\sum_{l=1}^{L} \tilde{\omega}_l}{L} \right] \tag{3.37}$$

(e) Calculate $\theta = \langle \tilde{\tilde{X}} \rangle_{m_s}$, where $m_s = \max\{m_l : l = 1, \ldots, L\}$.

(f) Let $\Omega = \{\lambda_{ks}^1 = \tilde{r}_{ks}, \lambda_{ks}^2 = \tilde{r}_{ks} - m_s, \lambda_{ks}^3 = \tilde{r}_{ks} + m_s | k = 1, \ldots, K\}$. Select elements from Ω satisfying $|\theta - \lambda_{ks}^i| < m_s$. Those elements from the set $\{\kappa_1, \ldots, \kappa_h\}$ in an ascending order.

(g) Find $\Phi_\gamma = \sum_{i=\gamma}^{K+\gamma-1} \kappa_i = \tilde{Q}\Gamma + \tilde{\omega}_s - K\tilde{\tilde{X}} + K\lambda$ for some $\gamma \in \{1, 2, \ldots, h - K + 1\}$.

(h) Estimate desired frequencies as $\tilde{X}_k = \tilde{\tilde{X}} - \theta + \kappa_{\gamma+k-1}$.

The latest robust CRT technique was proposed in [55]. It follows the same assumptions on undersampling frequencies $\{m_l = \Gamma M_l, l = 1, \ldots, L\}$ as in [54]. However, its upper error bound of the remainders is tighter than that in [54] $\delta < \frac{\Gamma}{4K}$.
Given the erroneous remainders $\{\tilde{r}_{kl} | k = 1, \ldots, K.l = 1, \ldots, L\}$, unknown frequency components were estimated in [55] by applying the following steps:

(a) Calculate $\tilde{r^c}_{kl} = \langle \tilde{r}_{kl} \rangle_\Gamma$ and arrange then in ascending order in the set $\Omega = \{\gamma_1, \dots, \gamma_\kappa\}$, where $\kappa \leq KL$

(b) According to Lemma 4 in [55], there exists $\zeta \in \{1, \dots, \kappa\}$ such that:

$$\gamma_{\langle \zeta+1 \rangle} - \gamma_{\langle \zeta \rangle} + \Gamma \mathbf{1}(\zeta = \kappa) > 2\delta \tag{3.38}$$

Obtain $\hat{r}^c_{kl}$ from $\tilde{r^c}_{kl}$ as following:
If $\zeta = \kappa$, then

$$\hat{r}^c_{kl} = \tilde{r^c}_{kl} \tag{3.39}$$

when $\zeta \neq \kappa$. If $\tilde{r^c}_{kl} \leq \gamma_\zeta$ then

$$\hat{r}^c_{kl} = \tilde{r^c}_{kl} \tag{3.40}$$

otherwise

$$\hat{r}^c_{kl} = \tilde{r^c}_{kl} \tag{3.41}$$

(c) Calculate $\tilde{q}_{kl} = \langle \frac{\tilde{r}_{kl} - \hat{r^c}_{kl}}{\Gamma} \rangle_{M_l}$.

(d) Calculate $\langle \sum_{k=1}^{K} \tilde{q}_{kl} \rangle_{M_l}$ for $l \in \{1, \dots, L\}$ and recover $\tilde{q} = \frac{\sum_{k=1}^{K} \tilde{q}_k}{K}$ using conventional CRT.

(e) Calculate $\langle \tilde{q} \rangle_{M_l}$

(f) Calculate $\langle \{\alpha_1, \dots, \alpha_K\} \rangle_{M_l}$ for $l \in \{1, \dots, L\}$, where $\{\alpha_k\}$ are the Viete coefficients of $\tilde{q}_k - \tilde{q}$ for $k \in \{1, \dots, K\}$. They can be calculated using Equations (3.15 - 3.18).

(g) Recover $\{\alpha_1, \dots, \alpha_K\}$ with CRT.

(h) Let $\alpha_0 = 1$. Construct the polynomial $P(x) = \sum_{k=0}^{K} = (-1)^k \alpha_k x^{K-k}$.

(i) Solve $P(x) = 0$ and get the roots $\{\tilde{q}_k - \tilde{q}$, for $k \in \{1, \dots, K\}\}$. Therefore $\tilde{q}_k$ are obtained.

(j) The desired components are: $\tilde{f}_k = \tilde{q}_k \Gamma + \frac{\sum_{l=1}^{L} \hat{r}^c_{kl}}{L}$

In [56], the same estimation approach is used with relaxation on the number of errors upper bounded by δ. It is allowed to have unbounded errors under the following conditions:

- Given L co-prime moduli, $(m_1, \dots, m_L)$ sorted in ascending order, the unknown frequencies are in the range $[0, \prod_{l=1}^{K} m_l,\ K \leq L$.
- The upper bound of the number of unbounded errors is: $\lfloor \frac{L-K}{2} \rfloor$

Under these conditions, unknown frequencies can be estimated using the robust CRT for multiple numbers (RCRTMN) for arbitary errors in [56].

Chapter 4

Proposed Estimation Technique

In this chapter, our proposed signal estimation technique will be explained, and thereafter, we will discuss the simulation results. This technique is an expansion of that published in [20], where we divide the full signal into multi bands inspired from the Time Interleaved ADC technique in [57].

