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NON-VANISHING OF HECKE L-FUNCTIONS OF CUSP FORMS OF INTEGER AND HALF-INTEGER WEIGHT

by

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An Abstract of the Thesis of

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Title: Non-Vanishing of Hecke L-Functions of Cusp Forms of Integer and Half-Integer Weight

We show a non-vanishing result for L-functions of cuspidal Hecke eigenforms of integer weight in the full modular group and of half integer weight in the plus space. In chapter 1, we review definitions of modular forms of integer weight, their Hecke operators and their corresponding L-functions. In chapter 2, we introduce modular forms of half integer weight and some related properties. In chapter 3, will show that the average of the normalized L-functions $L^*(f,s)$ with f a cusp form of weight k in $SL_2(\mathbf{Z})$, running over a basis of Hecke eigenforms, does not vanish inside the critical strip. A similar result will be presented in chapter 4 for cusp forms of half-integer weight in the plus space.

Contents

Chapter 1

Introduction

In this chapter, we define modular forms of integer weight on the full group along with Hecke operators and the associated L-functions. We continue to discuss modular forms on subgroups of finite index in the full modular groups.

1.1 Basic Definitions

Let $\mathbb{H}$ be the upper half plane defined by $\mathbb{H} = \{z \in \mathbb{C}, Imz > 0\}$.

We define the full modular group

$$SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$$

This group is generated by $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Proof. Consider the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$. Without loss of generality, suppose $c \geq 0$ and consider two cases. The first case is when $c = 0$. We have $ad - bc = 0$ so $ad = 1$ i.e. $a = d = \pm 1$, so that $A = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ or $A = \begin{pmatrix} -1 & n \\ 0 & -1 \end{pmatrix}$. Both matrices are similar to $T^{\pm n}$. The claim is proved for $c = 0$.

Now for $c > 0$, let $c = 1$ first; so that $A = \begin{pmatrix} a & ad-1 \\ 1 & d \end{pmatrix} = T^a S T^d$. Assume the claim is true for all A where the lower left-hand element of A is less than c. Note that $d = cl + r$ where $0 <\leq r < c$ since $gcd(c, d) = 1$ $(ad - bc = 1)$: Then $AT^{-l}S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & -l \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -al+b & -a \\ r & -c \end{pmatrix}$

Using the induction hypothesis, the last matrix in generated by S and T (because $r < c$). We get

$$AT^{-l}S = T^{n_1}S...ST^{n_k}$$

so that

$$A = T^{n_1}S...ST^{n_k}ST^l$$

$\square$

Now, we define an action of $SL_2(\mathbb{Z})$ into $\mathbb{C}$ by:

$$\gamma.z := \frac{az+b}{cz+d} \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}).$$

Note that, for $\Im(z)$ representing the imaginary part of z, we have:

$$\Im(\gamma z) = \frac{\Im(z)}{|cz+d|^2} \tag{1.1}$$

In particular, the action of $SL_2(\mathbb{Z})$ preserves $\mathbb{H}$.

Definition 1.1.1. Let G be a group. A fundamental domain R of $\mathbb{H}$ is an open set that satisfies:

1. No two distinct points in R are equivalent under G.

2. If $z \in \mathbb{H}$, then there exists $\gamma \in$ G such that $\gamma z \in \bar{R}$.

Proposition 1.1.1. *The fundamental domain of $SL_2(\mathbb{Z})$ is given by:*

$$\mathfrak{R} = \{z \in \mathbb{H} : -\frac{1}{2} < Re(z) < \frac{1}{2} and |z| > 1\}$$

Proof. Let $z \in \mathbb{H}$. $Im(\gamma z) = \frac{Imz}{|cz+d|^2}$ with $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$. To prove the first condition, suppose that $z \in R$ and that there exists $z' \in R$ such that $z' = \gamma z$ so $Imz' = \frac{Imz}{|cz+d|^2}$. For $c \neq 0$, $|cz+d|^2 = (cz+d)(c\bar{z}+d) =$

$c^2 z\bar{z} + cd(z+\bar{z}) + d^2 > c^2 - |cd| + d^2$. If $d = 0$, $|cz+d|^2 > c^2 \geq 1$. If $d \neq 0$, $|cz+d|^2 > (|c|-|d|)^2 + |cd| \geq |cd| \geq 1$. Therefore, $Imz' < Imz$ for $c \neq 0$. However, by the samme argument in reverse, $z = \gamma^{-1}z'$, so $Imz < Imz'$. We get a contradiction. We deduce that $c = 0$. To satisfy $det\gamma = 1$, we should have $a = d = \pm 1$, so that $\gamma = T^{\pm b}$ but then $b = 0$. Therefore, $\gamma = \pm I$.

Now for the second condition, given z, there are finitely many c, d such that $|cz+d| < 1$. This implies that there exists some $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ such that $|cz+d| \leq |c'z+d'|$ for all $\gamma' = \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ or equivalently $Im(\gamma z) \geq Im(\gamma' z)$. Now we can translate γz i.e. multiplying from the left by a power of T for a specific γ so that $-1/2 \leq Re(\gamma z) \leq 1/2$. We have, by the choice of γ, $I(\gamma z) \geq Im(S\gamma z) = \frac{Im(\gamma z)}{|\gamma z|^2}$ so that $|\gamma z| \geq 1$, and hence $\gamma z \in \mathfrak{R}$. $\square$

1.2 Modular Forms of Integer weights

Definition 1.2.1. Let $k \in \mathbb{Z}$. A modular form of weight k for $\Gamma = SL_2(\mathbb{Z})$ is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ that satisfies:

1. $f(\gamma z) = (cz+d)^k f(z)$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$
2. f is holomorphic at ∞ (or $f(z) = \sum_{n=0}^{\infty} c(n) e^{2\pi i n z}$).

Note that for $\gamma = -I$,

$$f(z) = f(-Iz) = (-1)^k f(z).$$

Therefore, one can easily show that nonzero modular forms are of even integer weight.

Definition 1.2.2. We define M_k to be the complex vector space of modular forms of weight k.

Definition 1.2.3. If $f(z) = \sum_{n=0}^{\infty} c(n)e^{2\pi inz} \in M_k$, f vanishes at infinity i.e. $c(0) = 0$, then $f(z)$ is called a cusp-form of weight k. We denote the space of such functions by S_k.

Theorem 1.2.1. *The spaces M_k and S_k are of finite dimension for all k, and we have:*

$$dim M_k = dim S_k + 1$$

and

$$dim M_k = \begin{cases} \lfloor \frac{k}{12} \rfloor & k \equiv 2 \ mod\ 12 \\ \\ \lfloor \frac{k}{12} \rfloor + 1 & k \not\equiv 2 mod\ 12 \end{cases}$$

Proof. To prove the first equality, consider the following exact sequence:

$$0 \to S_k \xrightarrow{\alpha} M_k \xrightarrow{\beta} \mathbb{C} \to 0$$

where α is the inclusion map and $\beta(f) = c(0)$ for $f = \sum_{n=0}^{\infty} c(n)e^{2\pi i n z} \in M_k$.

We have $ker\beta = S_k = Im\alpha$. Therefore,

$$M_k = S_k \oplus \mathbb{C}E_k$$

where E_k is the normalized Eisenstein series(cf. next section).

Then

$$dimM_k = dimS_k + 1$$

Now for the second equality, we will prove it by induction. Note that Iwaniec[1] studied the case for $k \in \{0, 2, 4, 6, 8, 10\}$.

Suppose

$$dimM_{k-1} = \begin{cases} \lfloor \frac{k-1}{12} \rfloor & k-1 \equiv 2 \ mod \ 12 \\ \\ \lfloor \frac{k-1}{12} \rfloor + 1 & k-1 \not\equiv 2 mod 12 \end{cases}$$

If $k \equiv 2 \ mod \ 12$, then $k - 12 \equiv 2 \ mod \ 12 \Rightarrow dimM_{k-12} = \lfloor \frac{k-12}{12} \rfloor = \lfloor \frac{k}{12} \rfloor - 1 \Rightarrow$ $dimM_k = dimM_{k-12} + 1 = \lfloor \frac{k}{12} \rfloor$

If $k \not\equiv 2mod \ 12$, then $k - 12 \not\equiv 2mod \ 12 \Rightarrow dimM_k = dimM_{k-12} + 1 =$ $\lfloor \frac{k}{12} \rfloor + 1$. $\square$

Note that Proposition 1.3.1 will give an additional result concerning dimensions.

1.3 Examples of Modular Forms of Integer Weight

1.3.1 Eisenstein Series

Let k be an even integer, $k \geq 4$. We define the Eisenstein Series of weight k, by $G_k : \mathbb{H} \longrightarrow \mathbb{C}$:

$$G_k(z) = \sum_{\substack{(m,n)\in\mathbb{Z}^2\\(m,n)\neq(0,0)}} \frac{1}{(mz+n)^k}$$

It converges absolutely and uniformly on subsets of $\mathbb{H}$ of the form $R_{r,s} = \{x+iy, |x| \leq r, y \geq s\}$, therefore G_k defines a holomorphic function. Also, for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$

$$\begin{aligned} G_k(\gamma z) &= \sum_{\substack{(m,n)\in\mathbb{Z}^2\\(m,n)\neq(0,0)}} \frac{1}{(m\left(\frac{az+b}{cz+d}\right)+n)^k} \\ &= (cz+d)^k \sum_{\substack{(m,n)\in\mathbb{Z}^2\\(m,n)\neq(0,0)}} \frac{1}{(m(az+b)+n(cz+d))^k} \\ &= (cz+d)^k \sum_{\substack{(m,n)\in\mathbb{Z}^2\\(m,n)\neq(0,0)}} \frac{1}{((ma+nc)z+mb+nd)^k} \\ &= (cz+d)^k \sum_{\substack{(m',n')\in\mathbb{Z}^2\\(m',n')\neq(0,0)}} \frac{1}{(m'z+n')^k} \end{aligned}$$

where $m' = ma + nc$ and $n' = mb + nd$. As (m, n) runs through all pairs in $\mathbb{Z} \setminus \{(0,0)\}$ so does (m', n').

And finally, one can show that the Fourier expansion of G_k is as follows:

$$G_k(z) = 2\zeta(k) + 2\frac{(2\pi k)^k}{(k-1)!}\sum_{t=1}^{\infty} \sigma_{k-1}(t)e^{2\pi itz}$$

where $\zeta(k)$ is the Riemann-Zeta function and $\sigma_{k-1}(t)$ is the divisor sum function. cf.[2]

Now we can say that for $k \geq 4$, G_k is a modular form of weight k.

Definition 1.3.1. The *normalized Eisenstein series* of even weight $k \geq 4$ is given by

$$E_k(z) = \frac{(k-1)!}{2.(2\pi i)^k} G_k(z)$$

1.3.2 The Modular Discriminant Δ

We define the function

$$\Delta = \frac{E_4^3 - E_6^2}{1728}.$$

This function is a modular form of weight 12. Also, one can prove that Δ is a cusp form. cf.[2].

Proposition 1.3.1.

$$dim M_k = dim S_{k+12}$$

Proof. We consider the map $\phi : M_k \to S_{k+12}$ mapping f to Δf. This homomorphism is actually an isomorphism between M_k and S_{k+12}: let $g \in S_{k+12}$ and take $h = \frac{g}{\Delta}$. We claim that $h \in M_k$, because Δ has no zeros in $\mathbb{H}$ so h is holomorphic on $\mathbb{H}$, and it is also holomorphic at ∞ cf.[2], and one can show that $f(\gamma z) = (cz+d)^k f(z)$ for $\gamma \in SL_2(\mathbb{Z})$. Now since $\phi(h(z)) = g$, we can say that ϕ is surjective. Now it is also clear that ϕ is injective by proving $ker(\phi) = \{0\}$. Therefore, ϕ is an isomorphism, and $dimM_k = dimS_{k+12}$. $\square$

1.4 Hecke Operators On $SL_2(\mathbb{Z})$

We define

$$\Delta_n = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, ad = n, 0 \le b < d \right\}.$$

It is a complete set of coset representatives of $SL_2(\mathbb{Z})$ in $GL_2(\mathbb{Z})$,

i.e. $GL_2(\mathbb{Z}) = \bigcup_{\delta \in \Delta_n} SL_2(\mathbb{Z})\delta$.

Definition 1.4.1. For a fixed integer k and function $f \in M_k(SL_2(\mathbb{Z}))$, we define the Hecke operator on f to be:

$$T_n f = \sum_{\delta = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Delta_n} f|_k \delta = n^{k-1} \sum_{d|n} d^{-k} \sum_{b=0}^{d-1} f\left(\frac{nz+bd}{d^2}\right)$$

In particular, if p is prime

$$T_p f = p^{k-1} f(pz) + \frac{1}{p} \sum_{b=0}^{p-1} f\left(\frac{az+b}{p}\right)$$

Theorem 1.4.1. *Suppose f has the Fourier expansion at ∞*

$$f(z) = \sum_{m=0}^{\infty} c(m) e^{2\pi i m z}$$

then

$$T_n f(z) = \sum_{m=0}^{\infty} \gamma_n(m) e^{2\pi i m z}$$

with

$$\gamma_n(m) = \sum_{d|(n,m)} d^{k-1} c\left(\frac{mn}{d^2}\right)$$

Proof. We have

$$\begin{aligned} T_n f(z) &= \sum_{m=0}^{\infty} n^{k-1} \sum_{d|n} d^{-k} \sum_{b=0}^{d-1} c(m) e^{2\pi i m(nz+bd)/d^2} \\ &= \sum_{m=0}^{\infty} n^{k-1} \sum_{d|n} d^{-k-1} c(m) e^{2\pi i m n z/d^2} \frac{1}{d} \sum_{b=0}^{d-1} e^{2\pi i m (b/d)} \\ &= \sum_{m=0}^{\infty} \sum_{\substack{d|n \\ d|m}} \left(\frac{n}{d}\right)^{k-1} c(m) e^{2\pi i m n z/d^2} \end{aligned}$$

since $\frac{1}{d} \sum_{b=0}^{d-1} e^{2\pi i m(b/d)} = \begin{cases} 1 & d|m \\ 0 & d \nmid m \end{cases}$.

Now let $m = qd$, then we get

$$T_n f(z) = \sum_{q=0}^{\infty} \sum_{d|n} \left(\frac{n}{d}\right)^{k-1} c(qd) e^{2\pi i n q z/d}.$$

Since d runs over all divisors of n, so does $\frac{n}{d}$, we get

$$\begin{aligned} T_n f(z) &= \sum_{q=0}^{\infty} \sum_{d|n} d^{k-1} c\left(\frac{qn}{d}\right) e^{2\pi i q d z} \\ &= \sum_{m=0}^{\infty} \sum_{d|(m,n)} d^{k-1} c\left(\frac{mn}{d^2}\right) e^{2\pi i m z}. \end{aligned}$$

□

Theorem 1.4.2. *If $f \in M_k$ and $V = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(\mathbb{Z})$, then*

$$T_n f(Vz) = (\gamma z + \delta)^k T_n f(z).$$

Proof. Straightforward from the definition of $T_n f(z)$, e.g.[2]. □

The last two theorems will immediately give the following

Corollary 1.4.1. If $f \in M_k(SL_2(\mathbb{Z}))$, then $T_n f \in M_k(SL_2(\mathbb{Z}))$. Also, if $f \in S_k(SL_2(\mathbb{Z}))$, then $T_n f \in S_k(SL_2(\mathbb{Z}))$ $(\gamma_n(0) = \sigma_{k-1}(n)c(0))$.

Definition 1.4.2. A nonzero function $f(z)$ satisfying $T_n f = \lambda(n) f$ is called an *eigenfunction* of the Hecke operator T_n with *eigenvalue* $\lambda(n)$. If this equality holds for all n, then $f(z)$ is called *simultaneous eigenfunction* of all Hecke operators.

We call a simultaneous eigenfunction f *normalized* if $c(1) = 1$ where $f(z) = \sum_{m=0}^{\infty} c(m) e^{2\pi i m z}$.

Theorem 1.4.3. *Let $k > 0$, even. If the space M_k contains a simultaneous Hecke eigenform f with Fourier expansion $f(z) = \sum_{m=0}^{\infty} c(m)e^{2\pi imz}$, then $c(1) \neq 0$.*

Proof. Since f is a simultaneous Hecke eigenform, then

$$T_n f(z) = \sum_{m=0}^{\infty} \left(\sum_{d|(n,m)} d^{k-1} c\left(\frac{mn}{d^2}\right) \right) e^{2\pi imz} = \lambda(n) \sum_{m=0}^{\infty} c(m)e^{2\pi imz}$$

so that $c(n) = \lambda(n)c(1)$. If $c(1) = 0$, then $c(n) = 0$ for all $n \geq 0$. Therefore, $f(z) = c(0)$. However, the only constant modular form is the zero function, $f(z) = 0$. This is a contradiction because f needs to be nonvanishing in order to be an eigenform. $\square$

Remark. *For the case of a normalized simultaneous Hecke eigenform,*

$$c(n) = \lambda(n)c(1) = \lambda(n).$$

Hence, the n-th Fourier coefficient of f is the same as its n-th eigenvalue.

Theorem 1.4.4. *Let $k \in 2\mathbb{Z}$. Suppose $0 \neq f(z) \in S_k(\Gamma(1))$ with the Fourier expansion $f(z) = \sum_{m=0}^{\infty} c(n)e^{2\pi inz}$. Then f is a normalized simultaneous Hecke eigenform if and only if*

$$c(m)c(n) = \sum_{d|(n,m)} d^{k-1} c\left(\frac{mn}{d^2}\right).$$

Proof. The equation $T_n f(z) = \lambda(n) f(z)$ is equivalent to $\gamma_n(m) = \lambda(n)c(m)$ obtained by equating coefficients of x^m in the corresponding Fourier expansion. Now for $m = 1$, $\gamma_n(1) = \lambda(n)c(1) = \lambda(n)$ (normalized). But $\gamma_n(1) = c(n)$ which implies $\lambda(n) = c(n)$, so that $\gamma_n(m) = c(n)c(m)$ for $c(1) = 1$. $\square$

Corollary 1.4.2. For $k \in 2\mathbb{N}$, let $f \in M_k(\Gamma(1))$ be a normalized Hecke eigenform with Fourier coefficients $c(n)$. Let p be a prime. Then, the Fourier coefficients satisfy

- $c(p^n m) = c(p^n)c(m)$ for all $m, n \in \mathbb{N}$ with $(m, p) = 1$
- $c(p^{n+1}) = c(p^n)c(p) - p^{k-1}c(p^{n-1})$.

Proof. Applying the previous theorem, we can see that

$c(p^n)c(m) = \sum_{d|(p^n,m)=1} d^{k-1} c\left(\frac{p^n m}{d^2}\right) = c(p^n m)$ and

$c(p^n)c(p) = \sum_{d|(p^n,p)=p} d^{k-1} c\left(\frac{p^{n+1}}{d^2}\right) = c(p^{n+1}) + p^{k-1}c(p^{n-1})$. $\square$

Theorem 1.4.5. *There exists an orthonormal basis of S_k which consists of eigenfunction of all Hecke operators T_n.*

For the proof, please refer to Iwaniec[1].

1.5 L-functions of Eigenforms

Definition 1.5.1. If $f(z) = c(0) + \sum_{n=1}^{\infty} c(n)e^{2\pi inz}$, we define *the Dirichlet* $L-$ $Function$ of $f(z)$ by:

$$L(f,s) = \sum_{n=1}^{\infty} \frac{c(n)}{n^s}. \tag{1.2}$$

Proposition 1.5.1. *$L(f,s)$ converges absolutely.*

Proof. If $f(z) = \sum_{n=1}^{\infty} c(n)e^{2\pi inz} \in S_k(SL_2(\mathbb{Z}))$, then $c(n) = \theta(n^{\frac{k}{2}})$, i.e $c(n) \leq cn^{\frac{k}{2}}$ form some $c \in \mathbb{R}$ c.f.[1]. Then

$$|L(f,s)| = \left|\sum_{n=1}^{\infty} \frac{a(n)}{n^s}\right| \leq \sum_{n=1}^{\infty} \frac{|a(n)|}{n^{Re(s)}} \leq c\sum_{n=1}^{\infty} \frac{1}{n^{Re(s)-\frac{k}{2}}}.$$

Therefore, $L(f,s)$ converges absolutely for $Re(s) > 1 + \frac{k}{2}$.

Also, if$f(z) \in M_K \backslash S_k(SL_2(\mathbb{Z}))$, then $a(n) = \theta(n^{k-1})$, and $L(f,s)$ converges absolutely for $Re(s) > k$. $\square$

We define the completed L-function $L^*(f,s)$ by

$$L^*(f,s) = (2\pi)^{-s}\Gamma(s)L(f,s) \tag{1.3}$$

where $\Gamma(s) = \int_0^{\infty} e^{-t}t^{s-1}dt$ is the Gamma function defined for $Re(s) > 0$.

Proposition 1.5.2. *L^* satisties the functional equation*

$$L^*(f,k-s) = (-1)^{\frac{k}{2}}L^*(f,s). \tag{1.4}$$

Proof. We have

$$\Gamma(s) = \int_0^\infty e^{-t}t^s\frac{dt}{t} \overset{t\to 2\pi ny}{=\!=\!=} \int_0^\infty e^{-2\pi ny}(2\pi ny)^s\frac{dy}{y}$$
$$\Rightarrow (2\pi)^{-s}\frac{1}{n^s}\Gamma(s) = \int_0^\infty e^{-2\pi ny}y^s\frac{dy}{y}.$$

As a result, we have

$$\begin{aligned}
L^*(f,s) &= (2\pi)^{-s}L(f,s)\Gamma(s)\\
&= \int_0^\infty \sum_{n\geq 1} c(n)e^{-2\pi ny}y^s\frac{dy}{y}\\
&= \int_0^\infty \left(f(iy)-c(0)\right)y^s\frac{dy}{y}\\
&= \int_0^1 \left(f(iy)-c(0)\right)y^s\frac{dy}{y} + \int_1^\infty \left(f(iy)-c(0)\right)y^s\frac{dy}{y}\\
&= \int_1^\infty \left(f(\frac{i}{y})-c(0)\right)y^s\frac{dy}{y} + \int_1^\infty \left(f(iy)-c(0)\right)y^s\frac{dy}{y}
\end{aligned}$$

where we are allowed to interchange integration and summation using Fubini's theorem. But $f(\frac{i}{y}) = f(\frac{-1}{iy}) = (iy)^k f(iy) = (-1)^{k/2}y^k f(iy)$, This gives

$$\begin{aligned}
L^*(f,s) &= (-1)^{k/2}\int_1^\infty f(iy)y^{k-s}\frac{dy}{y} - \int_1^\infty c(0)y^{-s}\frac{dy}{y} + \int_1^\infty \left(f(iy)-c(0)\right)y^s\frac{dy}{y}\\
&= (-1)^{k/2}\int_1^\infty (f(iy)-c(0))y^{k-s}\frac{dy}{y} + (-1)^{k/2}\int_1^\infty c(0)y^{-s}\frac{dy}{y}\\
&- \int_1^\infty c(0)y^{-s}\frac{dy}{y} + \int_1^\infty (f(iy)-c(0))y^s\frac{dy}{y}\\
&= \int_1^\infty (f(iy)-c(0))((-1)^{k/2}y^{k-s}+y^s)\frac{dy}{y} - \frac{(-1)^{k/2}c(0)}{k-s} + \frac{c(0)}{s}\\
&= (-1)^{k/2}(2\pi)^{-(k-s)}\Gamma(k-s)L(f,k-s)\\
&= (-1)^{k/2}L^*(f,k-s).
\end{aligned}$$