4.1 System Model

Let $x(t)$ a noisy K-sparse signal containing K frequency components.

$$x(t) = \sum_{k=1}^{K} s_k e^{j2\pi f_k t} + w(t) \tag{4.1}$$

Consider a noisy complex signal $x(t)$ containing K frequency components with unknown amplitudes and phases, with additive white Gaussian noise $w(t)$ with zero mean and variance σ^2:

$$x(t) = \sum_{k=1}^{K} s_k e^{j2\pi f_k t} + w(t) \tag{4.2}$$

In equation (4.2), f_k is the k^{th} frequency and s_k is the corresponding amplitude. In practice, one of the major problems is to determine the frequency components of $x(t)$. Hence, if the bandwidth of the signal can be estimated or known, Nyquist sampling theorem will be imposed. In this case, the signal $x(t)$ has to be sampled at the sampling frequency (at least equal to Nyquist rate) $f_s = m\ Hz$ and then appropriate power spectral density methods have to be applied. The signal at the sampling frequency $f_s\ Hz$ yields:

$$x_m[n] = x(\frac{n}{m}) = \sum_{k=1}^{K} s_k e^{j2\pi f_k n/m} + w[n/m] \tag{4.3}$$

Without loss of generality, the frequency components of the signal $x_m[n]$ can be directly obtained from the m-point DFT or the PSD (Welch method) of $x_m[n]$. Definitely, this method works perfectly when the sampling frequency is above the Nyquist rate and the signal is a narrow-band signal resulting in a low frequency rate. However, for wide and ultra-wide bands, sampling at Nyquist rate becomes a bottleneck imposing some constraints on the quantizer when the latter is needed.
To overcome this problem, different approaches have been proposed to undersample the signal at a sampling frequency lower than the Nyquist rate resulting in processing fewer samples. However, undersampling creates frequency overlapping and a cross-correlation between the signal and the noise components. Hence, the recovery of the signal components is not straightforward approach anymore. From the technical point of view, the spectral schematic of a signal sampled at a rate lower than Nyquist frequency will include frequencies that have modulo m ambiguities. To recover the frequency components, the literature has shown that the signal has to be sampled at several undersampling frequencies m_l with $1 \leq l \leq L$. In this case, the undersampled signal can be written as:

$$x_{m_l}[n] = x(\frac{n}{m_l}) = \sum_{k=1}^{K} s_k e^{j2\pi f_k n/m_l} + w[n/m_l] \tag{4.4}$$

The output of the m_l- point DFT of x_{m_l} gives the residues $r_{k,l}$ of the frequency components f_k by the corresponding sampling frequencies m_l [19]:

$$\begin{aligned} f_k &= n_{k,l}\ m_l + r_{k,l} \\ r_{k,l} &= f_k \mod m_l \end{aligned}$$

4.2 Assumptions and Algorithm

Inspired by the TIADC architecture at sub-Nyquist rates in [57], we propose the following signal estimation approach (Schema in Figure 4.2):

1. The signal $x(t)$ is the input of M bandpass filters (BPF). There is no aliasing between the pass-bands of each of these filters. Filtered signals are then converted to baseband.

2. Each of signals $x_i(t)$ ($i = 1, \ldots, M$) is the input of L Analog-to-Digital Converters (ADCs).

3. The output of each ADC $y_{i,l}$ ($l = 1, \ldots, L$) is a sub-sampled signal. Then, in each band, these signals are the inputs of the CRT block, which is explained in details below.

4. The output of the CRT block is the estimated signal $x_{i,est}$ in each band

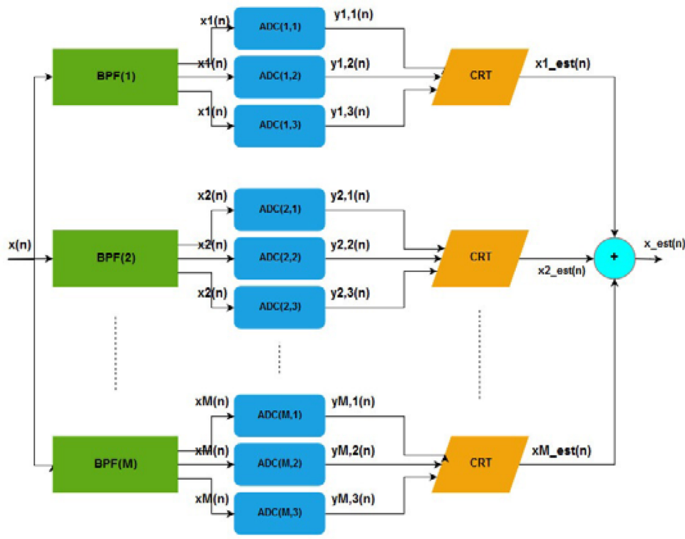

Figure 4.1: Schema of proposed estimation and reconstruction technique.

Since our objective is to estimate the signal components using the Chinese Remainder Theorem (CRT) at sub-Nyquist rates, and to fulfill this objective we made the following assumptions and conditions:

1. The sampling rates $m_l (l = 1, \ldots, L)$ must be pair-wisely co-prime to meet CRT requirements. Therefore, the dynamic range of the frequencies is: $DR = \prod_{l=1}^{L} m_l - 1$.