$\square$

Theorem 1.5.1. *For $k \in 2\mathbb{N}$ and let $f \in S_k(\Gamma(1))$ be a cusp form with associated L-series $L(f, s)$ defined in* (1.2)*. If f is a normalized Hecke eigenform with Fourier coefficients $c(n)$, then the associated L-series $L(f, s)$ admits the Euler product*

$$L(f, s) = \prod_{p \ prime} \frac{1}{(1 - c(p)p^{-s} + p^{k-1-2s})}.$$

Proof. Let $p_1^{i_1} \dots p_l^{i_l} = n$ be the prime factorization of n for some $l \in \mathbb{N}$ and $p_1 \dots p_l$ distinct primes. Now,

$$\begin{aligned} L(f, s) &= \sum_{n \in \mathbb{N}} \frac{c(n)}{n^s} \\ &= c(1) + \sum_{l \in \mathbb{N}} \sum_{\substack{i_1 \dots i_l \in \mathbb{N} \\ p_1 \dots p_l \ distinct \ primes}} \frac{c(p_1^{i_1} \dots p_l^{i_l})}{(p_1^{i_1} \dots p_l^{i_l})^s} \\ &= 1 + \sum_{l \in \mathbb{N}} \sum_{\substack{i_1 \dots i_l \in \mathbb{N} \\ p_1 \dots p_l \ distinct \ primes}} \frac{c(p_1^{i_1})}{p_1^{i_1}} \dots \frac{c(p_l^{i_l})}{p_l^{i_l}} \\ &= \prod_{p \ prime} \sum_{i \in \mathbb{N}} \frac{c(p^i)}{(p^i)^s} \end{aligned}$$

where we have used Corollary 1.4.2 because $\gcd(p_m^{i_m}, p_n^{i_n}) = 1$ for all distinct primes p_m and p_n. Now we claim that

$$\sum_{i \in \mathbb{N}} \frac{c(p^i)}{(p^i)^s} = \frac{1}{1 - c(p)p^s - p^{k-1}p^{-2s}}.$$

This claim can be proved for any $\sum_{i\in\mathbb{N}} c(p^i)\alpha^i$ where in our situation α is equal to p^{-s} :

$$
\begin{aligned}
&(1 - c(p)\alpha + p^{k-1}\alpha^2) \sum_{i\in\mathbb{N}} c(p^i)\alpha^i \\
&= \sum_{i\in\mathbb{N}} c(p^i)\alpha^i - \sum_{i\in\mathbb{N}} c(p)c(p^i)\alpha^{i+1} + \sum_{i\in\mathbb{N}} p^{k-1}c(p^i)\alpha^{i+2} \\
&= c(1) + c(p)\alpha + \sum_{i=2}^{\infty} c(p^i)\alpha^i - c(p)c(1)\alpha \\
&- \sum_{i=1}^{\infty} c(p)c(p^i)\alpha^{i+1} + \sum_{i\in\mathbb{N}} p^{k-1}c(p^i)\alpha^{i+2} \\
&= 1 + \sum_{i=2}^{\infty} c(p^i)\alpha^i - \sum_{i=1}^{\infty} \left(c(p)c(p^i) - p^{k-1}c(p^{i-1})\right)\alpha^{i+1}
\end{aligned}
$$

$$
\begin{aligned}
&= 1 + \sum_{i=2}^{\infty} c(p^i)\alpha^i - \sum_{i=1}^{\infty} c(p^{i+1})\alpha^{i+1} \\
&= 1 + \sum_{i=2}^{\infty} c(p^i)\alpha^i - \sum_{i=2}^{\infty} c(p^i)\alpha^i = 1
\end{aligned}
$$

where we have used again Corollary 1.4.2 and that $a(1) = 1$ holds for normalized Hecke eigenforms.

Therefore,

$$
L(f,s) = \prod_{p\ prime} \sum_{i\in\mathbb{N}} \frac{c(p^i)}{(p^i)^s} = \prod_{p\ prime} \frac{1}{1 - c(p)p^s - p^{k-1}p^{-2s}}.
$$

$\square$

1.6 Poincare Series and Petersson Inner Product

Definition 1.6.1. Let $m \in \mathbb{Z}_{>0}$, k an even integer, $k > 2$. **The Poincaré series** of weight k is given by:

$$P_{m,k}(z) = \sum_{M \in \Gamma_\infty \backslash SL_2(\mathbb{Z})} j(M,z)^{-k} e^{2\pi i m M z} = \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ gcd(c,d)=1}} (cz+d)^{-k} e^{2\pi i m\left(\frac{a_0 z + b_0}{cz+d}\right)}.$$

Proposition 1.6.1. *The Poincaré series is a modular form of weight k.*

Proof. Using the definition of Poincaré series, we can check that $P_{m,k}|_k V = P_{m,k}$ for all $V \in SL_2(\mathbb{Z})$. Also, as a holomorphic function, $P_{m,k}$ admits a Fourier expansion of the form $P_{m,k} = \sum_{n>0} a(n) e^{2\pi i n z}$.

For more details, please refere to [1] □

Definition 1.6.2. Let $\Gamma \subset \Gamma(1)$ be a subgroup of finite index μ with R_Γ its fundamental region. Let $k \in 2\mathbb{Z}$ and $z = x + iy$.

We define **the Petersson Inner Product** by

$$\langle .,. \rangle : M_k(\Gamma) \times S_k(\Gamma) \to \mathbb{C}$$

$$\langle f, g \rangle = \frac{1}{\mu} \int_{R_\Gamma} f(z)\overline{g(z)} \Im(z)^k \frac{dxdy}{\Im(z)^2}.$$

Proposition 1.6.2. *Let* $f(z) = \sum_{n=1}^{\infty} a(n) e^{2\pi i n z} \in S_k$. *Then*

$$\langle P_{m,k}, f \rangle = \frac{\Gamma(k-1)}{\mu (4\pi m)^{k-1}} \overline{a(m)}.$$

Proof.

$$\begin{aligned}\langle P_{m,k}, f\rangle &= \frac{1}{\mu}\int_{\mathfrak{R}} P_{m,k}(z)\overline{f(z)}\Im(z)^k \frac{dxdy}{\Im(z)^2}\\ &= \frac{1}{\mu}\int_{\mathfrak{R}} \sum_{M\in\Gamma_\infty\backslash SL_2(\mathbb{Z})} j(M,z)^{-k}e^{2\pi imMz}\overline{f(z)}\Im(z)^k \frac{dxdy}{\Im(z)^2}.\end{aligned}$$

But f is a modular form, so that $\overline{f}$ satisfies: $\overline{f(Mz)} = j(M,\bar{z})^k\overline{f(z)}$, and using (1.1), we get

$$\begin{aligned}\langle P_{m,k}, f\rangle &= \frac{1}{\mu}\int_{\mathfrak{R}} \sum_{M\in\Gamma_\infty\backslash SL_2(\mathbb{Z})} j(M,z)^{-k}e^{2\pi imMz}\overline{f(z)}j(M,z)^k\left(j(M,\bar{z})\Im(Mz)\right)^k \frac{dxdy}{\Im(z)^2}\\ &= \frac{1}{\mu}\int_{\mathfrak{R}} \sum_{M\in\Gamma_\infty\backslash SL_2(\mathbb{Z})} e^{2\pi imMz}\overline{f(Mz)}\Im(Mz)^k \frac{dxdy}{\Im(z)^2}\\ &\overset{Mz\mapsto z}{=} \frac{1}{\mu}\int_{M\mathfrak{R}} \sum_{M\in\Gamma_\infty\backslash SL_2(\mathbb{Z})} e^{2\pi imz}\overline{f(z)}\Im(z)^k \frac{dxdy}{\Im(z)^2}.\end{aligned}$$

Now because of the shape of $\mathfrak{R}$ (a vertical strip by Proposition 1.1.1), and replacing f by its Fourier expansion, we can rewrite the above expression as

$$\begin{aligned}\langle P_{m,k}, f\rangle &= \frac{1}{\mu}\int_0^\infty \int_{-1/2}^{1/2} e^{2\pi imz}\sum_{n=0}^{\infty}\overline{a(n)e^{2\pi inz}}\Im(z)^k\frac{dxdy}{\Im(z)^2}\\ &= \frac{1}{\mu}\sum_{n=0}^{\infty}\overline{a(n)}\int_0^\infty\int_{-1/2}^{1/2} e^{2\pi i(m-n)x}e^{-2\pi i(m+n)y}y^{k-2}dxdy\\ &= \frac{\overline{a(m)}}{\mu}\int_0^\infty e^{-4\pi imy}y^{k-2}dy.\end{aligned}$$

Now replacing $4\pi imy$ by t, we get:

$$\begin{aligned}\langle P_{m,k}, f\rangle &= \frac{\overline{a(z)}}{\mu(4\pi m)^{k-1}}\int_0^\infty e^{-t}t^{k-2}dt\\ &= \frac{\overline{a(z)}}{\mu(4\pi m)^{k-1}}\Gamma(k-1)\end{aligned}$$

□

Corollary 1.6.1. The space of cusp forms $S_k(SL_2(\mathbb{Z}))$ is spanned by the Poincaré series $P_{m,k}$, $m \geq 0$.

Proof. To show that $S_k(SL_2(\mathbb{Z}))$ is spanned by the Poincaré series, we have to show that each cusp form can be expressed as a linear combination of Poincaré series. Suppose $f \in S_k(SL_2(\mathbb{Z}))$ is orthogonal to all Poincaré series i.e. $\langle 0, P_{m,k} \rangle = 0$. But the previous proposition implies that all Fourier coefficients of f vanish. Hence, f must be the zero-function. This means that all cusp forms lie in the span of the Poincaré series. $\square$

1.7 Modular Forms on Congruence Subgroups

Definition 1.7.1. The principal subgroup of $SL_2(\mathbb{Z})$ of level N is given by

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ mod\ N \right\}.$$

Definition 1.7.2. A congruence subgroup of level N is a subgroup of $SL_2(\mathbb{Z})$ that contains $\Gamma(N)$.

Examples: $\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0 \ mod\ N \right\}$,

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \ mod\ N \right\}.$$

Remark. • *$\Gamma(N)$ is normal in $\Gamma(1)$ where $\Gamma(1)$ is equal to $SL_2(\mathbb{Z})$. Moreover, $\Gamma(N)$ has finite index in $\Gamma(1)$*

- *If $N'|N$, then $\Gamma(N) \subset \Gamma(N')$*

- *$\Gamma(N) \subset \Gamma_1(N) \subset \Gamma_0(N)$.*

Theorem 1.7.1. *Let γ be a congruence subgroup of $SL_2(\mathbb{Z})$ and let C be a set of coset representatives for $\Gamma \setminus SL_2(\mathbb{Z})$. Then*

$$R_\Gamma = \bigcup_C \gamma R$$

is the fundamental region of the congruence subgroup Γ where R is the fundamental region of $SL_2(\mathbb{Z})$.