2. The width of each subband $BW_i \leq DR \qquad (i = 1, \ldots, M)$

3. Consecutive signal components are separated by an interval Δ such as:

$$f_{k+1} = f_k + \Delta_i, \qquad 1 \leq k \leq K_i - 1 \tag{4.5}$$

   where K_i is the number of components within the subband i

4. The upper bound of Δ_i in subband i is: $\Delta_i \leq (\min m_l)/(K_i - 1)$

4.3 CRT block

The CRT block, shown in Figure 4.2, holds our proposed technique of signal estimation and reconstruction. It takes subsampled signals (ADCs output) as inputs and returns at the output the detected frequency components along with the reconstructed signal.
The CRT block consists of the following blocks: **SSA, Peaks Estimation, Delta Estimation**, and **CRT**. Each of which will be explained below:

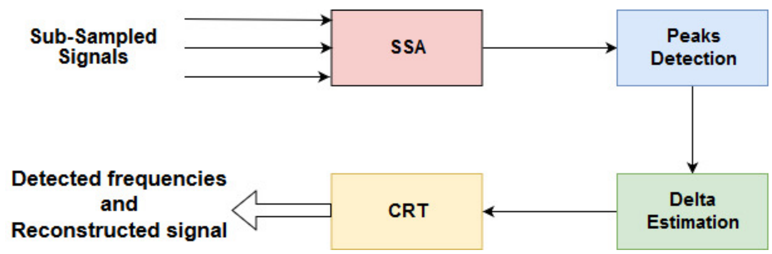

Figure 4.2: CRT block in details.

4.3.1 Singular Spectrum Analysis (SSA)

Input: Subsampled Signals.
Output: Number of frequency components.
The main role of the SSA block is to estimate the number of frequency components of the signal. The estimation process is done by applying the singular spectrum analysis technique cited in [58] as following:

1. For each subsampled signal $y_{i,l}$, create its Hankel matrix:

$$H_l = \begin{bmatrix} y_{i,l}[M-1] & y_{i,l}[M] & \dots & y_{i,l}[N-1] \\ y_{i,l}[M-2] & y_{i,l}[M-1] & \dots & y_{i,l}[N-2] \\ \vdots & \vdots & \ddots & \vdots \\ y_{i,l}[0] & y_{i,l}[1] & \dots & y_{i,l}[N-M] \end{bmatrix} \tag{4.6}$$

   H_l is $M \times B$, N is the signal length, M is the frame length, and $B = N - M + 1$

2. Calculate the normalized autocorrelation matrix:

$$R_{H,l} = H_l H_l^T / B \tag{4.7}$$

   and its eigenvalue decomposition:

$$R_{H,l} = U \Lambda_l U^T \tag{4.8}$$

   where $\Lambda_l = diag(\lambda_{1,l}, \dots, \lambda_{M,l})$. However, the normalized eigenvalues $\Lambda_{Norm,l} = diag(\lambda_{1,l}/M, \dots, \lambda_{M,l}/M)$ are used in this technique

3. Based on our assumptions, the maximum possible number of components in a signal K_{max} could be: $K_{max} = \min(m_l) \quad (L = 1, \dots, L)$. Then, to obtain the exact number of signal components we perform the following steps:

   - Take the maximum K_{max} eigenvalues from $\Lambda_{Norm,l}$, $(L = 1, \dots, L)$ and sort then in the descending order.

- Calculate the difference between each two consecutive eigenvalues in $\Lambda_{Norm,l}$ and put them in $\Lambda_{l,diff} = \{\lambda_{Norm,i} - \lambda_{Norm,i+1} \mid \lambda_{Norm,i} \in \Lambda_{Norm,l} \mid i = 1, \ldots, K_{max} - 1\}$.
- Calculate the ratio set $\Lambda_{ratio,l}$ of each two consecutive elements in $\Lambda_{l,diff}$.
  $\Lambda_{ratio,l} = \{\lambda_{ratio,d,l} = \lambda_{j,diff}/\lambda_{j+1,diff} \mid \lambda_j \in \Lambda_{diff,l} \mid j = 1, \ldots, K_{max} - 2\}$.
- To separate between the signal subspace and the noise subspace: Let $d = 1$. If $\lambda_{ratio,d,l} <<< \lambda_{ratio,d+1,l}$ $(l = 1, \ldots, L)$, then, the estimated number of components $\hat{K} = d + 1$. Otherwise, make $d = d + 1$ and repeat the process until the condition is satisfied.

4.3.2 Peaks Detection

Input: Subsampled Signals and Estimated Number of components $\hat{K}$.
Output: Remainders and Magnitude.
After estimating the number of components in the signal, the role of the **Peaks Detection** block is to detect the remainders from each sub-sampled signal $y_l[n]$. To do so, the following steps are applied:

1. Get $Y_l(f)$ the FFT of each subsampled signal $y_l[n]$
2. Detect the indices of $\hat{K}$ peaks with maximum magnitude. The indices represent the remainders $\hat{R}_l = \{\hat{r}_{i,l} \mid i = 1, \ldots, \hat{K}\}$, $(L = 1, \ldots, L)$ and the magnitude sets $\hat{s}_l$ represents the power of the signal components.

4.3.3 Delta Estimation

Input: Remainders sets $\hat{R}_l$.
Output: Delta $\hat{\Delta}$, and Remainders difference $\hat{R}_{diff,l}$.
Since our assumption is based on the existence of a fixed separation between signal components Δ, the role of this block is to calculate the estimation $\hat{\Delta}$ of Δ by applying these steps:

1. Calculate the difference between each two consecutive remainders in each set $\hat{R}_l$ and put them into set $\hat{R}_{diff,l} = \{\hat{r}_{i,l} - \hat{r}_{i+1,l} \mid i = 1, \ldots, \hat{K} - 1\}$, $(L = 1, \ldots, L)$
2. Get the intersection $\cap_{l=1}^{L} \hat{R}_{diff,l}$.
3. If the intersection is singleton, then $\hat{\Delta} = \cap_{l=1}^{L} \hat{R}_{diff,l}$. Otherwise, $\hat{\Delta} = \min \cap_{l=1}^{L} \hat{R}_{diff,l}$

4.3.4 Chinese Remainder Theorem (CRT)

Input: Magnitude $\hat{s}_l$, Remainders $\hat{R}_l$, Remainders difference $\hat{R}_{diff,l}$, Delta $\hat{\Delta}$, and Number of components $\hat{K}$.
Output: Detected frequencies $\hat{f}$ and Reconstructed Signal $\hat{x}$.
In this block, the values of frequency components are estimated using our algorithm in [20]:

1. Apply the algorithm in [20] shown in Figure 3.1 above to sort the remainders $\hat{R}_l$ in order along with their magnitude $\hat{s}_l$. Remainders difference $\hat{R}_{diff,l}$ are used to determine the correct order.

2. Apply the conventional CRT to calculate the desired frequencies $\hat{f}$ from the sorted remainders sets.

3. Having the values of frequency components and their corresponding magnitudes, the signal $\hat{x}$ can be reconstructed and regenerated.

4.4 Results and Discussion

In this section, we show the simulation results to validate our proposed technique. Firstly, we studied the effect of the dividing the signal into sub-band through band-pass filters. Table 4.1 contains the simulation parameters. Three undersampling frequencies are used along with three signal components per sub-band. Two cases are considered, the first one is without filtering when the signal has one band only and therefore, no need for filtering. The second case is when filtering applied and the signal is divided into two sub-bands.

Parameters	**Values**
Undersampling Frequencies	$[19, 23, 31]$ Hz
Frequency components per sub-band	3
Number of Sub-bands	$1, 2$
SNR Values	$-5, -10$ dB

Table 4.1: CRT Simulation Parameters

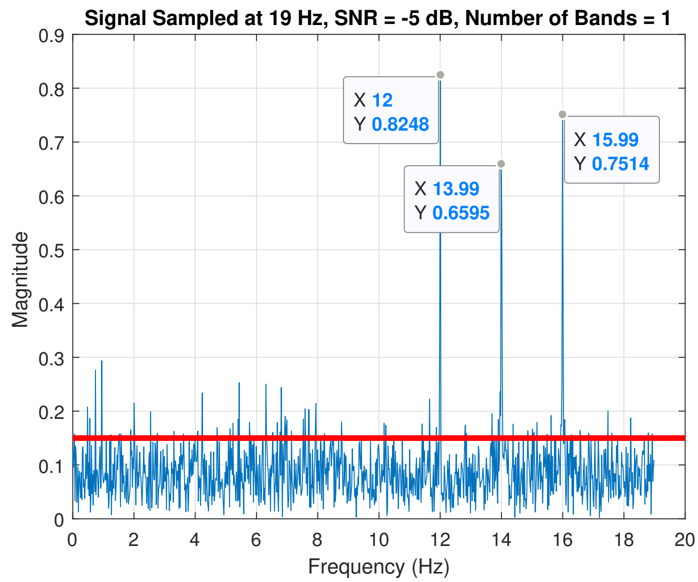

Figure 4.3: Signal Sampled at 19 Hz without filtering, SNR = -5 dB.

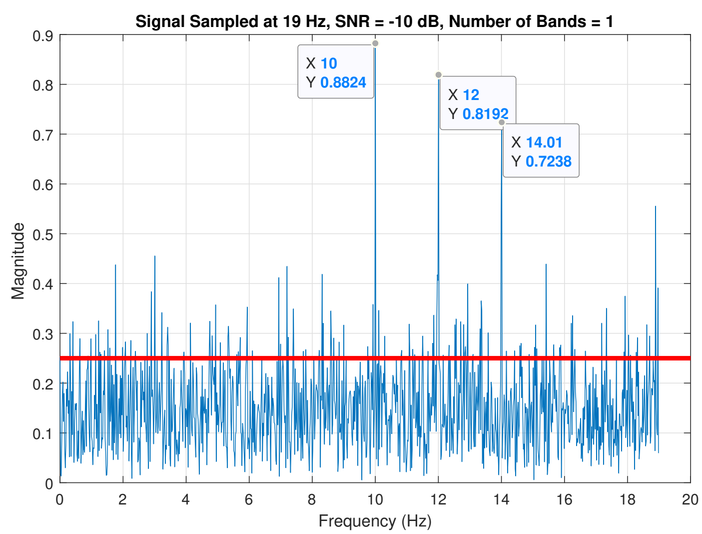

Figure 4.4: Signal Sampled at 19 Hz without filtering, SNR = -10 dB.

Figures 4.3 and 4.4 show the plot at the FFT output of the signal sampled at 19 Hz without filtering (One sub-band) for $SNR = [-5, -10]$ dB respectively, and Figures 4.5 and 4.6 show the same output at same SNR values using filtering

(Two sub-bands). The red line in each plot represents the same constant value per SNR. They are used to compare the effect of the filtering on the noise level. It is noticed that the noise level of a filtered signal is lower than that of non-filtered signal. Therefore, by filtering, the effect of noise is reduced since only a part of the signal is processed, and therefore, the peaks can be still detected correctly at lower SNR values much easier than unfiltered signal.

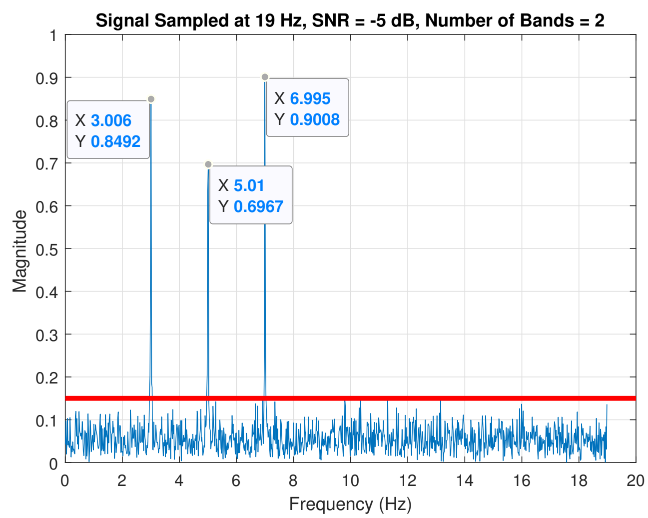

Figure 4.5: Signal Sampled at 19 Hz with filtering, SNR = −5 dB.