Proof. First, suppose z_1, $z_2 \in R_\Gamma$ are equivalent with respect to Γ, i.e. $z_2 = Vz_1$ for $V \in \Gamma$. Since Γ is a subgroup of $\Gamma(1)$, there exists $1 \leq i, j \leq \mu$ and $u, w \in R$ with $z_1 = A_i u$ and $z_2 = A_j w$. Hence $A_j w = VA_i u$. This implies $w = (A_j^{-1} V A_i) u \in R$. We get $A_j^{-1} V A_i = \pm 1$. Thus we have $V = \pm A_j A_i^{-1}$ which implies that $z_2 = A_j A_i^{-1} z_1$ and $A_j A_i^{-1} \in \Gamma$. Since A_i and A_j are representatives of right cosets of Γ in $\Gamma(1)$, we find that $A_j A_i^{-1} \in \Gamma$ holds only for $i = j$. Therefore $A_i = A_j$ and $z_1 = z_2$.

Now let $z \in \mathbb{H}$. We have to show that there exists an element $V \in \Gamma$ such that $Vz \in \overline{R_\Gamma}$. We know that there exists $V_1 \in \Gamma(1)$ such that $V_1 z \in \overline{R}$. On the other hand, there exists $M \in \Gamma$ and a representative A_i, $1 \leq i \leq \mu$ satisfying $V_1 = MA_i$. Hence, we have $V_1 z = (MA_i)z = M(A_i z) \in \overline{R}$. Multiplying M^{-1} from the left gives $M^{-1}V_1 z = A_i z \in A_i\overline{R} \subset \overline{R_\Gamma}$. □

Definition 1.7.3. A cusp z where $z \in P^1(\mathbb{R}) = \mathbb{R} \cup \{\infty\}$ is an element which is fixed by a parabolic element α of A.(α is parabolic if $|tr(\alpha)| = 2$)

Definition 1.7.4. Let Γ be a congruence subgroup of $SL_2(\mathbb{Z})$ of finite index. A modular form of weight k on Γ is a holomorphic function in $\mathbb{H}$ satisfying

- $f(\gamma z) = (cz+d)^k f(z)$ where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$
- f is holomorphic at all cusps.

Chapter 2

Modular Forms of Half-Integral Weight

In this chapter, we define modular forms of half integer weight on $\Gamma_0(4)$, in addition to the plus space and their associated Hecke operators. cf.[3]

2.1 Definitions and Examples

We define G:

$$G = \left\{ (\alpha, \phi(z)) : \alpha \in GL_2^+(Q), \phi^2(z) = t\frac{cz+d}{\sqrt{det\ \alpha}} \right\}.$$

where $t \in T^2 = \{\pm 1\}$ and such that:

$$(\alpha, \phi(z)).(\beta, \psi(z)) = (\alpha\beta, \phi(\beta z)\psi(z))$$

Proposition 2.1.1. *G is a group.*

Proof.

- Let $a = (\alpha, \phi(z)), b = (\beta, \psi(z)) \in G$. We need $a.b \in G$: It is obvious that $\alpha\beta \in GL_2^+(Q)$. Now,

$$\begin{aligned}(\phi(\beta z)\psi(z))^2 &= \left(\phi(\frac{a_2 z + b_2}{c_2 z + d_2})\right)^2 \psi^2(z) \\ &= t_1(\frac{c_1 \beta z + d_1}{\sqrt{det\alpha}})t_2(\frac{c_2\beta z + d_2}{\sqrt{det\beta}}) \\ &= t_1 t_2 \frac{(c_1 a_1 + d_1 c_2)z + c_1 b_2 + d_1 d_2}{\sqrt{det\alpha\beta}}\end{aligned}$$

  which finishes the proof.

- Associativity is immediate.

- Let $I = \left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), 1\right)$ where 1 is the identity function so that $I.(\alpha, \phi(z)) = (\alpha, \phi(z)).I = (\alpha, \phi(z))$.

- Let $(\alpha, \phi(z))^{-1} = (\alpha^{-1}, \frac{1}{\phi(\alpha^{-1}(z))})$.

  Now

$$\begin{aligned}(\alpha, \phi(z)).(\alpha, \phi(z))^{-1} &= \left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \phi(\alpha^{-1}(z))\frac{1}{\phi(\alpha^{-1}(z))}\right) \\ &= \left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), 1\right) \\ &= I\end{aligned}$$

$\square$

The group G acts on the space of complex valued functions on $\mathbb{H}$, by

$$f|_{k+1/2}[\zeta] = f|[\zeta] = \phi(z)^{-2k-1} f(\alpha z)$$

where $\zeta = (\alpha, \phi(z)) \in G$ and $f : \mathbb{H} \to \mathbb{C}$.

We define the automorphy factor $j(\gamma, z)$, for $\gamma \in \Gamma_0(4)$ and $z \in \mathbb{H}$, as follows:

$$j(\gamma, z) = \left(\frac{c}{d}\right) \varepsilon_d^{-1} \sqrt{cz+d} \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$$

and

$$\tilde{\Gamma}_0(4) := \{\tilde{\gamma} = (\gamma, j(\gamma, z)),\ \gamma \in \Gamma_0(4)\}.$$

We note that $\tilde{\Gamma}_0(4)$ is actually a subgroup of G. [3]

Recall the following

- $\Gamma_0(4) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0 \ mod\ 4 \right\}$
- $\varepsilon_d = \begin{cases} 1 & \text{if } d \equiv 1 \ mod\ 4 \\ i & \text{if } d \equiv -1 \ mod\ 4 \end{cases}$

- $\left(\frac{a}{p}\right)$ is the Legendre symbol given by

$$\left(\frac{a}{p}\right) = \begin{cases} 0 & \text{if } p|a \\ 1 & \text{if a is a quadratic residue mod p} \\ -1 & \text{if a is not a quadratic residue mod p} \end{cases}$$

(a is a quadratic residue mod p if there is an integer x, $0 < x < p$, such that $x^2 \equiv a \ mod \ p$ has a solution.)

Definition 2.1.1. Let k be a positive integer. A holomorphic function f on $\mathbb{H}$ is a modular form of weight $k + 1/2$ if f satisfies $f|_{k+1/2}[\tilde{\gamma}] = f$ for all $\tilde{\gamma} \in \tilde{\Gamma}_0(4)$, and is holomorphic at all the cusps of $\Gamma_0(4)$.

We denote such a space of modular forms by $M_{k+1/2}(4)$, and the space of cusp forms by $S_{k+1/2}(4)$ where, as before, a cusp form is a modular form that vanishes on all cusps.

Example. Let $\Gamma \subset \Gamma_0(4)$. Let $\Gamma_\infty = \{\gamma \in \Gamma : \gamma(i\infty) = i\infty\}$. Then Γ_∞ is an infinite subgroup of $SL_2(\mathbb{Z})$ generated by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Consider

$$E_{k/2}(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(4)} j(\gamma, z)^{-k}$$

where $\Gamma_\infty \backslash \Gamma_0(4)$ have coset representatives $\left\{ \begin{pmatrix} a & b \\ m & n \end{pmatrix} \in \Gamma_0(4) : 4|m, (m,n) = 1, n > 0 \right\}$.

Then $E_{k/2}(z)$ is a modular form of weight $k/2$ for $\Gamma_0(4)$.

2.2 L-functions on the plus space

We define $M^+_{k+1/2}(4)$ to be the subspace of $M_{k+1/2}(4)$ consisting of functions whose nth Fourier coefficients vanish whenever $(-1)^k n \equiv 2, 3 \bmod(4)$. We also put $S^+_{k+1/2}(4) = S_{k+1/2}(4) \cap M^+_{k+1/2}(4)$.

Kohnen proved, in [4] , that there is an isomorphism between $S_{2k}(1)$ and $S^+_{k+1/2}(4)$, or equivalently between $M_{2k}(1)$ and $M^+_{k+1/2}(4)$. This will help us, with Theorem 1.2.1, determine the dimension of the given plus space.

Let $L(f, s)$ be the L-function associated to cusp forms $f \in S^+_{k+1/2}(4)$ defined by $L(f, s) = \sum_{(-1)^k n \equiv 0,1 mod(4)} c(n)n^{-s}$ for Re s>1 where $c(n)$ is the n-th Fourier coefficient of f. The completed L-function is defined as

$$L^*(f, s) = (2\pi)^{-s} 2^s \Gamma(s) L(f, s).$$

Proposition 2.2.1. *The completed L-function $L^*(f, s)$ has the following functional equation:*

$$L^*(f|W_4, k + 1/2 - s) = L^*(f, s)$$

where W_4 is the Fricke involution on $S_{k+1/2}(4)$ defined by

$$f|W_4(z) = (-2iz)^{-k-1/2} f(-1/4z).$$

Proof. We have

$$\begin{aligned}\Gamma(s) &= \int_0^\infty e^{-t} t^s \frac{dt}{t} \\ &\overset{t \to 2\pi n y}{=\!=} \int_0^\infty e^{-2\pi n y} (2\pi n y)^s \frac{dy}{y}\end{aligned}$$

Thus we get, as in Chapter 1,

$$(2\pi)^{-s} \frac{1}{n^s} 2^s \Gamma(s) = \int_0^\infty e^{-2\pi n y} (2y)^s \frac{dy}{y}$$

$$\begin{aligned}L^*(f,s) &= \int_0^\infty f(iy) y^s \frac{dy}{y} \\ &= \int_0^{1/2} f(iy) y^s \frac{dy}{y} + \int_{1/2}^\infty f(iy) y^s \frac{dy}{y}\end{aligned}$$

Now we replace y by $\frac{1}{4y}$ in the first term

$$L^*(f,s) = \int_{1/2}^\infty f(\frac{i}{y}) y^s \frac{dy}{y} + \int_{1/2}^\infty f(iy) y^s \frac{dy}{y}.$$

But since

$$f|W_4(z) = (-2iz)^{-k-1/2} f(-1/4z)$$

then

$$f|W_4(iz) = (2z)^{-k-1/2} f(i/4z).$$

Therefore,

$$\begin{aligned}L^*(f,s) &= \int_{1/2}^\infty \left[f|W_4(iy)(2y)^{k+1/2-s} + f(iy)(2y)^s \right] \frac{dy}{y} \\ &= L^*(f|W_4, k+1/2-s).\end{aligned}$$

□

2.3 Hecke Operators on Forms of Half Integer Weight

As in the case of modular forms of integer weight, we can also define Hecke operators in the half-integral weight case. For $f(z) = \sum_{n=0}^{\infty} c(n)e^{2\pi inz}$ an element of $M^{+}_{k+1/2}(4)$ and a prime p, we define the Hecke operator by

$$f|T^{+}_{k+1/2}(p^2) = \sum_{(-1)^k n \equiv 0,1 mod(4)} \left(c(p^2 n) + \left(\frac{(-1)^k n}{p} \right) p^{k-1} c(n) + p^{2k-1} c\left(\frac{n}{p^2} \right) \right) e^{2\pi inz}$$

Theorem 2.3.1. *(Kohnen)*

1. *$S^{+}_{k+1/2}(4)$ (and $M^{+}_{k+1/2}(4)$) is preserved by Hecke operator.*

2. *The space $S^{+}_{k+1/2}(4)$ has an orthogonal basis of Hecke eigenforms with respect to all Hecke operators $T^{+}(p^2)$, p prime.*

Proof. [5] $\square$

2.4 Poincare Series and Petersson Inner Product

Definition 2.4.1. Let $m \in \mathbb{Z}_{>0}$. **The Poincaré series** of weight $k + 1/2$ is given by:

$$P_{m,k}(z) = \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ gcd(c,d)=1}} \left(\frac{c}{d}\right)\left(\frac{-4}{d}\right)^{k+1/2} (cz+d)^{-(k+1/2)} e^{2\pi i m\left(\frac{a_0 z + b_0}{cz+d}\right)}.$$

Definition 2.4.2. For $f \in M_{k+1/2}(4)$ and $g \in S_{k+1/2}(4)$, we define **the Petersson Inner Product** by

$$\langle f, g\rangle = \frac{1}{i_4}\int_{\Gamma_0(4)\backslash\mathbb{H}} f(z)\overline{g(z)} y^{k+1/2}\frac{dxdy}{y^2}$$

where i_4 is the index of $\Gamma_0(4)$ in $SL_2(\mathbb{Z})$.