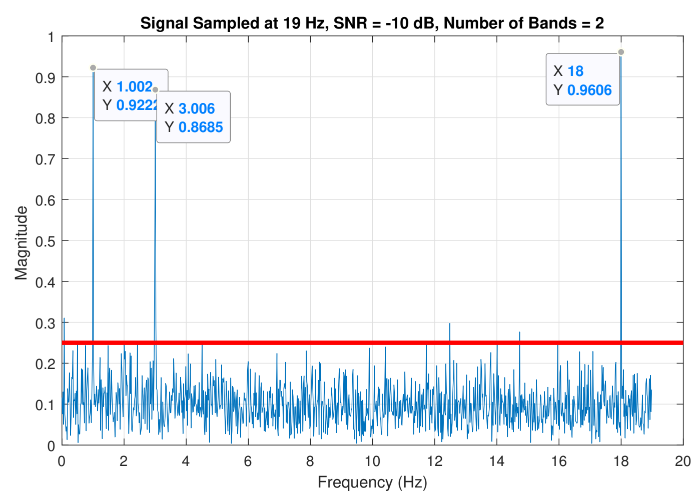

Figure 4.6: Signal Sampled at 19 Hz with filtering, SNR = -10 dB.

Parameters	Values
Undersampling Frequencies	$[29, 31, 37]$ Hz
Dynamic Range	$29 \times 31 \times 37 = 33263$ Hz
Sub-band Bandwidth	33263 Hz
Total Number of Components	12
Δ	2 Hz
Number of Sub-bands	$[1, 2, 3, 4]$

Table 4.2: Monte Carlo Simulation Parameters

Table 4.2 shows the parameters of the Monte Carlo simulation done to study the performance of the proposed technique. This simulation is done with support of the High Performance computing (HPC) at the American University of Beirut (AUB). The objective is to evaluate the probability of detection (POD) and the mean squared error (MSE) of the proposed technique with respect to several SNR values, and different number of sub-bands. Three undersampling frequencies are considered, with total number of signal components of 12. The number of components per sub-band depends of how many sub-bands the signal is divided to, i.e.: the number of components per sub-band $K - i$, $1 \leq i \leq N$ follows: $K_i = \frac{K}{N}$ where K is the total number of components, and N is the number of sub-bands. The separation Δ between consecutive signal components is 2 Hz.

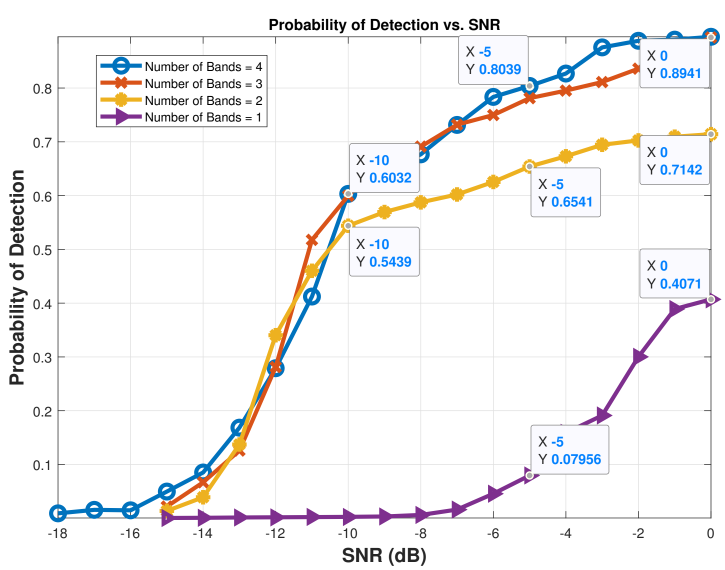

Figure 4.7: Probability of Detection against SNR values with different number of bands.

Figures 4.7 and 4.8 show the results of the Monte Carlo simulation of the proposed technique in terms of probability of detection (POD) and mean squared error (MSE) respectively. It is clearly noticed how filtering enhances the estimation process. Without filtering (One sub-band), the POD reaches 0.4 at $SNR = 0\ dB$, whereas, with filtering, this value is easily obtained at SNR values around $-12\ dB$. Moreover, increasing the number of sub-bands also improves the POD and reduces the MSE (i.e. Number of Bands = 4), since the larger the number of sub-bands means less components per sub-band, and then less number of samples are needed to perform correct estimation.

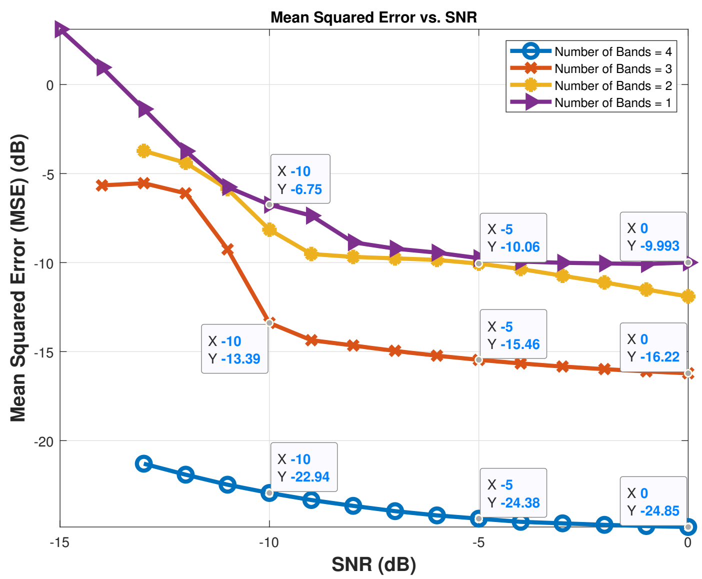

Figure 4.8: Mean Squared Error against SNR values with different number of bands.