Chapter 3

Non-Vanishing of L-functions Associated to Integer Weight Modular Forms

In this chapter, we prove that, given a real number t_0 and a positive real number ϵ, for all k large enough the average of $L^*(f,s)$ with f running over a basis of Hecke eigenforms of weight k, does not vanish on the line segments $Im(s) = t_0$, $(k-1)/2 < Re(s) < (k/2) - \epsilon$, $(k/2) + \epsilon < Re(s) < (k+1)/2$.

For this purpose, we define, for $z \in \mathbb{H}$, $s = \sigma + it \in \mathbb{C}$ where $1 < \sigma < k-1$:

$$R_{k,s}(z) := \gamma_k(s) \sum_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})} (cz+d)^{-k} \left(\frac{az+b}{cz+d}\right)^{-s} \tag{3.1}$$

where

$$\gamma_k(s) = \frac{1}{2} e^{\frac{\pi i s}{2}} \Gamma(s)\Gamma(k-s).$$

Note that the letter k always denotes an even integer ≥ 4.

Proposition 3.0.1. $R_{k,s} \in S_k$

Proof. First, we can see that this series converges absolutely uniformly whenever $z = x+iy$ satisfying $y \geq \epsilon$, $x \leq 1/\epsilon$ for a given $\epsilon > 0$, and s varying over a compact set, using standard convergence tests. For instant, for $2 < \sigma < k-2$, we have

$$\begin{aligned}
\left| \sum_{(a,b),(c,d)\in\mathbb{Z}^2\setminus\{0,0\}} (cz+d)^{-k} \left(\frac{az+b}{cz+d}\right)^{-s} \right| &\leq \sum_{(a,b),(c,d)\in\mathbb{Z}^2\setminus\{0,0\}} |cz+d|^{-k} \left| \left(\frac{az+b}{cz+d}\right)^{-s} \right| \\
&\leq \sum_{(a,b),(c,d)\in\mathbb{Z}^2\setminus\{0,0\}} |cz+d|^{-k} \left| \frac{az+b}{cz+d} \right|^{-\sigma} \\
&\leq \sum_{(a,b),(c,d)\in\mathbb{Z}^2\setminus\{0,0\}} |cz+d|^{-(k-\sigma)} |az+b|^{-\sigma} < \infty.
\end{aligned}$$

So combining this with the definition of $R_{k,s}$ we obtain our assertion: It is holomorphic on $\mathbb{H}$, invariant under $SL_2(\mathbb{Z})$ and holomorphic at ∞. □

Theorem 3.0.1. *We have*

$$R_{k,s}(z) = (2\pi)^s \Gamma(k-s) \sum_{n\geq 1} n^{s-1} P_{k,n}(z) \tag{3.2}$$

where

$$P_{k,n}(z) = \frac{1}{2} \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ gcd(c,d)=1}} (cz+d)^{-k} e^{2\pi i n \left(\frac{a_0 z + b_0}{cz+d}\right)}.$$

Proof. By holomorphic continuation, it is enough to prove the assertion say for $1 < \sigma < \frac{k-1}{2}$. For each coprime pair (c, d), we take a fixed choice $(a_0, b_0) \in \mathbb{Z}^2$ with $a_0 d - b_0 c = 1$ (we have $ad - bc = 1$ so that $ad \equiv 1[c]$, now we fix a_0 such that $a = a_0 + nc$ for some $n \in \mathbb{Z}$, same is for b_0). Hence, we get

$$R_{k,s}(z) = \gamma_k(s) \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ gcd(c,d)=1}} \sum_{n\in\mathbb{Z}} (cz+d)^{-k} \left(\frac{a_0 z + b_0}{cz+d} + n \right)^{-s}.$$

Now using Lipschitz's formula:

$$\sum_{n\in\mathbb{Z}} (\tau + n)^{-s} = \frac{e^{-\pi i s/2}(2\pi)^s}{\Gamma(s)} \sum_{n\geq 1} n^{s-1} e^{2\pi i n \tau} \qquad (\tau \in \mathbb{H}, Re(s) > 1) \quad (3.3)$$

for $\tau = \frac{a_0 z + b_0}{cz+d} + n$ in our case, we get

$$\begin{aligned} R_{k,s}(z) =& \gamma_k(s) \frac{e^{-\pi i s/2}(2\pi)^s}{\Gamma(s)} \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ gcd(c,d)=1}} \sum_{n\geq 1} (cz+d)^{-k} n^{s-1} e^{2\pi i n\left(\frac{a_0 z + b_0}{cz+d}\right)} \\ =& \frac{1}{2}(2\pi)^s \Gamma(k-s) \sum_{n\geq 1} n^{s-1} \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ gcd(c,d)=1}} (cz+d)^{-k} e^{2\pi i n\left(\frac{a_0 z + b_0}{cz+d}\right)} \end{aligned}$$

Finally, comparing this with Definition1.6.1 will finish the proof.

□

Lemma 3.1.

$$\langle\ f, R_{k,\bar{s}} \rangle\ = c_k L^*(f, s) \qquad\qquad \forall f \in S_k$$

where

$$c_k := \frac{(-1)^{\frac{k}{2}}\pi(k-2)!}{2^{k-2}}.$$

Proof. Using Theorem 3.0.1,we have:

$$\begin{aligned}\langle\, f, R_{k,\bar{s}}\rangle \;&= \langle\, f, (2\pi)^{\bar{s}}\Gamma(k-\bar{s})\sum_{n\geq 1} n^{\bar{s}-1}P_{k,n}(z)\rangle \\ &= \frac{1}{\mu}(2\pi)^{s}\Gamma(k-s)\sum_{n\geq 1} n^{s-1}\int_{R_\Gamma} f\overline{P_{k,n}(z)}y^k\frac{dxdy}{y^2} \\ &= (2\pi)^{s}\Gamma(k-s)\sum_{n\geq 1} n^{s-1}\langle\, f, P_{k,n}\rangle\end{aligned}$$

where we can interchange the summation and the integral because of absolute convergence.

Using Proposition 1.6.2, we get

$$\begin{aligned}\langle\, f, R_{k,\bar{s}}\rangle \;=&(2\pi)^{s}\Gamma(k-s)\sum_{n\geq 1} n^{s-1}\frac{(k-2)!}{(4\pi n)^{k-1}}a_f(n) \\ =&(2\pi)^{s-k+1}2^{1-k}\Gamma(k-s)(k-2)!\sum_{n\geq 1} a_f(n)n^{s-k} \\ =&2^{2-k}\pi(k-2)!L^*(f,k-s) \qquad\qquad (using(1.3)) \\ =&(-1)^{\frac{k}{2}}\frac{\pi(k-2)!}{2^{k-2}}L^*(f,s) \qquad\qquad (using(1.4)) \\ =&c_kL^*(f,s).\end{aligned}$$

□

Lemma 3.2.

$$R_{k,s}(z) = \sum_{n\geq 1} r_{k,s}(n)e^{2\pi inz}$$

where

$$\begin{aligned} r_{k,s}(n) =&(2\pi)^s\Gamma(k-s)n^{s-1} + (-1)^{\frac{k}{2}}(2\pi)^{k-s}\Gamma(s)n^{k-s-1} \\ &+ \frac{1}{2}(-1)^{\frac{k}{2}}(2\pi)^k n^{k-1}\frac{\Gamma(s)\Gamma(k-s)}{\Gamma(k)} \\ &\times \sum_{\substack{(a,c)\in\mathbb{Z}, ac>0 \\ gcd(a,c)=1}} c^{-k}(\frac{c}{a})^s[e^{\frac{2\pi i n a'}{c}}e^{\frac{\pi i s}{2}}{}_1F_1(s,k;\frac{-2\pi i n}{ac}) \\ &+ e^{\frac{-2\pi i n a'}{c}}e^{\frac{-\pi i s}{2}}{}_1F_1(s,k;\frac{2\pi i n}{ac})]; \end{aligned} \tag{3.4}$$

with $a' \in \mathbb{Z}$ is an inverse of a modulo c, and ${}_1F_1(\alpha, \beta; z)$ is Kummer's degenerate hypergeometric function [6].

Proof. Suppose $ac = 0$ i.e. $a = 0$ or $c = 0$. To satisfy $ad - bc = 1$, the matrices in $SL_2(\mathbb{Z})$ will have the forms $\pm\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)$ or $\pm\left(\begin{smallmatrix} 0 & -1 \\ 1 & n \end{smallmatrix}\right)$. So by (3.1),

$$R_{k,s}(z) = 2\gamma_k(s)\sum_{n\in\mathbb{Z}}\left((z+n)^{-s} + e^{-\pi i s}(z+n)^{-k+s}\right).$$

Using Lipschitz's formula again (3.3), the n-th Fourier coefficient of the terms with $ac = 0$ is equal to

$$2\gamma_k(s)e^{-\pi i s/2}\left(\frac{(2\pi)^s}{\Gamma(s)}\sum_{n\geq 1}n^{s-1}e^{2\pi i n z} + (-1)^{\frac{k}{2}}\frac{(2\pi)^{k-s}}{\Gamma(k-s)}\sum_{n\geq 1}n^{k-s-1}e^{2\pi i n z}\right). \tag{3.5}$$

Now for $ac \neq 0$, the n-th Fourier coefficient of the sum, is given by:

$$I = \int_{iC}^{iC+1}\left(\sum_{\substack{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\in SL_2(\mathbb{Z}) \\ ac\neq 0}}(cz+d)^{-k}\left(\frac{az+b}{cz+d}\right)^{-s}\right)e^{-2\pi i n z}dz \quad C > 0 \tag{3.6}$$

We replace z by $z+m$ which is equivalent to multiply the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ by $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and we fix integers b_0 and d_0 for each pair (a,c) as before such that $ad_0 - cb_0 = 1$, we get

$$I = \int_{iC}^{iC+1} \left(\sum_{m\in\mathbb{Z}} \sum_{\substack{(a,c)\in\mathbb{Z}^2 \\ ac\neq 0 \\ gcd(a,c)=1}} (c(z+m)+d_0)^{-k} \left(\frac{a(z+m)+b_0}{c(z+m)+d_0}\right)^{-s} \right) e^{-2\pi inz} dz$$

$$= \sum_{\substack{(a,c)\in\mathbb{Z}^2 \\ ac\neq 0 \\ gcd(a,c)=1}} \int_{iC-\infty}^{iC+\infty} (cz+d_0)^{-k} \left(\frac{az+b_0}{cz+d_0}\right)^{-s} e^{-2\pi inz} dz.$$

We now substitute z by $z - \frac{d_0}{c}$ and we get

$$I = \sum_{\substack{(a,c)\in\mathbb{Z}^2 \\ ac\neq 0 \\ gcd(a,c)=1}} \int_{iC-\infty}^{iC+\infty} (cz)^{-k} \left(-\frac{1}{c^2 z} + \frac{a}{c}\right)^{-s} e^{-2\pi in(z-\frac{d_0}{c})} dz$$

$$= \sum_{\substack{(a,c)\in\mathbb{Z}^2 \\ ac\neq 0 \\ gcd(a,c)=1}} c^{-k} e^{2\pi in\frac{a'}{c}} \int_{iC-\infty}^{iC+\infty} z^{-k+s} z^{-s} \left(-\frac{1}{c^2 z} + \frac{a}{c}\right)^{-s} e^{-2\pi inz} dz \qquad (3.7)$$

where

$$a' \in \mathbb{Z}, a'a \equiv 1 (mod c).$$

Now we suppose that $ac > 0$, so that

$$z^{-s} \left(-\frac{1}{c^2 z} + \frac{a}{c}\right)^{-s} = \left(-\frac{1}{c^2} + \frac{a}{c} z\right)^{-s}$$