			MSE(dB)			POD		
SNR(dB)			**-10**	**-5**	**0**	**-10**	**-5**	**0**
Paper [20] (3 Freqs)			-10	-19	-25	0	0.2	0.9
Proposed Technique	**Number of Bands (Freqs per Band)**	1 (12 Freqs)	-7	-10	-10	0	0.08	0.4
		2 (6 Freqs)	-8	-10	-12	0.54	0.65	0.71
		3 (4 Freqs)	-13	-15	-16	0.6	0.78	0.88
		4 (3 Freqs)	-23	-24	-25	0.6	0.8	0.89

Table 4.3: Performance comparison between the proposed technique and a previous work.

Table 4.3 shows a performance comparison between the proposed technique in this thesis, and these obtained in paper [20]. This paper is chosen because the same CRT estimation approach (existence of interval Δ) is used in both works. The comparison between similar situations (3 components) in terms of POD and MSE shows the huge improvements of our proposed technique.
At $SNR = -10$ and -5 dB, 60% and 80% of signals can be estimated correctly, whereas in [20] the POD was 0% and 20% respectively. Moreover, at same SNR values, the MSE improved from -10 dB and -19 dB to -23 dB and -24 dB respectively.

Chapter 5

Conclusion

In this thesis, a new signal estimation and reconstruction technique is proposed. It is based on dividing the signal into subbands, then sampling each subband at sub-Nyquist rates, and estimating the frequency components using Chinese Remainders Theorem (CRT) given that these components are organized in a particular pattern. This work is the first integrated signal estimation technique using Chinese Remainder Theorem (CRT).

Firstly, we started by a review about Residue Number Systems (RNS). Thereafter, we provided and extensive description of main CRT recovery algorithms covering all possible cases (one or multiple components, noise-free or noisy environments). After that, we explained our proposed technique in details, and we provided a description on the role of each block in our proposed system. Finally, we validated our approach by using Monte Carlo simulations that gave reasonable results in terms of probability of detection (POD) and mean squared error (MSE).

As future work, this technique can be enhanced from multiple perspectives. One interesting point to think about would be how to solve the overlap problem between the remainders and therefore upper bound of the number of components in one band.

Appendix A

Abbreviations

ADC	Analog to Digital Converter
CRT	Chinese Remainders Theorem
DFT	Discrete Fourier Transform
FFT	Fast Fourier Transform
MRC	Matrix Radix Conversion
MSE	Mean Squared Error
MUSIC	Multiple Signal Classification
POD	Probability of Detection
RNS	Residue Number Systems
SSA	Singular Spectrum Analysis
SNR	Signal to Noise Ratio
TIADC	Time Interleaved Analog to Digital Converter

Bibliography

[1] O. Besson, Y. Abramovich, and B. Johnson, "Direction-of-arrival estimation in a mixture of K-distributed and Gaussian noise," *Signal Processing*, vol. 128, pp. 512 – 520, 2016.

[2] R. Arablouei, K. Doanay, and S. Werner, "Adaptive frequency estimation of three-phase power systems," *Signal Processing*, vol. 109, pp. 290 – 300, 2015.

[3] T. Funada, "A method for the extraction of spectral peaks and its application to fundamental frequency estimation of speech signals," vol. 13, pp. 15–28, 07 1987.

[4] D. Belega and D. Petri, "Frequency estimation by two- or three-point interpolated Fourier algorithms based on cosine windows," *Signal Processing*, vol. 117, pp. 115 – 125, 2015.

[5] C. Candan, "Fine resolution frequency estimation from three DFT samples: Case of windowed data," *Signal Processing*, vol. 114, pp. 245 – 250, 2015.

[6] R. Schmidt, "Multiple emitter location and signal parameter estimation," *IEEE Transactions on Antennas and Propagation*, vol. 34, pp. 276–280, Mar 1986.

[7] R. Roy and T. Kailath, "ESPRIT-estimation of signal parameters via rotational invariance techniques," *IEEE Transactions on Acoustics, Speech, and Signal Processing*, vol. 37, pp. 984–995, Jul 1989.

[8] D. Zachariah, P. Wirflt, M. Jansson, and S. Chatterjee, "Line spectrum estimation with probabilistic priors," *Signal Processing*, vol. 93, no. 11, pp. 2969 – 2974, 2013.

[9] Y.-Q. Tu and Y.-L. Shen, "Phase correction autocorrelation-based frequency estimation method for sinusoidal signal," *Signal Processing*, vol. 130, pp. 183 – 189, 2017.

[10] S. Rangan, T. S. Rappaport, and E. Erkip, "millimeter-wave cellular wireless networks: Potentials and challenges," *Proceedings of the IEEE.*

[11] J. Mo, P. Schniter, N. G. Prelcic, and R. W. Heath, “Channel estimation in millimeter wave MIMO systems with one-bit quantization,” in *2014 48th Asilomar Conference on Signals, Systems and Computers*, pp. 957–961, Nov 2014.