Let's call the integral in (3.7) I_1, and replace z by $\frac{c}{a}it$, we obtain

$$\begin{aligned}
I_1 =& \int_{C+i\infty}^{C-i\infty} \left[\left(\frac{c}{a}\right) it\right]^{-k+s} \left(-\frac{1}{c^2} + \frac{a}{z}.\frac{z}{a}it\right)^{-s} e^{-2\pi in\frac{c}{a}it} \left(\frac{c}{a}i\right) dt \\
=& \int_{C+i\infty}^{C-i\infty} \left(\frac{c}{a}\right)^{-k+s+1} t^{-k+s} i^{-k+s+1} \left(\frac{i^2}{c^2} + it\right)^{-s} e^{-2\pi in\frac{c}{a}it} dt \\
=& \int_{C+i\infty}^{C-i\infty} \left(\frac{c}{a}\right)^{-k+s+1} t^{-k+s} i^{-k+1} \left(\frac{i}{c^2} + t\right)^{-s} e^{-2\pi in\frac{c}{a}it} dt \\
=& (-1)^{\frac{k}{2}} 2\pi \left(\frac{c}{a}\right)^{-k+s+1} \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} t^{-k+s} \left(\frac{i}{c^2} + t\right)^{-s} e^{2\pi n\frac{c}{a}t} dt.
\end{aligned}$$

Now using

$$\frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} (t+\alpha)^{-\mu}(t+\beta)^{-\nu} e^{pt} dt = \frac{1}{\Gamma(\mu+\nu)} p^{\mu+\nu-1} e^{-\beta p} {}_1F_1(\mu, \mu+\nu; (\beta-\alpha)p),$$

cf.[6] we get

$$I_1 = (-1)^{\frac{k}{2}} \frac{(2\pi)^k}{\Gamma(k)} n^{k-1} \left(\frac{c}{a}\right)^s {}_1F_1(s, k; \frac{-2\pi in}{ac}).$$

Therefore, the contribution of the n-th Fourier coefficients by terms with $ac > 0$, of $R_{k,s}(z)$ is given by

$$\gamma_k(s)(-1)^{\frac{k}{2}} \frac{(2\pi)^k}{\Gamma(k)} n^{k-1} \sum_{\substack{(a,c)\in\mathbb{Z}^2 \\ ac>0 \\ gcd(a,c)=1}} c^{-k} \left(\frac{c}{a}\right)^s e^{2\pi in\frac{a'}{c}} {}_1F_1(s, k; \frac{-2\pi in}{ac}). \tag{3.8}$$

If $ac < 0$, we write

$$z^{-s}\left(-\frac{1}{c^2 z} + \frac{a}{c}\right)^{-s} = z^{-s}\left(-\left(\frac{1}{c^2 z} - \frac{a}{c}\right)\right)^{-s} = e^{-\pi is}\left(\frac{1}{c^2} - \frac{a}{c}z\right)^{-s}.$$

We can see that in the same way as before, with replacing (a, c) by $(-a, c)$, the contribution of the terms with $ac < 0$ is given by

$$\gamma_k(s)e^{-\pi is}(-1)^{\frac{k}{2}}\frac{(2\pi)^k}{\Gamma(k)}n^{k-1}\sum_{\substack{(a,c)\in\mathbb{Z}^2\\ ac>0\\ gcd(a,c)=1}} c^{-k}\left(\frac{c}{a}\right)^s e^{-2\pi in\frac{a'}{c}}{}_1F_1(s,k;\frac{2\pi in}{ac}). \quad (3.9)$$

Finally, combining (3.5), (3.8) and (3.9) will finish the proof. $\square$

By Theorem 1.4.5, we let $\{f_{k,1},...f_{k,g_k}\}$ be the basis of normalized Hecke eigenforms of S_k, where g_k is the dimension of S_k.

Theorem 3.1. *Let $t_0 \in \mathbb{R}$ and $\epsilon > 0$. Then there exist a constant $C(t_0,\epsilon) > 0$ depending only on t_0 and ϵ such that for $k > C(t_0,\epsilon)$ the function*

$$\sum_{\nu=1}^{g_k}\frac{1}{\langle f_{k,\nu},f_{k,\nu}\rangle}L^*(f_{k,\nu},s)$$

does not vanish at any point $s = \sigma+it$ with $t = t_0$, and $\frac{k-1}{2} < \sigma < \frac{k}{2}-\epsilon$, $\frac{k}{2}+\epsilon < \sigma < \frac{k+1}{2}$.

Proof. Since $\{f_{k,1},...f_{k,g_k}\}$ is the orthogonal basis, and $R_{k,\bar{s}} \in S_k$, we have

$$R_{k,\bar{s}} = \sum_{\nu=1}^{g_k}\frac{\langle f_{k,\nu},R_{k,\bar{s}}\rangle}{\langle f_{k,\nu},f_{k,\nu}\rangle}f_{k,\nu}.$$

Now using Lemma 3.1, we get

$$R_{k,\bar{s}} = c_k\sum_{\nu=1}^{g_k}\frac{L^*(f_{k,\nu},s)}{\langle f_{k,\nu},f_{k,\nu}\rangle}f_{k,\nu}. \quad (3.10)$$

By Lemma 3.2, and since $f_{k,\nu} \in S_k$, comparing the first Fourier coefficient of the two sides of (3.10) will lead us to

$$\begin{aligned}
&(2\pi)^s\Gamma(k-s) + (-1)^{\frac{k}{2}}(2\pi)^{k-s}\Gamma(s) + \frac{1}{2}(-1)^{\frac{k}{2}}(2\pi)^k \frac{\Gamma(s)\Gamma(k-s)}{\Gamma(k)} \\
&\quad \times \sum_{\substack{(a,c)\in\mathbb{Z}, ac>0 \\ gcd(a,c)=1}} c^{-k}(\frac{c}{a})^s (e^{\frac{2\pi i a'}{c}} e^{\frac{\pi i s}{2}} {}_1F_1(s,k;\frac{-2\pi i}{ac}) \\
&\quad + e^{\frac{-2\pi i a'}{c}} e^{\frac{-\pi i s}{2}} {}_1F_1(s,k;\frac{2\pi i}{ac}) \\
&\quad = c_k \sum_{\nu=1}^{g_k} \frac{L^*(f_{k,\nu},s)}{\langle f_{k,\nu}, f_{k,\nu}\rangle}
\end{aligned} \tag{3.11}$$

Dividing by $(2\pi)^s\Gamma(k-s)$ and letting ${}_1f_1(s,k;z)$ to be equal to $\frac{\Gamma(s)\Gamma(k-s)}{\Gamma(k)}{}_1F_1(s,k;z)$, we can write (3.11) as

$$\begin{aligned}
&1+(-1)^{\frac{k}{2}}(2\pi)^{k-2s}\frac{\Gamma(s)}{\Gamma(k-s)} + \frac{(-1)^{\frac{k}{2}}(2\pi)^{k-s}}{2\Gamma(k-s)} \sum_{\substack{(a,c)\in\mathbb{Z}, ac>0 \\ gcd(a,c)=1}} c^{-k}\left(\frac{c}{a}\right)^s \\
&\quad \times (e^{\frac{2\pi i a'}{c}} e^{\frac{\pi i s}{2}} {}_1f_1(s,k;\frac{-2\pi i}{ac}) + e^{\frac{-2\pi i a'}{c}} e^{\frac{-\pi i s}{2}} {}_1f_1(s,k;\frac{2\pi i}{ac}) \\
&\quad = c_k' \sum_{\nu=1}^{g_k} \frac{L^*(f_{k,\nu},s)}{\langle f_{k,\nu}, f_{k,\nu}\rangle}.
\end{aligned} \tag{3.12}$$

What we need to prove is that the right-hand side of (3.12) does not vanish at $s = \sigma + it$ with $t = t_0$, and $\frac{k-1}{2} < \sigma < \frac{k}{2} - \epsilon$, $\frac{k}{2} + \epsilon < \sigma < \frac{k+1}{2}$, k large enough.

Note that it will be enough to prove this on the left half of the critical strip only because of the functional equation (1.4).

So take $s = \frac{k}{2} - \delta - it_0$ where $\epsilon < \delta < 1/2$, so that $\frac{k-1}{2} < \sigma < \frac{k}{2} - \epsilon$ and suppose that the right-hand side of (3.12) vanishes at s, we then obtain

$$\begin{aligned}-1 =&(-1)^{\frac{k}{2}}(2\pi)^{2\delta+2it_0}\frac{\Gamma(\frac{k}{2}-\delta-it_0)}{\Gamma(\frac{k}{2}+\delta+it_0)}+\frac{(-1)^{\frac{k}{2}}(2\pi)^{\frac{k}{2}+\delta+it_0}}{2\Gamma(\frac{k}{2}+\delta+it_0)}\\ &\sum_{\substack{(a,c)\in\mathbb{Z},ac>0\\ gcd(a,c)=1}} c^{\frac{-k}{2}-\delta-it_0}a^{\frac{-k}{2}+\delta+it_0}(e^{\frac{2\pi ia'}{c}}e^{\frac{\pi i}{2}(\frac{k}{2}-\delta-it_0)}{}_1f_1(\frac{k}{2}-\delta-it_0,k;\frac{-2\pi i}{ac})\\ &+e^{\frac{-2\pi ia'}{c}}e^{\frac{-\pi i}{2}(\frac{k}{2}-\delta-it_0)}{}_1f_1(\frac{k}{2}-\delta-it_0,k;\frac{2\pi i}{ac})).\end{aligned} \tag{3.13}$$

We claim that

$$|{}_1f_1(\alpha,\beta;z)|\leq 1$$

for $Re(\alpha)>1, Re(\beta-\alpha)>1$, and $|z|=1$ because of its definition for $Re(\beta)>Re(\alpha)>0$: ${}_1f_1(\alpha,\beta;z)=\int_0^1 e^{zu}u^{\alpha-1}(1-u)^{\beta-\alpha-1}du$. cf.[6] □

We now take the absolute value of (3.13) and thus we get

$$\begin{aligned}1\leq&(2\pi)^{2\delta}\left|\frac{\Gamma(\frac{k}{2}-\delta-it_0)}{\Gamma(\frac{k}{2}+\delta+it_0)}\right|+\left|\frac{(2\pi)^{\frac{k}{2}+\delta}}{2\Gamma(\frac{k}{2}+\delta+it_0)}\right|\\ &\times\left(\sum_{\substack{(a,c)\in\mathbb{Z},ac>0\\ gcd(a,c)=1}} c^{\frac{-k}{2}-\delta}a^{\frac{-k}{2}+\delta}\right)(e^{\pi t_0/2}+e^{-\pi t_0/2})\\ &=(2\pi)^{2\delta}\left|\frac{\Gamma(\frac{k}{2}-\delta-it_0)}{\Gamma(\frac{k}{2}+\delta+it_0)}\right|+\left|\frac{(2\pi)^{\frac{k}{2}+\delta}}{2\Gamma(\frac{k}{2}+\delta+it_0)}\right|A.B(t_0)\end{aligned} \tag{3.14}$$

where $A>0$ is an absolute constant and $B(t_0)$ is a constant depending only on t_0. However,

$$\lim_{n\to\infty} n^{b-a}\frac{\Gamma(n+a)}{\Gamma(n+b)}=1$$

cf. e.g.[6], so that

$$\lim_{k\to\infty} \frac{\Gamma(\frac{k}{2}-\delta-it_0)}{\Gamma(\frac{k}{2}+\delta+it_0)} = \frac{1}{\lim_{k\to\infty}(k/2)^{2\delta+2it_0}} = 0.$$

Also, clearly we have

$$\lim_{k\to\infty} \frac{(2\pi)^{\frac{k}{2}+\delta}}{2\Gamma(\frac{k}{2}+\delta+it_0)} = 0.$$

Finally we get $1 \leq 0$ for k large enough. Contradiction! This proves the theorem.

Corollary 3.1. Let $t_0 \in \mathbb{R}$ and $\epsilon > 0$. Then for $k > C(t_0, \epsilon)$ and any $s = \sigma + it$ with $t = t_0$, $(k-1)/2 < \sigma < (k/2) - \epsilon$, $(k/2) + \epsilon < \sigma < (k+1)/2$, there exists a Hecke eigenform $f \in S_k$ such that $L^*(f, s) \neq 0$.

Proof. By the previous theorem, any function $f \in S_k$, such that f is running over a basis of normalized Hecke eigenforms of weight k, will satisfy this condition. $\square$

Chapter 4

Non-Vanishing of L-functions for Half-Integer Weight Modular Forms

In this chapter, we prove a similar result as in chapter 3, for modular forms of half-integral weight in the plus space. So given a real number t_0 and a positive real number ϵ, we will prove that, for all k large enough, the average of the functions $L^*(f,s)$ with f running over a basis of Hecke eigenforms of weight $k+1/2$, does not vanish on the line segments $Im(s) = t_0$, $k/2-1/4 < Re(s) < k/2+3/4$.

For this purpose we need to determine the Fourier coefficients of the projected kernel function on the plus space. The kernel L-function of the map $f \to L^*(f,s)$ in the case of half integral weight on $\Gamma_0(4)$ is given in [7] by:

$$\begin{aligned} R_{s,k}(z) &= \gamma_k(s) \sum_A z^{-s}|_{k+1/2} A^* \\ &= \gamma_k(s) \sum_A \left(\frac{c}{d}\right) \left(\frac{-4}{d}\right) (cz+d)^{-(k+1/2)} \left(\frac{az+b}{cz+d}\right)^{-s} \end{aligned}$$

where

$$\gamma_k(s) = \frac{1}{2} e^{\pi i s/2} \Gamma(s) \Gamma(k+1/2-s)$$

and the sum runs over all matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$ with $A^* = (A, j(A,z))$. Now since we are working on the plus space, we need to define the kernel function acting on this space. To do so, we need to use the projection operator pr as given in [7]: for $g \in S_k(\Gamma_0(4))$, we have

$$g|pr = (-1)^{(k+1/2)/2} \frac{1}{3\sqrt{2}} \left(\sum_{\nu \ mod \ 4} g|\zeta A_\nu^* \right) + \frac{1}{3} g,$$

where

$$\eta = \left(\begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}, e^{(2k+1)\pi i/4} \right), \quad A_\nu^* = \left(\begin{pmatrix} 1 & 0 \\ 4\nu & 1 \end{pmatrix}, (4\nu z+1)^{-(k+1/2)} \right).$$

Lemma 4.1.

$$\langle\, f, R_{\bar{s},k}|pr\rangle \; = c_k L^*(f|W_4, s) \qquad\qquad \forall f \in S_{k+1/2}(4)$$

where

$$c_k := \frac{\pi\Gamma(k-1/2)}{i_4 2^{k-3/2}}.$$

Proof. We've seen in [7] that:

$$\langle\, f, R_{\bar{s},k}\rangle \; = \frac{\pi\Gamma(k-1/2)}{i_4 2^{k-3/2}} L^*(f, k+1/2-s)$$

However, since our projection is hermitian [7], we have for $f \in S^+_{k+1/2}(4)$,

$$\langle\, f, R_{\bar{s},k}\rangle \; = \langle\, f|pr, R_{\bar{s},k}\rangle \; = \langle\, f, R_{\bar{s},k}|pr\rangle$$

so that

$$\langle\, f, R_{\bar{s},k}|pr\rangle \; = \frac{\pi\Gamma(k-1/2)}{i_4 2^{k-3/2}} L^*(f, k+1/2-s).$$

Finally, using Proposition 2.2.1 will finish our proof.

□

The Fourier expansion of the projected function $g|pr$ for $g \in \Gamma_0(4N)$ is given in [8] as follows:

Proposition 4.0.1. *We set* $\alpha = \left(\frac{-4}{N}\right)$,

$$\eta^{(-\alpha/N)} = \left(\begin{pmatrix} 1 & 0 \\ -\alpha N & 1 \end{pmatrix}, (-\alpha N z + 1)^{(k+1/2)} \right)$$

and

$$\eta^{(1/2N)} = \left(\begin{pmatrix} 1 & 0 \\ 2N & 1 \end{pmatrix}, (2Nz+1)^{(k+1/2)} \right).$$

Let $g \in S_{k+1/2}(4N)$ and write

$$g(z) = \sum_{n\geq 1} a(n)q^n,$$

$$g|\eta^{(-\alpha/N)}(z) = \sum_{n\geq 1} a^{(-\alpha/N)}(n)q^{n/4}$$

$$g|\eta^{(1/2N)}(z) = \sum_{n\geq 1, (-1)^k n\equiv 1(4)} a^{(1/2N)}(n)q^{n/4}$$

where $q = e^{2\pi iz}$. Then

$$\begin{aligned} g|pr =& \frac{2}{3} \sum_{n\geq 1, n\equiv 0(4)} \left(a(n) + (1-(-1)^k i)2^{2k-1} i^{n/4} a^{(-\alpha/N)}(n/4)\right) q^n \\ &+ \frac{2}{3} \sum_{n\geq 1, (-1)^k n\equiv 1(4)} \left(a(n) + 2^{k-1} \left(\frac{(-1)^k n}{2}\right) a^{1/2N}(n)\right) q^n. \end{aligned}$$

Proof. [8] □

The next Lemma finds explicitly the Fourier coefficients of $R_{s,k}|pr$ at different cusps:

Lemma 4.2.

$$\begin{aligned} R_{s,k}|pr =& \frac{2}{3} \sum_{n\geq 1, n\equiv 0(4)} \left(a_s(n) + (1-(-1)^k i)2^{2k-1} i^{n/4} a_s^1(n/4)\right) q^n \\ &+ \frac{2}{3} \sum_{n\geq 1, (-1)^k n\equiv 1(4)} \left(a_s(n) + 2^{k-1} \left(\frac{(-1)^k n}{2}\right) a_s^{1/2}(n)\right) q^n, \end{aligned}$$

where

$$a_s(n) = (2\pi)^s\Gamma(k+1/2-s)n^{s-1} + \frac{1}{2}(2\pi i)^{k+1/2}n^{k-1/2}\frac{\Gamma(s)\Gamma(k+1/2-s)}{\Gamma(k+1/2)}$$
$$\times \sum_{\substack{ac>0\\(a,c)=1,4|c}} \left(\frac{c}{a}\right)\left(\frac{-4}{a}\right)^{k+1/2} c^{-(k+1/2)}\left(\frac{c}{a}\right)^s$$
$$\left(e^{2\pi i n a'/c}e^{\pi i s/2}{}_1F_1(s,k+1/2;\frac{-2\pi i n}{ac}) + e^{\frac{-2\pi i n a'}{c}}e^{\frac{-\pi i s}{2}}{}_1F_1(s,k+1/2;\frac{2\pi i n}{ac})\right),$$

$$a_s^1(n) = 2\times 4^s\Gamma(k+1/2-s)n^{s-1} + 4^s(2\pi i)^{k+1/2}n^{k-1/2}\frac{\Gamma(s)\Gamma(k+1/2-s)}{\Gamma(k+1/2)}$$
$$\times \sum_{\substack{c\geq 1, c\equiv 1 \ mod\ 2\\ d(c)*, ad-bc=1\\ d\equiv -c\ mod\ 4}} \left(\frac{4c}{d}\right)\left(\frac{-4}{-c}\right)^{k+1/2} c^{-(k+1/2)}\left(\frac{c}{a}\right)^s$$
$$\left(e^{2\pi i n d/c}e^{\pi i s/2}{}_1F_1(s,k+1/2;\frac{-2\pi i n}{ac}) + e^{\frac{-2\pi i n d}{c}}e^{\frac{-\pi i s}{2}}{}_1F_1(s,k+1/2;\frac{2\pi i n}{ac})\right),$$

and

$$a_s^{1/2}(n) = 2\times 4^s\Gamma(k+1/2-s)n^{s-1} + 4^s(2\pi i)^{k+1/2}n^{k-1/2}\frac{\Gamma(s)\Gamma(k+1/2-s)}{\Gamma(k+1/2)}$$
$$\times \sum_{\substack{c\geq 1, c\equiv 1 \ mod\ 2\\ d(c)*, ad-2bc=1\\ d\equiv -c\ mod\ 4}} \left(\frac{4c}{d}\right)\left(\frac{-4}{-c}\right)^{k+1/2} c^{-(k+1/2)}\left(\frac{c}{a}\right)^s$$
$$\left(e^{2\pi i n d/c}e^{\pi i s/2}{}_1F_1(s,k+1/2;\frac{-2\pi i n}{ac}) + e^{\frac{-2\pi i n d}{c}}e^{\frac{-\pi i s}{2}}{}_1F_1(s,k+1/2;\frac{2\pi i n}{ac})\right).$$

Proof. As for the expansion at the infinite cusp, we consider first the case where $ac = 0$. The matrices in $\Gamma_0(4)$ will have the form $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ so that

$$R_{s,k}(z) = \gamma_k(s)\sum_{n\in\mathbf{Z}}(z+n)^{-s}.$$

Now using (3.3) again, we get

$$R_{s,k}(z) = (2\pi)^s \Gamma(k+1/2-s) \sum_{n\geq 1} n^{s-1} e^{2\pi i n z}.$$

Now for the case when $ac \neq 0$, the nth Fourier coefficient of the sum is given by

$$\int_{iC}^{iC+1} \left(\sum_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4); ac\neq 0} \left(\frac{c}{d}\right) \left(\frac{-4}{d}\right)^{k+1/2} (cz+d)^{-(-k+1/2)} \left(\frac{az+b}{cz+d}\right)^{-s} \right) e^{-2\pi i n z} dz \quad C > 0$$

Note that this integral is equivalent to (3.6) with minor change due to the space we are working on; so following similar steps as in chapter 3 will lead us to

$$R_{s,k}(z) = \sum_{n\geq 1} a_s(n) e^{2\pi i n z}$$

where, for a' is an inverse of a modulo c and ${}_1F_1(\alpha, \beta; z)$ is the Kummer's degenerate hypergeometric function

$$\begin{aligned} a_s(n) =& (2\pi)^s \Gamma(k+1/2-s) n^{s-1} + \frac{1}{2} (2\pi i)^{k+1/2} n^{k-1/2} \frac{\Gamma(s)\Gamma(k+1/2-s)}{\Gamma(k+1/2)} \\ & \times \sum_{\substack{ac>0 \\ (a,c)=1, 4|c}} \left(\frac{c}{a}\right) \left(\frac{-4}{a}\right)^{k+1/2} c^{-(k+1/2)} \left(\frac{c}{a}\right)^s \\ & \left(e^{2\pi i n a'/c} e^{\pi i s/2} {}_1F_1(s, k+1/2; \frac{-2\pi i n}{ac}) + e^{\frac{-2\pi i n a'}{c}} e^{\frac{-\pi i s}{2}} {}_1F_1(s, k+1/2; \frac{2\pi i n}{ac}) \right). \end{aligned}$$

Now we have to determine the expansion at the cusp 1 of

$$R_{s,k}|\eta^1(z) = \sum_{n\geq 1} a_s^1(n) e^{2\pi i n z/4}$$

where $\eta^1(z) = \left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, (z+1)^{(k+1/2)} \right)$.