[12] Y. Dong, C. Chen, and Y. Jin, “AoAs and AoDs estimation for sparse millimeter wave channels with one-bit ADCs,” in *2016 8th International Conference on Wireless Communications Signal Processing (WCSP)*, pp. 1–5, Oct 2016.

[13] M. D. Zoltowski and C. P. Mathews, “Real-time frequency and 2-D angle estimation with sub-Nyquist spatio-temporal sampling,” in *1993 IEEE International Conference on Acoustics, Speech, and Signal Processing*, vol. 4, pp. 117–120 vol.4, April 1993.

[14] D. W. Tufts and H. Ge, “Digital estimation of frequencies of sinusoids from wide-band under-sampled data,” in *1995 International Conference on Acoustics, Speech, and Signal Processing*, vol. 5, pp. 3155–3158 vol.5, May 1995.

[15] X. G. Xia, “An efficient frequency-determination algorithm from multiple undersampled waveforms,” *IEEE Signal Processing Letters*, vol. 7, pp. 34–37, Feb 2000.

[16] M. Mishali and Y. C. Eldar, “From Theory to Practice: Sub-Nyquist Sampling of Sparse Wideband Analog Signals,” *IEEE Journal of Selected Topics in Signal Processing*, vol. 4, pp. 375–391, April 2010.

[17] L. Wang, L. Zhao, G. Bi, C. Wan, L. Zhang, and H. Zhang, “Novel Wideband DOA Estimation Based on Sparse Bayesian Learning With Dirichlet Process Priors,” *IEEE Transactions on Signal Processing*, vol. 64, pp. 275–289, Jan 2016.

[18] J. A. Tropp, J. N. Laska, M. F. Duarte, J. K. Romberg, and R. G. Baraniuk, “Beyond Nyquist: Efficient Sampling of Sparse Bandlimited Signals,” *IEEE Transactions on Information Theory*, vol. 56, pp. 520–544, Jan 2010.

[19] S. Huang, H. Zhang, H. Sun, L. Yu, and L. Chen, “Frequency estimation of multiple sinusoids with three sub-Nyquist channels,” *Signal Processing*, vol. 139, pp. 96 – 101, 2017.

[20] M. Saab, Y. Nasser, and K. Y. Kabalan, “Frequency Estimation of Multiple Components using Chinese Remainder Theorem,” in *2018 25th International Conference on Telecommunications (ICT)*, pp. 610–616, June 2018.

[21] N. S. Szabo and R. I. Tanaka, *Residue Arithmetic and Its Applications to Computer Technology.* New York: McGraw-Hill Book Company, 1967.

[22] M. A. Soderstrand, W. K. Jenkins, G. A. Jullien, and F. J. Taylor, eds., *Residue Number System Arithmetic: Modern Applications in Digital Signal Processing*. Piscataway, NJ, USA: IEEE Press, 1986.

[23] Y. Wang, "Residue-to-binary converters based on new chinese remainder theorems," *IEEE Transactions on Circuits and Systems II: Analog and Digital Signal Processing*, vol. 47, pp. 197–205, Mar 2000.

[24] A. Hiasat, "A residue-to-binary converter for the extended four-moduli set $2^n - 1, 2^n + 1, 2^{2n} + 1, 2^{2n+p}$," *IEEE Transactions on Very Large Scale Integration (VLSI) Systems*, vol. 25, pp. 2188–2192, July 2017.

[25] N. B. Chakraborti, J. S. Soundararajan, and A. L. N. Reddy, "An implementation of mixed-radix conversion for residue number applications," *IEEE Transactions on Computers*, vol. C-35, pp. 762–764, Aug 1986.

[26] C. H. Huang, "A fully parallel mixed-radix conversion algorithm for residue number applications," *IEEE Transactions on Computers*, vol. C-32, pp. 398–402, April 1983.

[27] H. M. Yassine and W. R. Moore, "Improved mixed-radix conversion for residue number system architectures," *IEE Proceedings G - Circuits, Devices and Systems*, vol. 138, pp. 120–124, Feb 1991.

[28] K. Gbolagade and S. Cotofana, "Mrc technique for rns to decimal conversion using the moduli set 2n + 2, 2n + 1, 2n," in *Proc. 19th Annual Workshop on Circuits, Systems and Signal Processing*, (Veldhoven, The Netherlands), pp. 318–321, November 2008.

[29] Y. Wang, "New chinese remainder theorems," in *Conference Record of Thirty-Second Asilomar Conference on Signals, Systems and Computers (Cat. No.98CH36284)*, vol. 1, pp. 165–171 vol.1, Nov 1998.

[30] J. A. Naranjo, J. Lopez-Ramos, and L. G. Casado, "Applications of the extended euclidean algorithm to privacy and secure communications," pp. 702–, 06 2010.

[31] B. Cao, C. H. Chang, and T. Srikanthan, "A residue-to-binary converter for a new five-moduli set," *IEEE Transactions on Circuits and Systems I: Regular Papers*, vol. 54, pp. 1041–1049, May 2007.

[32] C. H. Chang and J. Y. S. Low, "Simple, fast, and exact rns scaler for the three-moduli set $\{2^n - 1, 2^n, 2^n + 1\}$," *IEEE Transactions on Circuits and Systems I: Regular Papers*, vol. 58, pp. 2686–2697, Nov 2011.

[33] A. Hiasat, "A reverse converter and sign detectors for an extended rns five-moduli set," *IEEE Transactions on Circuits and Systems I: Regular Papers*, vol. 64, pp. 111–121, Jan 2017.

[34] A. Hiasat, "An efficient reverse converter for the three-moduli set ($2^{n+1} - 1, 2^n, 2^n - 1$)," *IEEE Transactions on Circuits and Systems II: Express Briefs*, vol. 64, pp. 962–966, Aug 2017.

[35] P. A. Ramamoorthy and B. Potu, "High-speed adc using residue number system," in *International Conference on Acoustics, Speech, and Signal Processing,*, pp. 1063–1066 vol.2, May 1989.

[36] C. H. Vun and A. B. Premkumar, "Rns encoding based folding adc," in *2012 IEEE International Symposium on Circuits and Systems*, pp. 814–817, May 2012.

[37] D.-M. Pham, A. Premkumar, D.-H. Ngo, and A. Madhukumar, "Preprocessing for analog signal conversion using novel number system," *Digital Signal Processing*, vol. 21, no. 1, pp. 150 – 161, 2011.

[38] J. A. Esparza and P. E. Pace, "A preprocessing architecture for resolution enhancement in high speed analog-to-digital converters," in *1993 IEEE International Symposium on Circuits and Systems*, pp. 1140–1143 vol.2, May 1993.

[39] P. E. Pace, D. Styer, and W. P. Ringer, "An optimum sns-to-binary conversion algorithm and pipelined field-programmable logic design," vol. 47, pp. 736 – 745, 09 2000.

[40] P. E. Pace, D. Styer, and I. A. Akin, "A folding adc preprocessing architecture employing a robust symmetrical number system with gray-code properties," *IEEE Transactions on Circuits and Systems II: Analog and Digital Signal Processing*, vol. 47, pp. 462–467, May 2000.

[41] E. Lin and L. Monte, "Joint frequency and angle of arrival estimation using the chinese remainder theorem," in *2017 IEEE Radar Conference (RadarConf)*, pp. 1547–1551, May 2017.

[42] X. Li, H. Liang, and X. Xia, "A robust chinese remainder theorem with its applications in frequency estimation from undersampled waveforms," *IEEE Transactions on Signal Processing*, vol. 57, pp. 4314–4322, Nov 2009.

[43] X. Li, W. Wang, W. Zhang, and Y. Cao, "Phase-detection-based range estimation with robust chinese remainder theorem," *IEEE Transactions on Vehicular Technology*, vol. 65, pp. 10132–10137, Dec 2016.

[44] L. Xiao, X. Xia, and W. Wang, "Multi-stage robust chinese remainder theorem," *IEEE Transactions on Signal Processing*, vol. 62, pp. 4772–4785, Sep. 2014.

[45] W. Wang, X. Li, W. Wang, and X. Xia, "Maximum likelihood estimation based robust chinese remainder theorem for real numbers and its fast algorithm," *IEEE Transactions on Signal Processing*, vol. 63, pp. 3317–3331, July 2015.

[46] L. Xiao and X.-G. Xia, "A new robust chinese remainder theorem with improved performance in frequency estimation from undersampled waveforms," *Signal Processing*, vol. 117, pp. 242 – 246, 2015.

[47] L. Xiao and X. Xia, "A generalized chinese remainder theorem for two integers," *IEEE Signal Processing Letters*, vol. 21, pp. 55–59, Jan 2014.

[48] W. Wang, X. Li, X. Xia, and W. Wang, "The largest dynamic range of a generalized chinese remainder theorem for two integers," *IEEE Signal Processing Letters*, vol. 22, pp. 254–258, Feb 2015.

[49] X. Li, X. Xia, W. Wang, and W. Wang, "A robust generalized chinese remainder theorem for two integers," *IEEE Transactions on Information Theory*, vol. 62, pp. 7491–7504, Dec 2016.

[50] G. Zhou and X.-G. Xia, "Multiple frequency detection in undersampled complex-valued waveforms with close multiple frequencies," *Electronics Letters*, vol. 33, pp. 1294–1295, July 1997.

[51] L. Xiao, X. Xia, and H. Huo, "New conditions on achieving the maximal possible dynamic range for a generalized chinese remainder theorem of multiple integers," *IEEE Signal Processing Letters*, vol. 22, pp. 2199–2203, Dec 2015.

[52] H. Xiao, C. Cremers, and H. K. Garg, "Symmetric polynomial & CRT based algorithms for multiple frequency determination from undersampled waveforms," in *2016 IEEE Global Conference on Signal and Information Processing (GlobalSIP)*, pp. 202–206, Dec 2016.

[53] X.-G. Xia and K. Liu, "A generalized chinese remainder theorem for residue sets with errors and its application in frequency determination from multiple sensors with low sampling rates," *IEEE Signal Processing Letters*, vol. 12, pp. 768–771, Nov 2005.

[54] H. Xiao and G. Xiao, "Notes on crt-based robust frequency estimation," *Signal Processing*, vol. 133, pp. 13–17, 2017.

[55] H. Xiao, Y. Huang, Y. Ye, and G. Xiao, "Robustness in chinese remainder theorem for multiple numbers and remainder coding," *IEEE Transactions on Signal Processing*, vol. 66, pp. 4347–4361, Aug 2018.

[56] H. Xiao and G. Xiao, "On Solving Ambiguity Resolution with Robust Chinese Remainder Theorem for Multiple Numbers," *IEEE Transactions on Vehicular Technology*, vol. PP, pp. 1–1, 03 2019.

[57] A. Pillai, *Signal Reconstruction Algorithms for Time-Interleaved ADCs*. Linköping Studies in Science and Technology. Dissertations, Linköping University Electronic Press, 2015.

[58] A. Tome, D. Malafaia, A. Teixeira, and E. Lang, "On the use of Singular Spectrum Analysis," 2017.