We have

$$\begin{aligned}
R_{s,k}|\eta^1(z) &= (z+1)^{-(k+1/2)} R_{s,k}\left(\frac{z}{z+1}\right) \\
&= \gamma_k(s)(z+1)^{-(k+1/2)} \sum_{n\in\mathbb{Z}} \left(\frac{z}{z+1}+n\right)^{-s} \\
&+ 2\gamma_k(s) \sum_{\substack{c>0,d \\ (c,d)=1, ad-bc=1 \\ c\equiv 0 \ mod \ 4}} \left(\frac{4}{d}\right)\left(\frac{c+d}{d}\right)\left(\frac{-4}{d}\right)^{k+1/2} \\
&\times ((c+d)z+d)^{-(k+1/2)} \left(\frac{(a+b)z+b}{(c+d)z+d}\right)^{-s} \\
&= 2\gamma_k(s) \sum_{\substack{c>0,d \\ (c,d)=1, ad-bc=1 \\ c\equiv d \ mod \ 4}} \left(\frac{4c}{d}\right)\left(\frac{-4}{d}\right)^{k+1/2} (cz+d)^{-(k+1/2)} \left(\frac{az+b}{cz+d}\right)^{-s}
\end{aligned} \tag{4.1}$$

where in the last equality we have replaced $a+b$ by a and $c+d$ by c. Thus we get

$$\begin{aligned}
R_{s,k}|\eta^1(z) = 2 \times 4^s \gamma_k(s) &\sum_{c\geq 1, c\equiv 1 \ mod \ 2} \left(\frac{-4}{-c}\right)^{k+1/2} \sum_{\substack{d(c)^*, ad-bc=1 \\ d\equiv -c \ mod \ 4}} \left(\frac{4c}{d}\right) \\
&\times (4c(z/4+r)+d)^{-(k+1/2)} \left(\frac{a(z/4+r)+b}{c(z/4+r)+d}\right)^{-s},
\end{aligned}$$

where $d(c)^*$ means that d runs through a primitive residue system modulo c. Now, we can determine the expansion at the cusp 1 as we we did for the case at infinity to get

$$\begin{aligned}
a_s^1(n) =&2 \times 4^s\Gamma(k+1/2-s)n^{s-1} + 4^s(2\pi i)^{k+1/2}n^{k-1/2}\frac{\Gamma(s)\Gamma(k+1/2-s)}{\Gamma(k+1/2)} \\
&\times \sum_{\substack{c\geq 1, c\equiv 1 \ mod\ 2 \\ d(c)*, ad-bc=1 \\ d\equiv -c \ mod\ 4}} \left(\frac{4c}{d}\right)\left(\frac{-4}{-c}\right)^{k+1/2} c^{-(k+1/2)}\left(\frac{c}{a}\right)^s \\
&\left(e^{2\pi ind/c}e^{\pi is/2}{}_1F_1(s,k+1/2;\frac{-2\pi in}{ac}) + e^{\frac{-2\pi ind}{c}}e^{\frac{-\pi is}{2}}{}_1F_1(s,k+1/2;\frac{2\pi in}{ac})\right).
\end{aligned}$$

And finally, for the expansion at the cusp 1/2, we have

$$R_{s,k}|\eta^{1/2}(z) = \sum_{n\geq 1, (-1)^k n\equiv 1(4)} a_s^{(1/2N)}(n)e^{2\pi inz/4}$$

where $\eta^{1/2}(z) = \left(\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, (2z+1)^{(k+1/2)}\right)$. Following the same steps exactly as for the case of the cusp 1, we get

$$\begin{aligned}
a_s^{1/2}(n) =&2 \times 4^s\Gamma(k+1/2-s)n^{s-1} + 4^s(2\pi i)^{k+1/2}n^{k-1/2}\frac{\Gamma(s)\Gamma(k+1/2-s)}{\Gamma(k+1/2)} \\
&\times \sum_{\substack{c\geq 1, c\equiv 1 \ mod\ 2 \\ d(c)*, ad-2bc=1 \\ d\equiv -c \ mod\ 4}} \left(\frac{4c}{d}\right)\left(\frac{-4}{-c}\right)^{k+1/2} c^{-(k+1/2)}\left(\frac{c}{a}\right)^s \\
&\left(e^{2\pi ind/c}e^{\pi is/2}{}_1F_1(s,k+1/2;\frac{-2\pi in}{ac}) + e^{\frac{-2\pi ind}{c}}e^{\frac{-\pi is}{2}}{}_1F_1(s,k+1/2;\frac{2\pi in}{ac})\right).
\end{aligned}$$

□

By Theorem 2.3.1, we let $\{f_{k,1}, ...f_{k,gk}\}$ be the basis of Hecke eigenforms of $S^+_{k+1/2}(4)$, where g_k is the dimension of $S^+_{k+1/2}(4)$.

Theorem 4.1. *Let $t_0 \in \mathbb{R}$ and $\epsilon > 0$. Then there exist a constant $C(t_0, \epsilon) > 0$ depending only on t_0 and ϵ such that for $k > C(t_0, \epsilon)$ and $k \in 2\mathbb{Z}$, the function*

$$\sum_{\nu=1}^{g_k} \frac{1}{\langle f_{k,\nu}, f_{k,\nu} \rangle} L^*(f_{k,\nu}|W_4, s)$$

does not vanish at any point $s = \sigma + it$ with $t = t_0$, and $\frac{k}{2} - \frac{1}{4} < \sigma < \frac{k}{2} + \frac{3}{4}$.

Proof. Since $\{f_{k,1}, ...f_{k,g_k}\}$ is the basis, and $R_{\bar{s},k}|pr \in S^+_{k+1/2}(4)$, we have

$$R_{\bar{s},k}|pr = \sum_{\nu=1}^{g_k} \frac{\langle f_{k,\nu}, R_{\bar{s},k}|pr \rangle}{\langle f_{k,\nu}, f_{k,\nu} \rangle} L^*(f_{k,\nu}, s).$$

Now using Lemma 4.1, we get

$$R_{\bar{s},k}|pr = c_k \sum_{\nu=1}^{g_k} \frac{L^*(f_{k,\nu}|W_4, s)}{\langle f_{k,\nu}, f_{k,\nu} \rangle} f_{k,\nu} \tag{4.2}$$

where c_k is a constant. By Lemma 4.2, and since $f_{k,\mu}$ is a cusp form, comparing the first Fourier coefficient of the two sides of will lead us to

$$\frac{2}{3}\left(a_s(1) + 2^{k-1}\left(\frac{(-1)^k n}{2}\right) a_s^{1/2}(1)\right) = c'_k \sum_{\nu=1}^{g_k} \frac{L^*(f_{k,\nu}|W_4, s)}{\langle f_{k,\nu}, f_{k,\nu} \rangle}.$$

Now we divide by $\frac{2}{3} 4^s (2\pi)^s \Gamma(k + \frac{1}{2} - s)$ and let ${}_1f_1(\alpha, \beta; z)$ be equal to

$\frac{\Gamma(\alpha)\Gamma(\beta-\alpha)}{\Gamma(\beta)}\,{}_1F_1(\alpha,\beta;z)$ as before, we get

$$4^{-s}+\frac{(2\pi i)^{k+1/2-s}}{2\times 4^s\Gamma(k+1/2-s)}\sum_{\substack{ac>0\\(a,c)=1,4|c}}\left(\frac{c}{a}\right)\left(\frac{-4}{a}\right)^{k+1/2}c^{-(k+1/2)}\left(\frac{c}{a}\right)^{s}$$

$$\left(e^{2\pi ina'/c}e^{\pi is/2}{}_1F_1(s,k+1/2;\frac{-2\pi in}{ac})+e^{\frac{-2\pi ina'}{c}}e^{\frac{-\pi is}{2}}{}_1F_1(s,k+1/2;\frac{2\pi in}{ac})\right)$$

$$+2^{k-1}(2\pi)^s(-1)^k+\frac{(2\pi)^{k+1/2-s}i^{k+1/2}}{\Gamma(k+1/2-s)}\sum_{\substack{c\geq 1,c\equiv 1\ mod\ 2\\ d(c)*,ad-2bc=1\\ d\equiv -c\ mod\ 4}}\left(\frac{4c}{d}\right)\left(\frac{-4}{-c}\right)^{k+1/2}c^{-(k+1/2)}\left(\frac{c}{a}\right)^{s}$$

$$\left(e^{2\pi ind/c}e^{\pi is/2}{}_1F_1(s,k+1/2;\frac{-2\pi in}{ac})+e^{\frac{-2\pi ind}{c}}e^{\frac{-\pi is}{2}}{}_1F_1(s,k+1/2;\frac{2\pi in}{ac})\right)$$

$$=c_k'\sum_{\nu=1}^{g_k}\frac{L^*(f_{k,\nu}|W_4,s)}{\langle f_{k,\nu},f_{k,\nu}\rangle}. \tag{4.3}$$

Suppose the right-hand side of (4.3) vanishes at $s=k/2+1/4-\delta+it_0$, where $0\leq\delta\leq 1/2$; thus taking the absolute value on both sides will give us

$$2^{k-1}-4^{-k/2-1/4+\delta}\leq|2^{k-1}+4^{-s}|$$

$$\leq\frac{(2\pi)^{k/2+1/4+\delta}}{2\times 4^{k/2+1/4-\delta}|\Gamma(\frac{k}{2}+\frac{1}{4}+\delta-it_0|}\sum_{\substack{ac>0\\(a,c)=1,4|c}}\frac{1}{c^{k+1/2}}(e^{-\pi t_0/2}+e^{\pi t_0/2})$$

$$+\frac{(2\pi)^{k/2+1/4+\delta}}{|\Gamma(\frac{k}{2}+\frac{1}{4}+\delta-it_0|}\sum_{\substack{c\geq 1,c\equiv 1\ mod\ 2\\ d(c)^*,ad-2bc=1\\ d\equiv -c\ mod\ 4}}\frac{1}{c^{k+1/2}}(e^{-\pi t_0/2}+e^{\pi t_0/2})$$

where as is Chapter 3, we have used that

$$|{}_1f_1(\alpha,\beta;z)|\leq 1.$$

Now if we tend k to ∞ on both sides and since

$$\lim_{k\to\infty} \frac{(2\pi)^{k/2+1/4+\delta}}{\alpha^k|\Gamma(\frac{k}{2}+\frac{1}{4}+\delta-it_0|} = 0$$

for $\alpha \in \mathbb{Z}$, then we get

$$\lim_{k\to\infty} 2^{k-1} = +\infty < 0$$

Contradiction! This proves the theorem. $\square$

Corollary 4.1. Let $t_0 \in \mathbb{R}$ and $\epsilon > 0$. Then for $k > C(t_0, \epsilon)$ and any $s = \sigma + it$ with $t = t_0$, $k/2 - 1/4 < Re(s) < k/2 + 3/4$, there exists a Hecke eigenform $f \in S^+_{k+1/2}(4)$ whose L-values do not vanish at s.

Proof. By the previous theorem, any function $f \in S^+_{k+1/2}(4)$, such that f is running over a basis of normalized Hecke eigenforms of weight $k + 1/2$, will satisfy this condition. $\square$

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